Chapter 13
Functions of an Operator

One of the most fruitful applications of the aforementioned analogy between operators and numbers is encountered in the study of differential equations. As it turns out, the solution of the equation $y' = Ay$ can be written in the form $y = e^{At} y_0$ not only for scalar-valued functions and a numerical parameter A, but also for vector-valued functions and an operator A, respectively. The apparatus of functions of an operator was created precisely to enable the free use of such analogies.

13.1 Continuous Functions of an Operator

13.1.1 Polynomials in an Operator

In this subsection we consider operators in an arbitrary complex Banach space X.

Definition 1. Given a polynomial $p = a_0 + a_1 t + \cdots + a_n t^n$ and an operator $T \in L(X)$, an operator of the form $p(T) = a_0 I + a_1 T + \cdots + a_n T^n$ is called a *polynomial in the operator* T.

Let us list some readily verifiable properties of polynomials in operators.

Theorem 1. *Let p_1, p_2 be polynomials, $T \in L(X)$, and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then*

(i) $(\lambda_1 p_1 + \lambda_2 p_2)(T) = \lambda_1 p_1(T) + \lambda_2 p_2(T)$;

(ii) $(p_1 p_2)(T) = p_1(T)\, p_2(T)$.

Further,
(iii) *suppose the operators $T_1, T_2 \in L(X)$ commute, and p_1, p_2 are polynomials. Then the operators $p_1(T_1)$ and $p_2(T_2)$ also commute.* □

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Theorem 2. *The operator $p(T)$ is invertible if and only if the polynomial p does not vanish at any of the points of the spectrum of the operator T.*

Proof. Let $t_1, \dots, t_n$ be the roots of the polynomial p, i.e., $p(t) = a_n(t - t_1) \cdots (t - t_n)$. Then $p(T) = a_n(T - t_1 I) \cdots (T - t_n I)$. By Lemma 1 of Subsection 11.1.2, the invertibility of a product of commuting operators is equivalent to the simultaneous invertibility of its factors. Therefore, the invertibility of the operator $p(T)$ is equivalent to the simultaneous invertibility of the factors $T - t_i I$, i.e., to the fact that none of the roots t_i of the polynomial p lie in the spectrum of the operator T. □

Theorem 3 (Spectral mapping theorem for polynomials in an operator). *The spectrum of the polynomial $p(T)$ consists of the values of the polynomial in the points of the spectrum of the operator T, i.e., $\sigma(p(T)) = p(\sigma(T))$.*

Proof. Let us show that $\lambda \in \sigma(p(T))$ if and only $\lambda \in p(\sigma(T))$. Indeed, the condition $\lambda \in \sigma(p(T))$ means that the operator $p(T) - \lambda I = (p - \lambda)(T)$ is not invertible. By the preceding theorem, this is equivalent to the polynomial $p - \lambda$ vanishing at some point of the spectrum: there exists a $t \in \sigma(T)$ such that $p(t) = \lambda$. This in turn is equivalent to the requisite condition $\lambda \in p(\sigma(T))$. □

Exercises

1. Let p_1, p_2 be a pair of coprime polynomials and assume that $p_1 p_2(T) = 0$. Prove that the whole space X decomposes into the direct sum of its subspaces $X_1 = \operatorname{Ker} p_1(T)$ and $X_2 = \operatorname{Ker} p_2(T)$.

2. By analogy with calculus, introduce the concepts of derivative and differentiability for functions $f: [0, 1] \to E$, where E is a Banach space. Verify for differentiable functions $f, g: [0, 1] \to E$, that $(f + g)' = f' + g'$.

3. Let $f: [0, 1] \to L(X)$ be a differentiable function. Prove that $\frac{d}{dt}\left[f^2(t)\right] = f'(t)f(t) + f(t)f'(t)$.

4. Prove that if all the values of a function $f: [0, 1] \to L(X)$ pairwise commute, then the values of f and f' also commute.

5. For any operator $A \in L(X)$, define e^A by the formula

$$e^A = 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots .$$

Is it true that if $f: [0, 1] \to L(X)$ is a differentiable function, then the function $y = e^{f(t)}$ is a solution of the differential equation $y' = f'(t)y$? Why is the particular case $y = e^{tA}$ of this formula successfully used for equations $y' = Ay$ with constant coefficients $A \in L(X)$?

6. Suppose that in some basis the matrix of the operator $A \in L(X)$ is diagonal. What will the matrix of the operator $p(A)$, where p is a polynomial, look like in this basis? How about the matrix of the operator e^A?

7. By analogy with the above, define polynomials in the elements of a Banach algebra $\mathbf{A}$. Prove that all properties of polynomials in an operator considered above carry over to polynomials in elements of a Banach algebra.

The reader interested in the theory of functions of elements of a Banach algebra in the general case is referred to W. Rudin's textbook [38].

13.1.2 Polynomials in a Self-adjoint Operator

From here on till the end of the chapter we will consider only operators in a Hilbert space.

Lemma 1. *Let $A, B \in L(H)$ be commuting self-adjoint operators. Then $\|A + iB\| = \sqrt{\|A^2 + B^2\|}$.*

Proof. Since A and B commute, their product is a self-adjoint operator. Hence, $\langle Ax, Bx\rangle = \langle BAx, x\rangle$ is a real number for all $x \in H$. Therefore,

$$\|(A + iB)x\|^2 = \|Ax\|^2 + 2\,\mathrm{Re}\,(-i)\langle Ax, Bx\rangle + \|Bx\|^2 = \|Ax\|^2 + \|Bx\|^2,$$

and so

$$\|A + iB\|^2 = \sup_{x\in S_H} \|(A + iB)x\|^2 = \sup_{x\in S_H} (\|Ax\|^2 + \|Bx\|^2)$$

$$= \sup_{x\in S_H} (\langle Ax, Ax\rangle + \langle Bx, Bx\rangle) = \sup_{x\in S_H} \langle (A^2 + B^2)x, x\rangle = \left\|A^2 + B^2\right\|. \qquad \square$$

Theorem 1. *Let $A \in L(H)$ be a self-adjoint operator and $p = a_0 + a_1 t + \cdots + a_n t^n$ be a polynomial. Then the operator $p(A)$ has the following properties:*

(i) $(p(A))^* = \overline{p}(A)$, *where $\overline{p} = \overline{a}_0 + \overline{a}_1 t + \cdots + \overline{a}_n t^n$. In particular, if all the coefficients of p are real, then $p(A)$ is a self-adjoint operator.*

(ii) $\|p(A)\| = \sup_{t\in\sigma(A)} |p(t)|$.

Proof.
(i) $(p(A))^* = \overline{a}_0(I)^* + \overline{a}_1(A)^* + \cdots + \overline{a}_n(A^n)^* = \overline{p}(A)$.

(ii) Consider first the case of a polynomial with real coefficients. By Corollary 2 in Subsection 12.4.5 and the spectral mapping theorem for polynomials in an operator (Theorem 3 of Subsection 13.1.1),

$$\|p(A)\| = \sup_{\tau\in\sigma(p(A))} |\tau| = \sup_{\tau\in p(\sigma(A))} |\tau|.$$

To obtain the required formula, it remains to define $\tau = p(t)$ and observe that as t runs through $\sigma(A)$, τ runs through $p(\sigma(A))$.

Now suppose that the coefficients of the polynomial p have the form $a_j = u_j + iv_j$, $u_j, v_j \in \mathbb{R}$. Put $p_1 = u_0 + u_1 t + \cdots + u_n t^n$ and $p_2 = v_0 + v_1 t + \cdots + v_n t^n$. Using the lemma and the case of real polynomials treated above, we have

$$\|p(A)\| = \|p_1(A) + ip_2(A)\| = \sqrt{\|p_1(A)^2 + p_2(A)^2\|}$$

$$= \sqrt{\|(p_1^2 + p_1^2)(A)\|} = \sqrt{\sup_{t\in\sigma(A)} |(p_1^2 + p_1^2)(t)|} = \sup_{t\in\sigma(A)} |p(t)|. \qquad \square$$

Exercises

1. Give an example of a pair of self-adjoint operators $A, B \in L(H)$, for which $\|A + iB\| \neq \sqrt{\|A^2 + B^2\|}$.

2. Give an example of a pair of commuting self-adjoint operators $A, B \in L(H)$, for which $\|A + iB\| \neq \sqrt{\|A\|^2 + \|B\|^2}$.

3. Let $A \in L(H)$ be a self-adjoint operator and p_1, p_2 be polynomials such that $p_1(t) = p_2(t)$ for all $t \in \sigma(A)$. Then $p_1(A) = p_2(A)$.

13.1.3 Definition of a Continuous Function of a Self-adjoint Operator

Lemma 1. *Let $K \subset \mathbb{R}$ be a compact subset, and let $[a, b]$ be the smallest interval containing K. Then every function $f \in C(K)$ can be extended to a continuous function on $[a, b]$.*

Proof. The set $[a, b] \setminus K$ can be written as a union of open intervals with endpoints in K. Now redefine the function f on each such interval $(c, d) \subset [a, b] \setminus K$ by linear interpolation: $f(t) = f(c) + (t - c)\dfrac{f(d) - f(c)}{d - c}$. $\square$

Lemma 2. *Let $K \subset \mathbb{R}$ be a compact subset. Then for any function $f \in C(K)$ there exists a sequence of polynomials (p_n) which converges to f uniformly on K.*

Proof. Let $[a, b]$ be the smallest interval containing K. Then by the preceding lemma, we may assume that f is defined on the whole interval $[a, b]$. By the Weierstrass theorem, there exists a sequence of polynomials (p_n) which converges uniformly to f on $[a, b]$. This sequence (p_n) will also converge to f on K, a subset of $[a, b]$. $\square$

Lemma 3. (a) *Let A be a self-adjoint operator, and let (p_n) be a sequence of polynomials which converges uniformly on $\sigma(A)$. Then the sequence of operators $p_n(A)$ converges in norm.*

(b) *If the sequences of polynomials (p_n) and (q_n) converge uniformly on $\sigma(A)$ to one and the same limit, then $p_n(A)$ and $q_n(A)$ also converge to one and the same limit.*

Proof. We use assertion (ii) of the theorem proved in the preceding subsection:

$$\|p_n(A) - p_m(A)\| = \sup_{t \in \sigma(A)} |(p_n - p_m)(t)| \to 0 \quad \text{as } n, m \to \infty.$$

Since the space of operators is complete, this proves assertion (a). Assertion (b) is proved in exactly the same way:

$$\|p_n(A) - q_n(A)\| = \sup_{t \in \sigma(A)} |(p_n - q_n)(t)| \to 0 \quad \text{as } n \to \infty. \qquad \square$$

Definition 1. Let A be a self-adjoint operator, and $f \in C(\sigma(A))$ be a continuous function given on the spectrum of the operator A. The *function f of the operator A* is defined as

$$f(A) = \lim_{n \to \infty} p_n(A),$$

where (p_n) is an arbitrary sequence of polynomials that converges uniformly on $\sigma(A)$ to f.

The relevance of this definition is justified by Lemmas 2 and 3 proved above.

Exercises

1. Deduce Lemma 1 from Tietze's extension theorem (Theorem 3 in Subsection 1.2.3).

2. Consider in $C(\sigma(A))$ the subspace $\mathcal{P}$ consisting of all polynomials. Define the operator $U : \mathcal{P} \to L(H)$ by the formula $U(p) = p(A)$. Verify that U is a continuous linear operator. What is the norm of U equal to?

3. Applying the theorem of extension by continuity (Subsection 6.5.1) to the operator U, extend it to the whole space $C(\sigma(A))$. Verify that the equality $U(p) = p(A)$ holds not only for polynomials, but also for arbitrary continuous functions.[1]

[1] We could have used the extension of the operator U to $C(\sigma(A))$ and *defined* continuous functions of the operator A by the equality $f(A) = U(f)$. However, such a definition would be unnecessarily abstract and require additional interpretation.

13.1.4 Properties of Continuous Functions of a Self-adjoint Operator

First we will present properties that are obtained by direct passage to the limit from polynomials to continuous functions of a self-adjoint operator.

Theorem 1. *Let A be a self-adjoint operator, $f_1, f_2 \in C(\sigma(A))$, and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then*

(1) $(\lambda_1 f_1 + \lambda_2 f_2)(A) = \lambda_1 f_1(A) + \lambda_2 f_2(A)$, *and*

(2) $(f_1 f_2)(A) = f_1(A) f_2(A)$.

Further, let $f \in C(\sigma(A))$. Then

(3) $(f(A))^* = \overline{f}(A)$. *In particular, if the function f takes only real values on $\sigma(A)$, then $f(A)$ is a self-adjoint operator.*

(4) $\|f(A)\| = \sup_{t\in\sigma(A)} |f(t)|$.

Finally,

(5) *suppose the operators A and B commute, and let f and g be continuous functions on the spectra of the operators A and B, respectively. Then $f(A)$ and $g(B)$ also commute.* □

The following property already needs justification.

Theorem 2 (Invertibility criterion). *Let f be a continuous function defined on the spectrum of the self-adjoint operator A. Then for the operator $f(A)$ to be invertible it is necessary and sufficient that the function f has no zeros on the spectrum of A.*

Proof. Suppose first that the function f has no zeros on $\sigma(A)$. Then $g := 1/f$ is also a continuous function, $gf = 1$ and by assertion (2) of the preceding theorem, the operator $g(A)$ is the inverse of the operator $f(A)$. Now suppose that f vanishes at some point $t_0 \in \sigma(A)$. Pick a sequence of polynomials (p_n) which converges to f uniformly on $\sigma(A)$. With no loss of generality, we can assume that $p_n(t_0) = 0$ (otherwise, we replace $p_n(t)$ by $\widetilde{p}_n(t) = p_n(t) - p_n(t_0)$. By Theorem 2 of Subsection 13.1.1, the operators $p_n(A)$ are not invertible. Hence, since the set of non-invertible operators is closed (see the corollary to Theorem 1 of Subsection 11.1.2), the operator $f(A) = \lim_{n\to\infty} p_n(A)$ is also non-invertible.

Theorem 3 (Spectral mapping theorem for continuous functions). *Let A be a self-adjoint operator and $f \in C(\sigma(A))$. Then $\sigma(f(A)) = f(\sigma(A))$.*

Proof. We repeat the argument used earlier for polynomials (Theorem 3 of Subsection 13.1.1). The condition $\lambda \in \sigma(f(A))$ means that the operator $f(A) - \lambda I = (f - \lambda)(A)$ is not invertible. By the preceding assertion, this is equivalent to the existence of a point $t \in \sigma(A)$ such that $f(t) - \lambda = 0$. In turn, this is equivalent to the required condition $\lambda \in f(\sigma(A))$. □

Theorem 4. *Under the conditions of the preceding theorem, if $f \geqslant 0$ on the spectrum of the operator A, then $f(A) \geqslant 0$.*

Proof. By the spectral mapping theorem, $\sigma(p(A)) \subset [0, +\infty)$. It remains to use Corollary 1 in Subsection 12.4.5. □

Exercises

1. Suppose that in some basis the matrix of the operator A is diagonal. What will the matrix of the operator $f(A)$, where f is a continuous function, look like in that basis?

2. Does the definition of the operator e^A, given above in Exercise 5 of Subsection 13.1.1, agree with the definition as a continuous function of a self-adjoint operator?

3. Suppose $A \in L(H)$ is self-adjoint, $f \in C(\sigma(A))$, $f(A) = (f(A))^*$, and the function g is continuous on the spectrum of the operator $f(A)$. Prove that $g(f(A)) = (g \circ f)(A)$.

13.1.5 Applications of Continuous Functions of an Operator

Theorem 1. *The product of two commuting positive operators is a positive operator.*

Proof. Let $A, B \in L(H)$ be a pair of commuting positive operators. Since the spectrum of any positive operator lies on the positive half-line, the function $\sqrt{t}$ is continuous on the spectra of both operators A and B and takes positive values there. Consequently, the operators $\sqrt{A}$ and $\sqrt{B}$ are self-adjoint, and by property (5) in Theorem 1 of Subsection 13.1.4, $\sqrt{A}$ and $\sqrt{B}$ commute. We have

$$\langle ABx, x\rangle = \left\langle\left(\sqrt{A}\sqrt{A}\right)\left(\sqrt{B}\sqrt{B}\right)x, x\right\rangle = \left\langle\left(\sqrt{A}\sqrt{B}\right)\left(\sqrt{A}\sqrt{B}\right)x, x\right\rangle$$

$$= \left\langle\left(\sqrt{A}\sqrt{B}\right)x, \left(\sqrt{A}\sqrt{B}\right)x\right\rangle = \left\|\left(\sqrt{A}\sqrt{B}\right)x\right\|^2 \geqslant 0. \qquad \square$$

Lemma 1. *Let $A \in L(H)$ be a self-adjoint operator, and let $f \in C(\sigma(A))$ be such that $f(\sigma(A)) = \{0, 1\}$. Then $f(A)$ is an orthogonal projector onto a non-trivial (i.e., different from $\{0\}$ and the whole H) subspace.*

Proof. Since the function f satisfies the condition $f^2 = f$, we have $f^2(A) = f(A)$, and so the operator $f(A)$ is a projector. Since A is self-adjoint, $f(A)$ is an orthogonal projector. Finally, since $\sigma(f(A)) = f(\sigma(A)) = \{0, 1\}$, $f(A)$ cannot coincide with the zero operator or with the identity operator. That is, the image $f(A)$ is a nontrivial subspace. □

Definition 1. Let $H = H_1 \oplus H_2, A_1 \in L(H_1), A_2 \in L(H_2)$. The operator $A \in L(H)$ that coincides with A_j on the space H_j, $j = 1, 2$, is called the *direct sum of the operators A_1 and A_2 with respect to the decomposition $H = H_1 \oplus H_2$*, and is denoted by $A = A_1 \oplus A_2$. In other words, if $h_1 \in H_1$ and $h_2 \in H_2$, then $(A_1 \oplus A_2)(h_1 + h_2) = A_1 h_1 + A_2 h_2$.

The reader is encouraged to verify on his own that the operator $A = A_1 \oplus A_2$ is invertible if and only if both operators A_1 and A_2 are invertible. This readily implies that $\sigma(A_1) \cup \sigma(A_2) = \sigma(A_1 \oplus A_2)$.

Theorem 2. *Let $A \in L(H)$ be a self-adjoint operator whose spectrum is the union of two disjoint closed sets: $\sigma(A) = K_1 \cup K_2$. Then the space H admits an orthogonal direct sum decomposition $H = H_1 \oplus H_2$ into two nontrivial A-invariant subspaces, and the operator A decomposes into the direct sum $A = A_1 \oplus A_2$ of two operators $A_1 \in L(H_1)$ and $A_2 \in L(H_2)$, such that $\sigma(A_1) = K_1$ and $\sigma(A_2) = K_2$.*

Proof. The functions $f_1 = \mathbb{1}_{K_1}$ and $f_2 = \mathbb{1}_{K_2}$ are continuous on $\sigma(A)$. Consider the operators $P_1 = f_1(A)$ and $P_2 = f_2(A)$. By Lemma 1, P_1 and P_2 are othogonal projectors. Since $f_1 + f_2 \equiv 1$ on $\sigma(A)$, we have $P_1 + P_2 = I$. Put $H_1 = P_1(H)$ and $H_2 = \text{Ker } P_1$. Then one has the direct sum decomposition $H = H_1 \oplus H_2$, with $H_2 = P_2(H)$ (Theorem 2 of Subsection 10.3.2); moreover, $H_1 \perp H_2$, because P_1 is an orthogonal projector. H_j is the eigensubspace of the operator P_j corresponding to the eigenvalue 1. Since a function of an operator commutes with the operator itself this implies (Theorem 1 of Subsection 11.1.5) that the subspaces H_j are invariant under A.

We define the sought-for operators $A_j \in L(H_j)$, $j = 1, 2$, as the restrictions of the operator A to the subspaces H_j. With this definition, we obviously have that $A = A_1 \oplus A_2$ and $\sigma(A_1) \cup \sigma(A_2) = \sigma(A) = K_1 \cup K_2$. To complete the proof, it remains to verify the inclusions $\sigma(A_j) \subset K_j$, $j = 1, 2$. By symmetry, it suffices to consider the case $j = 1$. Let $\lambda \notin K_1$. Consider the function $g(t)$ equal to $\frac{1}{t-\lambda}$ for $t \in K_1$ and to 0 on K_2. Then, for every $x \in H_1$,

$$g(A)(A_1 - \lambda I)x = g(A)(A - \lambda I)x = f_1(A)x = P_1 x = x.$$

(Here and below I denotes the identity operator in the whole space, as well as in the subspaces H_1 and H_2.)
The subspace H_1 is invariant under $g(A)$ (again by Theorem 1 of Subsection 11.1.5); hence, thanks to commutativity, the last equality means that the restriction of the operator $g(A)$ to the subspace H_1 is the inverse of $A_1 - \lambda I$. Thus, we have shown that $\lambda \notin K_1$ implies that $\lambda \notin \sigma(A_1)$, which is equivalent to the inclusion $\sigma(A_1) \subset K_1$. □

Corollary 1. *Let λ_0 be an isolated point of the spectrum of the self-adjoint operator A. Then λ_0 is an eigenvalue of A.*

Proof. Apply the preceding result, taking $K_1 = \{\lambda_0\}$ and $K_2 = \sigma(A) \setminus \{\lambda_0\}$. In this case $\sigma(A_1) = \{\lambda_0\}$, that is (Corollary 3 in Subsection 12.4.5), $A_1 - \lambda_0 I = 0$, and any element of the subspace H_1 provides the required eigenvector. □

Exercises

1. In the last corollary, why can't the subspace H_1 be equal to $\{0\}$?

2. The self-adjoint operator $A \in L(H)$ is positive if and only if there exists a self-adjoint operator $B \in L(H)$ such that $B^2 = A$.

3. Suppose $B \geqslant 0$ and $B^2 = A$. Then $B = \sqrt{A}$.

4. Suppose $\dim H \geqslant 2$. Then there exist infinitely many self-adjoint operators $B \in L(H)$ such that $B^2 = I$.

13.2 Unitary Operators and the Polar Representation

13.2.1 The Absolute Value of an Operator

Let $T \in L(H)$ be an arbitrary operator. Following the analogy with numbers, one can conjecture that T^*T will be a positive self-adjoint operator. Let us verify that this is the case. Since $(T^*T)^* = T^*(T^*)^* = T^*T$, self-adjointness holds. Positivity is a consequence of the scalar product axioms: $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle \geqslant 0$. Now since the operator T^*T is positive, the function $\sqrt{t}$ is continuous on its spectrum, which enables us to define the *absolute value of the operator* T as $|T| = \sqrt{T^*T}$. The absolute value of an operator is a positive operator.

Theorem 1. *For any element $x \in H$, $\||T|x\| = \|Tx\|$. In particular, $|T|x = 0$ if and only if $Tx = 0$.*

Proof. Indeed,

$$\||T|x\|^2 = \langle |T|x, |T|x\rangle = \langle |T|^2 x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2. \qquad \square$$

For the ensuing material we need the following reformulation.

Theorem 2 (Weak polar representation). *Let X be the image of the operator $|T|$, and Y the image of the operator T. Then there exists an isometric bijective operator $V \in L(X, Y)$ such that $T = V \circ |T|$. Moreover, the operator V is unique.*

Proof. First, the uniqueness. Let $x = |T|(h)$ be an arbitrary element of the space X. For the equality $T = V \circ |T|$ to hold when both terms are evaluated on h, it is necessary and sufficient that the operator V satisfies the condition $Vx = Th$. Hence, V is uniquely determined. By the preceding theorem, if the element x admits two representations $x = |T|(h_1) = |T|(h_2)$, then $\|Th_1 - Th_2\| = 0$, i.e., the condition $Vx = Th$ can be taken as the definition of the operator V. As h runs through[2] the entire Hilbert space H, the elements $x = |T|(h)$ and $Vx = Th$ run through the entire spaces X and Y, respectively. Therefore, the operator V is bijective. Finally, V is an isometry, because $\|Vx\| = \|Th\| = \| \, |T|(h) \, \| = \|x\|$. □

Exercises

Calculate the absolute values of the following operators:

1. The multiplication operator $A_g \in L(H)$ by a bounded function g: $(A_g f)(t) = g(t)f(t)$.

2. The right-shift operator $S_r \in L(\ell_2)$, acting as $S_r(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$;

3. The left-shift operator $S_l \in L(\ell_2)$, acting as $S_l(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$.

13.2.2 Definition and Simplest Properties of Unitary Operators

A complex number lying on the unit circle satisfies the equation $z \cdot \bar{z} = 1$. Developing further the analogy between operators and numbers, it is natural to introduce the corresponding class of operators.

Definition 1. The operator $U \in L(H)$ is called *unitary* if $UU^* = U^*U = I$. In other words, the operator U is unitary if it is invertible and $U^{-1} = U^*$.

Theorem 1. *Unitary operators preserve the scalar product: $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in H$. Consequently, unitary operators preserve orthogonality: if $x \perp y$, then $Ux \perp Uy$.*

Proof. Indeed, $\langle Ux, Uy \rangle = \langle x, U^*Uy \rangle = \langle x, y \rangle$. □

[2]If one ponders over this generally accepted expression, then one is struck by the disparity with the picture arising here. Indeed, to "run through" even a domain in the plane, an element requires considerable effort. It is true that in this case "he" could actually perform this task by moving along the Peano curve (though in his place I would look for a more interesting activity). As for the infinite-dimensional case, "running through" the entire space is in fact impossible. Indeed, prove that a continuous mapping $f : [0, +\infty) \to H$ cannot be surjective.

Theorem 2 (Unitarity criterion). *An operator is unitary if and only if it is a bijective isometry.*

Proof. Let U be unitary. Then U is invertible, and hence bijective. Further, $\|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, x\rangle = \|x\|^2$, i.e., U is an isometry. Conversely, suppose that U is an isometry. Then

$$\langle x, x\rangle = \|x\|^2 = \|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, U^*Ux\rangle.$$

Thus, the quadratic forms of the operators U^*U and I coincide, hence so do the operators themselves: $I = U^*U$. And since for a bijective operator the notions of right inverse and left inverse coincide, we conclude that $UU^* = I$, as needed. □

Theorem 3. *The spectrum of any unitary operator U lies on the unit circle.*

Proof. Since U is isometric, $\|U\| = \|U^{-1}\| = 1$. Therefore, if $|\lambda| < 1$, then, by the theorem on small perturbations of an invertible element (Theorem 1 in Subsection 11.1.2), the operator $U - \lambda I$ is invertible, while if $|\lambda| > 1$, the invertibility of the operator $U - \lambda I = \lambda(I - \lambda^{-1}U)$ is guaranteed by the lemma on the invertibility of small perturbations of the identity element (Lemma 2 in Subsection 11.1.2). Therefore, $U - \lambda I$ can be non-invertible only if $|\lambda| = 1$. □

In Sect. 13.3 we will pursue further the analogy between unitary operators and numbers of absolute magnitude 1: a chain of exercises culminating in Exercise 23 of that section will show that every unitary operator U can be represented as $U = e^{iA}$ with A self-adjoint.

Exercises

1. Under what conditions on the function g will the multiplication operator $A_g \in L(L_2[0,1])$, $(A_g f)(t) = g(t)f(t)$, be unitary?

2. Show that for every closed subset K of the unit circle there exists a unitary operator U such that $\sigma(U) = K$.

3. Suppose the operator $U \in L(H)$ is an isometric embedding (i.e., $\|Ux\| = \|x\|$ for all $x \in H$) with dense image. Then U is unitary.

13.2.3 Polar Decomposition

A *polar decomposition* of the operator T is a representation of the operator as $T = UA$, where U is a unitary operator and A is a positive self-adjoint operator. That is, the polar decomposition of an operator is the analogue of the polar decomposition for complex numbers, $z = e^{i \arg z} \cdot |z|$. In contrast to the scalar case, for operators such a decomposition is not always possible. To find conditions for the existence of the polar decomposition, we regard it as an equation in the unknown operators U and A.

So, suppose that U and A are solutions of the equation $T = UA$ with the required properties. Then $T^* = AU^*$ (we used the self-adjointness of A) , and thanks to the unitarity of U, we have $T^*T = A^2$. Extracting the square root, we obtain the value of one of the unknowns:

$$A = |T|.$$

To determine the second unknown, we have the equation $T = U \circ |T|$. How does this condition differ from the analogous condition on the operator V in Theorem 2 of Subsection 13.2.1? Only by the fact that the operator U needs to be defined not merely on the subspace $X = |T|(H)$, but on the whole space H, with preservation of the isometry and bijectivity properties that the operator V enjoyed. Let us examine when such an extension is possible. To formulate the result, we need to define more precisely what the dimension of a Hilbert space means. For a finite-dimensional space, the dimension was defined as the number of elements in a basis of the space. Generalizing this definition to the infinite-dimensional case, the *dimension of a Hilbert space* is the cardinality of an orthonormal basis of the space (see Exercise 7 in Subsection 12.3.4). Two Hilbert spaces have the same dimension if and only if they are isomorphic (Exercise 12 in Subsection 12.3.5).

Lemma 1. *Let X, Y be linear subspaces of the space H, and $V \in L(X, Y)$ be a bijective isometry. Then the following conditions are equivalent:*

(1) *the mapping V can be extended to a unitary operator $U \in L(H)$.*

(2) $\dim X^\perp = \dim Y^\perp$, *where the dimensions may be finite or infinite.*

Proof. (1) $\Longrightarrow$ (2). Since on X the operators U and V coincide, $U(X) = Y$. By Theorem 1 of Subsection 13.2.2, a unitary operator preserves orthogonality, and so $U(X^\perp) = Y^\perp$. In view of the injectivity of the operator, this yields the required equality of the dimensions.

(2) $\Longrightarrow$ (1). With no loss of generality, we can assume that the subspaces X and Y are closed (otherwise we extend the operator V by continuity to the closure of the subspace X). Thanks to the equality of dimensions, there exists a bijective isometry $W\colon X^\perp \to Y^\perp$. Given an arbitrary $x \in H$, we decompose it as $x = x_1 + x_2$, with $x_1 \in X$ and $x_2 \in X^\perp$. Now define the requisite operator U by the rule $Ux = Vx_1 + Wx_2$. □

Theorem 1. *For the existence of a polar decomposition of the operator T it is necessary and sufficient that the equality of dimensions*

$$\dim \operatorname{Ker} T = \dim \operatorname{Ker} T^*$$

holds.

Proof. By the last lemma above and the arguments that precede it, the required necessary and sufficient condition is the equality $\dim(|T|(H))^\perp = \dim(T(H))^\perp$. To reduce this condition to the one in the lemma, we observe that $(T(H))^\perp = \operatorname{Ker} T^*$. On the other hand, self-adjointness implies that $(|T|(H))^\perp = \operatorname{Ker}|T|$. In turn, by Theorem 1 of Subsection 13.2.1, $\operatorname{Ker}|T| = \operatorname{Ker} T$. □

Let us mention several useful sufficient conditions for the existence of the polar decomposition.

Corollary 1.

1. *For the operator T to admit a polar decomposition it is sufficient that T be invertible.*
2. *Let T be a* normal operator, *i.e., T commutes with T^*. Then T admits a polar decomposition.*
3. *Let T be a scalar + compact operator. Then T admits a polar decomposition.*

Proof.

1. If T in invertible, then so is T^*. Consequently, $\dim \operatorname{Ker} T = \dim \operatorname{Ker} T^* = 0$.
2. $\operatorname{Ker} T = \operatorname{Ker} |T| = \operatorname{Ker} \sqrt{T^*T} = \operatorname{Ker} \sqrt{TT^*} = \operatorname{Ker} |T^*| = \operatorname{Ker} (T^*)$.
3. This follows from the Fredholm theorem (see Exercise 2 in Subsection 11.3.3).

Exercises

1. Prove that any operator $T \in L(H)$ is representable, and in fact in a unique way, as $T = A + iB$, where A and B are self-adjoint operators. Moreover, the operator T will be normal if and only if A and B commute. The stated representation serves as the starting point of one of the ways of constructing functions of a normal operator (see [4]).

2. Show that the operator T is normal if and only if it admits a polar decomposition $T = UA$ with commuting operators A and U.

3. Show that if an operator T has a non-commuting polar decomposition, then T is not normal.

4. Describe the operators for which the polar decomposition is unique.

5. Justify the following fact that was already used, without drawing attention to it, in the present subsection: if two positive operators A and B satisfy the equality $A^2 = B^2$, then $A = B$. Does this assertion remain true if we discard the positivity assumption? Where specifically did we use this fact?

13.3 Borel Functions of an Operator

Using the fact that any continuous function can be approximated by polynomials, we were able to construct continuous functions of a self-adjoint operator. Below we show that functions of a self-adjoint operator can also be defined in a considerably more general situation, namely, for any bounded Borel-measurable function. The construction is based on the possibility of unique extension of linear functionals from the space of continuous functions to the wider space of bounded Borel-measurable functions.

Let $A \in L(H)$ be a fixed self-adjoint operator and K be its spectrum.

Given two arbitrary elements $x, y \in H$, define the linear functional $F_{x,y} \in C(K)^*$ by the formula $F_{x,y}(f) = \langle f(A)x, y\rangle$. Clearly, in addition to the linearity in f, the following relations, characteristic of bilinear forms, hold: $F_{a_1x_1+a_2x_2,y} = a_1 F_{x_1,y} + a_2 F_{x_2,y}$ and $F_{y,x} = \overline{F}_{x,y}$. It is also readily verified that $\|F_{x,y}\| \leqslant \|x\| \cdot \|y\|$; indeed,

$$|F_{x,y}(f)| = |\langle f(A)x, y\rangle| \leqslant \|f(A)\| \cdot \|x\| \cdot \|y\| = \|f\| \cdot \|x\| \cdot \|y\|.$$

By the theorem on the general form of continuous linear functionals on the space $C(K)$, there exists a regular Borel charge $\sigma_{x,y}$ on K such that

$$F_{x,y}(f) = \int_K f \, d\sigma_{x,y}.$$

Since the indicated correspondence between functionals on $C(K)$ and charges is a bijective isometry, the relations for functionals written above remain valid for charges:

$$\|\sigma_{x,y}\| \leqslant \|x\| \cdot \|y\|, \quad \sigma_{a_1x_1+a_2x_2,y} = a_1\sigma_{x_1,y} + a_2\sigma_{x_2,y}, \text{ and } \sigma_{y,x} = \overline{\sigma}_{x,y}.$$

Definition 1. Let f be a bounded Borel function on K, and $\sigma_{x,y}$ be the Borel charges defined above. Define the operator $f(A)$ by the equality

$$\langle f(A)x, y\rangle = \int_K f \, d\sigma_{x,y}.$$

In view of the theorem in Subsection 12.4.1 (with a change in the order of factors) the above definition is correct: the expression on the right-hand side of the equality is a continuous bilinear form.

Note that the last definition is consistent with the definition of a continuous function of an operator (i.e., the two definitions give the same result), and many of the properties of continuous functions of an operator listed in Subsection 13.1.4 remain valid in the more general situation.

Theorem 1. *For bounded Borel functions on K the following relations hold:*

1. $(\lambda_1 f_1 + \lambda_2 f_2)(A) = \lambda_1 f_1(A) + \lambda_2 f_2(A)$*;*

2. $(f_1 f_2)(A) = f_1(A) f_2(A)$;
3. $(f(A))^* = \overline{f}(A)$. *In particular, if on* $\sigma(A)$ *the function* f *takes only real values, then* $f(A)$ *is a self-adjoint operator.*

Proof. Property 1. follows from the linearity of the integral.

To verify property 3, we use the relation $\sigma_{y,x} = \overline{\sigma}_{x,y}$:

$$\langle (f(A))^* x, y\rangle = \langle x, f(A)y\rangle = \overline{\langle f(A)y, x\rangle}$$
$$= \int_K \overline{f}\, d\overline{\sigma}_{y,x} = \int_K \overline{f}\, d\sigma_{x,y} = \langle \overline{f}(A)x, y\rangle.$$

It remains to verify property 2. The equality $(f_1 f_2)(A) = f_1(A) f_2(A)$ is already known to hold for continuous functions. Threfore, for $f_1, f_2 \in C(K)$ we have the equality of bilinear forms

$$\langle (f_1 f_2)(A)x, y\rangle = \langle f_1(A)(f_2(A)x), y\rangle. \tag{1}$$

Using the definition of the charges $\sigma_{x,y}$, we recast (1) as

$$\int_K f_1 f_2\, d\sigma_{x,y} = \int_K f_1\, d\sigma_{f_2(A)x,y}.$$

Since the integrals above are equal for any continuous function f_1, they will also be equal for any bounded Borel-measurable function.[3] Going in the opposite direction, we deduce that the equality (1) again holds not only for a continuous, but also for an arbitrary bounded Borel function f_1. Rewriting (1) in the form

$$\langle (f_1 f_2)\,(A)x, y\rangle = \langle f_2(A)x, \overline{f}_1(A)y\rangle$$

and using the definition, we conclude that for any bounded Borel function f_1 the equality of integrals

$$\int_K f_1 f_2 d\sigma_{x,y} = \int_K f_2 d\sigma_{x,\overline{f}_1(A)y} \tag{2}$$

is valid for any continuous function f_2. Extending equality (2) to the more general class of bounded Borel-measurable functions and passing again to bilinear forms, we see that relation (1) is valid for all bounded Borel-measurable functions f_1 and f_2. The coincidence of the bilinear forms implies the coincidence of the corresponding operators. Thus, the required mutiplicativity relation is proved. □

The remaining properties of continuous functions of operators discussed in Subsection 13.1.4 are not fully valid for Borel functions. The main reason for this is that two different functions, f_1, f_2, that coincide almost everywhere with respect to all charges $\sigma_{x,y}$, generate the same operator: $f_1(A) = f_2(A)$.

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[3] To prove this, take a measure μ that dominates the variations of both charges figuring in the equality; represent the Borel function as the limit of a μ-almost everywhere convergent sequence of uniformly bounded continuous functions and apply the dominated convergence theorem.

Theorem 2 (Sufficient conditions for invertibility). *If the bounded Borel function f is separated away from 0 on $\sigma(A)$ (i.e., there exists an $\varepsilon > 0$ such that $|f(t)| \geqslant \varepsilon$ for all $t \in \sigma(A)$), then the operator $f(A)$ is invertible.*

Proof. The operator $\frac{1}{f}(A)$ is the inverse to $f(A)$. □

Theorem 3 (Spectral mapping theorem for bounded Borel functions of an operator). *The spectrum of a function of an operator is contained in the closure of the image of the spectrum of the original operator: $\sigma(f(A)) \subset \overline{f(\sigma(A))}$.*

Proof. Let $\lambda \notin \overline{f(\sigma(A))}$. Then the function $f - \lambda$ satisfies the conditions of the preceding theorem. Hence, the operator $(f - \lambda)(A) = f(A) - \lambda I$ is invertible, i.e., $\lambda \notin \sigma(f(A))$. The theorem is proved. □

Theorem 4 (Estimate of the norm of a function of an operator). *Let f be a bounded Borel function on $K = \sigma(A)$. Then*

$$\|f(A)\| \leqslant \sup_{t \in \sigma(A)} |f(t)|.$$

Proof. We use the condition $|\sigma_{x,y}|(K) = \|\sigma_{x,y}\| \leqslant \|x\| \cdot \|y\|$ and the estimate of the integral through the variation of the charge (Theorem 4 in Subsection 8.4.5) to obtain

$$\|f(A)\| = \sup_{x,y \in S_H} |\langle f(A)x, y\rangle| \leqslant \sup_{x,y \in S_H} \int_{\sigma(A)} |f| \, d\left|\sigma_{x,y}\right| \leqslant \sup_{t \in \sigma(A)} |f(t)|.$$ □

Theorem 5. *Let (f_n) be a monotonically increasing uniformly bounded sequence of real-valued Borel functions on $K = \sigma(A)$ that converges at each point to the function f. Then the sequence of operators $(f_n(A))$ converges pointwise to the operator $f(A)$.*

Proof. The operators $f_n(A)$ form a monotone bounded sequence. By Theorem 2 of Subsection 12.4.4, there exists the pointwise limit of the sequence $f_n(A)$, which we denote by T. To establish the claimed equality $f(A) = T$, we compare the bilinear forms of the operators:

$$\langle Tx, y\rangle = \lim_{n\to\infty} \langle f_n(A)x, y\rangle = \lim_{n\to\infty} \int_{\sigma(A)} f_n \, d\sigma_{x,y} = \int_{\sigma(A)} f \, d\sigma_{x,y} = \langle f(A)x, y\rangle.$$

Here we used the Lebesgue dominated convergence theorem. □

Discontinuous Borel functions of an operator can be calculated, rather than using the (quite abstract) definition, by using approximation (in one sense or another) by continuous functions. For example, if the bounded Borel function f on $\sigma(A)$ is representable as the pointwise limit of an increasing sequence (f_n) of continuous functions,[4] the last theorem enables us to calculate $f(A)$ as the pointwise limit of the sequence $(f_n(A))$. For more details on such an approach to functions of an operator, we refer to the exercises below.

[4]Such a representation is possible only for lower-semicontinuous functions.

Exercises

1. Suppose the operators A and B commute. Then each of them commutes with the bounded Borel functions of the other.

2. Suppose the operators A and B commute, and let f and g be Borel functions on the spectra of the operators A and B, respectively. Then the operators $f(A)$ and $g(B)$ also commute.

3. Suppose that in some basis the matrix of the self-adjoint operator A is diagonal. If f is a bounded Borel function, what will the matrix of the operator $f(A)$ look like in that basis?

4. Describe the functions of the multiplication operator A_g by a bounded function g: $A_g \in L(L_2[0, 1])$, $(A_g f)(t) = g(t) f(t)$.

Definition. We say that the sequence of operators $A_n \in L(H)$ *form converges*[5] to the operator $A \in L(H)$ if the corresponding bilinear forms converge:

$$\langle A_n x, y\rangle \to \langle Ax, y\rangle \quad \text{as } n \to \infty$$

for all $x, y \in H$. Notation: $A_n \xrightarrow{\text{form}} A$.

5. Pointwise convergence of operators implies form convergence.

6. Let $\{e_n\}_1^\infty$ be an orthormal system in H. Then the operators A_n, acting by the formula $A_n x = \langle x, e_1\rangle e_n$, form converge to 0, but do not converge pointwise.

7. If $A_n \xrightarrow{\text{form}} A$, then $\|A\| \leqslant \sup_n \|A_n\| < \infty$.

8. If $A_n \xrightarrow{\text{form}} A$, $B_n \xrightarrow{\text{form}} B$, and $a, b \in \mathbb{C}$, then $aA_n + bB_n \xrightarrow{\text{form}} aA + bB$.

9. If $A_n \xrightarrow{\text{form}} A$ and $B \in L(H)$, then $A_n B \xrightarrow{\text{form}} AB$ and $BA_n \xrightarrow{\text{form}} BA$.

10. If $A_n \xrightarrow{\text{form}} A$, then $A_n^* \xrightarrow{\text{form}} A^*$. Does the analogous property hold for pointwise convergence?

11. Provide an example in which $A_n \xrightarrow{\text{form}} A$ and $B_n \xrightarrow{\text{form}} B$, but $A_n B_n$ does not form converge to AB.

Let $A \in L(H)$ be a self-adjoint operator and K be its spectrum. The Borel measure μ on K is said to be a *control measure* of the operator A if all the charges $\sigma_{x,y}$ generated by the operator A are absolutely continuous with respect to μ.

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[5]The generally accepted name for this type of convergence of operators is "weak pointwise convergence", because in this case $A_n x$ weakly converge to Ax for all $x \in H$. The meaning of word "weakly" here is "when evaluated by every continuous linear functional". We will speak a lot about weak convergence and weak topology in Chap. 17.

12. For any self-adjoint operator in a separable Hilbert space H there exists a control measure. Hint: choose a sequence of pairs (x_n, y_n) that is dense in $S_H \times S_H$. Define the control measure by the formula $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \left|\sigma_{x_n, y_n}\right|$.

13. Let $A \in L(H)$ be a self-adjoint operator and K be its spectrum. Let μ be a control measure for A, and let (f_n) be a uniformly bounded sequence of Borel functions on K which converges μ-almost everywhere to f. Then $f_n(A) \xrightarrow{\text{form}} f(A)$.

14. Let $A \in L(H)$ be a self-adjoint operator and K be its spectrum. Then for any bounded Borel function f on K there exists a sequence (f_n) of continuous functions on K such that $f_n(A) \xrightarrow{\text{form}} f(A)$.

The next chain of exercises will enable the reader to construct on her/his own a theory of functions of a unitary operator. Throughout this part U will denote a fixed unitary operator and S the spectrum of U.

The main difference compared to the case of self-adjoint operators is that the polynomials are not dense in the space of continuous functions on the unit circle: the closure of the set of polynomials in the uniform metric contains only the boundary values of functions analytic in the unit disc. For instance, the function $1/z$ does not belong to this closure.

15. Find the distance of the function $1/z$ to the set of polynomials in the space of continuous functions on the unit circle.

To circumvent this difficulty, we introduce the set of generalized polynomials, which also include negative powers of the indeterminate:

$$\mathcal{P}_* = \{p \in C(S) : \ p(z) = \sum_{k=-n}^{n} a_k z^k\}.$$

By analogy with the case of ordinary polynomials, we put

$$p(U) = \sum_{k=-n}^{n} a_k U^k.$$

16. Verify that the mapping $p \mapsto p(U)$ enjoys the linearity and multiplicativity properties.

17. Establish an invertibility criterion for the operator $p(U)$. Prove the spectral mapping theorem.

A generalized polynomial $p(z) = \sum_{k=-n}^{n} a_k z^k$ is said to be symmetric,[6] if $a_{-k} = \overline{a}_k$ for all indices k.

[6]This term is used by convention and has nothing to do with the symmetric polynomials of several variables.

18. Prove that a symmetric polynomial takes only real values on the unit circle. Conversely, if S is an infinite subset of the unit circle and the polynomial p takes only real values on S, then p is symmetric.

19. For any generalized polynomial p on the unit circle, the functions $\operatorname{Re} p$ and $\operatorname{Im} p$ are symmetric polynomials.

20. Prove that every real-valued continuous function on the unit circle can be arbitrarily well approximated in the uniform metric by symmetric polynomials.

21. Prove that a symmetric polynomial of a unitary operator is a self-adjoint operator. Deduce for this case the formula $\|p(U)\| = \sup_{t\in\sigma(U)} |p(t)|$.

22. From this point on, the entire scheme for constructing functions of a self-adjoint operator carries over with no modifications to functions of a unitary operator. Verify this!

23. Prove that every unitary operator U can be represented as $U = e^{iA}$, where A is a self-adjoint operator. (Hint: Pick a branch f of the argument on the unit circle. Take $f(U)$ for A.) Is this representation unique?

24. Suppose the generalized polynomial p takes on the unit circle $\mathbb{T}$ only positive values. Then there is another generalized polynomial g such that $p(z) = |g(z)|^2$ for all $z \in \mathbb{T}$.

13.4 Functions of a Self-adjoint Operator and the Spectral Measure

13.4.1 The Integral with Respect to a Vector Measure

Suppose given a set Ω, an algebra Σ of subsets of Ω, and a Banach space X. A mapping $\mu\colon \Sigma \to X$ is called an X-*valued measure* if it has the finite additivity property: $\mu(D_1 \cup D_2) = \mu(D_1) + \mu(D_2)$ for all disjoint subsets $D_1, D_2 \in \Sigma$. Measures with values in Banach spaces are also called *vector measures*.

The basic case we will be dealing with is that of complex scalars. The real case is practically identical.

Example. Let $X = \mathbb{C}^n$ be the space of rows. Then every X-valued measure μ can be written as $\mu(D) = (\mu_1(D), \mu_2(D), \ldots, \mu_n(D))$, where μ_j are finitely-additive complex charges.

We define the integral of a scalar function with respect to a vector measure by analogy with how we proceeded in Sect. 4.2 for the ordinary integral. The difference here is not only that we are dealing with a vector measure, but also that the measure is only finitely — and not countably — additive. For this reason, in all definitions we will work only with finite partitions of sets into subsets.

So, let $\Delta \in \Sigma$ and $f : \Delta \to \mathbb{C}$ be a function. Let $D = \{\Delta_k\}_{k=1}^n$ be a finite partition of the set Δ into subsets $\Delta_k \in \Sigma$, and $T = \{t_k\}_1^n$ be a collection of marked points. The *integral sum* of the function f on the set Δ with respect to the pair (D, T) is the vector

$$S_\Delta(f, D, T) = \sum_{k=1}^{n} f(t_k)\mu(\Delta_k) \in X.$$

The element $x \in X$ is called the *integral* of the function f on the set Δ with respect to the vector measure μ (notation: $x = \int_\Delta f\, d\mu$) if for any $\varepsilon > 0$ there exists a finite partition D_ε of the set Δ such that for any finite partition D refining D_ε and any choice of marked points T for D, it holds that $\|x - S_\Delta(f, D, T)\| \leqslant \varepsilon$. The function $f : \Delta \to \mathbb{C}$ is said to be *integrable* on the set Δ with respect to the measure μ, or *μ-integrable*, if the corresponding integral exists.

In other words, the function f is integrable on Δ if its integral sums have a limit along the directed set of finite partitions with marked points, analogous to that described in Subsection 4.1.3.

Let us list, with no proofs, a number of simple properties of the integral.

(1) Linearity in the function: if the functions f and g are integrable on Δ and a, b are scalars, then the function $af + bg$ is also integrable, and $\int_\Delta (af + bg)d\mu = a\int_\Delta f\, d\mu + b\int_\Delta g\, d\mu$.

(2) Set additivity: if $\Delta_1 \cap \Delta_2 = \emptyset$ and f is integrable on both sets Δ_1 and Δ_2, then f is integrable on their union, and $\int_{\Delta_1} f\, d\mu + \int_{\Delta_2} f\, d\mu = \int_{\Delta_1 \sqcup \Delta_2} f\, d\mu$.

(3) The characteristic function of any set $\Delta \in \Sigma$ is integrable, and $\int_\Omega \mathbb{1}_\Delta\, d\mu = \mu(\Delta)$.

(4) For any collection $\{\Delta_k\}_1^n$ of measurable subsets and any collection of scalars $\{a_k\}_1^n$ the step function $f = \sum_{k=1}^n a_k \mathbb{1}_{\Delta_k}$ is integrable, and $\int_\Omega f\, d\mu = \sum_{k=1}^n a_k \mu(\Delta_k)$.

(5) Let $G \in L(X, Y)$ be a continuous linear operator and $\mu : \Sigma \to X$ be an X-valued measure. Then the composition $G \circ \mu$ is a Y-valued measure. Every μ-integrable function f is also $(G \circ \mu)$-integrable, and $G\left(\int_\Delta f\, d\mu\right) = \int_\Delta f\, d(G \circ \mu)$.

13.4.2 Semivariation and Existence of the Integral

Definition 1. Let $\mu : \Sigma \to X$ be a vector measure. For each $\Delta \in \Sigma$ we define the *semivariation of the measure* μ on the set Δ, denoted by $\|\mu\|(\Delta)$, as the supremum of the quantity $\left\|\sum_{k=1}^n a_k \mu(\Delta_k)\right\|$ over all finite partitions $\{\Delta_k\}_{k=1}^n$ of the set Δ into measurable subsets and all finite collections of scalars $\{a_k\}_{k=1}^n$ that satisfy the condition $|a_k| \leqslant 1$. We define $\|\mu\| = \|\mu\|(\Omega)$. The measure μ is said to be *bounded* if $\|\mu\| < \infty$. Throughout the remaining part of this subsection the measure μ will be assumed to be bounded.

Lemma 1. *For any bounded function f on the set $\Delta \in \Sigma$, any finite partition $D = \{\Delta_k\}_{k=1}^n$ of Δ into subsets $\Delta_k \in \Sigma$, and any collection $T = \{t_k\}_1^n$ of marked points $t_k \in \Delta_k$, one has the estimate*

$$\|S_\Delta(f, D, T)\| \leqslant \|\mu\|(\Delta) \cdot \sup_{t \in \Delta} |f(t)|.$$

Proof. Denote $\sup_{t\in\Delta} |f(t)|$ by M and $a_k = f(t_k)/M$. Since $|a_k| \leqslant 1, k = 1, 2, \dots, n$, the required estimate follows:

$$\left\| \sum_{k=1}^n f(t_k)\mu(\Delta_k) \right\| = M \left\| \sum_{k=1}^n a_k \mu(\Delta_k) \right\| \leqslant M\|\mu\|(\Delta) = \|\mu\|(\Delta) \cdot \sup_{t\in\Delta} |f(t)|. \quad \square$$

Letting the integral sums converge to the integral, we obtain the following assertion.

Theorem 1. *The inequality*

$$\left\| \int_\Delta f\, d\mu \right\| \leqslant \|\mu\|(\Delta) \cdot \sup_{t\in\Delta} |f(t)|$$

holds for all bounded integrable functions f on Δ. □

By analogy with Subsection 4.3.2 we prove the following uniform limit theorem.

Theorem 2. *Let f and f_n be scalar-valued functions on Δ and μ be a bounded vector measure. Suppose that the functions f_n are μ-integrable on Δ and the sequence (f_n) converges uniformly on Δ to f. Then f is integrable and $\int_\Delta f\, d\mu = \lim_{n\to\infty} \int_\Delta f_n\, d\mu$.*

Proof. Let $x_n = \int_\Delta f_n d\mu$. The sequence (x_n) is Cauchy: indeed,

$$\|x_n - x_m\| = \left\| \int_\Delta (f_n - f_m) d\mu \right\| \leqslant \sup_{t\in\Delta} \|f_n(t) - f_m(t)\| \cdot \|\mu\|(\Delta) \to 0 \quad \text{as } n, m \to \infty.$$

Denote the limit of the sequence x_n by x. Fix $\varepsilon > 0$ and choose an $n \in \mathbb{N}$ such that $\sup_{t\in\Delta} \|f_n(t) - f(t)\| < \varepsilon/(3\|\mu\|(\Delta))$ and $\|x - x_n\| < \varepsilon/3$. Further, let D_ε be a partition such that, starting with D_ε, we have $\|x_n - S_\Delta(f_n, D, T)\| \leqslant \varepsilon$. Then for any partition $D \succ D_\varepsilon$ and any collection of marked points T corresponding to D,

$$\begin{aligned} \|x - S_\Delta(f, D, T)\| &\leqslant \|x - x_n\| + \|x_n - S_\Delta(f_n, D, T)\| \\ &\quad + \|S_\Delta(f_n - f, D, T)\| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, f is integrable and $\int_\Delta f\, d\mu = x$. It remains to recall that, by construction,

$$x = \lim_{n\to\infty} \int_\Delta f_n\, d\mu. \quad \square$$

Theorem 3. *Let $\mu\colon \Sigma \to X$ be a bounded vector measure and $\Delta \in \Sigma$. Then every bounded measurable function f is integrable on Δ.*

Proof. The function f can be represented as the limit of a uniformly convergent sequence f_n of finitely-valued functions (Corollary 1 in Subsection 3.1.4). Since any finitely-valued measurable function is integrable, it remains to apply Theorem 2 on uniform limit. □

We have provided the basic definitions and simplest properties of vector measures that are required for the theory of operators. To simplify the exposition, we did not aim at maximal generality in definitions and statements. The theory of vector measures is itself an extensive domain of functional analysis, rich in deep results and applications. For an introduction to the theory of vector measures we refer to the monograph of J. Diestel and J.J. Uhl [13].

Exercises

1. Prove that for real-valued charges the semivariation coincides with the variation of the charge familiar from Definition 1 in Subsection 7.1.1.

2. Verify that the expression $\|\mu\| = \|\mu\|(\Omega)$ gives a norm on the space $M(\Omega, \Sigma, X)$ of all bounded X-valued measures on Σ. Prove that the normed space $M(\Omega, \Sigma, X)$ is complete.

3. A vector measure is bounded if and only if its range (set of all its values) is bounded.

4. Prove that if a vector measure is given on a σ-algebra and is countably-additive, then it is bounded.

5. Let Σ be a σ-algebra, $\mu\colon \Sigma \to X$ be a vector measure, and let μ be weakly countably-additive (i.e., $x^* \circ \mu$ be a countably-additive charge for every $x^* \in X^*$), then μ is also countably-additive in the ordinary sense.

Let (Ω, Σ, ν) be a space with (ordinary finite scalar-valued positive) measure. We say that the vector measure $\mu\colon \Sigma \to X$ is *absolutely continuous with respect to* ν if $\mu(\Delta) = 0$ for all sets $\Delta \in \Sigma$ such that $\nu(\Delta) = 0$.

6. Prove that if the vector measure $\mu\colon \Sigma \to X$ is countably-additive on the σ-algebra Σ and absolutely continuous with respect to ν, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mu\|(\Delta) < \varepsilon$ for all $\Delta \in \Sigma$ with $\nu(\Delta) < \delta$.

7. Under the conditions of the preceding exercise, the following analogue of the dominated convergence theorem holds true: If the uniformly bounded sequence (f_n) of measurable functions converges ν-almost everywhere to the function f, then $\int_\Omega f\,d\mu = \lim_{n\to\infty} \int_\Omega f_n\,d\mu$.

8. Suppose the vector measure $\mu\colon \Sigma \to X$ is countably-additive on the σ-algebra Σ. Then on Σ there exists a scalar-valued measure ν with respect to which μ is absolutely continuous.

13.4.3 The Spectral Measure and Spectral Projectors

Let $A \in L(H)$ be a fixed self-adjoint operator, and let $\mathfrak{B}$ be the σ-algebra of Borel sets on its spectrum $\sigma(A)$.

Definition 1. The *spectral measure* of the operator A is the vector measure $\mu_A : \mathfrak{B} \to L(H)$ defined by the rule $\mu_A(\Delta) = \mathbb{1}_\Delta(A)$.

We note that using the term "measure" here is correct, since $\mathbb{1}_{\Delta_1} + \mathbb{1}_{\Delta_2} = \mathbb{1}_{\Delta_1 \cup \Delta_2}$ for any pair of disjoint sets Δ_1 and Δ_2.

Lemma 1. *Let $f = \sum_{k=1}^n \alpha_k \mathbb{1}_{\Delta_k}$ be a finitely-valued Borel function on $\sigma(A)$. Then $f(A) = \int_{\sigma(A)} f\, d\mu_A$.*

Proof. Indeed,

$$\int\limits_{\sigma(A)} f\, d\mu_A = \sum_{k=1}^n \alpha_k \mu_A(\Delta_k) = \sum_{k=1}^n \alpha_k \mathbb{1}_{\Delta_k}(A) = f(A). \qquad \square$$

Theorem 1. *The spectral measure of any self-adjoint operator A is bounded, and $\|\mu_A\|(\Delta) \leqslant 1$ for all Borel subsets $\Delta \subset \sigma(A)$.*

Proof. By definition, $\|\mu_A\|(\Delta) = \sup \left\| \sum_{k=1}^n a_k \mu_A(\Delta_k) \right\|$, where the supremum is taken over all finite partitions $\{\Delta_k\}_{k=1}^n$ of the set Δ into Borel subsets and all finite collections of scalars $\{a_k\}_{k=1}^n$ that satisfy the condition $|a_k| \leqslant 1$. Consider the function $f = \sum_{k=1}^n a_k \mathbb{1}_{\Delta_k}$. By Theorem 4 of Sect. 13.3,

$$\left\| \sum_{k=1}^n a_k \mu_A(\Delta_k) \right\| = \|f(A)\| \leqslant \sup_{t \in \sigma(A)} |f(t)| = \sup_{1 \leqslant k \leqslant n} |a_k| \leqslant 1. \qquad \square$$

Theorem 2 (Main identity for the spectral measure). *For every bounded Borel function f on $\sigma(A)$,*

$$f(A) = \int\limits_{\sigma(A)} f\, d\mu_A.$$

Proof. For finitely-valued functions the required relation was proved in Lemma 1 above. Now let f be a bounded Borel function and let the sequence (f_n) of finitely-valued functions converge uniformly to f on $\sigma(A)$. Then

$$\|f_n(A) - f(A)\| = \|(f_n - f)(A)\| \leqslant \sup_{t \in \sigma(A)} |(f_n - f)(t)| \to 0 \quad \text{as } n \to \infty,$$

i.e., $f_n(A) \to f(A)$. On the other hand, by the uniform limit theorem (Theorem 2 of Subsection 13.4.2),

$$f_n(A) = \int\limits_{\sigma(A)} f_n \, d\mu_A \to \int\limits_{\sigma(A)} f \, d\mu_A \quad \text{as } n \to \infty.$$

The theorem is proved. □

Corollary 1. $A = \int\limits_{\sigma(A)} t \, d\mu_A(t)$. □

If an operator has a complete system of eigenvectors (i.e., the matrix of the operator is diagonalizable), then the structure of the operator becomes clear once its eigenvectors are calculated. For operators that are often encountered in various problems, it proves quite reasonable to carry out this work of calculating the eigenvalues and eigenvectors, even if not simple, so that subsequently the results of this investigation could be applied whenever needed. The spectral measure and the integral decompositions with respect to this measure play for self-adjoint operators the same role as the eigenvector expansions do for diagonalizable operators.

Exercises: Properties of the Spectral Measure

1. All values of the spectral measure are orthogonal projectors (called *spectral projectors*).

2. $\mu_A(\sigma(A)) = I$.

3. If the operator T commutes with A, then the spectral projectors of the operator A commute with T.

4. The image of every spectral projector is an invariant subspace of A.

5. $\int_{\sigma(A)} f_1 \, d\mu_A \cdot \int_{\sigma(A)} f_2 \, d\mu_A = f_1(A) f_2(A) = (f_1 f_2)(A) = \int_{\sigma(A)} (f_1 f_2) \, d\mu_A$; in particular, $\mu_A(D_1)\mu_A(D_2) = \mu_A(D_1 \cap D_2)$.

6. We note that the last property looks rather unusual: say, for a regular scalar-valued Borel charge μ on a compact space it can be satisfied (prove this as an exercise) only if μ is a probability measure concentrated at a single point.

7. Denote by X the image of the operator $\mu_A(D)$. Prove that $\sigma(A|_X) \subset \overline{D}$.

8. The point $\lambda \in \sigma(A)$ is an eigenvalue of the operator A if and only if $\mu_A(\{\lambda\}) \neq 0$.

9. The spectral measure of an operator with infinite spectrum does not possess the countable additivity property in the sense of norm convergence. At the same time, pointwise countable additivity does hold.

10. Let A be a diagonalizable operator. Prove that for any set $D \subset \sigma(A)$ the image of the projector $\mu_A(D)$ is the closure of the linear span of the set of eigenvectors of A associated to the eigenvalues that lie in D.

11. Let A be the multiplication operator in $L_2[0, 1]$ by the function $g(t) = t$: $(Af)(t) = tf(t)$. Show that for any set $D \subset \sigma(A)$ the image of the projector $\mu_A(D)$ is the set of all functions from $L_2[0, 1]$ that vanish identically in the complement of D.

12. Prove that the operator $\mu_A(\{0\})$ is the orthogonal projector onto the kernel of the operator A.

13. Prove that the set of invertible operators in $L(H)$ is connected.

13.4.4 Linear Equations

If $T \in L(H)$ is an invertible operator and $y \in H$, then the problem of solving the equation $Tx = y$ is equivalent to the same problem for the equation $T^*Tx = T^*y$, where T^*T, as we know, is a positive self-adjoint operator. For non-invertible T possessing a polar decomposition $T = UA$ with $A \geqslant 0$, the equation $Tx = y$ is equivalent to $Ax = U^*b$. These are some of the reasons why the most important linear equations in Hilbert space are of the form $Ax = b$, where $A \in L(H)$ is a given positive operator, $b \in H$ is a given element, and $x \in H$ is the unknown. So, in this subsection we consider linear equations with a positive self-adjoint operator A. As in the previous subsection, μ_A denotes the spectral measure of A.

Lemma 1. *Denote by P the orthogonal projector $\mu_A(\sigma(A) \setminus \{0\})$. Then $PA = A$.*

Proof. The equality $\mathbb{1}_{\sigma(A)\setminus\{0\}}(t) \cdot t = t$ holds everywhere on $\sigma(A)$. It remains to plug the operator A in it. □

Corollary 1. *For the equation $Ax = b$ to be solvable it is necessary that the element b satisfies the condition $Pb = b$.*

Proof. Indeed, if $Ax = b$ for some $x \in H$, then $Pb = PAx = Ax = b$. □

If one observes that the operator $Q = I - P = \mu_A(\{0\})$ is the orthogonal projector onto the kernel of the operator A, the condition $Pb = b$ can be written in the more familiar form $b \perp \operatorname{Ker} A$.

Lemma 2. *Let (f_n) be a non-decreasing sequence of bounded Borel functions which converge pointwise on the set $\sigma(A) \setminus \{0\}$ to the function $1/t$. Then the sequence of operators $(Af_n(A))$ converges pointwise to the operator P from the preceding lemma.*

Proof. Apply Theorem 5 of Sect. 13.3. □

Theorem 1. *Let (f_n) be a non-decreasing sequence of bounded Borel functions that converges pointwise on $\sigma(A) \setminus \{0\}$ to the function $1/t$, and let $b \in H$ be an element which satisfies the condition $Pb = b$. Then for the solvability of the equation $Ax = b$ it is necessary and sufficient that the sequence of elements $(f_n(A)(b))$ converges. In this case, the limit of the sequence will be one of the solutions of the equation.*

Proof. Suppose the equation $Ax = b$ is solvable and let x_0 be a solution. Then, by Lemma 2, $f_n(A)(b) = f_n(A)(Ax_0) = Af_n(A)x_0 \to Px_0$, so the convergence is proved. Conversely, suppose the sequence $f_n(A)(b)$ converges to some element x_0. Then by the same lemma, $Ax_0 = \lim_{n\to\infty} Af_n(A)(b) = Pb = b$. □

We note that if the operator A is injective, then $P = I$ and the condition $Pb = b$ is automatically satisfied. Further, if the operator A is invertible, then 0 does not lie in the spectrum of A and the function $1/t$ is continuous on $\sigma(A)$. Accordingly, one can take for (f_n) a uniformly convergent sequence of polynomials, and the rate of convergence of the elements $f_n(A)(b)$ to a solution will be estimated by the rate of convergence of the polynomials f_n (in this case no monotonicity of the sequence (f_n) is required). If the operator is given by an explicit expression, then the polynomials in this operator can also be written explicitly. Therefore, in the case of an invertible operator A, Theorem 1 provides a completely feasible method for solving the equation $Ax = b$ approximately. Needless to say, the lower the degree of the polynomial, the easier is to compute its value on an operator. Hence, here the most appropriate approach is to take as f_n the best approximation polynomials of the function $1/t$ on $\sigma(A)$.

In the case of an non-invertible operator the problem of finding approximate solutions to the equation $Ax = b$ is considerably more difficult. This problem belongs to the class of so-called *ill-posed problems*: arbitrarily small perturbations of the right-hand side can make the problem unsolvable, or strongly modify its solution. Since in approximate calculations all the initial data are usually also known only approximately, this issue is quite crucial. Help in solving this problem can come from exploiting a priori information about the solution which is not contained in the equation. In so doing, in any case the accuracy of the solution depends on the magnitude of the error in the right-hand side of the equation, and letting $n \to \infty$ in the sequence $(f_n(A)(b))$ does not lead to convergence to the solution. Moreover, as a rule, the approximating sequence approaches the solution only up to some moment, after which its behavior is in no way related to the true solution. Using the a priori information in order to find a reasonable step in the approximation is one of the available ideas for *regularizing* an ill-posed problem. For details on this subject one can consult the monograph by A. Tikhonov and V. Arsenin [41].

Exercises

1. What property of the spectrum of the operator A ensures the existence of a sequence (f_n) satisfying the conditions of Lemma 2 and Theorem 1?

2. Using Exercises 7 and 8 of Subsection 13.4.2 and the pointwise countable additivity of the spectral measure (Exercise 9 in Subsection 13.4.3), replace the monotonicity condition in Theorem 1 by the condition that the sequence $tf_n(t)$ is uniformly bounded on the spectrum of the operator A.

3. Let $\mu\colon \Sigma \to X$ be a vector measure. The set $D \in \Sigma$ is said to be *negligible* with respect to the measure μ if $\|\mu\|(D) = 0$. Is the set D being μ-negligible equivalent to the equality $\mu(D) = 0$? Does the answer change if μ is the spectral measure of a self-adjoint operator?

4. Prove Theorem 1 when pointwise convergence of the functions f_n is replaced by convergence almost everywhere with respect to the measure μ_A.

5. Is the set of noninvertible operators in $L(H)$ connected?

Comments on the Exercises

Subsection 13.1.1

Exercise 1. Use the following algebraic result: if p_1, p_2 is a pair of coprime polynomials, then there are polynomials q_1, q_2 such that $p_1q_1 + p_2q_2 = 1$. Substituting T one obtains $p_1(T)q_1(T) + p_2(T)q_2(T) = I$. For every $x \in X$ one gets the decomposition $x = x_1 + x_2$, where $x_1 = p_1(T)q_1(T)x$ and $x_2 = p_2(T)q_2(T)x$. It remains to show that $x_1 \in \operatorname{Ker} p_2(T)$, $x_2 \in \operatorname{Ker} p_1(T)$ and that $\operatorname{Ker} p_2(T) \cap \operatorname{Ker} p_1(T) = \{0\}$.

Exercise 5. For the differentiation formula $(e^f(t))' = f'(t)e^{f(t)}$ to hold, it is necessary that all values of the function f pairwise commute. In particular, this condition is satisfied when $f(t) = tA$, where $A \in L(X)$ is a fixed operator. In this case the differential equation $y' = f'(t)y$ becomes the equation with the constant operator coefficient $y' = Ay$.

Subsection 13.1.5

Exercise 3. By assumption, $B^2 = A$, and so B commutes with A. Then B also commutes with $\sqrt{A}$. We have $(B - \sqrt{A})(B + \sqrt{A}) = B^2 - A = 0$. Define the subspace $X \subset H$ as the closure of the image of the operator $B + \sqrt{A}$. The equality $(B - \sqrt{A})(B + \sqrt{A}) = 0$ means that on X the operator $B - \sqrt{A}$ is equal to zero. It remains to show that $B - \sqrt{A}$ is equal to zero on $X^\perp$. Since $B + \sqrt{A}$ is self-adjoint, the orthogonal complement of its image is its kernel: $X^\perp = \operatorname{Ker}(B + \sqrt{A})$. Again due to the commutativity, $\operatorname{Ker}(B + \sqrt{A})$ is an invariant subspace for the operators B and $\sqrt{A}$. Since B and $\sqrt{A}$ are positive operators and since on $X^\perp$ the operator $B + \sqrt{A}$ vanishes, it follows that $B = \sqrt{A} = 0$ on $X^\perp$ (Exercise 5 in Subsection 12.4.4). Therefore, $B - \sqrt{A}$ is indeed the zero operator on $X^\perp$.

Another way to solve this exercise is to apply Exercise 3 of Subsection 13.1.4 to the operator B and the functions $f(t) = t^2$ and $g(t) = \sqrt{t}$.

Subsection 13.2.3

Exercise 1. See the related Exercise 10 in Subsection 12.4.3.

Exercise 5. See Exercise 3 in Subsection 13.1.5 and its solution. This fact was used at the very beginning of the subsection to pass from the equality $T^*T = A^2$ to the equality $A = |T|$.

Section 13.3

Exercise 1. Use the fact that this property has already been established for continuous functions of an operator.

Exercise 2. By Exercise 1 in this subsection, A also commutes with $g(B)$. Applying again Exercise 1, but now to the operators $g(B)$ and A, we obtain the required assertion.

Subsection 13.4.2

Exercise 5. This classical Orlicz–Pettis theorem can be found, for example, in [13, p. 22].

Exercise 8. See [13, p. 14].

Subsection 13.4.3

Exercise 1. The equality $(\mathbb{1}_\Delta)^2 = \mathbb{1}_\Delta$ means that $\mu_A(\Delta)$ is a projector; thanks to self-adjointness, $\mu_A(\Delta)$ is an orthogonal projector.

Exercise 3. Use Exercise 1 of Sect. 13.3.

Exercise 4. Apply Exercise 3 and Theorem 1 of Subsection 11.1.5.

Exercises 7 and 8. Argue as in Theorem 2 of Subsection 13.1.5.

Exercises 9. The pointwise countable additivity follows from Theorem 5 of Sect. 13.3.

Exercise 13. Let $T \in L(H)$ be an invertible operator and $T = e^{iA}|T|$ be its polar decomposition (see Subsection 13.2.3 and Exercise 24 in Sect. 13.3). Define the continuous curve $F\colon [0, 1] \to L(H)$ by the formula $F(t) = e^{itA}|T|$. This curve passes only through invertible operators and connects the operator $|T|$ to the operator T. Further, the curve $G\colon [0, 1] \to L(H)$ given by the formula $G(t) = (1 - t)|T| + tI$ connects $|T|$ with the identity operator. Hence, in the set of invertible operators every operator can be joined to the identity operator by a continuous curve.