Introduction to Stability Conditions

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1 Motivation

Let X be a smooth projective Calabi–Yau variety over $\mathbb{C}$. Then $\mathcal{D}^b(X)$, the derived category of coherent sheaves on X, is equivalent to the category of D-branes on X [9]. In [10], Douglas defined a notion of stability for D-branes on X called Π-stability. This notion of stability was meant to pick out BPS-branes on X. In [7], Bridgeland aimed to define a notion of stability directly for objects in $\mathcal{D}^b(X)$ which would correspond to Π-stability for D-branes. Bridgeland's stability can be defined on any triangulated category, and hence has been studied in other cases, such as for varieties which are not Calabi–Yau.

2 Definition of Stability

2.1 *Example:* $\mathbb{P}^1$

Consider the example of $\mathrm{Coh}(\mathbb{P}^1)$, the category of coherent sheaves on $\mathbb{P}^1$. The objects in this category are all direct sums of the following building blocks:

1. Line bundles $\mathcal{O}(n)$, $n \in \mathbb{Z}$.
2. Torsion sheaves $\mathcal{O}_{nx}$, $x \in \mathbb{P}^1$.

There are two invariants which can be assigned to each type of sheaf. First, there is the rank of the sheaf. The rank of a line bundle is 1, and the rank of a skyscraper sheaf is 0. Further, there is the degree of the sheaf. The degree of the line bundle $\mathcal{O}(n)$ is n, and the degree of the torsion sheaf $\mathcal{O}_{nx}$ is n.

Both the rank and degree functions can be defined more generally for any sheaf on $\mathbb{P}^1$. Both invariants are additive on short exact sequences. So, for example, the

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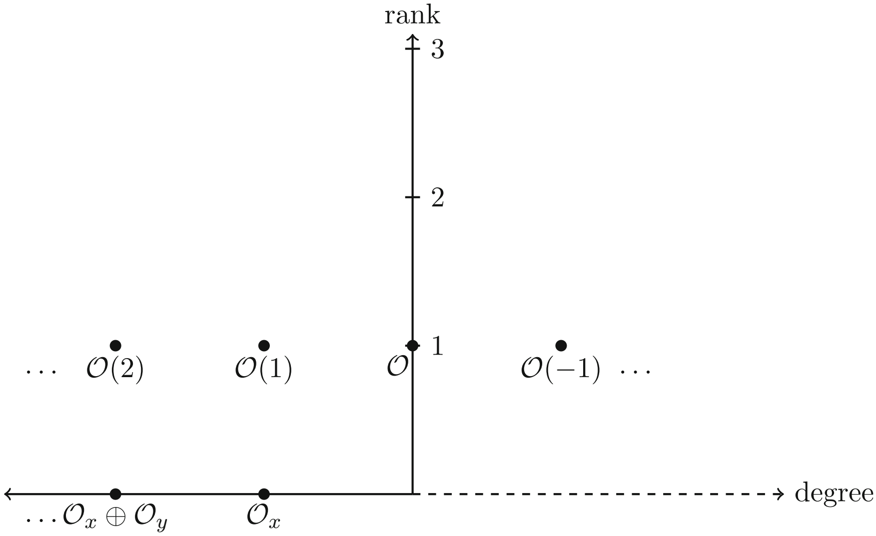
rank and degree of $\mathcal{O}(2) \oplus \mathcal{O}(4) \oplus \mathcal{O}_x$ are 2 and 7 respectively. Similarly, all objects in $\mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1))$ are extensions or shifts of line bundles and torsion sheaves, hence we could define the degree and rank functions for any objects in $\mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1))$.

We can define a group homomorphism $Z\colon K(\mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1))) \to \mathbb{C}$ as

$$Z(E^{\cdot}) = -\mathrm{degree}(E^{\cdot}) + i\,\mathrm{rank}(E^{\cdot})$$

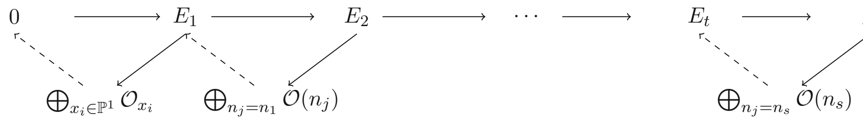
for $E^{\cdot} \in \mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1))$. This is well-defined since degree and rank are additive on short exact sequences. Further, if we consider the subcategory $\mathrm{Coh}(\mathbb{P}^1)$ inside $\mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1))$, the image is the upper half plane.



For each $E^{\cdot} \in \mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1))$ we can write $Z(E^{\cdot}) = m(E^{\cdot})e^{\pi i \phi(E)}$ for some $m(E^{\cdot}) > 0$. We call $m(E^{\cdot})$ the mass of $E^{\cdot}$ and $\phi(E^{\cdot})$ the phase of $E^{\cdot}$. Note that for objects E in $\mathrm{Coh}(\mathbb{P}^1)$, the phase lies in the range $0 < \phi(E) \leq 1$.

For $E \in \mathrm{Coh}(\mathbb{P}^1)$, we say E is Z-stable if for all subsheaves $F \subsetneq E$, $\phi(F) < \phi(E)$. We say E is semistable if for all subsheaves $F \subsetneq E$, $\phi(F) \leq \phi(E)$. It is easy to check that the only stable sheaves are line bundles and skyscraper sheaves, and that a sheaf is semistable if and only if it is either a direct sum of skyscraper sheaves or a direct sum of line bundles all of the same degree.

We can use this fact to construct a filtration of a sheaf $E \in \mathrm{Coh}(\mathbb{P}^1)$ whose successive quotients are semistable sheaves of strictly decreasing phase as follows. We write $E = \bigoplus_{x_i \in \mathbb{P}^1} \mathcal{O}_x \oplus \bigoplus_{j=1}^{s} \mathcal{O}(n_j)$ for a collection of points $x_i \in \mathbb{P}^1$ and $n_1 \geq n_2 \geq \cdots \geq n_s$. Then we can construct a filtration

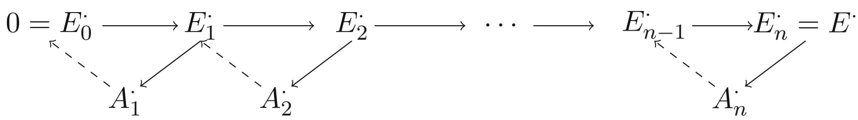


by building E out of its summands, one type at a time. Such a filtration is called a Harder-Narasimhan, or HN, filtration.

2.2 Definition

Definition 2.1 Let $\mathcal{D}$ be a triangulated category. A heart of a bounded t-structure is a full additive subcategory $\mathcal{A}$ of $\mathcal{D}$ satisfying

1. $\mathrm{Hom}^i(A, B) = 0$ for $i < 0$ and $A, B \in \mathcal{A}$.
2. Objects in $\mathcal{D}^b(X)$ have filtrations by cohomology objects in $\mathcal{A}$. That is, for all nonzero $E^\cdot \in \mathcal{D}^b(X)$, there is a sequence of exact triangles



   such that $A_i[-k_i] \in \mathcal{A}$ for integers $k_1 > \cdots > k_n$.

Definition 2.2 ([7, Proposition 5.3]) A Bridgeland stability condition is a pair $\sigma = (Z, \mathcal{A})$ where $Z\colon K_0(\mathcal{D}^b(X)) \to \mathbb{C}$ is a group homomorphism and $\mathcal{A}$ is a heart of a bounded t-structure. The pair must further satisfy that

1. $Z(\mathcal{A} \setminus \{0\}) \subseteq \{re^{i\pi\phi} \mid r > 0,\ 0 < \phi \le 1\}$. Define the phase of $0 \neq E \in \mathcal{A}$ to be $\phi(E) := \phi$. We say $E \in \mathcal{A}$ is Z-semistable if for all nonzero subobjects $F \in \mathcal{A}$ of E, $\phi(F) \le \phi(E)$. E is Z-stable if for all nonzero subobjects $F \in \mathcal{A}$ of E, $\phi(F) < \phi(E)$.
2. The objects of $\mathcal{A}$ have Harder-Narasimhan filtrations with respect to Z. That is, for every $E \in \mathcal{A}$ there is a unique sequence of inclusions

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{n-1} \subseteq E_n = E$$

   such that the successive quotients E_i/E_{i-1} are Z-semistable, and the phases $\phi(E_1/E_0) > \phi(E_2/E_1) > \cdots > \phi(E_{n-1}/E_{n-2}) > \phi(E_n/E_{n-1})$.

There is an alternate definition of a Bridgeland stability condition, given in [7, Definition 5.1]. I will give this definition as well. First, we must define a slicing of a triangulated category.

Definition 2.3 A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$ satisfying

1. For all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
2. If $\phi_1 > \phi_2$, $A_1 \in \mathcal{P}(\phi_1)$, and $A_2 \in \mathcal{P}(\phi_2)$, then $\mathrm{Hom}(A_1, A_2) = 0$.
3. For every $E \in \mathcal{D}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

so that there is a sequence of exact triangles

$$\begin{array}{ccccccccccc} 0 = E^{\cdot}_0 & \longrightarrow & E^{\cdot}_1 & \longrightarrow & E^{\cdot}_2 & \longrightarrow & \cdots & \longrightarrow & E^{\cdot}_{n-1} & \longrightarrow & E^{\cdot}_n = E^{\cdot} \\ & \nwarrow \swarrow & & \nwarrow \swarrow & & & & & & \nwarrow \swarrow & \\ & A^{\cdot}_1 & & A^{\cdot}_2 & & & & & & A^{\cdot}_n & \end{array}$$

such that $A_i \in \mathcal{P}(\phi_i)$ for each $i = 1, \ldots, n$.

Definition 2.4 A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z\colon K(\mathcal{D}) \to \mathcal{C}$ and a slicing $\mathcal{P}$ such that if $0 \neq E^{\cdot} \in \mathcal{P}(\phi)$, then $Z(E^{\cdot}) = m(E)e^{\pi i \phi(E^{\cdot})}$ for some $m(E^{\cdot}) \in \mathbb{R}_{>0}$.

In this definition, the semistable objects of phase ϕ are defined to be the objects of $\mathcal{P}(\phi)$. Note that the phase of an arbitrary $E^{\cdot} \in \mathcal{D}$ is not well-defined, only the objects of slicings $\mathcal{P}(\phi)$ have well-defined phase.

This definition is equivalent to the previous definition. The heart $\mathcal{A}$ is replaced by the category $\mathcal{P}(0, 1]$, the extension closure of the collection of objects in $\mathcal{P}(\phi)$ for $0 < \phi \leq 1$. That this category is necessarily abelian is shown in [7, Proposition 5.3]. In fact, one can show that all the subcategories $\mathcal{P}(\phi)$ are abelian [7, Lemma 5.2].

3 Examples

3.1 Curves

For a smooth projective curve C of genus g, stability conditions can be constructed of the type described for $\mathbb{P}^1$, with heart Coh(S) and central charge $Z = -\text{degree} + i\,\text{rank}$. Note that for $g > 0$, sheaves are more complicated, and vector bundles are no longer necessarily direct sums of line bundles. Hence HN filtrations must be constructed more carefully.

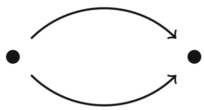
There is an action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ [7, Lemma 8.2] on the space of stability conditions on C (or on any smooth projective variety). If we consider an element of this group to be a pair (T, f) where T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is an orientation preserving isomorphism, and $f : \mathbb{R} \to \mathbb{R}$ is increasing, and satisfies

$f(\phi + 1) = f(\phi) + 1$ for all $\phi \in \mathbb{R}$, then it acts on a stability condition $(Z, \mathcal{P})$ by replacing Z with $T^{-1}Z$, and replacing $\mathcal{P}(\phi)$ with $\mathcal{P}(f(\phi))$.

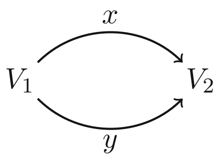
In fact, if $g > 0$, then up to the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ the previous construction of stability conditions on a curve in terms of rank and degree gives all possible stability conditions on C [11]. There are other possible stability conditions on $\mathbb{P}^1$ described in [5, 12].

3.2 Quivers

First, consider the following quiver, Q.



A representation $\overline{V}$ of this quiver consists of a choice of two vector spaces, V_1 and V_2, and two linear maps x and y from V_1 to V_2.

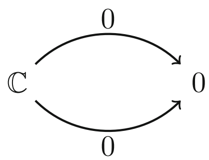


Suppose we wish to define a stability condition on Q. We may start with the abelian category $\mathrm{Rep}(Q)$. If we pick any two numbers $z_1, z_2 \in \mathbb{C}$ which lie in the upper half plane or along the negative real axis, we can define a central charge

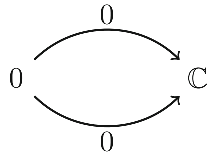
$$Z(\overline{V}) = z_1 \mathrm{dim}(V_1) + z_2 \mathrm{dim}(V_2).$$

In other words, we choose the images of the two simple representations, S_1 and S_2, pictured below.

S_1 :



S_2 :



We can then extend our central charge to complexes of representations by requiring it to be additive on exact triangles. We claim that the pair $(\text{Rep}(Q), Z)$ is a Bridgeland stability condition on $\mathcal{D}^b(\text{Rep}(Q))$.

The fact that the image of $\text{Rep}(Q)$ lies in the upper half plane follows from the fact that dimensions are positive, and from the choice of z_1 and z_2. It remains to show that each representation $\overline{V}$ of Q has an HN filtration. This is argued nicely in [2, Theorem 2.1.6]

It is interesting to note in this example how the choice of z_1 and z_2 controls which representations are stable. Suppose first that we pick z_2 so that its phase is larger than z_1. S_2 is a subobject of any representation for which $V_2 \neq 0$, and S_1 is a quotient of any representation for which $V_1 \neq 0$. Hence no object can be stable besides the simple representations.

On the other hand, suppose we choose z_1 so that its phase is larger than z_2. Then again, S_1 and S_2 are necessarily stable. Now, however, so is any representation for which V_1 and V_2 are one-dimensional. Hence, these stable objects are parameterized by the choice of linear maps x, y. Up to scaling, we can suppose $x = 1$. In this way, we see a one-to-one correspondence between stable representations of Q and points of $\mathbb{P}^1$.

Reference [6] shows that there is an equivalence of categories $\mathcal{D}^b(\text{Rep}(Q)) \cong \mathcal{D}^b(\text{Coh}(\mathbb{P}^1))$. This equivalence is given explicitly by the functor $R\text{Hom}(\mathcal{O} \oplus \mathcal{O}(1), -)\colon \mathcal{D}^b(\text{Coh}(\mathbb{P}^1)) \to \mathcal{D}^{\text{b}}(\text{Rep(Q)})$. Such an equivalence always sends a heart of a bounded t-structure to a heart of a bounded t-structure. Hence if we consider the stability conditions we have constructed here on $\text{Rep}(Q)$, there should be corresponding stability conditions on a heart in $\mathcal{D}^b(\text{Coh}(\mathbb{P}^1))$. Note that the inverse image of $\text{Rep}(Q)$ under this equivalence is not $\text{Coh}(\mathbb{P}^1)$, so this already gives an example of a stability condition on $\mathbb{P}^1$ with a heart that is not $\text{Coh}(\mathbb{P}^1)$. The heart on $\mathbb{P}^1$ we get via this map can also be constructed by the process of tilting, described below.

3.3 Surfaces, Threefolds, and Higher Dimensional Varieties

Let X be a smooth projective variety of dimension n. In order to define a central charge, we may wish to start with the example of curves and generalize the ideas of degree and rank. In order to do this, we may choose an ample divisor ω on X, and use the Chern characters of sheaves on X to define the central charge. This is convenient, since these quantities are once again additive on short exact sequences.

For a sheaf $E \in \mathrm{Coh(X)}$ (or an object $E^{\cdot} \in \mathcal{D}^b(\mathrm{Coh(X)})$, our numerical invariants are now $\omega^n \cdot \mathrm{ch}_0(\mathrm{E}^{\cdot}), \omega^{n-1} \cdot \mathrm{ch}_1(\mathrm{E}^{\cdot}), \ldots, \omega^0\mathrm{ch}_\mathrm{n}(\mathrm{E}^{\cdot})$. Note that if $n = 1$, this does not depend on the choice of ω, and gives us exactly the rank and degree of E.

However, it is not possible to define a stability condition on the heart Coh(X) with central charge in terms of these quantities for $n > 1$. Hence we must find a different choice of abelian subcategory of $\mathcal{D}^b(\mathrm{Coh(X)})$. One technique for constructing new hearts inside $\mathcal{D}^b(\mathrm{Coh(X)})$ is called tilting. In order to perform the process of tilting, one chooses two full additive subcategories $\mathcal{T}$ and $\mathcal{F}$ in Coh(X) which form what is called a torsion pair.

Definition 3.1 A torsion pair in a heart $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories of $\mathcal{A}$ such that

1. If $T \in \mathcal{T}$ and $F \in \mathcal{F}$, then $\mathrm{Hom}(T, F) = 0$.
2. For all $E \in \mathcal{A}$ there is an object $T \in \mathcal{T}$ and $F \in \mathcal{F}$ so that the sequence $0 \to T \to E \to F \to 0$ is exact.

We then replace our category Coh(X) with the tilt

$$\mathcal{A}^{\#} = \{E^{\cdot} \in \mathcal{D}^b(X) \mid H^0_{\mathcal{A}}(E^{\cdot}) \in \mathcal{T},\ H^{-1}_{\mathcal{A}}(E^{\cdot}) \in \mathcal{F},\ H^i_{\mathcal{A}}(E^{\cdot}) = 0 \text{ for } i \neq 0, -1\}$$

whose elements are 2-term complexes with restrictions on cohomology. This process can then be repeated to construct more hearts.

If X is a surface, it is shown in [1, 8] that this process can be used to construct stability conditions on X. In particular, we choose another class $B \in \mathrm{NS}_{\mathbb{R}}(X)$, and then can write the central charge formula explicitly as

$$Z(E^{\cdot}) = -\int_X e^{B+i\omega}\mathrm{ch}(\mathrm{E}^{\cdot}).$$

In particular, [1] shows that this central charge, paired with a heart which is a single tilt of Coh(X) given explicitly in terms of ω and B, give a stability condition on X.

For $n > 2$, one might hope a similar process might work. We might hope that the same central charge formula, and a heart constructed in terms of ω and B by tilting Coh(X) perhaps $n - 1$ times could give a stability condition on X. Unfortunately, this has been difficult to prove. It is conjectured true for threefolds in [4], with the heart given explicitly, although the exact conjecture in [4] has been shown not to hold for certain threefolds in [14]. It is shown only for certain threefolds, in [3, 4, 13].

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