# On Cantor's Theorem for Fuzzy Power Sets

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**Abstract.** The aim of the paper is to introduce the concept of fuzzy power set in a universe of sets and investigate its basic properties. We focus here on an analysis of Cantor's theorem for fuzzy sets, which states in the set theory that the cardinality of a set is strictly smaller then the cardinality of its power set. For our investigation of Cantor's theorem we chose two types of equipollency of fuzzy sets, particularly, the binary Cantor's equipollence and its graded version.

**Keywords:** Cardinal theory · Fuzzy sets · Universe of sets
Fuzzy power sets · Cantor's theorem

## 1 Introduction

In the elementary set theory, the cardinality of the power set of a set $x$ is strictly greater than the cardinality of the original set $x$. Symbolically, we write $|x| < |\mathrm{P}(x)|$, where $|x|$ denotes the cardinality of the set $x$ and $|\mathrm{P}(x)|$ the cardinality of the power set of $x$. This fundamental result is known as Cantor's theorem and has been used to demonstrate that there are sets having cardinality greater than the infinite cardinality of the set of natural numbers. In literature on the set theory, Cantor's theorem is sometimes formulated as there is no function from $x$ onto $\mathrm{P}(x)$ or $x$ is not equipollent $\mathrm{P}(x)$, which is also referred as a more general form of Cantor's theorem. For the purpose of this contribution, we consider the last formulation of Cantor's theorem.

In the standard fuzzy set theory, we can distinguish two concepts: the *power set* and the *fuzzy power set* of a fuzzy set $A$ over a universe of discourse $x$. The power set of a fuzzy set $A : x \longrightarrow [0,1]$ is the classical set, denoted by $\mathscr{F}(A)$, consisting of all fuzzy subsets of $A$, where a fuzzy set $B : x \longrightarrow [0,1]$ is a fuzzy subset of $A$ if $B(z) \leq A(z)$ holds for any $z \in x$. A generalization of this concept can be found in the theory of categories under the name of (fuzzy) powerset operator [10,12]. An extension of the power set of a fuzzy set to the fuzzy power set has been proposed by Bandler and Kohout in [1]. In this paper, the fuzzy power set of a fuzzy set $A$, denoted by $\mathscr{P}(A)$, is defined as a fuzzy set $\mathscr{P}(A) : \mathscr{F}(x) \longrightarrow [0,1]$, where $\mathscr{F}(x)$ is the power set of the fuzzy set $x$ (each


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classical set can be considered as a special fuzzy set), and $\mathscr{P}(A)(B)$ expresses the membership degree in which $B$ belongs to the power set of $A$, or equivalently, the truth degree of the statement saying that $B$ is a fuzzy subset of $A$. Using a fuzzy implication operator $\to$ on $[0,1]$,[1] Bandler and Kohout defined the value of $\mathscr{P}(A)(B)$ as

$$\mathscr{P}(A)(B) = \bigwedge_{z \in x} (B(x) \to A(x)), \tag{1}$$

where $\bigwedge$ denotes the infimum operation in $[0,1]$. One can see that $B \in \mathscr{F}(A)$ if and only if $\mathscr{P}(A)(B) = 1$. The fuzzy power sets can be extended also for fuzzy sets whose membership degrees are interpreted in more general algebras of truth values. As an example, let us mention the development of lattice-valued set theory provided by Takeuti and Titani in [11] (see also [4]).

In this contribution, we deal with fuzzy sets whose universes of discourse belong to a given universe of sets (e.g., the class of all sets or finite sets; or a set known as a Grothendieck universe). Note that the universe of sets has been introduced in [8] to form a framework for development of fuzzy set theory. The concept of fuzzy power set, which is sound in each universe of sets, has been introduced in [7] and admits only classical (crisp) sets in the universe of discourse of the fuzzy power set. This restriction to crisp sets ensures that each fuzzy power set becomes a fuzzy set in the given universe of sets, which is not true in general, if one admits also fuzzy sets as in the case of Bandler-Kohout definition. A typical example is the fuzzy power set of a fuzzy set over a finite set with the membership degrees interpreted in an infinite algebraic structure of truth values, which does not belong to the universe of all finite sets. For our analysis of Cantor's theorem within the fuzzy set theory, we introduce two types of equipollence for fuzzy sets. The first type of equipollence is a binary class relation on the class of all fuzzy sets in a universe of sets stating that two fuzzy sets have or have not the same cardinality. The second type of equipollence is a graded version of the first type (a fuzzy class relation) and its definition has been proposed in [8] and further developed in [5,6] (see also [7] for finite fuzzy sets).

The main goal of this contribution is to show that Cantor's theorem is valid (valid in a weaker form) for fuzzy sets and proposed fuzzy power sets in each universe of sets if the first (second) type of equipollence is considered.

The paper is structured as follows. The next section introduces basic concepts that are used in the main part of the contribution. The third section is devoted to Cantor's theorem whose validity is verified for Cantor's equipollence. The fourth section provides the proof of Cantor's theorem for graded Cantor's equipolence.

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[1] The fuzzy implication operator on $[0,1]$ is often modeled in fuzzy logic as a residuum operation on a complete residuated lattice on $[0,1]$ (see Subsect. 2.1).

## 2 Preliminaries

### 2.1 Algebraic Structures of Truth Values

A complete linearly ordered residuated lattice is considered as a structure of membership degrees for fuzzy sets. Recall that a *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow \bot, \top\rangle$ with four binary operations and two constants, for which it holds that

(i) $\langle L, \wedge, \vee, \bot, \top\rangle$ is a bounded lattice, where $\bot$ is the least element and $\top$ is the greatest element of $L$, respectively,
(ii) $\langle L, \otimes, \top\rangle$ is a commutative monoid,
(iii) the pair $\langle \otimes, \rightarrow\rangle$ forms an adjoint pair, i.e.,

$$a \leq b \rightarrow c \quad \text{if and only if} \quad a \otimes b \leq c \tag{2}$$

holds for each $a, b, c \in L$ ($\leq$ denotes the corresponding lattice ordering).

A residuated lattice is said to be *complete* (*linearly ordered*) if the corresponding lattice $\langle L, \wedge, \vee, \bot, \top\rangle$ is a complete (linearly ordered) lattice. Details and examples of residuated lattices can be found in [2,9].

### 2.2 Fuzzy Sets in a Universe of Sets

A fuzzy set is usually defined as a function from a fixed non-empty universe of discourse to a set (lattice) of truth values. Nevertheless, the fuzzy set constructions like fuzzy power sets or exponentiation of fuzzy sets requires a system of universes of discourse rather than one fixed universe (cf., [3]). This motivated us to introduce a universe of sets over a complete residuated lattice as a basic framework for our fuzzy set theory [8]. In what follows, we use $x \in y$ to denote that the set $x$ is a member of set $y$, further, we use $\mathrm{P}(x)$, $\mathscr{D}(f)$ and $\mathscr{R}(f)$ to denote the power set of a set $x$, the domain and the range of a function $f$, respectively.

**Definition 1.** *Let* $\mathbf{L}$ *be a complete linearly ordered residuated lattice. A universe of sets over* $\mathbf{L}$ *is a non-empty class* $\mathfrak{U}$ *of sets in the Zermelo–Fraenkel set theory with the axiom of choice (ZFC) satisfying the following properties:*

*(U1)* $x \in y$ *and* $y \in \mathfrak{U}$, *then* $x \in \mathfrak{U}$,
*(U2)* $x, y \in \mathfrak{U}$, *then* $\{x, y\} \in \mathfrak{U}$,
*(U3)* $x \in \mathfrak{U}$, *then* $\mathrm{P}(x) \in \mathfrak{U}$,
*(U4)* $x \in \mathfrak{U}$ *and* $y_i \in \mathfrak{U}$ *for any* $i \in x$, *then* $\bigcup_{i \in x} y_i \in \mathfrak{U}$,
*(U5)* $x \in \mathfrak{U}$ *and* $f : x \longrightarrow L$, *then* $\mathscr{R}(f) \in \mathfrak{U}$,

*where* $L$ *denotes the support of* $\mathbf{L}$.

Basic examples of the universes of sets are the classes of all or finite sets. If the ZFC is extended by the axiom admitting the existence of strongly inaccessible cardinals, one can introduce a universe of sets over $\mathbf{L}$ to be a Grothendieck universe.

**Definition 2.** *Let $\mathfrak{U}$ be a universe of sets over* $\mathbf{L}$. *A function $A : z \longrightarrow L$ (in ZFC) is called a* fuzzy set *in $\mathfrak{U}$ if $z \in \mathfrak{U}$.*

Let $A : z \longrightarrow L$ be a fuzzy set in $\mathfrak{U}$. The domain $z = \mathscr{D}(A)$ is called the *universe of discourse of* $A$, and the set $\mathscr{S}(A) = \{x \in z \mid A(x) > \bot\}$ the *support of fuzzy set* $A$. Further, for $\alpha \in L$, the sets $A_\alpha = \{x \in z \mid A(x) \geq \alpha\}$ and $A^\alpha = \{x \in z \mid A(x) = \alpha\}$ are called the $\alpha$*-cut* and $\alpha$*-level* of $A$, respectively. An element $x \in z$ is said to be *negligible* in $A$ whenever $x \notin \mathscr{S}(A)$. A fuzzy set $A$ is said to be *crisp* and referred to a *crisp set* if $A(x) \in \{\bot, \top\}$ for any $x \in z$. The empty function $\emptyset : \emptyset \longrightarrow L$ is called the *empty fuzzy set*. One can see that the empty function as a vacuous fuzzy set is crisp, since the assumption on a crisp set is trivially satisfied. If $x \subseteq y$ are sets in $\mathfrak{U}$, we use $\chi_x$ to denote the *characteristic function of $x$ on $y$*, i.e., $\chi_x : y \longrightarrow L$, which is defined by $\chi_x(z) = \top$ if $z \in x$, and $\chi_x(z) = \bot$, otherwise. A fuzzy set $A$ is a *fuzzy subset* of $B$ in $\mathfrak{U}$ provided that $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ and $A(a) \leq B(a)$ for any $a \in \mathscr{D}(A)$. It is easy to see that $\subseteq$ is a partial ordering on the class $\mathfrak{F}(\mathfrak{U})$ of all fuzzy sets in $\mathfrak{U}$.

We say that two fuzzy sets $A$ and $B$ in $\mathfrak{U}$ are *identical* (symbolically, $A = B$) if $\mathscr{D}(A) = \mathscr{D}(B)$ and $A(a) = B(a)$ for any $a \in \mathscr{D}(A)$. Moreover, $A$ and $B$ are *identical up to negligibility* (symbolically, $A \equiv B$) if $\mathscr{S}(A) = \mathscr{S}(B)$ and $A(a) = B(a)$ for any $a \in \mathscr{S}(A)$. One can observe that the relation "to be identical up to negligibility" is an equivalence on $\mathfrak{F}(\mathfrak{U})$. We use $\mathrm{cls}(A)$ to denote the equivalence class of all fuzzy sets from $\mathfrak{U}$ being identical with $A$ up to negligibility.

### 2.3 Functions Between Fuzzy Sets

Let $\mathfrak{Fcs}$ and $\mathfrak{Fcs}(x, y)$ denote the class of all functions in $\mathfrak{U}$ and the set of all functions from $x$ to $y$, respectively. Let $x, y, a, b \in \mathfrak{U}$ such that $a \subseteq x$ and $b \subseteq y$. By the definition, a function $f : x \longrightarrow y$ is a function from $a$ to $b$ if $f(z) \in b$ for any $z \in a$ or

$$\chi_a(z) \leq \chi_b(f(z)) \quad (\text{or } \chi_a(z) \to \chi_b(f(z)) = \top) \tag{3}$$

for any $z \in a$, if we consider the characteristic functions of the sets $a$ and $b$. Replacing the characteristic functions in condition (3) by fuzzy sets, we obtain a natural definition of a function between fuzzy sets.

**Definition 3.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f \in \mathfrak{Fcs}$. We say that $f$ is a* function from $A$ to $B$ *(symbolically $f : A \longrightarrow B$) if $f \in \mathfrak{Fcs}(\mathscr{D}(A), \mathscr{D}(B))$ and*

$$A(z) \leq B(f(z)) \quad (\textit{or equivalently } A(z) \to B(f(z)) = \top) \tag{4}$$

*for any $z \in \mathscr{D}(A)$.*

The set of all functions from $A$ to $B$ is denoted by $\mathfrak{Fcfs}(A, B)$. Note that the empty function from the empty fuzzy set to an arbitrary fuzzy set trivially satisfies condition (4) and thus belongs to $\mathfrak{Fcfs}(A, B)$. Obviously, the composition of functions $g \circ f \in \mathfrak{Fcfs}(A, C)$, whenever $f \in \mathfrak{Fcfs}(A, B)$ and $g \in \mathfrak{Fcfs}(B, C)$.

A function $f : x \longrightarrow y$ in $\mathfrak{U}$ is a 1-1 correspondence between $x$ and $y$ if there exists a function $f^{-1} : y \longrightarrow x$ (an inverse function) for which $f^{-1} \circ f = 1_x$ and $f \circ f^{-1} = 1_y$, where $1_x$ and $1_y$ denote the identity functions on $x$ and $y$, respectively. Similarly, we define the 1-1 correspondence between fuzzy sets.

**Definition 4.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f \in \mathfrak{Fcfs}(A, B)$. We say that $f : A \longrightarrow B$ is a* 1-1 correspondence *(symbolically $f : A \xrightarrow[\text{corr}]{\text{1-1}} B$) if there exists $f^{-1} : B \longrightarrow A$ such that $f^{-1} \circ f = 1_{\mathscr{D}(A)}$ and $f \circ f^{-1} = 1_{\mathscr{D}(B)}$.*

The set of all 1-1 correspondences between fuzzy sets $A$ and $B$ in $\mathfrak{U}$ is denoted by $\mathfrak{Cfs}(A, B)$. Later, we introduce a graded version of 1-1 correspondences that play a fundamental role in the definition of graded equipollence. An equivalent definition in terms of 1-1 and onto functions is the following. Denote $\mathfrak{Fcs}^{\text{1-1}}_{\text{corr}}(x, y)$ the set of all 1-1 correspondences between $x$ and $y$.

**Theorem 1.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$. A function $f : A \longrightarrow B$ is a 1-1 correspondence between fuzzy sets if and only if $f \in \mathfrak{Fcs}^{\text{1-1}}_{\text{corr}}(\mathscr{D}(A), \mathscr{D}(B))$ and $A(a) = B(f(a))$ for any $a \in \mathscr{D}(A)$.*

*Proof.* ($\Rightarrow$) Let $f : A \longrightarrow B$ be a function such that there exists $f^{-1} : B \longrightarrow A$ such that $f^{-1} \circ f = 1_{\mathscr{D}(A)}$ and $f \circ f^{-1} = 1_{\mathscr{D}(B)}$. Then, $f$ is a 1-1 function of $A$ onto $B$. Since $A(a) \rightarrow B(f(a)) = \top$ for any $a \in \mathscr{D}(A)$ and simultaneously $B(b) \rightarrow A(f^{-1}(b)) = \top$ for any $b \in \mathscr{D}(B)$, we find that

$$(A(a) \rightarrow B(f(a))) \wedge (B(f(a)) \rightarrow A(a)) = A(a) \leftrightarrow B(f(a)) = \top$$

for any $a \in \mathscr{D}(A)$; therefore, $A(a) = B(f(a))$ for any $a \in \mathscr{D}(A)$.

($\Leftarrow$) Since $f$ is a 1-1 function of $\mathscr{D}(A)$ onto $\mathscr{D}(B)$, there exists $f^{-1} : \mathscr{D}(B) \longrightarrow \mathscr{D}(A)$ such that $f^{-1} \circ f = 1_{\mathscr{D}(A)}$ and $f \circ f^{-1} = 1_{\mathscr{D}(B)}$. To finish the proof, we have to prove that $f^{-1}$ is a function of $B$ to $A$, i.e., (4) is satisfied for $f^{-1}$. Let $b \in \mathscr{D}(B)$, and let $a \in \mathscr{D}(A)$ such that $f(a) = b$. Then, we find that

$$B(b) \rightarrow A(f^{-1}(b)) = B(f(a)) \rightarrow A(a) = A(a) \rightarrow A(a) = \top,$$

where we used $A(a) = B(f(a))$. $\square$

Hence, its easy to see that the composition of functions $g \circ f \in \mathfrak{Cfs}(A, C)$, whenever $f \in \mathfrak{Cfs}(A, B)$ and $g \in \mathfrak{Cfs}(B, C)$.

Let $f : x \longrightarrow y$ be a function between sets, and let $z \subseteq x$. The image of $z$ under $f$ is defined by $f^{\rightarrow}(z) := \{b \in y \mid \exists a \in x \;\&\; f(a) = b\}$. The image of a fuzzy set under a function is a straightforward extension of the previous definition and is given by Zadeh's extension principle as follows.

**Definition 5.** *Let $x, y \in \mathfrak{U}$, and let $f : x \longrightarrow y$ be a function. Let $A : x \longrightarrow L$ be a fuzzy set in $\mathfrak{U}$. The* image of $A$ under $f$ *is denoted by $f^{\rightarrow}(A)$ and defined by*

$$f^{\rightarrow}(A)(b) := \bigvee_{a \in x; f(a) = b} A(a) \tag{5}$$

*for any $y \in y$.*

### 2.4 Functions Between Fuzzy Sets in a Certain Degree

Let $\varphi$ be a formula in fuzzy set theory. Then $[\varphi]$ denotes the truth degree in which the formula $\varphi$ is true, which is interpreted in the residuated lattice $\mathbf{L}$. For example, the truth degree $[f \in \mathfrak{Fcs}(x,y)]$ expresses how it is true that the function $f$ is a member of the set $\mathfrak{Fcs}(x,y)$. Of course, in this case, the truth degree becomes $\bot$ or $\top$.

**Definition 6.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f \in \mathfrak{Fcs}$. We say that $f$ is a function of $A$ to $B$ in the degree $\alpha$ provided that*

$$\alpha = [f \in \mathfrak{Fcs}(\mathscr{D}(A), \mathscr{D}(B))] \otimes \bigwedge_{(a,f(a)) \in \mathscr{D}(A) \times \mathscr{D}(B)} (A(a) \rightarrow B(f(a)). \qquad (6)$$

By our convention, $[f : A \longrightarrow B]$ denotes the truth degree in which the function $f$ can be considered as a function from $A$ to $B$. Let us emphasize that if $f$ is not a function from $\mathscr{D}(A)$ to $\mathscr{D}(B)$, then $[f : A \longrightarrow B] = \bot$ even if the infimum value in (6) is greater than $\bot$. Similarly we define the truth degree of a correspondences between fuzzy set.

**Definition 7.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f \in \mathfrak{Fcs}$. We say that $f$ is approximately a one-to-one correspondence between $A$ and $B$ in the degree $\alpha$ provided that*

$$\alpha = [f \in \mathfrak{Fcs}^{\text{1-1}}_{\text{corr}}(\mathscr{D}(A), \mathscr{D}(B))] \otimes \bigwedge_{(a,f(a)) \in \mathscr{D}(A) \times \mathscr{D}(B)} (A(a) \leftrightarrow B(f(a)). \qquad (7)$$

The value $[f : A \xrightarrow[\text{corr}]{\text{1-1}} B]$ denotes the truth degree in which the function $f$ can be considered as a one-to-one correspondence between fuzzy sets $A$ and $B$.

### 2.5 Fuzzy Power Sets

As we have mentioned in Introduction, the fuzzy power set for fuzzy sets is considered to be a fuzzy set over the set of appropriate fuzzy sets. Here, we propose an alternative definition that straightforwardly generalizes the classical approach to the power set and it is sound in our fuzzy set theory.

**Definition 8.** *Let $A \in \mathfrak{F}(\mathfrak{U})$, and $x = \mathrm{P}(\mathscr{D}(A))$. The fuzzy set $\mathscr{P}(A) : x \longrightarrow L$ defined by*

$$\mathscr{P}(A)(y) = \bigwedge_{z \in \mathscr{D}(A)} (\chi_y(z) \rightarrow A(z)) \qquad (8)$$

*is called the* fuzzy power set of $A$, *where $\chi_y$ is characteristic function of $y$ on $\mathscr{D}(A)$.*

One can see that the previous definition copies the Bandler-Kohout definition (1) with the restriction to crisp sets. As a simple consequence of (8), we obtain a simple expression of the membership degrees of fuzzy power set

$$\mathscr{P}(A)(y) = \bigwedge_{z \in y} A(z). \qquad (9)$$

The following statement shows that the fuzzy power sets preserve the class equivalence of being identical up to negligibility.

**Theorem 2.** *$\mathscr{P}(A) \equiv \mathscr{P}(B)$, whenever $A \equiv B$.*

*Proof.* It can be found in [7]. □

*Example 1.* Let $\mathbf{L}_{\text{Ł}}$ be the Łukasiewicz algebra, and let $A = \{1/a, 0.4/b\}$. Then,

$$\mathscr{P}(A) = \{1/\emptyset, 1/\{a\}, 0.4/\{b\}, 0.4/\{a,b\}\}.$$

Moreover, $\mathscr{P}(\emptyset) = \{1/\emptyset\}$, since $\mathscr{P}(\emptyset)(\emptyset) = \bigwedge \emptyset = 1$.

*Example 2.* Let $\mathbf{L}$ be a complete residuated lattice on $[0,1]$, let the set of all natural numbers $\omega$ belong to $\mathfrak{U}$, and let $A : \omega \longrightarrow L$ be defined by $A(n) = 1/n$. Then, curiously, it holds that $|\mathscr{S}(A)| = |\mathscr{S}(\mathscr{P}(A))|$. Indeed, one can see that $\mathscr{P}(A)$ assigns the zero truth degree to each infinite subset of $\omega$. Hence, we obtain that $x \in \mathscr{S}(\mathscr{P}(A))$ if and only if $x$ is a finite subset of $\omega$. It is well-known that the set of all finite subsets of $\omega$ is countable.[2] The statement follows from the fact that the support of $A$ is a countable set.

**Theorem 3.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f : A \longrightarrow B$ be a function between fuzzy sets. Then, the following diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle i_A}\big\downarrow & & \big\downarrow{\scriptstyle i_B} \\ \mathscr{P}(A) & \xrightarrow[\quad f^{\rightarrow} \quad]{} & \mathscr{P}(B), \end{array}$$

*where $i_A, i_B$ are the inclusion functions, i.e., $i_A(a) = \{a\}$ for any $a \in \mathscr{D}(A)$ and similarly $i_B$, and $f^{\rightarrow}$ is the image function of sets.*

*Proof.* Obviously, $i_A : \mathscr{D}(A) \longrightarrow \mathrm{P}(\mathscr{D}(A))$ given by $i_A(a) = \{a\}$ is a function from $A$ into $\mathscr{P}(A)$, since $A(a) = \mathscr{P}(A)(\{a\})$, and similarly $i_B$ is a function from $B$ into $\mathscr{P}(B)$. Obviously, the diagram commutes. To finish the proof, we show that $f^{\rightarrow}$ is a function from $\mathscr{P}(A)$ to $\mathscr{P}(B)$. If $x \subseteq \mathscr{D}(A)$, then

$$\mathscr{P}(A)(x) = \bigwedge_{a \in x} A(a) \le \bigwedge_{a \in x} B(f(a)) = \bigwedge_{b \in f^{\rightarrow}(x)} B(b) = \mathscr{P}(B)(f^{\rightarrow}(x)),$$

and the proof is finished. □

[2] For example, we can put $\lambda(n) := \{1, \ldots, n\}$. Then

$$|\mathscr{S}(\mathscr{P}(A))| = |\bigcup_{n \in \omega} \mathrm{P}(\lambda(n))| \le |\bigcup_{n \in \omega} (\mathrm{P}(\lambda(n)) \times \{n\})| \le |\omega \times \omega| = |\omega|,$$

where $\mathrm{P}(\lambda(n)$ is the power set of $\lambda(n)$ and we used that $|\mathrm{P}(\lambda(n))| < |\omega|$ for any $n \in \omega$.

### 2.6 Fuzzy Classes

Although the fuzzy sets in $\mathfrak{U}$ are the major objects in our theory, it is useful, similarly to the set theory, to introduce the concept of fuzzy class in $\mathfrak{U}$.

**Definition 9.** *Let $\mathfrak{U}$ be a universe of sets over $\mathbf{L}$. A class function $\mathcal{A} : \mathfrak{Z} \longrightarrow L$ (in ZFC) is called a fuzzy class in $\mathfrak{U}$ if $\mathfrak{Z} \subseteq \mathfrak{U}$.*

Note that each fuzzy set is a fuzzy class because of (U1), but not vice versa. Hence, a fuzzy class $\mathcal{A}$ is said to be *proper* if there is no fuzzy set which is identical to $\mathcal{A}$ up to negligibility (the relation $\equiv$ is extended here to fuzzy classes).

Fuzzy class relations are defined similarly to fuzzy set relations, only fuzzy sets are replaced by fuzzy classes. For the purpose of this paper, we introduce the fuzzy class equivalence and fuzzy class partial ordering.

**Definition 10.** *A fuzzy class relation $\mathcal{R} : \mathfrak{Z} \times \mathfrak{Z} \longrightarrow L$ is called a fuzzy class equivalence if for any $a, b, c \in \mathfrak{Z}$, it satisfies*

*(FE1)* $\mathcal{R}(a, a) = \top$,
*(FE2)* $\mathcal{R}(a, b) = \mathcal{R}(b, a)$,
*(FE3)* $\mathcal{R}(a, b) \otimes \mathcal{R}(b, c) \leq \mathcal{R}(a, c)$.

## 3 Cantor's Equipollence

In set theory, two sets are equipollent (equipotent, equivalent, bijective, or have the same cardinality, etc.) if there exists a 1-1 correspondence between them. This definition was proposed by G. Cantor. Formally, the class relation of equipollence denoted by $\sim$ is introduced on the class of all sets as follows:

$$x \sim y \quad \text{iff} \quad \exists f : x \xrightarrow[\text{corr}]{\text{1-1}} y. \tag{10}$$

Obviously, the equipollence of sets is a class relation extending the relation to be identical sets. One can see that the substitution of fuzzy sets for the sets in (10) does not reflect the idea that fuzzy sets being identical up to negligibility should be also equipollent. Furthermore, the restriction to particular fuzzy sets in (10) the consistency of our theory is broken as the following simple examples demonstrate.

*Example 3.* Let $x = \{a, b\}$ and $y = \{c, d, e\}$. Let $A = \chi_x$ and $B = \chi_z$, where $z = \{c, d\} \subset y$. Obviously, the set $\mathfrak{Fcfs}(A, B)$ is empty because there is no 1-1 correspondence between the domains of $A$ and $B$; hence, $A \not\sim B$. On the other hand, there is a function $f$ such that $f : x \xrightarrow[\text{corr}]{\text{1-1}} z$; therefore, naturally it should be $A \sim B$.

*Example 4.* Let $\omega$ be the set of natural numbers, and assume that $\omega \in \mathfrak{U}$. Let $\mathbf{L}_{Ł}$ be the Łukasiewicz algebra, and let $N, O : \omega \longrightarrow [0, 1]$ be fuzzy sets defined by

$$N(n) = 1 \quad \text{and} \quad O(n) = \begin{cases} 1, \text{ if } n \text{ is an odd number,} \\ 0, \text{ otherwise,} \end{cases}$$

for any $n \in \omega$. Obviously, the set $\mathfrak{Fcfs}(N, O)$ is empty even if there exists a 1-1 correspondence $f : \omega \longrightarrow \omega$; hence, $N \not\sim O$. On the other hand, the sets of odd numbers and natural numbers are equipollent; therefore, it should be $N \sim O$.

To overcome the aforementioned difficulties and simultaneously to accept fuzzy sets that differ up to negligible elements to be identical we propose the following definition of the equipollence of fuzzy sets.

**Definition 11.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$. We say that* $A$ and $B$ are Cantor's equipollent *(symbolically $A \overset{c}{\sim} B$) provided that there exist $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$, and $f \in \mathfrak{Fcs}$ such that $f : A' \xrightarrow[\text{corr}]{\text{1-1}} B'$.*

The following theorem states a necessary and sufficient condition reducing the verification of Cantor's equipollence to two specific fuzzy sets that are identical to the original ones up to negligibility.

**Theorem 4.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $C \in \mathrm{cls}(A)$ and $D \in \mathrm{cls}(B)$ such that $\mathscr{S}(C) = \mathscr{D}(C)$ and $\mathscr{S}(D) = \mathscr{D}(D)$. Then $A \overset{c}{\sim} B$ if and only if there exists $f : C \xrightarrow[\text{corr}]{\text{1-1}} D$.*

*Proof.* If $A \overset{c}{\sim} B$, then there exist $A' \in \mathrm{cls}(A)$, $B'\mathrm{cls}(B)$ and $f : A' \xrightarrow[\text{corr}]{\text{1-1}} B'$. A simple consequence of Cantor's equipollence, we find that $A'(x) = \perp$ if and only if $B'(f(x)) = \perp$. Hence, $f$ restricted to $\mathscr{S}(A)$ must be a 1-1 correspondence between the supports of $A$ and $B$ such that $A(x) = B(f(x))$ for any $x \in \mathscr{S}(A)$. Therefore, $f \restriction \mathscr{S}(A) : C \xrightarrow[\text{corr}]{\text{1-1}} D$, and the sufficient part is proved. Since the necessary part follows immediately from the definition of Cantor's equipollence, the statement is proved. □

The equipollence of sets is a class equivalence. The same holds for the Cantor's equipollence of fuzzy sets.

**Theorem 5.** *The class relation $\overset{c}{\sim}$ is a class equivalence on $\mathfrak{U}$.*

The following lemma provides an equivalent definition of Cantor's equipollence of fuzzy sets based on the classical equipollence of $\alpha$-levels.

**Lemma 1.** *$A \overset{c}{\sim} B$ if and only if $|A^\alpha| = |B^\alpha|$ for any $\alpha \in L \setminus \{\perp\}$.*

*Proof.* ($\Rightarrow$) Let $A \overset{c}{\sim} B$. By Theorem 4, we may assume that the domains and the supports of $A$ and $B$ coincide. If $f : A \xrightarrow[\text{corr}]{\text{1-1}} B$ is the respective 1-1 correspondence, then, for any $\alpha \in L \setminus \{\perp\}$, we simply find that $A^\alpha = \emptyset$ if and only if $B^\alpha = \emptyset$; otherwise, we have $f^{\rightarrow}(A^\alpha) = B^\alpha$. Hence, $|A^\alpha| = |B^\alpha|$ for $A^\alpha = \emptyset$. If $A^\alpha \neq \emptyset$, then $f \restriction A^\alpha : A^\alpha \xrightarrow[\text{corr}]{\text{1-1}} B^\alpha$, which implies that $|A^\alpha| = |B^\alpha|$.

($\Leftarrow$) Let $|A^\alpha| = |B^\alpha|$ for any $\alpha \in L \setminus \{\perp\}$. Since

$$\mathscr{S}(A) = \bigcup_{\alpha \in L \setminus \{\perp\}} A^\alpha$$

and $A(x) = \alpha$ if and only if $x \in A^\alpha$ (note that $A^\alpha \cap A^\beta = \emptyset$ whenever $\alpha \neq \beta$), and similarly for $B$, we find that a 1-1 correspondence $f$ between $A$ and $B$ can be derived as

$$f := \bigcup_{\alpha \in L \setminus \{\perp\}} f_\alpha,$$

where $f_\alpha$ is an arbitrary 1-1 correspondence of $A^\alpha$ onto $B^\alpha$ for any $\alpha \in L \setminus \{\perp\}$. Note that $f(x) = f_\alpha(x)$ if and only if $x \in A^\alpha$; therefore, $f : \mathscr{S}(A) \xrightarrow[\text{corr}]{\text{1-1}} \mathscr{S}(B)$ such that $A(x) = B(f(x))$; hence, we obtain $A \overset{c}{\sim} B$. □

The following theorem is Cantor's theorem for fuzzy sets based on the equipollence relation $\overset{c}{\sim}$.

**Theorem 6 (Cantor's theorem).** *$A \overset{c}{\nsim} \mathscr{P}(A)$.*

*Proof.* Let $A \in \mathfrak{F}(\mathfrak{U})$. First, we show that $\mathrm{P}(A^\alpha) \subseteq \mathscr{P}(A)^\alpha$ for any $\alpha \in L \setminus \{\perp\}$. From the fuzzy power set definition, if $y \subseteq A^\alpha$ (including $y = \emptyset$), then $\mathscr{P}(A)(y) = \bigwedge_{x \in y} A(x) = \alpha$; therefore, $y \in \mathscr{P}(A)^\alpha$, which means that $\mathrm{P}(A_\alpha) \subseteq \mathscr{P}(A)_\alpha$. The statement is a simple consequence of Lemma 1 and the fact that $|A^\alpha| < |\mathrm{P}(A^\alpha)| \leq |\mathscr{P}(A)^\alpha|$. □

## 4 Graded Cantor's Equipollence

We say that fuzzy sets $A, B \in \mathfrak{F}(\mathfrak{U})$ have *cardinal separable supports* if

$$|\mathscr{S}(A)| \leq |\mathscr{D}(B) \setminus \mathscr{S}(B)| \text{ and } |\mathscr{S}(B)| \leq |\mathscr{D}(A) \setminus \mathscr{S}(A)|. \tag{11}$$

In [8], we have introduced the graded version of Cantor's equipollence. The following definition of graded Cantor's equipollence has been presented in [6].

**Definition 12.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $C \in \mathrm{cls}(A)$ and $D \in \mathrm{cls}(B)$ be fuzzy sets that have cardinal separable supports and $|\mathscr{D}(C)| = |\mathscr{D}(D)|$. We say that $A$ and $B$ are Cantor's equipollent in the degree $\alpha$ provided that*

$$\alpha = \bigvee_{f \in \mathfrak{Fcs}(\mathscr{D}(C), \mathscr{D}(D))} [f : C \xrightarrow[\text{corr}]{\text{1-1}} D]. \tag{12}$$

We use $\overset{c}{\approx}$ to denote the fuzzy class relation of being Cantor's equipollent in a certain degree and the value $[A \overset{c}{\approx} B]$ denotes the truth degree in which the fuzzy sets $A$ and $B$ are Cantor's equipollent.

**Definition 13.** *The fuzzy class relation $\overset{c}{\approx}$ is called the graded Cantor's equipollence of fuzzy sets.*

It is well known that $a \sim b$ implies $\mathrm{P}(a) \sim \mathrm{P}(b)$ in set theory. The following theorem is a natural extension of this statement for fuzzy sets.

**Theorem 7.** *Let $A, B \in \mathfrak{F}(\mathfrak{U})$. Then,*

$$[A \stackrel{c}{\approx} B] \leq [\mathscr{P}(A) \stackrel{c}{\approx} \mathscr{P}(B)]. \tag{13}$$

*Proof.* Let $A, B \in \mathfrak{F}(\mathfrak{U})$. Without lost of generality (due to Theorem 2), we assume that $A$ and $B$ have cardinal separable supports and $|\mathscr{D}(A)| = |\mathscr{D}(B)|$. For $A = B = \emptyset$, the statement is a trivial consequence of Theorem 2. Let $A \neq \emptyset$ or $B \neq \emptyset$. Recall that $\mathscr{D}(\mathscr{P}(A)) = \mathrm{P}(\mathscr{D}(A))$. For any $f \in \mathfrak{Fcs}^{1\text{-}1}_{\mathrm{corr}}(\mathscr{D}(A), \mathscr{D}(B))$, let us define $f^{\rightarrow} : \mathscr{D}(\mathscr{P}(A)) \longrightarrow \mathscr{D}(\mathscr{P}(B))$ by

$$f^{\rightarrow}(y) = \{f(x) \mid x \in y\}, \quad y \in \mathscr{D}(\mathscr{P}(A)). \tag{14}$$

Obviously, $f^{\rightarrow} \in \mathfrak{Fcs}^{1\text{-}1}_{\mathrm{corr}}(\mathscr{D}(\mathscr{P}(A)), \mathscr{D}(\mathscr{P}(B)))$ and

$$[\mathscr{P}(A) \stackrel{c}{\approx} \mathscr{P}(B)] \geq \bigwedge_{y \in \mathscr{D}(\mathscr{P}(A))} \big(\mathscr{P}(A)(y) \leftrightarrow \mathscr{P}(B)(f^{\rightarrow}(y))\big)$$

$$= \bigwedge_{y \in \mathscr{D}(\mathscr{P}(A))} \Big( (\bigwedge_{x \in y} A(x)) \leftrightarrow (\bigwedge_{z \in f^{\rightarrow}(y)} B(z)) \Big) \geq \bigwedge_{y \in \mathscr{D}(\mathscr{P}(A))} \bigwedge_{x \in y} (A(x) \leftrightarrow B(f(x)))$$

$$= \bigwedge_{x \in \mathscr{D}(A)} (A(x) \leftrightarrow B(f(x))) = [f : A \xrightarrow[\mathrm{corr}]{1\text{-}1} B].$$

Since the previous inequality holds for any $f \in \mathfrak{Fcs}^{1\text{-}1}_{\mathrm{corr}}(\mathscr{D}(A), \mathscr{D}(B))$, we obtain

$$[\mathscr{P}(A) \stackrel{c}{\approx} \mathscr{P}(B)] \geq \bigvee_{f \in \mathfrak{Fcs}^{1\text{-}1}_{\mathrm{corr}}(\mathscr{D}(A), \mathscr{D}(B))} [f : A \xrightarrow[\mathrm{corr}]{1\text{-}1} B] = [A \stackrel{c}{\approx} B],$$

and the proof is finished. □

One can observe that $A \not\stackrel{c}{\approx} \mathscr{P}(A)$ does not imply $[A \stackrel{c}{\approx} \mathscr{P}(A)] < \top$. In other words, if there is no 1-1 correspondence between $A$ and $\mathscr{P}(A)$, we cannot immediately exclude that the fuzzy sets $A$ and $\mathscr{P}(A)$ are equipollent in degree $\top$, where $\top$ is a result of the supremum operation in (12). Nevertheless, this claim is true and can be considered as a graded version of Cantor's theorem.

**Theorem 8 (Graded version of Cantor's theorem).** $[A \stackrel{c}{\approx} \mathscr{P}(A)] < \top$.

Since the proof is long we left it out in the paper. The following example demonstrates the graded version of Cantor's theorem on the fuzzy set from Example 2.

*Example 5.* Assume that $\mathbf{L}$ is the Łukasiewicz algebra, and let $A : \omega \longrightarrow [0, 1]$ be the fuzzy set defined by $A(n) = 1/n$ . Since $\mathscr{P}(A)(\emptyset) = 1$, the evaluation of $[A \stackrel{c}{\approx} \mathscr{P}(A)]$ is based on one-to-one correspondences $f$, for which $f(1) = \emptyset$ and $f(2) = \{1\}$ or $f(1) = \{1\}$ and $f(2) = \emptyset$. Obviously, $[f : A \xrightarrow[\mathrm{corr}]{1\text{-}1} \mathscr{P}(A)] = 1/2$, which follows from $A(1) \leftrightarrow \mathscr{P}(A)(\emptyset) = 1 = A(1) \leftrightarrow \mathscr{P}(A)(\{1\})$ and $A(2) \leftrightarrow \mathscr{P}(A)(\emptyset) = 1/2 = A(2) \leftrightarrow \mathscr{P}(A)(\{1\})$. Since there is no one-to-one correspondence in a degree $\alpha$, which is greater than $1/2$, we obtain $[A \stackrel{c}{\approx} \mathscr{P}(A)] = 1/2$.

## 5 Conclusion

In this contribution, we proposed a novel concept of fuzzy power sets of a fuzzy set defined over the set of crisp subsets of the universe of discourse and analyzed the validity of Cantor's theorem for it with respect to types of equipollences of fuzzy sets. We gave preference to this simpler definition over the Bandler-Kohout concept of fuzzy power set to ensure the soundness of fuzzy set theory, which is built in the framework of a universe of fuzzy sets. Nevertheless, if the Bandler-Kohout fuzzy power sets exist in a universe of sets, a similar analysis can be provided, but this is a subject of our future research.

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