Introduction to Topological String Theories

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1 Mathematical Background

In this section, we give mathematical definitions and some results that physicists might not be familiar with, but these are necessary to understand the topological A-model and B-model and mirror symmetry. See [1, 2] for more details.

1.1 Complex Manifolds

An m-dimensional complex manifold is defined to be a $2m$-dimensional real manifold which locally looks $\mathbb{C}^m$ with holomorphic transition functions, hence any complex manifold is a real manifold. The converse is however not always true and we need to introduce the concept of *complex structure*. Let M be an $2m$-dimensional real manifold with tangent bundle TM and cotangent bundle T^*M, and we denote by $\Gamma(\otimes^k TM \otimes^l T^*M)$ the space of tensor fields of rank (k, l). We define an *almost complex structure* $J \in \Gamma(TM \otimes T^*M)$ to be a smooth tensor field of rank (1,1) on M satisfying $J_c^a J_b^c = -\delta_b^a$. Then we define the *Nijenhuis tensor* N which locally takes the form

$$N_{bc}^a = J_b^d(\partial_d J_c^a - \partial_c J_d^a) - J_c^d(\partial_d J_b^a - \partial_b J_d^a). \tag{1}$$

If $N = 0$ everywhere, J is called a *complex structure*. It is proven that an $2m$-dimensional real manifold can be considered to be an m-dimensional complex manifold only if it admits a complex structure J. Roughly speaking, a complex structure tells how to mix local coordinates $(z_i, \bar{z}_i)$.

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1.2 Calabi–Yau Manifold

A *Hermitian metric h* on a complex vector bundle E over a complex manifold M is a smooth section $h \in \Gamma(E \otimes \overline{E})$ satisfying

$$h(u, \bar{v}) = h(\bar{u}, v), \qquad h(u, \bar{u}) \geq 0, \qquad u, v \in E. \tag{2}$$

Locally h can be written as

$$h = h_{a\bar{b}} dz^a \otimes dz^{\bar{b}}. \tag{3}$$

The Riemannian metric g on complexified cotangent bundle $T^*_C M$ is defined to be the real part of the Hermitian metric

$$g = \frac{1}{2}(h + \bar{h}). \tag{4}$$

On the other hand, the imaginary part

$$\omega = \frac{i}{2}(h - \bar{h}), \tag{5}$$

is called the *Hermitian form*. All h, g, ω are compatible with a complex structure J, i.e., $h(u, \bar{v}) = h(Ju, J\bar{v})$. Also note that any of these three uniquely determines the other two. A *Kähler manifold* is defined to be a complex manifold with the non-degenerate closed Hermitian form $d\omega = 0$ and ω in this case is called a *Kähler form*. It is known that locally a Kähler form is given by a so-called Kähler potential K as

$$\omega_{a\bar{b}} = i g_{a\bar{b}} = i \partial_a \bar{\partial}_{\bar{b}} K. \tag{6}$$

One can calculate the Riemann tensors by the Riemannian metric g and it turns out that all $R_{ab} = R_{\bar{a}\bar{b}} = 0$ on a Kähler manifold. A *Calabi–Yau manifold* is defined to be a compact Ricci flat Kähler manifold, which is our interest in topological string theories.

1.3 Cohomologies

On a complex manifold M, we define a (p,q)-form[1] which locally is given as

$$A = A_{a_1 \cdots a_p b_1 \cdots b_q}(z, \bar{z}) dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge d\bar{z}^{b_1} \wedge \cdots \wedge d\bar{z}^{b_q}. \tag{7}$$

[1]Do not be confused with type (k,l) tensors.

The space of p-forms on M is the direct sum of the space of $(p-q,q)$ forms over q. Accordingly there exists three different exterior derivatives, namely d which maps p-forms to $(p+1)$-forms, ∂ which maps (p,q)-forms to $(p+1,q)$-forms and $\bar{\partial}$ which maps (p,q) to $(p,q+1)$-froms. They are all nilpotent.

Let us denote d-cohomology, ∂-cohomology and $\bar{\partial}$-cohomology by $H^p(M)$, $H_\partial^{p,q}(M)$, and $H_{\bar{\partial}}^{p,q}(M)$ respectively. Then a Kähler form ω is, for example, in $H^2(M)$ and also in $H^{1,1}(M)$. There is no relation among these cohomologies in general but if M is a Kähler manifold, it is known that $H_\partial^{p,q}(M)=H_{\bar{\partial}}^{p,q}(M)$ and

$$H^p(M)=\bigoplus H^{p-q,q}(M). \tag{8}$$

We call $h^{p,q}=\dim H^{p,q}(M)$ the *Hodge numbers* of M and for a Kähler manifold they satisfy the following relations

$$h^{(p,q)}=h^{(q,p)}, \qquad h^{(p,q)}=h^{(m-p,m-q)}. \tag{9}$$

1.4 Chern Class

Let us consider a connection form[2] $\tilde{\omega}$ on M and define the curvature form Ω as

$$\Omega=d\tilde{\omega}+\tilde{\omega}\wedge\tilde{\omega}. \tag{10}$$

Then the *Chern class* is defined as

$$c(M)=\det\left(I+\frac{i\Omega}{2\pi}\right)=c_0(M)+c_1(M)+c_2(M)+\cdots, \tag{11}$$

where n^{th} Chern class $c_n(M)$ is given by the term with n powers of Ω.

In particular, we find

$$c_0(M)=1, \qquad c_1(M)=\frac{i\mathrm{Tr}(\Omega)}{2\pi}. \tag{12}$$

For a Calabi–Yau manifold, it is known that $c_1(M)=0$, which becomes a key to define the B-model in Sect. 6. Note that the requirement of the Ricci flatness is indeed equivalent to the requirement $c_1(M)=0$.

One may be concerned that this definition relies on a connection $\tilde{\omega}$ so different choices of a connection gives different Chern classes. However, this turns out to be an overthinking. The Chern classes are independent of the choice of connection.

[2]This is called a spin connection in supergravity.

1.5 Moduli Spaces of Calabi–Yau manifolds

We denote a Calabi–Yau manifold by M_C within this subsection. There is a theorem by Calabi and Yau that given a complex manifold with vanishing first Chern class, there is precisely one Calabi–Yau manifold in each Kähler class. We thus define the moduli space $\mathscr{M}_C$ of Calabi–Yau manifolds to be a space of all possible Kähler classes and complex structures on M_C. It is shown that $h^{2,0} = h^{0,2}$ are fixed by $\dim(M_C)$, and especially they are zero if $\dim(M_C) \geq 3$.

It is suggested that the Kähler part of $\mathscr{M}$ is related to $H^{1,1}(M_C)$ since $\omega \in H^{1,1}(M_C)$. More precisely, the tangent space of it is isomorphic to $H^{1,1}(M_C)$. On the other hand, the complex structure part of $\mathscr{M}$ is more complicated in general so let us stick on M of dimensions three. In this case, the tangent space of infinitesimal deformation of complex structure of $\mathscr{M}$ is shown to be isomorphic to $H^{2,1}(M)$. In fact one can explicitly calculate some of the Hodge numbers for the Calabi–Yau 3-fold and the hodge diamond becomes

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{13}$$

The *mirror symmetry* between two Calabi–Yau threefolds is a duality under reflection along the diagonal line, i.e. by interchanging $h^{1,1}$ and $h^{2,1}$, or in other words $H^{1,1}(M)$ (Kähler classes) and $H^{2,1}(M)$ (complex structures) of two mirror pair theories.

2 Topological and Cohomological Field Theory

Discussions in this section and here after are based on [2, 3]. Our interests in quantum field theories are correlation functions, or observables, of physical operators in some certain background. Here a background includes a choice of a manifold, metric and coupling constants. A *topological field theory*, TFT, is defined to be a theory if all physical observables are independent of the choice of the metric. So obviously any observable has no explicit dependence of the metric, though it implicitly can.

This is a quite powerful requirement. Since one can freely change the metric and coordinates in TFT, and these two end up with changing insertion points of local operators without varying observables. That is, order of operators does not matter and observables are independent of insertion points in contrast to standard quantum field theories.

A *cohomological field theory*, CohFT or sometimes called a Witten's type TFT, is a TFT with the following properties.

- There is a global Grassmann scalar symmetry operator Q such that $Q^2 = 0$.
- All physical operators $\mathcal{O}_i$ are Q-closed, $\{Q, \mathcal{O}_i\} = 0$.
- The vacuum state is Q-symmetric, $Q|0 = 0$.
- The EM tensor $T_{\mu\nu}$ is Q-exact, $T_{\mu\nu} = \{Q, G_{\mu\nu}\}$, where $G_{\mu\nu}$ is some operator.

The first three are properties of any quantum field theory with a BRST charge. These three ensure that observables of physical operators $\mathcal{O}_i$ are invariant under a Q-exact shift with some operator Λ

$$\mathcal{O}_i \sim \mathcal{O}_i + \{Q, \Lambda_i\}. \tag{14}$$

The fourth one is crucial to make a theory topological. Let us consider a functional derivative of an observable with respect to the metric g. Since physical operators are required to be independent of the metric, we have

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle &= i \int D\phi \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \frac{\delta S}{\delta g^{\mu\nu}} e^{iS[\phi]}, \\ &= i \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \{Q, G_{\mu\nu}\} \rangle , \\ &= 0, \end{aligned} \tag{15}$$

where ϕ is a field in the theory.

As a simple example, if the action is Q-exact, the theory is a CohFT since the functional derivative with respect to the metric should be also Q-exact. Further since the action is given by

$$\exp \frac{i}{h} \Big\{ Q, \int_M V \Big\}, \tag{16}$$

we have

$$\frac{d}{dh} \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle = 0. \tag{17}$$

Therefore in this case, all correlation functions are given in the classical limit $(h \to 0)$.

2.1 Nonlocal Operators

Consider an observable of physical local operators $\{\mathcal{O}_i(x_i)\}$. Since topological invariance of the theory implies that it is independent of insertion points x_i, the derivative with respect to, for example, x_1 vanishes

$$d_{x1} \langle \mathscr{O}_1(x_1) \cdots \mathscr{O}_k(x_k) \rangle = \langle d\mathscr{O}_1(x_1) \cdots \mathscr{O}_k(x_k) \rangle = 0. \tag{18}$$

This means that $d\mathscr{O}$ must be Q-exact

$$d\mathscr{O}^{(0)}(x_1) = \{Q, \mathscr{O}^{(1)}(x_1)\}, \tag{19}$$

where we denote the original physical operator by $\mathscr{O}^{(0)}$ and the associated local operator by $\mathscr{O}^{(1)}$. If C is a closed circle in M, then

$$U(C) = \oint_C \mathscr{O}^{(1)} \tag{20}$$

is Q-closed. In fact,

$$\{Q, U(C)\} = \oint_C d\mathscr{O}^{(0)} = 0. \tag{21}$$

Topological invariance implies that $\delta U(C)$ under small displacements of C should be Q-exact. Since Stoke's theorem gives for a small area A with two boundaries C_1, C_2,

$$\oint_{C_1} \mathscr{O}^{(1)} - \oint_{C_2} \mathscr{O}^{(1)} = \int_A d\mathscr{O}^{(1)}, \tag{22}$$

it is again shown that $d\mathscr{O}^{(1)}$ must be Q-exact $d\mathscr{O}^{(1)} = \{Q, \mathscr{O}^{(2)}\}$. Thus we have another nonlocal operator for a closed two-dimensional surface S.

$$U(S) = \int_S \mathscr{O}^{(2)}. \tag{23}$$

One can of course repeat this procedure and eventually obtain a Q-closed nonlocal operator

$$U(M) = \int_M \mathscr{O}^{(m)}, \tag{24}$$

where m is dimensions of M. Since this is independent of the metric by construction and Q-closed, one can freely add $\mathscr{O}^{(m)}$ into the action S with some coupling constants without spoiling the cohomological property.

3 Twisted $N = 2$ Supersymmetry

Let us consider a two-dimensional $N = 2$ supersymmetric theory in the superfield formulation. There are two bosonic local variables $z, \bar{z}$ and two Grassmann variables $\theta_\pm$ and their complex conjugate $\bar{\theta}_\pm$. In our conventions, we define $\bar{\theta}_-$ to be a complex conjugate of θ_+. This is because under the Lorentz transformation $z \mapsto e^{2i\alpha}z$, Grassmann variables are changed as

$$\theta_\pm \mapsto e^{\pm i\alpha}\theta_\pm, \qquad \bar{\theta}_\mp \mapsto e^{\pm i\alpha}\bar{\theta}_\pm. \tag{25}$$

In the superfield formulation, the set of supersymmetry generators is represented by

$$\begin{aligned}
H &= -\frac{d}{idx^0} = -i(\partial_z - \partial_{\bar{z}}), \\
P &= -\frac{d}{dx^1} = -i(\partial_z + \partial_{\bar{z}}), \\
M &= 2z\partial_z - 2\bar{z}\partial_{\bar{z}} + \theta_+\frac{\partial}{\partial\theta_+} + \bar{\theta}_+\frac{\partial}{\partial\bar{\theta}_+} - \theta_-\frac{\partial}{\partial\theta_-} - \bar{\theta}_-\frac{\partial}{\partial\bar{\theta}_-}, \\
Q_\pm &= \frac{\partial}{\partial\theta_\pm} + i\bar{\theta}_\pm\partial_\pm, \\
\overline{Q}_\pm &= -\frac{\partial}{\partial\bar{\theta}_\pm} - i\theta_\pm\partial_\pm,
\end{aligned} \tag{26}$$

where $z = x^1 + ix^0$. Note that the complex conjugate of $\partial_{\bar{\theta}_+}$ is $-\partial_{\theta_-}$. Their commutators are

$$\begin{aligned}
[M, H] &= -2P, \qquad [M, P] = -2H, \\
[M, Q_\pm] &= \mp Q_\pm, \qquad [M, \overline{Q}_\pm] = \mp\overline{Q}_\pm, \\
\{Q_\pm, \overline{Q}_\pm\} &= P \pm H,
\end{aligned} \tag{27}$$

and others are zero. Note that transformations of supercharges under the Lorentz operator M are a half of those of H, P which represents that they are spinorial quantities.

Let Φ be a superfield, then a super transformation is given by a Grassmann parameter $\varepsilon_\pm$ as $\delta\Phi = (\varepsilon_- Q_+ + \varepsilon_+ Q_-)\Phi$. However since there are no constant covariant spinors $\varepsilon_\pm$ for arbitrary manifolds, these supersymmetries $Q_\pm$ are not global in general.

Fortunately since we have two supersymmetries, there is an additional $U(1)$ symmetry, called an *R-symmetry*, between them and it plays an important role in constructing a CohFT. In particular, consider the following two independent R_V, R_A transformations

$$\begin{aligned}
R_V &: (\theta_+, \bar{\theta}_+) \mapsto (e^{-i\beta}\theta_+, e^{i\beta}\bar{\theta}_+), \qquad (\theta_-, \bar{\theta}_-) \mapsto (e^{-i\beta}\theta_-, e^{i\beta}\bar{\theta}_-), \\
R_A &: (\theta_+, \bar{\theta}_+) \mapsto (e^{-i\beta}\theta_+, e^{i\beta}\bar{\theta}_+), \qquad (\theta_-, \bar{\theta}_-) \mapsto (e^{i\beta}\theta_-, e^{-i\beta}\bar{\theta}_-),
\end{aligned} \tag{28}$$

and they leave $z, \bar{z}$ invariant. That is, these are rotation among Grassmann variables. Their generators in terms of θ and nonzero commutators are given by

$$\begin{aligned} F_V &= -\theta_+ \frac{\partial}{\partial\theta_+} + \bar{\theta}_+ \frac{\partial}{\partial\bar{\theta}_+} - \theta_- \frac{\partial}{\partial\theta_-} - \bar{\theta}_+ \frac{\partial}{\partial\bar{\theta}_-}, \\ F_A &= -\theta_+ \frac{\partial}{\partial\theta_+} + \bar{\theta}_+ \frac{\partial}{\partial\bar{\theta}_+} + \theta_- \frac{\partial}{\partial\theta_-} - \bar{\theta}_- \frac{\partial}{\partial\bar{\theta}_-}, \end{aligned} \tag{29}$$

$$\begin{aligned} [F_V, Q_\pm] &= +Q_\pm, \qquad [F_V, \overline{Q}_\pm] = -\overline{Q}_\pm, \\ [F_A, Q_\pm] &= \pm Q_\pm, \qquad [F_A, \overline{Q}_\pm] = \mp\overline{Q}_\pm. \end{aligned} \tag{30}$$

Note that these commutators are different from those in (27), though they are similar as F_V, F_A are basically generators of $U(1)$.

Now we get to a crucial argument. Let us define a new Lorentz operator as

$$M_A = M - F_V, \quad \text{or} \quad M_B = M - F_A, \tag{31}$$

then their commutators with H, P remains unchanged while those with supercharges are

$$\begin{aligned} [M_A, Q_+] &= -2Q_+, & [M_B, Q_+] &= -2Q_+, \\ [M_A, Q_-] &= \quad 0, & [M_B, Q_+] &= +2Q_+, \\ [M_A, \overline{Q}_+] &= \quad 0, & [M_B, \overline{Q}_+] &= \quad 0, \\ [M_A, \overline{Q}_-] &= +2\overline{Q}_-, & [M_B, \overline{Q}_-] &= \quad 0. \end{aligned} \tag{32}$$

A theory with the Lorentz generator M_A is called A-*twisted* and one with M_B is B-*twisted*. In an A-twisted theory, if one defines $Q_A = \overline{Q}_+ + Q_-$ then we have

$$[M_A, Q_A] = 0, \qquad \{Q_A, Q_A\} = 0, \tag{33}$$

The first commutator shows that Q_A transforms as a scalar under the Lorentz transformation M_A so does its associated parameter ε_A. That is, Q_A-symmetry is globally defined. The second equation suggests that one can construct a CohFT with Q_A.

Similarly in a B-twisted model, commutators with $Q_B = \overline{Q}_+ + \overline{Q}_-$ are

$$[M_B, Q_B] = 0, \qquad \{Q_B, Q_B\} = 0, \tag{34}$$

which suggests the existence of another CohFT. Note that these observations only guarantee the first condition for a CohFT. We will explicitly see in the next section that we can indeed construct two CohFTs from $N = 2$ supersymmetric theory.

4 Sigma Model and R-anomalies

We study a supersymmetric nonlinear sigma model in two dimensions, which gives the topological A-model and B-model after twisting. Let Σ be a Riemann surface of two dimensions and M be a target space of complex dimensions m with the metric g then the sigma model governs maps $\Phi : \Sigma \to M$. The on-shell action of this model is given as

$$S = 2t \int_{\Sigma} d^2z \bigg(\frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + \frac{i}{2} g_{IJ} \psi_-^I \Delta_z \psi_-^J + \frac{i}{2} g_{IJ} \psi_+^I \Delta_{\bar{z}} \psi_+^J + \frac{1}{4} R_{IJKL} \psi_+^I \psi_+^J \psi_-^K \psi_-^L \bigg), \tag{35}$$

where t is a coupling constant and Δ is the covariant derivative with respect to the metric on both Σ and M.

Let K, $\overline{K}$ be the canonical, and anti-canonical line bundles of Σ and $T^{1,0}M$, $T^{0,1}M$ be the complexified tangent bundles of M respectively. Then each field lives in

$$\phi^a \in \Phi^*(T^{1,0}M), \tag{36}$$
$$\phi^{\bar{a}} \in \Phi^*(T^{0,1}M), \tag{37}$$
$$\psi_+^a \in K^{1/2} \otimes \Phi^*(T^{1,0}M), \tag{38}$$
$$\psi_+^{\bar{a}} \in K^{1/2} \otimes \Phi^*(T^{0,1}M), \tag{39}$$
$$\psi_-^a \in \overline{K}^{1/2} \otimes \Phi^*(T^{1,0}M), \tag{40}$$
$$\psi_-^{\bar{a}} \in \overline{K}^{1/2} \otimes \Phi^*(T^{0,1}M). \tag{41}$$

If M is Kähler, then it has the supersymmetry transformations listed below. (If M is not Kähler, it is still supersymmetric, just not with (2,2) supersymmetry, only (1,1).) Let $\varepsilon_-, \bar{\varepsilon}_- \in K^{-1/2}$ and $\varepsilon_-, \bar{\varepsilon}_- \in \overline{K}^{-1/2}$. Then the super transformation laws with these parameters are respectively

$$\begin{aligned}
\delta\phi^a &= i\varepsilon_- \psi_+^a + i\varepsilon_+ \psi_-^a, \\
\delta\phi^{\bar{a}} &= i\bar{\varepsilon}_- \psi_+^{\bar{a}} + i\bar{\varepsilon}_+ \psi_-^{\bar{a}}, \\
\delta\psi_+^a &= -\bar{\varepsilon}_- \partial_z \phi^a - i\varepsilon_+ \psi_-^b \Gamma_{bc}^a \psi_+^c, \\
\delta\psi_+^{\bar{a}} &= -\varepsilon_- \partial_z \phi^{\bar{a}} - i\bar{\varepsilon}_+ \psi_-^{\bar{b}} \Gamma_{\bar{b}\bar{c}}^{\bar{a}} \psi_+^{\bar{c}}, \\
\delta\psi_-^a &= -\bar{\varepsilon}_+ \partial_{\bar{z}} \phi^a - i\varepsilon_- \psi_+^b \Gamma_{bc}^a \psi_-^c, \\
\delta\psi_-^{\bar{a}} &= -\varepsilon_+ \partial_{\bar{z}} \phi^{\bar{a}} - i\bar{\varepsilon}_- \psi_+^{\bar{b}} \Gamma_{\bar{b}\bar{c}}^{\bar{a}} \psi_+^{\bar{c}}.
\end{aligned} \tag{42}$$

Now let us discuss the R-anomalies. For simplicity, we drop the a-indices and $\pm$-indices. In quantum field theories with fermions, one needs to be careful about their zero modes. In our model (35), the following part is problematic

$$\int D\psi D\overline{\psi} \exp(\overline{\psi} \Delta \psi). \tag{43}$$

If ψ is expanded as $\psi = \sum \psi^{(k)}$, it is given by

$$\prod_{k,l} \int d\psi^{(k)} d\overline{\psi}^{(l)} \exp\left(\overline{\psi}^{(l)} \Delta \psi^{(k)}\right). \tag{44}$$

Thus if ψ has some zero modes, the path integral vanishes since $\int d\theta = 0$ for any Grassmann variable θ. We give some facts below which we omit proofs since they are too technical:

- Except in some special cases, one can show that the number of zero modes of $\psi_\pm$ is $|k_\pm|$ and that of $\overline{\psi}_\pm$ is zero if $k_\pm$ is positive, while the number of zero modes of $\psi_\pm$ is zero and that of $\overline{\psi}_\pm$ is $|k_\pm|$ if $k_\pm$ is negative.
- $k_\pm$ satisfy $k_+ = -k_-$. Thus one can choose $k = k_-$ then there are k zero modes of ψ_-, $\overline{\psi}_+$, and no zero modes of ψ_+, $\overline{\psi}_-$ if $k \geq 0$.
- k is given by the first Chern class of the target space as

$$k = \int_{\phi(\Sigma)} c_1(M). \tag{45}$$

- The last term is small perturbation in string scale. However since small perturbative effect is not expected to give some change of the integer number $k_\pm$ in topological theories, we ignore the contribution from this term to $k_\pm$.

Therefore in this model, we need to insert local operators to have nonzero observables and those are given in the following form

$$\int D\psi_+ D\overline{\psi}_+ D\psi_- D\overline{\psi}_- W_{a_1\cdots a_k \bar{b}_1 \cdots \bar{b}_k} \left(\prod_{i=1}^{k} \psi_-^{a_i} \overline{\psi}_+^{\bar{b}_i}\right) e^{iS}, \tag{46}$$

where we have assumed $k \geq 0$. Equation (28) shows that the product of ψ_- and $\overline{\psi}_+$ is invariant under the R_V-symmetry, while it is not under R_A. Thus the R_A-symmetry is broken unless $k = 0$. We arrive at the same conclusion if $k \leq 0$. This implies that the A-twisting is defined for any Kähler target space, but the B-twisting can only be defined for Calabi–Yau target spaces since otherwise the R_A symmetry is not well-defined. In the next section, we twist this sigma model into the A-model and B-model to investigate more details.

5 The A-model

In the A-model, the spinors live in the following bundles

$$\begin{aligned}
\psi_z^a &:= \psi_+^a \in K \otimes \Phi^*(T^{1,0}(M)),\\
\chi^a &:= \psi_-^a \in \Phi^*(T^{1,0}(M)),\\
\chi^{\bar a} &:= \psi_+^{\bar a} \in \Phi^*(T^{0,1}(M)),\\
\psi_{\bar z}^{\bar a} &:= \psi_-^{\bar a} \in \overline{K} \otimes \Phi^*(T^{0,1}(M)).
\end{aligned} \tag{47}$$

By (42), by setting $\varepsilon_- = \bar\varepsilon_+ = 0$ and $\varepsilon_+ = \varepsilon$, $\bar\varepsilon_- = \bar\varepsilon$ to be constants, we have

$$\begin{aligned}
\delta\phi^a &= i\varepsilon\chi^a\\
\delta\phi^{\bar a} &= i\bar\varepsilon\chi^{\bar a}\\
\delta\chi^a &= \delta\chi^{\bar a} = 0\\
\delta\psi_z^a &= -\bar\varepsilon\partial_z\phi^a - i\varepsilon\chi_-^b\Gamma^a_{bc}\psi_z^c\\
\delta\psi_{\bar z}^{\bar a} &= -\varepsilon\partial_{\bar z}\phi^{\bar a} - i\bar\varepsilon\chi_-^{\bar b}\Gamma^{\bar a}_{\bar b\bar c}\psi_{\bar z}^{\bar c}
\end{aligned} \tag{48}$$

Note that δ^2, or Q_A^2 vanishes up to the equations of motion. We can of course consider the off-shell formalism and then $Q^2 = 0$ without using the equations of motion.

The on-shell action becomes

$$\begin{aligned}
S = 2t\int_\Sigma d^z\Big(&\frac{1}{2}g_{IJ}\partial_z\phi^I\partial_{\bar z}\phi^J + ig_{\bar a b}\psi_{\bar z}^{\bar a}\Delta_z\chi^b\\
&+ig_{a\bar b}\psi_z^a\Delta_{\bar z}\chi^{\bar b} + \frac{1}{2}R_{a\bar b c\bar d}\psi_z^a\psi_{\bar z}^{\bar b}\chi^c\chi^{\bar d}\Big),\\
\sim it\int_\Sigma d^z\{Q_A, V\} &+ t\int_\Sigma \Phi^*(\omega),
\end{aligned} \tag{49}$$

where the second term is the pull back of the target space Kähler form and the last line is true up to terms vanishing by the ψ-equations of motion. This does not make any difference in the A-model as shown shortly. V is given by

$$V = g_{a\bar b}\left(\psi_z^a\partial_z\phi^{\bar b} + \psi_{\bar z}^{\bar b}\partial_{\bar z}\phi^a\right). \tag{50}$$

The second term of (49) depends only on the cohomology class of ω and the homotopy class of the map Φ. Let us denote it by $it\beta\cdot\omega$ then the action is given by

$$S = -t\beta\cdot\omega + \int_\Sigma\{Q_A, V\}. \tag{51}$$

We can put $it\beta \cdot \omega$ out from the path integral so a physical observable takes the form

$$\langle \prod_a \mathscr{O}_a \rangle = e^{-t\beta\cdot\omega} \int D\phi D\chi D\psi \prod_a \mathscr{O}_a e^{it\{Q_A, \int V\}}. \tag{52}$$

As discussed in Sect. 2, the path integral part is independent of t hence we can calculate it in the classical limit as long as $\Re(t\beta \cdot \omega) > 0$. The t-dependence factor is called an instanton number.

The remaining terms in the Lagrangian can be written as

$$\{Q_A, V\} = L - 2tg_{a\bar{b}} \left(\partial_z \phi^a \partial_{\bar{z}} \phi^{\bar{b}} - \partial_z \phi^{\bar{b}} \partial_{\bar{z}} \phi^a \right), \tag{53}$$

In particular, it includes only $\partial_{\bar{z}}\phi^a$ and $\partial_z \phi^{\bar{a}}$. Then one realizes that L is minimized (classical limit) when ϕ is holomorphic

$$\partial_{\bar{z}}\phi^a = \partial_z \phi^{\bar{a}} = 0. \tag{54}$$

Thus the A-model sums over holomorphic maps from $\Sigma \to M$. In general, the space of such maps is finite hence the path integral reduces to a finite dimensional integral and it is known to be $m(1-g)$ for a Calabi–Yau manifold where g is the number of genus of Σ.

Note that the instanton factor obviously depends on the choice of Kähler classes while it is independent of complex structures of M hence all information about complex structures of M is embedded in the definition of V. If one modifies a complex structure of M, the variation of the action gives

$$\delta S = \{Q_A, \int_M \delta V\}, \tag{55}$$

which is irrelevant in CohFTs. Therefore, the A-model depends on the Kähler classes on M but not their complex structures.

5.1 Local Operators

In order to construct local operators independent of both the worldsheet metric and diffeomorphism, one can use only ϕ, χ but not ψ because ψ behaves as a vector and the z-indices should be either contracted by the metric or integrated out into a nonlocal operator. χ is on the other hand a fermion even after twisting hence a well-defined local operator is in the form

$$\mathscr{O}_A = A_{a_1\cdots a_p \bar{b}_1 \cdots \bar{b}_q}(\phi) \chi^{a_1} \cdots \chi^{a_p} \chi^{\bar{b}_1} \cdots \chi^{\bar{b}_q}. \tag{56}$$

By using (48), a simple calculation shows

$$\{Q_A, \mathscr{O}_A\} = -\mathscr{O}_{dA} \tag{57}$$

with d the exterior derivative acting on A. Indeed the de Rham cohomology on M turns out to be isomorphic to the Q_A-cohomology of the A-model as long as we consider only local operators.

Let us come back to the question why there is no need of terms vanishing by the ψ-equations of motion in (49). One can define a new operator $\tilde{Q}_A$ such that the second line of (49) is given by equality but instead the transformation law for ψ in (48) is modified. Then one can of course consider another topological theory with $\tilde{Q}_A$ and what potentially changes is only the $\tilde{Q}_A$-cohomology, i.e. the form of local operators. However since $\tilde{Q}_A$-operations on ϕ, χ are precisely the same as Q_A-operations, local operators given in (56) is not modified at all. Therefore there is no topological difference between the A-model with Q_A and $\tilde{Q}_A$.

After twisting, spinors are not in the same bundle as before so that the number of zero modes is also different. For example, it is known that the number of zero modes of χ is

$$k = m(1 - g), \tag{58}$$

if the target space is a Calabi–Yau manifold, which is the same as the dimensions of the space of the map ϕ[3] and no zero modes of ψ for $k \geq 0$. If k is negative, one can regard that there is no zero mode of χ but $|k|$ zero modes of ψ. However in this case, one cannot construct local topological theories because ψ should be inserted which is either nonlocal or is contracted by the worldsheet metric. Thus nonzero observables in the local A-model are only the partition function if $g = 1$ and (m, m)-point functions if $g = 0$.

6 The *B*-model

In the B-model, the spinors live in the following bundles

$$\begin{aligned}
\psi_+^a &\in K \otimes \Phi^*(T^{1,0}(M)),\\
\psi_-^a &\in \overline{K} \otimes \Phi^*(T^{1,0}(M)),\\
\psi_+^{\bar{a}} &\in \Phi^*(T^{0,1}(M)),\\
\psi_-^{\bar{a}} &\in \Phi^*(T^{0,1}(M)).
\end{aligned} \tag{59}$$

[3] This is not a coincidence. One can intuitively see from the isomorphism between d-cohomology and Q_A-cohomology by using the transformation laws of ϕ, χ.

It is convenient to define spinors as

$$\begin{aligned}
\eta^{\bar{a}} &= \psi_+^{\bar{a}} + \psi_-^{\bar{a}}, \\
\theta_a &= g_{a\bar{b}}(\psi_+^{\bar{a}} - \psi_-^{\bar{a}}), \\
\rho_z^a &= \psi_+^a, \qquad \rho_{\bar{z}}^a = \psi_-^a,
\end{aligned} \tag{60}$$

then by setting $\varepsilon_- = \varepsilon_+ = 0$ and $\bar{\varepsilon}_- = \bar{\varepsilon}_+ = \varepsilon$ to be constants, the super transformations become much simpler than those of the A-model

$$\begin{aligned}
\delta\phi^{\bar{a}} &= i\varepsilon\eta^{\bar{a}}, \\
\delta\rho &= -\varepsilon d\phi^a, \\
\delta\phi^a &= \delta\eta^{\bar{a}} = \delta\theta_a = 0.
\end{aligned} \tag{61}$$

The B-model Lagrangian is

$$\begin{aligned}
L = t\int_\Sigma d^2z\Big(& g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + ig_{a\bar{b}}\eta^{\bar{a}}(\Delta_z\rho_{\bar{z}}^b + \Delta_{\bar{z}}\rho_z^b) \\
& + i\theta_a(\Delta_{\bar{z}}\rho_z^a - \Delta_z\rho_{\bar{z}}^a) + R_{a\bar{b}c\bar{d}}\rho_z^a\rho_{\bar{z}}^c\eta^{\bar{b}}g^{\bar{d}e}\theta_e\Big), \\
= it\int_\Sigma & d^2z\{Q_B, V\} + tW,
\end{aligned} \tag{62}$$

where

$$V = g_{a\bar{b}}\left(\rho_z^a\partial_{\bar{z}}\phi^{\bar{b}} + \rho_{\bar{z}}^a\partial_z\phi^{\bar{b}}\right), \tag{63}$$

$$W = -\int_\Sigma\left(\theta_a D\rho^a + \frac{i}{2}R_{a\bar{b}c\bar{d}}\rho^a\wedge\rho^c\eta^{\bar{b}}g^{\bar{d}e}\theta_e\right), \tag{64}$$

where Δ is the extended exterior derivative on Σ. Note that this is an equality and we did not use any equations of motion, unlike the A-model. Note that since W is an integral of a (1,1)-form over Σ, it is independent of the worldsheet metric. Therefore this model satisfies the requirements to be a CohFT.

Just like the A-model, any local operator should be consisted of ϕ, θ and η thus it takes the form

$$\mathscr{O}_B = B_{\bar{b}_1\cdots\bar{b}_q}^{a_1\cdots a_p}(\phi)\theta_{a_1}\cdots\theta_{a_p}\eta^{\bar{b}_1}\cdots\eta^{\bar{b}_q}, \tag{65}$$

and by (61), one simply gets

$$\{Q_B, \mathscr{O}_B\} = -\mathscr{O}_{\bar{\partial}B}. \tag{66}$$

In contrast to the A-model, local operators include θ so that we cannot use the θ equations of motion. Note that since $B^{a_1\cdots a_p}_{\bar{b}_1\cdots\bar{b}_q}$ in (65) has not only subscripts but also superscripts, the Q_B-cohomology is isomorphc to $\oplus_{p,q} H^q(M, \wedge^p T^{1,0}M)$.

Fortunately this θ-dependence of W makes the B-model rather simpler. Since θ is linear in W, and V is independent of it, one can redefine $\theta \mapsto \theta/t$ hence we can get rid of the t-dependence of the W term. The remaining V term is Q_B-exact hence it is independent of t. Accordingly any observable in B-model is proportional to some power of t coming from the path-integral measure and local operators.

The path integral part is calculated in the classical limit similar to the A-model. In the B-model, the V term has both $(\partial_{\bar{z}}\phi^a, \partial_z\phi^{\bar{a}})$ and $(\partial_z\phi^a, \partial_{\bar{z}}\phi^{\bar{a}})$ so that the Lagrangian is minimized when

$$\partial_{\bar{z}}\phi^a = \partial_z\phi^{\bar{a}} = \partial_z\phi^a = \partial_{\bar{z}}\phi^{\bar{a}} = 0. \tag{67}$$

This is just a set of constant maps $\Phi : \Sigma \to M$. The space of such maps is a copy of M hence the path integral simply reduces to an integral over M.

The number of fermion zero modes again changes after twisting. If the target space is a Calabi–Yau, it is known that the difference of the number of η, θ zero modes and that of ρ zero modes is $k = m(1-g)$. Note that the objects integrated over M is not a $(0, m)$-form but (m, m)-form thus it is natural to contract with a holomorphic $(m, 0)$-form Ω

$$B^{a_1\cdots a_m}_{\bar{b}_1\cdots\bar{b}_m} \mapsto B^{a_1\cdots a_p}_{\bar{b}_1\cdots\bar{b}_q}\Omega_{a_1\cdots a_p}\Omega_{a'_1\cdots a'_p}. \tag{68}$$

Therefore, an observable of the B-model is an integral of wedge products of forms B and Ω over M, which one can classically calculate. It is shown for Calabi–Yau manifolds that the space of holomorphic (m,0)-forms is isomorphic to that of $\wedge^d T^{1,0}M$.

All properties of the B-model discussed so far are much simpler than those of the A-model. The only thing which is not so clear yet to see is that it is independent of Kähler classes. In fact a tedious calculation shows that a modification of the Kähler metric on M changes W in (62) by $\{Q, \cdots\}$. On the other hand, it is easy to see by the above argument that it depends on complex structures since observables are determined by the choice of the holomorphic $(m, 0)$-form Ω. As a conclusion the A-model and B-model are a mirror pair under interchange of their Kähler class and complex structure.

7 The Fixed Point Theorem

We explain why the maps ϕ reduce to holomorphic maps in the A-model and constant maps in the B-model here in an alternative way.

Consider an arbitrary quantum field theory with a group of symmetry G. Let F be the configuration space of all fields in the theory then the path integral of some operator $\mathscr{O}$ is

$$\int_F \mathscr{O} e^{-S} = \mathrm{Vol}(G) \cdot \int_{F/G} \mathscr{O} e^{-S} + \int_{F_0} \mathscr{O} e^{-S}, \tag{69}$$

where F_0 is a subspace invariant under the G-action. Notice that if G is Grassmann, $\mathrm{Vol}(F)$ should be also Grassmann and which implies that it vanishes because $\int d\theta = 0$. Therefore the first term is zero.

Now let us consider F to be a nilpotent group then F_0 is defined by fields such that $\delta\Phi = 0$. In the A-model (48) gives

$$\partial_{\bar{z}}\phi^a = \partial_z \phi^{\bar{a}} = 0, \tag{70}$$

while in the B-model we have by (61)

$$d\phi^a = 0. \tag{71}$$

Thus F_0 is the space of holomorphic maps in the A-model and the space of constants maps in the B-model respectively.

8 Topological String Theories

We only focus on closed string theories so that there is no need to worry about boundary conditions. The main difference between topological field theories and topological string theories is whether or not we path-integrate over the worldsheet metric $h_{\mu\nu}$. This makes theories more interesting.

8.1 R-anomalies

Note that the sigma model given in Sect. 4 becomes a super-conformal field theory once we couple the worldsheet metric in the action. There are three (bosonic) local symmetries, namely two diffeomorphisms and the Weyl symmetry, in two-dimensional CFT and the number of independent components of the metric is also three. Thus one can always locally gauge-fix the metric in the flat form

$$h_{\mu\nu} = \eta_{\mu\nu}. \tag{72}$$

However this is globally impossible since there are parameters that cannot be gauged away. In general, there is no parameter for a sphere, one parameter for a torus, the famous modular parameter τ, and for higher genus there are $m_g = 3(g-1)$ parameters left.

Let us first consider the $g > 1$ case, for which the number of parameters is $3(g-1)$. Conformal transformations in two dimensions are equivalent to holomorphic transformations hence these modular parameters in fact describe change of complex structure on Σ, which can be parametrized by $\mu_{\bar{z}}^{z}$, $\bar{\mu}_{z}^{\bar{z}}$ as

$$dz \mapsto dz + c\mu_{\bar{z}}^{z} d\bar{z}, \qquad d\bar{z} \mapsto d\bar{z} + \bar{c}\bar{\mu}_{z}^{\bar{z}} dz \tag{73}$$

where $c, \bar{c}$ are infinitesimal constants. After gauge-fixing the metric, one still need to integrate over this $3(g-1)$-dimensional moduli space $\tilde{M}_g$. The measure of $\tilde{M}_g$ should be invariant under coordinate transformations of Σ so $\mu_{\bar{z}}^{z}$, $\bar{\mu}_{z}^{\bar{z}}$ should be contracted by G_{zz}, $\overline{G}_{\bar{z}\bar{z}}$ and integrated over Σ, where G is the Q-partner of the EM tensor[4]. Thus, let $(m_i, \bar{m}_i)$ be coordinates on $\tilde{M}_g$ then it is natural to guess the form of the measure as

$$\int_{\tilde{M}_g} \prod_{i}^{3(g-1)} dm_i d\bar{m}_i \int_{\Sigma} G_{zz}\mu_{\bar{z}(i)}^{z} \int_{\Sigma} \overline{G}_{\bar{z}\bar{z}}\bar{\mu}_{z(i)}^{\bar{z}}. \tag{74}$$

It is indeed proven that this is correct, that is, this is invariant under coordinate transformations on $\tilde{M}_g$. We have arrived at the first crucial point of topological string theories. Even though the metric itself is independent of R-transformation, its path integral measure is not invariant under R-transformation because of fermionic fields G, $\overline{G}$. One can then see that the product of these two G, $\overline{G}$ has no R_V-charge while the R_A-charge is 2, thus the total R_A-charge is $6(g-1)$. On the other hand as discussed before, the fermion zero mode requires $2m(1-g)$ R_A-charges after twisting to be nonzero. Therefore the partition function vanishes for any genus $g > 1$ unless $m = 3$, a Calabi–Yau threefold.

For a sphere, there is no modular parameter so one can copy all results from topological filed theories and observables of (3, 3)-forms are evaluated. The only difference is that one needs to fix three rotational symmetries of a sphere. In particular for a Calabi–Yau threefold, this can be done to consider three -point functions with three marked points. That is, these points are fixed as a gauge choice.

For a torus, there is one modular parameter so we need to insert one local (1,1)-form to have nonzero observables because then the R_A-charge is consistent. Similar to the case for a sphere, there is a axial symmetry on a tours hence the insertion point of the local operator should be fixed.

As a summary for Calabi–Yau threefolds, nonzero observables of local operators are a three-point function of (3,3)-forms on a sphere, a one-point function of (1,1)-forms on a torus and partition functions for any higher genus.

[4]This result should be rigorously achieved by the Fadeev-Popov method so that the contracting tensor is suggested to be fermionic and G_{zz}, $\bar{G}_{\bar{z}\bar{z}}$ are the most natural choice.

8.2 Weyl Anomaly

In CFT, there is another anomaly we have to consider carefully, which is the *Weyl anomaly* coming from the central charge of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}, \tag{75}$$

and similarly for the right moving modes $\bar{L}_m$.

To twist a theory, we need to use the R-symmetry, which is a $U(1)$ symmetry. Thus by the Noether theorem, there exist associated conserved currents $J(z)$, $\bar{J}(\bar{z})$. For open strings, $\bar{J}(\bar{z})$ is the complex conjugate of $J(z)$, on the other hand for closed strings, they are independent. The modes of J satisfy

$$[L_m, J_n] = -nJ_{m+n}, \qquad [J_m, J_n] = \frac{c}{3}m\delta_{m+n}, \tag{76}$$

and similarly for $\overline{J}_n$. Thus by using these currents J, $\overline{J}$, we can define the new stress tensors as

$$\tilde{T}^{\pm} = T \pm \frac{1}{2}\partial J, \qquad \tilde{\overline{T}}^{\pm} = \overline{T} \pm \frac{1}{2}\bar{\partial}\,\overline{J}, \tag{77}$$

and we denote their modes as $\tilde{L}^{\pm}_m$, $\tilde{\bar{L}}^{\pm}_m$. This is another important point of topological string theories that the new modes simply obey the Witt algebra, i.e. no central charge.

$$[\tilde{L}^{\pm}_m, \tilde{L}^{\pm}_n] = (m-n)\tilde{L}^{\pm}_{m+n}, \qquad [\tilde{\bar{L}}^{\pm}_m, \tilde{\bar{L}}^{\pm}_n] = (m-n)\tilde{\bar{L}}^{\pm}_{m+n}. \tag{78}$$

As a result, there is no Weyl anomaly.

It turns out that this shift of the stress tensors is equivalent to the A-twisting or B-twisting. In this sense, twisted string theories are somewhat more fundamental to define anomaly-free consistent theories. For simplicity, let us choose $+$ for both definitions in (77) then the zero modes are

$$\tilde{L}_0 = L_0 - \frac{1}{2}J_0, \tag{79}$$

and similarly for the left-moving mode. The generators of the R-symmetry and the new Lorentz symmetry $\tilde{M}$ are defined as

$$F_L = 2\pi i J_0, \qquad F_R = 2\pi i \bar{J}_0, \tag{80}$$

$$\tilde{M} = 2\pi i(\tilde{L}_0 - \tilde{\bar{L}}_0) = M - \frac{1}{2}(F_L - F_R), \tag{81}$$

where M is the generator of the Lorentz symmetry before the shift. (29) implies that $F_V + F_A$ only acts on $+$-indices, i.e. left-moving indices and $F_V - F_A$ on left-moving indices. That is, they are the generators of left- and right-moving currents and it is natural to identify them with F_L and F_R. More accurately, they are identified as

$$F_V = \frac{1}{2}(F_L + F_R), \qquad F_A = \frac{1}{2}(F_L - F_R). \tag{82}$$

Thus (81) is none other than the Lorentz generator for the B-model and a similar argument works for the A-model if one choose $-$ sigh in the $\tilde{\bar{T}}$ shift (77).

Further discussions about topological string theories, in particular nonlocal operators, are left to [2].

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