A Typology of Mathematical Diagrams

Mikkel Willum Johansen[1(✉)], Morten Misfeldt[2], and Josefine Lomholt Pallavicini[1]

[1] University of Copenhagen, Copenhagen, Denmark
mwj@ind.ku.dk, pallavicini@hum.ku.dk
[2] Aalborg University, Copenhagen, Denmark
misfeldt@learning.aau.dk

Abstract. In this paper, we develop and discuss a classification scheme that allows us to distinguish between the types of diagrams used in mathematical research based on the cognitive support offered by diagrams. By cognitive support, we refer to the gain that research mathematicians get from using diagrams. This support transcends the specific mathematical topic and diagram type involved and arises from the cognitive strategies mathematicians tend to use. The overall goal of this classification scheme is to facilitate a large-scale quantitative investigation of the norms and values governing the publication style of mathematical research, as well as trends in the kinds of cognitive support used in mathematics. This paper, however, focuses only on the development of the classification scheme.

The classification scheme takes its point of departure from case studies known from the literature, but in this paper, we validate the scheme using examples from a preliminary investigation of developments in the use of diagrams. Building on these results, we discuss the potential and pitfalls in using one generic classification scheme, as done in this analysis. This approach is contrasted with attempts that respect and build on individual diagram types, and as part of this discussion, we report the problems we experienced when using that strategy. The paper ends with a description of possible next steps in using text corpora as an empirical approach to understanding the nature of mathematical diagrams and their relation to mathematical culture.

Keywords: Classification of diagrams · Corpus analysis
Mathematical cognition

1 Introduction: Why Classify Mathematical Diagrams from a Cognitive Perspective

With this paper, we wish to develop a framework that allows us to characterize mathematical diagrams based on the kind of cognitive support they offer to mathematics. Furthermore, we aim to construct a classification scheme that bracket both the mathematical content represented by the diagram and the practice of specific mathematicians and to consider the diagram only as it presents itself on the page. Such a classification scheme will enable large-scale quantitative analysis of corpora of published mathematics papers that would otherwise be difficult to process.

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There are a number of good reasons for analyzing the use of diagrams in mathematical publications. The prominence of social norms governing the use of diagrams in publications was one result of an interview study with working mathematicians the two first authors of this paper conducted [1]. As a general trend, the mathematicians to whom we spoke considered various forms of diagrams and drawings to be vital to their work practices, but nevertheless, they tended to leave out such representations in their published papers.

Such results are hardly surprising to research mathematicians. Indeed, they generally confirm the point made nearly three decades ago by Hersh [2]: mathematics has a frontstage and a backstage; there is in mathematics a huge difference between the way a dish is prepared and the way it is presented to the public. However, recent work in the philosophy of mathematical practice indicates that the value and validity of inferences based on diagrams and figures should not be neglected, and the formalistic claim that such inferences are insecure and obsolete (e.g. [3: 43]) has been questioned by a number of scholars in the field [4, 5]. This vindication of diagrammatic reasoning was also confirmed in the interview study mentioned above as several of the interviewed mathematicians expressed the view that diagrams and other visual representations can constitute good arguments, and that it was a shame they were not allowed to be used in published papers.

Furthermore, investigations of historical cases have shown that diagrams and figures have played (and still play) a crucial role both in the conceptual development of mathematics [6–8] and in the framing and development of new areas of mathematics research [9, 10]. Such results might be seen as a call for a revision of the norms and values that govern publication practice. If inferences based in diagrams are (in some cases) valid, and if diagrams play a crucial part in the conceptual development of mathematics, perhaps we should allow diagrams in published papers to a greater extent.

To the working mathematician, the more pressing question is: are the norms and values of publication changing in mathematics? There are indications that we are in the midst of major changes in publication practice. In the year 1992 the mathematics journal *Experimental Mathematics* was launched with the direct intention to bring the behind-the-scene processes leading to mathematical discovery to the fore [11]. While *Experimental Mathematics* is mainly aimed at methodological developments connected to the use of computers as experimental tools, there are signs that a similar change in attitudes toward diagrams is underway. At least, diagrams seem to be brought more frequently to the frontstage of mathematics. It is not unusual for textbook expositions, even on advanced topics, to rely heavily on diagrams and drawings (e.g. [12]), and it also seems that mathematics journals contain more diagrams of still more varied types than they did at the height of the formalist movement in the middle of the 20^{th} century. This, however, is merely an impression. We do not have substantial evidence to support the claim that the use of diagrams in mathematics journals is on the rise, and if it is, we do not know the shape or size of the change, when it started, or if all or only certain types of diagrams are being published more frequently. One way to address such questions in a thorough, empirical way is to track developments in the use of diagrams in mathematics research papers over time. This could be done by counting the number of diagrams included in research papers in selected influential mathematical journal over a long period of time (e.g., a century). However, in mathematics, a diagram is not simply a diagram. Diagrams are different and play varied cognitive roles in mathematical research; therefore, it would be interesting to track not only the number of diagrams but also the types of diagrams used.

To carry out such a large-scale empirical investigation of the use of diagrams in mathematics journals, an instrument that allows for counting and classifying diagrams with relative ease needs to be constructed. The objective of this paper is to describe and discuss a first attempt at building such an instrument. This leads to the following research question:

> *How can we classify mathematical diagrams published during the past century in a way that distinguishes among their different cognitive functions?*

To answer this question, we build on work by the first author of this paper. In [13] Johansen distinguishes between diagrams, symbols, and figures based in the different cognitive roles these representational forms play in mathematical practice. In the following, we introduce Johansen's distinction, adapt it by adding the category of Cartesian diagrams, and discuss the possible inclusion of a fourth category of matrices and tables. We demonstrate the classification scheme by describing how it worked and the problems we encountered when we used it in a systematic pilot investigation of the journal *Annals of Mathematics*. We should note that the focus of this paper is the development of the classification scheme, not developments in the use of diagrams. For this reason, we discuss only how well the classification scheme worked and not the preliminary results we obtained from using it.

2 Mathematical Diagrams and Figures

Although concepts such as diagram and figure are frequently discussed in the philosophy of mathematics, there is no consensus on precise definitions of the terms. Definitions range from the extreme, such as Peirce's [14: 90] functional definition of diagrams as any representation that allows the deduction of new information not used in its construction, to more restrictive definitions that come closer to the everyday use of the word, such as Larkin and Simon's [15] definition requiring a representation to involve an element of two-dimensionality to be considered a diagram.

Although Peirce's functional definition captures and justifies a central aspect of mathematical practice, namely, that mathematical work frequently involves translation between different representational types, it is too broad and does not allow for making cognitively and practically meaningful distinctions in the category of diagrams [16]. Moreover, Peirce's definition does not make it possible to distinguish between algebraic and diagrammatic representations (in the pre-theoretical sense of the words), so it is impossible to track the very changes in norms and publication practices that we are interested in here. Here, we are interested in actual mathematical practice and the cognitive role diagrams may play herein. For that reason, we use the less inclusive definition, which understands diagrams as representations that require two-dimensionality and uses this two-dimensionality in a non-trivial way (e.g. [15: 68, 17: 162]; see also [18] for a discussion). This definition comes closer to the everyday use of the word but also inherits some of the fuzziness of the pre-theoretical concept. When confronted with real-life examples of two-dimensional representations, choices have to be made to establish the precise boundaries of the concept, as we will see in the following.

Furthermore, when diagrams are defined in this way, the concept covers a rather inhomogeneous category of representations. To make useful distinctions in the concept of diagram, we will take departure in Johansen [13], where an attempt is made (a) to divide the class of diagrams into two subclasses (called ‘diagrams’ and ‘figures’) on the basis of cognitive function and (b) to explain how the cognitive functions provided by these two subclasses differ from those provided by symbols. It thus is pointed out that mathematical symbols can produce new knowledge by being subjected to purely syntactic manipulations. Some diagrams and figures may in the same way be subjected to purely syntactic manipulation, but what sets these representations apart from symbols from a cognitive perspective is that they can also be used for contentual and intuitive reasoning by relating mathematical objects to the everyday experience of the mathematician. The point in [13] is that members of the two subclasses of diagrams create this relation in different ways. One subclass (which Johansen calls ‘figures’) consists of representations such as Euclidian diagrams. When operating on a Euclidian diagram, the representation has a direct resemblance to the physical objects that constitute the abstraction class for the corresponding mathematical concept. When operating on a drawing of a triangle, for example, the representation has a direct resemblance to other perceptible triangles that make up the abstraction class for the general mathematical concept of ‘triangle’. With the other subclass (which Johansen calls ‘diagrams’), it is not possible to establish this kind of direct resemblance. Instead, diagrams display the mathematical objects they represent only if the objects are understood within a particular conceptualization. To give an example, a commutative diagram cannot be said to resemble neither the mathematical objects it represents (typically mathematical sets and maps between sets), nor the abstraction class of such objects. However, if the mathematical sets are conceptualized as objects located in space and the maps between them are conceptualized as movements between locations in space, the commutative diagram can be said to resemble such a situation. Similarly, Venn diagrams do not function via simple resemblance. The objects represented by such diagrams are typically not located in bounded regions of space and may not even have inherent spatial properties. The resemblance is only established it the objects are conceptualized as if they were bounded regions of space [c.f. 13: 100]. In other words, Johansen claims that diagrams in this second subclass function as material anchors for conceptual maps that relate mathematical content to concrete experiences in the physical world. We further refer to [19] for direct empirical evidence that research mathematicians in actual practice conceptualize the mathematical objects represented by diagrams in the way described (see also [17] for a further discussion of commutative diagrams’ function as maps).

2.1 A Classification Scheme

In the following, we build on the basic idea from [13] but refine it. In particular, we do not follow Johansen in renaming a subclass of diagrams ‘figures’. Instead, we introduce the following subclasses of diagrams:

1. Resemblance diagrams: This class roughly corresponds to Johansen’s first sub-class of diagrams. That is, it includes diagrams that have a direct likeness to the physical objects that the corresponding mathematical concepts are supposed to model. This likeness can be geometric (as in Euclidian diagrams) or topological (as in knot diagrams).
2. Abstract diagrams: This class corresponds to Johansen’s second class of diagrams. That is, it includes diagrams that are only meaningful if the mathematical content they represent is understood through a particular conceptual map.

In addition to these two subclasses, we will introduce a third subclass:

3. Cartesian diagrams: This subclass includes shapes and figures drawn in a coordinate system.

The reason for introducing this third subclass is largely pragmatic and grounded in our exploratory work with corpora of mathematical texts. The use of coordinate systems builds on conceptual blending allowing mathematical objects to simultaneously possess geometric and numerical properties [20: 385]. This function is not clearly included in the two categories introduced so far; therefore, we believe it necessary to propose this third subclass.

Furthermore, various forms of schemas, tables, and matrices can be said to constitute a fourth, independent category of two-dimensional visual inscriptions, and we discuss whether to include such representations as a fourth subclass. In this subclass, mathematical symbols are arranged in a way so that patterns in the physical layout of the representations may reveal mathematically relevant information (c.f. [13, 19]). This way of creating and using representations opens cognitive possibilities other than the three subclasses of diagrams described above, and such schemas fall outside the categories we have introduced. Tables and matrices, however, are not encompassed by the everyday concept of diagrams, and including such representations in the concept of diagrams might force us to broaden it further to include, say, the two-dimensional arrangement of numeral symbols in pen-and-paper multiplication. Here, therefore, a choice will clearly have to be made.

2.2 Applying the Scheme

The motivation for the development of the classification scheme is the need to carry out large-scale, empirical investigations of the use of diagrams in mathematics journals. Applying the scheme for such a purpose leads to a trade-off between reliability and quantity: on one hand, reliable classification of a given diagram in the scheme requires in-depth case studies exploring how mathematicians conceive of and use the diagram in actual practice, but on the other hand, an investigation of the general trends in diagram use requires the classification of large quantities of diagrams. Because the suggested classification scheme takes the way mathematicians perceive diagrams as its point of departure we believe that an examination of the typographical features of a given diagram and a minimum of mathematical context will give a clear indication of how the diagram should be classified in the scheme. Of course, misclassifications are inevitable, but we are primarily interested in general trends in the frequency and types of diagrams used, so a few misclassifications will not be detrimental to achieving the overall objective of the investigation. However, it is clear that transparency and reflections about reliability and error margins must be a constant, important concern in quantitative analysis of this type.

Finally, it should be noted that the move to quantitative analysis of diagram use suggested here is new in the philosophy of mathematical diagrams. In certain respects, it stands in contrast to core values of the paradigm, where careful case studies are the norm. We do hope the reader will appreciate the possible benefits of deviating from this norm.

3 Examples

In the following, we give a few representative examples of the diagrams we encountered during our pilot study on the diagrams used in *Annals of Mathematic*. The selection represents the different kinds of diagrams we found by investigating the journal over ten-year intervals in the period from 1885 to 2005. For each year (e.g. 1885, 1895), we analyzed all the research papers published in the journal, marked all the two-dimensional representations, and tried to apply the classification scheme to them. All the diagrams we present in the following sections are reproduced with kind permission from *Annals of Mathematics*.

3.1 Prototypical Diagrams

Most diagrams we encountered were easily classified using the suggested scheme. We begin with three prototypical examples illustrating how the subclasses of the scheme function when confronted with representations from real-life publications.

As a first example, we consider a figure from [21] (Fig. 1). Notice that the representation is classified as a figure in the original text (and not as a diagram, Euclidian diagram, or similar terms). This illustrates the confusion in nomenclature discussed above. In the classification scheme suggested here, however, the representation clearly belongs to the subclass of resemblance diagrams as it displays a circle and geometric constructions involving chords within the circle. From a cognitive point of view, the diagram supports reasoning with these objects by anchoring the conceptual structure in a stable medium (the paper) so that the mathematical objects are represented by shapes that geometrically resemble the class of physical objects modelled by the mathematical concepts. In other words, in this case, our categories allow for a clear and obvious classification of the diagram.

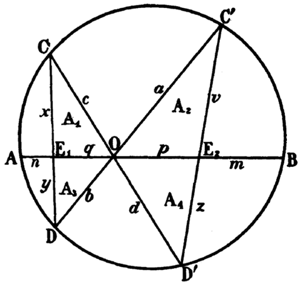

Fig. 1. A resemblance diagram. Reproduced with permission from [21: 175].

As our second example, we consider a representation from [22] (Fig. 2). In this case, the representation is not labeled, but the accompanying text refers to it as a commutative diagram.

$$\begin{array}{ccccccc} 0 & \rightarrow & C(\mathfrak{V}, \mathfrak{A}) & \rightarrow & C(\mathfrak{V}, \mathfrak{B}) & \rightarrow & C(\mathfrak{V}, \mathfrak{C}) \\ & & \tau\downarrow & & \tau\downarrow & & \tau\downarrow \\ 0 & \rightarrow & C(\mathfrak{U}, \mathfrak{A}) & \rightarrow & C(\mathfrak{U}, \mathfrak{B}) & \rightarrow & C(\mathfrak{U}, \mathfrak{C}) \end{array}$$

Fig. 2. An abstract diagram. Reproduced with permission from [22: 216].

We consider the diagram to belong to the class of abstract diagrams and claim that, in contrast to the diagram in Fig. 1, this diagram does not directly resemble any aspect of physical reality modelled by the mathematical objects involved. Rather, the diagram only resembles the mathematical objects if they are viewed from a particular conceptualization. We, however, refrain from identifying or describing the conceptualization because that requires the kind of in-depth contextual analysis we are trying to avoid here (see [13, 19] for examples).

As a third example, we look at the following representation from [23] (Fig. 3):

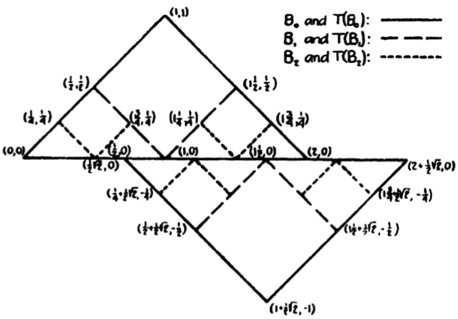

Fig. 3. A Cartesian diagram. Reproduced with permission from [23: 43].

This diagram involves a construction in a coordinate system, and we interpret it as representing a conceptual blend, where points are simultaneously considered to be locations on the geometric plane and elements in a set of ordered pairs of numbers (x, y). In other words, we see the diagram as belonging to the subclass of Cartesian diagrams.

3.2 Is This a Diagram?

Most of the diagrams we inspected fall neatly into one of the three categories of the classification scheme, but we also encountered diagrams challenging the scheme. In this and the following section, we discuss two types of challenges: cases where the representations in question are on the borderline of being considered a (mathematical) diagram at all and cases where a diagram is on the borderline of two of the subclasses in our classification scheme.

Beginning with the first type of concern, we return to the commutative diagram presented in Fig. 2. Interestingly, this diagram is preceded by the following representation (Fig. 4):

$$\cdots \to H^q(\mathfrak{U}, \mathfrak{B}) \to H_0^q(\mathfrak{U}, \mathfrak{C}) \xrightarrow{d} H^{q+1}(\mathfrak{U}, \mathfrak{A}) \to H^{q+1}(\mathfrak{U}, \mathfrak{B}) \to \cdots,$$

Fig. 4. Exact sequence. Reproduced with permission from [22: 216].

In many ways, this representation seems similar to the diagram presented in Fig. 2. Letters representing mathematical objects are connected with arrows representing maps between these objects. Why is this not a diagram? The answer is simply that the representation is not two dimensional but one dimensional (in the sense that it can be read linearly). Although this distinction between one- and two-dimensional representations may seem arbitrary, it, at least in this case, connects to a real distinction in the language used in mathematical practice. In the text, Fig. 2 is referred to as a (commutative) diagram, while the representation in Fig. 4 is referred to as something different: an exact sequence. This furthermore illustrates a fundamental difficulty connected to delaminating diagrams as a distinct type of representations. Functional categories, such as the one introduced by Peirce [14] or the one attempted in [13], tend to clash with the pre-theoretical language used by practitioners. From a cognitive perspective, it is very difficult to point to the operative difference between an exact sequence and a commutative diagram, but in mathematical practice, the two representations appear to have different statuses, at least judging from the language (one is labeled a diagram, and the other is not). It is our goal in this paper to create an instrument that makes it possible to track and understand aspects of mathematical practice, so we have to make pragmatic compromises. One such compromise is to adopt the criterion that a diagram has to be two dimensional, even if this criterion may seem arbitrary from a theoretical perspective (as the classification of commutative diagrams as diagrams and exact sequences as non-diagrams illustrates).

As a second example of a representation that is on the borderline of being considered a mathematical diagram, we can look at the following representation (Fig. 5):

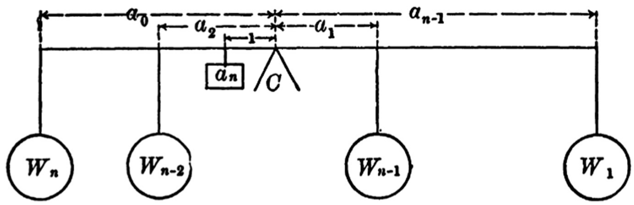

Fig. 5. Cartesian diagram. Reproduced with permission from [24: 73].

The circles marked W_n to W_1 represent water-filled vessels, C represents a fulcrum, and the overall diagram represents a mathematical problem expressed in terms of balancing weights around a fulcrum. The diagram pictures the structure of an (imagined) physical mechanism. As a first approximation, it might be taken to be a resemblance diagram, similar to Euclidian diagrams picturing geometric structures. However, mathematical resemblance diagrams typically involve only geometric and/or topological features of the represented objects, whereas the diagram in Fig. 5 also involves physical features, such as weight, equilibrium, and movement. For this reason, it is questionable whether the diagram is a *mathematical* diagram. Perhaps it is a physics diagram, which might not be the same as those used in mathematics. In other words, the diagram in Fig. 5 challenges us to draw another (more or less arbitrary) border between the diagrams used in mathematics and those used in physics. Although the diagram is on the borderline, we need a clear border; we would not consider, say, Feynman diagrams to be mathematical diagrams, although they are clearly diagrams.

During the pilot study, we also frequently encountered schemas and matrices such as that reproduced in Fig. 6:

$$(C^{\Delta})^{-1} = \frac{1}{x\delta}\begin{pmatrix} 1 & -x & 0 & 0 & \cdots \\ -x & 1+x^2 & -x & 0 & \cdots \\ 0 & -x & 1+x^2 & -x & \cdots \\ & & \ddots & \ddots & -x \\ & \cdots & 0 & -x & 1 \end{pmatrix}.$$

Fig. 6. Matrix. Reproduced with permission from [25: 198].

As noted, it can be argued from a cognitive perspective that such representations play a different role in mathematical practice than diagrams, in particular, they allow abstract mathematical structures to be visualized as patterns in the physical arrangements of symbols on the paper. However, although such representations clearly exploit their two-dimensionality, they are traditionally not considered to be diagrams, and for that reason, we decided not to include them in our pilot study. This can again be seen as a somewhat arbitrary choice. However, as argued, including matrices and similar schemas in our category of diagrams would introduce a demarcation problem: if we consider matrices and schemas to be diagrams, we would also have to consider, say, the arrangement of numerals used for pen-and-paper multiplication as diagrams. Although this might make sense from a purely functional perspective, important aspects of the distinction between mathematical discourse and diagrams would cease to exist. The distinctions we consider here are based on both the two-dimensional aspect of diagrams and established practice among mathematicians, which usually does not consider matrices and schemas to be diagrams. The example of matrices and schemas, however, clearly and interestingly illustrates once again that the pre-theoretical concept of diagrams found in mathematical practice does not correspond with purely functional or theoretical categories.

3.3 What Kind of Diagram Is This? Examples that Challenge the Classification Scheme

Another type of challenge came from diagrams that did not fall clearly into the categories in the classification scheme. We begin by considering the following representation from [24] (Fig. 7):

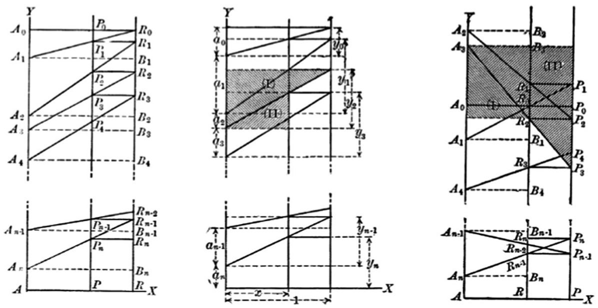

Fig. 7. Cartesian diagram. Reproduced with permission from [24: 66].

We see this as a Cartesian diagram, but the identification of the diagram as something involving a coordinate system is less straightforward than in the prototypical example above (Fig. 3). The construction of unity in the part of the diagram marked "Fig. 2" indicates that we are dealing with a Cartesian construction, but to make sure that it is not a resemblance diagram, we have to consider the textual context. Here, it is made clear that the diagram indicates the pointwise construction of the graph of a polynomial. It, therefore, is most sensible to classify the diagram as Cartesian because the overall framework of the diagram involves a Cartesian blend.

As a second example of a diagram challenging the subclasses of our scheme, we can look at the following representation (Fig. 8) from [26]:

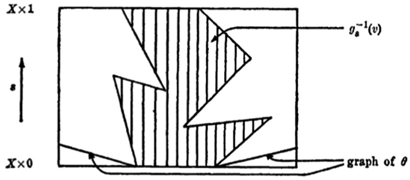

Fig. 8. Abstract diagram. Reproduced with permission from [26: 107].

Once more, the categorization is not straightforward, but as an educated guess, we consider the diagram to be what we call an abstract diagram. The reason for this categorization is partly positive and partly negative. The diagram does seem to involve abstract elements, such as sets represented as locations in space, and although it also includes geometric elements, we do not believe these to resemble the mathematical objects in any straightforward way. In other words, the diagram seems to be conceptualized, although a contextual analysis is necessary to identify the exact conceptualization. A similar analysis can be extended to other diagrams, such as the following (Fig. 9):

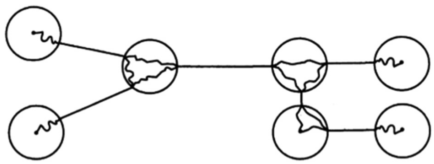

Fig. 9. Abstract diagram. Reproduced with permission from [27: 223].

As our final example, we look at two diagrams from [28]. The first one is the following diagram (Fig. 10):

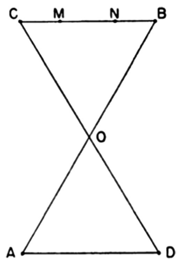

Fig. 10. Resemblance diagram. Reproduced with permission from [28: 609].

As we see it, this is a resemblance diagram displaying a particular situation. If we look at the textual context, we furthermore can see that, in this case, the resemblance is topological, not geometric, although the particular type of resemblance is not consequential in our classification scheme. However, if we continue to read the paper [28], we encounter the following figure (Fig. 11):

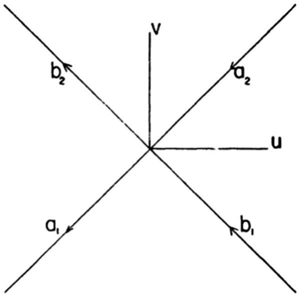

Fig. 11. Resemblance diagram. Reproduced with permission from [28: 613].

If we know that we are dealing with topology, it is not difficult to decode the diagrams as resemblance diagrams (based on topological resemblance), but without such knowledge, we believe it would be very difficult to make a meaningful classification of these diagrams in our scheme. In difficult cases such as this, it is, in other words, not possible to completely bracket the intellectual context of the diagram and the classification cannot be more than an educated guess.

4 Discussion: Balancing the Resolution of Diagram Classification

Our experience from the pilot study indicates that the classification scheme we have presented makes it possible to handle most of the diagrams published over a century in a leading mathematical journal. Although most classifications are straightforward, we also encountered cases where we had to involve the textual or intellectual context of diagrams to classify them, and in other cases, we could only give educated guesses.

It, therefore, might be worthwhile contemplating possible alternatives. When we began the pilot study, we initially set out to use a classification based on the standard names given to various diagram types (e.g., commutative diagrams, Dynkin diagrams, and simplex diagrams). Although such a classification gave a more fine-grained resolution in the diagram classification—which allowed tracking trends in the use of specific diagram types—it turned out to be difficult to carry out in practice, at least on a large scale. Mathematicians do not consistently name diagrams, and often diagrams are not referred to in the text by their standard names. Consequently, counting the number of diagrams of a specific type requires making judgements based on the visual appearance of diagrams to classify them correctly. Furthermore, although mathematicians tend to express themselves using a relatively small number of standard types of diagrams, they also occasionally use their own idiosyncratic representations. Although such cases are difficult to handle in the proposed classification scheme, they are in principle out of the reach of a classification taking departure in standard names.

In another alternative, one might consider counting the number of diagrams without attempting to classify them further. This would reduce some of the uncertainty we encountered, but at the price of a similar reduction in information. The subclasses in our classification scheme delineate three very different kinds of diagrams, ranging from direct geometric representations (e.g., Euclidian diagrams) to completely abstract representations (e.g., commutative diagrams). The ability to track the development not only of the frequency of diagrams but also of the relative frequency of the different subclasses of diagrams could yield valuable information about the norms governing the use of diagrams in mathematical practice. Furthermore, even a simple count of the number of diagrams without attempting to classify them further, is not without uncertainty. As we have seen, the concept of mathematical diagrams is not well defined, and many of the challenges we faced concerned not how to classify diagrams within the scheme but whether to count a given representation as a mathematical diagram.

Finally, it should be noted that the classification of diagrams can be performed using dimensions other than those suggested here, such as Stenning's [18] distinction between directness and indirectness or de Toffoli's [17] distinctions among expressiveness, calculability, and transparency. Our choice of classification scheme is mainly pragmatic. We believe that it will work because its point of departure is mathematicians' conception of diagrams and for that reason it will provide relevant information about the changes and trends in the use of diagrams in mathematics publications. This, of course, does not rule out the possibility of including other dimensions in the categorization and quantitative analysis of diagram use. Furthermore, we cannot rule out that the classification scheme will have to be augmented or adjusted in order to be applied to other corpora of mathematical texts; we have tested the scheme only on a single journal.

5 Conclusion

In this paper, we have argued for a change in the philosophy of mathematical diagrams. We believe it will be productive to augment in-depth, qualitative case-studies of diagrams with quantitative investigations tracking overall trends in diagram use. We have developed the first version of an instrument that can be used to classify diagrams in large-scale, quantitative studies, and we have demonstrated the function of the instrument by reporting on a pilot investigation on a mathematical journal. The instrument generally seems to give reliable results, but as we have also seen, the quantitative approach proposed here also leads to a number of dilemmas. A choice between the reliability of the investigation and the relevance of the generated knowledge must be made. Furthermore, the pre-theoretical concept of mathematical diagram is not well defined, and choices have to be made between the use of a well-defined theoretical concept of diagrams and the ability to track and speak into actual mathematical practice and mathematicians' understanding of this practice. However, as we see it—and as our pilot study confirms—the challenges confronting a quantitative approach are manageable, and the full-scale application of the proposed instrument has the potential to generate valuable information about long-term developments in the norms governing the use and publication of mathematical diagrams.

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