

On the Longtime Behavior of Almost Periodic Entropy Solutions to Scalar Conservation Laws



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Abstract We found the precise condition for the decay as $t \rightarrow \infty$ of Besicovitch almost periodic entropy solutions of multidimensional scalar conservation laws. Moreover, in the case of one space variable we establish asymptotic convergence of the entropy solution to a traveling wave (in the Besicovitch norm). Besides, the flux function turns out to be affine on the minimal segment containing the essential range of the limit profile while the speed of the traveling wave coincides with the slope of the flux function on this segment.

Keywords Almost periodic entropy solutions · Decay property
Scalar conservation laws · Spectrum · Traveling waves

1 Introduction

In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, where $\mathbb{R}_+ = (0, +\infty)$, we consider a conservation law

$$u_t + \operatorname{div}_x \varphi(u) = 0, \quad u = u(t, x), \quad (t, x) \in \Pi. \quad (1)$$

The flux vector $\varphi(u) = (\varphi_1(u), \dots, \varphi_n(u))$ is supposed to be merely continuous: $\varphi(u) \in C(\mathbb{R}, \mathbb{R}^n)$. Recall the notion of Kruzhkov entropy solution of the Cauchy problem for Eq. (1) with initial condition

$$u(0, x) = u_0(x) \in L^\infty(\mathbb{R}^n). \quad (2)$$

Definition 1 ([6]). A bounded measurable function $u = u(t, x) \in L^\infty(\Pi)$ is called an entropy solution (e.s.) of (1), (2) if for all $k \in \mathbb{R}$

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$$\frac{\partial}{\partial t}|u - k| + \operatorname{div}_x[\operatorname{sign}(u - k)(\varphi(u) - \varphi(k))] \leq 0 \tag{3}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$), and

$$\operatorname{ess\,lim}_{t \rightarrow 0+} u(t, \cdot) = u_0 \quad \text{in } L^1_{loc}(\mathbb{R}^n). \tag{4}$$

Here $\operatorname{sign} u = \begin{cases} 1, & u > 0, \\ -1, & u \leq 0 \end{cases}$ and relation (3) means that for each test function $h = h(t, x) \in C^1_0(\Pi), h \geq 0$,

$$\int_{\Pi} [|u - k|h_t + \operatorname{sign}(u - k)(\varphi(u) - \varphi(k)) \cdot \nabla_x h] dt dx \geq 0,$$

where \cdot denotes the inner product in \mathbb{R}^n .

Taking in (3) $k = \pm R$, where $R \geq \|u\|_{\infty}$, we obtain that $u_t + \operatorname{div}_x \varphi(u) = 0$ in $\mathcal{D}'(\Pi)$; that is, an e.s. $u = u(t, x)$ is a weak solution of this equation as well.

The existence of e.s. of (1), (2) follows from the general result of [12, Theorem 3]. In the case under consideration when the flux vector is only continuous the effect of infinite speed of propagation appears, which may even lead to the nonuniqueness of e.s. if $n > 1$, and see examples in [7, 8, 12], where exact sufficient conditions of the uniqueness were also found. Nevertheless, if an initial function u_0 is periodic in \mathbb{R}^n (at least in $n - 1$ independent directions), then the e.s. of (1), (2) is unique and x -periodic; see [11], as well as the more general result [12, Theorem 11].

We will study problem (1), (2) in the class of Besicovitch almost periodic functions. Let C_R be the cube

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x|_{\infty} = \max_{i=1, \dots, n} |x_i| \leq R/2\}, \quad R > 0.$$

We define the seminorm

$$N_1(u) = \limsup_{R \rightarrow +\infty} R^{-n} \int_{C_R} |u(x)| dx, \quad u(x) \in L^1_{loc}(\mathbb{R}^n).$$

Recall (see [1, 9]) that the Besicovitch space $\mathcal{B}^1(\mathbb{R}^n)$ is the closure of trigonometric polynomials, i.e., finite sums $\sum a_{\lambda} e^{2\pi i \lambda \cdot x}$ with $i^2 = -1, \lambda \in \mathbb{R}^n$, in the quotient space $B^1(\mathbb{R}^n)/B^1_0(\mathbb{R}^n)$, where

$$B^1(\mathbb{R}^n) = \{u \in L^1_{loc}(\mathbb{R}^n) \mid N_1(u) < +\infty\}, \quad B^1_0(\mathbb{R}^n) = \{u \in L^1_{loc}(\mathbb{R}^n) \mid N_1(u) = 0\}.$$

The space $\mathcal{B}^1(\mathbb{R}^n)$ is equipped with the norm $\|u\|_1 = N_1(u)$ (we identify classes in the quotient space $B^1(\mathbb{R}^n)/B^1_0(\mathbb{R}^n)$ and their representatives). The space $\mathcal{B}^1(\mathbb{R}^n)$ is a Banach space, and it is isomorphic to the completeness of the space $AP(\mathbb{R}^n)$ of

Bohr almost periodic functions with respect to the norm N_1 . It is known (see, for instance, [1]) that for each function $u \in \mathcal{B}^1(\mathbb{R}^n)$ there exists the mean value

$$\bar{u} = \int_{\mathbb{R}^n} u(x)dx \doteq \lim_{R \rightarrow +\infty} R^{-n} \int_{C_R} u(x)dx$$

and, more generally, the Bohr–Fourier coefficients

$$a_\lambda = \int_{\mathbb{R}^n} u(x)e^{-2\pi i\lambda \cdot x} dx, \quad \lambda \in \mathbb{R}^n.$$

The set

$$Sp(u) = \{ \lambda \in \mathbb{R}^n \mid a_\lambda \neq 0 \}$$

is called the spectrum of an almost periodic function $u(x)$. It is known [1] that the spectrum $Sp(u)$ is at most countable.

Now we assume that the initial function $u_0(x) \in \mathcal{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $I = \int_{\mathbb{R}^n} u_0(x)dx$, and M_0 be the smallest additive subgroup of \mathbb{R}^n containing $Sp(u_0)$.

It was shown in [17] that an e.s. $u(t, x)$ of (1), (2) is almost periodic with respect to spatial variables. Moreover, $u(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$ (after possible correction on a set of null measure) and $Sp(u(t, \cdot)) \subset M_0$, $\int_{\mathbb{R}^n} u(t, x)dx = I$ for all $t \geq 0$. The uniqueness of e.s. $u(t, x)$ in the space $C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$ is a consequence of the following general result [17, Proposition 1.3], which holds for arbitrary bounded and measurable initial functions.

Theorem 1. *Let $u(t, x), v(t, x) \in L^\infty(\Pi)$ be e.s. of (1), (2) with initial functions $u_0(x), v_0(x) \in L^\infty(\mathbb{R}^n)$, respectively. Then for a.e. $t > 0$*

$$N_1(u(t, \cdot) - v(t, \cdot)) \leq N_1(u_0 - v_0). \tag{5}$$

For completeness, we reproduce the proof.

Proof. Applying Kruzhkov doubling of variables method, we obtain the relation (see [6, 12])

$$|u - v|_t + \operatorname{div}_x[\operatorname{sign}(u - v)(\varphi(u) - \varphi(v))] \leq 0 \text{ in } \mathcal{D}'(\Pi). \tag{6}$$

We choose a function $g(y) \in C_0^1(\mathbb{R}^n)$ such that $0 \leq g(y) \leq 1$, and $g(y) \equiv 1$ in the cube C_1 , $g(y) \equiv 0$ in the complement of the cube C_k , $k > 1$, and a function $h = h(t) \in C_0^1(\mathbb{R}_+)$, $h \geq 0$. Applying (6) to the test function $f = R^{-n}h(t)g(x/R)$ with $R > 0$, we obtain

$$\begin{aligned} & \int_0^\infty \left(R^{-n} \int_{\mathbb{R}^n} |u(t, x) - v(t, x)|g(x/R)dx \right) h'(t)dt + \\ & R^{-n-1} \int_\Pi \operatorname{sign}(u - v)(\varphi(u) - \varphi(v)) \cdot \nabla_y g(x/R)h(t)dt dx \geq 0. \end{aligned} \tag{7}$$

Making the change $y = x/R$ in the last integral in (7), we derive the estimate

$$R^{-n-1} \left| \int_{\Pi} \text{sign}(u - v)(\varphi(u) - \varphi(v)) \cdot \nabla_y g(x/R)h(t)dt dx \right| \leq R^{-1} \|\varphi(u) - \varphi(v)\|_{\infty} \int_{\Pi} |\nabla_y g|(y)h(t)dt dy \leq \frac{A}{R} \int_0^{+\infty} h(t)dt, \tag{8}$$

where $A = \|\varphi(u) - \varphi(v)\|_{\infty} \int_{\mathbb{R}^n} |\nabla_y g|(y)dy$. Here and below we use the notation $|z|$ for the Euclidean norm of a finite-dimensional vector z . Let

$$I_R(t) = R^{-n} \int_{\mathbb{R}^n} |u(t, x) - v(t, x)|g(x/R)dx.$$

From (7) and (8), it follows that

$$\int_0^{+\infty} (I_R(t) - At/R)h'(t)dt = \int_0^{+\infty} I_R(t)h'(t)dt + \frac{A}{R} \int_0^{+\infty} h(t)dt \geq 0$$

for all $h(t) \in C_0^1((0, +\infty))$, $h(t) \geq 0$. This means that the generalized derivative $\frac{d}{dt}(I_R(t) - At/R) \leq 0$, which readily implies that there exists a set $F \subset (0, +\infty)$ of full Lebesgue measure (which can be defined as the set of common Lebesgue points of functions $I_R(t)$, $R \in \mathbb{Q}$) such that $\forall t_2, t_1 \in F, t_2 > t_1, \forall R \in \mathbb{Q} I_R(t_2) - At_2/R \leq I_R(t_1) - At_1/R$, that is $I_R(t_2) \leq I_R(t_1) + A(t_2 - t_1)/R$. By the evident continuity of $I_R(t)$ with respect to R , the latter relation remains valid for all $R > 0$. In the limit as $F \ni t_1 \rightarrow 0$ we obtain, taking into account the initial conditions for e.s. u, v , that $\forall t_2 = t \in F$ for all $R > 0$

$$I_R(t) \leq I_R(0) + At/R, \tag{9}$$

where $I_R(0) = R^{-n} \int_{\mathbb{R}^n} |u_0(x) - v_0(x)|g(x/R)dx$. By the properties of $g(y)$, we find the inequalities

$$R^{-n} \int_{C_R} |u(t, x) - v(t, x)|dx \leq I_R(t) \leq R^{-n} \int_{C_{kR}} |u(t, x) - v(t, x)|dx = k^n (kR)^{-n} \int_{C_{kR}} |u(t, x) - v(t, x)|dx,$$

which imply that

$$N_1(u(t, \cdot) - v(t, \cdot)) \leq \limsup_{R \rightarrow +\infty} I_R(t) \leq k^n N_1(u(t, \cdot) - v(t, \cdot)). \tag{10}$$

In view of (10), we derive from (9) in the limit as $R \rightarrow +\infty$ that $N_1(u(t, \cdot) - v(t, \cdot)) \leq k^n N_1(u_0 - v_0)$ for all $t \in F$. To complete the proof, it only remains to notice that $k > 1$ is arbitrary.

Remark 1. As was established in [13, Corollary 7.1], after possible correction on a set of null measure any e.s. $u(t, x) \in C(\mathbb{R}_+, L^1_{loc}(\mathbb{R}^n))$. In particular, without loss of generality, we may claim that relation (9) holds for all $t > 0$. This implies in the limit as $R \rightarrow +\infty$ that the statement of Theorem 1 holds for all $t > 0$ as well. The continuity property allows also to replace the essential limit in initial condition (4) by the usual one.

Our main results are contained in Theorems 2, 4, indicated below.

Theorem 2. *Assume that the following non-degeneracy condition holds for the flux components in “resonant” directions $\xi \in M_0$:*

$$\forall \xi \in M_0, \xi \neq 0 \text{ the functions } u \rightarrow \xi \cdot \varphi(u) \text{ are not affine in any vicinity of } I = \overline{u_0}. \tag{11}$$

Then, an e.s. $u(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$ satisfies the decay property

$$\lim_{t \rightarrow +\infty} u(t, \cdot) = I \text{ in } \mathcal{B}^1(\mathbb{R}^n). \tag{12}$$

Condition (11) is precise: if it fails, then there exists an initial data $u_0 \in \mathcal{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with the properties $Sp(u_0) \subset M_0, \overline{u_0} = I$, such that the corresponding e.s. $u(t, x)$ of (1), (2) does not satisfy (12).

Remark 2. The decay of almost periodic e.s. was firstly studied by H. Frid [5] in the class of Stepanov almost periodic function. This class is natural for the case of smooth flux vector $\varphi(u)$, when an e.s. $u(t, x)$ of (1), (2) exhibits the property of finite speed of propagation. The decay of such solutions was established in the stronger Stepanov norm but under rather restrictive assumptions on the dependence of the length of inclusion intervals for ε -almost periods of u_0 on the parameter ε .

Notice that in the case of a periodic function u_0 the group M_0 coincides with the dual lattice \mathcal{L}' to the lattice \mathcal{L} of periods of u_0 , and in this case, Theorem 2 reduces to the following result [15] (see also the earlier paper [14]):

Theorem 3. *Under the condition*

$$\forall \xi \in \mathcal{L}', \xi \neq 0 \text{ the functions } u \rightarrow \xi \cdot \varphi(u) \text{ are not affine in any vicinity of } I = \int_{\mathbb{T}^n} u_0(x) dx \tag{13}$$

an e.s. $u(t, x) \in C([0, +\infty), L^1(\mathbb{T}^n))$ satisfies the decay property

$$\lim_{t \rightarrow +\infty} \int_{\mathbb{T}^n} |u(t, x) - I| dx = 0. \tag{14}$$

Here $\mathbb{T}^n = \mathbb{R}^n / \mathcal{L}$ is the n -dimensional torus, and dx is the normalized Lebesgue measure on \mathbb{T}^n .

Remark that in the case $\varphi(u) \in C^2(\mathbb{R}, \mathbb{R}^n)$ the assertion of Theorem 3 was established in [3]. Now we consider the case of one space variable $n = 1$ when (1) has the form

$$u_t + \varphi(u)_x = 0, \tag{15}$$

where $\varphi(u) \in C(\mathbb{R})$. As above, we assume that $u_0 \in \mathcal{B}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and that M_0 is the additive subgroup of \mathbb{R} generated by $Sp(u_0)$. For an almost periodic function $v(x) \in \mathcal{B}^1(\mathbb{R})$, we denote by $S(v)$ the minimal segment $[a, b]$ containing essential values of $v(x)$. This segment can be defined by the relations

$$b = \min\{ k \in \mathbb{R} \mid (v - k)^+ = \max(v - k, 0) = 0 \text{ in } \mathcal{B}^1(\mathbb{R}) \},$$

$$a = \max\{ k \in \mathbb{R} \mid (k - v)^+ = 0 \text{ in } \mathcal{B}^1(\mathbb{R}) \}.$$

As is easy to verify, the above minimal and maximal values exist and $a \leq b$.

Our second result is the following unconditional asymptotic property of convergence of an e.s. $u(t, x)$ to a traveling wave:

Theorem 4. *There is a constant $c \in \mathbb{R}$ (speed) and a function $v(y) \in \mathcal{B}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (profile) such that*

$$\lim_{t \rightarrow +\infty} (u(t, x) - v(x - ct)) = 0 \text{ in } \mathcal{B}^1(\mathbb{R}). \tag{16}$$

Moreover, $Sp(v) \subset M_0$, $\bar{v} = I = \overline{u_0}$, and $\varphi(u) - cu = \text{const}$ on the segment $S(v)$.

We remark, in addition to Theorem 4, that the profile $v(y)$ of the traveling wave and, if $v \neq \text{const}$, its speed c are uniquely defined. Indeed, if (16) holds with $v = v_1, v_2, c = c_1, c_2$, respectively, then $v_1(x - c_1t) - v_2(x - c_2t) \rightarrow 0$ in $\mathcal{B}^1(\mathbb{R})$ as $t \rightarrow +\infty$, which implies the relation

$$\lim_{t \rightarrow +\infty} (v_1(y) - v_2(y + (c_1 - c_2)t)) = 0 \text{ in } \mathcal{B}^1(\mathbb{R}). \tag{17}$$

By the known property of almost periodic functions (see, for example, [1]), there exists a sequence $t_r \rightarrow +\infty$ such that $v_2(y + (c_1 - c_2)t_r) \xrightarrow{r \rightarrow \infty} v_2(y)$ in $\mathcal{B}^1(\mathbb{R})$ (this is evident if $c_1 = c_2$). On the other hand, in view of (17) $v_2(y + (c_1 - c_2)t_r) \xrightarrow{r \rightarrow \infty} v_1(y)$ in $\mathcal{B}^1(\mathbb{R})$ and hence $v_1 = v_2$ in $\mathcal{B}^1(\mathbb{R})$. Further, if $\Delta c = c_1 - c_2 \neq 0$, then it follows from (17) in the limit as $t = t_r + h/\Delta c \rightarrow +\infty$ that $v_2(y) = v_2(y + h)$ in $\mathcal{B}^1(\mathbb{R})$ for each $h \in \mathbb{R}$. Therefore,

$$v_2(y) = \int_{\mathbb{R}} v_2(y + h)dh = \int_{\mathbb{R}} v_2(h)dh = \bar{v}_2 = \text{const}.$$

Thus, for the nonconstant profile $v = v_2$ the speed $c_1 = c_2 = c$ is uniquely determined. We also remark that $\|v\|_\infty \leq \|u_0\|_\infty$ because by the maximum principle $|u(t, x)| \leq \|u_0\|_\infty$ a.e. in Π .

Theorem 4 defines the nonlinear operator T on $\mathcal{B}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, which associates an initial function u_0 with the profile $v(y) = T(u_0)(y)$ of the limit traveling wave for the corresponding e.s. of problem (15), (2). In Theorem 5 below, we establish that T does not increase the distance in $\mathcal{B}^1(\mathbb{R})$.

Remark 3. In the case $n = 1$, the statement of Theorem 2 follows from Theorem 4. Indeed, under the assumptions of Theorem 2, $v(y) = I$ in $\mathcal{B}^1(\mathbb{R})$. Otherwise, $a < I < b$, where $[a, b] = S(v)$ and, by Theorem 4, $\varphi(u) = cu + \text{const}$ in the vicinity (a, b) of I . But the latter contradicts to assumption (11) of Theorem 2.

Note that in the periodic case Theorems 4, 5 were proved in [16].

2 Proof of Theorem 2

We assume firstly that the initial function is a trigonometric polynomial $u_0(x) = \sum_{\lambda \in \Lambda} a_\lambda e^{2\pi i \lambda \cdot x}$. Here $\Lambda = Sp(u_0) \subset \mathbb{R}^n$ is a finite set. The minimal additive subgroup $M_0 \doteq M(u_0)$ of \mathbb{R}^n containing Λ is a finite generated torsion-free abelian group, and therefore, it is a free abelian group of finite rank (see [10]). Therefore, there is a basis $\lambda_j \in M_0$, $j = 1, \dots, m$, so that every element $\lambda \in M_0$ can be uniquely represented as $\lambda = \lambda(\bar{k}) = \sum_{j=1}^m k_j \lambda_j$, $\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$. In particular, the vectors λ_j , $j = 1, \dots, m$, are linearly independent over the field of rational numbers \mathbb{Q} . We introduce the finite set $J = \{ \bar{k} \in \mathbb{Z}^m \mid \lambda(\bar{k}) \in \Lambda \}$ and represent the initial function as

$$u_0(x) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \sum_{j=1}^m k_j \lambda_j \cdot x}, \quad a_{\bar{k}} \doteq a_{\lambda(\bar{k})}.$$

By this representation $u_0(x) = v_0(y(x))$, where

$$v_0(y) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \bar{k} \cdot y}$$

is a periodic function on \mathbb{R}^m with the standard lattice of periods \mathbb{Z}^m while $y(x)$ is a linear map from \mathbb{R}^n to \mathbb{R}^m defined by the equalities $y_j = \lambda_j \cdot x = \sum_{i=1}^n \lambda_{ji} x_i$, λ_{ji} , $i = 1, \dots, n$, being coordinates of the vectors λ_j , $j = 1, \dots, m$. We consider the conservation law

$$v_t + \text{div}_y \tilde{\varphi}(v) = 0, \quad v = v(t, y), \quad t > 0, \quad y \in \mathbb{R}^m, \tag{18}$$

$\tilde{\varphi}(v) = (\tilde{\varphi}_1(v), \dots, \tilde{\varphi}_m(v))$, where

$$\tilde{\varphi}_j(v) = \lambda_j \cdot \varphi(u) = \sum_{i=1}^n \lambda_{ji} \varphi_i(v) \in C(\mathbb{R}), \quad j = 1, \dots, m.$$

As was shown in [11, 12], there exists a unique e.s. $v(t, y) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^m)$ of the Cauchy problem for Eq.(18) with initial function $v_0(y)$ and this e.s. is y -periodic, i.e., $v(t, y + e) = v(t, y)$ a.e. in $\mathbb{R}_+ \times \mathbb{R}^m$ for all $e \in \mathbb{Z}^m$. Besides, in view of [13, Corollary 7.1], we may suppose that $v(t, \cdot) \in C([0, +\infty), L^1(\mathbb{T}^m))$, where $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$ is an m -dimensional torus (which may be identified with the fundamental cube $[0, 1)^m$). Formally, for $u(t, x) = v(t, y(x))$

$$u_t + \operatorname{div}_x \varphi(u) = v_t + \sum_{i=1}^n \sum_{j=1}^m (\varphi_i(v))_{y_j} \frac{\partial y_j(x)}{\partial x_i} =$$

$$v_t + \sum_{i=1}^n \sum_{j=1}^m (\varphi_i(v))_{y_j} \lambda_{ji} = v_t + \sum_{j=1}^m (\tilde{\varphi}_j(v))_{y_j} = 0.$$

However, these reasons are correct only for classical solutions. In the general case $v(t, y) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^m)$, the range of $y(x)$ may be a proper subspace of \mathbb{R}^m (for example, this is always true if $m > n$), and the composition $v(t, y(x))$ is not even defined. The situation is saved by introduction of additional variables $z \in \mathbb{R}^m$. Namely, the linear change $(z, x) \rightarrow (z + y(x), x)$ is not degenerated; i.e., it is a linear automorphism of $\mathbb{R}^m \times \mathbb{R}^n$. Since $v(t, y)$ is an e.s. of Eq.(18) considered in the extended half-space $t > 0$, $(y, x) \in \mathbb{R}^{m+n}$, then the function $u(t, z, x) = v(t, z + y(x))$ satisfies the relations

$$|u - k|_t + \operatorname{div}_x [\operatorname{sign}(u - k)(\varphi(u) - \varphi(k))] =$$

$$|v - k|_t + \sum_{i=1}^n \sum_{j=1}^m [\operatorname{sign}(v - k)(\varphi_i(v) - \varphi_i(k))]_{y_j} \frac{\partial y_j(x)}{\partial x_i} =$$

$$|v - k|_t + \sum_{j=1}^m \sum_{i=1}^n [\operatorname{sign}(v - k)(\varphi_i(v) - \varphi_i(k))]_{y_j} \lambda_{ji} =$$

$$|v - k|_t + \sum_{j=1}^m [\operatorname{sign}(v - k)(\tilde{\varphi}_j(u) - \tilde{\varphi}_j(k))]_{y_j} \leq 0 \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^{m+n}).$$

Evidently, the initial condition

$$\lim_{t \rightarrow 0^+} u(t, z, x) = u_0(z, x) \doteq v_0(z + y(x)) \text{ in } L^1_{loc}(\mathbb{R}^{m+n})$$

is also satisfied; therefore, $u(t, z, x)$ is an e.s. of (1), (2) in the extended domain $\mathbb{R}_+ \times \mathbb{R}^{m+n}$. Since Eq.(1) does not contain the auxiliary variables $z \in \mathbb{R}^m$, then (cf. [17, Theorem 2.1]) for all $z \in E \subset \mathbb{R}^m$, where E is a set of full measure, the function $v(t, z + y(x))$ is an e.s. of (1), (2) with initial data $v_0(z + y(x)) \in \mathcal{B}^1(\mathbb{R}^n)$. Therefore, $v(t, z + y(x)) = u^z(t, x)$ a.e. in Π , where, in accordance with [17, Theorem 1.6], $u^z(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$ is a unique almost periodic e.s. of (1),

(2). Therefore, we may find a countable dense set $S \subset \mathbb{R}_+$ and a subset $E_1 \subset E$ of full measure such that $u^z(t, x) = v(t, z + y(x))$ in $\mathcal{B}^1(\mathbb{R})$ for all $t \in S, z \in E_1$.

Further, as follows from independence of the vectors $\lambda_j, j = 1, \dots, m$, over \mathbb{Q} , the action of the additive group \mathbb{R}^n on the torus \mathbb{T}^m defined by the shift transformations $T_x z = z + y(x), x \in \mathbb{R}^n$ is ergodic; see [17] for details. By the variant of Birkhoff individual ergodic theorem [4, Chap. VIII] for every $w(y) \in L^1(\mathbb{T}^m)$ for a.e. $z \in \mathbb{T}^m$ there exists the mean value

$$\int_{\mathbb{R}^n} w(z + y(x))dx = \int_{\mathbb{T}^m} w(y)dy. \tag{19}$$

In view of (19), there exists a set $E_2 \subset E_1$ of full measure such that for $z \in E_2$ and all $t \in S$

$$\int_{\mathbb{R}^n} |u^z(t, x) - I|dx = \int_{\mathbb{R}^n} |v(t, z + y(x)) - I|dx = \int_{\mathbb{T}^m} |v(t, y) - I|dy.$$

Since $u^z(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n)), v(t, \cdot) \in C([0, +\infty), L^1(\mathbb{T}^m))$, while the set S is dense in $[0, +\infty)$, we find that property

$$\int_{\mathbb{R}^n} |u^z(t, x) - I|dx = \int_{\mathbb{T}^m} |v(t, y) - I|dy \tag{20}$$

remains valid for all $t \geq 0$. Observe that $v_0(z + y(x)) \rightarrow v_0(y(x)) = u_0(x)$ as $z \rightarrow 0$ in $\mathcal{B}^1(\mathbb{R}^n)$ (and even in $AP(\mathbb{R}^n)$). Hence, by Theorem 1 in the limit as $E_2 \ni z \rightarrow 0$ $u^z(t, x) \rightarrow u(t, x)$ in $C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$, where $u(t, x)$ is the e.s. of original problem (1), (2). Therefore, relation (20) in the limit as $z \rightarrow 0$ implies the equality

$$\int_{\mathbb{R}^n} |u(t, x) - I|dx = \int_{\mathbb{T}^m} |v(t, y) - I|dy. \tag{21}$$

Further, for every $\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$

$$\bar{k} \cdot \tilde{\varphi}(u) = \sum_{j=1}^m \sum_{i=1}^n k_j \lambda_{ji} \varphi_i(u) = \lambda(\bar{k}) \cdot \varphi(u),$$

where $\lambda(\bar{k}) = \sum_{j=1}^m k_j \lambda_j \in M_0$. By condition (11), the functions $u \rightarrow \bar{k} \cdot \tilde{\varphi}(u)$ are not affine in any vicinity of $I = \bar{u}_0 = \int_{\mathbb{T}^m} v_0(y)dy$. We see that non-degeneracy requirement (13) is satisfied, and by [15, Theorem 1.3]

$$\lim_{t \rightarrow +\infty} \int_{\mathbb{T}^m} |v(t, y) - I|dy = 0.$$

Now it follows from (21) that

$$\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^n} |u(t, x) - I| dx = 0,$$

i.e., (12) holds.

In the general case $u_0 \in \mathcal{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we choose a sequence $u_{0m}, m \in \mathbb{N}$, of trigonometric polynomials converging to u_0 in $\mathcal{B}^1(\mathbb{R}^n)$ and such that $Sp(u_{0m}) \subset M_0, \overline{u_{0m}} = I$ (for instance, we may choose the Bochner–Fejér trigonometric polynomials; see [1]). Let $u_m(t, x)$ be the corresponding sequence of e.s. of (1), (2) with initial data $u_{0m}(x), m \in \mathbb{N}$. By Theorem 1 and Remark 1, this sequence converges as $m \rightarrow \infty$ to the e.s. $u(t, x)$ of the original problem in $C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$. We have already established that under condition (11) e.s. $u_m(t, x)$ satisfy the decay property

$$\lim_{t \rightarrow +\infty} u_m(t, \cdot) = I \text{ in } \mathcal{B}^1(\mathbb{R}^n).$$

Passing to the limit as $m \rightarrow \infty$ in this relation and taking into account the uniform convergence $u_m(t, \cdot) \xrightarrow{m \rightarrow \infty} u(t, \cdot)$ in $\mathcal{B}^1(\mathbb{R}^n)$, we obtain (12).

In conclusion, we demonstrate that condition (11) is precise. Indeed, if this condition is violated, then there is a nonzero vector $\xi \in M_0$ such that $\xi \cdot \varphi(u) = \tau u + c$ on some segment $[I - \delta, I + \delta]$, where $\tau, c, \delta \in \mathbb{R}$, and $\delta > 0$. Obviously, the function

$$u(t, x) = I + \delta \sin(2\pi(\xi \cdot x - \tau t))$$

is an e.s. of (1), (2) with the periodic initial function $u_0(x) = I + \delta \sin(2\pi(\xi \cdot x))$. We see that $\overline{u_0} = I, Sp(u_0) \subset \{-\xi, 0, \xi\} \subset M_0$ but the e.s. $u(t, x)$ does not converge to a constant in $\mathcal{B}^1(\mathbb{R}^n)$ as $t \rightarrow +\infty$.

The proof of Theorem 2 is complete.

3 Proof of Theorem 4

If the flux function $\varphi(u)$ is not affine in any vicinity of I , then by Theorem 2 the function $v(y) \equiv I$, and the segment $S(v) = [I, I] = \{I\}$. Otherwise, suppose that the function $\varphi(u)$ is affine in a certain maximal interval (a, b) , where $-\infty \leq a < I < b \leq +\infty: \varphi(u) - cu = \text{const}$ in (a, b) .

Assuming that $b < +\infty$, we define $u_+ = u_+(t, x)$ as the e.s. of (15), (2) with initial function $u_0(x) + b - I > u_0$. By the comparison principle [7, 8, 11, 12] $u_+ \geq u$ a.e. in Π . We note that $\int_{\mathbb{R}} (u_0(x) + b - I) dx = b$ while $\varphi(u)$ is not affine in any vicinity of b (otherwise, $\varphi(u)$ is affine on a larger interval $(a, b'), b' > b$, which contradicts the maximality of (a, b)). By Theorem 2 $u_+(t, \cdot) \rightarrow b$ in $\mathcal{B}^1(\mathbb{R})$ as $t \rightarrow +\infty$, and it follows from the inequality $u \leq u_+$ that $(u(t, \cdot) - b)^+ \rightarrow 0$ as $t \rightarrow +\infty$ in $\mathcal{B}^1(\mathbb{R})$. Similarly, if $a > -\infty$, then $u \geq u_-$, where $u_- = u_-(t, x)$ is an e.s. of (15), (2) with initial function $u_0(x) + a - I < u_0$. By Theorem 2 again the function $u_-(t, \cdot) \rightarrow a$ as $t \rightarrow +\infty$ in $\mathcal{B}^1(\mathbb{R})$ because $\int_{\mathbb{R}} (u_0(x) + a - I) dx = a$ while the

function $\varphi(u)$ is not affine in any vicinity of a . Therefore, $(a - u(t, \cdot))^+ \xrightarrow{t \rightarrow +\infty} 0$ in $\mathcal{B}^1(\mathbb{R})$. The obtained limit relations can be represented in the form

$$u(t, \cdot) - s_{a,b}(u(t, \cdot)) \xrightarrow{t \rightarrow +\infty} 0 \text{ in } \mathcal{B}^1(\mathbb{R}), \tag{22}$$

where $s_{a,b}(u) = \min(b, \max(a, u))$ is the cut-off function at the levels a, b (it is possible that $a = -\infty$ or $b = +\infty$).

We set $w(t, x) = s_{a,b}(u(t, x))$ and choose a strictly increasing sequence $t_k > 0$ such that $t_k \rightarrow +\infty$ and $N_1(u(t_k, \cdot) - w(t_k, \cdot)) \leq 2^{-k}$. Since $a \leq w(t, x) \leq b$ while $\varphi(u) = cu + \text{const}$ on (a, b) , then the e.s. of (15) with initial data $w(t_k, x)$ at $t = t_k$ has the form $u = w(t_k, x - c(t - t_k))$. By Theorem 1 (with the initial time t_k) for all $t > t_k$

$$\begin{aligned} \int_{\mathbb{R}} |w(t, x) - w(t_k, x - c(t - t_k))| dx &= \int_{\mathbb{R}} |s_{a,b}(u(t, x)) - s_{a,b}(w(t_k, x - c(t - t_k)))| dx \\ &\leq \int_{\mathbb{R}} |u(t, x) - w(t_k, x - c(t - t_k))| dx \leq \int_{\mathbb{R}} |u(t_k, x) - w(t_k, x)| dx \leq 2^{-k}. \end{aligned}$$

Substituting $t = t_l$, where $l > k$, into this inequality, we obtain

$$\int_{\mathbb{R}} |w(t_l, x + ct_l) - w(t_k, x + ct_k)| dx = \int_{\mathbb{R}} |w(t, x) - w(t_k, x - c(t_l - t_k))| dx \leq 2^{-k}.$$

Thus, $w(t_k, x + ct_k)$, $k \in \mathbb{N}$, is a Cauchy sequence in $\mathcal{B}^1(\mathbb{R})$. Therefore, this sequence converges as $k \rightarrow \infty$ to some function $v(x) \in \mathcal{B}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ in $\mathcal{B}^1(\mathbb{R})$. It is clear that the segment $S(v) \subset [a, b]$ and therefore $\varphi(u) - cu = \text{const}$ on $S(v)$. Since $Sp(w(t_k, x + ct_k)) = Sp(w(t_k, \cdot)) \subset Sp(u(t_k, \cdot)) \subset M_0$, the same inclusion holds for the limit function: $Sp(v) \subset M_0$. Finally, as follows from Theorem 1, for $t > t_k$

$$\begin{aligned} \int_{\mathbb{R}} |u(t, x) - v(x - ct)| dx &\leq \int_{\mathbb{R}} |u(t_k, x) - w(t_k, x)| dx + \int_{\mathbb{R}} |w(t_k, x) - v(x - ct_k)| dx = \\ &\int_{\mathbb{R}} |u(t_k, x) - w(t_k, x)| dx + \int_{\mathbb{R}} |w(t_k, x + ct_k) - v(x)| dx \leq \\ &2^{-k} + N_1(w(t_k, \cdot + ct_k) - v) \rightarrow 0 \end{aligned}$$

as $t \rightarrow +\infty$ (then also $k = \max\{l \mid t > t_l\} \rightarrow +\infty$). We see that relation (16) is satisfied. To complete the proof of Theorem 4, it only remains to notice that

$$\forall t > 0 \quad \overline{u(t, \cdot)} = \int_{\mathbb{R}} u(t, x) dx = I, \quad \bar{v} = \int_{\mathbb{R}} v(x - ct) dx$$

and (16) implies that $\bar{v} = I$.

In conclusion, we show that the operator $u_0 \rightarrow v = T(u_0)$, defined in the Introduction, does not increase the distance in $\mathcal{B}^1(\mathbb{R})$.

Theorem 5. *Let $u_{01}(x), u_{02}(x) \in \mathcal{B}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v_1 = T(u_{01})(x), v_2 = T(u_{02})(x)$. Then*

$$\int_{\mathbb{R}} |v_1(x) - v_2(x)|dx \leq \int_{\mathbb{R}} |u_{01}(x) - u_{02}(x)|dx. \tag{23}$$

Proof. Let $u_1(t, x), u_2(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R})) \cap L^\infty(I)$ be e.s. of (15), (2) with initial data u_{01}, u_{02} , respectively. By Theorem 4

$$\delta(t) = \int_{\mathbb{R}} |u_1(t, x) - v_1(x - c_1t)|dx + \int_{\mathbb{R}} |u_2(t, x) - v_2(x - c_2t)|dx \xrightarrow{t \rightarrow +\infty} 0,$$

where c_1, c_2 are constants. We can choose a sequence $t_k > 0$ such that $t_k \rightarrow +\infty$ as $k \rightarrow \infty$, and $N_1(v_2(x + (c_1 - c_2)t_k) - v_2(x)) \leq 1/k$. Then, with property (5) taken into account,

$$\begin{aligned} \int_{\mathbb{R}} |v_1(x) - v_2(x)|dx &= \int_{\mathbb{R}} |v_1(x - c_1t_k) - v_2(x - c_1t_k)|dx \leq \\ &\int_{\mathbb{R}} |v_1(x - c_1t_k) - v_2(x - c_2t_k)|dx + \int_{\mathbb{R}} |v_2(x - c_2t_k) - v_2(x - c_1t_k)|dx = \\ &\int_{\mathbb{R}} |v_1(x - c_1t_k) - v_2(x - c_2t_k)|dx + \int_{\mathbb{R}} |v_2(x + (c_1 - c_2)t_k) - v_2(x)|dx \leq \\ &\int_{\mathbb{R}} |u_1(t_k, x) - u_2(t_k, x)|dx + \delta(t_k) + 1/k \leq \int_{\mathbb{R}} |u_{01}(x) - u_{02}(x)|dx + \delta(t_k) + 1/k. \end{aligned}$$

In the limit as $k \rightarrow \infty$, this inequality implies (23).

Remark 4. In view of Theorem 1 the map F , which associates an initial data $u_0 \in \mathcal{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with the e.s. $u(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$ of problem (1), (2), is a uniformly continuous map from $\mathcal{B}^1(\mathbb{R}^n)$ to $C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$. Therefore, it admits the unique continuous extension on the whole space $\mathcal{B}^1(\mathbb{R}^n)$. By analogy with [2], the corresponding function $F(u_0) = u(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$ may be called a renormalized solution of (1), (2) with possibly unbounded almost periodic initial data u_0 . By the approximation techniques, all our results can be extended to the case of renormalized almost periodic solutions.

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