

The SYZ Conjecture via Homological Mirror Symmetry



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1 Introduction

These are expanded notes based on a talk given at the Superschool on Derived Categories and D -branes held at the University of Alberta in July of 2016. The goal of these notes is to give a motivated introduction to the Strominger-Yau-Zaslow (SYZ) conjecture from the point of view of homological mirror symmetry.

The SYZ conjecture was proposed in [35] and attempts to give a geometric explanation for the phenomena of mirror symmetry. To date, it is still the best template for constructing mirrors \check{X} to a given Calabi–Yau n -fold X . We aim to give the reader an idea of why one should believe some form of this conjecture and a feeling for the ideas involved without getting into the formidable details. We assume some background on classical mirror symmetry and homological mirror symmetry as covered for example in the relevant articles in this volume.

Should the readers appetite be sufficiently whet, she is encouraged to seek out one of the many more detailed surveys such as [2, 3, 10–12, 18–20] etc.

2 From Homological Mirror Symmetry to Torus Fibrations

Suppose X and \check{X} are mirror dual Kähler Calabi–Yau n -folds. Kontsevich’s homological mirror symmetry conjecture [29] posits that there is an equivalence of categories

$$\mathcal{Fuk}(X) \cong D^b(\mathrm{Coh}(\check{X}))$$

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between¹ the Fukaya category of X and the derived category of \check{X} . This should make precise the physical expectation that “the A -model on X is equivalent to the B -model on \check{X} .” The basic idea of the correspondence is summarized by the following table:

	A -model on X	B -model on \check{X}
Objects	Lagrangians with flat $U(m)$ -connection (L, ∇)	(complexes) of coherent sheaves \mathcal{F}
Morphisms	Floer cohomology groups $HF^*(L, M)$	Ext groups $Ext^*(\mathcal{F}, \mathcal{G})$
Endomorphism algebra	$HF^*(L, L) = H^*(L)$	$Ext^*(\mathcal{F}, \mathcal{F})$

Now we can now try to understand how this correspondence should work in simple cases. The simplest coherent sheaves on \check{X} are structure sheaves of points \mathcal{O}_p and indeed \check{X} is the moduli space for such sheaves:

$$\{\mathcal{O}_p : p \in \check{X}\} \cong \check{X}.$$

Therefore there must be a family of Lagrangians with flat connections (L_p, ∇_p) parametrized by $p \in \check{X}$ and satisfying

$$H^*(L_p) \cong Ext^*(\mathcal{O}_p, \mathcal{O}_p).$$

Let us compute the right hand side explicitly.

This question is local so we can reduce to an affine neighborhood U of p . Since U is smooth at p , then p is the zero set of a section $\mathcal{O}_U \rightarrow V \cong \mathcal{O}_U^{\oplus n}$. Dualizing, we obtain an exact sequence

$$V^* \xrightarrow{s} \mathcal{O}_U \longrightarrow \mathcal{O}_p \longrightarrow 0$$

that we can extend by the *Koszul resolution*

$$0 \longrightarrow \wedge^n V^* \xrightarrow{s_n} \wedge^{n-1} V^* \xrightarrow{s_{n-1}} \dots \xrightarrow{s_2} V^* \xrightarrow{s} \mathcal{O}_U \longrightarrow \mathcal{O}_p \longrightarrow 0 .$$

where

$$s_k(v_1 \wedge \dots \wedge v_k) = \sum_{i=1}^k s(v_i)v_1 \wedge \dots \wedge \hat{v}_i \wedge \dots \wedge v_k.$$

Truncating and applying $\text{Hom}(-, \mathcal{O}_p)$ gives us

$$0 \longleftarrow \wedge^n V_p \longleftarrow \wedge^{n-1} V_p \longleftarrow \dots \longleftarrow V_p \longleftarrow k_p \longleftarrow 0$$

¹One should work with the dg/ A_∞ enhancements of these categories but we ignore that here.

where k_p is the skyscraper sheaf at p , V_p is the fiber of V , and all the morphisms are 0 since $s(w)$ vanishes at p for any w . It follows that

$$\text{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) = \bigoplus_{k=0}^n \bigwedge^k V_p$$

where V_p is an n -dimensional vector space (in fact isomorphic by the section s to $T_p U$).

Therefore we are looking for Lagrangians L_p in X with

$$H^*(L_p) \cong \bigoplus_{k=0}^n \bigwedge^k V_p$$

where V_p is an n -dimensional vector space. If we stare at this for a while, we realize this is exactly the cohomology of an n -torus; $H^*(L_p) \cong H^*(T^n)$. This suggests that points $p \in \check{X}$ might correspond to Lagrangian tori in X with flat connections.

We are led to consider the geometry of Lagrangian tori in the symplectic manifold (X, ω) . The first thing to note is that under the isomorphism $TX \cong T^*X$ induced by the symplectic form, the normal bundle of a Lagrangian L is identified with its cotangent bundle:

$$N_L X \cong T^*L.$$

In fact, more is true. There is always a tubular neighborhood of $N_\epsilon(L)$ in X isomorphic to a neighborhood of L in $N_L X$, and under this identification we get that $N_\epsilon(L)$ is *symplectomorphic* to a neighborhood of the zero section in T^*L with the usual symplectic form by the Weinstein neighborhood theorem [38, Corollary 6.2].

On the other hand, if $L \cong T^n$ is an n -torus then $T^*L \cong \mathbb{R}^n \times T^n$ is the trivial bundle. Therefore we can consider the projection

$$\mu : \mathbb{R}^n \times T^n \rightarrow \mathbb{R}^n.$$

This is a *Lagrangian torus fibration* over T^*L over an affine space. The restriction of μ to the tubular neighborhood $N_\epsilon(L)$ under the aforementioned identification equips X with the structure of a Lagrangian torus fibration, at least locally around a Lagrangian torus.

The SYZ conjecture predicts that this is true globally: given a Calabi–Yau manifold X for which we expect mirror symmetry to hold, then X should be equipped with a global Lagrangian torus fibration $\mu : X \rightarrow B$ which locally around smooth fibers looks like the fibration $T^*T^n \rightarrow \mathbb{R}^n$ over a flat base. By the previous discussion, \check{X} should be the moduli space of pairs (L, ∇) where L is a Lagrangian torus fiber of μ and ∇ is a flat unitary connection on the L . However μ can, and often will, have singular Lagrangian fibers (see Remark 2.1.ii) and understanding how these singular fibers affect \check{X} is the greatest source of difficulty in tackling the SYZ conjecture.

Let us momentarily restrict to the open locus $B_0 \subset B$ over which μ has smooth torus fibers and denote the restriction $\mu_0 : X_0 \rightarrow B_0$. Then there is an open subset $\check{X}_0 \subset \check{X}$ for which the description as a moduli space of pairs (L, ∇) of a smooth Lagrangian torus fiber of μ_0 equipped with a flat unitary connection makes sense. We can ask what structure does \check{X}_0 gain from the existence of $\mu : X \rightarrow B$?

Viewing B_0 as the space of smooth fibers of μ , there is a natural map $\check{\mu}_0 : \check{X}_0 \rightarrow B_0$ given by $(L, \nabla) \mapsto L$. Now a flat unitary connection ∇ is equivalent to a homomorphism

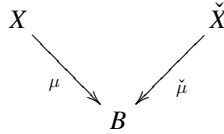
$$\text{Hom}(\pi_1(L), U(m)).$$

Since \check{X}_0 must be $2n$ real dimensional and $\check{\mu}_0$ is a fibration over an n real dimensional base, the fibers must be n real dimensional and so $m = 1$. That is, the fibers of $\check{\mu}_0$ are given by

$$\text{Hom}(\pi_1(L), U(1)) \cong (L)^*$$

the dual torus of L . Ignoring singular Lagrangians, $\check{X}_0 \subset \check{X}$ is equipped with a dual Lagrangian torus fibration $\check{\mu}_0 : \check{X}_0 \rightarrow B_0 \subset B$!

Conjecture 1 (Strominger-Yau-Zaslow [35]) *Mirror Calabi–Yau manifolds are equipped with special Lagrangian fibrations*



such that μ and $\check{\mu}$ are dual torus fibrations over a dense open locus $B_0 \subset B$ of the base.

Remark 2.1 (i) We will discuss the notion of a *special Lagrangian* and the reason for this condition in 2.1.

(ii) Note that unless $\chi(X) = 0$, then the fibration μ must have singularities. Indeed the only compact CY manifolds with smooth Lagrangian torus fibrations are tori.

(iii) From the point of view of symplectic geometry, Lagrangian torus fibrations are natural to consider. Indeed a theorem of Arnol’d and Liouville states that the smooth fibers of *any* Lagrangian fibration of a symplectic manifold are tori [8, Sect. 49].

This conjecture suggests a recipe for constructing mirror duals to a given Calabi–Yau X . Indeed we pick a $\mu : X \rightarrow B$ and look at the restriction $\mu_0 : X_0 \rightarrow B_0$ to the smooth locus. Then μ_0 is a Lagrangian torus fibration which we may dualize to obtain $\check{\mu}_0 : \check{X}_0 \rightarrow B_0$. Then we compactify X_0 by adding back the boundary $X \setminus X_0 =: D$ and hope that this suggests a way to compactify the dual fibration to obtain a mirror \check{X} .

It turns out the story is not so simple and understanding how to compactify \check{X}_0 and endow it with a complex structure leads to many difficulties arising from *instanton corrections* and convergence issues for Floer differentials. Furthermore this strategy to construct the dual depends not only on X but also on the chosen fibration μ and indeed we can obtain different mirrors by picking different fibrations, or even from the same fibration by picking a different “compactification” recipe. This leads to mirrors that are *Landau-Ginzburg models* and allows us to extend the statement of mirror symmetry outside of the Calabi–Yau case ([9, 28], etc). Finally, there are major issues in constructing Lagrangian torus fibrations in general. Indeed it is not known if they exist for a general Calabi–Yau, and in fact they are only expected to exist in the *large complex structure limit* (LCSL) [24, 30]. This leads to studying SYZ mirror symmetry in the context LCSL degenerations of CY manifolds as in the *Gross-Siebert program* [20, 21]. We discuss these ideas in more detail in Sect. 5.

2.1 Some Remarks on Special Lagrangians

As stated, the SYZ conjecture is about *special* Lagrangian (sLag) torus fibrations rather than arbitrary torus fibrations. Recall that a Calabi–Yau manifold has a non-vanishing holomorphic volume form $\Omega \in H^0(X, \Omega_X^n)$.

Definition 2.2 A Lagrangian $L \subset X$ is special if there exists a choice of Ω such that

$$\mathrm{Im}(\Omega)|_L = 0.$$

There are several reasons to consider special Lagrangians:

- SLags minimize the volume within their homology class. In physics this corresponds to the fact these are the *BPS branes* (see Sect. 2.2). Mathematically, this corresponds to the existence of a conjectural Bridgeland–Douglas stability condition on the Fukaya category whose stable objects are the special Lagrangians (see for example [27]).
- SLags give canonical representatives within a Hamiltonian isotopy class of Lagrangians. Indeed a theorem of Thomas and Yau [37, Theorem 4.3] states that under some assumptions, there is a unique sLag within each Hamiltonian deformation class.
- The deformation theory of sLag tori is well understood and endows the base B of a sLag fibration with the structures needed to realize mirror symmetry, at least away from the singularities. We will discuss this in more detail in Sect. 4.1.

However, it is much easier to construct torus fibrations than it is to construct sLag torus fibrations and in fact its an open problem whether the latter exist for a general Calabi–Yau. Therefore for many partial results and in many examples, one must get by with ignoring the special condition and considering only Lagrangian torus fibrations.

2.2 A Remark on D -branes and T -Duality

Strominger-Yau-Zaslow's original motivation in [35] differed slightly from the story above. Their argument used the physics of D -branes, that is, boundary conditions for open strings in the A - or B -model.²

They gave roughly the following argument for Calabi–Yau threefolds. The moduli space of $D0^3$ B -branes on \check{X} must be the moduli space of some BPS A -brane on X . The BPS condition and supersymmetry necessitate that this is a $D3$ brane consisting of a special Lagrangian L equipped with a flat $U(1)$ connection. Topological considerations force $b_1(L) = 3$ and so the space of flat $U(1)$ connections

$$\mathrm{Hom}(\pi_1(L), U(1)) \cong T^3$$

is a 3 torus. Thus \check{X} must be fibered by $D3$ A -branes homeomorphic to tori and by running the same argument with the roles of X and \check{X} reversed, we must get a fibration by tori on X as well.

The connection with homological mirror symmetry, which was discovered later, comes from the interpretation of the Fukaya category and the derived category as the categories of topological D -branes for the A - and B -model respectively. The morphisms in the categories correspond to massless open string states between two D -branes.

Now one can consider what happens if we take a $D6$ B -brane given by a line bundle \mathcal{L} on \check{X} . By using an argument similar to the one above, or computing

$$\mathrm{Ext}^*(\mathcal{L}, \mathcal{O}_p) \cong k[0],$$

we⁴ see that there is a one dimensional space of string states between \mathcal{L} and \mathcal{O}_p . Therefore the Lagrangian S in X dual to \mathcal{L} must satisfy

$$HF^*(S, L) = k[0].$$

Remembering that the Floer homology groups count intersection points of Lagrangians, this suggests that S must be a section of the fibration μ .

In summary, the SYZ Conjecture states that mirror symmetry interchanges $D0$ B -branes on \check{X} with $D3$ Lagrangian torus A -branes on X and $D6$ B -branes on \check{X} with $D3$ Lagrangian sections on X . On a smooth torus fiber of the fibration, this is interchanging $D0$ and $D3$ branes on dual 3-tori. This duality on each torus is precisely what physicists call T -duality and one of the major insights of [35] is that in the presence of dual sLag fibrations, mirror symmetry is equivalent to fiberwise T -duality.

²For background on D -branes see for example [2] or the other entries in this volume.

³ $D0, D3, \dots$ denote 0-dimensional, 3-dimensional, \dots D -branes.

⁴That is, k in degree zero and 0 in other degrees.

3 Hodge Symmetries from SYZ

The first computational evidence that led to mirror symmetry was the interchange of Hodge numbers

$$\begin{aligned} h^{1,1}(X) &= h^{1,2}(\check{X}) \\ h^{1,2}(X) &= h^{1,1}(\check{X}) \end{aligned} \tag{1}$$

for compact simply connected mirror Calabi–Yau threefolds X and \check{X} . Thus the first check of the SYZ conjecture is if it implies the interchange of Hodge numbers. We will show this under a simplifying assumption on the SYZ fibrations.

Let $f : X \rightarrow B$ be a proper fibration and let $i : B_0 \subset B$ be the locus over which f is smooth so that $f_0 : X_0 \rightarrow B_0$ is the restriction. Then the higher direct image of the constant sheaf $R^p f_* \mathbb{R}$ is a constructible sheaf with

$$i^* R^p f_* \mathbb{R} \cong R^p (f_0)_* \mathbb{R}$$

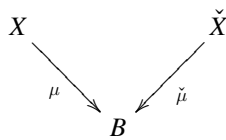
for each $p \geq 0$. Furthermore, $R^p (f_0)_* \mathbb{R}$ is the local system on B_0 with fibers the cohomology groups $H^p(X_b, \mathbb{R})$ for $b \in B_0$ since f_0 is a submersion.

Definition 3.1 We say that f is simple if we can recover the constructible sheaf $R^p f_* \mathbb{R}$ by the formula

$$i_* R^p (f_0)_* \mathbb{R} \cong R^p f_* \mathbb{R}$$

for all $p \geq 0$.

Proposition 3.2 Suppose X and \check{X} are compact simply connected Calabi–Yau threefolds with dual sLag fibrations



such that μ and $\check{\mu}$ are simple. Assume further that μ and $\check{\mu}$ admit sections. Then the Hodge numbers of X and \check{X} are interchanged as in (1).

Before the proof, we will review some facts about tori. If T is an n -torus, there is a canonical identification

$$T \cong H_1(T, \mathbb{R}) / \Lambda_T$$

where Λ_T denotes the lattice $H_1(T, \mathbb{Z}) / tors \subset H_1(T, \mathbb{R})$. Then the isomorphism $H^1(T, \mathbb{R}) \cong H_1(T, \mathbb{R})^*$ induces an identification

$$T^* \cong H^1(T, \mathbb{R}) / \Lambda_T^*$$

where $\Lambda_T^* = H^1(T, \mathbb{Z})/tors \subset H^1(T, \mathbb{R})$. It follows that $H_1(T^*, \mathbb{R}) = H^1(T, \mathbb{R})$ and $\Lambda_{T^*} = \Lambda_T^*$. More generally, denoting $V = H_1(T, \mathbb{R})$, there are isomorphisms

$$H^p(T, \mathbb{R}) \cong \bigwedge^p V^*,$$

$$H^p(T^*, \mathbb{R}) \cong \bigwedge^p V.$$

After fixing an identification $\bigwedge^n V \cong \mathbb{R}$, Poincaré duality gives rise to isomorphisms

$$H^p(T, \mathbb{R}) \cong H^{n-p}(T^*, \mathbb{R})$$

compatible with the identification $\Lambda_T^* = \Lambda_{T^*}$.

Proof of Proposition 3.2 Applying the above discussion fiber by fiber to the smooth torus bundle $\mu_0 : X_0 \rightarrow B_0$, we obtain an isomorphism of torus bundles

$$R^1(\mu_0)_*(\mathbb{R}/\mathbb{Z}) := (R^1(\mu_0)_*\mathbb{R})/(R^1(\mu_0)_*\mathbb{Z}/tors) \cong \check{X}_0$$

over B . Similarly $X_0 \cong R^1(\check{\mu}_0)_*(\mathbb{R}/\mathbb{Z})$ and Poincaré duality gives rise to

$$R^p(\mu_0)_*\mathbb{R} \cong R^{3-p}(\check{\mu}_0)_*\mathbb{R}.$$

By the simple assumption on μ and $\check{\mu}$ it follows that

$$R^p\mu_*\mathbb{R} \cong R^{3-p}\check{\mu}_*\mathbb{R}. \tag{2}$$

We want to use this isomorphism combined with the Leray spectral sequence to conclude the relation on Hodge numbers.

Let us analyze the cohomology of X and \check{X} . First, $H^1(X, \mathbb{R}) = 0$ by the simply connected assumption and so $H^5(X, \mathbb{R}) = 0$ by Poincaré duality. This implies the Hodge numbers $h^{0,1}(X)$, $h^{1,0}(X)$, $h^{2,3}(X)$ and $h^{3,2}(X)$ are all zero. By Serre duality, $h^{2,0} = h^{0,2}(X) = h^{0,1}(X) = 0$. Furthermore, $h^{1,3} = h^{3,1} = h^1(X, \Omega_X^3) = h^{0,1} = 0$ by the Calabi–Yau condition. Finally, $h^{3,3} = h^{0,0} = 1$ is evident and $h^{0,3} = h^{3,0} = h^0(X, \Omega_X^3) = 1$ again by the Calabi–Yau condition. Putting this together gives us the following relation between Hodge numbers and Betti numbers:

$$h^{1,1}(X) = b_2(X) = b_4(X) = h^{2,2}(X)$$

$$b_3(X) = 2 + h^{1,2}(X) + h^{2,1}(X) = 2(1 + h^{1,2}(X))$$

Of course the same is also true for \check{X} . Thus it would suffice to show

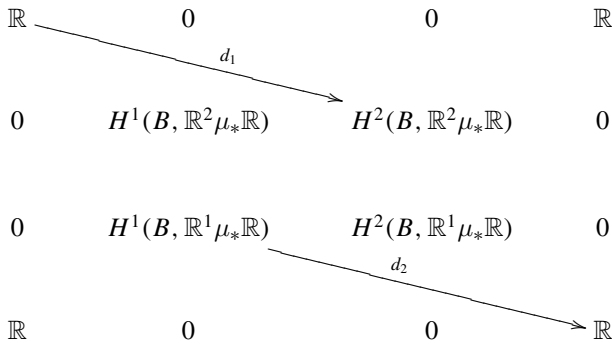
$$b_3(\check{X}) = 2 + h^{1,1}(X) + h^{2,2}(X) = 2(1 + h^{1,1}(X)) \tag{3}$$

from which it follows that $h^{1,1}(X) = h^{1,2}(\check{X})$ as well as $h^{1,1}(\check{X}) = h^{1,2}(X)$ by applying the same argument to X .

The sheaves $R^3\mu_*\mathbb{R}$ and $R^0\mu_*\mathbb{R}$ are both isomorphic to the constant sheaf \mathbb{R} . As X is simply connected, so is B so we deduce $H^1(B, \mathbb{R}) = 0$ and similarly $H^2(B, \mathbb{R}) = 0$ by Poincaré duality. Thus $H^1(B, R^0\mu_*\mathbb{R}) = H^2(B, R^0\mu_*\mathbb{R}) = H^1(B, R^3\mu_*\mathbb{R}) = H^2(B, R^3\mu_*\mathbb{R}) = 0$ and $H^i(B, R^j\mu_*\mathbb{R}) = \mathbb{R}$ for $i, j = 0, 3$. Next the vanishing $H^1(X, \mathbb{R}) = H^5(X, \mathbb{R})$ imply that $H^0(B, R^1\mu_*\mathbb{R}) = H^3(B, R^2\mu_*\mathbb{R}) = 0$. Applying the same reasoning to $\check{\mu}$ and using the isomorphism (2), we get

$$\begin{aligned} H^0(B, R^2\mu_*\mathbb{R}) &= H^0(B, R^1\check{\mu}_*\mathbb{R}) = 0, \\ H^3(B, R^1\mu_*\mathbb{R}) &= H^3(B, R^2\check{\mu}_*\mathbb{R}) = 0. \end{aligned}$$

Putting this all together, the E_2 page of the Leray spectral sequence for μ becomes



with the only possibly nonzero differentials depicted above. We claim in fact that d_1 and d_2 must also be zero.

Indeed let $S \subset X$ be a section of μ . Then S induces a nonzero section $s \in \mathbb{R} \cong H^0(B, R^3\mu_*\mathbb{R})$ since it intersects each fiber in codimension 3. Furthermore S must represent a nonzero cohomology class on X and so $s \in \ker(d_1)$. This forces d_1 to be the zero map since $H^0(B, R^3\mu_*\mathbb{R})$ is one dimensional. Similarly, the fibers of μ give rise to a nonzero class in $f \in H^3(B, R^0\mu_*\mathbb{R}) \cong \mathbb{R}$. Since the class of a fiber is also nonzero in the cohomology of X as the fibers intersect the section, then f must remain nonzero in $\text{coker}(d_2)$; that is, d_2 must be zero.

This means the Leray spectral sequence for μ degenerates at the E_2 page and similarly for $\check{\mu}$. In particular, we can compute

$$\begin{aligned} h^{1,1}(X) &= b_2(X) = h^1(B, R^1\mu_*\mathbb{R}) = h^1(B, R^2\check{\mu}_*\mathbb{R}), \\ h^{2,2}(X) &= b_4(X) = h^2(B, R^2\mu_*\mathbb{R}) = h^2(B, R^1\check{\mu}_*\mathbb{R}), \end{aligned}$$

where we have again used (2). Therefore we can verify

$$b_3(\check{X}) = 2 + h^1(B, R^2\check{\mu}_*\mathbb{R}) + h^2(B, R^1\check{\mu}_*\mathbb{R}) = 2 + h^{1,1}(X) + h^{2,2}(X)$$

as required. □

Remark 3.3 The argument above (originally appearing in [16]) was generalized by Gross in [17] to obtain a relation between the integral cohomologies of X and \check{X} .

The reader may object that there are several assumptions required in the above result. The existence of a section isn't a serious assumption. Indeed all that was required in the proof is the existence of a cohomology class that behaves like a section with respect to cup products. As we already saw in 2.2, mirror symmetry necessitate the existence of such Lagrangians on X dual to line bundles on \check{X} and vice versa. The simplicity assumption, on the other hand, is serious and isn't always satisfied. However, this still gives us a good heuristic check of SYZ mirror symmetry.

4 Semi-flat Mirror Symmetry

In this section we will consider the case where μ and $\check{\mu}$ are smooth sLag fibrations so that $B_0 = B$. This is often called the *semi-flat* case.

In this case we will see that the existence of dual sLag fibrations endows B with the extra structure of an integral affine manifold which results in a toy model of mirror symmetry on B . In fact, we will see that the dual SYZ fibrations can be recovered from this integral affine structure. Finally, we will discuss an approach to realize HMS conjecture in the semi-flat case.

4.1 The Moduli Space of Special Lagrangians

The starting point is the following theorem of McLean:

Theorem 4.1 (McLean [32, Sect. 3]) *Let (X, J, ω, Ω) be a Kähler Calabi–Yau n -fold. Then the moduli space \mathcal{M} of special Lagrangian submanifolds is a smooth manifold. Furthermore, there are natural identifications*

$$H^{n-1}(L, \mathbb{R}) \cong T_L \mathcal{M} \cong H^1(L, \mathbb{R})$$

of the tangent space to any sLag submanifold $L \subset X$.

The idea is that a deformation of L is given by a normal vector field $v \in C^\infty(N_L X, \mathbb{R})$. Then we obtain a 1-form $\alpha \in \Omega^1(L, \mathbb{R})$ and an $n - 1$ -form $\beta \in \Omega^{n-1}(L, \mathbb{R})$ by contraction with ω and $\text{Im}\Omega$ respectively:

$$\begin{aligned} \alpha &= -i_v \omega, \\ \beta &= i_v \text{Im}\Omega. \end{aligned}$$

It turns out that α and β determine each other and that v induces a sLag deformation of L if and only if α and β are both closed. This gives the above isomorphisms by the maps $v \mapsto [\alpha] \in H^1(L, \mathbb{R})$ and $v \mapsto [\beta] \in H^{n-1}(L, \mathbb{R})$ respectively.

Note in particular that the isomorphism $T_L\mathcal{M} \cong H^1(L, \mathbb{R})$ depends on the symplectic structure ω and the isomorphism $T_L\mathcal{M} \cong H^{n-1}(L, \mathbb{R})$ depends on the complex structure through the holomorphic volume form Ω .

Definition 4.2 An integral affine manifold M is a smooth manifold equipped with transition functions in the affine group $\mathbb{R}^n \rtimes \text{GL}_n(\mathbb{Z})$. Equivalently it is a manifold M equipped with a local system of integral lattices $\Lambda \subset TM$.

The equivalence in Definition 4.2 can be seen by noting that if the transition functions of M are affine transformations, they preserve the integral lattice defined in local coordinates by

$$\Lambda := \text{Span}_{\mathbb{Z}} \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right) \subset TU. \tag{4}$$

On the other hand, if there exists a local system of integral lattice $\Lambda \subset TM$ with a compatible flat connection ∇ on TM , then on a small enough coordinate patch we can choose coordinates such that Λ is the coordinate lattice and the transition functions must be linear isomorphisms on this lattice.

The vector spaces $H^1(L, \mathbb{R})$ and $H^{n-1}(L, \mathbb{R})$ glue together to form vector bundles on \mathcal{M} . Explicitly, if $\mathcal{L} \subset X \times \mathcal{M}$ is the universal family of sLags over \mathcal{M} with projection $\pi : \mathcal{L} \rightarrow \mathcal{M}$ then these bundles are $R^1\pi_*\mathbb{R}$ and $R^{n-1}\pi_*\mathbb{R}$ respectively. Similarly, the integral cohomology groups $H^1(L, \mathbb{Z})/\text{tors} \subset H^1(L, \mathbb{R})$ and $H^{n-1}(L, \mathbb{Z})/\text{tors} \subset H^{n-1}(L, \mathbb{R})$ glue together into local systems of integral lattices $R^1\pi_*\mathbb{Z}/\text{tors} \subset R^1\pi_*\mathbb{R}$ and $R^{n-1}\pi_*\mathbb{Z}/\text{tors} \subset R^{n-1}\pi_*\mathbb{R}$. Applying Theorem 4.1 fiber by fiber yields two integral affine structures on \mathcal{M} :

Corollary 4.3 *There are isomorphisms $R^1\pi_*\mathbb{R} \cong T\mathcal{M} \cong R^{n-1}\pi_*\mathbb{R}$ which endow \mathcal{M} with two integral affine structures given by the integral lattices*

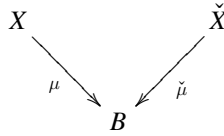
$$\begin{aligned} R^1\pi_*\mathbb{Z}/\text{tors} &\subset R^1\pi_*\mathbb{R} \cong T\mathcal{M}, \\ R^{n-1}\pi_*\mathbb{Z}/\text{tors} &\subset R^{n-1}\pi_*\mathbb{R} \cong T\mathcal{M}. \end{aligned}$$

Poincare duality induces an isomorphism $T\mathcal{M} \cong T^\mathcal{M}$ exchanging the lattices and their duals.*

4.2 Mirror Symmetry for Integral Affine Structures

4.2.1 From SYZ Fibrations to Integral Affine Structures

Now let us return to the case of dual SYZ fibrations



where both μ and $\check{\mu}$ are smooth. Then $\dim B = n = \dim H^1(L, \mathbb{R})$ is the dimension of the moduli space of sLag n -tori in X and so B must be an open subset of the moduli space \mathcal{M} .

In particular, by Corollary 4.3, the symplectic form ω and the holomorphic volume form Ω on X induces two integral affine structures on B explicitly given by

$$\begin{aligned}
 \Lambda_\omega &:= R^1\mu_*\mathbb{Z}/tors \subset R^1\mu_*\mathbb{R} \cong TB, \\
 \Lambda_\Omega &:= R^{n-1}\mu_*\mathbb{Z}/tors \subset R^{n-1}\mu_*\mathbb{R} \cong TB.
 \end{aligned}$$

We call these the Kähler and complex integral affine structures respectively. Similarly the symplectic and holomorphic forms $\check{\omega}$ and $\check{\Omega}$ on \check{X} induce two other integral affine structures

$$\begin{aligned}
 \Lambda_{\check{\omega}} &:= R^1\check{\mu}_*\mathbb{Z}/tors \subset R^1\check{\mu}_*\mathbb{R} \cong TB, \\
 \Lambda_{\check{\Omega}} &:= R^{n-1}\check{\mu}_*\mathbb{Z}/tors \subset R^{n-1}\check{\mu}_*\mathbb{R} \cong TB,
 \end{aligned}$$

on B . The fact that these torus fibrations are dual implies natural isomorphisms

$$\begin{aligned}
 R^1\mu_*\mathbb{R} &\cong R^{n-1}\check{\mu}_*\mathbb{R}, \\
 R^{n-1}\mu_*\mathbb{R} &\cong R^1\check{\mu}_*\mathbb{R}.
 \end{aligned}$$

The top isomorphism exchanges Λ_ω and $\Lambda_{\check{\Omega}}$ while the bottom isomorphism exchanges $\Lambda_{\check{\omega}}$ and Λ_Ω . We can summarize this as follows: *SYZ mirror symmetry for smooth sLag torus fibrations interchanges the complex and Kähler integral affine structures on the base B .*

4.2.2 From Integral Affine Structures to SYZ Fibrations

We can go in the other direction and recover the mirror SYZ fibrations μ and $\check{\mu}$ from the integral affine structures on the base B . The key is the following proposition:

Proposition 4.4 *Let $(B, \Lambda \subset TB)$ be an integral affine manifold. Then the torus fibration $TB/\Lambda \rightarrow B$ has a natural complex structure and the dual torus fibration $T^*B/\Lambda^* \rightarrow B$ has a natural symplectic structure.*

Proof Locally we can find a coordinate chart $U \subset B$ with coordinates y_1, \dots, y_n such that Λ is a coordinate lattice as in (4). Then the coordinate functions on TU are given by y_1, \dots, y_n and $x_1 = dy_1, \dots, x_n = dy_n$ and we can define holomorphic

coordinates on TU by $z_j = x_j + \sqrt{-1}y_j$. Since the transition functions on B preserve the lattice, they induce transition functions on TB that are holomorphic with respect to these coordinates giving TB the structure of a complex manifold.

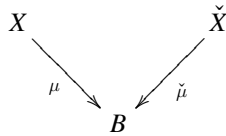
Consider the holomorphic functions defined locally by

$$q_j := e^{2\pi\sqrt{-1}z_j}.$$

These functions are invariant under integral affine transition functions as well as global translations by Λ and so they give a compatible system of holomorphic coordinates for TB/Λ .

Similarly, in local coordinates U where Λ is the coordinate lattice, then $\Lambda^* \subset T^*U$ is generated by dy_1, \dots, dy_n as a lattice in T^*U . Therefore the standard symplectic structure on T^*B is invariant by Λ^* and descends to T^*B/Λ^* . \square

Now suppose B is a smooth manifold equipped with two integral affine structures $\Lambda_0, \Lambda_1 \subset TB$ as well as an isomorphism $TB \cong T^*B$ such that $\Lambda_0 \cong (\Lambda_1)^*$ and $\Lambda_1 \cong (\Lambda_0)^*$. Then we have dual torus fibrations



where $X := TB/\Lambda_0 \cong T^*B/(\Lambda_1)^*$ and $\check{X} := T^*B/(\Lambda_0)^* \cong TB/(\Lambda_1)$. This construction satisfies the following properties:

- (a) if Λ_0 and Λ_1 are the integral affine structures associated to SYZ dual torus fibrations as in Sect. 4.2.1, then this construction recovers the original fibrations;
- (b) Λ_0 determines the complex structure of X and the symplectic structure of \check{X} ;
- (c) Λ_1 determines the symplectic structure of X and the complex structure of \check{X} .

As a result we recover one of the main predictions of mirror symmetry: *deformations of the complex structure on X are the same as deformations of the symplectic structure on \check{X} and vice versa.*

Remark 4.5 There is an extra piece of structure on B that we haven't discussed. This is a Hessian metric g realizing the identification $TB \cong T^*B$. Recall that a Hessian metric is a Riemannian metric that is locally the Hessian of some smooth potential function K . The two integral affine structures on B endow it with two different sets of local coordinates and the potential functions in these coordinates are related by the Legendre transform. In fact the complex and symplectic structures constructed in Proposition 4.4 can be recovered from the potential function so mirror symmetry in this context is governed by the Legendre transform [25] [2, Sect. 6.1.2].

4.3 The SYZ Transform

To finish off the discussion of semi-flat mirror symmetry, we turn our attention to the HMS conjecture. The goal is to construct a geometric functor

$$\Phi : \mathcal{Fuk}(X) \rightarrow D^b(\text{Coh}(\check{X}))$$

from the Fukaya category of X to the derived category of coherent sheaves on \check{X} using the geometry of the dual fibrations. The first step is to produce an object of $D^b(\text{Coh}(\check{X}))$ from a Lagrangian $L \subset X$ equipped with a flat unitary connection. We will attempt to do this by exploiting the interpretation of a point $p \in X$ as a flat $U(1)$ -connection on the dual fiber.

Let $L \subset X$ be a Lagrangian section of μ corresponding to a map $\sigma : B \rightarrow X$, equipped with the trivial connection. By restricting L to each fiber of μ , we obtain a family of flat $U(1)$ -connections

$$\{\nabla_{\sigma(b)}\}_{b \in B}$$

on the fibers of $\check{\mu} : \check{X} \rightarrow B$. These glue together to give a flat $U(1)$ -connection on a complex line bundle \mathcal{L} on \check{X} . It turns out this connection gives \mathcal{L} the structure of a holomorphic line bundle on \check{X} (endowed with the complex structure constructed in the last subsection).

This construction was generalized by [7] (see also [31]) as follows. As X is the moduli space of flat $U(1)$ -connections on the fibers of $\check{\mu} : \check{X} \rightarrow B$, there exists a universal bundle with connection $(\mathcal{P}, \nabla^{\mathcal{P}})$ on $X \times_B \check{X}$. Now given (L, \mathcal{E}, ∇) where $L \subset X$ is a multisection transverse to the fibers of μ and (\mathcal{E}, ∇) is a flat unitary vector bundle on L , define the SYZ transform by

$$\Phi^{SYZ}(L, \mathcal{E}, \nabla) := (pr_{\check{X}})_*((pr_L)^*\mathcal{E} \otimes (i \times id)^*\mathcal{P})$$

where $pr_L, pr_{\check{X}} : L \times_B \check{X} \rightarrow L, \check{X}$ are the projections and $(i \times id) : L \times_B \check{X} \rightarrow X \times_B \check{X}$ is the inclusion. Note that $\Phi^{SYZ}(L, \mathcal{E}, \nabla)$ comes equipped with a connection we denote $\nabla_{(L, \mathcal{E}, \nabla)}$.

Theorem 4.6 ([7, Theorem 1.1]) *If $L \subset X$ is Lagrangian, then $\nabla_{(L, \mathcal{E}, \nabla)}$ endows $\Phi^{SYZ}(L, \mathcal{E}, \nabla)$ with the structure of a holomorphic vector bundle on \check{X} . When X and \check{X} are dual elliptic curves fibered over S^1 , then every holomorphic vector bundle on \check{X} is obtained this way.*

Viewing holomorphic vector bundles as objects in $D^b(\text{Coh}(\check{X}))$, we hope to extend the SYZ transform to an equivalence $\Phi : \mathcal{Fuk}(X) \rightarrow D^b(\text{Coh}(\check{X}))$, thus realizing the HMS conjecture. While this hope hasn't been realized in general, it has in some special cases.

When X and \check{X} are dual elliptic curves fibered over S^1 , a HMS equivalence Φ is constructed by hand in [34]. One can check that their functor Φ does indeed extend the SYZ transform Φ^{SYZ} . In fact, assuming Theorem 4.6, it is not so hard

to construct Φ at least on the level of objects. Each coherent sheaf on the curve X can be decomposed as a direct sum of a torsion sheaf and a vector bundle. Vector bundles are taken care of by Theorem 4.6. Torsion sheaves are successive extensions of skyscrapers at points which correspond to S^1 fibers of $\mu : X \rightarrow B$. For more recent work on understanding the SYZ transform see [12] and the references therein.

5 Constructing Mirrors

We now move on to the general problem of constructing mirrors. Given a Kähler Calabi–Yau n -fold (X, J, ω, Ω) , the SYZ conjecture suggests the following strategy for constructing a mirror.

5.0.1 Strategy

- (i) produce a special Lagrangian fibration $\mu : X \rightarrow B$;⁵
- (ii) dualize the smooth locus $\mu_0 : X_0 \rightarrow B_0$ to obtain a semi-flat mirror $\check{\mu}_0 : \check{X}_0 \rightarrow B_0$;
- (iii) compactify \check{X}_0 to obtain a CY n -fold with a dual SYZ fibration $\check{\mu} : \check{X} \rightarrow B$;
- (iv) use the geometry of the dual fibrations to construct a HMS equivalence

$$\Phi : Fuk(X) \rightarrow D^b(Coh(\check{X})).$$

5.0.2 Obstacles

There are many obstacles to carrying out 5.0.1 and (ii) is the only step where a totally satisfactory answer is known as we discussed in Sect. 4.

Producing sLag fibrations on a compact Calabi–Yau n -folds is a hard open problem in general. Furthermore, work of Joyce [26] suggests that even when sLag fibrations exist, they might be ill-behaved. The map μ is not necessarily differentiable and may have real codimension one discriminant locus in the base B . In this case B_0 is disconnected and one needs to perform steps (ii) and (iii) on each component and then glue.

Compactifying \check{X}_0 to a complex manifold also poses problems. There are obstructions to extending the semi-flat complex structure on \check{X}_0 to any compactification. To remedy this, one needs to take a small deformation of \check{X}_0 by modifying the complex structure using *instanton corrections*.

⁵This choice is the reason that X may have several mirrors.

Step (iv) has been realized in some special cases (e.g. [1, 3–5, 30] and references therein) but a general theory for producing an equivalence Φ given an SYZ mirror is still elusive.

5.1 Instanton Corrections

The small deformation of the complex structure on the dual \check{X}_0 is necessitated by the existence of obstructed Lagrangians. The point is that the Fukaya category of X doesn't contain all pairs (L, ∇) of Lagrangians with flat connection but only those pairs where L is *unobstructed*.

A Lagrangian L is unobstructed if certain counts of holomorphic discs bounded by L cancel out so that the Floer differential satisfies $d^2 = 0$. In particular, if L doesn't bound any nonconstant holomorphic discs, then it is unobstructed. A problem arises if $\mu : X \rightarrow B$ has singular fibers because then the smooth torus fibers may bound nontrivial holomorphic discs known as *disc instantons*. For example, any vanishing 1-cycle on a nearby fiber sweeps out such a disc.

To construct the dual \check{X} as a complex moduli space of objects in the Fukaya category we need to account for the effect of these instantons on the objects in the Fukaya category. This is done by modifying the semi-flat complex structure using counts of such disc instantons.

In fact, one can explicitly write down the coordinates for the semi-flat complex structure described in Sect. 4 in terms of the symplectic area of cylinders swept out by isotopy of nearby smooth Lagrangian fibers as in Sect. 5.3. Then the discs bounded by obstructed Lagrangians lead to nontrivial monodromy of the semi-flat complex on \check{X}_0 which is an obstruction to the complex structure extending to a compactification \check{X} . The instanton corrections are given by multiplying these coordinates by the generating series for virtual counts of holomorphic discs bounded by the fibers.

For more details on instanton corrections, see for example [1, 10, 36].

5.2 From Torus Fibrations to Degenerations

Heuristics from physics suggest that X will admit an SYZ fibration in the limit toward a maximally unipotent degeneration.⁶ It was independently conjectured in [24, 30] that if $\mathcal{X} \rightarrow \mathbb{D}$ is such a degeneration over a disc (where $X = \mathcal{X}_\epsilon$ for some small $\epsilon \ll 1$) and g_t is a suitably normalized metric on \mathcal{X}_t , then the Gromov-Hausdorff limit of the metric spaces (\mathcal{X}_t, g_t) collapses the Lagrangian torus fibers onto the base B of an SYZ fibration. Furthermore, this base should be recovered as the dual complex of the special fiber of $\mathcal{X} \rightarrow \mathbb{D}$ endowed with the appropriate singular integral affine

⁶That is, a degeneration with maximally unipotent monodromy. These are sometimes known as large complex structure limits (LCSL).

structure. Then one can hope to reconstruct the instanton corrected SYZ dual directly from data on B .

This allows one to bypass the issue of constructing a sLag fibration by instead constructing a maximally unipotent degeneration. Toric degenerations are particularly well suited for this purpose. This is the point of view taken in the Gross-Siebert program [20, 21] and gives rise to a version of SYZ mirror symmetry purely within algebraic geometry. In this setting the instanton corrections should come from logarithmic Gromov-Witten invariants of the degeneration as constructed in [6, 13, 23] and these invariants can be computed tropically from data on the base B . For more on this see for example [18, 19, 22].

5.3 Beyond the Calabi–Yau Case

The SYZ approach can also be used to understand mirror symmetry beyond the case of Calabi–Yau manifolds. The most natural generalization involves log Calabi–Yau pairs (X, D) where $D \subset X$ is a boundary divisor and the sheaf $\omega_X(D)$ of top forms with logarithmic poles along D is trivial. That is, D is a section of the anticanonical sheaf ω_X^{-1} and $X \setminus D$ is an open Calabi–Yau.

In this case the mirror should consist of a pair (M, W) consisting of a complex manifold M with a holomorphic function $W : M \rightarrow \mathbb{C}$. The pair (M, W) is known as a *Landau-Ginzburg model* and the function W is the *superpotential* [28]. Homological mirror symmetry takes the form of an equivalence

$$\Phi : \mathcal{Fuk}(X, D) \rightarrow MF(M, W)$$

between a version of the Fukaya category for pairs (X, D) and the category of *matrix factorizations* of (M, W) . Recall that a matrix factorization is a 2-periodic complex

$$\left(\dots \longrightarrow P_0 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \longrightarrow \dots \right)$$

of coherent sheaves on M satisfying $d^2 = W$. By a theorem of Orlov [33], the category $MF(M, W)$ is equivalent to the derived category of singularities $D_{sing}^b(\{W = 0\})$.⁷

The SYZ conjecture gives a recipe for constructing the Landau-Ginzburg dual (M, W) . Here we give the version as stated in [9]:

Conjecture 2 *Let (X, J, ω) be a compact Kähler manifold and D a section of K_X^{-1} . Suppose $\mu : U = X \setminus D \rightarrow B$ is an SYZ fibration where U is equipped with a holomorphic volume form Ω . Then the mirror to (X, D) is the Landau-Ginzburg model (\check{U}, W) where*

⁷Here we’ve assumed for simplicity that the only critical value of W is at $0 \in \mathbb{C}$.

$$\check{\mu} : \check{U} \rightarrow B$$

is the SYZ dual fibration equipped with the instanton corrected complex structure and the superpotential W is computed by counting holomorphic discs in (X, D) .

We briefly recall the construction of the superpotential. Let $\mu_0 : U_0 \rightarrow B_0$ be the smooth locus of the fibration so that \check{U}_0 is the semi-flat dual. Consider a family of relative homology classes $A_L \in H_2(X, L; \mathbb{Z})$ as the Lagrangian torus fiber L varies. Then the function

$$z^A : \check{U}_0 \rightarrow \mathbb{C} \quad z^A(L, \nabla) = \exp\left(-\int_{A_L} \omega\right) \text{hol}_{\nabla}(\partial A_L).$$

is a holomorphic local coordinate on \check{U}_0 .

Let

$$m_0(L, \nabla) = \sum_{\beta \in H_2(X, L; \mathbb{Z})} n_{\beta}(L) z^{\beta}$$

where $n_{\beta}(L)$ is Gromov-Witten count of holomorphic discs in X bounded by L and intersecting D transversally.⁸ This is a holomorphic function on \check{U}_0 when it is defined but in general it only becomes well defined after instanton correcting the complex structure. The idea is that the number $n_{\beta}(L)$ jumps across an obstructed Lagrangian L that bounds disc instantons in $X \setminus D$. Instanton corrections account for this and so m_0 should extend to a holomorphic function W on the instanton corrected dual \check{U} .

In fact m_0 is the obstruction to Floer homology constructed in [15]. That is, $d^2 = m_0$ where d is the Floer differential on the Floer complex $CF^*(L, L)$. This explains why the Landau-Ginzburg superpotential W should be given by m_0 . If one believes homological mirror symmetry, then obstructed chain complexes in the Fukaya category should lead to matrix factorizations with $W = m_0$ on the mirror.

Example 5.1 Let $X = \mathbb{P}^1$ with anticanonical divisor $\{0, \infty\} = D$. Then $U = \mathbb{C}^*$ admits a sLag fibration $\mu : U \rightarrow B$ where B is the open interval $(0, \infty)$ and $\mu^{-1}(r) = \{|z| = r\}$ is a circle. The dual is $\check{U} = \mathbb{C}^*$ is also an algebraic torus and there are no instanton corrections since all the fibers of μ are smooth. Each sLag circle $L \subset U \subset X$ cuts X into two discs D_0 and D_{∞} whose classes satisfy $[D_0] + [D_{\infty}] = [\mathbb{P}^1]$ in $H_2(X, L; \mathbb{Z})$ so that the corresponding coordinate functions z_0 and z_{∞} on \check{U} satisfy $z_0 z_{\infty} = 1$. Furthermore,

$$\exp\left(-\int_{D_0} \omega\right) \exp\left(-\int_{D_{\infty}} \omega\right) = e^{-A}$$

where $A = \int_{\mathbb{P}^1} \omega$ is the symplectic area. Furthermore, it is easy to see that $n_{[D_0]}(L) = n_{[D_1]}(L) = 1$. Putting it together and rescaling by a factor, we obtain the superpotential

⁸More precisely, the sum is over curve classes β with Maslov index $\mu(\beta) = 2$.

$$W = z_0 + \frac{e^{-A}}{z_0} : \mathbb{C}^* \rightarrow \mathbb{C}.$$

A similar argument works for any Fano toric pair (X, D) where μ is the moment map, B is the interior of the moment polytope P , $\check{U} = (\mathbb{C}^*)^n$ is an algebraic torus, and W is given as a sum over facets of P [9, 14].

References

1. Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov. Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces. *Publ. Math. Inst. Hautes Études Sci.*, 123:199–282, 2016.
2. Paul S. Aspinwall, Tom Bridgeland, Alastair Craw, Michael R. Douglas, Mark Gross, Anton Kapustin, Gregory W. Moore, Graeme Segal, Balázs Szendrői, and P. M. H. Wilson. *Dirichlet branes and mirror symmetry*, volume 4 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2009.
3. Mohammed Abouzaid. Family Floer cohomology and mirror symmetry. *ArXiv e-prints*, April 2014.
4. Mohammed Abouzaid. The family Floer functor is faithful. *ArXiv e-prints*, August 2014.
5. Mohammed Abouzaid. Homological mirror symmetry without corrections. *ArXiv e-prints*, March 2017.
6. Dan Abramovich and Qile Chen. Stable logarithmic maps to Deligne-Faltings pairs II. *Asian J. Math.*, 18(3):465–488, 2014.
7. D. Arinkin and A. Polishchuk. Fukaya category and Fourier transform. In *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, volume 23 of *AMS/IP Stud. Adv. Math.*, pages 261–274. Amer. Math. Soc., Providence, RI, 2001.
8. V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, [1989?]. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.
9. Denis Auroux. Mirror symmetry and T -duality in the complement of an anticanonical divisor. *J. Gökova Geom. Topol. GGT*, 1:51–91, 2007.
10. Denis Auroux. Special Lagrangian fibrations, wall-crossing, and mirror symmetry. In *Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry*, volume 13 of *Surv. Differ. Geom.*, pages 1–47. Int. Press, Somerville, MA, 2009.
11. Matthew Robert Ballard. Meet homological mirror symmetry. In *Modular forms and string duality*, volume 54 of *Fields Inst. Commun.*, pages 191–224. Amer. Math. Soc., Providence, RI, 2008.
12. K. Chan. Homological mirror symmetry for local Calabi-Yau manifolds via SYZ. *ArXiv e-prints*, July 2016.
13. Qile Chen. Stable logarithmic maps to Deligne-Faltings pairs I. *Ann. of Math. (2)*, 180(2):455–521, 2014.
14. Cheol-Hyun Cho and Yong-Geun Oh. Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds. *Asian J. Math.*, 10(4):773–814, 2006.
15. Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction.*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.
16. Mark Gross. Special Lagrangian fibrations. I. Topology. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 156–193. World Sci. Publ., River Edge, NJ, 1998.

17. Mark Gross. Special Lagrangian fibrations. II. Geometry. A survey of techniques in the study of special Lagrangian fibrations. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 341–403. Int. Press, Boston, MA, 1999.
18. Mark Gross. *Tropical geometry and mirror symmetry*, volume 114 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2011.
19. Mark Gross. Mirror symmetry and the Strominger-Yau-Zaslow conjecture. In *Current developments in mathematics 2012*, pages 133–191. Int. Press, Somerville, MA, 2013.
20. Mark Gross and Bernd Siebert. Affine manifolds, log structures, and mirror symmetry. *Turkish J. Math.*, 27(1):33–60, 2003.
21. Mark Gross and Bernd Siebert. Mirror symmetry via logarithmic degeneration data. *I. J. Differential Geom.*, 72(2):169–338, 2006.
22. Mark Gross and Bernd Siebert. From real affine geometry to complex geometry. *Ann. of Math. (2)*, 174(3):1301–1428, 2011.
23. Mark Gross and Bernd Siebert. Logarithmic Gromov-Witten invariants. *J. Amer. Math. Soc.*, 26(2):451–510, 2013.
24. Mark Gross and P. M. H. Wilson. Large complex structure limits of $K3$ surfaces. *J. Differential Geom.*, 55(3):475–546, 2000.
25. Nigel J. Hitchin. The moduli space of special Lagrangian submanifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(3-4):503–515 (1998), 1997. Dedicated to Ennio De Giorgi.
26. Dominic Joyce. Singularities of special Lagrangian fibrations and the SYZ conjecture. *Comm. Anal. Geom.*, 11(5):859–907, 2003.
27. Dominic Joyce. Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow. *EMS Surv. Math. Sci.*, 2(1):1–62, 2015.
28. Anton Kapustin and Yi Li. Topological correlators in Landau-Ginzburg models with boundaries. *Adv. Theor. Math. Phys.*, 7(4):727–749, 2003.
29. Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139. Birkhäuser, Basel, 1995.
30. Maxim Kontsevich and Yan Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 203–263. World Sci. Publ., River Edge, NJ, 2001.
31. Naichung Conan Leung, Shing-Tung Yau, and Eric Zaslow. From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform. *Adv. Theor. Math. Phys.*, 4(6):1319–1341, 2000.
32. Robert C. McLean. Deformations of calibrated submanifolds. *Comm. Anal. Geom.*, 6(4):705–747, 1998.
33. D. O. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):240–262, 2004.
34. Alexander Polishchuk and Eric Zaslow. Categorical mirror symmetry: the elliptic curve. *Adv. Theor. Math. Phys.*, 2(2):443–470, 1998.
35. Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. Mirror symmetry is T -duality. *Nuclear Phys. B*, 479(1–2):243–259, 1996.
36. Junwu Tu. On the reconstruction problem in mirror symmetry. *Adv. Math.*, 256:449–478, 2014.
37. R. P. Thomas and S.-T. Yau. Special Lagrangians, stable bundles and mean curvature flow. *Comm. Anal. Geom.*, 10(5):1075–1113, 2002.
38. Alan Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Advances in Math.*, 6:329–346 (1971), 1971.