Introduction to Symplectic Geometry and Fukaya Category

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1 Symplectic Geometry

Definition 1 (*symplectic form*) Given a vector space V, a **symplectic form** ω is a non-degenerate, anti-symmetric bilinear form. namely, $\forall\, v \in V, \omega(v, w) = 0, \forall\, w \in V \iff v = 0$ and $\omega(v, w) = -\omega(w, v)$. Such a vector space V is called a symplectic vector space.

We know from linear algebra that all symplectic vector spaces must have even dimensions. Let $W \subseteq V$, $W^\omega := \{v \in V, \omega(v, w) = 0\ \forall w \in W\}$,

Definition 2 Given a symplectic vector space (V, ω), a subspace $W \subseteq V$ is called isotropic if $W \subseteq W^\omega$, $i.e. \omega|_W = 0$;

W is coisotropic if $W \supseteq W^\omega$,

W is symplectic if $\omega|_W$ is also a symplectic form on W.

W is Lagrangian if it is isotropic and $\dim W = \frac{1}{2}\dim V$.

We have $\forall\ W \subset V$, $\dim\ W +\ \dim\ W^\omega = \dim\ V$, therefore, $W^{\omega\omega} = W$. The Euclidean space $\mathbb{R}^{2n}$ is a symplectic vector space equipped with the standard symplectic form $\omega_0 = \sum_{i=1}^n x_i \wedge y_i$. Also, for any symplectic vector space, we have s symplectic basis $u_1, \ldots u_n; v_1, \ldots, v_n$ such that $\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \omega(u_, v_k) = \delta_{kj}$. Namely, we have a map $\Phi : \mathbb{R}^{2n} \to V$ such that $\Phi^*\omega = \omega_0$.

Definition 3 (*symplectomorphism*) $\mathrm{Sp}(V, \omega) = \{\Phi \in Gl(V) \big| \Phi^*\omega = \omega\}$, the linear isomorphisms that preserves the symplectic structure are called symplectomorphisms. Since we know that $V \simeq \mathbb{R}^{2n}$ by the paragraph above, we can identify

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$Sp(V, w)$ as the maps $\{\Phi \big| \Phi^* \omega_0 = \omega_0\} = \{A \big| A^t J_0 A = J_0\}$. If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, then J_0 acts as i.

Lemma 4 *$Sp(2n) \cap O(2n) = Sp(2n) \cap Gl(n, \mathbb{C}) = O(2n) \cap Gl(n, \mathbb{C}) \simeq U(n)$, and $U(n)$ is a maximal compact subgroup of the symplectic group and $Sp(2n)$ is homotopy equivalent to $U(n)$.*

Sketch of proof. The first equation is a matter of writing down explicitly the definitions and calculate. We have a polar decomposition $\forall\ \Phi \in Sp(2n)$, $\Phi = UP$ where $U := \Phi \cdot (\Phi^t \Phi)^{-\frac{1}{2}} \in U(n)$. $P = (\Phi^t \Phi)^{\frac{1}{2}}$ is symplectic symmetric and positive definite. Let $U_t := (\Phi^t \Phi)^{-\frac{1}{2}t} \in Sp(2n)$ for $t \in [0, 1]$, this gives a deformation retract from $Sp(2n)$ to $U(n)$. (Further details may be found in Chap. 2 of [1].) □

Corollary 5 *$\pi_1(Sp(2n)) = \pi_1(U(n)) = \pi_1(S^1) \simeq \mathbb{Z}$, where the second equality is induced by the complex determinant function.*

Now we try to associate an integer μ to any loop in the Lagrangian Grassmannian $\Lambda : \mathbb{R}/\mathbb{Z} \to LGr(n)$ such that $\mu(\Lambda_1) = \mu(\Lambda_2)$ if and only if Λ_1 and Λ_2 are homotopic. It should also satisfy $\mu(\Lambda \oplus \Lambda') = \mu(\Lambda) + \mu(\Lambda')$, and $\lambda_0(t) = e^{\pi i t}$ has the number 1 associated to it. This integer is the **Maslov index** of the loop. Actually $LGr(n) \simeq U(n)/O(n)$, therefore, $\pi_1(LG(n)) = \pi_1(U(n)/O(n)) \simeq \mathbb{Z}$, which is induced by μ.

More generally, we have the **Maslov index** for any 2nd relative homotopy group:

$$\mu : \pi_2(M, L) \to \mathbb{Z}$$

defined as follows: if a map $u : (\mathbb{D}^2, S^1) \to (M, L)$ represent a class $[u] \in \pi_2(M, L)$, we trivialize the pullback of the tangent bundle u^*TMon $\mathbb{D}^2$ and get the trivial rank $2n$ bundle. Take the tangent bundle TL restricted to S^1 along this trivialization which gives a loop in $LGr(n)$. Then we define the Maslov index of $[u]$ as the Maslov index for as above.

The **minimal Maslov number** N_L is defined as the smallest positive integer that the image of the map μ hits in $\mathbb{Z}$. We set $N_L = \infty$ if the Maslov index μ vanishes.

Definition 6 (*symplectic manifold*) Now a symplectic structure on a smooth manifold M is a non degenerate closed 2-form ω, namely $(T_q M, \omega_p)$ is a symplectic vector space $\forall p \in M$. Non-degeneracy implies that $\omega^n = \omega \wedge \omega \wedge \ldots \omega$ doesn't vanish, which implies that M is oriented.

A symplectomorphism of $(M, \omega) = Symp(M, \omega) := \{\phi \in Diff(M) \big| \phi^* \omega = \omega\}$.

There is a systematic way to construct a symplectomorphism from a function $H : M \to \mathbb{R}$. First define a vector field $X_H \in \mathcal{X}(M, \omega)$ by $i_{X_H} \omega = dH$, the non-degeneracy of ω implies the existence of such a vector field. Note since $d(i_{X_H} \omega) = ddH = 0$, we have $\mathcal{L}_{X_H} \omega = (di_X + i_X d)\omega = 0$. Let ψ_t be the local flow generated by X_H, namely $\frac{d\psi_t}{dt} = X_H(\psi_t)$, then ψ_t is a symplectomorphism. In this case, we call ψ_t the **Hamiltonian flow** generated by H.

Now $dH(X_h) = i_{X_H}\omega X_H = 0$, thus X_H is tangent to the level sets of H. For example, if we were to have the height function on the sphere, then with the standard symplectic form on the sphere induced volume form on $\mathbb{R}^3$, we have $\omega = d\theta \wedge dz$, then $X_H = \frac{\partial}{\partial\theta}$, the flow ϕ^t is rotation of S^2 at constant speed.

Basic examples:

(1) $(\mathbb{R}^{2n}, \omega_0)$
(2) any oriented Riemann surface with area form;
(3) $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the standard form ω_0 on the quotient space.
(4) Cotangent bundle of any manifold. T^*M with canonical 1-form $\lambda_{can} \in \Omega^1(T^*M)$, $\omega = -d\lambda_{can}$, where $\lambda_{can} = \sum_1^n y_i dx_i$. Here the y_i are the coordinates for dx_i, namely we have coordinate charts $T^*U \to \mathbb{R}^n \times \mathbb{R}^n$, $(q, v^*) \mapsto (x(q), y(q, v^*))$, and $T_{(q,0)}(T^*M) \simeq TM \oplus T_q^*M$.

Proposition 7 *λ_{can} is characterized by the property that $\sigma^*\lambda_{can} = \sigma, \forall\, \sigma \in \Omega^1(M)$.*

This is because if we write out $\sigma = \sum a_j(x)dx_j$, then as a map in local charts, we should get $(x_1, \ldots, x_n, a_1(x), \ldots a_n(x))$, and $\sigma^*(\sum y_j dx_j) = \sigma$.

Proposition 8 *The image of a 1-form σ is Lagrangian in $T^*M \iff \sigma$ is closed.*

Proof $d\sigma = d\sigma^*\lambda_{can} = \sigma^*(d\lambda_{can}) = d\lambda_{can}\big|\Gamma_\sigma$. □

(5) $\mathbb{CP}^n$ and Fubini-Study form: Consider the function ρ on $\mathbb{C}^n$: $z \mapsto \log\, (|z|^2 + 1)$. This function is strictly plurisubharmonic, with $\partial\bar{\partial}\rho = \frac{1}{(|z|^2+1)^2}$; therefore $\omega_{FS} := \frac{1}{2}\partial\bar{\partial}\rho$ is Kähler.
Now on a chart $U_0 = (z_1, \ldots, z_n) \subseteq \mathbb{C}P^n$, the transition function on $\mathcal{U} = U_0 \cap U_1$ looks like $\varphi_{0,1}(z_1, \ldots, z_n) = (\frac{1}{z_1}, \cdots \frac{z_n}{z_1})$, this map maps (U) biholomorphically onto itself with $\varphi^*(\log(|z|^2 + 1)) = \log(|z|^2 + 1) + \log(|z_1|^{-2})$. Thus, $\partial\bar{\partial}\varphi^*(\log(|z|^2 + 1)) = \partial\bar{\partial}\varphi^*(\log(|z|^2 + 1)) + \partial\bar{\partial}\log(|z_1|^{-2}) = \partial\bar{\partial}\varphi^*(\log(|z|^2 + 1))$. So we can "glue" $\varphi_i^*\omega_{FS}$ together to give a Kähler structure on $\mathbb{CP}^n$.

Now we introduce a very important property of symplectic manifold, which claims that locally, all symplectic manifolds look the same; however, the global structure would be different.

Theorem 9 (Darboux) *Given a symplectic manifold (M, ω), $\forall\, p \in M$, there exists a neighborhood $U_p \subseteq M$ such that ω restricted to U_p is symplectomorphic to the standard ω_0 in $\mathbb{R}^{2n}$, where $\dim M = 2n$.*

The proof of Darboux's theorem uses the so call Moser's trick, details can be found in Chap. 2 of [1].

2 Lagrangian Floer Colomology

Definition 10 (*Lagrangian*) Now let (M, ω) be a symplectic manifold, $N \subset M$ is isotopic if $\omega|_N = 0$.This implies that $\dim N \leq \frac{1}{2}\dim M$ as ω is non-degenerate.If L is isotopic and $\dim L = \frac{1}{2}\dim M$, then we say L is **Lagrangian**.

Now suppose we have compact lagrangians $L_0, L_1 \subset (M, \omega)$, $L_0 \pitchfork L_1 \Rightarrow L_0 \cap L_1$ is a finite set of points.

Definition 11 (*Monotone*) We say a Lagrangian submanifold $L \subseteq M$ is **Monotone** if $\forall\, A \in \pi_2(M, L)$ we have a fixed $\lambda \in \mathbb{R}^+$ such that:

$$\int_A \omega = \lambda \cdot \mu_L(A)$$

From now on we work over monotone Lagrangians with minimal Maslov number N_L at least 2.

Definition 12 The Floer complex

$$CF^*(L_0, L_1) := \Lambda < L_0 \cap L_1 >$$

Which is a Λ- vector space with basis $= L_0 \cap L_1$ $\Lambda := \{\Sigma a_i T^{\lambda_i} | a_i \in \mathbb{K}, \lim_{i\to\infty} \lambda_i = +\infty\}$ is the **Novikov field** with coefficient in $\mathbb{K}$.

If we have $2c_1(TM) = 0$ and the maslov class μ_L vanishes, then we can make $CF^*(L_0, L_1)$ a $\mathbb{Z}-$graded complex, else it is Z_2-graded.

Definition 13 Now given $p, q \in L_0 \cap L_1$, define $\widehat{\mathcal{M}}(p, q, J) = \{u : \mathbb{R} \times [0, 1] \to M \big| D_u \circ j = J \circ D_u, u(s, 0) \in L_0, u(s, 1) \in L_1, \lim_{s\to\infty} u(s, t) = p, \lim_{s\to-\infty} u(s, t) = q\}$.

Then we have an $\mathbb{R}$ action on $\widehat{M}(p, q, J)$ by $r \cdot u(s, t) = u(s + r, t)$, the moduli space $\mathcal{M}(p, q, J) := \widehat{\mathcal{M}}(p, q, J)/\mathbb{R}$.

Remark 14 The equation $D_u \circ j = J \circ D_u$ is just saying $\overline{\partial_J} u = 0$.

Now we define differential on the complex:

Definition 15 $\forall p \in CF^*(L_0 \cap L_1)$,

$$\partial p := \sum_{q \in L_0 \cap L_1, ind(\beta)=1} (\#\mathcal{M}(p, q, \beta, J))T^{\omega(\beta)} \cdot q, \tag{1}$$

Where $\omega(\beta)$ is the **energy** of the J-holomorphic map u which is represented by β in $\pi_2(M, L_0 \cup L_1)$, it is defined as

$$\omega(u) := \int_{\mathbb{R}\times[0,1]} u^*\omega = \int\int |\frac{\partial u}{\partial s}|^2 ds dt \geq 0$$

Remark 16 If the linearized operator $D_{\bar{\partial}_{J,u}}$ is surjective at $\forall u \in \widehat{\mathcal{M}}(p, q, J)$, then we have $\widehat{\mathcal{M}}(p, q, J)$ is a manifold of dimension $\mu_{L_0 \cup L_1}(u)$ (the Maslov index of u, note that $\pi_1((LGr)) = \pi_1(U(n)/O(n)) \simeq \mathbb{Z}$.)

Remark 17 Gromov's compactness claims that given any positive upper bound E_0 on energy, there are only finitely many homotopy class $\beta = [u]$ such that $\omega(u) \leq E_0$, therefore, we know that the RHS of Eq. 1 is well defined. Namely, for any fixed energy E, $\#\mathcal{M}(p, q, \beta, J))$ is finite.

Proposition 18 *Assume* $[\omega] \cdot \pi_2(M, L_i) = 0$ *for* $i = 0, 1$ *and* L_0, L_1 *are oriented, compact Lagrangians equipped with spin structure, then* ∂ *is well defined and satisfies* $\partial^2 = 0$*, and the* ***Lagrangian Floer cohomology*** $HF^*(L_0, L_1) := H^*(CF(L_0, L_1; \partial))$ *is independent of the almost complex structure* J *and invariant under Hamiltonian isotopes of* L_0 *or* L_1.

The idea of the proof of $\partial^2 = 0$ is to look at a J-holomorphic map u with $\mu(u) = 2$, then Gromov's compactness say that for a sequence of $J-$holomorphic maps with bounded energy, there exists subsequence that converges to nodal configurations. In the case when $\mu(u) = 2$, we have three possible configurations.

(1) Sphere bubbles, a J-holomorphic sphere is connected to the J-holomorphic strip at an interior point of the strip. This is the case when some energy concentrates at the interior point.
(2) Disc bubble: a J-holomorphic disc connected with the J-holomorphic strip at a point on L_0 or L_1, this is the case when some energy concentrates at a point on the boundary.
(3) Broken strip, there are energy concentrates at $\pm\infty$.

Proposition 19 $\omega \cdot \pi_2(M, L_i) = 0$ *implies there are no disc bubbles or sphere bubbles.*

Proof The idea is the energy of the bubbles have to be zero, which implies that they are constant. Look at the long exact sequence of homology groups

$$\cdots \to \pi_2(L) \to \pi_2(M) \to \pi_2(M, L) \xrightarrow{\partial} \pi_1(L) \to \cdots$$

Note that $\omega \cdot \pi_2(M, L_i) = 0$ automatically implies that $\forall\ \beta \in \pi_2(M, L)$, $\int_\beta \omega = 0$, thus no disc bubbles with boundary on L_0 or L_1. Since $\omega|_L = 0$ by definition of Lagrangian manifolds, we have $\forall\ \eta \in \pi_2(L)$, we have $\int_\eta \omega = 0$. By the exactness at $\pi_2(M)$, $\forall\ \alpha \in \pi_2(M)$, we have $\int_\alpha \omega = 0$. Thus no sphere bubbles. □

Gromov compactness claims that after adding the 3 possible configurations, $\overline{\mathcal{M}}(p, q, J)$ is compact. However, since $\omega \cdot \pi_2(M, L_i) = 0$, we are only allowed to have broken strips.

However, the signed count of the number of boundary points of a 1-dimensional manifold is zero. A gluing theorem states that any broken strip is locally the limit of a sequence of index 2 J-holomorphic strips. And

$$\partial\overline{\mathcal{M}}(p,q,[u],J) = \coprod_{\substack{r\in L_0\cap L_1\\ [u']+[u'']=[u]\\ \mathrm{ind}([u'])=\mathrm{ind}([u''])=1}} \Big(\mathcal{M}(p,r;[u'],J)\times\mathcal{M}(r,q;[u''],J)\Big) \quad (2)$$

$HF^*(L, L)$ is defined as $HF^*(L_0, \varphi_H(L))$ where $\varphi_H(L)$ is Hamiltonian isotopic to L because HF^* is invariant under Hamiltonian perturbation. namely the original J-holomorphic equation is replaced with

$$\frac{\partial u}{\partial s} + J(u,t)\left(\frac{\partial u}{\partial t} - X_H(t,u)\right) = 0 \quad (3)$$

Example 20 Consider $L \subseteq T^*L$ as the zero section of the cotangent bundle, suppose $f : L \to \mathbb{R}$ is a Morse function, let $H = \pi^* f$, then $\varphi_H(f) = \Gamma_{df} \subseteq T^*L$. Thus $L \cap \varphi_H(L) =$ critical points of f, and $CF^*(L, \varphi(L)) \simeq CM^*(f)$ (the Morse complex) as vector space. The Moduli space of J-holomorphic strips from p to q corresponds $1-1$ to the Moduli space of Morse flow liness from p to q. So we have an iso of chain complex $(CF^*(L, \varphi(L), \partial) \simeq (CM^*(f), d_M)$

The main idea is that under "good" conditions, we have Lagrangian Floer homology is isomorphic to the Morse homology which is isomorphic to the singular homology.

Theorem 21 (Albers, 2007) *For a $2n$-dimensional, closed, symplectic manifold M and a closed, monotone, Lagrangian sub manifold $L \subset M$ of minimal Maslov number $N_L \geq 2$, there exist homomorphisms*

$$\varphi_k : HF_k(L, \phi_H(L)) \to H^{n-k}(L; \mathbb{Z}/2) \text{ for } k \leq N_L - 2$$

Where $H : S^1 \times M \to \mathbb{R}$ is a Hamiltonian function and ϕ_H the corresponding Hamiltonian diffeomorphism. For $n - N_L + 2 \leq k \leq N_L - 2$, φ_k is an isomorphism.

See [2] for more details.

Remark 22 This morphism above is not always an isomorphism, a counterexample can be found in [3] where a construction by Audin and Polterovich provides Lagrangian embeddings of spheres S^k into $\mathbb{R}^{2n}$.

Remark 23 We might imagine that every Lagrangian can be embedded locally in T^*L in a neighborhood by Weinstein's Lagrangian neighborhood theorem below and use the idea of zero section in Example 20 to think of Lagrangian Floer homology as Morse homology; however, Weinstein's Lagrangian neighborhood theorem is a local result, so we don't always have a rigorous isomorphism globally.

Theorem 24 (Weinstein's Lagrangian neighborhood theorem) *$\forall L \subseteq M$ a Lagrangian sub manifold, there exists a neighborhood U that is symplectic to a neighborhood of $L \subseteq T^*L$.*

Details of the proof can be found in Chap. 3 of [1]

3 Product Structure and Fukaya Category

Definition 25 We define $\mu^1 : CF^*(L_0, L_1) \to CF^*(L_0, L_1)[1]$ as the differential ∂. We can also define

$$\mu^2 : CF^*(L_0, L_1) \otimes_\Lambda CF^*(L_1, L_2) \to CF^*(L_0, L_2)$$

by the following equation:

$$\mu^2(p, q) := \sum_{\substack{q \in L_0 \cap L_2 \\ [u]:\mathrm{ind}([u])=0}} (\#\mathcal{M}(p, q, r; [u], J))T^{\omega([u])} r. \tag{4}$$

Where $\mathcal{M}(p, p, r; [u], J))$ denotes, for a disc with three given points z_0, z_1, z_2 on the boundary, a J-holomorphic map from $\mathbb{D}$ to M that represents $[u]$ in $\pi_2(M)$ and extends continuously to the closed disc, mapping the boundary arcs from z_0 to z_1, z_1 to z_2, z_2 to z_3 to L_0, L_1, L_2 respectively, while the z_0, z_1, z_2 are mapped to p, q, r respectively.

Proposition 26 *If* $\omega \cdot \pi_2(M, L_i) = 0, \forall i \in \{0, 1, 2\}$*, then* μ^2 *satisfies the Leibniz rule with proper signs with respect to* ∂*; in particular,*

$$\partial(\mu^2(p, q)) = \pm\mu^2(\partial p, q) \pm \mu^2(p, \partial q) \tag{5}$$

The idea of the proof is similar to that of $\partial^2 = 0$, we look at the index 1 moduli spaces of J-holomorphic discs and their compactification. Still assuming transversality, $\mathcal{M}(p, q, r; [u], J)$ is a smooth 1-dimensional manifold and admits a compactification $\overline{\mathcal{M}}(p, q, r; [u], J)$ by adding nodal trees (there is no disc or sphere bubble by the assumption that the symplectic form vanishes on relative homotopy classes). and there can be strip breaking happening at any of the three points p, q, r. If it breaks at p, it represents $\mu^2(\partial p, q)$; at q then represents $\mu^2(p, \partial q)$; if at r, then represents $\partial\mu^2(p, q)$. Since the signed count of the boundary of a 1-dimensional manifold is 0, we have Eq. 5.

Therefore, μ^2 defines a product in Floer cohomology as well, namely

$$[\mu^2] : HF^*(L_0, L_1) \otimes HF^*(L_1, L_2) \to HF^*(L_0, L_2).$$

If $L_0 = L_1 = L_2$, then $[\mu^2]$ is the cup product on $HF^*(L)$.

Proposition 27 (Associativity of μ^2) *We have*

$$\mu^2(p, \mu^2(q, r)) \pm \mu^2(\mu^2(p, q), r) = \pm\mu^3(\partial p, q, r) \pm \mu^3(p, \partial q, r) \pm \mu^3(p, q, \partial r) \\ \pm \partial\mu^3(p, q, r) \tag{6}$$

This is because $\mu^3(p, q, r)$ is defined similar as the sum of the number of J-holomorphic maps of a disc (with four points z_0, z_1, z_2, z_3 on its boundary to M, with the map converges to the points $p, q, r, s \in M$ near the four points and the arcs in between each adjacent pair of z_i to L_i), weighted with the symplectic energy. Then by Gromov compactness, the boundary of 1-dimensional moduli spaces are of two kinds:

(1) Those with a broken strip on the boundary of $\mathbb{D}$ at a nodal point of $\mathbb{D}$ while the other three marked points remain on $\partial\mathbb{D}$, there are four of these, corresponding to the four summands on the RHS of Eq. 6.

(2) Those that corresponds to a degeneration of the domain to the boundary of $\bar{\mathcal{M}}_{0,4}$, namely to a pair of discs, each of whose boundary carries two marked points, and the disc connects to the J-holomorphic strip with a nodal point, there are two marked points left on the disc. There are two of these, corresponding to the two summands on the LHS of Eq. 6.

Thus the singed count of the number of boundary points of a 1-dimensional manifold with boundary give Eq. 6.

More generally, consider $L_0, \ldots L_k \subseteq M$, compact, oriented Lagrangians with spin structure. $p_i \in L_{i-1} \cap L_i$, we define

$\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_1, L_2) \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_k)$

$$\mu^k(p_k, \ldots, p_1) = \sum_{\substack{q \in L_0 \cap L_k) \\ [u]:\mathrm{ind}([u])=2-k}} (\#\mathcal{M}(p_1, \ldots, p_k, q; [u], J))T^{\omega([u])} q, \tag{7}$$

where the dimension of the moduli spaces are

$$\dim \mathcal{M}(p_1, \ldots, p_k, q; [u], J) = k - 2 + \mathrm{ind}([u]) = k - 2 + \deg(q) - \sum_{i=1}^{k} \deg(p_i). \tag{8}$$

The special case is when $k = 1$. We had

$$\mu^1 = \partial : CF^*(L_0, L_1) \to CF^*(L_0, L_1),$$

$$\partial p = \sum_{\substack{q \in L_0 \cap L_1) \\ [u]:\mathrm{ind}([u])=1}} (\#\mathcal{M}(p, q; [u], J))T^{\omega([u])} q$$

Proposition 28 *If $\omega \cdot \pi_2(M, L_i) = 0,\ \forall i$, then the operations μ^k satisfy the A_∞-relations*

$$\sum_{\ell=1}^{k} \sum_{j=0}^{k-\ell} (-1)^* \mu^{k+1-\ell}(p_k, \ldots, p_{j+\ell+1}, \mu^\ell(p_{j+\ell}, \ldots, p_{j+1}), p_j, \ldots, p_1) = 0, \tag{9}$$

where $* = j + \deg(p_1) + \cdots + \deg(p_j)$.

Example 29 (1) $k = 1$, Eq. 9 is the same as $\mu^2 = 0$,
(2) $k = 2$, Eq. 9 is the Leibniz' rule
(3) $k = 3$, Eq. 9 is the associativity law of $[\mu^2]$ in HF^*.

For higher k, this gives an explicit homotopy for certain compatibility property among the preceding ones.

The proof is similar to that of the associativity law, we study dimension-1 moduli spaces of J-holomorphic discs and their compactification, fix $p_1, \ldots p_k$ and q, and $[u]$ such that $\text{ind}[u] = 3 - k$, assume J is chosen generically so we have transversality and then $\mathcal{M}(p_1, \ldots p_k, q; [u], J)$ compactifies to a 1-dimensional manifold with boundary, and the boundary points are either of an index 1 J-holomorphic strip breaking off at one of the $(k + 1)$ points or a pair of discs each contain at least two marked points. Those consists of the summands of the Eq. 9.

Definition 30 (*Fukaya Category*) Given a symplectic manifold (M, ω) such that $2c_1(TM) = 0$, consider the category consisting of the following data:

(1) Objects: compact, oriented Lagrangians L_i equipped with spin structure, such that $[\omega] \cdot \pi_2(M, L_i) = 0$ with vanishing Maslov index, together with a spin structure.
(2) hom-spaces: $hom^*_{\mathcal{F}(M)}(L_0, L_1) := CF^*(L_0, L_1)$, with differential μ^1 and composition μ^2
(3) higher operations and A_∞ relations (9) for μ^s.

See [4, 5] for more details.

Remark 31 In our previous definition, we may allow $c_1(TM)$, μ_L to be nonzero if we only need a $\mathbb{Z}/2$-grading; and we may also drop the spin structure if we are content to work over characteristic 2.

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