Splash Singularity for a Free-Boundary Incompressible Viscoelastic Fluid Model

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Abstract Numerical computations in viscoelasticity show the failure of many numerical schemes when the Weissenberg number is beyond a critical value Keunings (J Non-Newtonian Fluid Mech 20:209–226, 1986, [6]). The existence of singularities in the continuum model could be the way to explain instability appearing in numerical simulations. We consider here a 2D Oldroyd-B type model at high Weissenberg number, and we show the existence of the so-called splash singularities (namely, points where the free boundary remains smooth but self-intersects). In our case, we assume physically realistic boundary conditions given by the static equilibrium of all the force fields acting at the interface. Our strategy is based on local existence and stability results applied to a family of smooth suitable initial configurations, we show they will evolve into a self-intersecting configuration, and then necessarily there exists a positive time $t = t^*$, where the configuration has a splash singularity. To prove local existence and stability, we first apply a conformal transformation to the 2D domain, in order to separate the contact point with splash, and then we pass into Lagrangian coordinates to fix our domain, inspired by a Thomas Beale's paper on the initial value problem for the Navier–Stokes equations with a free surface.

Keywords Splash singularity · Viscoelasticity · Oldroyd model

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1 Introduction

In this paper, we study a viscoelastic fluid model of Oldroyd-B type at high Weissenberg number. The goal of this paper is to study the existence of splash singularity for the following system:

$$\begin{cases} \partial_t F + u \cdot \nabla F = \nabla u F \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = \operatorname{div}(F F^T) \\ \operatorname{div} u = 0, \end{cases} \tag{1}$$

with appropriate boundary conditions, which will be explained later.

Incompressible viscoelastic fluids are described classically by the following momentum balance equations:

$$\bar{\rho}(\partial_t u + (u \cdot \nabla)u) + \nabla p = \operatorname{div}\tau,$$

where $\tau = \nu_s(\nabla u + \nabla u^T) + \tau_p$ denotes the stress, ν_s is the solvent viscosity, and τ_p is the stress related to the elastic part. From now on, we assume $\bar{\rho} = 1$.

The stress tensor satisfies the well-known Oldroyd-B model

$$\tau + \lambda \partial_t^{uc} \tau = \nu_0((\nabla u + \nabla u^T) + \lambda_s \partial_t^{uc}(\nabla u + \nabla u^T)), \tag{2}$$

where

- $\partial_t^{uc} \tau = \partial_t \tau + (u \cdot \nabla)\tau - \nabla u^T \tau - \tau \nabla u$ denotes the upper convective time derivative,
- $\nu_0 = \nu_s + \nu_p$ denotes the material viscosity, ν_s the solvent viscosity, and ν_p the polymeric viscosity, respectively,
- λ the relaxation time, and
- $\lambda_s = \frac{\nu_s}{\nu_0}\lambda$.

By separating the solvent and the polymeric contributions to the stress, we get that the stress τ_p satisfies

$$\lambda \partial_t^{uc} \tau_p + \tau_p = \nu_p(\nabla u + \nabla u^T).$$

Moreover, combining the equations for the stress together with the equations for the mass and the momentum balance, it follows

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu_s \Delta u + \operatorname{div}\tau_p \\ \partial_t^{uc} \tau_p = -\frac{1}{\lambda}\tau_p + \frac{\nu_p}{\lambda}(\nabla u + \nabla u^T) \\ \operatorname{div} u = 0. \end{cases} \tag{3}$$

In the study of this type of non-Newtonian fluids, an important role is played by the relaxation time λ, which in our case turns out to be proportional to the Weissenberg number We, a number which measures the ratio between the viscous and the elastic forces, see [11]. For very high Weissenberg number (We $\to \infty$), the system (3) reduces to the following one:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu_s \Delta u + \mathrm{div}\tau_p \\ \partial_t \tau_p + (u \cdot \nabla)\tau_p - \nabla u^T \tau_p - \tau_p \nabla u = 0 \\ \mathrm{div} u = 0. \end{cases} \tag{4}$$

Let us denote with $\alpha \in \mathbb{R}^2$ the Lagrangian coordinate and let $X(t, \alpha)$ be the flux associated with the velocity u. Then, by applying the chain rule, the deformation gradient $F(t, X) = \frac{\partial X}{\partial \alpha}$ satisfies the following transport equation:

$$\partial_t F + u \cdot \nabla F = \nabla u F.$$

If the initial condition $\tau_p(0, X) = \tau_0(X)$ is **positive definite**, then $\tau_p(t, X) = F\tau_0 F^T$ is also **positive definite** and satisfies the equation

$$\partial_t \tau_p + (u \cdot \nabla)\tau_p - (\nabla u)\tau_p - \tau_p(\nabla u)^T = 0.$$

This allows us to solve the system (1), instead of (4). By imposing that the physical boundary conditions are given by the static equilibrium of the force fields at the interface, the free-boundary problem for the system (1) is

$$\begin{cases} \partial_t F + u \cdot \nabla F = \nabla u F & \text{in } \Omega(t) \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = \mathrm{div}(FF^T) & \\ \mathrm{div} u = 0, & \\ (-p\mathrm{Id} + (\nabla u + \nabla u^T) + (FF^T - \mathrm{Id}))n = 0 & \text{on } \partial\Omega(t) \\ u(t)_{|t=0} = u_0,\ F(t)_{|t=0} = F_0 & \text{in } \Omega_0. \end{cases} \tag{5}$$

We also assume $\mathrm{div} F_0 = 0$, therefore $\mathrm{div} F = 0$, for all t. The variable domain $\Omega(t) \subset \mathbb{R}^2$ denotes the region occupied by the fluid. Our main result is stated in the following theorem.

Theorem 1. *There exists a time $t^* \in [0, T]$ such that the interface $\partial\Omega(t^*)$ self-intersects in one point.*

Similar results for the Navier–Stokes equations are obtained by Castro, Córdoba, Fefferman, Gancedo and Gómez-Serrano in [2] and by Coutand and Shkoller in [3]. In our paper, the main problem is the presence of the elastic components, which could prevent the development of splash singularity. We prove that this is not the case.

One of the important ingredients in the proof is the use of a conformal map that has been introduced for this specific problem in [2]. The map $P(z) = \tilde{z}$, for $z \in \mathbb{C} \setminus \Gamma$, is defined as a branch of $\sqrt{z}$, where Γ is a line, passed through the splash point. We take $z \in \mathbb{C} \setminus \Gamma$ in order to make $\sqrt{z}$ an analytic function. The key idea, to prove our theorem is to make the analysis into the Lagrangian framework, in order to have a fixed boundary, as done in the paper of Beale [1] to analyze the free boundary of the

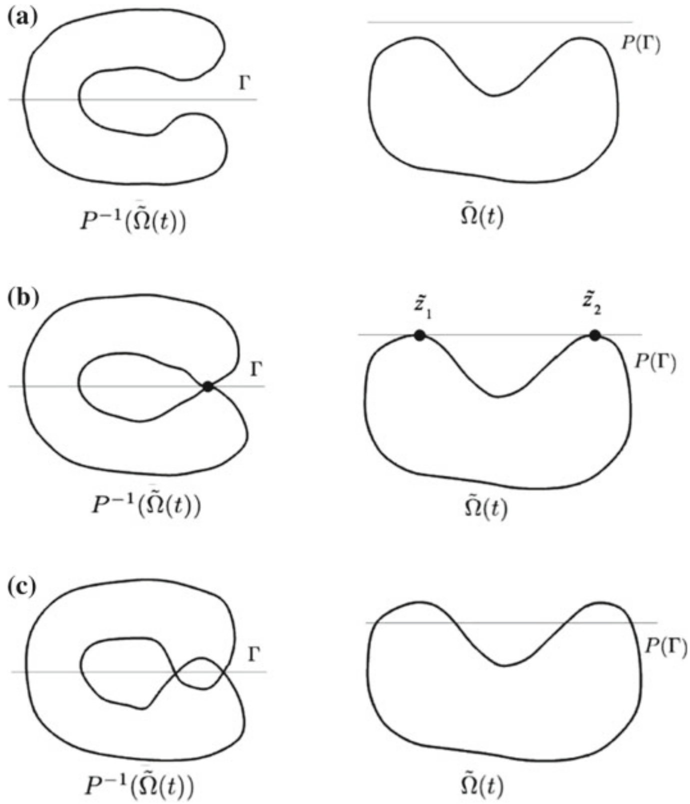

Fig. 1 Possibilities for $P^{-1}(\tilde{\Omega}(t))$

Navier–Stokes equations. The geometric ideas behind the proof are inspired from the construction in [2]. The main steps are explained below with the help of Fig. 1.

- Let the initial domain Ω_0 be a non-regular domain as (b); for this reason, we use the conformal map P and by projection we get $\tilde{\Omega}_0$, a non-splash domain.
- If $\{\tilde{\Omega}_0, \tilde{u}(0,\cdot), \tilde{p}(0,\cdot), \tilde{F}(0,\cdot)\}$ are smooth, we can prove the existence of a local solution $\{\tilde{\Omega}(t), \tilde{u}(t,\cdot), \tilde{p}(t,\cdot), \tilde{F}(t,\cdot)\}$, $t \in [0,T]$.
- By a suitable choice of the initial velocity, in particular $\tilde{u}(0,\tilde{z}_1)\cdot n > 0$, $\tilde{u}(0,\tilde{z}_2)\cdot n > 0$ such that there exists $\bar{t} > 0$ and $P^{-1}(\tilde{\Omega}(\bar{t}))$ is as (c). This solution lives in the tilde complex plane and cannot be transformed, by P^{-1}, into a solution in the non-tilde complex plane.
- To solve the problem in the non-tilde domain, we take a one-parameter family $\{\tilde{\Omega}_\varepsilon(0), \tilde{u}_\varepsilon(0), \tilde{F}_\varepsilon(0)\}$, with $\tilde{\Omega}_\varepsilon(0) = \tilde{\Omega}_0 + \varepsilon b$ and $|b| = 1$, such that $P^{-1}(\tilde{\Omega}_\varepsilon(0))$ is regular, and there exists a local in time smooth solution $\{\tilde{\Omega}_\varepsilon(t), \tilde{u}_\varepsilon(t,\cdot), \tilde{p}_\varepsilon(t,\cdot), \tilde{F}_\varepsilon(t,\cdot)\}$, which can be inverted in the non-tilde complex plane.
- By stability we get

$$\mathrm{dist}(\partial\tilde{\Omega}_\varepsilon(\bar{t}), \partial\tilde{\Omega}(\bar{t})) \leq C\varepsilon$$

hence $P^{-1}(\tilde{\Omega}_\varepsilon(\bar{t})) \sim P^{-1}(\tilde{\Omega}(\bar{t}))$ and so $P^{-1}(\tilde{\Omega}_\varepsilon(\bar{t}))$ self-intersects.

- Since $P^{-1}(\tilde{\Omega}_\varepsilon(0))$ is regular of type (a) and $P^{-1}(\tilde{\Omega}_\varepsilon(\bar{t}))$ is self-intersecting domain of type (c), and then there exists a time t^* such that $P^{-1}(\tilde{\Omega}_\varepsilon(t^*))$ has a splash singularity.

2 Conformal and Lagrangian Transformations

The free-boundary incompressible viscoelastic fluid model in Eulerian coordinates that we intend to study is given in (5). Because of the geometrical singularity induced by the self-intersection point, to start with an initial domain Ω_0 which is a splash domain, as in Fig. 1b, we use the conformal transformation $P(z) = \sqrt{z}$ to set our problem inside a regular domain. The new velocity field is defined as follows

$$\tilde{u}(t, \tilde{X}) = u(t, P^{-1}(\tilde{X})) \quad \text{then} \quad u(t, X) = \tilde{u}(t, P(X)).$$

The same for the deformation gradient F

$$\tilde{F}(t, \tilde{X}) = F(t, P^{-1}(\tilde{X})) \quad \text{then} \quad F(t, X) = \tilde{F}(t, P(X)).$$

Remark 1. Defining $J^P_{kj} = \partial_{X_j} P_k(P^{-1}(\tilde{X}))$, we have the following derivation formulas:

$$\partial_{X_j} u_i(t, X) = \partial_{\tilde{X}_k} \tilde{u}_i(t, P(X))\partial_{X_j} P_k(X), \quad \text{hence} \quad \partial_{X_j} u_i(t, P^{-1}(\tilde{X})) = J^P_{kj}\partial_{\tilde{X}_k}\tilde{u}_i(t, \tilde{X}).$$

The system in $\tilde{\Omega}$ takes the following form:

$$\begin{cases} \partial_t \tilde{F} + (J^P \tilde{u} \cdot \nabla_{\tilde{X}})\tilde{F} = J^P \nabla_{\tilde{X}} \tilde{u}\tilde{F} & \text{in } \tilde{\Omega}(t) \\ \partial_t \tilde{u} + (J^P \tilde{u} \cdot \nabla_{\tilde{X}})\tilde{u} - Q^2 \Delta \tilde{u} + J^P \nabla_{\tilde{X}} \tilde{p} = (J^P \tilde{F} \cdot \nabla_{\tilde{X}})\tilde{F} & \\ \mathrm{Tr}(\nabla \tilde{u} J^P) = 0 & \\ (-\tilde{p}\mathrm{I}d + (\nabla \tilde{u} J^P + (\nabla \tilde{u} J^P)^T) + (\tilde{F}\tilde{F}^T - \mathrm{Id}))(J^P)^{-1}\tilde{n} = 0 & \text{on} \partial\tilde{\Omega}(t) \\ \tilde{u}(t)_{|t=0} = \tilde{u}_0, \tilde{F}(t)_{|t=0} = \tilde{F}_0 & \text{for} \in \tilde{\Omega}_0. \end{cases}$$

The next step is to move form Eulerian into Lagrangian coordinates so that we transform a free-boundary problem into a fixed boundary problem. Then, the equation for the flux becomes

$$\begin{cases} \dfrac{d}{dt}\tilde{X}(t, \tilde{\alpha}) = J^P(\tilde{X}(t, \tilde{\alpha}))\tilde{u}(t, \tilde{X}(t, \tilde{\alpha})) & \text{in } \tilde{\Omega}(t) \\ \tilde{X}(0, \tilde{\alpha}) = \tilde{\alpha} & \text{in } \tilde{\Omega}(0). \end{cases} \tag{6}$$

Therefore, the Lagrangian variables are given by

$$\begin{cases} \tilde{v}(t,\tilde{\alpha}) = \tilde{u}(t,\tilde{X}(t,\tilde{\alpha})) \\ \tilde{q}(t,\tilde{\alpha}) = \tilde{p}(t,\tilde{X}(t,\tilde{\alpha})) \\ \tilde{G}(t,\tilde{\alpha}) = \tilde{F}(t,\tilde{X}(t,\tilde{\alpha})). \end{cases}$$

The new system in $[0,T] \times \tilde{\Omega}_0$ that we are going to study becomes

$$\begin{cases} \partial_t \tilde{G} = J^P(\tilde{X})\tilde{\zeta}\nabla_{\tilde{\alpha}}\tilde{v}\tilde{G} \\ \partial_t \tilde{v} - Q^2(\tilde{X})\tilde{\zeta}\nabla_{\tilde{\alpha}}(\tilde{\zeta}\nabla_{\tilde{\alpha}}\tilde{v}) + (J^P(\tilde{X}))^T\tilde{\zeta}\nabla_{\tilde{\alpha}}\tilde{q} = J^P(\tilde{X})\tilde{G}\tilde{\zeta}\nabla_{\tilde{\alpha}}\tilde{G} \\ \mathrm{Tr}(\nabla_{\tilde{\alpha}}\tilde{v}(\nabla_{\tilde{\alpha}}\tilde{X})^{-1}J^P(\tilde{X})) = 0 \\ \big[-\tilde{q}\mathrm{Id} + ((\nabla_{\tilde{\alpha}}\tilde{v}(\nabla_{\tilde{\alpha}}\tilde{X})^{-1}J^P(\tilde{X})) + (\nabla_{\tilde{\alpha}}\tilde{v}(\nabla_{\tilde{\alpha}}\tilde{X})^{-1}J^P(\tilde{X}))^T + \\ \qquad\qquad\qquad + (\tilde{G}\tilde{G}^T - \mathrm{Id})\big](J^P)^{-1}(\tilde{X})\nabla_\Lambda \tilde{X}\tilde{n}_0 = 0 \\ \tilde{v}(0,\tilde{\alpha}) = \tilde{v}_0(\tilde{\alpha}) = \tilde{u}_0(\tilde{\alpha}),\ \tilde{G}(0,\tilde{\alpha}) = \tilde{G}_0(\tilde{\alpha}) = \tilde{F}_0(\tilde{\alpha}), \end{cases} \tag{7}$$

where $\tilde{\zeta} = (\nabla\tilde{X})^{-1}$ and $\nabla_\Lambda \tilde{X} = -\Lambda\nabla\tilde{X}\Lambda$, with $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, since $\tilde{n} = -\Lambda J^P_{|\partial\tilde{\Omega}(t)}\Lambda n$.

3 Local Existence of Smooth Solutions

The idea to prove the local existence is based on a fixed point argument. The iteration will separate the equation in $\tilde{v}$ from the equation in $\tilde{G}$. In particular, $\tilde{G}$ satisfies an ODE, and then it will not interfere in the boundary conditions.

3.1 Iterative Scheme

The iterative scheme is given by the following two steps:

STEP 1

$$\begin{cases} \partial_t \tilde{G}^{(n+1)} = J^P(\tilde{X}^{(n)})\tilde{\zeta}^{(n)}\nabla_{\tilde{\alpha}}\tilde{v}^{(n)}\tilde{G}^{(n)} \\ \tilde{G}(0,\tilde{\alpha}) = \tilde{G}_0(\tilde{\alpha}). \end{cases} \tag{8}$$

STEP 2

$$\begin{cases} \partial_t \tilde{v}^{(n+1)} - Q^2\Delta\tilde{v}^{(n+1)} + (J^P)^T\nabla\tilde{q}^{(n+1)} = \tilde{f}^{(n)} \\ \mathrm{Tr}(\nabla\tilde{v}^{(n+1)}J^P) = \tilde{g}^{(n)} \\ (-\tilde{q}^{(n+1)}\mathrm{Id} + ((\nabla\tilde{v}^{(n+1)}J^P) + (\nabla\tilde{v}^{(n+1)}J^P)^T))(J^P)^{-1}\tilde{n}_0) = \tilde{h}^{(n)} \\ \tilde{v}(0,\tilde{\alpha}) = \tilde{v}_0(\tilde{\alpha}). \end{cases} \tag{9}$$

where

$$\tilde{f}^{(n)} = -Q^2\Delta\tilde{v}^{(n)} + (J^P)^T\nabla\tilde{q}^{(n)} + Q^2(\tilde{X}^{(n)})\tilde{\zeta}^{(n)}\nabla(\tilde{\zeta}^{(n)}\nabla\tilde{v}^{(n)}) - (J^P(\tilde{X}^{(n)}))^T\tilde{\zeta}^{(n)}\nabla\tilde{q}^{(n)}$$
$$+ J^P(\tilde{X}^{(n)})\tilde{G}^{(n)}\tilde{\zeta}^{(n)}\nabla\tilde{G}^{(n)},$$
$$\tilde{g}^{(n)} = \mathrm{Tr}(\nabla\tilde{v}^{(n)}J^P) - \mathrm{Tr}(\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)}J^P(\tilde{X}^{(n)})),$$
$$\tilde{h}^{(n)} = -\tilde{q}^{(n)}(J^P)^{-1}\tilde{n}_0 + \tilde{q}^{(n)}(J^P(\tilde{X}^{(n)}))^{-1}\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0$$
$$-[\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)}J^P(\tilde{X}^{(n)}) + (\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)}J^P(\tilde{X}^{(n)}))^T](J^P(\tilde{X}^{(n)}))^{-1}\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0$$
$$+((\nabla\tilde{v}^{(n)}J^P) + (\nabla\tilde{v}^{(n)}J^P)^T)(J^P)^{-1}\tilde{n_0} - (\tilde{G}^{(n)}\tilde{G}^{T(n)} - \mathrm{Id})(J^P(\tilde{X}^{(n)}))^{-1}\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0.$$

The system regarding the flux $\tilde{X}$ is given by

$$\begin{cases} \dfrac{d}{dt}\tilde{X}^{(n+1)}(t,\tilde{\alpha}) = J^P(\tilde{X}^{(n)}(t,\tilde{\alpha}))\tilde{v}^{(n)}(t,\tilde{\alpha}) & \\ \tilde{X}(0,\tilde{\alpha}) = \tilde{\alpha} & \text{in } \tilde{\Omega}_0, \end{cases} \tag{10}$$

thus $\tilde{X}^{(n+1)}$ satisfies

$$\tilde{X}^{(n+1)}(t,\tilde{\alpha}) = \tilde{\alpha} + \int_0^t \left(J^P(\tilde{X}^{(n)})\tilde{v}^{(n)}\right)(\tau,\tilde{\alpha})\,d\tau \tag{11}$$

3.2 Analysis of the System (9)

As we mentioned before, we will analyze the linearized system related to STEP 2 by means of the techniques introduced by T. Beale in [1] and their improvements in [2]. Namely, we study

$$\begin{cases} \partial_t\tilde{v} - Q^2\Delta\tilde{v} + (J^P)^T\nabla\tilde{q} = \tilde{f} \\ \mathrm{Tr}(\nabla\tilde{v}J^P) = \tilde{g} \\ (-\tilde{q}\mathrm{Id} + (\nabla\tilde{v}J^P) + (\nabla\tilde{v}J^P)^T)(J^P)^{-1}\tilde{n} = \tilde{h} \\ \tilde{v}(0,\tilde{\alpha}) = \tilde{v}_0, \end{cases} \tag{12}$$

where the compatibility conditions for the initial data are as follows:

$$\begin{cases} \mathrm{Tr}(\nabla\tilde{v}_0J^P) = \tilde{g}(0) & \text{in } \tilde{\Omega}_0 \\ ((J^P)^{-1}\tilde{n})^\perp(\nabla\tilde{v}_0J^P + (\nabla\tilde{v}_0J^P)^T)(J^P)^{-1}\tilde{n} = \tilde{h}(0)((J^P)^{-1}\tilde{n})^\perp & \text{on } \partial\tilde{\Omega}_0. \end{cases} \tag{13}$$

We are going to use the following functional spaces:

$$X_0 := \left\{(\tilde{v}, \tilde{q}) \in \mathscr{K}^{s+1} \times \mathscr{K}^{s}_{pr} : \tilde{v}(0) = 0, \partial_t \tilde{v}(0) = 0, \tilde{q}(0) = 0\right\},$$
$$Y_0 := \{(\tilde{f}, \tilde{g}, \tilde{h}, 0) \in \mathscr{K}^{s-1} \times \bar{\mathscr{K}}^{s} \times \mathscr{K}^{s-\frac{1}{2}}([0, T] \times \partial\tilde{\Omega}) :$$
$$\tilde{f}(0) = 0, \tilde{g}(0) = 0, \partial_t \tilde{g}(0) = 0, \tilde{h}(0) = 0 \text{ and (13) are satisfied}\}.$$

Therefore on the linearized problem, we recall the following result obtained by Beale, used later on in order to prove the local existence.

Theorem 2. *Let $2 < s < \frac{5}{2}$ and let $L : X_0 \to Y_0$ be the operator associated with the system (12), then L is invertible and the norm of the inverse is uniformly bounded for any $0 < T < \bar{T}$.*

3.3 The Fixed Point Argument

In order to apply Theorem 2 and hence to get bounds independent of T, we need $\tilde{v}_{|t=0} = 0$ and $\partial_t \tilde{v}_{|t=0} = 0$. For this reason, we replace the initial condition $\tilde{v}_0$ as follows:

$$\phi = \tilde{v}_0 + t \exp(-t^2)(Q^2 \Delta \tilde{v}_0 - (J^P)^T \nabla \tilde{q}_\phi),$$

where $\tilde{q}_\phi$ is chosen in such a way that for all n, $\partial_t \tilde{v}^{(n)}_{|t=0} = \partial_t \phi_{|t=0} = 0$. The velocity is defined by

$$\tilde{w}^{(n)} = \tilde{v}^{(n)} - \phi. \tag{14}$$

Therefore, we can rewrite the system (9) in the following way:

$$\begin{cases} \partial_t \tilde{w}^{(n+1)} - Q^2 \Delta \tilde{w}^{(n+1)} + (J^P)^T \nabla \tilde{q}^{(n+1)}_w = \tilde{f}^{(n)} - \partial_t \phi \\ \qquad + Q^2 \Delta \phi - (J^P)^T \nabla \tilde{q}_\phi \\ \mathrm{Tr}(\nabla \tilde{w}^{(n+1)} J^P) = \tilde{g}^{(n)} - \mathrm{Tr}(\nabla \phi J^P) \\ [-\tilde{q}^{(n+1)}_w \mathrm{Id} + ((\nabla \tilde{w}^{(n+1)} J^P) + (\nabla \tilde{w}^{(n+1)} J^P)^T)](J^P)^{-1} \tilde{n}_0 = \\ \qquad = \tilde{h}^{(n)} + \tilde{q}_\phi (J^P)^{-1} \tilde{n}_0 - ((\nabla \phi J^P) + (\nabla \phi J^P)^T)(J^P)^{-1} \tilde{n}_0 \\ \tilde{w}^{(n+1)}_{|t=0} = 0. \end{cases} \tag{15}$$

Our main local existence result is then equivalent to show the following theorem.

Theorem 3. *Let $2 < s < \frac{5}{2}$ and $1 < \gamma < s - 1$. If $(\tilde{v}(0), \partial_t \tilde{v}(0)) = (0, 0)$ and $(\tilde{q}(0), \tilde{f}(0), \tilde{g}(0), \partial_t \tilde{g}(0), \tilde{h}(0)) = (0, 0, 0, 0, 0)$, moreover $\tilde{G}(0) = \tilde{G}_0 \in H^s$, then there exist T (sufficiently small) and a solution $\{\tilde{X}(\cdot), \tilde{v}(\cdot), \tilde{q}(\cdot), \tilde{G}(\cdot)\} \in \mathscr{F}^{s+1,\gamma} \times \mathscr{K}^{s+1} \times \mathscr{K}^{s}_{pr} \times \mathscr{F}^{s,\gamma-1}$ on $[0, T]$.*

In order to prove this theorem, we need the following technical results.

Proposition 1. *1. Let* $\tilde{G}^{(n)} - \tilde{G}_0 \in \mathscr{F}^{s,\gamma-1}$, $\tilde{X}^{(n)} - \tilde{\alpha} \in \mathscr{F}^{s+1,\gamma}$, *and* $\tilde{w}^{(n)} \in \mathscr{K}^{s+1}$ *and such that*

(i) $\tilde{G}^{(n)} - \tilde{G}_0 \in \Big\{ \tilde{G} - \tilde{G}_0 \in \mathscr{F}^{s,\gamma-1}$:

$$\left\| \tilde{G} - \tilde{G}_0 - \int_0^t J^P \nabla\phi \tilde{G}_0 \, d\tau \right\|_{\mathscr{F}^{s,\gamma-1}} \leq \left\| \int_0^t J^P \nabla\phi \tilde{G}_0 \, d\tau \right\|_{\mathscr{F}^{s,\gamma-1}} \Big\} \equiv B,$$

(ii) $\left\| \tilde{w}^{(n)} \right\|_{\mathscr{K}^{s+1}} \leq N.$

Then, for $T > 0$ *small enough, depending only on* $N, \tilde{v}_0, \tilde{G}_0$,

$$\tilde{G}^{(n+1)} - \tilde{G}_0 \in B.$$

2. Let $\tilde{G}^{(n)} - \tilde{G}_0, \tilde{G}^{(n-1)} - \tilde{G}_0 \in \mathscr{F}^{s,\gamma-1}$, *with* $\tilde{X}^{(n)} - \tilde{\alpha}, \tilde{X}^{(n-1)} - \tilde{\alpha} \in \mathscr{F}^{s+1,\gamma}$ *and* $\tilde{w}^{(n)}, \tilde{w}^{(n-1)} \in \mathscr{K}^{s+1}$ *and such that*

(i) $\left\| \tilde{w}^{(n)} \right\|_{\mathscr{K}^{s+1}} \leq M$, $\left\| \tilde{w}^{(n-1)} \right\|_{\mathscr{K}^{s+1}} \leq M$,
(ii) $\left\| \tilde{X}^{(n)} - \tilde{\alpha} \right\|_{\mathscr{F}^{s+1,\gamma}} \leq M$, $\left\| \tilde{X}^{(n-1)} - \tilde{\alpha} \right\|_{\mathscr{F}^{s+1,\gamma}} \leq M$,
(iii) $\left\| \tilde{G}^{(n)} - \tilde{G}_0 \right\|_{\mathscr{F}^{s,\gamma-1}} \leq M$, $\left\| \tilde{G}^{(n-1)} - \tilde{G}_0 \right\|_{\mathscr{F}^{s,\gamma-1}} \leq M$,

for some $M > 0$. *Then*

$$\left\| \tilde{G}^{(n+1)} - \tilde{G}^{(n)} \right\|_{\mathscr{F}^{s,\gamma-1}} \leq CT^{\delta} \left(\left\| \tilde{G}^{(n)} - \tilde{G}^{(n-1)} \right\|_{\mathscr{F}^{s,\gamma-1}} + \left\| \tilde{w}^{(n)} - \tilde{w}^{(n-1)} \right\|_{\mathscr{K}^{s+1}} + \left\| \tilde{X}^{(n)} - \tilde{X}^{(n-1)} \right\|_{\mathscr{F}^{s+1,\gamma}} \right),$$

for a suitable $\delta > 0$.

Proof. The proof is given in [4].

Proposition 2. *1. Let* $\tilde{X}^{(n)} - \tilde{\alpha} \in \mathscr{F}^{s+1,\gamma}$, $\tilde{w}^{(n)} \in \mathscr{K}^{s+1}$ *and such that*

(i) $\tilde{X}^{(n)} - \tilde{\alpha} \in \left\{ \tilde{X} - \tilde{\alpha} \in \mathscr{F}^{s+1,\gamma} : \left\| \tilde{X} - \tilde{\alpha} - \int_0^t J^P \phi \, d\tau \right\|_{\mathscr{F}^{s+1,\gamma}} \leq \left\| \int_0^t J^P \phi \, d\tau \right\|_{\mathscr{F}^{s+1,\gamma}} \right\} \equiv B_{J^P\phi}$,
(ii) $\|\tilde{w}^{(n)}\|_{\mathscr{K}^{s+1}} \leq N.$

Then, for $T > 0$ *small enough, depending only on* N *and* $\tilde{v}_0$,

$$\tilde{X}^{(n+1)} - \tilde{\alpha} \in B_{J^P\phi}.$$

2. Let $\tilde{X}^{(n)} - \tilde{\alpha}, \tilde{X}^{(n-1)} - \tilde{\alpha} \in \mathscr{F}^{s+1,\gamma}$, *with* $\tilde{w}^{(n)}, \tilde{w}^{(n-1)} \in \mathscr{K}^{s+1}$ *and such that*

(i) $\left\| \tilde{w}^{(n)} \right\|_{\mathscr{K}^{s+1}} \leq M$, $\left\| \tilde{w}^{(n-1)} \right\|_{\mathscr{K}^{s+1}} \leq M$,

(ii) $\left\|\tilde{X}^{(n)} - \tilde{\alpha}\right\|_{\mathscr{F}^{s+1,\gamma}} \leq M$, $\left\|\tilde{X}^{(n-1)} - \tilde{\alpha}\right\|_{\mathscr{F}^{s+1,\gamma}} \leq M$

for some $M > 0$. *Then*

$$\left\|\tilde{X}^{(n+1)} - \tilde{X}^{(n)}\right\|_{\mathscr{F}^{s+1,\gamma}} \leq CT^{\delta}\left(\left\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\right\|_{\mathscr{F}^{s+1,\gamma}} + \left\|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\right\|_{\mathscr{K}^{s+1}}\right),$$

for a suitable $\delta > 0$.

Proof. The proof of this theorem is given in [2].

Proposition 3. *1. Let* $\tilde{X}^{(n)} - \tilde{\alpha} \in \mathscr{F}^{s+1,\gamma}$, $\tilde{q}_w^{(n)} \in \mathscr{K}^s_{pr}$ *and* $\tilde{w}^{(n)} \in \mathscr{K}^{s+1}$, *and such that*

(i) $\|\tilde{X}^{(n)} - \tilde{\alpha}\|_{\mathscr{F}^{s+1,\gamma}} \leq N$,

(ii) $\|\tilde{G}^{(n)} - \tilde{G}_0\|_{\mathscr{F}^{s,\gamma-1}} \leq N$,

(iii) $(\tilde{w}^{(n)}, \tilde{q}_w^{(n)}) \in \Big\{(\tilde{w}, \tilde{q}) \in \mathscr{K}^{s+1} \times \mathscr{K}^s_{pr} : \tilde{w}_{|t=0} = 0, \partial_t \tilde{w}_{|t=0} = 0,$

$$\|(\tilde{w}, \tilde{q}) - L^{-1}(\tilde{f}^L_\phi, \tilde{g}^L_\phi, \tilde{h}^L_\phi)\|_{\mathscr{K}^{s+1} \times \mathscr{K}^s_{pr}} \leq \|L^{-1}(\tilde{f}^L_\phi, \tilde{g}^L_\phi, \tilde{h}^L_\phi)\|_{\mathscr{K}^{s+1} \times \mathscr{K}^s_{pr}}\Big\}$$
$$\equiv B_{L^{-1}(\tilde{f}^L_\phi, \tilde{g}^L_\phi, \tilde{h}^L_\phi)}.$$

Then

$$(\tilde{w}^{(n+1)}, \tilde{q}_w^{(n+1)}) \in B_{L^{-1}(\tilde{f}^L_\phi, \tilde{g}^L_\phi, \tilde{h}^L_\phi)}.$$

2. *Let* $\tilde{X}^{(n)} - \tilde{\alpha}, \tilde{X}^{(n-1)} - \tilde{\alpha} \in \mathscr{F}^{s+1,\gamma}$, $\tilde{G}^{(n)} - \tilde{G}_0, \tilde{G}^{(n-1)} - \tilde{G}_0 \in \mathscr{F}^{s,\gamma-1}$, $\tilde{w}^{(n)}, \tilde{w}^{(n-1)} \in \mathscr{K}^{s+1}$, *with* $\tilde{w}^{(n)}_{|t=0} = \tilde{w}^{(n-1)}_{|t=0} = 0$, $\partial_t \tilde{w}^{(n)}_{|t=0} = \partial_t \tilde{w}^{(n-1)}_{|t=0} = 0$, $\tilde{q}_w^{(n)}$, $\tilde{q}_w^{(n-1)} \in \mathscr{K}^s_{pr}$, *and such that*

(i) $\left\|\tilde{w}^{(n)}\right\|_{\mathscr{K}^{s+1}} \leq M$, $\quad \left\|\tilde{w}^{(n-1)}\right\|_{\mathscr{K}^{s+1}} \leq M$,

(ii) $\left\|\tilde{X}^{(n)} - \tilde{\alpha}\right\|_{\mathscr{F}^{s+1,\gamma}} \leq M$, $\quad \left\|\tilde{X}^{(n-1)} - \tilde{\alpha}\right\|_{\mathscr{F}^{s+1,\gamma}} \leq M$,

(iii) $\left\|\tilde{G}^{(n)} - \tilde{G}_0\right\|_{\mathscr{F}^{s,\gamma-1}} \leq M$, $\quad \left\|\tilde{G}^{(n-1)} - \tilde{G}_0\right\|_{\mathscr{F}^{s,\gamma-1}} \leq M$,

(iv) $\|\tilde{q}_w^{(n)}\|_{\mathscr{K}^s_{pr}}, \leq M$, $\quad \|\tilde{q}_w^{(n-1)}\|_{\mathscr{K}^s_{pr}}, \leq M$.

for some $M > 0$. *Then*

$$\left\|\tilde{w}^{(n+1)} - \tilde{w}^{(n)}\right\|_{\mathscr{K}^{s+1}} + \left\|\tilde{q}_w^{(n+1)} - \tilde{q}_w^{(n)}\right\|_{\mathscr{K}^s_{pr}} \leq CT^{\delta}\left(\left\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\right\|_{\mathscr{F}^{s+1,\gamma}}\right.$$
$$\left. + \left\|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\right\|_{\mathscr{K}^{s+1}} + \left\|\tilde{q}_w^{(n)} - \tilde{q}_w^{(n-1)}\right\|_{\mathscr{K}^s_{pr}} + \left\|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\right\|_{\mathscr{F}^{s,\gamma-1}}\right),$$

for a suitable $\delta > 0$.

Proof. For the proof of this theorem, we will use Theorem 2.

By putting together the results of Proposition 1, Proposition 2 and Proposition 3 and by applying the contraction mapping principle we get the proof of Theorem 3.

4 Stability

In this section, we want to prove what we described in the Introduction. So we pick a one-parameter family $\tilde{\Omega}_\varepsilon(0) = \tilde{\Omega}_0 + \varepsilon b$, where $|b| = 1$ and such that $P^{-1}(\tilde{\Omega}_\varepsilon(0))$ is a regular domain as in Fig. 1a. We take the difference between the solution $(\tilde{w}, \tilde{q}, \tilde{X}, \tilde{G})$ and the perturbed solution $(\tilde{w}_\varepsilon, \tilde{q}_\varepsilon, \tilde{X}_\varepsilon, \tilde{G}_\varepsilon)$, which is as follows:

$$\begin{cases} \partial_t(\tilde{w} - \tilde{w}_\varepsilon) - Q^2\Delta(\tilde{w} - \tilde{w}_\varepsilon) + (J^P)^T\nabla(\tilde{q}_w - \tilde{q}_{w,\varepsilon}) = \tilde{F}_\varepsilon \\ \mathrm{Tr}(\nabla(\tilde{w} - \tilde{w}_\varepsilon)J^P) = \tilde{K}_\varepsilon \\ \left[-(\tilde{q}_w - \tilde{q}_{w,\varepsilon})\mathrm{Id} + \nabla(\tilde{w} - \tilde{w}_\varepsilon)J^P + (\nabla(\tilde{w} - \tilde{w}_\varepsilon)J^P)^T\right](J^P)^{-1}\tilde{n}_0 = \tilde{H}_\varepsilon \\ \tilde{w}_0 - \tilde{w}_{\varepsilon,0} = 0. \end{cases} \tag{16}$$

The main estimates we prove, for a suitable $\delta > 0$ and for $2 < s < \frac{5}{2}$, are

$$\bullet \|\tilde{G} - \tilde{G}_\varepsilon\|_{L^\infty H^s} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{H^2 H^{\gamma-1}} \le C\varepsilon + CT^\delta \Big(\|\tilde{w} - \tilde{w}_\varepsilon\|_{\mathscr{K}^{s+1}} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{L^\infty H^s} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{H^2 H^{\gamma-1}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{H^2 H^\gamma}\Big),$$

$$\bullet \|\tilde{w} - \tilde{w}_\varepsilon\|_{\mathscr{K}^{s+1}} + \|\tilde{q}_w - \tilde{q}_{w,\varepsilon}\|_{\mathscr{K}^s_{pr}} \le C\varepsilon + CT^\delta(\|\tilde{w} - \tilde{w}_\varepsilon\|_{\mathscr{K}^{s+1}} + \|\tilde{q}_w - \tilde{q}_{w,\varepsilon}\|_{\mathscr{K}^s_{pr}} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{L^\infty H^s} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{H^2 H^{\gamma-1}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{H^2 H^\gamma}),$$

$$\bullet \|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{H^2 H^\gamma} \le C\varepsilon + CT^\delta \Big(\|\tilde{w} - \tilde{w}_\varepsilon\|_{\mathscr{K}^{s+1}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{H^2 H^\gamma}\Big).$$

5 Existence of Splash Singularity (Proof of Theorem 1)

From the stability estimates above, we obtain

$$\|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}} \le 3C\varepsilon, \tag{17}$$

by choosing $0 < T < \frac{1}{(3C)^{\frac{1}{\delta}}}$.

The main arguments to prove the existence of splash singularity are the stability results and the choice of the initial velocity. As stated in the Introduction, we choose $\tilde{u}_0 \cdot n > 0$, and then we get a domain $\tilde{\Omega}(\bar{t})$ such that, by the inverse mapping, $P^{-1}(\tilde{\Omega}(\bar{t}))$ is a self-intersecting domain, for $\bar{t} > 0$. By using (17), it follows that $P^{-1}(\tilde{\Omega}_\varepsilon(\bar{t}))$ is also a self-intersecting domain. In conclusion, we have that $P^{-1}(\tilde{\Omega}_\varepsilon(0))$ is a regular domain of type (a) and $P^{-1}(\tilde{\Omega}_\varepsilon(\bar{t}))$ is a self-intersecting domain of type (c), for some $\bar{t} \in (0, T]$, then there exists a time $t^* \in (0, \bar{t})$ such that $P^{-1}(\tilde{\Omega}(t^*))$ self-intersects in one point, so it forms a splash singularity. Thus, Theorem 1 holds.

Appendix

The spaces we used in our proof are of the type

$$H^{s,r}([0,T];\Omega) = L^2_t H^s_x \cap H^r_t L^2_x.$$

For our purposes, we shall always take $r = \frac{s}{2}$, and we introduce the following notations:

$$\mathcal{K}^s([0,T];\Omega) = L^2_t H^s_x \cap H^{\frac{s}{2}}_t L^2_x,$$
$$\mathcal{K}^s_{pr}([0,T];\Omega)=\{q \in L^\infty_t \dot{H}^1_x : \nabla q \in \mathcal{K}^{s-1}([0,T];\Omega), q \in \mathcal{K}^{s-\frac{1}{2}}([0,T];\partial\Omega)\},$$
$$\bar{\mathcal{K}}^s([0,T];\Omega) = L^2_t H^s_x \cap H_t^{\frac{s+1}{2}} H^{-1}_x,$$
$$\mathcal{F}^{s+1,\gamma}([0,T];\Omega) = L^\infty_{\frac{1}{4},t} H^{s+1}_x \cap H^2_t H^\gamma_x,$$
$$\text{for } s-1-\varepsilon < \gamma < s-1,$$

with

$$\|f\|_{L^\infty_{\frac{1}{4},t} H^s_x} = \sup_{t\in[0,T]} t^{-\frac{1}{4}} \|f(t)\|_{H^s_x}.$$

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