Chapter 1
A Quantum Mechanic's Toolbox

1.1 Bits and Qubits

The basic unit of information is called a *bit*. It represents a binary-valued quantity such as a yes/no answer to a question, the position of the toggle for an on/off switch, or a stop/go decision. For example, a light bulb can either be in the on or off state and thus serve as a storage device for a single bit of information. All digital computing machines, no matter how large and complex, are constructed from indivisible bits. Typically, the integers 0 and 1 denote the value of a bit. The *qubit* is a similar but distinct concept. I will elaborate on the difference between the two in subsequent discussions, but for now, we differentiate qubits from bits by a simple change in notation so that

$$\begin{aligned} 0 &\rightarrow |0\rangle \\ 1 &\rightarrow |1\rangle \end{aligned} \tag{1.1}$$

where $|0\rangle, |1\rangle$ are the two possible states of the qubit.

1.1.1 Binary Arithmetic

The values $0, 1$ of a bit form the *letters* of an alphabet that we call the binary number system. Just as words in the English language are concatenations of the 26 letters that comprise the western alphabet, so can a two-character bit alphabet form "words". The Morse code, where the "dash" and "dot" intervals are the two letters of the alphabet, is a familiar example of that proposition. The "book" of life written on a DNA polymer has four letters in its alphabet.

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Bits are also used to represent numbers. Obviously, the integer 0 is represented by the bit whose state is 0 and the number 1 by the bit whose value is 1. How about the integers 2, 3 . . . ? Consider the string

$$1100010$$

which is interpreted by the following *algorithm*,

$$1100010 \rightarrow 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$

and has the value 98 in the base ten system. Each entry in the string represents a coefficient of consecutive powers of 2^n, where n is an integer. Note that

$$98 = 0 \times 10^2 + 9 \times 10^1 + 8 \times 10^0$$

conforms to this algorithm but powers of 10^n replace 2^n, and the coefficients of each term are the symbols 0, 1, 2 . . . 9.

Mathematica Notebook 1.1: The binary system and operations within it http://www.physics.unlv.edu/%7Ebernard/MATH_book/Chap1/chap1_link.html

Imagine a set, or *register*, of five light bulbs. As each light bulb can store a bit of information, a little thought shows that this register can store one of $2^5 = 32$ integers at any single instant of time. For example, reverting to qubit notation, the number 26 is represented by the physical state

$$|11010\rangle.$$

In this notation, the first light bulb, starting from the right-hand side (r.h.s), is off, the second one is on, the third is off, while the fourth and fifth bulbs are in the on position. Consider the following expression

$$|11010\rangle + |00101\rangle. \tag{1.2}$$

What should the + symbol signify here and how can we interpret this construct? At first sight, it might seem reasonable to define it as the arithmetic operation of addition so that $|11010\rangle + |00101\rangle$ is equal to $|11111\rangle$ which represents the integer 31 in binary form. It is apparent, under scrutiny, that this definition is unsatisfactory for the following reason. We agreed that $|11010\rangle$ represents a physical state in which the light bulbs have the corresponding on-off values at a single instance of time. Thus the expression $|11010\rangle + |01001\rangle$ suggests a register of light bulbs simultaneously in two different configurations at the same time, an absurd proposition and so construct (1.2) appears to be meaningless.

Let's posit a five-qubit register comprised of the quantum mechanical analog of a light bulb, that I call a $q$bulb. Because atoms/ions obey the laws of quantum mechanics (QM), it is plausible to define a $q$bulb register as an array of five atoms, each of which can be toggled on and off, in analogy with a light bulb.[1] A more precise definition is forthcoming in later chapters. As a quantum system, this array of atoms/ions must obey the postulates of the quantum theory. Within this theory, we will show how to make sense of expression (1.2). Before elaborating on this statement, let's first embark on a short mathematical detour.

1.2 A Short Introduction to Linear Vector Spaces

Consider a set of objects $\alpha, \beta, \gamma \cdots$. We say that these objects belong to a *linear vector space* V provided that

(i) There exists an operation, which we denote by the $+$ sign, so that if α, γ are any two members of the vector space V then so is the quantity $\alpha + \gamma$.
(ii) For scalar c, there exists a scalar multiplication operation defined so that if β is a vector in V then so is $c\,\beta = \beta\, c$. If $a,\ b$ are scalars $a\, b\, \beta = a(b\, \beta)$.
(iii) Scalar multiplication is distributive, i.e. $c(\alpha + \beta) = c\,\alpha + c\beta$, also for scalar a, b, $(a + b)\alpha = a\alpha + b\alpha$.
(iv) The $+$ operation is associative, i.e. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
(v) The $+$ operation is commutative, i.e. $\alpha + \beta = \beta + \alpha$.
(vi) There exists a null vector 0 which has the property $0 + \alpha = \alpha$ for very vector α in V.
(vii) For every α in V there exists an inverse vector $-\alpha$ that has the property

$$\alpha + -\alpha = 0.$$

Numerous mathematical structures form vector spaces. Perhaps the most familiar are the vectors that define a direction in space. Convince yourself that the set of numbers on the real number line form a vector space. Do the set of all integers constitute a vector space?

Armed with these definitions, we are ready to tackle, in the context of the five-$q$bulb register, the postulates of quantum mechanics. Because our discussion here is limited to the five-$q$bulb register, we stress the tentative nature of these postulates by labeling them as meta-postulates or m-postulates for short.

m-Postulate I Following a measurement (observation) of this quantum register, we observe only one out of the 32 possible $q$bulb-on/off configurations. Immediately after that measurement the register is found in the state corresponding to the values measured.

[1] David Wineland [7], shows that such a notion is not as fanciful as might first appear.

For example, if the measurement resulted in the first $q$bulb on, the second off, the last three on, the state corresponding to this measurement is $|11101\rangle$, and the postulate asserts that the system occupies this state immediately after measurement. The verity of this postulate appears to be self-evident. It almost seems not worthwhile stating as it surely applies to the classical counterpart of this system. After introducing the second postulate we will appreciate its significance and implications.

m-Postulate II a The possible 32 states of our qubit register are vectors in a linear vector space.

According to this postulate, and property (i) of the itemized list above, expression (1.2) must also be a vector in this space. Indeed so is

$$
\begin{aligned}
&|\Phi\rangle \equiv \\
&|00000\rangle + |00001\rangle + |00010\rangle + |00011\rangle + |00100\rangle + |00101\rangle + |00110\rangle + |00111\rangle \\
&|01000\rangle + |01001\rangle + |01010\rangle + |01011\rangle + |01100\rangle + |01101\ \rangle + |01110\rangle + |01111\rangle \\
&|10000\rangle + |10001\rangle + |10010\rangle + |10011\rangle + |10100\rangle + |10101\rangle + |10110\rangle + |10111\rangle \\
&|11000\rangle + |11001\rangle + |11010\rangle + |11011\rangle + |11100\rangle + |11101\rangle + |11110\rangle + |11111\rangle,
\end{aligned}
\tag{1.3}
$$

as is any combination defined by the + operation. This property, that a quantum state $|\Psi\rangle$ can be expressed as a linear combination of other quantum states, is sometimes called the *superposition principle* (Fig. 1.1).

m-Postulate II b A complete physical description of this quantum register is encapsulated by a vector $|\Psi\rangle$ in this vector space.

What is a complete physical description? A classical register requires an itemization of bulbs that are on/off to characterize its physical state. For the quantum version of this system, m-Postulate II states that an abstract quantity, a vector $|\Psi\rangle$, that is a linear combination of states in which the system finds itself after a measurement is made, defines the system. For example, suppose the complete physical description of our quantum register is given by the vector (state) (1.3), i.e. $|\Phi\rangle$. Construct $|\Phi\rangle$ suggests that all possible on/off configurations exist at the same time, a strange proposition that counters everyday experience. If we are going to make sense of this theory, the necessity for m-Postulate I becomes apparent. If $|\Phi\rangle$ does exist, m-Postulate I insures that we observe only one of the possible $q$bulb configurations. In addition, immediately after that observation $|\Phi\rangle$ "collapses" into that particular state. The above scenario should stimulate lots of questions, perhaps even healthy skepticism. m-Postulate II argues for a "reality" in which all possible outcomes exist at the same time. m-Postulate I grounds us because it prevents direct observation of

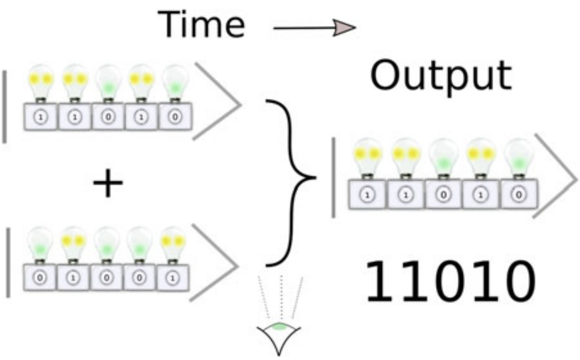

Fig. 1.1 The $|11010\rangle + |01001\rangle$ superposition state prior to measurement (left panel). The measured register value 11010 results in the collapse of the amplitude into state $|11010\rangle$. The normalization constant is not shown in this figure. The yellow bulbs are in the on-state, denoted by the binary digit 1

this "reality". The theory as expressed by these postulates reminds us of the child's conundrum; is the moon there when we are not watching it? [5].

We have not yet addressed the question; which of the possible 32 configurations do we observe after a measurement on a system described by $|\Phi\rangle$? m-Postulate I guarantees that one of the configurations is found but says nothing about which one. To answer this question we look toward another postulate, sometimes called the *Born rule*, after *Max Born* one of the founders of the quantum theory. But first we need to provide additional structure to our linear vector space. That discussion leads us to consider a special type of vector space called a Hilbert space.

1.3 Hilbert Space

We pointed out that if $|\alpha\rangle$ is a vector so is $c\,|\alpha\rangle$ where c is a scalar quantity. In Hilbert space the scalar c is generally a complex number. Consider the following linear combination of n vectors

$$c_1\,|\alpha_1\rangle + c_2\,|\alpha_2\rangle + \ldots c_n\,|\alpha_n\rangle. \tag{1.4}$$

If this sum equates to the null vector 0 only if $c_1 = c_2 \cdots = c_n = 0$ then the set of vectors $|\alpha_1\rangle, |\alpha_2\rangle, \ldots |\alpha_n\rangle$ are said to be *linearly independent*. A space that admits n linear independent vectors, but not $n+1$, is called an n-dimensional space. In general, a Hilbert space allows infinite dimension [2, 4] but we are primarily concerned, in this text, with Hilbert spaces spanned by a finite and denumerable set of basis vectors. The Hilbert space of the five-$q$bulb system has dimension 32.

1.3.1 *Dirac's Bra-Ket Notation*

The *inner, or scalar product* is a Hilbert space structure that provides a measure of the degree of "overlap" between two vectors. The dot product of two vectors in ordinary three-dimensional Euclidian space is an example of the latter. Here we will be a little bit more abstract and introduce the bra-ket notation to define an inner product. Indeed, we already made use of this notation in the previous sections where we employed the symbol $|\ldots\rangle$ to denote a vector in Hilbert space. It is called a ket and was introduced by the brilliant twentieth century physicist *Paul Dirac*, an architect of the modern quantum theory. Throughout this text, we will employ ket notation to describe quantum states. For example, ket $|\Psi\rangle$ is a vector in Hilbert space where the symbol Ψ nested within the ket is an identification parameter.

Having established ket notation, we introduce a new linear vector space called *dual space*. For our purposes, it is convenient to think of dual space as a mirror image of the vectors (or kets) in Hilbert space. For example, the ket $|\alpha\rangle$ has a mirror counterpart in dual space. The label α should also parameterize it, but we cannot use ket notation as this vector "lives" in a different linear vector space. Dirac suggested the notation $\langle\alpha|$ to denote it and called the symbol $\langle\ldots|$ a bra. Keeping in mind that every ket $|\beta\rangle$ has a corresponding bra $\langle\beta|$, we require that the ket

$$|\Psi\rangle = c_1|\alpha_1\rangle + c_2\,|\alpha_2\rangle + \ldots c_n\,|\alpha_n\rangle \tag{1.5}$$

has a bra counterpart. The rule for generating a bra $\langle\Psi|$ from ket $|\Psi\rangle$ is

$$\langle\Psi| = c_1^*\langle\alpha_1| + c_2^*\,\langle\alpha_2|, \ldots c_n^*\,\langle\alpha_n|. \tag{1.6}$$

The expansion coefficients for the bra vector are complex conjugates of the corresponding coefficients in ket space. Kets and bras are added to each of their kind, so the following expression

$$|\Psi\rangle + \langle\Phi|$$

is meaningless, but we are allowed to build additional structures by certain combinations of kets and bras. Two constructs which use kets and bras as its building blocks are expressions that have the form

$$\langle\Phi|\Psi\rangle \tag{1.7}$$

and

$$|\Psi\rangle\langle\Phi|. \tag{1.8}$$

The former (1.7), denotes an inner product and evaluates to a complex number. The latter expression (1.8) is called an *outer product*, and it is neither a scalar or vector. Later we show how it can be interpreted as an *operator*. Both structures enable various transformations and manipulations of vectors in Hilbert space. Let's first discuss the inner, or scalar, product. The scalar quantity defined by expression (1.7) provides a measure of overlap between the vectors $|\Phi\rangle, |\Psi\rangle$.[2] According to Dirac notation, the inner product of two vectors in Hilbert space $|\Phi\rangle, |\Psi\rangle$ is formed by taking the dual of $|\Phi\rangle$, i.e. the bra $\langle\Phi|$, and placing it side-by-side, as if a Lego® block, with the ket $|\Psi\rangle$. We also require that

$$\langle\Phi|\Psi\rangle = \langle\Psi|\Phi\rangle^*, \tag{1.9}$$

i.e. they are complex conjugates of each other. According to condition (1.9), we recognize that $\langle\Psi|\Psi\rangle = \langle\Psi|\Psi\rangle^*$ and so the inner product of a vector with itself must be a real number. An additional postulate requires

$$\langle\Psi|\Psi\rangle \geq 0. \tag{1.10}$$

Property (1.10) allows us to affix a "length" $\sqrt{\langle\Psi|\Psi\rangle}$ to any vector, in particular, if $\langle\Psi|\Psi\rangle = 1$ that vector is said to be of unit length or *normalized*.

We now state Dirac's distributive axiom for inner products. Given two vectors $|\Psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle$ and $|\Phi\rangle = d_1|\alpha_1\rangle + d_2|\alpha_2\rangle$ the inner product of $|\Psi\rangle$ with $|\Phi\rangle$ is

Dirac's distributive axiom for scalar products

$$\langle\Psi|\Phi\rangle = \Big(c_1^*\langle\alpha_1| + c_2^*\langle\alpha_2|\Big)\Big(d_1|\alpha_1\rangle + d_2|\alpha_2\rangle\Big) =$$
$$c_1^* d_1\langle\alpha_1|\alpha_1\rangle + c_1^* d_2\langle\alpha_1|\alpha_2\rangle + c_2^* d_1\langle\alpha_2|\alpha_1\rangle + c_2^* d_2\langle\alpha_2|\alpha_2\rangle. \tag{1.11}$$

The axiom shows how to distribute the inner product of a compound expression with all its components. Two vectors $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are said to be *orthogonal* if they have a null inner product, i.e.

$$\langle\alpha_1|\alpha_2\rangle = \langle\alpha_2|\alpha_1\rangle = 0.$$

Suppose we have a set, in an n-dimensional Hilbert space, of n normalized linear independent vectors $|\alpha_1\rangle, |\alpha_2\rangle, \ldots |\alpha_n\rangle$ that are mutually orthogonal. That is if for each $|\alpha_i\rangle$

$$\langle\alpha_i|\alpha_j\rangle = \delta_{ij} \tag{1.12}$$

for all i, j (Here, δ_{ij} is the *Kronecker delta* which has the property $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$). The set of vectors $|\alpha_i\rangle$ that satisfies those properties are said to form a *basis* for the vector space. You are probably familiar with the unit vectors, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ that constitute a basis for three-dimensional Euclidean space.

[2] In older textbooks the definition for the inner product is given by the alternative notation $(|\Phi\rangle, |\Psi\rangle)$ and does not rely on the use of dual space.

m-Postulate III a	The 32 vectors itemized on the right-hand side of (1.3) form a basis that spans the Hilbert space of the five-qubit register.

According to this postulate any of these vectors, for example, $|10110\rangle$, is orthogonal to all other vectors itemized on the r.h.s of (1.3) and each vector has unit length. Basis vectors are *orthonormal*, meaning that they are both orthogonal and have unit length. According to m-Postulate II b

$$|\Psi\rangle = \sum_{i=1}^{n} c_i |\alpha_i\rangle \tag{1.13}$$

and if we require that $|\Psi\rangle$ has unit length, i.e. $\langle\Psi|\Psi\rangle = 1$, we find that

$$\langle\Psi|\Psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i^* c_j \langle\alpha_i|\alpha_j\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i^* c_j \delta_{ij} = \sum_{i=1}^{n} |c_i|^2 = 1. \tag{1.14}$$

Taking the inner product of $|\alpha_m\rangle$ with $|\Psi\rangle$, we obtain

$$\langle\alpha_m|\Psi\rangle = \sum_{j=1}^{n} c_j \langle\alpha_m|\alpha_j\rangle = \sum_{j=1}^{n} c_j \delta_{mj} = c_m \tag{1.15}$$

where we used the orthonormality condition (1.12).

Before proceeding further it's useful to consolidate our understanding of this formalism with more familiar examples from vector calculus. We are all accustomed to expressions like,

$$\vec{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}.$$

It is a vector in 3-d Euclidian space expressed in terms of the unit basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. The latter are a set of orthonormal vectors since

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0$$

and

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1.$$

The scalars A_x, A_y, A_z are the components of vector $\vec{A}$ so that, using the orthonormality properties of the basis,

$$A_x = \vec{A}\cdot\hat{\mathbf{i}} \quad A_y = \vec{A}\cdot\hat{\mathbf{j}} \quad A_z = \vec{A}\cdot\hat{\mathbf{k}}.$$

Similarly, the Hilbert space vector

$$|\Psi\rangle = c_1\,|\alpha_1\rangle + c_2\,|\alpha_2\rangle + \cdots c_n\,|\alpha_n\rangle \tag{1.16}$$

is a linear combination of orthonormal basis vectors $|\alpha_i\rangle$. Using orthonormality we find,

$$c_1 = \langle\alpha_1\,|\psi\rangle\,,\ c_2 = \langle\alpha_2\,|\psi\rangle\,,\ \ldots c_n = \langle\alpha_n\,|\psi\rangle\,. \tag{1.17}$$

The similarity of expressions (1.16), (1.17) with the corresponding relations for vector $\vec{A}$ should be apparent. In the next section we discuss and elaborate on the Hilbert space outer product (1.8), but before going there let's review an analogous construct, the *dyadic* in Euclidean space. For the space covered by the basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, a dyadic is a bilinear expression such as $\hat{\mathbf{i}}\,\hat{\mathbf{j}}$ or any of the other eight combinations of basis vectors. A dyadic is positioned either before, or after, a vector $\vec{A}$ so that

$$\begin{aligned}\hat{\mathbf{j}}\,\hat{\mathbf{k}}\,\vec{A} &\equiv \hat{\mathbf{j}}(\hat{\mathbf{k}}\cdot\vec{A}) = \hat{\mathbf{j}}\,A_z\\ \vec{A}\,\hat{\mathbf{j}}\,\hat{\mathbf{k}} &\equiv (\vec{A}\cdot\hat{\mathbf{j}})\hat{\mathbf{k}} = \hat{\mathbf{k}}\,A_y.\end{aligned} \tag{1.18}$$

Table 1.1 summarizes some similarities between vector calculus constructs in Euclidean space and the Hilbert space of a n-qubit system.

Table 1.1 Examples and comparison of three-dimensional Euclidean space structures with n-dimensional Hilbert space analogues

Structure	Euclidean space	Hilbert space
Basis expansion	$\vec{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$	$\|\Psi\rangle = c_1\,\|\alpha_1\rangle + c_2\,\|\alpha_2\rangle + \cdots c_n\,\|\alpha_n\rangle$
Inner product	$\vec{A}\cdot\vec{B}$	$\langle\Phi\,\|\Psi\rangle$
Basis components	$A_x = \vec{A}\cdot\hat{\mathbf{i}}, \ldots$etc.	$c_i = \langle\alpha_i\,\|\Psi\rangle \quad i = 1, 2\ldots n$
Outer product	(dyadic) $\hat{\mathbf{j}}\,\hat{\mathbf{k}}, \ldots$etc.	$\|\alpha_i\rangle\,\langle\alpha_j\|$

At this stage of the narrative, it is useful to introduce a new, more economical, notation to represent basis vectors. In this alternative notation $|00000\rangle \equiv |0\rangle_5$, $|00010\rangle \equiv |2\rangle_5$, $|01001\rangle \equiv |9\rangle_5$, etc., i.e., we label each basis ket by the Roman numeral value of the binary number label. The subscript 5 reminds us that we are labeling a five-qubit register. Therefore $|\Phi\rangle = \sum_{i=0}^{31}|i\rangle_5$ and $\langle\Phi|\Phi\rangle = \sum_{i=0}^{31}\sum_{j=0}^{31}{}_5\langle i|j\rangle_5 = 32$ where the last identity follows from the fact that basis vectors are orthonormal. Because vectors in Hilbert space have a length, we attach an addendum to m-Postulate II b and require physical states to be of unit length, i.e. $\langle\Psi|\Psi\rangle = 1$. We are now in a position to state the Born rule for our five-qubit register.

m-Postulate III b (the Born rule) If the five-qubit register is in state $|\Psi\rangle = \sum_{i=0}^{31} c_i\,|i\rangle_5$ then a measurement yields the $q$bulb configuration corresponding to one of the 32 states $|i\rangle_5$, with probability $p_i = |c_i|^2 = |\langle i|\Psi\rangle_5|^2$. The condition $\langle\Psi|\Psi\rangle = 1$, insures that $\sum_i p_i = 1$.

The Born rule addresses the question not answered by m-Postulate I. To explore and appreciate its import we invoke a hypothetical *gedanken experiment*.[3] Imagine 1000 experimenters who each have a five-$q$bulb register in their respective laboratories. Furthermore, assume that each system is described by an identical state vector $|\Psi\rangle$. Each $q$bulb configuration is measured and m-Postulate I asserts that a measurement finds one of 32 possible configurations. The scientists tabulate the results of their measurements in lab books and convene a meeting later in the day to compare the data. A possible tally might look like that given in Table 1.2. For this trial, the scientists observed only four configurations out of the possible 32. In summary, 101 physicists found all $q$bulbs (atoms/ions) in the off position corresponding to collapsed state $|0\rangle_5$, 209 found that only the second $q$bulb (from the left) is on, corresponding to collapsed state $|8\rangle_5$ and 321 scientists found a configuration in which the first and last bulbs are on, corresponding to collapsed state $|17\rangle_5$. Finally, 369 found that the two left-most $q$bulbs are on, corresponding to the state $|24\rangle_5$. The frequency interpretation of probability tells us that, after a large number N of trials, the probability of obtaining the i'th result p_i = (# of trials resulting in choice i)/N. Clearly, $\sum_i p_i = 1$.

Table 1.2 Tally of results for the gedanken experiment

Measurement	00000	01000	10001	11000
# of observations	101	209	321	369

Mathematica Notebook 1.2: An Introduction to Probability Theory http://www.physics.unlv.edu/%7Ebernard/MATH_book/Chap1/chap1_link.html

With this tabulated data we venture an educated guess for state $|\Psi\rangle$. One reasonable hypothesis is

$$|\Psi\rangle = \sqrt{\frac{101}{1000}}\,|0\rangle_5 + \sqrt{\frac{209}{1000}}\,|8\rangle_5 + \sqrt{\frac{321}{1000}}\,|17\rangle_5 + \sqrt{\frac{369}{1000}}\,|24\rangle_5, \quad (1.19)$$

as it is consistent with the Born rule. However, it is not a unique choice. The expansion coefficients c_i are complex numbers and $|c_i|^2$, the probability measure, is un-changed if the coefficients are altered by an arbitrary phase β, i.e., $c_i \to \exp(i\beta)c_i$. For example, if the coefficient $c_0 = \sqrt{101/1000}$ of the first term in sum (1.19) is replaced by $-\mathrm{i}\sqrt{101/1000}$ the probability distribution is unchanged. Also, the frequency interpretation does not guarantee those outcomes unless $N \to \infty$. So repeating the experiment with another set of 1000 trials, a small number could, because of statistical fluctuations, instantiate configurations that do not appear in Table 1.2.

[3] Thought experiment.

m-Postulate-III b informs us that quantum mechanics is a probabilistic theory where full knowledge of the system, i.e., $|\Psi\rangle$, does not guarantee a definite outcome for a measurement. It does offer a probability distribution of outcomes for an ensemble of measurements. Knowing $|\Psi\rangle$ we can predict the relative outcome of any configuration, let's say 11100, over the other 31 possibilities. Born's rule and (1.15) tells us that evaluation of $|\langle 11100|\Psi\rangle|^2$ provides us with that information.

I live and work in Las Vegas, a city whose existence attests to the predictive power of probabilistic inference. With the Copenhagen interpretation of quantum mechanics, Born's rule illuminates the mystery behind expressions like $|10001\rangle + |10011\rangle$. In that interpretation, there is no reality[4] where configurations 10001 and 10011 are simultaneously in "existence". Born's rule simply provides an operational algorithm for calculating probabilities of possible outcomes in an experiment. However, once a measurement is made, m-Postulate I requires that the system "collapses" into the basis state corresponding to that measurement outcome. Immediately after that measurement, since all $c_i = 0$, save the state j in which the system "collapses" to, a subsequent measurement has probability 1, or certainty, for that outcome.

1.3.2 Outer Products and Operators

In this section, we use Dirac's bra-ket formalism to construct an outer product. We first state Dirac's distributive axiom for outer products. Given states $|\Psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle$ and $|\Phi\rangle = d_1|\alpha_1\rangle + d_2|\alpha_2\rangle$ the outer product of $|\Psi\rangle$ with $|\Phi\rangle$ is

Dirac's distributive axiom for outer products

$$|\Psi\rangle\langle\Phi| = \Big(c_1|\alpha_1\rangle + c_2|\alpha_2\rangle\Big)\Big(d_1^*\langle\alpha_1| + d_2^*\langle\alpha_2|\Big) = \tag{1.20}$$
$$c_1 d_1^*|\alpha_1\rangle\langle\alpha_1| + c_1 d_2^*|\alpha_1\rangle\langle\alpha_2| + c_2 d_1^*|\alpha_2\rangle\langle\alpha_1| + c_2 d_2^*|\alpha_2\rangle\langle\alpha_2|.$$

Consider the outer product $\mathbf{X} \equiv |\Phi\rangle\langle\Psi|$. According to Dirac notation we are allowed to place it in front of a ket $|\Gamma\rangle$, or to the right of a bra $\langle\Gamma|$, i.e. the constructs $\mathbf{X}\,|\Gamma\rangle$ and $\langle\Gamma|\,\mathbf{X}$ are valid expressions in Hilbert and dual space respectively. However, the expressions $|\Gamma\rangle\,\mathbf{X}$, $\mathbf{X}\langle\Gamma|$ are illegal.

Dirac's associative axiom for outer products states that

Dirac's associative axiom for outer products For an outer product $\mathbf{X}$ and ket $|\Gamma\rangle$

$$\mathbf{X}\,|\Gamma\rangle = \Big(|\Phi\rangle\langle\Psi|\Big)\,|\Gamma\rangle = |\Phi\rangle\Big(\langle\Psi|\Gamma\rangle\Big) = c\,|\Phi\rangle$$
$$\text{where} \quad c \equiv \langle\Psi|\Gamma\rangle, \quad \text{Also}$$

[4]There exist alternative descriptions, including the many-worlds interpretation of QM [1, 6] and the consistent histories approach [3, 6], but in this monograph we adhere to orthodox dogma.

$$\langle \Gamma | \mathbf{X} = \langle \Gamma | \Big(|\Phi\rangle\langle\Psi| \Big) = \Big(\langle \Gamma | \Phi \rangle \Big) \langle \Psi | = \langle \Psi | \, d$$
$$d \equiv \langle \Gamma | \Phi \rangle. \tag{1.21}$$

According to (1.21), placing an outer product to the left of a ket, "un-hinges" the bra vector $\langle \Psi |$ from $\mathbf{X}$ and "locks" it onto the right-most ket $|\Gamma\rangle$ forming the scalar product, i.e., the complex number $\langle \Psi | \Gamma \rangle$. The result is a new ket $c\,|\Phi\rangle$. In short, outer product $\mathbf{X}$ "operates" on vector $|\Gamma\rangle$ and transforms it to vector $c\,|\Phi\rangle$ in Hilbert space. When operating on bra vectors, it plays a similar role in dual space. Evidently, outer products play an essential role in facilitating transformations of vectors in Hilbert space. Thus the outer product allows us to construct *operators* in Hilbert space. We can think of operators as objects that map vectors to other vectors in Hilbert space.

According to (1.21) we find that the dual of the transformed vector $\mathbf{X}\,|\Gamma\rangle$ is $\langle \Phi |\, c^*$ where $c^* = \langle \Gamma | \Psi \rangle$. So the dual of $\mathbf{X}|\Gamma\rangle \neq \langle \Gamma | \, \mathbf{X}$. There is an outer product called $\mathbf{X}^\dagger$ that has the following property.

Definition 1.1 The adjoint operation For operator $\mathbf{X}$ and ket $|\Phi\rangle$, the dual of $\mathbf{X}\,|\Phi\rangle$ is given by the expression $\langle \Phi | \, \mathbf{X}^\dagger$ for all $|\Phi\rangle$. $\mathbf{X}^\dagger$ is called the *adjoint*, or conjugate transpose, operator to $\mathbf{X}$.

Definition 1.2 Hermitian operators Operators $\mathbf{X}$ that have the property

$$\mathbf{X} = \mathbf{X}^\dagger \tag{1.22}$$

are called Hermitian, or self-adjoint, operators.

Definition 1.3 Unitary operators Operator $\mathbf{U}$ is a unitary operator, if

$$\mathbf{U}\,\mathbf{U}^\dagger = \mathbf{U}^\dagger\,\mathbf{U} = \mathbb{1} \tag{1.23}$$

where $\mathbb{1}$ is a unit operator, i.e. $\mathbb{1}|\Psi\rangle = |\Psi\rangle$ for all $|\Psi\rangle$ in Hilbert space.

Both Hermitian and unitary operators play a central role in quantum computing and information (QCI) applications.

We know that operators map, or transform, a vector in Hilbert space to another vector in that same space. A special class of mappings, generated by operator $\mathbf{X}$, have the following property. For some vectors $|\Phi\rangle$,

$$\mathbf{X}\,|\Phi\rangle = \phi\,|\Phi\rangle \tag{1.24}$$

where ϕ is a scalar. An equation of this type is called an *eigenvalue* equation. The vector $|\Phi\rangle$ is called an *eigenvector*, and the constant ϕ is called the eigenvalue associated with that eigenvector. Two theorems [2] that concern important properties of Hermitian operators, are

Theorem 1.1 *The eigenvalues of a Hermitian operator are real numbers.*

Theorem 1.2 *If the eigenvalues of Hermitian operator are distinct, then the corresponding eigenvectors are mutually orthogonal. If some of the eigenvalues are not distinct, or* degenerate*, then a linear combination of that subset of eigenvectors can be made to be mutually orthogonal.*

Let's investigate how some of these concepts relate to our five-qubit register. Because we are performing all operations in 32-dimensional Hilbert space, kets $|j\rangle_5$ will simply be replaced with the symbol $|j\rangle$. Consider operator $\mathbf{N}_6 \equiv |00110\rangle\langle 01100|$, or in the alternate notation $|6\rangle\,\langle 6|$. First, note that $\mathbf{N}_6$ is Hermitian, and according to Theorem 1.1, its eigenvalues are real numbers. Indeed, the eigenvalues for $\mathbf{N}_6$ are 1 and 0, associated with eigenvectors $|6\rangle$ and kets $|j\rangle$ for $j \neq 6$ respectively.

Proof Using Dirac's associative axiom

$$\mathbf{N}_6\,|6\rangle = (|6\rangle\,\langle 6|)\,|6\rangle = |6\rangle\,(\langle 6|6\rangle) = 1\,|6\rangle$$

and

$$\mathbf{N}_6\,|j\rangle = |6\rangle\,(\langle 6|j\rangle) = 0\,|j\rangle \quad j \neq 6$$

where we have used the orthonormality property of the vectors $|j\rangle$. Note that the eigenvalue 0, is associated with each of the $|j \neq 6\rangle$ eigenvectors. It is 31-fold degenerate because each $|j \neq 6\rangle$ share the same eigenvalue. Convince yourself that the eigenvectors of $\mathbf{N}_6$ also satisfy Theorem 1.2.

Let's define yet another operator, $\mathbf{N} \equiv \sum_{j=0}^{31} j\,(|j\rangle\langle j|)$. Here, outer product $(|j\rangle\langle j|)$ is multiplied by the integer j that labels each ket. Now

$$\mathbf{N}^\dagger = \sum_{j=0}^{31} j\,(|j\rangle\langle j|)^\dagger = \sum_{j=0}^{31} j\,(|j\rangle\langle j|) = \mathbf{N} \tag{1.25}$$

and is therefore Hermitian.

Its eigenvectors $|j\rangle$ are labeled by j which happens to be an eigenvalue of $\mathbf{N}$. According to the Born rule, a measurement, with device $\mathbf{N}$ of the $q$bulb register in state $|\Psi\rangle$, yields the probability $p_j = |\langle j|\Psi\rangle|^2$ that the register is found in the j'th configuration. In other words, p_j is calculated by taking the inner product of the ket that is an eigenvector of $\mathbf{N}$, that is $|j\rangle$ with the system state vector $|\Psi\rangle$. The eigenvalue associated with this ket is the measured configuration index. For this reason, operator $\mathbf{N}$ is a configuration, or *occupation number*, *measurement operator*, as its eigenvalues identify each possible configuration label revealed by a measurement.

1.3.3 Direct and Kronecker Products

Up to this point, we introduced and discussed almost all foundational postulates, theorems, and definitions that are needed to describe a five-$q$bulb quantum register. To explore Hilbert spaces beyond the five-qubit example, we need to introduce an additional tool, that of the *direct, or tensor*, product. When we think of points in a plane as an ordered pair (x, y), we recognize it as the pairing of a x-axis coordinate with a coordinate on the y-axis number line. The pair is called a *Cartesian product* of two lower dimensional vector spaces, i.e., the real x and y-axis number lines. In the same way, we build higher-dimensional Hilbert spaces from direct products of single qubit Hilbert spaces. Below, we will show how to construct a register that contains an arbitrary number of qubits from a constituent, lower dimensional space via direct products.

First, let's discuss in more detail a single qubit system of Hilbert space dimension 2. The kets $|0\rangle_1$ and $|1\rangle_1$ are possible basis vectors as we require them to be linearly independent and orthonormal. The subscript reminds us that ket $|0\rangle_1$ differs from ket $|0\rangle_5$. The latter represents a five-qubit register in which all $q$bulbs are in the off-position, whereas the former denotes the off-position of a single $q$bulb. In the subsequent discussion, we will assume, unless otherwise stated, that vectors $|0\rangle$, $|1\rangle$ are single-qubit vectors and ignore the subscript. Any state vector $|\Psi\rangle$ in this Hilbert space can be expressed as a linear combination of those two basis kets. A single $q$bulb (e.g., a two-state atom) serves as a physical realization of this Hilbert space.

Definition 1.4 The direct product. Given kets $|a\rangle$, $|b\rangle$ their direct product is given by the expression

$$|a\rangle \otimes |b\rangle.$$

Suppose $a, b \in 0, 1$, i.e. both $|a\rangle$, $|b\rangle$ are single qubit basis kets. Itemize all distinct combinations of indices a, b to get

$$|0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle. \tag{1.26}$$

Note that the direct product of $|a\rangle$ with $|b\rangle$ is not the same as the direct product of $|b\rangle$ with $|a\rangle$, i.e.

$$|a\rangle \otimes |b\rangle \neq |b\rangle \otimes |a\rangle.$$

Without proof we state the following.

Theorem 1.3 *Given a Hilbert space of dimension d that is spanned by basis vectors $|b\rangle$, and the single qubit vector $|a\rangle$ for $a \in 0, 1$, the direct products $|a\rangle \otimes |b\rangle$ for all a, b are basis vectors in a Hilbert space of dimension $2d$. The dual vector for*

$$c\,|a\rangle \otimes |b\rangle$$

is

$$c^* \langle b| \otimes \langle a|$$

for all values of a, b.

Note the ordering of the bra vectors from that of the corresponding kets. According to this theorem vectors $|a\rangle \otimes |b\rangle$ allow inner products with all vectors in the direct product Hilbert space. Let's assume that $|a\rangle, |b\rangle, |c\rangle, |d\rangle$ are single qubit states. Given $|\Psi\rangle = c_1|a\rangle \otimes |b\rangle + c_2|c\rangle \otimes |d\rangle$, and $|\Phi\rangle = d_1|a\rangle \otimes |b\rangle + d_2|c\rangle \otimes |d\rangle$ the inner product $\langle\Psi|\Phi\rangle$ is obtained by application of Dirac's axiom in the following manner

$$\begin{aligned}
&\Big(c_1^*\langle b| \otimes \langle a| + c_2^*\langle d| \otimes \langle c|\Big)\Big(d_1|a\rangle \otimes |b\rangle + d_2|c\rangle \otimes |d\rangle\Big) = \\
&c_1^* d_1 \langle b|b\rangle \langle a|a\rangle + c_1^* d_2 \langle b|d\rangle \langle a|c\rangle + c_2^* d_1 \langle c|a\rangle \langle d|b\rangle + c_2^* d_2 \langle c|c\rangle \langle d|d\rangle = \\
&c_1^* d_1 + c_2^* d_2.
\end{aligned} \tag{1.27}$$

where the last identity follows from the orthonormality property of the single-qubit kets. Using (1.27) we find that the four vectors itemized in (1.26) are also mutually orthonormal, e.g. $(\langle 0| \otimes \langle 0|)(|0\rangle \otimes |0\rangle) = \langle 0|0\rangle \langle 0|0\rangle = 1$, $(\langle 0| \otimes \langle 0|)(|0\rangle \otimes |1\rangle = \langle 0|0\rangle \langle 0|1\rangle = 0$, etc. Because direct products are a common feature in our subsequent discussions, use of the $\otimes$ symbol becomes somewhat cumbersome. Let's drop the $\otimes$ symbol and re-express the four kets in (1.26) as follows

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle. \tag{1.28}$$

There should be no ambiguity with this notation as long as we accept an implicit understanding that it is shorthand for the direct product of two qubits. In this notation $|\Psi\rangle = c_1|ab\rangle + c_2|cd\rangle$, and $|\Phi\rangle = d_1|ab\rangle + d_2|cd\rangle$. The inner product $\langle\Psi|\Phi\rangle$ is given by the following expression

$$\Big(c_1^*\langle ba| + c_2^*\langle dc|\Big)\Big(d_1|ab\rangle + d_2|cd\rangle\Big)$$

and, when expanded, evaluates to $c_1^* d_1 + c_2^* d_2$, in harmony with result (1.27).

The kets itemized in (1.28) reminds us of the notation used to describe five-qubit kets. In that case, each ket was labeled by its binary number representation. Kets (1.28) are similarly labeled, except that the dimension of Hilbert space is 2^2 and so the kets are enumerated accordingly. In analogy with the five-qubit notation, we could label the basis vectors in (1.28) as

$$|0\rangle_2, \quad |1\rangle_2, \quad |2\rangle_2, \quad |3\rangle_2. \tag{1.29}$$

The subscript refers to the fact that the number of qubits $n = 2$, and the dimension of Hilbert space is $d = 2^n$. With this notation, it is straightforward to construct a higher dimensional Hilbert space from qubit constituents. For example, consider the direct product of qubit $|a\rangle$, $a \in 0, 1$ with kets (1.26). It should be clear, if we make use of Theorem 1.3, that this product generates the basis kets

$$\begin{aligned} &|0\rangle_3, \quad |1\rangle_3, \quad |2\rangle_3, \quad |3\rangle_3, \\ &|4\rangle_3, \quad |5\rangle_3, \quad |6\rangle_3, \quad |7\rangle_3 \end{aligned} \tag{1.30}$$

in a $d = 2^3$ dimensional Hilbert space. Generalization to direct products of n-qubits follows. We now recognize that the Hilbert space of our five-$q$bulb register can be expressed as a direct product of 5 individual qubits. For a modest number of n-qubits the dimension of Hilbert space $d = 2^n$ is huge.

We can think of direct products as qubit entries in different slots. For example, the register $|100\rangle$, or $|4\rangle_3$, is a direct product (starting from the right and going left) of ket $|0\rangle$ in slot 1, ket $|0\rangle$ in slot 2, and ket $|1\rangle$ in slot 3. According to Theorem 1.3 the bra vector associated with this ket is $\langle 001|$. So now the slot order for bra vectors is reversed from that of ket space, i.e., going from left to right bra $\langle 0|$ is in slot 1, $\langle 0|$ is in slot 2, and $\langle 1|$ is in slot 3. If we take an inner product of this three-qubit vector with $|abc\rangle$ ($a, b, c \in 0, 1$) then, according to (1.27), the inner product

$$\langle 001|abc\rangle = \langle 1|a\rangle \, \langle 0|b\rangle \, \langle 0|c\rangle.$$

Each qubit ket in a given slot "hooks" up with the bra in that same slot.

Using the slot analogy we can also express multi-qubit operators, such as $\mathbf{X}_6$ and $\mathbf{N}$, introduced in the previous section, as direct products of single qubit operators. Operators, or outer products, are not vectors and so we should not use the same symbol to define products of the latter. Instead, we employ the symbol $\tilde{\otimes}$ to denote products of operators. Given $|ab\rangle \equiv |a\rangle \otimes |b\rangle$ and $|cd\rangle \equiv |c\rangle \otimes |d\rangle$.

$$|ab\rangle\langle dc| = (|a\rangle \otimes |b\rangle)(\langle d| \otimes \langle c|) \equiv |a\rangle\langle c|\tilde{\otimes}|b\rangle\langle d| \tag{1.31}$$

The symbol $\tilde{\otimes}$ is called the *Kronecker product*. In this expression, the outer product $|b\rangle\langle d|$ "operates" only on a qubit in slot 1, whereas $|a\rangle\langle c|$ operates on qubits in slot 2. For example, consider the two-qubit state $|\Psi\rangle = c_1|01\rangle + c_2|10\rangle$ then, according to Dirac's axiom,

$$\begin{aligned} &(|ab\rangle\langle dc|)\,|\Psi\rangle = c_1(|a\rangle\langle c|\tilde{\otimes}|b\rangle\langle d|)|01\rangle + c_2(|a\rangle\langle c|\tilde{\otimes}|b\rangle\langle d|)|10\rangle = \\ &|ab\rangle\Big(c_1 \, \langle c|0\rangle\langle d|1\rangle + c_2\langle c|1\rangle\langle d|0\rangle\Big) = |ab\rangle\Big(c_1 \, \langle cd|10\rangle + c_2 \, \langle cd|01\rangle\Big). \end{aligned} \tag{1.32}$$

The two-qubit operator $|ab\rangle\langle dc|$, when operating on two-qubit state $|\Psi\rangle$, projects the latter into vector $c\,|ab\rangle$ where the constant $c = c_1 \, \langle cd|10\rangle + c_2 \, \langle cd|01\rangle$.

In the next chapter, we introduce matrix representations of states and operators and, in that framework, we will find that the operation $\tilde{\otimes}$ is equivalent to the direct product operation $\otimes$. With definitions for direct products of kets and the corresponding Kronecker products of operators, we generalize and re-frame the postulates itemized in the previous sections for an arbitrary n-qubit register.

Postulate I Kets $|0\rangle, |1\rangle$ constitute a basis for the qubit Hilbert space. An n-qubit register is spanned by basis vectors that are direct products of n-qubits $|a\rangle \otimes |b\rangle \otimes |c\rangle \dots |n\rangle$, where $a, b, c \cdots n \in 0, 1$.

Postulate II A full description of the system is encapsulated by a vector $|\Psi\rangle$, of unit length, in this 2^n dimensional Hilbert space.

Postulate III (Born's rule) The act of measurement associated with Hermitian operator $\mathbf{A}$ results in one of its eigenvalues. The probability for obtaining a nondegenerate eigenvalue a is given by the expression $|\langle a|\Psi\rangle|^2$ where $|a\rangle$ is an eigenvector of $\mathbf{A}$ that corresponds to eigenvalue a. If the eigenvalue a is degenerate, the probability to find that value is $\sum_i |\langle a_i|\Psi\rangle|^2$ where the sum is over all i in which $a_i = a$.

Postulate IV (collapse hypothesis) Immediately after measurement by $\mathbf{A}$ with result a, the system is described, up to an undetermined phase, by state vector $|a\rangle$. If a is degenerate, the system is in a linear combination of the corresponding eigenvectors.

We now recognize that a five-$q$bulb state, lets say $|22\rangle = |10110\rangle$ is shorthand for the direct product $|1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle$. It is an eigenstate, with eigenvalue $j = 22$, of operator $\mathbf{N}_{22} = 22\,|10110\rangle\langle 01101|$, equivalent to the Kronecker product

$$\mathbf{N}_{22} = 22\,|1\rangle\langle 1|\tilde{\otimes}|0\rangle\langle 0|\tilde{\otimes}|1\rangle\langle 1|\tilde{\otimes}|1\rangle\langle 1|\tilde{\otimes}|0\rangle\langle 0|. \tag{1.33}$$

Let's define the single qubit operators $\mathbf{n} \equiv |1\rangle\langle 1|$ and $\mathbf{1} \equiv |0\rangle\,\langle 0| + |1\rangle\,\langle 1|$. Since $\mathbf{n}|0\rangle = 0, \mathbf{n}|1\rangle = 1\,|1\rangle$, $\mathbf{n}$ is an occupation number operator for a single qubit, whereas $\mathbf{1}$ is an identity operator. It has the property $\mathbf{1}|0\rangle = |0\rangle, \mathbf{1}|1\rangle = |1\rangle$, and $(\mathbf{1} - \mathbf{n}) = |0\rangle\langle 0|$. With these definitions we re-express (1.33) as

$$\begin{aligned}
&\mathbf{N}_{22} = 22\,\mathbf{P}_{22} \\
&\mathbf{P}_{22} = \mathbf{n}\tilde{\otimes}(\mathbf{1}-\mathbf{n})\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}(\mathbf{1}-\mathbf{n}) = \\
&\mathbf{n}\tilde{\otimes}\mathbf{1}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{1} - \mathbf{n}\tilde{\otimes}\mathbf{1}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n} - \mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{1} + \mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}
\end{aligned} \tag{1.34}$$

The operator $\mathbf{P}_{22}$ is called a *projection operator* and has the properties

$$\mathbf{P}_{22}\mathbf{P}_{22} = \mathbf{P}_{22},$$
$$\mathbf{P}_{22}|22\rangle = |22\rangle \quad \text{and for } j \neq 22 \quad \mathbf{P}_{22}|j\rangle = 0.$$

Throughout this chapter, we focused primarily on one type of measurement, that for the configuration occupation number j of a register. Postulate III allows other measurement devices as there are no restrictions on operator $\mathbf{A}$ as long as it is Hermitian. So what other measurement possibilities arise? An obvious choice is a measurement for the occupation number of a $q$bulb in a given slot, disregarding the values for the other $q$bulbs. For example, if we are only interested on the state of the $q$bulb in slot 3 we can define the measurement operator

$$\mathbf{P}_3 \equiv \mathbf{1}\tilde{\otimes}\mathbf{1}\tilde{\otimes}\,\mathbf{n}\,\tilde{\otimes}\mathbf{1}\tilde{\otimes}\mathbf{1}$$

which also happens to be a projection operator. For any basis ket $|j\rangle$, $\mathbf{P}_3|j\rangle = |j\rangle$, if slot 3 is in the on position and it is zero otherwise. However, there are many kets with this property including

$$\begin{gathered}|00100\rangle, |00101\rangle, |00110\rangle, |00111\rangle \\ |01100\rangle, |01101\rangle, |01110\rangle, |01111\rangle \\ |10100\rangle, |10101\rangle, |10110\rangle, |10111\rangle \\ |11100\rangle, |11101\rangle, |11110\rangle, |1111\rangle.\end{gathered}$$

Each of these kets is an eigenstate of $\mathbf{P}_3$ with eigenvalue 1. They are 16-fold degenerate and Postulate III tells us that the probability to find the $q$bulb of slot 3 to be in the on position is $\sum_j |\langle j|\Psi\rangle|^2$ where the sum is over all states $|j\rangle$ itemized above. Obviously, because there are multiple qubits in a register, many possibilities for occupation type measurements arise. For example, we could measure the state of only a single $q$bulb in a given slot, or permutations for several $q$bulbs at a time. The various operators associated with all such measurements can ultimately be expressed, as in (1.34), by sums of direct products involving the single qubit operators $\mathbf{1}$, and $\mathbf{n}$. Are there other possibilities not involving occupation type measurements? To investigate this question let's consider the simplest system, the single $q$bulb or qubit. The $q$bulb can either be on or off and one might, incorrectly, conclude that the measurement operator $\mathbf{n}$ exhausts all possibilities in this Hilbert space. But consider the following operator

$$\mathbf{A} \equiv |0\rangle\langle 1| + |1\rangle\langle 0|.$$

$\mathbf{A}$ is Hermitian, i.e., $\mathbf{A} = \mathbf{A}^\dagger$ and so it is a measurement device candidate. But what does $\mathbf{A}$ measure since it is not expressed solely by $\mathbf{n}$ and $\mathbf{1}$? It turns out that, for a real qubit, there does exist a measurement device associated with this operator. We will discuss it, and others, in detail in the next chapter. That discussion will force us to reconsider the simple $q$bulb analogy employed in this chapter. Instead of a quantum light bulb having two distinct properties, we will learn that a real qubit is multi-faceted and forces us to broaden our conception of what it means for the $q$bulb to be in the on or off state. Nevertheless the $q$bulb analogy is still very useful as the eigenstates $|j\rangle_n$, also called the *computational basis*, of the configuration number operator $\mathbf{N} = \sum_{j=0}^{2^n-1} j\,(|j\rangle\,\langle j|)_n$ play a special role in quantum information theory.

Mathematica Notebook 1.3: The Born rule and projective measurements. http://www.physics.unlv.edu/%7Ebernard/MATH_book/Chap1/chap1_link.html

Before proceeding to that discussion in the next chapter, we need address two issues not yet discussed. In addition to the four postulates itemized above, there is a fundamental postulate that tells us how the state vector $|\Psi\rangle$ evolves in time. We defer that discussion to Chap. 3. Now, consider the state

$$|\Psi\rangle = \sum_{j=0}^{2^n-1} c_j|j\rangle \tag{1.35}$$

for an n-dimensional register. It is expressed as an expansion over the computational basis $|j\rangle$. For the sake of simplicity, we dropped the subscript index that defines the latter as n-dimensional basis kets. As the computational basis vectors are orthonormal, let's take the scalar product of both sides of (1.35) with ket $|m\rangle$, where m is an arbitrary index $0 \le m \le 2^n - 1$. We find

$$\langle m|\Psi\rangle = \sum_{j=0}^{2^n-1} c_j\delta_{mj} = c_m \tag{1.36}$$

where we used the fact $\langle i|j\rangle = \delta_{ij}$. Now relation (1.36) must be true for all values of m and so inserting this identity back into (1.35) we obtain

$$|\Psi\rangle = \sum_{j=0}^{2^n-1} \langle j|\Psi\rangle|j\rangle = \sum_{j=0}^{2^n-1} |j\rangle\langle j|\Psi\rangle. \tag{1.37}$$

Now according to Dirac's rule for inner products this sum should be the same as

$$|\Psi\rangle = \Big(\sum_{j=0}^{2^n-1} |j\rangle\langle j|\Big)\,|\Psi\rangle$$

and which only makes sense if the sum between the parenthesis has no effect on the n-dimensional qubit state $|\Psi\rangle$. In other words (1.37) implies that

$$\sum_{j=0}^{2^n-1} |j\rangle\langle j| = \mathbf{1}\tilde{\otimes}\mathbf{1}\tilde{\otimes}\mathbf{1}\tilde{\otimes}\cdots \equiv \mathbb{1}. \tag{1.38}$$

Relation (1.38) is called the *closure*, or *completeness*, property for any set of n qubits. We will often make use of this property in our subsequent discussions. The symbol $\mathbb{1}$ represents an identity operator in the full n-qubit Hilbert space. In subsequent chapters we will use this symbol to represent the identity without explicit reference to the dimensionality of the Hilbert space.

Problems

1.1 Evaluate the exercises in Mathematica Notebook 1.1.

1.2 Given the set of polynomials of degree 3 in variable x, $P_a = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_0 \dots a_3$ are real numbers. Let the binary operation $P_a + P_b$ denote ordinary addition. Show that set P_γ constitutes a linear vector space.

1.3 Answer the exercises in Mathematica Notebook 1.2

1.4 The state

$$|\psi\rangle = \frac{1}{\sqrt{6}}\,|01101\rangle + \sqrt{\frac{2}{3}}\,|11111\rangle + \frac{1}{\sqrt{6}}\,|00001\rangle$$

describes a register of five $q$bulbs. (a) Calculate the probability that the first $q$bulb is in the on position, after making a measurement with device $\mathbb{1}\tilde{\otimes}\mathbb{1}\tilde{\otimes}\mathbb{1}\tilde{\otimes}\mathbb{1}\tilde{\otimes}\mathbf{n}$ (b) What is the probability that all $q$bulbs are in the on position, after a measurement with device $\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}$. (c) Calculate the probability that at least three $q$bulbs are found in the off-position. (d) A measurement reveals the occupation configuration 01101. Immediately after that measurement another measurement with $\mathbf{N}$ is made. Calculate the probability that at least three $q$bulbs are in the off-position.

1.5 Given the states $|\psi\rangle = \frac{1}{\sqrt{6}}\,|01101\rangle + \sqrt{\frac{2}{3}}\,|11111\rangle + \frac{\mathrm{i}}{\sqrt{6}}\,|00001\rangle$ and $|\Phi\rangle = \frac{1}{\sqrt{2}}\,|01101\rangle + \frac{1}{\sqrt{2}}\,|01111\rangle$. Evaluate $\langle \Phi\,|\Psi\rangle$, $|\Phi\rangle\,\langle\Psi|$ and $|\Psi\rangle\,\langle\Phi|$.

1.6 Consider operator $\mathbf{X} = |\Phi\rangle\langle\Psi|$, show that $\mathbf{X}^\dagger = |\Psi\rangle\langle\Phi|$. Hint: use this expression for $\mathbf{X}^\dagger$ and operate it on bra $\langle\Gamma|$. Compare the result with that obtained by $\mathbf{X}\,|\Gamma\rangle$.

1.7 Re-express the following states $|17\rangle_5\,, |5\rangle_5\,, |12\rangle_5$ in binary notation, i.e., $|k_4k_3k_2k_1k_0\rangle$, $k_i \in \{0, 1\}$.

1.8 Consider the operator $\mathbf{X} \equiv exp(\mathrm{i}\alpha)\,|00110\rangle\,\langle 00100| + |00111\rangle\,\langle 11111|$. Evaluate $\mathbf{X}\,\,|\Phi\rangle$ where $|\Phi\rangle$ is given by Eq. (1.3) in the text.

1.9 Show that operator, in the Hilbert space of a single qubit, $\mathbf{X} \equiv |0\rangle\,\langle 1| + |1\rangle\,\langle 0|$ is Hermitian. Solve the following equation

$$\mathbf{X}\,\,|\psi\rangle = \lambda\,\,|\psi\rangle$$

for the state $|\psi\rangle$. Hint: express $|\psi\rangle = c_1 |0\rangle + c_2 |1\rangle$, and solve for the coefficients c_1, c_2. Show that this equation admits solutions only for select values of parameter λ.

1.10 In the Hilbert space of three qubits, consider the operator

$$\mathbf{A} \equiv |000\rangle\langle 000| + 2|001\rangle\langle 100| - 2|010\rangle\langle 010| +$$
$$3|100\rangle\langle 001| + |011\rangle\langle 110| - |101\rangle\langle 101|.$$

Find all the eigenvalues and eigenvectors of **A**. Identify the degenerate eigenvalues and show that any linear combination of the corresponding eigenvectors are also eigenstates of **A**.

1.11 In a two-qubit Hilbert space, consider the operator

$$\mathbf{A} = |00\rangle\langle 01| + |10\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01|.$$

Find the eigenvalues and eigenstates of **A**.

1.12 Given the single qubit operators

$$\mathbf{A} = |0\rangle\langle 0| + |1\rangle\langle 1|, \quad \mathbf{B} = \mathrm{i}\,|0\rangle\langle 1| - \mathrm{i}\,|1\rangle\langle 0|,$$

show that $[\mathbf{A}, \mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA} = 0$. Find the eigenstates for operator **B** and show that they are also eigenstates of operator **A**. What are the eigenvalues associated with operator **A**?

1.13 Prove the following: for two Hilbert space operators **A**, **B**, so that $[\mathbf{A}, \mathbf{B}] = 0$, show that, if **A** has non-degenerate eigenstates $|a_1\rangle, |a_2\rangle, \ldots$, then the kets $|a_i\rangle$ are also eigenstates of operator **B**.

1.14 Prove Theorems 1.1 and 1.2.

1.15 Given operator **A** in an n-dimensional Hilbert space with orthonormal eigenvectors $|a_1\rangle, |a_2\rangle, \ldots, |a_n\rangle$, prove that $\mathbf{A} = \sum_i^n a_i |a_i\rangle\langle a_i|$, where a_i is the eigenvalue associated with $|a_i\rangle$.

1.16 Show that operator

$$\mathbf{U} = \cos\theta\ |0\rangle\langle 0| + \exp(\mathrm{i}\phi)\sin\theta\ |0\rangle\langle 1| + \exp(-\mathrm{i}\phi)\sin\theta\ |1\rangle\langle 0| - \cos\theta\ |1\rangle\langle 1|,$$

where θ, ϕ are real parameters, is unitary.

1.17 Consider the operator $\mathbf{X} = |0\rangle\langle 1| + |1\rangle\langle 0|$, evaluate

$$\tilde{\mathbf{X}} = \mathbf{UXU}^\dagger$$

where **U** is given in problem 1.16.

1.18 Find the eigenvalues and eigenstates for operator $\tilde{\mathbf{X}}$ given in problem 1.17.

1.19 Evaluate $\tilde{\mathbf{Y}} = \mathbf{U}\mathbf{Y}\mathbf{U}^\dagger$ where, $\mathbf{Y} = -\mathrm{i}\,|0\rangle\,\langle 1| + \mathrm{i}\,|1\rangle\,\langle 0|$ and $\mathbf{U}$ is defined in problem 1.17. Demonstrate that

$$\left[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}\right] = 2\mathrm{i}\,\tilde{\mathbf{Z}}$$

where $\tilde{\mathbf{Z}} = \mathbf{U}(|0\rangle\,\langle 0| - |1\rangle\,\langle 1|)\mathbf{U}^\dagger$, and $\tilde{\mathbf{X}}$ is defined in problem 1.17.

1.20 Consider the operator

$$\mathbf{P} = 2\,\mathbf{n}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n} + \mathbb{1}\tilde{\otimes}\mathbb{1}\tilde{\otimes}\mathbf{n} - \mathbf{n}\tilde{\otimes}\mathbb{1}\tilde{\otimes}\mathbf{n} - \mathbb{1}\tilde{\otimes}\mathbf{n}\tilde{\otimes}\mathbf{n}.$$

Show that $\mathbf{P}$ is a projection operator.

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