



A Novel Approach to the Discrete Fuzzy Transform of Higher Degree

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Abstract. In this paper, we propose a new approach to the discrete fuzzy transform of higher degree based on the piecewise constant representation of discrete functions and the application of the continuous fuzzy transform. We show how a given discrete function can be reconstructed by using the discrete higher degree fuzzy transform and how convenient the latter is computed by the novel approach. Finally, we illustrate and compare the proposed technique with the original discrete fuzzy transform of higher degree.

Keywords: Continuous fuzzy transform · Discrete fuzzy transform
F-transform · Fuzzy partition

1 Introduction

The fuzzy transform (F-transform, for short) has been introduced by Perfilieva in [10]. In this paper, Perfilieva proposed both the continuous F-transform for continuous (and later locally integrable) functions and the discrete F-transform for discrete functions defined over sets of finite points. The continuous F-transform has been extended to higher degrees in [11] to improve its ability to approximate functions whose domains are connected subsets of the real line. The continuous higher degree F-transform has been reformulated for discrete functions by Holčapek and Tichý in [4]. It is well known that the F-transform consists of two phases, namely, direct and inverse. The direct phase transforms a (locally integrable or discrete) function into a set of its local approximations, which are called the direct F-transform components and are determined with respect to basic functions, i.e., fuzzy sets, that form a fuzzy partition of the domain of the given function. On the other hand, the inverse phase provides an approximate reconstruction of the original function from its direct fuzzy transform components. Due to its good reconstruction ability, low computational complexity and successful reduction of noise, the F-transform has become a popular alternative in various fields of application, e.g., data analysis, time series analysis, image processing, non-parametric regression, numerical solution of differential equations, (see, e.g., [3, 5, 7, 9, 13]).

In practice, all applications of F-transform are designed in a discrete form; therefore, the importance of discrete (higher degree) F-transform increases, when one solves practical tasks. Besides a discretization of integral formulas used in the computation of the continuous F-transform (see, e.g., [12]), it seems to be reasonable to apply the discrete F-transform of higher degree in the form as has been introduced in [4]. Recall that if p is a real function defined on a finite set $D = \{t_i \in [a, b] \mid i = 1, \dots, n, t_i < t_{i+1}\}$, then the k -th component of the direct F-transform of degree m ($m \in \mathbb{N}$) of p with respect to a fuzzy partition $\{A_k \mid k = 1, \dots, \ell\}$ of D is a polynomial

$$F_k^m[p](t) = C_{k,0} + C_{k,1}(t - c_k) + \dots + C_{k,m}(t - c_k)^m,$$

where c_k denotes the node of the k -th basic function, and

$$(C_{k,0}, \dots, C_{k,m})^T = (\mathbf{X}_k^T \mathbf{A}_k \mathbf{X}_k)^{-1} \mathbf{X}_k^T \mathbf{A}_k \mathbf{Y} \quad (1)$$

with

$$\mathbf{X}_k = \begin{bmatrix} 1 & t_1 - c_k & \dots & (t_1 - c_k)^m \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_n - c_k & \dots & (t_n - c_k)^m \end{bmatrix},$$

$\mathbf{A}_k = \text{diag}\{A_k(t_1), \dots, A_k(t_n)\}$ and $\mathbf{Y} = (p(t_1), \dots, p(t_n))^T$. The reconstruction, providing an approximation of the original function p , is then given by the linear like combinations

$$F^m[p](t_i) = \sum_{k=1}^{\ell} F_k^m[p](t_i) \cdot A_k(t_i). \quad (2)$$

One of the disadvantages of the presented approach is the setting of a fuzzy partition to ensure the invertibility of $\mathbf{X}_k^T \mathbf{A}_k \mathbf{X}_k$ for any k . Indeed, each setting of a fuzzy partition has to control that the number of elements in which each basic function gives a non-zero value is greater than or equal to $m + 1$. Another disadvantage of the presented approach is the computation of the inverse matrix in (1) for each k , if discrete functions are defined over non-uniformly distributed elements and the nodes of a fuzzy partition do not coincide with some of them. Obviously, these updates make the computation of discrete higher degree F-transform more time consuming, especially, if the number of basic functions is large, which appears in the case of higher dimensions.

The recent theory of the higher degree F-transform is mostly developed in the continuous design [1, 2, 8]. Particularly, we proposed in [1] an efficient approach to the computation of the direct higher degree F-transform components based on various bases of polynomials. The aim of this paper is to introduce a novel (alternative) approach to the computation of a discrete higher degree F-transform with the use of benefits of the continuous design, which can overcome the mentioned disadvantages. To justify the usefulness of our approach, we analyze the quality of reconstruction of the original discrete function provided by the proposed novel approach. Furthermore, we compare our novel approach

with the original approach to the computation of discrete F-transform. For the purpose of this paper, we restrict ourselves to discrete functions defined over integers. Nevertheless, the proposed technique can be simply modified for discrete functions defined over sets with uniformly distributed elements.

The paper is structured as follows. The next section provides a brief introduction to the continuous F-transform of higher degree. In Sect. 3, we introduce the novel approach to the discrete higher degree F-transform and discuss the reconstruction of discrete functions. An illustration and comparison of the novel approach and the original approach is presented in Sect. 4. The last section is a conclusion.

2 Preliminaries

Let \mathbb{N} , \mathbb{Z} , \mathbb{R} and \mathbb{C} denote the set of natural numbers, integers, reals and complex numbers, respectively.

2.1 Fuzzy Partition

Fuzzy partition is a fundamental concept of the theory of F-transform of higher degree. In this paper, we restrict our analysis to a particular type of fuzzy partitions, which is called a simple uniform fuzzy partition. This type of fuzzy partition consists of fuzzy sets, determined by a generating function and uniformly spread along the real line.

Definition 1. A real-valued function $K : \mathbb{R} \rightarrow [0, 1]$ is said to be a generating function if it is continuous, even, non-increasing on $[0, 1]$ and vanishing outside of $(-1, 1)$.

Basic examples of generating functions that are frequently used in applications of F-transform are the triangle and raised cosine functions.

Example 1. The functions $K^{tr}, K^{rc} : \mathbb{R} \rightarrow [0, 1]$ defined by

$$K^{tr}(t) = \max(1 - |t|, 0) \quad \text{and} \quad K^{rc}(t) = \begin{cases} \frac{1}{2}(1 + \cos(\pi t)), & -1 \leq t \leq 1; \\ 0, & \text{otherwise,} \end{cases}$$

for any $t \in \mathbb{R}$, are called the *triangle* and *raised cosine* generating functions, respectively.

Definition 2. Let K be a generating function, let h and r be positive real constants, and let $t_0 \in \mathbb{R}$. For any $k \in \mathbb{Z}$, let

$$A_k(t) = K\left(\frac{t - t_0 - c_k}{h}\right),$$

where $c_k = kr$. The set $\mathcal{A} = \{A_k \mid k \in \mathbb{Z}\}$ is said to be a simple uniform fuzzy partition of the real line determined by the quadruplet (K, h, r, t_0) if, for any $t \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such that $A_k(t) > 0$. The parameters h , r and t_0 are called the *bandwidth*, *shift* and *central node* of the fuzzy partition \mathcal{A} , respectively.

Since the setting of the central node has no effect on the theoretical results concerning the fuzzy transform, for the sake of simplicity, we restrict our investigation to the simple uniform fuzzy partitions with $t_0 = 0$. Moreover, we omit the reference to $t_0 = 0$ in the quadruplet (K, h, r, t_0) and simply write (K, h, r) .

2.2 Continuous F-transform of Higher Degree

Let $L^2_{loc}(\mathbb{R})$ be a set of all complex-valued functions that are square Lebesgue integrable on any closed subinterval of \mathbb{R} . As we have mentioned in Introduction, the F-transform of higher degree consists of two phases: *direct* and *inverse*. In what follows, we briefly recall their definitions in the form presented in [1, 8].

Definition 3. Let $f \in L_{loc}(\mathbb{R})$, $m \in \mathbb{N}$, and let \mathcal{A} be a simple uniform fuzzy partition of \mathbb{R} determined by the triplet (K, h, r) . The direct continuous fuzzy transform of degree m (F^m -transform) of f with respect to \mathcal{A} is the family

$$F^m_{\mathcal{A}}[f] = \{F^m_k[f] \mid k \in \mathbb{Z}\}$$

where, for any $k \in \mathbb{Z}$,

$$F^m_k[f](t) = C_{k,0} + C_{k,1} \left(\frac{t - c_k}{h}\right) + \dots + C_{k,m} \left(\frac{t - c_k}{h}\right)^m, \quad t \in [c_k - h, c_k + h],$$

such that

$$(C_{k,0}, C_{k,1}, \dots, C_{k,m})^T = (\mathcal{Z}_m)^{-1} \cdot \mathcal{Y}_{m,k} \tag{3}$$

with $\mathcal{Z}_m = (Z_{ij})$ is the $(m + 1) \times (m + 1)$ invertible matrix defined by

$$Z_{ij} = \int_{-1}^1 t^{i+j-2} K(t) dt, \quad i, j = 1, \dots, m + 1,$$

and $\mathcal{Y}_{m,k} = (Y_{k,1}, \dots, Y_{k,m+1})^T$ is defined by

$$Y_{k,\ell} = \int_{-1}^1 f(h \cdot t + c_k) \cdot t^{\ell-1} K(t) dt, \quad \ell = 1, \dots, m + 1. \tag{4}$$

The polynomial $F^m_k[f]$ is called the k -th component of the direct continuous F^m -transform of f .

Note that the k -th component $F^m_k[f]$ receives the interval $[c_k - h, c_k + h]$ as its support, so $F^m_k[f](t)$ is not defined for $t \notin [c_k - h, c_k + h]$. From the linearity property that holds for the Lebesgue integral, it is easy to see that the direct F^m -transform satisfies the linearity property, i.e.,

$$F^m_{\mathcal{A}}[a \cdot f + b \cdot g] = a \cdot F^m_{\mathcal{A}}[f] + b \cdot F^m_{\mathcal{A}}[g],$$

for any $a, b \in \mathbb{C}$ and $f, g \in L^2_{loc}(\mathbb{R})$. Moreover, the direct F^m -transform naturally preserves polynomials up to degree m . This fact is stated in the following lemma.

Lemma 1. *Let P be a polynomial of degree n , $n \in \mathbb{N}$, and let $F_{\mathcal{A}}^m[P] = \{F_k^m[P] \mid k \in \mathbb{Z}\}$ be the direct F^m -transform of P with respect to a simple uniform fuzzy partition of \mathbb{R} . If $m \geq n$, then, for any $k \in \mathbb{Z}$, it holds that*

$$F_k^m[P](t) = P(t), \quad t \in [c_k - h, c_k + h].$$

Proof: See [1] or [8].

Definition 4. *Let $f \in L_{loc}^2(\mathbb{R})$, and let $\mathcal{A} = \{A_k \mid k \in \mathbb{Z}\}$ be a simple uniform fuzzy partition of \mathbb{R} determined by the triplet (K, h, r) . Let the family $F_{\mathcal{A}}^m[f] = \{F_k^m[f] \mid k \in \mathbb{Z}\}$ be the direct F^m -transform of f with respect to \mathcal{A} . The inverse continuous fuzzy transform of degree m (F^m -transform) of f with respect to $F_{\mathcal{A}}^m[f]$ and \mathcal{A} is defined by*

$$\hat{f}_{\mathcal{A}}^m(t) = \frac{\sum_{k \in \mathbb{Z}} F_k^m[f](t) \cdot A_k(t)}{\sum_{z \in \mathbb{Z}} A_k(t)}, \quad t \in \mathbb{R}. \tag{5}$$

By the linearity property of the direct F^m -transform, one can simply demonstrate that the inverse F^m -transform also satisfies the linearity property. Additionally, the inverse F^m -transform approximates the original function f , where the quality of the approximation is controlled mainly by the setting of the bandwidth parameter h . The details can be found in [1, 8, 11].

In the next part, we assume that a simple uniform fuzzy partition \mathcal{A} determined by a triplet (K, h, r) is fixed. Moreover, if no confusion can appear, we simply write the (direct or inverse) F^m -transform of a function f , whereas the reference to a simple uniform fuzzy partition \mathcal{A} determined by a triplet (K, h, r) is omitted.

3 Higher Degree Fuzzy Transform for Discrete Functions

3.1 Novel Definitions of the Discrete F^m -transform

Let $p : \mathbb{Z} \rightarrow \mathbb{C}$ be a complex-valued discrete function defined on the set of all integers.¹ Note that we chose the set of integers for a simple description of our approach, but the same idea can be applied also for an arbitrary discrete set. Let us extend the function p to a piecewise constant function \bar{p} defined on the real line \mathbb{R} as follows. For any $t \in \mathbb{R}$, we define

$$\bar{p}(t) = p(z) \text{ if and only if } t \in [z - 1/2, z + 1/2).$$

An example of the extension of discrete function p is depicted in Fig. 1. Since piecewise constant functions belong to the linear space $L_{loc}^2(\mathbb{R})$, the continuous F^m -transform, which has been defined in the previous section, can be directly applied to them.

The introduced conversion from the discrete to the continuous space is the core of our novel approach to the higher degree fuzzy transform for discrete functions. The following definition introduces the direct F^m -transform of a discrete function.

¹ Complex-valued functions are frequently used in analysis of stochastic processes or signal processing (see [14]).

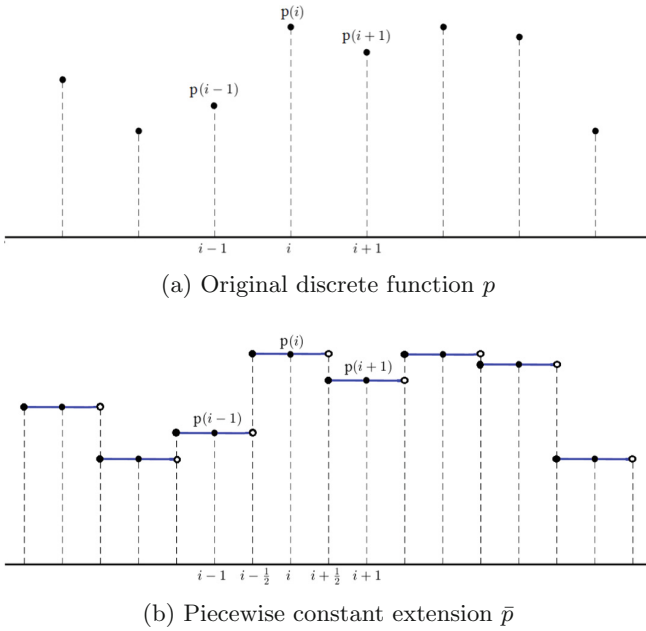


Fig. 1. The extension of a discrete function.

Definition 5. Let $p : \mathbb{Z} \rightarrow \mathbb{C}$ be a complex-valued discrete function, $m \in \mathbb{N}$, and let \mathcal{A} be a simple uniform fuzzy partition of \mathbb{R} determined by the triplet (K, h, r) . The direct discrete F^m -transform of p with respect to \mathcal{A} is defined as follows:

$$F_{\mathcal{A}}^m[p] = \{F_k^m[\bar{p}] \mid k \in \mathbb{Z}\}, \tag{6}$$

where \bar{p} is the extension of p defined above.

By the previous definition, the components of direct discrete F^m -transform of a function p with respect to a fuzzy partition can be simply computed using the product of matrices introduced in Definition 3. As a consequence of our conversion to the continuous case the verification of the invertibility of matrices in (1) is no longer required. Moreover, in spite of using integrals in the computation, the speed of the computation of the novel approach is completely comparable with the original. Indeed, the matrix \mathcal{Z}_m in formula (3) is fixed. Moreover, assuming that the parameters h and r of a given fuzzy partition are natural numbers, the components of vector $\mathcal{Y}_{m,k} = (Y_{k,1}, \dots, Y_{k,m+1})^T$ can be simply obtained as the product $Y_{k,\ell} = \mathcal{P}_k \cdot \mathcal{I}_{\ell}$, where

$$\mathcal{P}_k = (p(c_k - h), p(c_k - h + 1), \dots, p(c_k), \dots, p(c_k + h - 1), p(c_k + h))$$

and $\mathcal{I}_{\ell} = (I_{\ell,1}, \dots, I_{\ell,2h+1})^T$ is determined by

$$I_{\ell,j} = \int_{a_{j-1}}^{a_j} t^{\ell-1} K(t) dt, \quad j = 1, \dots, 2h + 1,$$

where $a_0 = -1, a_{2h+1} = 1$ and $a_j = -1 + \frac{2(j-1)+1}{2h}, j = 1, \dots, 2h$. Since the matrix \mathcal{I}_ℓ is independent on the choice of k and the indefinite integral of $t^{\ell-1}K(t)$ can be found for the standard generating functions, the derivation of vector $\mathcal{Y}_{m,k}$ can be obtained in a very short time, which consequently accelerates the computation of the direct discrete F^m -transform.

The inverse F^m -transform of a discrete function is analogously defined as in Definition 4. The only difference is the use of \mathbb{Z} instead of \mathbb{R} as the domain of reconstructed functions.

Definition 6. Let $p : \mathbb{Z} \rightarrow \mathbb{C}$ be a complex-valued discrete function, and let $\mathcal{A} = \{A_k \mid k \in \mathbb{Z}\}$ be a simple uniform fuzzy partition of \mathbb{R} determined by the triplet (K, h, r) . Let the family $F_{\mathcal{A}}^m[p] = \{F_k^m[p] \mid k \in \mathbb{Z}\}$ be the direct discrete F^m -transform of p with respect to \mathcal{A} . The inverse discrete F^m -transform of p with respect to $F_{\mathcal{A}}^m[p]$ and \mathcal{A} is determined as follows:

$$\hat{p}_{\mathcal{A}}^m(z) = \frac{\sum_{k \in \mathbb{Z}} F_k^m[p](z) \cdot A_k(z)}{\sum_{k \in \mathbb{Z}} A_k(z)}, \quad z \in \mathbb{Z}. \tag{7}$$

Obviously, the direct and inverse discrete F^m -transform preserves the linearity property similarly to the continuous case.

3.2 Estimation of the Reconstruction Error

Let $p : \mathbb{Z} \rightarrow \mathbb{C}$ be a discrete complex-valued function. We use $\|c\|$ to denote the size of the complex number c , i.e. $\|c\| = \sqrt{c \cdot \bar{c}}$, where \bar{c} is the complex conjugate of c . Let $z_0 \in \mathbb{Z}$ and $\delta > 0$. Then, the value

$$\omega_{z_0}(p, \delta) = \sup \{ \|\bar{p}(z_0) - \bar{p}(z_0 + \varepsilon)\| \mid \varepsilon \in \mathbb{R}, |\varepsilon| \leq \delta \}, \tag{8}$$

provides us a measure of how much the function values $p(z)$ differ from each other in a neighborhood of z_0 . Obviously, formula (8) imitates the definition of modulus of continuity. If p is bounded, then we define

$$\omega(p, \delta) = \sup \{ \omega_z(p, \delta) \mid z \in \mathbb{Z} \}$$

which measures the changes in the shape of function p with respect to the parameter δ .

Below, we consider the quality of reconstruction of a given function by the proposed discrete F^m -transform. First, we have to evaluate how the components of the direct F^m -transform are locally close (i.e., close in specific neighborhoods) to the original function.

Theorem 1. Let $p : \mathbb{Z} \rightarrow \mathbb{C}$ be a complex-valued discrete function. Let $\mathcal{A} = \{A_k \mid k \in \mathbb{Z}\}$ be a simple uniform fuzzy partition of \mathbb{R} determined by the triplet (K, h, r) , and let $F_{\mathcal{A}}^m[p]$ be the direct discrete F^m -transform of p with respect to \mathcal{A} . Then, for any $k \in \mathbb{Z}$, it holds that

$$\|p(z) - F_k^m[p](z)\| \leq \omega_z(p, 2h) \cdot \Theta(m, K),$$

for any $z \in \mathbb{Z}$ such that $z \in [c_k - h, c_k + h]$, where

$$\Theta(m, K) = \sum_{j,\ell=1}^{m+1} |V_{j\ell}| \cdot \int_{-1}^1 |t|^{\ell-1} K(t) dt$$

with $(V_{j\ell})_{j,\ell=1,\dots,m+1} = (\mathcal{Z}_m)^{-1}$.

Proof: Let $k \in \mathbb{Z}$, and let $z \in \mathbb{Z}$ such that $z \in [c_k - h, c_k + h]$. Since $F_k^m[c](t) = c$ holds for any complex-valued constant function c , we obtain

$$\begin{aligned} \|p(z) - F_k^m[p](z)\| &= \|F_k^m[p](z) - F_k^m[\bar{p}](z)\| = \\ &= \left\| (C_{k,0} - D_{k,0}) + \dots + (C_{k,m} - D_{k,m}) \left(\frac{t - c_k}{h} \right)^m \right\| \leq \\ &= \sum_{j=0}^m \|C_{k,j} - D_{k,j}\| \cdot \left| \frac{t - c_k}{h} \right|^j \leq \sum_{j=0}^m \|C_{k,j} - D_{k,j}\|, \end{aligned} \quad (9)$$

where we used $|(t - c_k)/h| \leq 1$, and $(C_{k,0}, \dots, C_{k,m})$ and $(D_{k,0}, \dots, D_{k,m})$ are determined by

$$(C_{k,0}, \dots, C_{k,m})^T = (\mathcal{Z}_m)^{-1} \cdot \mathcal{Y}_{m,k}, \quad (10)$$

$$(D_{k,0}, \dots, D_{k,m})^T = (\mathcal{Z}_m)^{-1} \cdot \mathcal{W}_{m,k} \quad (11)$$

with $\mathcal{Y}_{m,k} = (Y_{k,j})_{j=1,\dots,m+1}$ and $\mathcal{W}_{m,k} = (W_{k,j})_{j=1,\dots,m+1}$, which are the column matrices given by

$$Y_{k,j} = \int_{-1}^1 p(z) \cdot t^{j-1} K(t) dt, \quad \text{and} \quad W_{k,j} = \int_{-1}^1 \bar{p}(h \cdot t + c_k) \cdot t^{j-1} K(t) dt.$$

From (10) and (11), we obtain

$$(C_{k,0} - D_{k,0}, \dots, C_{k,m} - D_{k,m})^T = (\mathcal{Z}_m)^{-1} \cdot (\mathcal{Y}_{m,k} - \mathcal{W}_{m,k}).$$

Hence, for any $j = 0, \dots, m$, we find that

$$\begin{aligned} \|C_{k,j} - D_{k,j}\| &\leq \sum_{\ell=1}^{m+1} |V_{j+1\ell}| \cdot \|Y_{z\ell} - W_{z\ell}\| \\ &= \sum_{\ell=1}^{m+1} |V_{j+1\ell}| \cdot \int_{-1}^1 \|p(z) - \bar{p}(h \cdot t + c_k)\| \cdot |t|^{\ell-1} K(t) dt \\ &= \sum_{\ell=1}^{m+1} |V_{j+1\ell}| \cdot \int_{-1}^1 \|\bar{p}(z) - \bar{p}(h \cdot t + c_k)\| \cdot |t|^{\ell-1} K(t) dt \\ &\leq \omega_z(p, 2h) \cdot \sum_{\ell=1}^{m+1} |V_{j+1\ell}| \cdot \int_{-1}^1 |t|^{\ell-1} K(t) dt, \end{aligned}$$

where we used $\|\bar{p}(z) - \bar{p}(h \cdot t + c_k)\| \leq \omega_z(p, 2h)$. By this inequality and (9), we obtain

$$\begin{aligned} \|p(z) - F_k^m[p](z)\| &\leq \omega_z(p, 2h) \cdot \sum_{j=0}^m \sum_{\ell=1}^{m+1} |V_{j+\ell}| \cdot \int_{-1}^1 |t|^{\ell-1} K(t) dt \\ &\leq \omega_z(p, 2h) \cdot \sum_{j,\ell=1}^{m+1} |V_{j\ell}| \cdot \int_{-1}^1 |t|^{\ell-1} K(t) dt \\ &= \omega_z(p, 2h) \cdot \Theta(m, K), \end{aligned}$$

and the proof is finished. □

The upper bound of the error of reconstruction of a function by its inverse discrete F^m -transform is established in the following theorem.

Theorem 2. *Let $p : \mathbb{Z} \rightarrow \mathbb{C}$ be a complex-valued discrete function. Let $\mathcal{A} = \{A_k \mid k \in \mathbb{Z}\}$ be a simple uniform fuzzy partition of \mathbb{R} determined by the triplet (K, h, r) . Let $\hat{p}_{\mathcal{A}}^m$ be the inverse discrete F^m -transform of p with respect to $F_{\mathcal{A}}^m[p]$ and \mathcal{A} . Then,*

$$\|p(z) - \hat{p}_{\mathcal{A}}^m(z)\| \leq \omega_z(p, 2h) \cdot \Theta(m, K), \tag{12}$$

for any $z \in \mathbb{Z}$, where $\Theta(m, K)$ is defined in Theorem 1.

Proof: Let $F_{\mathcal{A}}^m[p] = \{F_k^m[p] \mid k \in \mathbb{Z}\}$ be the direct F^m -transform of p with respect to \mathcal{A} . For any $z \in \mathbb{Z}$, we have

$$\begin{aligned} \|p(z) - \hat{p}_{\mathcal{A}}^m(z)\| &= \left\| p(z) - \frac{\sum_{k \in \mathbb{Z}} F_k^m[p](z) \cdot A_k(z)}{\sum_{k \in \mathbb{Z}} A_k(z)} \right\| \\ &= \left\| \frac{\sum_{k \in \mathbb{Z}} (p(z) - F_k^m[p](z)) \cdot A_k(z)}{\sum_{k \in \mathbb{Z}} A_k(z)} \right\| \leq \frac{\sum_{k \in \mathbb{Z}} |p(z) - F_k^m[p](z)| \cdot A_k(z)}{\sum_{k \in \mathbb{Z}} A_k(z)}. \end{aligned}$$

It follows from Theorem 1 that

$$\|p(z) - F_k^m[p](z)\| \leq \omega_z(p, 2h) \cdot \Theta(m, K),$$

for any $k \in \mathbb{Z}$. Consequently,

$$\|p(z) - \hat{p}_{\mathcal{A}}^m(z)\| \leq \frac{\sum_{k \in \mathbb{Z}} \omega_z(p, 2h) \cdot \Theta(m, K) \cdot A_k(z)}{\sum_{k \in \mathbb{Z}} A_k(z)} = \omega_z(p, 2h) \cdot \Theta(m, K)$$

and the proof is finished. □

The following corollary is a straightforward consequence of Theorem 2 in the case that the original discrete function is bounded.

Corollary 1. *Let the assumptions of Theorem 2 be satisfied. If the function p is bounded, then*

$$\|p(z) - \hat{p}_{\mathcal{A}}^m(z)\| \leq \omega(p, 2h) \cdot \Theta(m, K),$$

where $\Theta(m, K)$ is defined in Theorem 1.

It is easy to see from (12) that for the bandwidth $h \leq 1/4$, we obtain $\omega_z(p, 2h) = 0$ for any $z \in \mathbb{Z}$; hence, the reconstruction of p is ideal. Note that, in practice, the bandwidth is chosen higher than $1/4$ with respect to specific tasks.

4 Illustration Examples

In this section, we illustrate the novel approach to the discrete F^m -transform on functions representing two time series. Particularly, we compare the inverse F^m -transform functions obtained by the newly defined discrete F^m -transform with that provided by the original approach presented in [4]. For the comparison, we use the MAPE and the time-consumption in computation for both approaches.

First, we consider the time series data “Monthly closings of the Dow–Jones industrial index, Aug. 1968–Aug. 1981” stored on the website <http://datamarket.com> that form a discrete function p with the domain $\{1, 2, \dots, 291\}$. In Fig. 2, we display the inverse discrete F^2 - and F^5 -transform of p , obtained by the proposed approach, with respect to the simple uniform fuzzy partition determined by the triplet $(K^{tr}, 20, 10)$. Note that the results obtained by the novel and the standard approach are negligible; particularly, the differences between them are $\text{MAPE} = 2.1137 \times 10^{-5}$ for the F^2 -transform and $\text{MAPE} = 5.7637 \times 10^{-5}$ for the F^5 -transform. In Fig. 3, we depict the newly defined inverse discrete

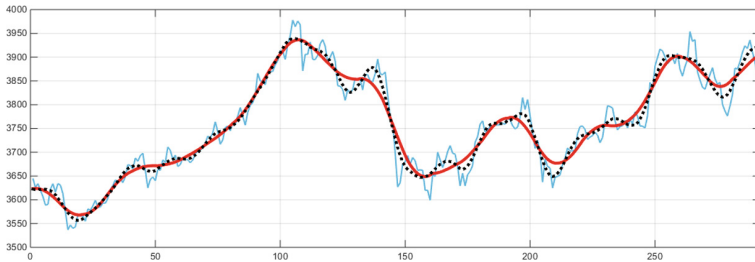


Fig. 2. The inverse F^2 - (red line) and F^5 -transform (dotted black line). (Color figure online)

F^3 -transform of p with respect to varying fuzzy partitions \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 determined by the triplets $(K^{tr}, 20, 10)$, $(K^{tr}, 10, 5)$ and $(K^{tr}, 1, 1)$, respectively. Among others, the results presented in Figs. 2 and 3 demonstrate the fact, which is well-known for the continuous F^m -transform, saying that a better reconstruction of the original function may be attained either by shortening the length of bandwidth h , if it is possible, or by enlarging the degree of the F -transform.

Below, we compare the time-consumption of the novel and the original approach in computation of inverse F^m -transform. For the comparison, we choose a long time series “Daily minimum temperatures in Melbourne, Australia, 1981–1990” with the dimension $d = 3650$, stored on the website <http://datamarket.com>. The both approaches to the discrete F^m -transform are programmed by

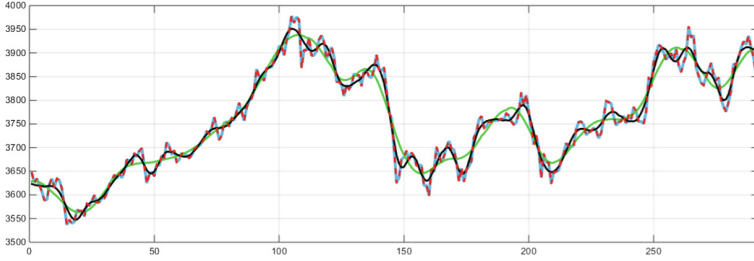


Fig. 3. The inverse F^3 -transform functions (green, black and dashed red lines) with respect to \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 , respectively. (Color figure online)

Matlab 2014 on the notebook with CPU Intel CoreTM i5-3320M, Ram 8Gb and OS Windows 10. The computation time is also measured by Matlab. The considered simple uniform fuzzy partitions were determined by triplets $(K^{tr}, h, h/2)$, where h varies from 20 to 100 with the step 20. From the results depicted in Table 1, one can see that the computational times for both approaches are low and very similar, which is a consequence of our restriction to discrete functions defined over uniformly distributed elements (integers). This restriction actually enables us to optimize and speed up the algorithm of the original approach. Moreover, one can see that the computation time depends on the size of the bandwidth.

Table 1. Time-consumption (second) in computation of inverse F^m -transform.

F^m -transform \	Bandwidth	$h = 20$	$h = 40$	$h = 60$	$h = 80$	$h = 100$
F^0	Original	0.11479	0.07451	0.06506	0.05151	0.04940
	Novel	0.11826	0.07386	0.05214	0.04461	0.04081
F^1	Original	0.14210	0.08974	0.07528	0.06187	0.05556
	Novel	0.14657	0.08249	0.06088	0.05216	0.04426
F^2	Original	0.16912	0.09855	0.08097	0.06433	0.06021
	Novel	0.16797	0.09163	0.07021	0.05520	0.04861
F^3	Original	0.27800	0.16698	0.12056	0.09538	0.08498
	Novel	0.28627	0.16790	0.11868	0.09233	0.07524

5 Conclusions

In this paper, we introduced a novel approach to the discrete fuzzy transform of higher degree. We analyzed the quality of the reconstruction of an original discrete function provided by the inverse discrete F^m -transform. We compared the novel approach with the original one proposed in [4] on two examples of time series. The restriction to the discrete functions defined over uniformly distributed

elements (integers) results in similar computation times for both approaches. The computation times are low in all tests and naturally depend on sizes of the bandwidth. Intuitively, the novel approach should provide better time consumption in computation for discrete functions defined over non-uniformly distributed values, because, in contrast to the novel approach, the original one needs to recompute the inverse matrices to get the respective F-transform components. A verification of our conjecture is a subject of our future research.

Acknowledgments. This work was supported by the project LQ1602 IT4Innovations excellence in science. The additional support was also provided by the Czech Science Foundation through the project of No. 16-09541S.

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