Lagrangian Representation for Systems of Conservation Laws: An Overview

Stefano Modena

Abstract We present an overview on some recent works in collaboration with S. Bianchini (see Bianchini and Modena in Lagrangian representation for solution to general systems of conservation laws [9] and the Ph.D. thesis Modena in Interaction functionals, Glimm approximations and Lagrangian structure of BV solutions for hyperbolic systems of conservation laws [15]), in which we propose a way to describe BV solutions to hyperbolic systems of conservation laws in one space dimension from a *Lagrangian* point of view.

Keywords Conservation laws · Hyperbolic systems · Interaction functional Lagrangian representation

1 Introduction

One of the key observations in fluid dynamics is that the fluid flow can be described from two different (and in some sense complementary) points of view: the Lagrangian points of view (in which the trajectory in space–time of each single fluid particle is tracked) and the Eulerian point of view (in which one looks at fluid motion focusing on fixed locations in the space through which the fluid flows as time passes).

From a mathematical perspective, such duality between the Lagrangian and the Eulerian approach can be seen, for instance, in the framework of the continuity equation:

$$\begin{cases} \partial_t v(t,x) + \mathrm{div}_x\big(v(t,x)b(t,x)\big) &= 0, \\ v(0,x) &= \bar{v}(x), \end{cases} \tag{1}$$

where $v : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is the unknown and $b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field. It is well known that, under suitable regularity assumptions, the solution

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to (1) can be written, for any time $t \in [0, \infty)$, as

$$v(t, \cdot)\mathscr{L}^1 = \mathtt{X}(t)_\sharp\big(\bar{v}\mathscr{L}^1\big), \tag{2}$$

where $\mathtt{X} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is the flow generated by the ODE

$$\begin{cases} \dfrac{\partial \mathtt{X}}{\partial t}(t, y) & = b(t, \mathtt{X}(t, y)), \\ \mathtt{X}(0, y) & = y, \end{cases} \tag{3}$$

$\mathscr{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $\sharp$ denotes the push-forward in the sense of measures.[1]

In the framework of the continuity equation, the Lagrangian and the Eulerian approach help each other: For instance, in the smooth setting, one can use the ODE (3) (Lagrangian approach) to solve the PDE (1) (Eulerian approach), while in the non-smooth setting one can use the PDE to solve the ODE (see [13]). The duality between the two approaches can be used not only to prove the existence of solutions, but also to prove their uniqueness and their stability and to investigate further properties of them, like their fine structure, their regularity, and so on. In few words, we could say that *two is better than one*: what cannot be done using the Lagrangian approach could be hopefully done using the Eulerian one, and vice versa.

For these reasons, it is an interesting question whether systems of conservation laws

$$\begin{cases} \partial_t u + \partial_x F(u) & = 0, \\ u(0, x) & = \bar{u}(x), \end{cases} \qquad u = u(t, x) \in \mathbb{R}^n, \;\; t \geq 0, \;\; x \in \mathbb{R}, \tag{4}$$

can be analyzed from a Lagrangian point of view. Here, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a generic smooth function, which is only assumed to be *strictly hyperbolic*; i.e. its differential $DF(u)$ has n distinct real eigenvalues in each point of its domain. We restrict our analysis to one space dimension, since this is the setting where a satisfying well-posedness theory for entropic solutions is available.[2]

For a scalar conservation law with a smooth initial datum, the method of characteristics provides a reasonable Lagrangian approach to the problem. Such method was extended by C. Dafermos (through the notion of *generalized characteristics* in [12]) to systems whose characteristic fields are either genuinely nonlinear or linearly degenerate,[3] and to initial data which are just BV. However, Dafermos' approach can not be further generalized to systems where the flux F has no convexity properties.

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[1]If A, B are sets, $\mathscr{A}, \mathscr{B}$ are σ-algebras on A, B, respectively, and $f : A \to B$ is a measurable function, then for any measure μ on $(A, \mathscr{A})$, the push-forward $f_\sharp\mu$ is the measure on $(B, \mathscr{B})$, defined by $f_\sharp\mu(E) = \mu(f^{-1}(E))$ for any $E \in \mathscr{B}$.

[2]By *entropic solution*, we mean a solution obtained as limit of vanishing viscosity approximations; see [3].

[3]See [11] for the definition of genuinely nonlinear or linearly degenerate characteristic fields. Roughly speaking, it amounts to say that the flux F has some strong *convexity* property.

Another Lagrangian approach in the analysis of conservation laws was proposed by T.-P. Liu in [14], where he introduced the notion of *wave tracing* for the waves present in an approximate solution to the system (4), constructed by means of the Glimm scheme. However, in [14], only approximate solutions (which in some sense are just piecewise constant functions) are considered.

Recently, some papers appeared in which a Lagrangian analysis is developed for the exact (and not approximate) entropic solution to conservation laws with a flux F which does not satisfy any convexity assumption. In particular,

- in [4, 5] (see also for a previous, slightly different approach [10]) S. Bianchini and E. Marconi develop a Lagrangian approach for the solution to the Cauchy problem associated to a scalar conservation law ($n = 1$), whose flux $F : \mathbb{R} \to \mathbb{R}$ is any smooth function and whose initial datum $\bar{u} \in L^\infty(\mathbb{R})$ is any bounded function;
- in [9, 15] S. Bianchini and the author develop a Lagrangian approach for the solution to the Cauchy problem associated to a system of conservation laws ($n \geq 1$), whose flux is any smooth strictly hyperbolic function and whose initial datum $\bar{u} \in BV(\mathbb{R})$ is a function of bounded variation.

In both cases, the starting point is the analysis of BV entropic solutions to scalar conservation laws. The extension to L^∞ initial data (for the scalar equation) [4, 5] or to systems [9, 15] requires, however, several new ideas. The goal of this notes is to present the notion of *Lagrangian representation* for BV entropic solutions to systems of conservation laws (4), proposed in [9, 15], and to present the main ideas behind the construction of such Lagrangian representation, focusing in particular on the difficulties in extending the scalar BV analysis to the system case.

As a final remark, we would like to stress that both in the scalar case and in the system one, the Lagrangian analysis is done in the same setting in which the well-posedness of the Cauchy problem is already know. We do not want to use Lagrangian methods to prove such well-posedness again. Rather, the aim of our new *Lagrangian tools* is to analyze in a more precise way the solution u to the Cauchy problem (4), in order to prove further properties of it. As an example, in the scalar case, the Lagrangian approach can be used to prove the *concentration of entropies* (see the papers by Bianchini and Marconi [4, 5]); in the system case, the Lagrangian tools can be used to study the fine structure of the solution (see the paper by Bianchini and the author [9] and the Ph.D. thesis of the author [15]).

2 Analysis of BV Solutions to Scalar Conservation Laws

The starting point of our analysis is the study of entropic BV solutions to scalar conservation laws

$$\begin{cases} \partial_t u + \partial_x F(u) = 0, \\ u(0,x) = \bar{u}(x), \end{cases} \qquad \bar{u} \in BV(\mathbb{R}) \text{ with compact support, } F : \mathbb{R} \to \mathbb{R} \text{ smooth.} \tag{5}$$

The first question we have to answer is: What is a good notion of *Lagrangian representation* of the solution to (5)? A hint in this direction is given by the following observation: By Vol'pert's rule[4] the distributional derivative $\partial_x u$ (which is a measure, being $u \in BV$) satisfies the $1D$ continuity equation

$$\partial_t(\partial_x u) + \partial_x(\hat{\lambda}(t,x)\partial_x u) = 0 \quad \text{in a distributional sense,} \tag{6}$$

where

$$\hat{\lambda}(\bar{t},\bar{x}) := \begin{cases} F'(u(\bar{t},\bar{x})) & \text{if } x \mapsto u(\bar{t},x) \text{ is continuous at } \bar{x}, \\ \frac{F(u(\bar{t},\bar{x}+))-F(u(\bar{t},\bar{x}-))}{u(\bar{t},\bar{x}+)-u(\bar{t},\bar{x}-)} & \text{if } x \mapsto u(\bar{t},x) \text{ has a jump at } \bar{x}. \end{cases} \tag{7}$$

Mimicking (1)–(2)–(3), we give the following definition.

Definition 1. A *Lagrangian representation* for the entropic solution u to (5) is a triple $(W, \mathtt{X}, \rho)$, where

1. $W \subseteq \mathbb{R}$ is a bounded interval; its elements are denoted by w and are called *waves*;
2. $\mathtt{X} : [0,\infty) \times W \to \mathbb{R}$ is $\|F'\|_{L^\infty}$-Lipschitz in t for fixed w and increasing in w for fixed t, and it is called *flow* or *position function*;
3. $\rho : W \to [-1, 1]$ is called *density function*,

such that for a.e. time $t \in [0,\infty)$

$$\partial_x u(t,\cdot) = \mathtt{X}(t,\cdot)_\sharp\big(\rho\mathscr{L}^1|_W\big) \text{ in the sense of measures} \tag{8}$$

and

$$\frac{\partial \mathtt{X}}{\partial t}(t,w) = \hat{\lambda}(t,\mathtt{X}(t,w)) \text{ for } |\rho|\mathscr{L}^1 - \text{a.e.} w \in W. \tag{9}$$

Remark 1. Notice that (8) is the analog of (2) and (9) is the analog of (3); only two differences must be observed:

- In (8), the term which is transported is an absolute continuous measure w.r.t. $\mathscr{L}^1$, even if the initial datum $\partial_x \bar{u}$ has a jump part or a Cantor part;
- In (9), in general $\mathtt{X}(0,w) \neq w$; i.e., w is just the label of a particle with no relationship with its starting point.

Definition 1 provides a (hopefully) good notion of Lagrangian representation. How can we now explicitly construct the objects W, $\mathtt{X}$, ρ satisfying the properties above?

As usual in the theory of conservation laws, the idea is to consider a sequence of approximate solutions $(u^q)_{q\in\mathbb{N}}$ solving the approximate Cauchy problem

$$\begin{cases} \partial_t u^q + \partial_x F^q(u^q) = 0, \\ u^q(0,x) = \bar{u}^q(x), \end{cases} \tag{10}$$

[4] The Vol'pert's rule (see, for instance, [1, Theorem 3.96]) is the chain rule for the derivative of the composition $F(u(x))$ of a Lipschitz function F with a BV function u.

where F^q is the piecewise affine interpolation of F with grid size 2^{-q} and $\bar{u}^q$ is a piecewise constant function taking values in $2^{-q}\mathbb{Z}$ such that $\|\bar{u}^q - \bar{u}\|_{L^1} \to 0$ as $q \to \infty$. The solution u^q to (10) can be constructed by means of the wavefront tracking algorithm (see [11, Chap. 4]), and it is a piecewise constant function with values in $2^{-q}\mathbb{Z}$ for any time t which converges strongly in L^1 to the entropic solution of (5), as $q \to \infty$.

Since $u^q(t, \cdot)$ is piecewise constant, it is not difficult to construct by hand a Lagrangian representation of it.[5] Now, the family $\{\mathtt{X}^q\}_q$ is pre-compact in $L^1([0, \infty) \times \mathbb{R})$, since, by Definition 1, each $\mathtt{X}^q$ is $\|F'\|$-Lipschitz in t for fixed w and increasing in w for fixed t; the family $\{\rho^q\}$ is weakly* pre-compact in $L^\infty(W)$. Therefore, up to subsequences, $\mathtt{X}^q \to \mathtt{X}$ strongly in L^1 and $\rho^q \to \rho$ weakly* in L^∞.

Equation (8) is then easily obtained passing to the limit in the corresponding equation for approximations

$$\partial_x u^q(t) = \mathtt{X}^q(t)_\sharp\big(\rho^q \mathscr{L}^1|_W\big) \tag{11}$$

and using that $u^q \to u$ in L^1.

On the contrary, Eq. (9) cannot be deduced directly from the corresponding equation for the approximations

$$\partial_t \mathtt{X}^q(t, w) = \hat{\lambda}^q(t, \mathtt{X}^q(t, w)), \tag{12}$$

since, in general, for fixed t, $\hat{\lambda}^q(t) \circ \mathtt{X}^q(t) \not\to \hat{\lambda}(t) \circ \mathtt{X}(t)$, as the following example shows.

Example 1. Assume that u is a solution of the scalar conservation law $\partial_t u + \partial_x F(u) = 0$, taking values in the finite set $\{u^L, u^M, u^R\}$, with $u^L, u^M, u^R \in \mathbb{R}$ and $u^L < u^M < u^R$, as described in Fig. 1.

Assume that the sequence of approximations $(u^q)_q$ is given by $u^q(t, x) := u(t - 1/q, x)$. Notice now that at time $\bar{t}$, $u^q(\bar{t}, \cdot)$ is made by two consecutive jumps, while

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[5]This can be done, for instance, in the following way. Assume for simplicity $u^q(t, \cdot)$ is right continuous. Set $\bar{U}^q(x) := \text{Tot.Var.}(\bar{u}^q; (-\infty, x])$. Set $W^q := (0, \text{Tot.Var.}(\bar{u}^q)]$,

$$\mathtt{X}^q(0, w) := (\bar{U}^q)^{-1}(w), \quad \rho^q(w) := \begin{cases} 1 & \text{if } u^q \text{ has a positive jump at } \mathtt{X}^q(0, w), \\ -1 & \text{if } u^q \text{ has a negative jump at } \mathtt{X}^q(0, w). \end{cases}$$

Set also for simplicity $\mathtt{u}^q(w) := \int_0^w \rho^q(w')dw'$. Denote by $\{(t_j, x_j)\}_j$ the points in the (t, x)-plane where two wavefronts in u^q collide (the discontinuity points at $t = 0$ are treated as collision points). By recursion, assume $\mathtt{X}^q(t, \cdot)$ is defined on $[0, t_j]$ and let us define it on $(t_j, t_{j+1}]$. Assume that at (t_j, x_j) the outgoing Riemann problem is (u^L, u^R) with $u^L < u^R$ (the case $u^R < u^L$ is completely similar). Set $\mathtt{A}(w) := \min\{\max\{\mathtt{u}^q(w') \mid w' \leq w\}, u^R\}$ for any $w \in \mathtt{X}^q(t_j)^{-1}(x_j)$ and then

$$\mathtt{X}^q(t, w) := x_j + \left[\frac{d\text{conv}_{[u^L, u^R]} F^q}{du}(\mathtt{A}(w))\right](t - t_j) \quad \text{for any } w \in \mathtt{X}^q(t_j)^{-1}(x_j) \text{ and any } t \in (t_j, t_{j+1}].$$

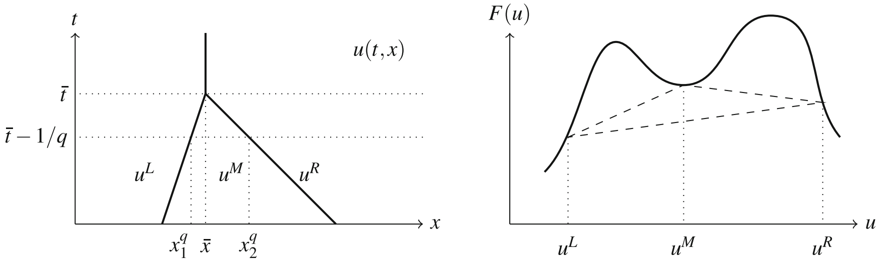

Fig. 1 The solution $u(t, x)$ to the scalar conservation law $\partial_t u + \partial_x F(u) = 0$ and the function $F(u)$

$u(\bar t, \cdot)$ is made by a single jump, which is given, roughly speaking, by the juxtaposition of the two jumps in the approximations.

By (9), all the waves w located in x_1^q have speed

$$\sigma_1 := \partial_t \mathrm{X}^q(\bar t, w) = \hat\lambda^q(\bar t, x_1^q) = \frac{F(u^M) - F(u^L)}{u^M - u^L},$$

all the waves w located in x_2^q have speed

$$\sigma_2 := \partial_t \mathrm{X}^q(\bar t, w) = \hat\lambda^q(\bar t, x_2^q) = \frac{F(u^R) - F(u^M)}{u^R - u^M},$$

while in the exact solution all the waves w should have speed

$$\sigma := \partial_t \mathrm{X}(\bar t, w) = \hat\lambda(\bar t, \bar x) = \frac{F(u^R) - F(u^L)}{u^R - u^L}.$$

Unfortunately, in general $\sigma_1, \sigma_2 \neq \sigma$ and thus $\hat\lambda^q(\bar t) \circ \mathrm{X}^q(\bar t) \nrightarrow \hat\lambda(\bar t) \circ \mathrm{X}(\bar t)$ as $q \to \infty$.

To overcome this problem and recover (9), we can proceed as follows (the argument is taken from [4]). From (6) and (8), we get for every $\varphi \in C_c^\infty((0, \infty) \times \mathbb{R})$

$$\iint u \partial_t \partial_x \varphi dxdt = \iint \partial_x \varphi(t, x) \hat\lambda(t, x) \partial_x u(t, dx) dt = \iint \partial_x \varphi(t, \mathrm{X}(t, w)) \hat\lambda(t, \mathrm{X}(t, w)) \rho(w) dwdt.$$

On the other side, testing (8) against $\partial_t \varphi$, we get

$$\iint u \partial_t \partial_x \varphi dxdt = -\iint \partial_t \varphi(t, \mathrm{X}(t, w)) \rho(w) dwdt = \iint \partial_x \varphi(t, \mathrm{X}(t, w)) \partial_t \mathrm{X}(t, w) \rho(w) dwdt.$$

Therefore,

$$\partial_x \Big[\mathrm{X}(t)_\sharp \Big(\rho (\hat\lambda(t, \mathrm{X}(t, \cdot)) - \partial_t \mathrm{X}(t, \cdot)) \mathscr{L}^1|_W \Big) \Big] = 0.$$

Equation (9) follows, just observing that $\mathtt{X}(t,\cdot)$ takes values in a compact set and that $\partial_t \mathtt{X}(t,w)$ is constant on waves having the same position (since $w \mapsto \mathtt{X}(t,w)$ is increasing).

3 Analysis of Linear Systems of Conservation Laws

We wish now to extend the scalar analysis done in the previous section to the system case. As a first step in this direction, let us study the linear system of conservation laws

$$\partial_t u + A\partial_x u = 0, \quad \text{where } A \text{ is a } n \times n \text{ strictly hyperbolic matrix,} \tag{13}$$

together with an initial datum $u(0,\cdot) = \bar{u} \in BV(\mathbb{R})$.

Let $\lambda_1, \dots, \lambda_n$ be the n distinct real eigenvalues of A, $r_1, \dots, r_n$ be the right eigenvectors (i.e., $Ar_k = \lambda_k r_k$) normalized such that $|r_k| = 1$, $l_1, \dots, l_n$ be the left eigenvectors (i.e., $l_k A = \lambda_k l_k$), normalized such that $l_k \cdot r_h = \delta_{kh}$.

Our aim is to find a good definition of *Lagrangian representation* for the solution to the linear system (13) and to explicitly construct such Lagrangian representation. This is easily done, observing that the scalar product of (13) with l_k gives the n scalar equations $\partial_t(l_k \cdot u) + \lambda_k \partial_x(l_k \cdot u) = 0$, with constant field λ_k.

Therefore, by the analysis in Sect. 2, for each k we can find a set W_k (called the *set of k-waves*), a flow $\mathtt{X}_k : [0,\infty) \times W_k \to \mathbb{R}$ and a density $\rho_k : W_k \to [-1,1]$, as in Definition 1, such that

$$\partial_x(l_k \cdot u) = \mathtt{X}_k(t)_\sharp\big(\rho_k \mathscr{L}^1|_{W_k}\big)$$

and

$$\partial_t \mathtt{X}_k(t,w) \equiv \lambda_k. \tag{14}$$

Definition 2. A *Lagrangian representation* of the solution to the linear system (13) is thus defined as a family of n triples $(W_k, \mathtt{X}_k, \rho_k)$, $k = 1, \dots, n$, (with the same regularity properties as the ones described in Definition 1) such that

$$\partial_x u(t) = \sum_{k=1}^{n} \partial_x(l_k \cdot u) r_k = \sum_{k=1}^{n} \mathtt{X}_k(t)_\sharp\big(\rho_k \mathscr{L}^1|_{W_k}\big) r_k \tag{15}$$

and the ODE (14) holds for every $k = 1, \dots, n$.

The existence of such a Lagrangian representation for the solution to the linear system (13) is then an immediate consequence of the scalar analysis done in Sect. 2.

4 The Riemann Problem

Before moving to the analysis of the nonlinear system (4), we need to recall some basic facts about the entropic solution to the *Riemann problem*, i.e., the Cauchy problem (4) with a piecewise constant initial datum

$$u(0,x) = \bar{u}(x) = \begin{cases} u^L & \text{if } x < 0, \\ u^R & \text{if } x \geq 0, \end{cases} \quad \text{with } u^L, u^R \in \mathbb{R}^n \text{ close enough to } 0. \tag{16}$$

It is shown in [3] that for any $k = 1, \dots, n$ it is possible to define a neighborhood $D_k \subseteq \mathbb{R}^{n+2}$ of the point $(0, 0, \lambda_k(0)) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and two functions $\tilde{r}_k : D_k \to \mathbb{R}^n$, $\tilde{\lambda}_k : D_k \to \mathbb{R}$; $r_k(u_k, v_k, \sigma_k)$ (resp. $\lambda_k(u_k, v_k, \sigma_k)$) is called the *kth generalized eigenvector* (resp. the *kth generalized eigenvalue*) at $(u_k, v_k, \sigma_k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$.

It is also shown in [3] that, given $u^L, u^R \in \mathbb{R}^n$ close enough to 0, one can find n curves $\gamma_k : I_k \to D_k \subseteq \mathbb{R}^{n+2}$, $k = 1, \dots, n$, defined on the intervals

$$I_k := \begin{cases} [0, s_k] & \text{if } s_k \geq 0, \\ [s_k, 0] & \text{if } s_k \leq 0, \end{cases}$$

satisfying the fixed point problem (if $s_k > 0$[6])

$$\begin{cases} u_k(\tau) = u_k^L + \int_0^\tau \tilde{r}_k(u_k(\varsigma), v_k(\varsigma), \sigma_k(\varsigma))d\varsigma, \quad u_k^L = \begin{cases} u^L & \text{if } k = 1, \\ u_{k-1}(s_{k-1}) & \text{if } k > 1, \end{cases} \\ v_k(\tau) = f_k(\tau) - \operatorname{conv}_{[0,s_k]} f_k(\tau), \\ \sigma_k(\tau) = \frac{d}{d\tau} \operatorname{conv}_{[0,\sigma_k]} f_k(\tau), \end{cases} \tag{17}$$

with f_k defined by $f_k(\tau) := \int_0^\tau \tilde{\lambda}_k(u_k(\varsigma), v_k(\varsigma), \sigma_k(\varsigma))d\varsigma$ and the $\operatorname{conv}_{[a.b]} g$ denotes the *convex envelope* of a function g on the interval $[a, b]$, i.e., the biggest convex function which stays below g.

The right-continuous solution to the Riemann problem (4), (16) is now given by the BV function

$$u(t,x) = \begin{cases} u^L & \text{if } x/t \leq \sigma_1(0), \\ u_k(\tau) & \text{if } x/t = \sigma_k(\tau), \\ u^R & \text{if } x/t \geq \sigma_n(\tau). \end{cases}$$

[6]If $s_k < 0$ the convex envelope $\operatorname{conv}_{[0,s_k]} f_k(\tau)$ must be substituted by the concave envelope $\operatorname{conv}_{[s_k,0]} f_k(\tau)$.

5 Definition of Lagrangian Representation for Systems

We can now finally move to the analysis of the nonlinear system (4). As in the linear case, let $\lambda_1(u), \dots, \lambda_n(u)$ be the n distinct real eigenvalues of $A(u) := DF(u)$, $r_1(u), \dots, r_n(u)$ (resp. $l_1(u), \dots, l_n(u)$) be the right (resp. left) eigenvectors.

Trying to extend Definition 2 (and, in particular, Eqs. (14), (15)) from the linear to the nonlinear case, the first problem we have to face is that λ_k and r_k are not constant anymore, but they depend on u. As in the scalar case (see (7)), we have thus to find a good definition of kth eigenvalue $\hat{\lambda}_k(\bar{t}, \bar{x})$ and kth eigenvector $\hat{r}_k(\bar{t}, \bar{x})$ at a given point $(\bar{t}, \bar{x})$.

If $x \mapsto u(\bar{t}, x)$ is continuous at $\bar{x}$, the natural choice is to set $\hat{r}_k(\bar{t}, \bar{x}) := r_k(u(\bar{t}, \bar{x}))$ and $\hat{\lambda}_k(\bar{t}, \bar{x}) := \lambda_k(u(\bar{t}, \bar{x}))$.

If $x \mapsto u(\bar{t}, x)$ has a jump at $\bar{x}$ between $u^L := u(\bar{t}, x-)$ and $u^R := u(\bar{t}, x+)$, we solve the Riemann problem (u^L, u^R), defining the curves $(u_k(\cdot), v_k(\cdot), \sigma_k(\cdot))$ as in (17), and we set

$$\hat{r}_k(\bar{t}, \bar{x}) := \fint \tilde{r}_k(u_k(\varsigma), v_k(\varsigma), \sigma_k(\varsigma))d\varsigma, \quad \hat{\lambda}_k(\bar{t}, \bar{x}) := \fint \tilde{\lambda}_k(u_k(\varsigma), v_k(\varsigma), \sigma_k(\varsigma))d\varsigma.$$

Notice that, in the case of a scalar equation, the definition of $\hat{\lambda}(\bar{t}, \bar{x})$ given above coincides with (7).

After this preparation, we can now propose the following definition of Lagrangian representation for the solution to the nonlinear system (4). Compare it with Definitions 1 and 2.

Definition 3. A *Lagrangian representation* for the entropic solution u to (4) is a family of n triples $(W_k, \mathtt{X}_k, \rho_k)$, $k = 1, \dots, n$, where

1. $W_k \subseteq \mathbb{R}$ is a bounded interval, whose elements are called *waves of the kth family*; we also assume for simplicity that $W_k \cap W_h = \emptyset$ for $k \neq h$;
2. $\mathtt{X}_k : [0, \infty) \times W_k \to \mathbb{R}$ is $\|DF\|_{L^\infty}$-Lipschitz in t for fixed w and increasing in w for fixed t, and it is called *kth flow* or *kth position function*;
3. $\rho_k : [0, \infty) \times W_k \to [-1, 1]$ is uniformly BV in time for a.e. w, and it is called *kth density function*;

such that for a.e. $t \in [0, \infty)$

$$\partial_x u(t) = \sum_{k=1}^{n} \mathtt{X}_k(t)_\sharp \big(\rho_k(t)\mathscr{L}^1|_{W_k}\big)\hat{r}_k(t) \text{ in the sense of measures} \tag{18}$$

and

$$\frac{\partial \mathtt{X}_k}{\partial t}(t, w) = \hat{\lambda}_k(t, \mathtt{X}_k(t, w)) \text{ for } |\rho_k(t)|\mathscr{L}^1\text{-a.e.} w \in W_k. \tag{19}$$

Remark 2. The main difference between Definition 3 and Definitions 1 and 2 is that the density function $\rho = \rho(t, w)$ is now allowed to be a function of time. This seems strange in comparison with Formula (2) for the continuity equation. However, this dependence on time cannot be avoided: It comes from the well-known fact that nonlinear interactions between wavefronts, taking place at times $t > 0$, can *create* new wavefronts.

Nevertheless, the total amount of created waves can be bounded a priori (see [2]): This implies that the length of the set of waves W_k can be bounded by $C(F)$ Tot.Var.$(\bar{u})$ and that ρ can be chosen uniformly BV in time for a.e. wave. Here, $C(F)$ is a constant which depends only on F.

6 Construction of a Lagrangian Representation

In Sect. 5, we proposed a possible definition of Lagrangian representation for the entropic solution u to the system (4). In this section, we state the main theorem of these notes, i.e., the existence of such a Lagrangian representation, and we present a sketch of its proof.

Theorem 1. *There exists a Lagrangian representation for the entropic solution to the system* (4)*, in the sense of Definition 3.*

Sketch of the proof. The proof follows a path similar to the one we used in the scalar case. We start by taking a sequence of piecewise constant approximate solutions $(u^q)_q$ (constructed through the wavefront tracking algorithm or the Glimm scheme) which converges in L^1 to the exact entropic solution u to (4).

For each u^q, it is not difficult to construct by hand a Lagrangian representation (as we did for the scalar conservation law in Sect. 2), i.e., for each $k = 1, \dots, n$, a set of k-waves W_k (which we assume to be independent of q, without restriction), a flow $\mathtt{X}_k^q : [0, \infty) \times W_k \to \mathbb{R}$ and a density $\rho_k^q : [0, \infty) \times W_k \to [-1, 1]$ such that:

- for a.e. time t $\partial_x u^q(t) = \sum_{k=1}^n \mathtt{X}_k^q(t)_\sharp \big(\rho_k^q(t)\mathscr{L}^1|_{W_k}\big)\hat{r}_k^q(t, \cdot)$ i.e. for any $\varphi \in C_c^\infty(\mathbb{R})$,

$$-\int \varphi'(x) u^q(t, x) dx = \sum_{k=1}^n \int_{W_k} \varphi(\mathtt{X}_k^q(t, w)) \rho_k^q(t, w) \hat{r}_k^q(t, \mathtt{X}_k^q(t, w)) dw; \quad (20)$$

- for a.e. time t and for $|\rho_k^q|\mathscr{L}^1$ almost every $w \in W_k$

$$\partial_t \mathtt{X}_k^q(t, w) = \lambda_k^q(t, \mathtt{X}_k^q(t, w)). \quad (21)$$

Exactly as in the scalar case, the regularity properties of $\mathtt{X}_k^q$, ρ_k^q imply that there exist $\mathtt{X}_k : [0, \infty) \times W_k \to \mathbb{R}$, $\rho_k : [0, \infty) \times W_k \to [-1, 1]$ such that, up to subsequences, $\mathtt{X}_k^q(t) \to \mathtt{X}_k(t)$ strongly in $L^1(W_k)$ and $\rho_k^q(t) \to \rho_k(t)$ weakly* in $L^\infty(W_k)$, for a.e. time t. To complete the proof of Theorem 1, we have thus to pass to the limit in Formulae (20), (21) to get (18), (19), respectively.

In the scalar case, first we passed to the limit in (11) (corresponding here to (20)) to obtain (8) (corresponding here to (18)); then, we used (8) to prove (9) (corresponding here to Eq. (19)). Example 1 showed that it is not possible to obtain (9) directly passing to the limit in its approximate version (12), because in general $\hat{\lambda}^q(t) \circ \mathtt{X}^q(t) \not\rightarrow \hat{\lambda}(t) \circ \mathtt{X}(t)$.

In the system case, we cannot repeat the same argument (i.e., first passing to the limit in (20) to get (18) and then use (18) to prove (19)), because in (20) there is already a term $\hat{r}_k^q(t) \circ \mathtt{X}_k^q(t)$ which most likely does not converge in general to $\hat{r}_k(t) \circ \mathtt{X}_k(t)$, exactly as $\hat{\lambda}^q(t) \circ \mathtt{X}^q(t)$ did not converge in general to $\hat{\lambda}(t) \circ \mathtt{X}(t)$ in the scalar case. We thus need some new ideas to pass to the limit in (20), (21).

Example 1 shows that the are times (as time $\bar{t}$ in that example) for which there is no hope for $\hat{r}_k^q(t) \circ \mathtt{X}_k^q(t)$ (resp. $\hat{\lambda}_k^q(t) \circ \mathtt{X}_k^q(t)$) to converge to $\hat{r}_k(t) \circ \mathtt{X}_k(t)$ (resp. $\hat{\lambda}_k(t) \circ \mathtt{X}_k(t)$). However, the same example suggests that these times are *strong interaction times*, i.e., roughly speaking, times when many waves undergo a major change of their speed. For instance, in Example 1, $\hat{\lambda}^q(t) \circ \mathtt{X}^q(t) \rightarrow \hat{\lambda}(t) \circ \mathtt{X}(t)$ for every time, except the time $\bar{t}$ where a strong interaction between wavefronts takes place.

The strategy is thus to find a way to identify a priori those times of *strong interaction* in the solution u, to show that the set of such times has zero Lebesgue measure (or even that it is countable), and to prove that, up to those times, we can pass to the limit in (20), (21).

To identify such bad times, we introduce, for each approximate solution u^q, the Radon measure $\mu^q := \sum_{k=1}^n |\partial_t(\rho^q \partial_t \mathtt{X}_k^q)|$, which measure the change of the speed of the waves. Being u^q a piecewise constant function with a finite number of discontinuity lines, μ^q is just a finite sum of Dirac's deltas. For instance, for the configuration described in Example 1, μ^q is just a single Dirac's delta, located in the point $(\bar{t}, \bar{x})$, with size $|\sigma_1 - \sigma||u^M - u^L| + |\sigma_2 - \sigma||u^R - u^M|$.

Notice that, by construction of the Lagrangian representation in the approximations, for each u^q the times where waves can change their speed, i.e., times of strong interaction, are exactly those times t for which $\mu^q(\{t\} \times \mathbb{R}) > 0$.

Next, we prove that there is a Radon measure μ such that $\mu^q \rightarrow \mu$ weakly* in the sense of measures (see Remark 3 below for a comment about the existence of μ).

To conclude the proof, it is now enough to prove that if t is not a *time of strong interaction*; i.e., by definition, if t is a time such that

$$\mu\big(\{t\} \times \mathbb{R}\big) = 0 \tag{22}$$

(and this happens for all but a countable number of times), then we can pass to the limit in (20), (21) to get (18), (19), respectively. This would conclude the proof of Theorem 1.

Proving this last fact (i.e., passing to the limit in (20), (21)) is a major part of the proof of Theorem 1, which, however, requires the introduction of several *ad hoc* notations and contains rather technical steps. Therefore, in these notes, it is omitted. We just spend some words about the general strategy.

For each approximate solution u^q at each time t, through a fixed point procedure similar to the one described in Sect. 4 for solving the Riemann problem, we associate to each wave $w \in W_k$, a point

$$(\hat{u}_k^q(t,w), \hat{v}_k^q(t,w), \hat{\sigma}_k^q(t,w)) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R},$$

such that for each time t and each point $x \in \mathbb{R}$ for which $\mathtt{X}_k^q(t)^{-1}(x) \neq \emptyset$,

$$\hat{r}_k^q(t,x) \approx \fint_{\mathtt{X}(t)^{-1}(x)} \tilde{r}_k(\hat{u}_k^q(t,w'), \hat{v}_k^q(t,w'), \hat{\sigma}_k^q(t,w'))\rho(t,w')dw'$$

and, similarly, for the exact solution u at each time t, we associate to each $w \in W_k$ a point

$$(\hat{u}_k(t,w), \hat{v}_k(t,w), \hat{\sigma}_k(t,w)) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R},$$

such that for each time t and each point $x \in \mathbb{R}$ for which $\mathtt{X}_k(t)^{-1}(x) \neq \emptyset$,

$$\hat{r}_k(t,x) \approx \begin{cases} \tilde{r}_k(\hat{u}_k(t,w), \hat{v}_k(t,w), \hat{\sigma}_k(t,w)) & \text{if } u(t,\cdot) \text{ is continuous at } x = \mathtt{X}(t,w), \\ \fint_{\mathtt{X}(t)^{-1}(x)} \tilde{r}_k(\hat{u}_k(t,w'), \hat{v}_k(t,w'), \hat{\sigma}_k(t,w'))\rho(t,w')dw' & \text{if } u(t,\cdot) \text{ has a jump at } x. \end{cases}$$

Similar expressions hold for $\lambda_k^q(t,x)$, $\lambda_k(t,x)$. We then prove that if t is not a time of strong interaction, i.e. if (22) holds, then $\hat{u}_k^q \to \hat{u}_k$, $\hat{v}_k^q \to \hat{v}_k$, $\hat{\sigma}_k^q \to \hat{\sigma}_k$ in some appropriate topologies. Using this fact, we finally show that $\hat{r}_k^q(t) \circ \mathtt{X}^q(t) \to \hat{r}_k(t) \circ \mathtt{X}(t)$ and $\hat{\lambda}_k^q(t) \circ \mathtt{X}^q(t) \to \hat{\lambda}_k(t) \circ \mathtt{X}(t)$, thus concluding the proof of Theorem 1. □

Remark 3. Proving that the sequence $(\mu^q)_q$ is weakly* pre-compact in the sense of measure, i.e., proving that $|\mu^q| \leq C(f,\bar{u})$, where C is a constant which depends on f and the initial datum $\bar{u}$, but not on q, is not trivial at all. It amounts to prove that the total amount of change of speed of the waves present in an approximate solution u^q

$$\mu^q\big([0,\infty) \times \mathbb{R}\big) = \sum_{k=1}^n \int_{W_k} \text{Tot.Var.}\big(\rho^q(\cdot,w)\partial_t \mathtt{X}_k^q(\cdot,w); [0,\infty)\big)dw$$

is uniformly bounded by $C(f,\bar{u})$. Such estimate is proved in [6–8], using a *quadratic interaction potential*.

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