Study of the Relevance of Objects and Attributes of *L*-fuzzy Contexts Using Overlap Indexes

Cristina Alcalde[1](✉) and Ana Burusco[2]

[1] Department of Applied Mathematics, University of the Basque Country - UPV/EHU, Plaza de Europa 1, 20018 San Sebastián, Spain
c.alcalde@ehu.eus

[2] Departamento de Automática y Computación, Institute of Smart Cities, Universidad Pública de Navarra, Campus de Arrosadía, 31006 Pamplona, Spain
burusco@unavarra.es

Abstract. Objects and attributes play an important role in an L-fuzzy context. From the point of view of the L-fuzzy concepts, some of them can be more relevant than others. Besides, the number of objects and attributes of the L-fuzzy context is one of the most important factors that influence in the size of the L-fuzzy concept lattice. In this paper, we define different rankings for the objects and the attributes according to their relevance in the L-fuzzy concept lattice and using different overlap indexes. These rankings can be useful for the reduction of the L-fuzzy context size.

Keywords: L-fuzzy contexts · L-fuzzy concepts · Overlap indexes

1 Introduction

The L-fuzzy concept analysis [15–17] is a theory that studies the information arising from an L-fuzzy context using the L-fuzzy concepts as tools. An L-fuzzy context is a tuple (L, X, Y, R) with L a complete lattice, X and Y sets of objects and attributes respectively and $R \in L^{X \times Y}$ a fuzzy relation between the objects and the attributes. An L-fuzzy concept is a pair of L-fuzzy sets that can be interpreted as a group of elements (objects) that shares some characteristics (attributes). The set of these L-fuzzy concepts has the structure of complete lattice.

Most of the times, not all the objects neither the attributes have the same relevance from the point of view of the L-fuzzy concepts. Some of them hardly play any role in the L-fuzzy concepts.

Besides, when the cardinality of this L-fuzzy concept lattice is large, the obtained result may not be easy to handle. One of the factors that determines the size of the L-fuzzy concept lattice is the cardinality of the lattice L. The other is the size of the L-fuzzy context. The latter is analyzed in this paper.

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Over the past, several researchers have developed models in order to reduce the size of this lattice. In [13], Bělohlávek and Vychodil use hedges to control the size of the concept lattice. Also Wei and Qi [34], and Medina [27] have published works from the point of view of the attributes for fuzzy oriented concept lattices. Other methods to reduce the complexity of the lattice using fuzzy similarity [14] or block relations [24] have also been developed.

In [8] we have study the possibility of aggregating rows or columns of the L-fuzzy context. Sometimes, the L-fuzzy context values are independent and we can use usual aggregations as weighted means [20,21], OWA operators [22,33] and WOWA operators [31]. However, these studies are incomplete when we have values that present dependencies among them. In these situations the use of Choquet integrals [23] can be very useful as a tool for doing a proper analysis without lost of information as can be seen in [8].

This paper addresses the study of the objects and attributes of the L-fuzzy context when $L = [0,1]$. We define different rankings for both sets according to their relevance in the L-fuzzy concept lattice and using different overlap indexes. These rankings will allow us to decide which objects and which attributes are less relevant. They will be the candidates for the elimination.

First, we are going to remember some important results about L-Fuzzy concept analysis and overlap indexes [19].

1.1 *L*-Fuzzy Concept Analysis

The Wille's Formal Concept Analysis [32] extracts information from a binary table that represents a formal context (X, Y, R) with X and Y finite sets of objects and attributes respectively and $R \subseteq X \times Y$. The hidden information consists of pairs (A, B) with $A \subseteq X$ and $B \subseteq Y$, called formal concepts, verifying $A^* = B$ and $B^* = A$, where $(\cdot)^*$ is the derivation operator that associates the attributes related to the elements of A to every object set A, and the objects related to the attributes of B to every attribute set B. These formal concepts can be interpreted as a group of objects A that shares the attributes of B.

In previous works [15,16] we have defined the L-fuzzy contexts (L, X, Y, R), with L a complete lattice, X and Y sets of objects and attributes respectively and $R \in L^{X \times Y}$ a fuzzy relation between the objects and the attributes. This is an extension of Wille's formal contexts to the fuzzy case when we want to study the relations between the objects and the attributes with values in a complete lattice L, instead of binary ones.

In our case, to work with these L-fuzzy contexts, we have defined the derivation operators 1 and 2 given by means of these expressions [16,17]:

$$\forall A \in L^X, \forall B \in L^Y, x \in X, y \in Y:$$
$$A_1(y) = \inf_{x \in X}\{I(A(x), R(x,y))\}$$
$$B_2(x) = \inf_{y \in Y}\{I(B(y), R(x,y))\}$$

with I a fuzzy implication operator defined in $(L, \leq)$.

The information stored in the context is visualized by means of the L-fuzzy concepts that are pairs $(A, A_1) \in L^X \times L^Y$ with $A \in fix(\varphi)$, set of fixed points of the operator φ, being defined from the derivation operators 1 and 2 as $\varphi(A) = (A_1)_2 = A_{12}$. These pairs, whose first and second components are said to be the fuzzy extension and intension respectively, represent a group of objects that share a group of attributes.

Using the usual order relation between fuzzy sets, that is, $\forall A, C \in L^X, A \leq C \iff A(x) \leq C(x), \forall x \in X$, we define the set $\mathcal{L} = \{(A, A_1) \mid A \in fix(\varphi)\}$ with the order relation $\preceq$ defined as: $\forall (A, A_1), (C, C_1) \in \mathcal{L}, (A, A_1) \preceq (C, C_1)$ if $A \leq C$ (or $A_1 \geq C_1$).

As φ is an order preserving operator, then the set $fix(\varphi)$ is a complete lattice and $(\mathcal{L}, \preceq)$ is also a complete lattice that is said to be the L-fuzzy concept lattice [15,16].

In addition, in the case of using a residuated implication ($I(a, b) = \sup\{x \mid T(a, x) \leq b\}$, with T a t-norm), given $A \in L^X$, (or $B \in L^Y$) we can obtain the associated L-fuzzy concept applying twice the derivation operators. In this case, the L-fuzzy concept associated to A is (A_{12}, A_1) (or (B_2, B_{21})). In the paper, residuated implication of Łukasiewicz will be used for the practical case.

Our last results are related to the use of two relations in the definition of the L-fuzzy context [2], the study of fuzzy context sequences [3,4,6,7] or the composition of L-fuzzy contexts [5]. We have also developed this theory in different areas as the treatment of incomplete information [1,10] or Mathematical Morphology [9].

Other important works that generalize the Formal Concepts Analysis using residuated implication operators are due to Bělohlávek [11,12] and Pollandt [28]. Moreover, extensions of Formal Concept Analysis to the interval-valued case are in [1,29,30] and to the fuzzy property-oriented and multi-adjoint concept lattices framework in [25,26].

1.2 Overlap Indexes

We start by recalling some basic notions about the idea of an overlap index.

Given a referential set U and $L = [0, 1]$, let L^U be the fuzzy sets of U. Bustince [19] define an overlap index as a mapping $O : L^U \times L^U \longrightarrow [0, 1]$, such that:

(i) $O(A, B) = 0$ if and only if in A and B have disjoint supports; that is, $A(i)B(i) = 0$ for every $i \in U$, and $A, B \in L^U$.
(ii) $O(A, B) = O(B, A), for every A, B \in L^U$.
(iii) If $B \subseteq C$, then $O(A, B) \leq O(A, C)$, for every $A, B, C \in L^U$.

An overlap index such that:

(iv) $O(A, B) = 1$ if there exists $i \in U$ such that $A(i) = B(i) = 1$ is called a normal overlap index.

Examples of overlap indexes are the following ones:

(1) Zadeh's consistency index:

$$O_Z(A,B) = \max_{i=1}^{n}(min(A(i),B(i)))$$

(2) Let $M : [0,1]^2 \longrightarrow [0,1]$ be a symmetric aggregation function such that $M(x,y) = 0$ if and only if $xy = 0$. We have that:

$$O_{M,Z}(A,B) = \max_{i=1}^{n}(M(A(i),B(i)))$$

is a normal overlap index that generalizes the Zadeh's index.

(3) If in the previous example, we consider a symmetric, increasing function $M : [0,1]^2 \longrightarrow [0,1]$ such that $M(1,1) < 1$ and $M(x,y) = 0$ if and only if $xy = 0$, then we obtain an overlap index which is not normal. For instance, when taking $M(x,y) = (xy)^p/2$ with $p > 0$, we arrive at the overlap index:

$$O(A,B) = \max_{i=1}^{n}\left(\frac{(A(i),B(i))^p}{2}\right)$$

(4) The following is also an example of overlap index:

$$O_\pi(A,B) = \frac{1}{n}\sum_{i=1}^{n}A(i)B(i)$$

Remark 1. Formally, overlap indexes can be seen as generalized measures of fuzzy intersection of considered fuzzy sets.

Let $E \in L^U$ be a fixed non-empty fuzzy set. Given $A \subseteq U$, we define:

$$E_A(i) = \begin{cases} E(i) & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}$$

Observe that E_A is the intersection of the fuzzy set E and the crisp set A.

Now we are ready to introduce the definition of a fuzzy measure in terms of a fuzzy set and an overlap index.

Theorem 1 *[19]. If $E \in L^U$ is a fixed, non-empty fuzzy set, then the mapping $m_{O,E} : U \longrightarrow [0,1]$ given by:*

$$m_{O,E}(A) = \frac{O(E,E_A)}{O(E,E)}$$

is a fuzzy measure for every overlap index O.

2 Ranking of Objects and Attributes Using Overlap Indexes

As it has been explained in the introduction, it is interesting to establish a ranking in the object and attribute sets from the point of view of the L-fuzzy concepts. Moreover the size of an L-fuzzy context is one of the factors that determines the size of the L-fuzzy concept lattice and its manageability. The possible reduction of the size of the object and attribute sets is an interesting problem of study.

So far, some of our research lines in relation to reducing the size of the L-fuzzy context have used the method of removing rows or columns in the relation (eliminating objects or attributes). In [18] we removed the objects and/or attributes of little significance, that is, that did not appear as relevant in any L-fuzzy concept. To do this, we first obtained the L-fuzzy concept lattice, a quite laborious task.

In another different field and in order to work with missing values, in [10] infrequently appearing objects and attributes were studied. We removed them when they did not exceed a minimum support. To do this, we defined support of an L-fuzzy set $A \in L^Z$ as $supp(A) = \sum_{z \in Z} A_1(z)/|Z|$. The aim was to eliminate some rows or columns of missing values.

This definition allow us to assign a support value to every object (or attribute). To do this, for every $x_i \in X, i \in \{1, \dots, n\}$, let $\mathbf{x_i}$ be the L-fuzzy set defined by the characteristic function $\mathbf{x_i}(x_i) = 1$ and $\mathbf{x_i}(x) = 0$, for any $x \neq x_i$. Analogously for $\mathbf{y_j}, j \in \{1, \dots, m\}$.

Next, we are going to define the L-fuzzy concepts associated with the objects and the attributes of the L-fuzzy context. To do this, we will use a residuated implication operator for the definition of operators denoted by the subindexes 1 and 2.

Definition 1. *For every $x_i \in X, i \in \{1, \dots, n\}$, the pair $\mathcal{C}_{\mathbf{x_i}} = ((\mathbf{x_i})_{12}, (\mathbf{x_i})_2)$ is said to be the L-fuzzy concept derived from $\mathbf{x_i}$. Analogously $\mathcal{C}_{\mathbf{y_j}} = ((\mathbf{y_j})_2, (\mathbf{y_j})_{21}), j \in \{1, \dots, m\}$ is the L-fuzzy concept derived from $\mathbf{y_j}$.*

These concepts are the closest to the departure sets represented by $\mathbf{x_i}$ or $\mathbf{y_j}$ (study of a single object or attribute).

Then, the definitions of support of an object or an attribute are:

Definition 2. *For every $x_i \in X, i \in \{1, \dots, n\}$ and $y \in Y$:*

$$supp(x_i) = \frac{\sum_{i=1}^{n} (\mathbf{x_i})_2(y)}{|Y|},$$

and for every $y_j \in Y, j \in \{1, \dots, n\}$ and $x \in X$:

$$supp(y_j) = \frac{\sum_{j=1}^{n} (\mathbf{y_j})_1(x)}{|X|}$$

Now, given a fuzzy set E and an overlap index O, we can use Theorem 1 to associate a fuzzy measure with every object and attribute: $m_{O,E}(x_i)$ and $m_{O,E}(y_j)$ are the fuzzy measures for every $x_i \in X$ and $y_j \in Y$.

These values establish a measure of the overlap between the L-fuzzy concepts derived from the L-fuzzy sets $\mathbf{x_i}$ or $\mathbf{y_j}$ and the fuzzy set E.

We can define relations between the objects and the attributes using these fuzzy measures:

Definition 3. *For every $x_i, x_j \in X$, let C_{x_i}, C_{x_j} be the L-fuzzy concepts associated with x_i, x_j. Let E be a fuzzy set and O an overlap index.*

$$x_i \geq_{O,E} x_j \ if \ m_{O,E}(x_i) \geq m_{O,E}(x_j)$$

Analogously, $y_i \geq_{O,E} y_j$ if $m_{O,E}(y_i) \geq m_{O,E}(y_j)$, for every $y_i, y_j \in Y$.

This is a preorder relation that establishes for every fuzzy set E and overlap index O, the *Object* and the *Attribute Rankings* associated with E and O.

The election of E and O are important points. In the case of E, it represents the model we want to look like. Taking into account that the support of an object can be understood as a measure or its relevance, we take $E(x_i) = supp(x_i)$ for every $x_i \in X$. From the point of view of the attributes, we take $E(y_i) = supp(y_i)$ for every $y_i \in Y$.

3 Reducing the Size of an L-Fuzzy Context by Means of the Elimination of Rows or Columns

In previous section, we have define rankings of objects and attributes taking into account different overlap indexes. These rankings order the objects (or attributes) by means of the overlap between their derived L-fuzzy concepts and the support of the objects (or attributes). Then, our proposal is the elimination of those objects and attributes that are in the last positions of those rankings.

The advantage of this method over the one described in [18] is that it is not necessary to obtain the total L-fuzzy complete lattice. In addition, we can define a model E and eliminate the objects and attributes that do not give L-fuzzy concepts close to this model. This fact improves the idea proposed in [7].

Furthermore, if we remove a row or column in the context, we can see that the L-fuzzy concepts obtained from the non modified objects or attributes have the same membership degrees.

It is not difficult to prove the following proposition:

Proposition 1. *Let (L, X, Y, R) and $(L, X\backslash\{x_0\}, Y, \bar{R})$ be L-fuzzy contexts such that $\bar{R}(x, y) = R(x, y), \forall x \in X\backslash\{x_0\}, \forall y \in Y$. Consider $x_l \in X\backslash\{x_0\}$ and let $\mathcal{C}_{\mathbf{x_l}}$ and $\bar{\mathcal{C}}_{\mathbf{x_l}}$ be the L-fuzzy derived concepts in (L, X, Y, R) and $(L, X, Y, \bar{R})$ respectively. For any $x \in X\backslash\{x_0\}$ and for any $y \in Y$, the membership degrees in both L-fuzzy concepts are coincident.*

An analogous proposition can be proved in the case of eliminating one attribute:

Proposition 2. *Let* (L, X, Y, R) *and* $(L, X, Y\backslash\{y_0\}, \bar{R})$ *be L-fuzzy contexts such that* $\bar{R}(x, y) = R(x, y), \forall x \in X, \forall y \in Y\backslash\{y_0\}$. *Consider* $y_l \in Y\backslash\{y_0\}$ *and let* $\mathcal{C}_{\mathbf{y_l}}$ *and* $\bar{\mathcal{C}}_{\mathbf{y_l}}$ *be the L-fuzzy derived concepts in* (L, X, Y, R) *and* $(L, X, Y, \bar{R})$. *For any* $y \in Y\backslash\{y_0\}$ *and for any* $x \in X$, *the membership degrees are coincident in both L-fuzzy concepts.*

4 Practical Case

Let us see below a practical case where we will apply the results obtained in the previous sections.

Suppose that we want to do a market survey about the consumption of soft drinks in some of the major cities in Spain. To do this, we have an L-Fuzzy context (L, X, Y, R) with $L = \{0, 0.1, 0.2, 0.3, ... 1\}$, the object and attribute sets X={cola1, cola2, orangeade1, orangeade2, orangeade3, lemonade1, lemonade2, lemonade3, tonic1, tonic2} (the commercial brands are avoided) and Y={Barcelona, Bilbao, Granada, Madrid, Malaga, San Sebastian, Santander, Sevilla, Valencia, Zaragoza} and $R \in L^{X\times Y}$ the L-Fuzzy relation of Table 1 that represents the consumption of soft drinks in the different cities.

The values of the table belong to L. For instance, $R(x_1, y_4) = R(\text{cola1}, \text{Madrid}) = 0.8$ means that *cola1 is consumed in large quantities in Madrid*, but *this does not hold for tonic2 in Granada* ($R(x_{10}, y_3) = R(\text{tonic2}, \text{Granada}) = 0.1$).

In the rest of the section, the objects and the attributes will be denoted by x_i and y_j, $i, j \in \{1 ... 10\}$, respectively.

Table 1. L-fuzzy context

R	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0.3	0.6	0.5	0.8	1.0	0.3	0.3	0.2	0.0	0.6
x_2	0.3	0.0	0.1	0.5	0.2	0.1	0.4	0.4	0.0	0.2
x_3	0.3	0.9	0.4	1.0	0.3	0.5	0.5	0.9	1.0	0.2
x_4	0.9	0.6	0.5	0.2	0.0	0.0	0.9	1.0	1.0	0.3
x_5	0.2	0.1	0.0	0.6	0.2	0.5	0.1	0.0	0.3	0.5
x_6	0.4	0.3	0.0	0.5	0.3	0.1	0.4	0.3	0.5	0.4
x_7	0.9	0.5	0.4	0.1	0.3	0.1	0.2	0.3	0.4	0.3
x_8	0.5	0.3	0.5	0.4	0.1	0.1	0.2	0.5	0.2	0.2
x_9	0.1	0.6	0.4	0.2	0.0	0.0	0.3	0.0	0.1	0.1
x_{10}	0.4	0.2	0.1	0.3	0.0	0.0	0.2	0.6	0.1	1.0

We are going to study which is the relationship among the objects and among the attributes. After this study, we will be able to reduce the size of the L-fuzzy context.

The construction of the whole L-fuzzy concept lattice has a high computational cost. So, for every $x_i, i \in \{1 \dots n\}$ and using the Łukasiewicz implication operator, we can obtain its derived L-fuzzy concept $\mathcal{C}_{\mathbf{x_i}}$. For instance, for $\mathbf{x_3}$ we have:

$$\mathcal{C}_{\mathbf{x_3}} = (\{x_1/0, x_2/0, x_3/1, x_4/0.2, x_5/0.1, x_6/0.4, x_7/0.1, x_8/0.2, x_9/0.1, x_{10}/0.1\}, \{y_1/0.3, y_2/0.9, y_3/0.4, y_4/1, y_5/0.3, y_6/0.5, y_7/0.5, y_8/0.9, y_9/1, y_{10}/0.2\})$$

We can say that *Orangeade1*(x_3) *is consumed mainly in Bilbao* (y_2), *Madrid* (y_4), *Sevilla* (y_8), *and Valencia* (y_9).

For every object x_i, the fuzzy extension of the derived L-fuzzy concepts are:
$\mathcal{C}_{\mathbf{x_1}} : \{x_1/1, x_2/0.2, x_3/0.3, x_4/0, x_5/0.2, x_6/0.3, x_7/0.3, x_8/0.1, x_9/0, x_{10}/0\}$
$\mathcal{C}_{\mathbf{x_2}} : \{x_1/0.8, x_2/1, x_3/1, x_4/0.7, x_5/0.6, x_6/0.9, x_7/0.6, x_8/0.8, x_9/0.6, x_{10}/0.8\}$
$\mathcal{C}_{\mathbf{x_3}} : \{x_1/0, x_2/0, x_3/1, x_4/0.2, x_5/0.1, x_6/0.4, x_7/0.1, x_8/0.2, x_9/0.1, x_{10}/0.1\}$
$\mathcal{C}_{\mathbf{x_4}} : \{x_1/0, x_2/0, x_3/0.4, x_4/1, x_5/0, x_6/0.3, x_7/0.3, x_8/0.2, x_9/0, x_{10}/0.1\}$
$\mathcal{C}_{\mathbf{x_5}} : \{x_1/0.7, x_2/0.6, x_3/0.7, x_4/0.5, x_5/1, x_6/0.6, x_7/0.5, x_8/0.6, x_9/0.5, x_{10}/0.5\}$
$\mathcal{C}_{\mathbf{x_6}} : \{x_1/0.5, x_2/0.5, x_3/0.8, x_4/0.7, x_5/0.7, x_6/1, x_7/0.6, x_8/0.7, x_9/0.6, x_{10}/0.6\}$
$\mathcal{C}_{\mathbf{x_7}} : \{x_1/0.4, x_2/0.4, x_3/0.4, x_4/0.7, x_5/0.3, x_6/0.5, x_7/1, x_8/0.6, x_9/0.2, x_{10}/0.5\}$
$\mathcal{C}_{\mathbf{x_8}} : \{x_1/0.7, x_2/0.6, x_3/0.8, x_4/0.8, x_5/0.5, x_6/0.5, x_7/0.7, x_8/1, x_9/0.5, x_{10}/0.6\}$
$\mathcal{C}_{\mathbf{x_9}} : \{x_1/0.9, x_2/0.4, x_3/1, x_4/1, x_5/0.5, x_6/0.6, x_7/0.9, x_8/0.7, x_9/1, x_{10}/0.6\}$
$\mathcal{C}_{\mathbf{x_{10}}} : \{x_1/0.6, x_2/0.2, x_3/0.2, x_4/0.3, x_5/0.4, x_6/0.4, x_7/0.3, x_8/0.2, x_9/0.1, x_{10}/1\}$

and for the attributes:
$\mathcal{C}_{\mathbf{y_1}} : \{y_1/1, y_2/0.6, y_3/0.5, y_4/0.2, y_5/0.1, y_6/0.1, y_7/0.3, y_8/0.4, y_9/0.5, y_{10}/0.4\}$
$\mathcal{C}_{\mathbf{y_2}} : \{y_1/0.4, y_2/1, y_3/0.5, y_4/0.6, y_5/0.4, y_6/0.4, y_7/0.6, y_8/0.4, y_9/0.4, y_{10}/0.3\}$
$\mathcal{C}_{\mathbf{y_3}} : \{y_1/0.7, y_2/0.8, y_3/1, y_4/0.7, y_5/0.5, y_6/0.5, y_7/0.7, y_8/0.6, y_9/0.5, y_{10}/0.7\}$
$\mathcal{C}_{\mathbf{y_4}} : \{y_1/0.3, y_2/0.5, y_3/0.4, y_4/1, y_5/0.3, y_6/0.5, y_7/0.5, y_8/0.4, y_9/0.2, y_{10}/0.2\}$
$\mathcal{C}_{\mathbf{y_5}} : \{y_1/0.3, y_2/0.6, y_3/0.5, y_4/0.8, y_5/1, y_6/0.3, y_7/0.3, y_8/0.2, y_9/0, y_{10}/0.6\}$
$\mathcal{C}_{\mathbf{y_6}} : \{y_1/0.7, y_2/0.6, y_3/0.5, y_4/1, y_5/0.7, y_6/1, y_7/0.6, y_8/0.5, y_9/0.7, y_{10}/0.7\}$
$\mathcal{C}_{\mathbf{y_7}} : \{y_1/0.8, y_2/0.6, y_3/0.6, y_4/0.3, y_5/0.1, y_6/0.1, y_7/1, y_8/0.7, y_9/0.6, y_{10}/0.4\}$
$\mathcal{C}_{\mathbf{y_8}} : \{y_1/0.4, y_2/0.6, y_3/0.5, y_4/0.2, y_5/0, y_6/0, y_7/0.6, y_8/1, y_9/0.5, y_{10}/0.3\}$
$\mathcal{C}_{\mathbf{y_9}} : \{y_1/0.3, y_2/0.6, y_3/0.4, y_4/0.2, y_5/0, y_6/0, y_7/0.5, y_8/0.7, y_9/1, y_{10}/0.2\}$
$\mathcal{C}_{\mathbf{y_{10}}} : \{y_1/0.4, y_2/0.2, y_3/0.1, y_4/0.3, y_5/0, y_6/0, y_7/0.2, y_8/0.5, y_9/0.1, y_{10}/1\}$

We are now in condition of calculate the support values for the objects. The obtained values are shown in Table 2.

Table 2. Object support values

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
supp	0.46	0.22	0.6	0.54	0.25	0.32	0.35	0.3	0.18	0.29

In Table 3 we can see the values obtained for the attributes.

Table 3. Attribute support values

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
supp	0.43	0.41	0.2	0.46	0.24	0.17	0.35	0.42	0.36	0.38

These supports can be taken into account to study the relevance of the objects and the attributes.

Let be $U = X, Card(U) = n$ and $E(x_i) = supp(x_i)$, for every $x_i \in X$. Using overlap index O_π, we can calculate by Theorem 1 the associated fuzzy measure for each object (See Table 4).

Table 4. Fuzzy measure for objects associated with O_π and E

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$m_{O_\pi,E}$	0.119	0.058	0.165	0.132	0.070	0.115	0.106	0.099	0.049	0.087

These values define a relevance ranking for the objects:

$$x_3 \geq_{O,E} x_4 \geq_{O,E} x_1 \geq_{O,E} x_6 \geq_{O,E} x_7 \geq_{O,E} x_8 \geq_{O,E} x_{10} \geq_{O,E} x_5 \geq_{O,E} x_2 \geq_{O,E} x_9$$

The same classification is obtained for overlap index O_Z since the obtained fuzzy measure values are those shown in Table 5.

Table 5. Fuzzy measure for objects associated with O_Z and E

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$m_{O_Z,E}$	0.848	0.591	1.000	0.894	0.652	0.833	0.803	0.773	0.545	0.727

From the point of view of the attributes, with overlap index O_π, we obtain the values in Table 6.

And in Table 7 we show the values obtained with O_Z.

The same ranking is also obtained with both overlap indexes:

$$y_2 >_{O,E} y_8 >_{O,E} y_7, y_1, y_4 >_{O,E} y_3 >_{O,E} y_{10} >_{O,E} y_9 >_{O,E} y_5 >_{O,E} y_6$$

In this case, if we consider that the size of the L-fuzzy context is large and we choose as the fuzzy set E defined by the support (as the model we want to look like), we can conclude that objects x_9 and x_2 and attributes y_6 and y_5 are the candidates to be removed.

Table 6. Fuzzy measure for attributes associated with O_π and E

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
$m_{O_\pi,E}$	0.119	0.157	0.106	0.119	0.041	0.035	0.119	0.123	0.085	0.097

Table 7. Fuzzy measure for attributes associated with O_Z and E

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
$m_{O_Z,E}$	0.869	1.000	0.820	0.869	0.508	0.475	0.869	0.885	0.738	0.787

5 Conclusions

In this work, we have seen that overlap indexes can be useful tools to analyze the relevance of the objects and the attributes from the point of view of the L fuzzy concepts. We define different ranking associated with the different overlap indexes. These rankings can help us to remove some objects or attributes when the size of the L-fuzzy context is large.

In future works, we will study how this reduction affects to the structure of the L-fuzzy concept lattice.

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