Chapter 8
Comparison on Solving a Class of Nonlinear Systems of Partial Differential Equations and Multiple Solutions of Second Order Differential Equations

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8.1 Introduction

Many problems in science and engineering such as problems posed in solid state physics, fluid mechanics, chemical physics, plasma physics, optics, etc. are modelled as nonlinear partial differential equations (PDEs) or systems of nonlinear PDEs. Nonlinear systems of PDEs have taken much interest in working evolution equations. Many researchers have investigated the analytical and approximate solutions of nonlinear systems of PDEs by utilizing different techniques [7].

In this paper, a general technique is shown in the reproducing kernel space for searching the following class of nonlinear systems of PDEs:

$$\begin{aligned} A_1(f_1(\eta,\tau)) &= P_1(\eta,\tau,F(\eta,\tau)) + M_1(\eta,\tau), \\ &\quad\cdots \qquad\qquad\qquad\qquad , \\ A_k(f_k(\eta,\tau)) &= P_k(\eta,\tau,F(\eta,\tau)) + M_k(\eta,\tau), \\ (\eta,\tau) \in \Omega &= [0,1]\times[0,1] \end{aligned}$$

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with the initial and boundary conditions

$$f_i(\eta,\tau)=0, \quad \text{for } \tau\le 0, \tag{8.1}$$

$$f_i(0,\tau)=0, \quad f_i(1,\tau)=0, \quad \text{for } \tau>0. \tag{8.2}$$

where A_i and P_i are linear and nonlinear differential operators for $i=1,2,\dots,k$. $M_i(\eta,\tau)$ are given functions and $F(\eta,\tau)=[f_1(\eta,\tau), f_2(\eta,\tau),\dots, f_k(\eta,\tau)]^T$ is an unknown vector function to be determined. Suppose this equation is of one-order derivative in τ, and has a unique solution. We only take into consideration the homogeneous initial and boundary conditions, because the non-homogeneous initial and boundary conditions can be easily transformed to the homogeneous ones. The reproducing method has been implemented to several nonlinear problems [1]. For more details of this method, see [3, 6, 8].

We take into consideration the boundary value problems:

$$\begin{cases} u''(x)=\lambda\exp(\mu u(x)), & 0\le x\le 1,\\ u(0)=u(1)=0, \end{cases} \tag{8.3}$$

and

$$\big(\exp(x)v'(x)\big)'+|\ln x|=0, \forall x\in(0,\infty), x\ne 1, \Delta v'\,|_{x_1=1}=v^2(1),$$
$$v(0)=0, \quad v(\infty)=0. \tag{8.4}$$

The problem (8.3) shows up in implementations containing the diffusion of heat produced by positive temperature-dependent sources. If $\mu=1$, it springs in the analysis of Joule losses in electrically conducting solids, with λ returning the square of constant current and $\exp(u)$ the temperature-dependent resistance, or frictional heating with λ projecting the square of the constant shear stress and $\exp(u)$ the temperature-dependent fluidity. In particular if $\lambda=1$ and $\mu=-1$ the boundary value problem (8.3) has two solutions $u_1(x)$ and $u_2(x)$. Solution $u_1(x)$ drops below up to $-0.14050941\dots$ and solution $u_2(x)$ up to $-4.0916146\dots$.

Boundary value problem (8.4) has at least two positive solutions v_1, v_2 satisfying $0\le\|v_1\|\le\frac{1}{2}\le\|v_2\|$.

This work is ordered as follows. Section 8.2 presents some useful reproducing kernel functions. The representation of solutions and a related linear operator are given in Sect. 8.3. This section shows the main results. Examples are shown in Sect. 8.4. The final section contains some conclusions.

8.2 Preliminaries

Definition 1 We present $G_2^1[0,1]$ by

$$G_2^1[0,1] = \{f \in AC[0,1] : \ f' \in L^2[0,1]\}.$$

The inner product and the norm in $G_2^1[0,1]$ are defined by

$$\langle f, g\rangle_{G_2^1} = f(0)g(0) + \int_0^1 f'(\eta)g'(\eta)\mathrm{d}\eta, \quad f, g \in G_2^1[0,1]$$

and

$$\|f\|_{G_2^1} = \sqrt{\langle f, f\rangle_{G_2^1}}, \quad f \in G_2^1[0,1].$$

Theorem 1 *Reproducing kernel function* $\widetilde{Q}_\tau$ *of* $G_2^1[0,1]$ *is obtained as:*

$$\widetilde{Q}_\tau(\eta) = \sum_{i=1}^{2} c_i(\tau)\eta^{i-1}, \quad 0 \le \eta \le \tau \le 1, \sum_{i=1}^{2} d_i(\tau)\eta^{i-1}, \quad 0 \le \tau < \eta \le 1.$$

Proof By Definition 1, we have

$$\left\langle u, \widetilde{Q}_\tau\right\rangle_{G_2^1} = u(0)\widetilde{Q}_\tau(0) + \int_0^1 u'(\eta)\widetilde{Q}'_\tau(\eta)\mathrm{d}\eta, \tag{8.5}$$

We get

$$\left\langle u, \widetilde{Q}_\tau\right\rangle_{G_2^1} = u(0)\widetilde{Q}_\tau(0) + u(1)\widetilde{Q}'_\tau(1) - u(0)\widetilde{Q'}_\tau(0) - \int_0^1 u(\eta)\widetilde{Q}''_\tau(\eta)\mathrm{d}\eta,$$

by integrating by parts. Note that property of the reproducing kernel is

$$\langle u(\eta), \widetilde{Q}'_\tau(\eta)\rangle_{G_2^1} = u(\tau). \tag{8.6}$$

If

$$\begin{cases} \widetilde{Q}_\tau(0) - \widetilde{Q}'_\tau(0) = 0, \\ \widetilde{Q}'_\tau(1) = 0, \end{cases} \tag{8.7}$$

then, we get

$$-\widetilde{Q}''_\tau(\eta) = \delta(\eta - \tau).$$

If $\eta \neq \tau$, then we obtain

$$\widetilde{Q}''_{\tau}(\eta) = 0.$$

Thus, we have

$$\widetilde{Q}_{\tau}(\eta) = \begin{cases} c_1(\tau) + c_2(\tau)\eta, & 0 \leq \eta \leq \tau \leq 1, \\ \\ d_1(\tau) + d_2(\tau)\eta, & 0 \leq \tau < \eta \leq 1. \end{cases} \tag{8.8}$$

Since

$$-\widetilde{Q}''_{\tau}(\eta) = \delta(\eta - \tau),$$

we get

$$\widetilde{Q}_{\tau^+}(\tau) = \widetilde{Q}_{\tau^-}(\tau) \tag{8.9}$$

and

$$\widetilde{Q}'_{\tau^+}(\tau) - \widetilde{Q}'_{\tau^-}(\tau) = -1. \tag{8.10}$$

The unknown coefficients $c_i(\tau)$ and $d_i(\tau)$ $(i = 1, 2)$ can be obtained. Thus $\widetilde{Q}_{\tau}$ is acquired as

$$\widetilde{Q}_{\tau}(\eta) = 1 + \eta, \quad 0 \leq \eta \leq \tau \leq 1, \ 1 + \tau, \quad 0 \leq \tau < \eta \leq 1.$$

Definition 2 We present the space $H_2^2[0, 1]$ as:

$$H_2^2[0, 1] = \{f \in AC[0, 1] : \ f' \in AC[0, 1], \ f'' \in L^2[0, 1], \ f(0) = 0\}.$$

The inner product and the norm in $H_2^2[0, 1]$ are presented as:

$$\langle f, g \rangle_{H_2^2} = f(0)g(0) + f'(0)g'(0) + \int_0^1 f''(\eta)g''(\eta)\mathrm{d}\eta, \quad f, g \in H_2^2[0, 1]$$

and

$$\|f\|_{H_2^2} = \sqrt{\langle f, f \rangle_{H_2^2}}, \quad f \in H_2^2[0, 1].$$

Theorem 2 *Reproducing kernel function $\widetilde{T}_\tau$ of $H_2^2[0,1]$ is obtained by:*

$$\widetilde{T}_\tau(\eta) = \sum_{i=1}^{4} c_i(\tau)\eta^{i-1}, \quad 0 \le \eta \le \tau \le 1, \ \sum_{i=1}^{4} d_i(\tau)\eta^{i-1}, \quad 0 \le \tau < \eta \le 1.$$

Proof By Definition 2, we have

$$\langle f, \widetilde{T}_\tau \rangle_{H_2^2} = f(0)\widetilde{T}_\tau(0) + f'(0)\widetilde{T}'_\tau(0) + \int_0^1 f''(\eta)\widetilde{T}''_\tau(\eta)\mathrm{d}\tau. \tag{8.11}$$

Integrating this equation by parts two times, we get

$$\begin{aligned}\langle f, \widetilde{T}_\tau \rangle_{H_2^2} &= f(0)\widetilde{T}_\tau(0) + f'(0)\widetilde{T}'_\tau(0) + f'(1)\widetilde{T}''_\tau(1) - f'(0)\widetilde{T}''_\tau(0) \\ &\quad - f(1)\widetilde{T}'''_\tau(1) + f(0)\widetilde{T}'''_\tau(0) + \int_0^1 f(\eta)\widetilde{T}^{(4)}_\tau(\eta)\mathrm{d}\eta.\end{aligned}$$

We have

$$\langle f(\eta), \widetilde{T}_\tau(\eta) \rangle_{H_2^2} = f(\tau) \tag{8.12}$$

by reproducing property. Since $\widetilde{T}_\tau \in H_2^2[0,1]$, we have

$$\widetilde{T}_\tau(0) = 0. \tag{8.13}$$

If

$$\begin{cases} \widetilde{T}'_\tau(0) - \widetilde{T}''_\tau(0) = 0, \\ \widetilde{T}''_\tau(1) = 0, \\ \widetilde{T}'''_\tau(1) = 0, \end{cases} \tag{8.14}$$

then, we get

$$\widetilde{T}^{(4)}_\tau(\eta) = \delta(\eta - \tau).$$

When $\eta \ne \tau$, we get

$$\widetilde{T}^{(4)}_\tau(\eta) = 0.$$

Thus

$$\widetilde{Q}_\tau(\eta) = c_1(\tau) + c_2(\tau)\eta + c_3(\tau)\eta^2 + +c_4(\tau)\eta^3, \quad 0 \le \eta \le \tau \le 1,$$

$$d_1(\tau) + d_2(\tau)\eta + d_3(\tau)\eta^2 + d_4(\tau)\eta^3, \quad 0 \le \tau < \eta \le 1.$$

Since

$$\widetilde{T}_{\tau}^{(4)}(\eta) = \delta(\eta - \tau),$$

we obtain

$$\widetilde{T}_{\tau^+}^{(k)}(\tau) = \widetilde{T}_{\tau^-}^{(k)}(\tau), \quad k = 0, 1, 2 \tag{8.15}$$

and

$$\widetilde{T}_{\tau^+}'''(\tau) - \widetilde{T}_{\tau^-}'''(\tau) = 1. \tag{8.16}$$

The unknown coefficients $c_i(\tau)$ and $d_i(\tau)$ $(i = 1, 2, 3, 4)$ can be obtained. Thus $\widetilde{T}_{\tau}$ is achieved as

$$\begin{aligned}\widetilde{T}_{\tau}(\eta) = \eta\tau + \frac{(\eta)(\tau)^2}{2} + \frac{(\tau-\eta)^3}{6} - \frac{(\tau)^3}{6}, \quad 0 \le \eta \le \tau \le 1,\\ \eta\tau + \frac{(\tau)(\eta)^2}{2} + \frac{(\eta-\tau)^3}{6} - \frac{(\tau)^3}{6}, \quad 0 \le \tau < \eta \le 1.\end{aligned}$$

Definition 3 We give $W_2^3[0, 1]$ as:

$$\begin{aligned}W_2^3[0, 1] = \{f \in AC[0, 1] :\ f', f'' \in AC[0, 1],\ f^{(3)} \in L^2[0, 1],\\ f(0) = f(1) = 0\}.\end{aligned}$$

The inner product and the norm in $W_2^3[0, 1]$ are defined by

$$\langle f, g\rangle_{W_2^3} = \sum_{i=0}^{2} f^{(i)}(0)g^{(i)}(0) + \int_0^1 f^{(3)}(\eta)g^{(3)}(\eta)\mathrm{d}\eta, \quad f, g \in W_2^3[0, 1]$$

and

$$\|f\|_{W_2^3} = \sqrt{\langle f, f\rangle_{W_2^3}}, \quad f \in W_2^3[0, 1].$$

Theorem 3 *Reproducing kernel function R_τ of $W_2^3[0, 1]$ is obtained as:*

$$R_\tau(\eta) = \begin{cases} \sum_{i=1}^{5} c_i(\tau)\eta^i, & 0 \le \eta \le \tau \le 1, \\ \sum_{i=0}^{5} d_i(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases} \tag{8.17}$$

where

$$\begin{aligned}
c_1(\tau) &= -\frac{1}{156}\tau^5 + \frac{5}{156}\tau^4 - \frac{5}{78}\tau^3 - \frac{5}{26}\tau^2 + \frac{3}{13}\tau,\\
c_2(\tau) &= -\frac{1}{624}\tau^5 + \frac{5}{624}\tau^4 - \frac{5}{312}\tau^3 + \frac{21}{104}\tau^2 - \frac{5}{26}\tau,\\
c_3(\tau) &= -\frac{1}{1872}\tau^5 + \frac{5}{1872}\tau^4 - \frac{5}{936}\tau^3 + \frac{7}{104}\tau^2 - \frac{5}{78}\tau,\\
c_4(\tau) &= \frac{1}{3744}\tau^5 - \frac{5}{3744}\tau^4 + \frac{5}{1872}\tau^3 + \frac{5}{624}\tau^2 - \frac{1}{104}\tau,\\
c_5(\tau) &= -\frac{1}{18720}\tau^5 + \frac{1}{3744}\tau^4 - \frac{1}{1872}\tau^3 - \frac{1}{624}\tau^2 - \frac{1}{156}\tau + \frac{1}{120},\\
d_0(\tau) &= \frac{1}{120}\tau^5,\\
d_1(\tau) &= -\frac{1}{156}\tau^5 - \frac{1}{104}\tau^4 - \frac{5}{78}\tau^3 - \frac{5}{26}\tau^2 + \frac{3}{13}\tau,\\
d_2(\tau) &= -\frac{1}{624}\tau^5 + \frac{5}{624}\tau^4 + \frac{7}{104}\tau^3 + \frac{21}{104}\tau^2 - \frac{5}{26}\tau,\\
d_3(\tau) &= -\frac{1}{1872}\tau^5 + \frac{5}{1872}\tau^4 - \frac{5}{936}\tau^3 - \frac{5}{312}\tau^2 - \frac{5}{78}\tau,\\
d_4(\tau) &= \frac{1}{3744}\tau^5 - \frac{5}{3744}\tau^4 + \frac{5}{1872}\tau^3 + \frac{5}{624}\tau^2 + \frac{5}{156}\tau,\\
d_5(\tau) &= -\frac{1}{18720}\tau^5 + \frac{1}{3744}\tau^4 - \frac{1}{1872}\tau^3 - \frac{1}{624}\tau^2 - \frac{1}{156}\tau.
\end{aligned}$$

Proof Let $f \in W_2^3[0, 1]$ and $0 \le \tau \le 1$. Note that

$$R'_\tau(\eta) = \begin{cases} \sum_{i=0}^{4}(i+1)c_{i+1}(\tau)\eta^i, & 0 \le \eta < \tau \le 1, \\ \sum_{i=0}^{4}(i+1)d_{i+1}(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases}$$

$$R''_\tau(\eta) = \begin{cases} \sum_{i=0}^{3}(i+1)(i+2)c_{i+2}(\tau)\eta^i, & 0 \le \eta < \tau \le 1, \\ \sum_{i=0}^{3}(i+1)(i+2)d_{i+2}(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases}$$

$$R^{(3)}_\tau(\eta) = \begin{cases} \sum_{i=0}^{2}(i+1)(i+2)(i+3)c_{i+3}(\tau)\eta^i, & 0 \le \eta < \tau \le 1, \\ \sum_{i=0}^{2}(i+1)(i+2)(i+3)d_{i+3}(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases}$$

$$R_\tau^{(4)}(\eta) = \begin{cases} \sum_{i=0}^{1}(i+1)(i+2)(i+3)(i+4)c_{i+4}(\tau)\eta^i, & 0 \le \eta < \tau \le 1, \\ \\ \sum_{i=0}^{1}(i+1)(i+2)(i+3)(i+4)d_{i+4}(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases}$$

and

$$R_\tau^{(5)}(\eta) = \begin{cases} 120c_5(\tau), & 0 \le \eta < \tau \le 1, \\ \\ 120d_5(\tau), & 0 \le \tau < \eta \le 1. \end{cases}$$

By Definition 3 and integrating by parts, we obtain

$$\begin{aligned} \langle f, R_\tau \rangle_{W_2^3} &= \sum_{i=0}^{2} f^{(i)}(0)R_\tau^{(i)}(0) + \int_0^1 f^{(3)}(\eta)R_\tau^{(3)}(\eta)\mathrm{d}\eta \\ &= f'(0)R'_\tau(0) + f''(0)R''_\tau(0) + f''(1)R_\tau^{(3)}(1) - f''(0)R_\tau^{(3)}(0) \\ &\quad -f'(1)R_\tau^{(4)}(1) + f'(0)R_\tau^{(4)}(0) + \int_0^1 f'(\eta)R_\tau^{(5)}(\eta)\mathrm{d}\eta \\ &= c_1(\tau)f'(0) + 2c_2(y)f''(0) \\ &\quad +6(d_3(\tau) + 4d_4(\tau) + 10d_5(\tau))f''(1) - 6c_3(\tau)f''(0) \\ &\quad -24(d_4(\tau) + 5d_5(\tau))f'(1) + 24c_4(\tau)f'(0) \\ &\quad + \int_0^\tau 120c_5(\tau)f'(\eta)\mathrm{d}\eta + \int_\tau^1 120d_5(\tau)f'(\eta)\mathrm{d}\tau \\ &= (c_1(\tau) + 24c_4(\tau))f'(0) + 2(c_2(\tau) - 3c_3(\tau))f''(0) \\ &\quad +6(d_3(\tau) + 4d_4(\tau) + 10d_5(\tau))f''(1) - 24(d_4(\tau) + 5d_5(\tau))f'(1) \\ &\quad +120(c_5(\tau) - d_5(\tau))f(\tau) \\ &= f(\tau). \end{aligned}$$

Definition 4 We give the binary space $W(\Omega)$ as:

$$W(\Omega) = \left\{ f : \frac{\partial^3 f}{\partial \eta^2 \partial t} \in CC(\Omega), \quad \frac{\partial^5 f}{\partial \eta^3 \partial t^2} \in L^2(\Omega), \right. \\ \left. f(\eta, 0) = f(0, t) = f(1, t) = 0 \right\},$$

where CC denotes the space of completely continuous functions. The inner product and the norm in $W(\Omega)$ are obtained as:

$$\langle f, g\rangle_W = \sum_{i=0}^{1}\int_0^1\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial \eta^i}u(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial \eta^i}g(0,t)\right]\mathrm{d}t$$
$$+\sum_{j=0}^{1}\left\langle\frac{\partial^j}{\partial t^j}f(\cdot,0), \frac{\partial^j}{\partial t^j}g(\cdot,0)\right\rangle_{W_2^3}$$
$$+\int_0^1\int_0^1\left[\frac{\partial^3}{\partial \eta^3}\frac{\partial^2}{\partial t^2}f(\eta,t)\frac{\partial^3}{\partial \eta^3}\frac{\partial^2}{\partial t^2}g(\eta,t)\right]\mathrm{d}t\mathrm{d}\eta, \quad f, g \in W(\Omega)$$

and

$$\|g\|_W = \sqrt{\langle g, g\rangle_W}, \quad g \in W(\Omega).$$

Lemma 1 (See [4, page 148]) *Reproducing kernel function* $K_{(\tau,s)}$ *of* $W(\Omega)$ *is given by:*

$$K_{(\tau,s)} = R_\tau r_s.$$

Definition 5 We define the binary space $\widehat{W}(\Omega)$ by

$$\widehat{W}(\Omega) = \left\{f \in CC(\Omega) : \quad \frac{\partial^2 f}{\partial \eta \partial t} \in L^2(\Omega)\right\}.$$

The inner product and the norm in $\widehat{W}(\Omega)$ are obtained as:

$$\langle f, g\rangle_{\widehat{W}} = \int_0^1\left[\frac{\partial}{\partial t}f(0,t)\frac{\partial}{\partial t}g(0,t)\right]\mathrm{d}t + \langle f(\cdot,0), g(\cdot,0)\rangle_{G_2^1}$$
$$+\int_0^1\int_0^1\left[\frac{\partial}{\partial \eta}\frac{\partial}{\partial t}f(\eta,t)\frac{\partial}{\partial \eta}\frac{\partial}{\partial t}g(\eta,t)\right]\mathrm{d}t\mathrm{d}\eta, \quad f, g \in \widehat{W}(\Omega)$$

and

$$\|g\|_{\widehat{W}} = \sqrt{\langle g, g\rangle_{\widehat{W}}}, \quad g \in \widehat{W}(\Omega).$$

Lemma 2 (See [4, page 23]) *Reproducing kernel function* $G_{(\tau,s)}$ *of* $\widehat{W}(\Omega)$ *is given as:*

$$G_{(\tau,s)} = (\widetilde{Q}_\tau)^2.$$

Definition 6 We define the space $W_2^1[0,1]$ by

$$W_2^1[0,1] = \{u \in AC[0,1] : \; u' \in L^2[0,1]\}.$$

The inner product and the norm in $W_2^1[0,1]$ are given as:

$$\langle u, g\rangle_{W_2^1} = \int_0^1 u(x)g(x) + u'(x)g'(x)\mathrm{d}x, \quad u, g \in G_2^1[0,1] \tag{8.18}$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}, \quad u \in W_2^1[0,1]. \tag{8.19}$$

The space $W_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function T_x is obtained as [4]

$$T_x(y)= \frac{1}{2\sinh(1)}[\cosh(x+y-1)+\cosh(|x-y|-1)]. \tag{8.20}$$

8.3 Analytical and Approximate Solutions

We consider

$$A_1(f_1(\eta,t)) = P_1(\eta, t, F(\eta,t)) + M_1(\eta,t), \tag{8.21}$$

where $A_1 : W(\Omega) \to \widehat{W}(\Omega)$ is a bounded linear operator, P_1 is a nonlinear operator, $M_1(\eta,t)$ is an arbitrary function, and $F(\eta,t) = [f_1(\eta,t), f_2(\eta,t), \ldots, f_k(\eta,t)]^T$. The spaces $W(\Omega)$ and $\widehat{W}(\Omega)$ are reproducing kernel spaces which are defined according to the highest derivatives. We pick a countable dense subset $\{(\eta_j, t_j)\}_{j=1}^{\infty}$ in Ω, and describe $\rho_j(\eta,t) = G_{(\eta_j,t_j)}(\eta,t)$, $\vartheta_{j_1}(\eta,t) = A_1^*\rho_j(\eta,t)$, where A_1^* is the adjoint operator of A_1. It is simple to show that [2]

$$\vartheta_{j_1}(\eta,t) = A_1 K_{(\tau,s)}(\eta,t).$$

The solutions of (8.3) and (8.4) are considered in the reproducing kernel space $W_2^3[0,1]$. On defining the linear operator $L : W_2^3[0,1] \to W_2^1[0,1]$ as

$$\mathrm{Lu}(x) = u''(x) \tag{8.22}$$

the problem changes the form:

$$\begin{cases} \mathrm{Lu} = f(x,u), & x \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{8.23}$$

where $f(x,u) = \lambda\exp(\mu u(x))$.

In Eq. (8.23) since $u(x)$ is sufficiently smooth $L : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator. For model problem (8.4) similar things can be done.

Theorem 4 *Assume that* $\{(\eta_j, t_j)\}_{j=1}^{\infty}$ *is dense in* Ω*, then the solution of* (8.21) *can be shown as*

$$f_1(\eta, t) = \sum_{j=1}^{\infty} \sigma_{j_1} \vartheta_{j_1}(\eta, t), \tag{8.24}$$

where the σ_{j_1} *are found by*

$$\begin{aligned} Z_1 \times \sigma_{j_1} = [&P_1(\eta_1, t_1, F(\eta_1, t_1)) + M_1(\eta_1, t_1), P_1(\eta_2, t_2, F(\eta_2, t_2)) \\ &+ M_1(\eta_2, t_2), \ldots]^T, \end{aligned} \tag{8.25}$$

$$Z_1 = [A_1 \vartheta_{j_1}(\eta, t) \mid_{(\eta,t)=(\eta_i,t_i)}]_{i,j=1,2,\ldots}, \quad \sigma_{j_1} = [\sigma_{11}, \sigma_{21}, \ldots]^T.$$

Proof $\{(\eta_j, t_j)\}_{j=1}^{\infty}$ is dense in Ω. Therefore, $\vartheta_{j_1}(\eta, t)$ is complete system in $W(\Omega)$ [2]. We get

$$\begin{aligned} \langle \vartheta_{i_1}, \vartheta_{j_1} \rangle_{W(\Omega)} &= \langle A_1^* \rho_i(\eta, t), \vartheta_{j_1} \rangle_{W(\Omega)} = \langle \rho_i(\eta, t), A_1 \vartheta_{j_1} \rangle_{\widehat{W}(\Omega)} \\ &= A_1 \vartheta_{j_1}(\eta, t) = \langle f_1(\eta, t), \vartheta_{i_1} \rangle_{W(\Omega)} = \langle f_1(\eta, t), A_1^* \rho_{i_1} \rangle_{W(\Omega)} \\ &= \langle A_1 f_1(\eta, t), \rho_{i_1} \rangle_{\widehat{W}(\Omega)} = P_1(\eta_i, t_i, F(\eta_i, t_i)) + M_1(\eta_i, t_i). \end{aligned}$$

This completes the proof.

Remark 1 If $P_\eta(\eta, t, F(\eta, t)) = 0$ for $\eta = 1, 2, \ldots, k$, then the analytical solution of each equation can be achieved and the approximate solution of each equation is the m-term intercept of the analytical solution which can be obtained by solving an $m \times m$ system of linear equations. If $P_\eta(\eta, t, F(\eta, t)) \neq 0$, then we need to construct an iterative method. We select the number of points m, the number of iterations n and put the initial vector function $F_{0,m}(\eta, t) = [0, 0, \ldots, 0]^T$. Then the approximate solution is presented as:

$$F_{n,m}(\eta, t) = [f_{n,m,1}(\eta, t), f_{n,m,2}(\eta, t), \ldots, f_{n,m,k}(\eta, t)]^T,$$

where

$$\begin{cases} f_{n,m,1}(\eta, t) = \sum_{j=1}^{m} \sigma_{j_1} \vartheta_{j_1}(\eta, t), \\ \ldots \\ f_{n,m,k}(\eta, t) = \sum_{j=1}^{m} \sigma_{j_k} \vartheta_{j_k}(\eta, t). \end{cases}$$

Theorem 5 *Suppose that* $\{(\eta_j, t_j)\}_{j=1}^{\infty}$ *is dense in* Ω. *Then the approximate solution* $F_{n,m}(\eta, t)$ *converges to the analytical solution* $F(\eta, t)$.

Proof We have

$$A_\eta(f_{n,m,\eta}(\eta_j, t_j)) = P_\eta(\eta_j, t_j, F_{n-1,m}(\eta_j, t_j)) + M_\eta(\eta_j, t_j) \tag{8.26}$$

for

$$\eta = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, m, \quad \text{and } n = 1, 2, \ldots.$$

There exists a convergent subsequence $\{f_{n_\epsilon,m,\eta}(\eta, t)\}_{\epsilon=1}^{\infty}$ of $\{f_{n,m,\eta}(\eta, t)\}_{n=1}^{\infty}$ such that $f_{n_\epsilon,m,\eta}(\eta, t) \to u_\varpi(\eta, t)$ as $\epsilon \to \infty$, $m \to \infty$, for $\varpi = 1, 2, \ldots, k$. Then, we acquire

$$A_\varpi(f_{n_\epsilon,m,\varpi}(\eta_j, t_j)) = P_\varpi(\eta_j, t_j, F_{n_\epsilon-1,m}(\eta_j, t_j)) + M_\varpi(\eta_j, t_j). \tag{8.27}$$

The operators A_ϖ and P_ϖ are both continuous. Therefore it can be concluded that $F(\eta, t) = [f_1(\eta, t), \ldots, f_k(\eta, t)]^T$ is the analytical solution of (8.21) and $F_{n_\epsilon,m}(\eta, t) = [f_{n_\epsilon,m,1}(\eta, t), \ldots, f_{n_\epsilon,m,k}(\eta, t)]^T$ is the approximate solution of (8.21) after taking limit from both sides. This completes the proof.

It is obvious that $L : W_2^3[0, 1] \to W_2^1[0, 1]$ is a bounded linear operator. Put $\varphi_i(x) = T_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is conjugate operator of L. The orthonormal system $\left\{\widehat{\Psi}_i(x)\right\}_{i=1}^{\infty}$ of $W_2^3[0, 1]$ can be obtained from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\widehat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots) \tag{8.28}$$

Lemma 3 (See [5]) *Let* $\{x_i\}_{i=1}^{\infty}$ *be dense in* $[0, 1]$ *and* $\psi_i(x) = L_y R_x(y)\big|_{y=x_i}$. *Then the sequence* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is a complete system in* $W_2^3[0, 1]$.

Theorem 6 *If* u_1 *and* u_2 *are the exact solutions of* (8.3), *then*

$$u_1(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k})\widehat{\Psi}_i(x) \tag{8.29}$$

and

$$u_2(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \gamma_{ij} f(x_j, u_{2_j})\widehat{\Psi}_i(x), \tag{8.30}$$

where $\{(x_i)\}_{i=1}^{\infty}$ *is dense in* $[0, 1]$.

Proof We have

$$
\begin{aligned}
u_1(x) &= \sum_{i=1}^{\infty} \left\langle u_1(x), \widehat{\Psi}_i(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u_1(x), \Psi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u_1(x), L^* \varphi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle \mathrm{Lu}_1(x), \varphi_k(x) \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle f(x, u_1), T_{x_k} \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i(x).
\end{aligned}
$$

Similar things can be done for u_2.

The approximate solutions $u_n(x)$ and $u_m(x)$ can be acquired from the n and m terms truncation of the exact solutions u_1 and u_2 as

$$
u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i(x), \tag{8.31}
$$

and

$$
u_m(x) = \sum_{i=1}^{m} \sum_{j=1}^{i} \gamma_{ij} f(x_k, u_{2_j}) \widehat{\Psi}_i(x). \tag{8.32}
$$

Theorem 7 *For any fixed $u_{1_0}(x) \in W_2^3[0, 1]$ assume that the following conditions are hold:*

(i)

$$
u_n(x) = \sum_{i=1}^{n} A_i \widehat{\psi}_i(x), \tag{8.33}
$$

$$A_i = \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_{k-1}}(x_k)), \tag{8.34}$$

(ii) $\|u_n\|_{W_2^3}$ *is bounded;*
(iii) $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[0, 1]$*;*
(iv) $f(x, u_1) \in W_2^1[0, 1]$ *for any* $u_1(x) \in W_2^3[0, 1]$.

Then $u_n(x)$ *converges to the exact solution of* (8.3) *in* $W_2^3[0, 1]$ *and*

$$u_1(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i(x),$$

where A_i *is given by* (8.34).

Proof We will show the convergence of $u_n(x)$. We get

$$u_{n+1}(x) = u_n(x) + A_{n+1}\widehat{\Psi}_{n+1}(x), \tag{8.35}$$

from the orthonormality of $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$, it follows that

$$\|u_{n+1}\|^2 = \|u_n\|^2 + A_{n+1}^2 = \|u_{n-1}\|^2 + A_n^2 + A_{n+1}^2 = \ldots = \sum_{i=1}^{n+1} A_i^2, \tag{8.36}$$

from boundedness of $\|u_n\|_{W_2^3}$, we obtain

$$\sum_{i=1}^{\infty} A_i^2 < \infty,$$

i.e.,

$$\{A_i\} \in l^2 \quad (i = 1, 2, \ldots).$$

Let $p > n$, in view of $\left(u_p - u_{p-1}\right) \perp \left(u_{p-1} - u_{p-2}\right) \perp \ldots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned} \left\|u_p - u_n\right\|_{W_2^3}^2 &= \left\|u_p - u_{p-1} + u_{p-1} - u_{p-2} + \ldots + u_{n+1} - u_n\right\|_{W_2^3}^2 \\ &\leq \left\|u_p - u_{p-1}\right\|_{W_2^3}^2 + \ldots + \|u_{n+1} - u_n\|_{W_2^3}^2 \\ &= \sum_{i=n+1}^{p} A_i^2 \to 0, \quad p, n \to \infty. \end{aligned}$$

Considering the completeness of $W_2^3[0, 1]$, there exists $u_1(x) \in W_2^3[0, 1]$, such that

$$u_n(x) \to u_1(x) \quad as\ n \to \infty.$$

(ii) Taking limits,

$$u_1(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i(x).$$

Since

$$\begin{aligned}(\mathrm{Lu}_1)\left(x_j\right) &= \sum_{i=1}^{\infty} A_i \left\langle L\widehat{\psi}_i(x), \varphi_j(x)\right\rangle_{W_2^1} = \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), L^*\varphi_j(x)\right\rangle_{W_2^3} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \psi_j(x)\right\rangle_{W_2^3},\end{aligned}$$

it follows that

$$\begin{aligned}\sum_{j=1}^{n} \beta_{nj}(\mathrm{Lu}_1)(x_j) &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \sum_{j=1}^{n} \beta_{nj}\psi_j(x)\right\rangle_{W_2^3} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \widehat{\psi}_n(x)\right\rangle_{W_2^3} = A_n.\end{aligned}$$

If $n = 1$, then

$$\mathrm{Lu}_1(x_1) = f(x_1, u_{1_0}(x_1)). \tag{8.37}$$

If $n = 2$, then

$$\beta_{21}(\mathrm{Lu}_1)(x_1) + \beta_{22}(\mathrm{Lu}_1)(x_2) = \beta_{21} f(x_1, u_{1_0}(x_1)) + \beta_{22} f(x_2, u_{1_1}(x_2)). \tag{8.38}$$

From (8.37) and (8.38), we have

$$(\mathrm{Lu}_1)(x_2) = f(x_2, u_{1_1}(x_2)).$$

We get

$$(\mathrm{Lu}_1)(x_j) = f(x_j, u_{1_{j-1}}(x_j)), \tag{8.39}$$

by induction. By the convergence of $u_n(x)$ we get

$$(\mathrm{Lu}_1)(y) = f(y, u_1(y)),$$

that is, $u_1(x)$ is the solution of (8.3) and

$$u_1(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i,$$

where A_i are given by (8.34). It can be shown in a similar way that $u_2(x)$ is a solution of (8.4).

Theorem 8 *If* $u_1 \in W_2^3[0, 1]$, *then*

$$\|u_n - u_1\|_{W_2^3} \to 0, \quad n \to \infty.$$

Moreover a sequence $\|u_n - u_1\|_{W_2^3}$ *is monotonically decreasing in* n.

Proof We have

$$\|u_n - u_1\|_{W_2^3} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i \right\|_{W_2^3}.$$

Thus

$$\|u_n - u_1\|_{W_2^3} \to 0, \quad n \to \infty.$$

In addition

$$\|u_n - u_1\|_{W_2^3}^2 = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i \right\|_{W_2^3}^2$$
$$= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i \right)^2.$$

Clearly, $\|u_n - u_1\|_{W_2^3}$ is monotonically decreasing in n. In a similar way $\|u_m - u_2\|_{W_2^3}$ is monotonically decreasing in m. This completes the proof.

Remark 2 Let us consider countable dense set $\{x_1, x_2, \ldots\} \in [0, 1]$ and define

$$\varphi_i = T_{x_i}, \quad \Psi_i = L^* \varphi_i, \quad \widehat{\Psi}_i = \sum_{k=1}^{1} \beta_{ik} \Psi_k.$$

Then β_{ik} coefficients can be found by

$$\beta_{11} = \frac{1}{\|\Psi_1\|}, \quad \beta_{ii} = \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}},$$

$$\beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad c_{ik} = \left\langle \Psi_i, \widehat{\Psi}_k \right\rangle.$$

In a similar way γ_{ij} can be defined by using Q_{x_i} .

8.4 Numerical Results

We conceive the following nonlinear system of partial differential equations by RKM:

$$\frac{\partial f}{\partial t} = \alpha + \frac{1}{Re}\frac{\partial}{\partial \tau}\left(\mu(T)\frac{\partial f}{\partial \tau}\right) - \frac{Ha^2}{Re} f - \frac{R}{Re}(f - f_p),$$

$$\frac{\partial f_p}{\partial t} = \frac{1}{ReG}(f - f_p),$$

$$\frac{\partial T}{\partial t} = \frac{1}{RePr}\frac{\partial}{\partial \tau}\left(K(T)\frac{\partial T}{\partial \tau}\right) + \frac{Ec}{Re}\mu(T)\left(\frac{\partial f}{\partial \tau}\right)^2 + \frac{Ec}{Re}Ha^2u^2 + \frac{2R}{3Pr}(T_p - T),$$

$$\frac{\partial T_p}{\partial t} = -L(T_p - T),$$

$$f(\tau, t) = f_p(\tau, t) = T(\tau, t) = T_p(\tau, t) = 0, \quad \text{for} \quad t \leq 0,$$
$$f_p(0, t) = f_p(1, t) = T_p(0, t) = T(0, t) = 0, \quad \text{for} \quad t > 0,$$

$$\beta\frac{\partial f}{\partial \tau} = f, \quad \text{for } \tau = 0, 1,\ t > 0,$$

$$T_p(1, t) = T(1, t) = 1, \quad \text{for } t > 0.$$

where

$$\mu(T) = \exp(-aT), \quad K(T) = \exp(bT),$$

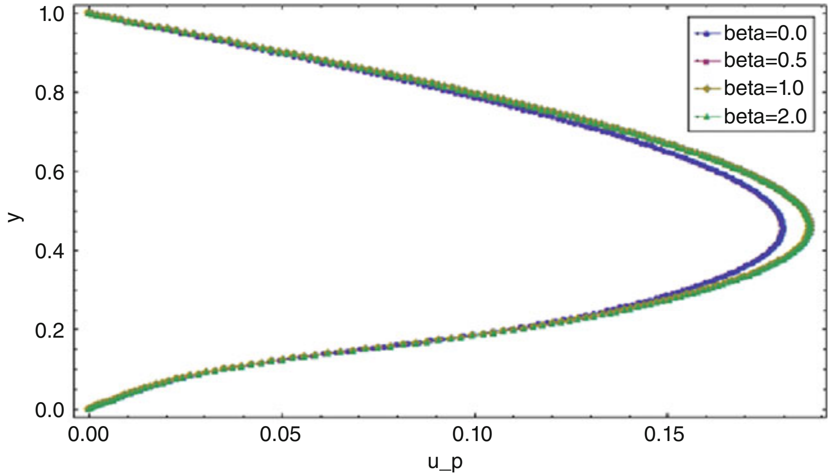

Fig. 8.1 Approximate solutions of $u_p = f_p$ for various β

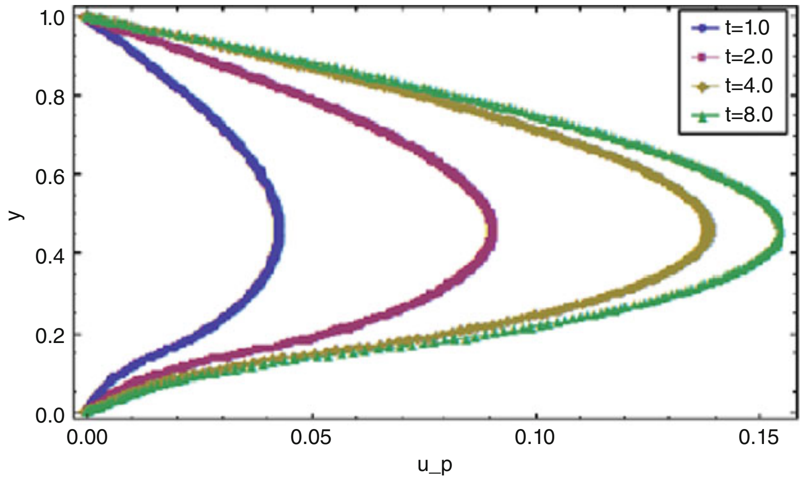

Fig. 8.2 Approximate solutions of $u_p = f_p$ for various t

$$R = 0.5,\ G = 0.8,\ \alpha = 1,\ a = 1,\ b = 0.01,\ Pr = 7.1,$$
$$Ec = 0.2,\ Re = 1,\ L = 0.7,\ \beta = 1,\ t = 10.$$

We obtained the numerical results and demonstrated them in Figs. 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11, 8.12, 8.13, 8.14, 8.15, 8.16, 8.17, and 8.18.

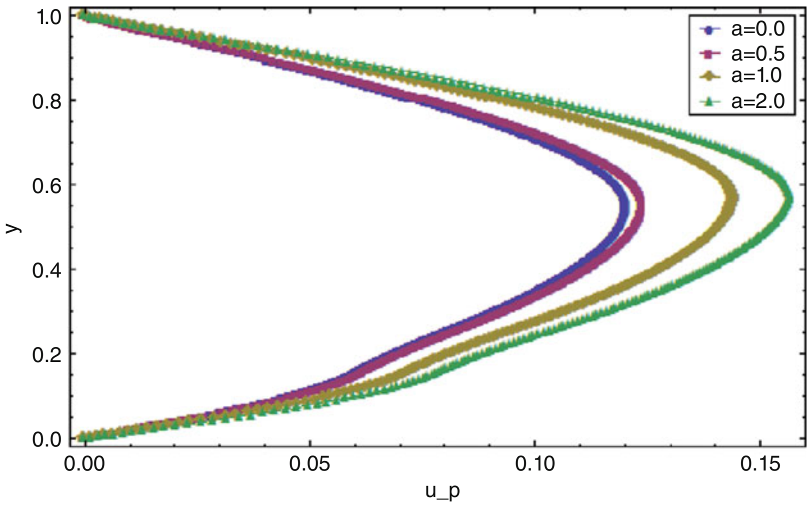

Fig. 8.3 Approximate solutions of $u_p = f_p$ for various a

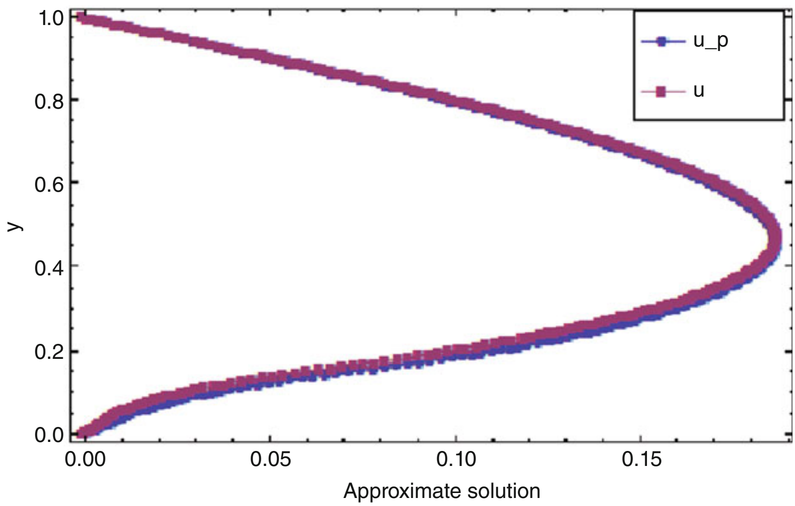

Fig. 8.4 Comparison of approximate solutions of $u = f$ and $u_p = f_p$

Example 1 We now consider (8.3). If $\mu\lambda < 0$, the problem (8.3) has as many solutions as the number of roots of the equation

$$\theta = \sqrt{2|\mu\lambda|} \cosh\left(\frac{\theta}{4}\right),$$

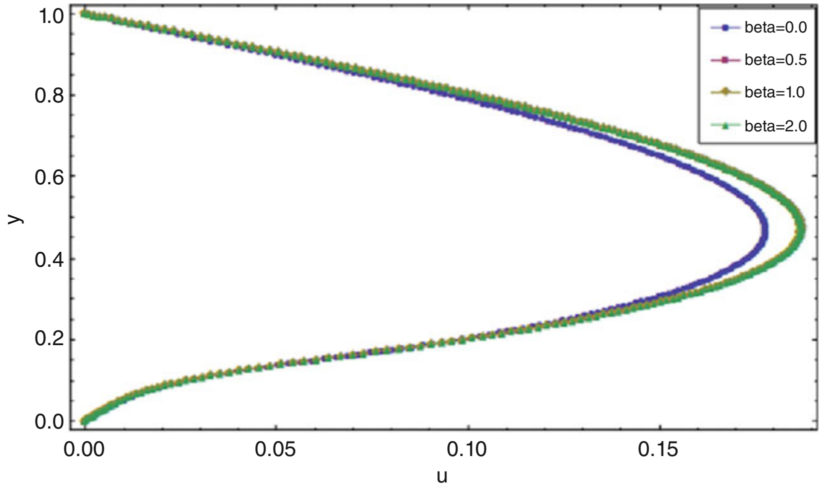

Fig. 8.5 Approximate solutions of $u = f$ for different values of β

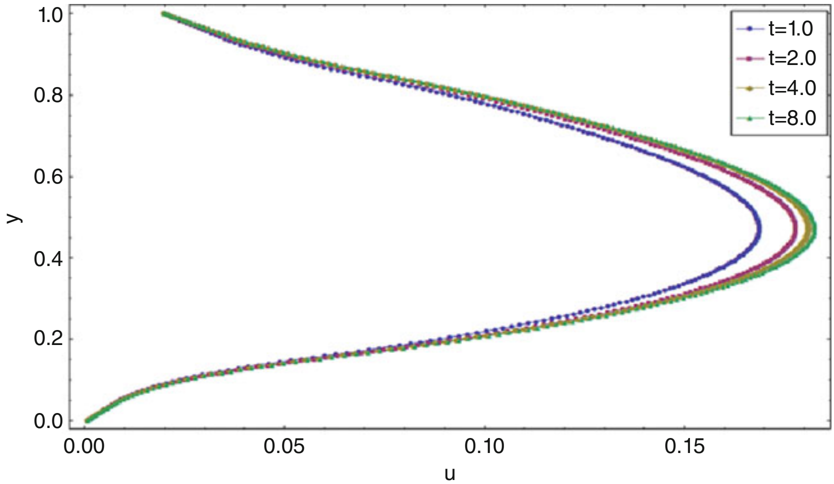

Fig. 8.6 Approximate solutions of $u = f$ for various t

also for each such θ_i

$$u_i(x) = -\frac{2}{\mu}\left\{\ln\left[\cosh\left(\frac{\theta_i}{2}\left(x-\frac{1}{2}\right)\right)\right] - \ln\left[\cosh\left(\frac{\theta_i}{4}\right)\right]\right\}.$$

We obtain Tables 8.1, 8.2, 8.3, 8.4, 8.5, and 8.6 by RKM.

Example 2 We consider (8.4) for the second example. We obtain Tables 8.7, 8.8, and 8.9 by RKM.

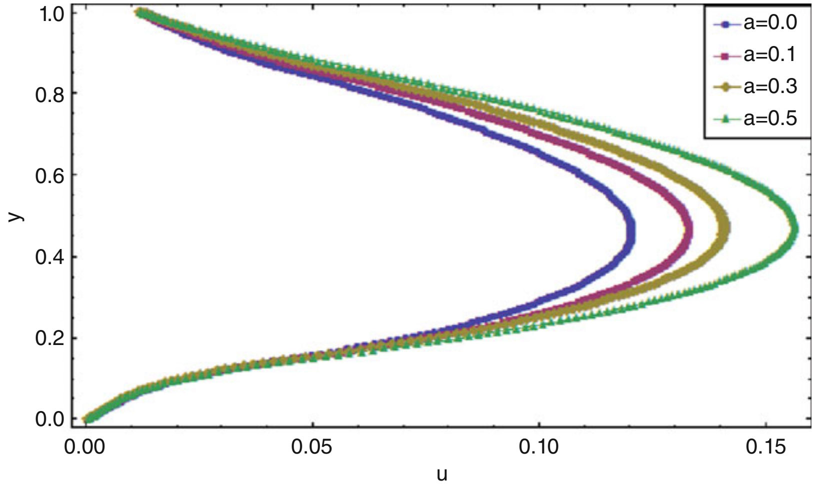

Fig. 8.7 Approximate solutions of $u = f$ for various a

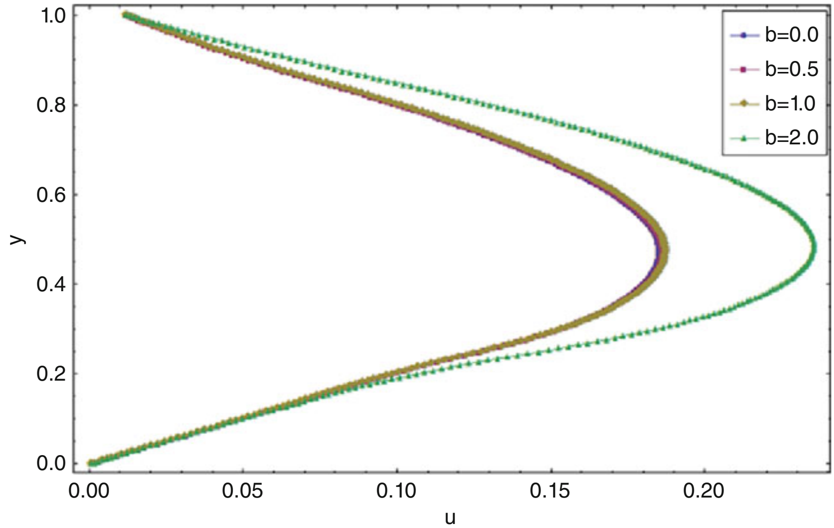

Fig. 8.8 Approximate solutions of $u = f$ for various b

8.5 Conclusion

We studied approximate solutions of nonlinear systems of partial differential equations and multiple solutions of differential equations in the reproducing kernel space in this paper. We demonstrated our results with Tables 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, and 8.9 and Figs. 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11,

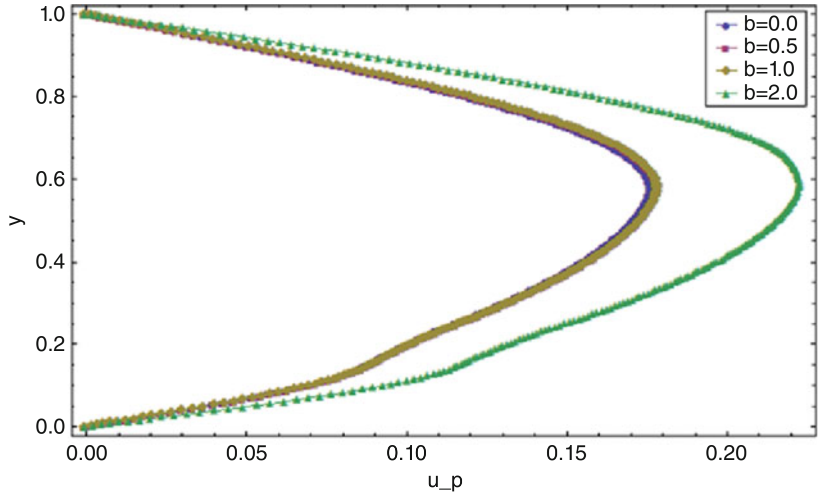

Fig. 8.9 Approximate solutions of $u_p = f_p$ for various b

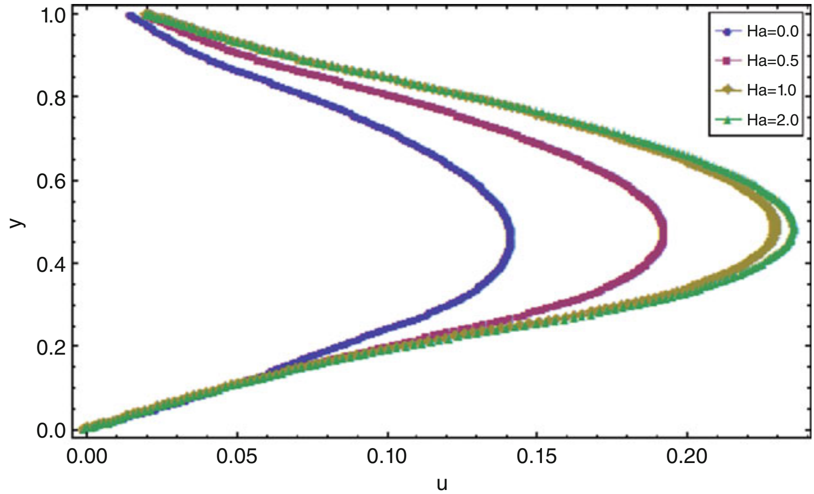

Fig. 8.10 Approximate solutions of $u = f$ for various Ha

8.12, 8.13, 8.14, 8.15, 8.16, 8.17, and 8.18. We proved that the reproducing kernel method is an accurate technique for solving nonlinear systems of partial differential equations and second order differential equations.

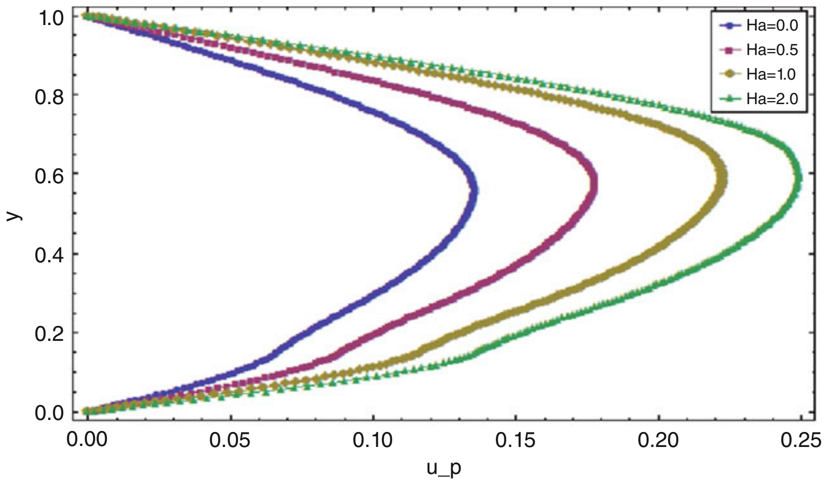

Fig. 8.11 Approximate solutions of $u_p = f_p$ for various Ha

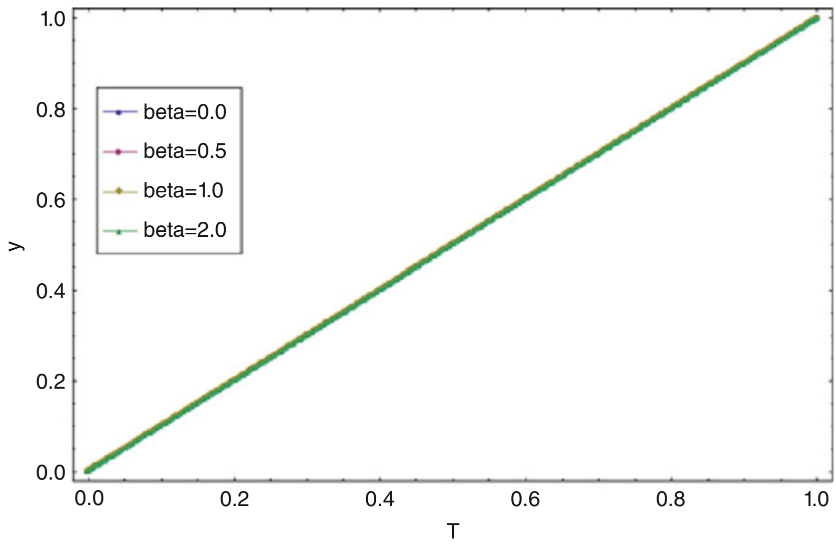

Fig. 8.12 Approximate solutions of T for various β

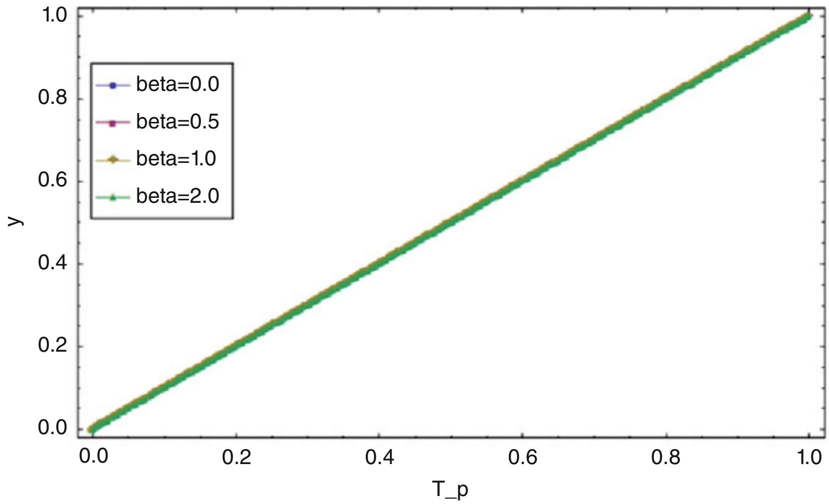

Fig. 8.13 Approximate solutions of T_p for various β

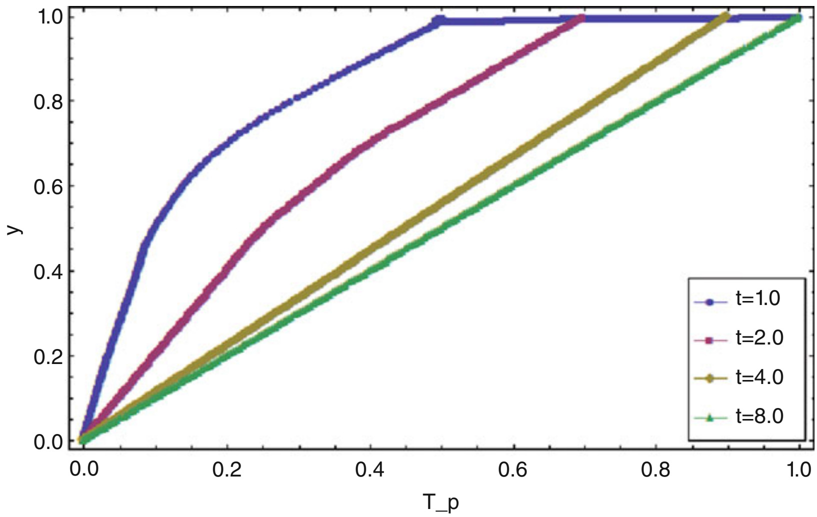

Fig. 8.14 Approximate solutions of T_p for various t

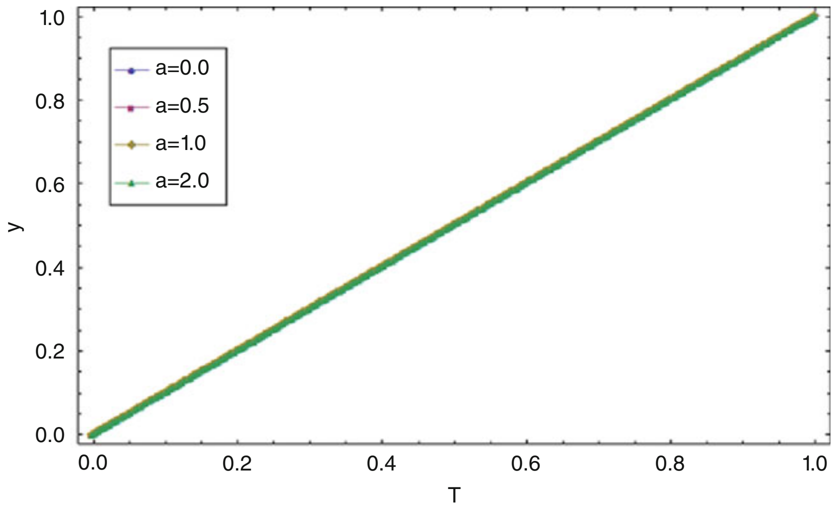

Fig. 8.15 Approximate solutions of T for various a

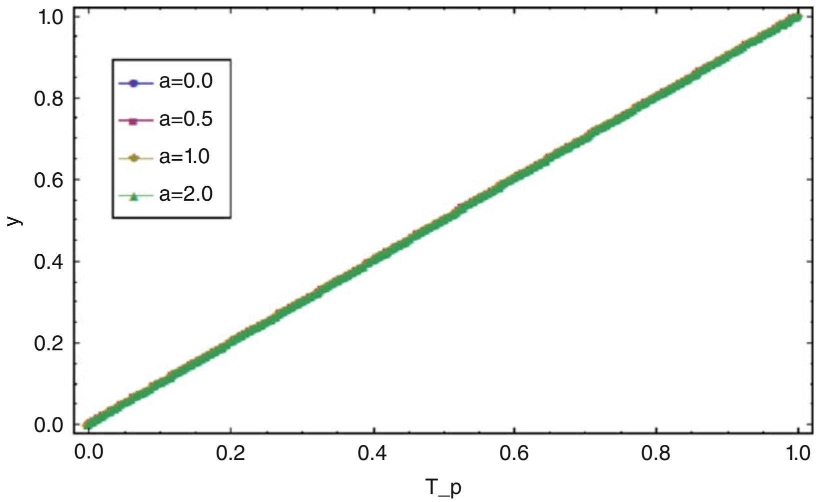

Fig. 8.16 Approximate solutions of T_p for various a

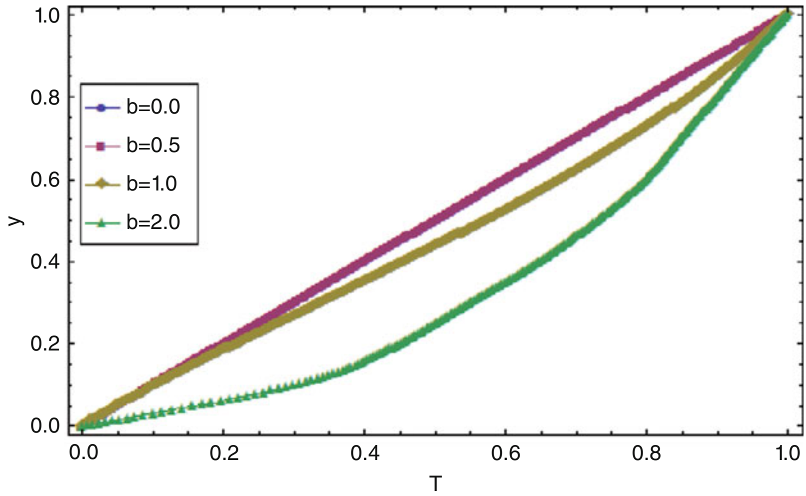

Fig. 8.17 Approximate solutions of T for various b

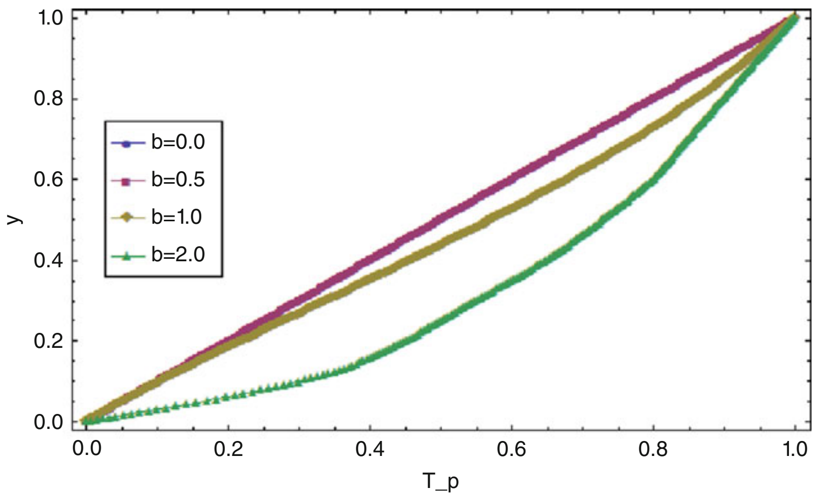

Fig. 8.18 Approximate solutions of T_p for various b

Table 8.1 The numerical results of Example 1 for first solutions when $\theta = 1.51716$, $t_1 = 1.389$, $t_2 = 8.362$, $\lambda = 1$, and $\mu = -1$

x	Exact solution	Approximate solution ($m = 20$)
0.1	−0.049846791242201656122	−0.049300593023852969510
0.2	−0.089189934623040384704	−0.088590959369428596421
0.3	−0.11760909576028070931	−0.11718659278945286749
0.4	−0.13479025387538511030	−0.13459000029373223235
0.5	−0.14053921439128292991	−0.14057905280854341332
0.6	−0.13479025387538511030	−0.13506729939580710291
0.7	−0.11760909576028070931	−0.11811210960560610009
0.8	−0.089189934623040384704	−0.089894391109067316779
0.9	−0.049846791242201656122	−0.050520853813557519252

x	Approximate solution ($m = 40$)
0.1	−0.049615470754930670101
0.2	−0.088955868014683035 65
0.3	−0.11742857480052264011
0.4	−0.1346979968120595907
0.5	−0.14054409043574415912
0.6	−0.134882969385795952
0.7	−0.11777439140332240699
0.8	−0.089415830538767154698
0.9	−0.050108239594912442539

Table 8.2 Absolute errors of Example 1 for first solutions when $\theta = 1.51716$, $t_1 = 1.389$, $t_2 = 8.362$, $\lambda = 1$, and $\mu = -1$

x	Absolute error ($m = 20$)	Absolute error ($m = 40$)
0.1	0.00054619821834868 6612	0.00023132048727098 6021
0.2	0.00059897525361178 8283	0.00023406660835734 9054
0.3	0.00042250297082784182	0.0001805209597580692
0.4	0.00020025358165287795	0.00009225706332551960
0.5	0.00003983841726048341	0.0000048760444612292
0.6	0.00027704552042199261	0.0000927155104108417
0.7	0.00050301384532539078	0.00016529564304169768
0.8	0.00070445648602693 2075	0.00022589591572676 9994
0.9	0.00067406257135586313	0.00026144835271078 6417

Table 8.3 Relative errors of Example 1 for first solutions when $\theta = 1.51716$, $t_1 = 1.389$, $t_2 = 8.362$, $\lambda = 1$, and $\mu = -1$

x	Relative error ($m = 20$)	Relative error ($m = 40$)
0.1	0.01095754018939257757	0.004640629446878895043
0.2	0.0067157270172172023205	0.0026243612504777277994
0.3	0.0035924344804845508337	0.0015349234563117461783
0.4	0.0014856681094911677931	0.00068444906566324999152
0.5	0.0002834683361013182336	0.000034695259129978821097
0.6	0.0020553824364640181301	0.00068785025434077810681
0.7	0.0042769978127428992563	0.0014054664902671738089
0.8	0.0078983854961246965039	0.0025327512199836777494
0.9	0.013522687309613288213	0.0052450387717120915581

Table 8.4 The numerical results of Example 1 for second solutions when $\theta = 10.9387$, $t_1 = 2.247$, $t_2 = 8.034$, $\lambda = 1$, and $\mu = -1$

x	Exact solution	Approximate solution ($m = 20$)
0.1	−1.0772733167967386889	−1.4723999999999999691
0.2	−2.1223923410527448670	−2.6175999999999998767
0.3	−3.0773951005699656976	−3.4355999999999997226
0.4	−3.8061519589366269394	−3.9263999999999995068
0.5	−4.0914672451371118790	−4.0899999999999992295
0.6	−3.8061519589366269394	−3.9263999999999988905
0.7	−3.0773951005699656976	−3.4355999999999984898
0.8	−2.1223923410527448670	−2.6175999999999980275
0.9	−1.0772733167967386889	−1.4723999999999975035

x	Approximate solution ($m = 40$)
0.1	−1.4705999999999999679
0.2	−2.6143999999999998716
0.3	−3.4313999999999997114
0.4	−3.9215999999999994866
0.5	−4.0849999999999991979
0.6	−3.9215999999999988456
0.7	−3.4313999999999984283
0.8	−2.614399999999997947
0.9	−1.4705999999999974016

Table 8.5 Absolute error of Example 1 for second solutions when $\theta = 10.9387$, $t_1 = 2.247$, $t_2 = 8.034$, $\lambda = 1$, and $\mu = -1$

x	Absolute error ($m = 20$)	Absolute error ($m = 40$)
0.1	0.3951266832032612802	0.393326683203261279
0.2	0.4952076589472550097	0.4920076589472550046
0.3	0.358204899430034025	0.3540048994300340138
0.4	0.1202480410633725674	0.1154480410633725472
0.5	0.0014672451371126495	0.0064672451371126811
0.6	0.1202480410633719511	0.1154480410633719062
0.7	0.3582048994300327922	0.3540048994300327307
0.8	0.4952076589472531605	0.49200765894725308
0.9	0.3951266832032588146	0.3933266832032587127

Table 8.6 Relative error of Example 1 for second solutions when $\theta = 10.9387$, $t_1 = 2.247$, $t_2 = 8.034$, $\lambda = 1$, and $\mu = -1$

x	Relative error ($m = 20$)	Relative error ($m = 40$)
0.1	0.36678406217111779588	0.36511317700953941112
0.2	0.23332521954994573834	0.23181748700770863839
0.3	0.11639873585412894767	0.11503394522350042442
0.4	0.031593074149611144625	0.030331957922044377369
0.5	0.00035861099434598545167	0.0015806664821279665282
0.6	0.031593074149610982703	0.030331957922044208957
0.7	0.11639873585412854707	0.11503394522350000748
0.8	0.23332521954994486706	0.23181748700770773158
0.9	0.36678406217111550714	0.3651131770095370289

Table 8.7 The first approximate solutions of Example 2 when $t_1 = 3.026$, $t_2 = 17.629$

x	Approximate solution ($m = 20$)	Approximate solution ($m = 40$)
0.10	0.046249255609840261107	0.047107959922349180118
0.20	0.067769610201103198051	0.069020525808629500991
0.25	0.072542992200350418849	0.073992569159451718608
0.30	0.074434206914517408745	0.076077916140538342689
0.40	0.071867699646151598001	0.07388068138764640163
0.50	0.063644001668100548478	0.065997331390535922972
0.60	0.052169172941129235241	0.054833380565893847229
0.70	0.039097646775136232426	0.042044528121172720677
0.75	0.03233520286288125028	0.035413465852768821882
0.80	0.02557036321912830349	0.02877375910539847071
0.90	0.012365702227148662923	0.015801844129696190555

Table 8.8 The first approximate solutions of Example 2 when $t_1 = 3.026$, $t_2 = 17.629$

x	Approximate solution ($m = 20$ *and* $m = 40$)
$\frac{1}{8}$	0.053459133264424318687
	0.054411793915706876268
$\frac{3}{8}$	0.073154996223772261394
	0.075078284156485197160
$\frac{5}{8}$	0.048999016113865678553
	0.051736471222444720848
$\frac{7}{8}$	0.015604606874545517774
	0.018984660238594126715

Table 8.9 The second approximate solutions of Example 2 when $t_1 = 2.246$, $t_2 = 8.642$

x	Approximate solution ($m = 20$)	Approximate solution ($m = 40$)
1.01	0.99882359294951381045	0.99882235929452273651
1.02	0.99765900933602205292	0.99765655449537351669
1.03	0.99650625041189210164	0.99650258720875611371
1.04	0.99536531943327912521	0.99536046161080033598
1.05	0.99423622266201998919	0.99423018573252504798
1.06	0.99311897036752715301	0.99311177274480118408
1.07	0.99201357782868256676	0.99200524224331477294
1.08	0.99092006633573157252	0.99091062153352994887
1.09	0.98983846419217680040	0.98982794691565197597
1.1	0.98876880771667206865	0.98875726496959026110

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