Nonlinear Systems and Complexity 23

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Kenan Taş · Dumitru Baleanu
J. A. Tenreiro Machado *Editors*

# Mathematical Methods in Engineering

## Theoretical Aspects

Springer

# Nonlinear Systems and Complexity

Volume 23

**Series editor**

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# Preface

This work, organized into two volumes, publishes a selection of most relevant contributions, presented at the International Symposium, MME2017 Mathematical Methods in Engineering, held at Çankaya University, Ankara, Turkey, during April 27–29, 2017.

The first volume of the book *Mathematical Methods in Engineering - Theoretical Aspects*" is divided into two parts, namely, "Fixed point theory and applications" and "New mathematical ideas."

The reader starts by finding an excellent review about the fixed point results on partial metric spaces. This book also includes 13 high-quality contributions. The book expands the works entitled: *Fixed points results for mixed multivalued mappings of Feng-Liu type on Mb-metric spaces, Hyers-Ulam and Hyers-Ulam-Rassias stability for a class of integro-differential equations, Exact solutions, Lie symmetry analysis and conservation laws of the time fractional diffusion-absorption equation, Integral balance approach to 1-D space-fractional models: approximate solutions and analysis, Fractional order filter discretization by particle swarm optimization method, and On the existence of solution for a sum fractional finite difference inclusion.*

In addition, the book contains *Comparison on solving a class of nonlinear systems of partial differential equations and multiple solutions of second order differential equations, Effect of edge deletion and addition on Zagreb indices of graphs, The limit Q-Bernstein operators with varying Q, Localization of the spectral expansions associated with the partial differential operators, Energy decay in a quasilinear system with finite and infinite memories, A new method for solving two-dimensional Bratu differential equation, and A note on the upper bound of average distance via irregularity index.*

The symposium provided a forum for discussing recent developments about theoretical and applied areas of mathematics and engineering with emphasis on the topics fractional calculus and nonlinear analysis.

The members of the organizing committee were Kenan Taş (Turkey), J. A. Tenreiro Machado (Portugal), and Yangjian Cai (China).

All local organizing committee members with leadership of Dumitru Baleanu and all members of Çankaya University, Mathematics Department, as well as the organizers of Special Sessions, Plenary and Invited Speakers, and International Scientific Committee deserve heartfelt thanks.

The editors of this book are grateful to the President of the board of trustees of Çankaya University, Sitki Alp, and to the Rector, Prof. Dr. Hamdi Mollamahmutoglu, for their continuous support of the symposium activities.

We would like to thank all the referees and other colleagues who helped in preparing this book for publication.

Finally, our special thanks are due to Kiruthika Kumar and Michael Luby from Springer, for their continuous help and work in connection with this book.

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# Part I
# Fixed Point Theory and Applications

# Chapter 1
# Advances on Fixed Point Results on Partial Metric Spaces

**Erdal Karapınar, Kenan Taş, and Vladimir Rakočević**

## 1.1 Introduction and Preliminaries

It is a well-accepted fact that the notion of metric space axiomatically formulated first by Maurice René Frechét [52] under the name of $L$-space. The name, metric space, has been used after F. Hausdorff. This notion has crucial roles in various branches of mathematics and quantitative sciences, such as economics, statistics, theoretical physics, engineering, neuroscience, computer vision, computational biology, networking, and so on (see, e.g., [38, 41, 51, 53, 56, 82–85, 116, 117, 121, 128, 129] and the references therein).

**Definition 1** Let $X$ be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called metric if it fulfills the following conditions:

$(d_1)$ $d(x, x) = 0$;
$(d_2)$ $d(x, y) = d(y, x) = 0 \Longrightarrow x = y$;
$(d_3)$ $d(x, y) = d(y, x)$;
$(d_4)$ $d(x, z) \leq d(x, y) + d(y, z)$;

for every $x, y, z \in X$. In addition, the pair $(X, d)$ is called metric space.

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On account of the problems of applications, the researchers need to extend and improve the notion of metric spaces in various ways, like modular metric space, quasi-metric spaces, ultra metric space, uniform spaces, $D$-metric space, symmetric spaces, fuzzy metric space, cone metric spaces, TVS-metric space, complex valued metric space, $G$-metric space, partial metric space, statistical metric space, rectangular (Branciari) metric space, $b$-metric space, and so on (see, e.g., [22, 23, 28–30, 35, 36, 40, 46, 54, 57, 61–64, 86, 88, 90, 91]). Among them, we shall focus on very interesting generalization metric spaces, namely, partial metric spaces.

The concept of partial metric spaces was proposed by Matthews [85] in 1992 to solve the problems of computer science, especially, to domain theory and semantics, by transferring the structure of metric space. The most important difference of partial metric rather than the standard metric is the existing possibility of non-zero self-distance. In other words, in partial metric, self-distance, $p(x, x)$, need not to be zero.

At the first glance, it seems that such axiom is superfluous and it is not a natural axiom. Particularly considering the well-known examples of metric space, such axiom seems impossible. On the other hand, by regarding Baire metric, we can visualize how self-distance can be non-zero in a very natural way. More precisely, consider a distance function on the set of all infinite sequences $\omega$ as follows:

$$p : \omega \times \omega : [0, \infty) \text{ with } p(x, y) = 2^{-\sup\{n | \forall i < n \text{ such that } x_i = y_i\}}. \tag{1.1}$$

One can easily check that the function $p(x, y)$, defined above, forms a standard metric on $\omega$. This metric is known as Baire metric in the literature. Matthews [85] extended the domain of the above function $\omega$ by replacing with a more general set, the set of all finite and infinite sequences, $\omega_f$. For instance, let $x \in \omega_f$ such that $x = (x_1, x_2, x_3, x_4, x_5)$, thus, $p(x, x) = \frac{1}{2^5} \neq 0$ (for more details, see, e.g. [41]). Indeed, in computer programming, finite sequences have been used rather than infinite sequences. That is why Matthews [85] proposed the notion of partial metric regarding the similarity with standard metric. After that, the topological structure and various applications of them have been investigated by several authors independently (see, e.g., [51, 56, 81, 83, 116, 117, 121, 128, 129] and the references therein).

In this study, we focus on the fixed point problems of certain mappings in the context of partial metric. This direction was also initiated by Matthews [85], by proving the analog of celebrated Banach Contraction Mapping Principle in the setting of partial metric spaces. Following this pioneer result, several papers have been reported on the existence and uniqueness of various operators in the frame of the partial metric spaces (see, e.g., [1, 3–13, 19–21, 24, 41, 47, 66–72, 75, 76, 93–111, 119, 120] and the references therein).

Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers, respectively. Furthermore, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ represent the set of reals, positive reals, and nonnegative reals, respectively.

We recall the notion of a partial metric introduced by Matthews [85].

**Definition 2 ([85])** A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}_0^+$ such that for all $x, y, z \in X$,

$(p_1)$ $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
$(p_2)$ $p(x, x) \leq p(x, y)$;
$(p_3)$ $p(x, y) = p(y, x)$;
$(p_4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A pair $(X, p)$ is called a partial metric space.

*Remark 1* If $p(x, y) = 0$, then from $(p_1)$ and $(p_2)$, we have $x = y$. The converse may not hold. For example, let $X = \mathbb{R}_0^+$ and $p : X \times X \to \mathbb{R}_0^+$ be $p(x, y) = \max\{x, y\}$. Then $(X, p)$ is a partial metric space and $p(x, x) \neq 0$ for all $x \in X \setminus \{0\}$.

*Example 1 (See, e.g., [76, 120])* Let $(X, d)$ be standard metric space and $(X, p)$ be a partial metric space. Consider the mappings $\rho_i : X \times X \longrightarrow \mathbb{R}_0^+$ $(i \in \{1, 2, 3\})$ defined by

$$\rho_1(x, y) = p(x, y) + p(x, y)$$
$$\rho_2(x, y) = p(x, y) + \max\{\omega(x), \omega(y)\}$$
$$\rho_3(x, y) = p(x, y) + a$$

It is clear that the functions $\rho_1, \rho_2, \rho_3$ form a partial metrics on $X$, where $\omega : X \longrightarrow \mathbb{R}_0^+$ is an arbitrary function and $a \geq 0$.

*Example 2 (See [85])* Let $X = \{[a, b] : a, b, \in \mathbb{R},\ a \leq b\}$ and define $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then $(X, p)$ is a partial metric space.

*Example 3 (See [85])* Let $X := [0, 1] \cup [2, 3]$ and define $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \begin{cases} \max\{x, y\} & \text{if } \{x, y\} \cap [2, 3] \neq \emptyset, \\ |x - y| & \text{if } \{x, y\} \subset [0, 1]. \end{cases}$$

Then, $(X, p)$ is a complete partial metric space.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x, \gamma) : x \in X, \gamma > 0\}$, where $B_p(x, \gamma) = \{y \in X : p(x, y) < p(x, x) + \gamma\}$ for all $x \in X$ and $\gamma > 0$. If $p$ is a partial metric on $X$, then the function $d_p : X \times X \to \mathbb{R}_0^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on $X$.

The functions $d_m^p$ and $p_0$ defined on $X \times X$ by

$$\begin{aligned} d_m^p(x, y) &= \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\} \\ &= p(x, y) - \min\{p(x, x), p(y, y)\}, \end{aligned} \tag{1.2}$$

and

$$\begin{cases} p_0(x, x) = 0, & \text{for all } x \in X \\ p_0(x, y) = p(x, y), & \text{for all } x \neq y \end{cases} \tag{1.3}$$

are also metrics on $X$ (see [9] and [118], respectively).

Observe that if $p$ is a metric on $X$ then $p = d_m^p$.

The following topological inclusions are well-known and easy to check: $\tau_p \subseteq \tau_{d_p} = \tau_{d_p^m} \subseteq \tau_{p_0}$.

Pay attention to the fact that in the partial metric space $(X, p)$ mentioned in Remark 1 both $d_p$ and $d_p^m$ are the Euclidean metric on $X$.

**Definition 3 ([21, 85])** Let $(X, p)$ be a partial metric space. Then

(1) a sequence $\{x_n\}$ in $(X, p)$ converges to $x \in X$ if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$;
(2) a sequence $\{x_n\}$ in $(X, p)$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists (and is finite);
(3) $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{m,n\to\infty} p(x_m, x_n)$;
(4) a subset $A$ of a partial metric space $(X, p)$ is closed in $(X, p)$ if it contains its limit points, that is, if a sequence $\{x_n\}$ in $A$ converges to some $x \in X$, then $x \in A$.
(5) a subset $A$ of a partial metric space $(X, p)$ is bounded in $(X, p)$ if there exist $x_0 \in X$ and $M \in \mathbb{R}$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(a, a) + M$.

**Definition 4** Let $(X, p)$ be a partial metric space. A self-mapping $T$ on $X$ is called continuous, if for each sequence $\{x_n\}$ in $X$ converges to $u \in X$, that is,

$$\lim_{n\to\infty} p(x_n, u) = \lim_{n\to\infty} p(x_n, x_{n+k}) = p(u, u) \tag{1.4}$$

provides

$$\lim_{n\to\infty} p(Tx_n, Tu) = \lim_{n\to\infty} p(Tx_n, Tx_{n+k}) = P(Tu, Tu). \tag{1.5}$$

Notice that the equality (1.5) can be expressed as

$$\begin{aligned} &\lim_{n\to\infty} p(Tx_n, Tu) \\ &\quad = \lim_{n\to\infty} p(x_{n+1}, Tu) = \lim_{n\to\infty} p(x_{n+1}, x_{n+k+1}) = p(u, u) \\ &\quad = \lim_{n\to\infty} p(Tx_n, Tx_{n+k}) = P(Tu, Tu). \end{aligned} \tag{1.6}$$

*Remark 2* The limit in a partial metric space may not be unique. For example, consider the sequence $\left\{\frac{1}{n^2+n}\right\}_{n\in\mathbb{N}}$ in the partial metric space $(X, p)$ where $p(x, y) = \max\{x, y\}$. Note

$$p(1, 1) = \lim_{n\to\infty} p\left(1, \frac{1}{n^2+n}\right) \quad \text{and} \quad p(2, 2) = \lim_{n\to\infty} p\left(2, \frac{1}{n^2+n}\right).$$

**Lemma 1 ([85, 92])**

*(i)* $\{x_n\}$ *is a Cauchy sequence in a partial metric space* $(X, p)$ *if and only if it is a Cauchy sequence in the metric space* $(X, d_p)$*;*
*(ii)* *A partial metric space* $(X, p)$ *is complete if and only if the metric space* $(X, d_p)$ *is complete. Furthermore,* $\lim_{n\to\infty} d_p(x_n, x) = 0$ *if and only if* $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_m)$.
*(iii)* *If* $\{x_n\}$ *is a convergent sequence in* $(X, d_p)$*, then it is a convergent sequence in the partial metric space* $(X, p)$.

**Lemma 2 ([1])** *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in a partial metric space* $X$ *such that*

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x), \tag{1.7}$$

*and*

$$\lim_{n\to\infty} p(y_n, y) = \lim_{n\to\infty} p(y_n, y_n) = p(y, y), \tag{1.8}$$

*then* $\lim_{n\to\infty} p(x_n, y_n) = p(x, y)$*. In particular,* $\lim_{n\to\infty} p(x_n, z) = p(x, z)$ *for every* $z \in X$.

**Lemma 3 (See e.g.[67])** *Let* $(X, p)$ *be a partial metric space. Then*

*(A)* *If* $p(x, y) = 0$*, then* $x = y$*,*
*(B)* *If* $x \neq y$*, then* $p(x, y) > 0$.

Fixed point theory in partial metric spaces was studied by many authors in the literature (see [1, 2, 4, 9–21, 24, 25, 43, 44, 47, 55, 59, 60, 66–73, 75–77, 81, 83–85, 92, 94, 100, 105–111, 113, 115–117, 119, 120, 124–129] and the reference therein). On the other hand, we should underline that all fixed point results in partial metric spaces do not bring a novelty. For instance, in [55], the authors proved that certain fixed point results in the context of partial metric spaces are equivalent to corresponding results in the frame of the standard metric spaces. More precisely, for a mapping $T : X \to X$ where $X \neq \emptyset$ it was observed that

$$M_d^T(x, y) = M_p^T(x, y), \text{ with}$$

$$M_\rho^T(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(Ty, y), \rho(Tx, y), \rho(x, Ty)\}$$

where $\rho = d, p$ are metric, partial metric, respectively. For the presented results in this work, the approaches and techniques used in [55] are not applicable.

Let $\Psi$ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

($\Psi_1$) $\psi$ is nondecreasing;
($\Psi_2$) there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \le a\psi^k(t) + v_k, \tag{1.9}$$

for $k \ge k_0$ and any $t \in \mathbb{R}^+$.

In the literature such functions are called as $(c)$-comparison functions (see [112] and also [37, 101, 102]).

**Lemma 4 (See, e.g., [112])** *If $\psi \in \Psi$, then the following hold:*

*(i)* $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* $0$ *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$*;*
*(ii)* $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$;
*(iii)* $\psi$ *is continuous at* $0$*;*
*(iv)* *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

Recently, Samet et al. [114] suggested a new contraction type self-mapping to unify several existing results in the literature by auxiliary functions.

**Definition 5** Let $\alpha : X \times X \to [0, \infty)$. A self-mapping $T : X \to X$ is called $\alpha$-admissible if the condition

$$\alpha(x, y) \ge 1 \Longrightarrow \alpha(Tx, Ty) \ge 1,$$

is satisfied for all $x, y \in X$.

**Definition 6** Let $T$ be a self-mapping defined on a $b$-metric space $(X, d)$. Then, $T$ is called an $\alpha - \psi$ contractive mapping if there exist two auxiliary mappings $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)p(Tx, Ty) \le \psi(p(x, y)), \quad \text{for all } x, y \in X.$$

The main results in [114] are the following fixed point theorems.

**Theorem 1** *Let $T : X \to X$ be an $\alpha - \psi$ contractive mapping where $(X, d)$ is a complete b-metric space. Suppose that*

*(i)* *$T$ is $\alpha$-admissible;*
*(ii)* *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
*(iii)* *either $T$ is continuous, or*
*(iii)'* *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$ for all $n$.*

*Then, there exists $u \in X$ such that $Tu = u$.*

**Theorem 2** *Adding to the hypotheses of Theorem 1 the condition: For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, we obtain uniqueness of the fixed point.*

Note that Banach fixed point theorem is covered by Theorem 2 by letting $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, where $k \in (0, 1)$. For more interesting consequence of Theorem 2, see, e.g., [74, 114].

Popescu [100] suggested the concept of $\alpha$-orbital admissible as a refinement of the alpha-admissible notion, defined in [78, 114].

**Definition 7 ([100])** Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. We say that $T$ is an $\alpha$-orbital admissible if

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

Notice that each $\alpha$-admissible mapping is an $\alpha$-orbital admissible. For more details and counterexamples, see, e.g., [79, 100].

## 1.2 Basic Fixed Point Results

In this section, we shall state and prove a fixed point theorem that covers and unifies several existing results in the literature.

**Definition 8** Let $T$ be a self-mapping defined on a partial metric space $(X, p)$. We say that $T$ is an $(\alpha - \psi)$-type *$K$-contraction* if there exist mappings $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)p(Tx, Ty) \leq \psi(K(x, y)) \text{ for all } x, y \in X, \tag{1.10}$$

where

$$K(x, y) := a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Ty) + a_4[p(x, Ty) + p(y, Tx)], \tag{1.11}$$

where $0 \leq a_i \leq 1,\ i = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + 2a_4 \leq 1$.

**Theorem 3** *Let $(X, p)$ be a complete partial metric space and let $T : X \to X$ be an $(\alpha - \psi)$-type $K$-contraction. Suppose that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) $T$ is continuous.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$.*

*Proof* On account of the condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We shall construct an iterative sequence $\{x_n\}$ in $X$ by

$$x_{n+1} = Tx_n \text{ for all } n \geq 0.$$

Observe that if $x_{n_0} = x_{n_0+1}$ for some $n_0$, then $u = x_{n_0}$ is a fixed point of $T$. Thus, throughout the proof, we shall assume that $x_n \neq x_{n+1}$ for all $n$. Regarding that $T$ is $\alpha$-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Inductively, we find

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } \ n = 0, 1, \ldots \tag{1.12}$$

Due to (1.10) and (1.77), it follows, for all $n \geq 1$, that

$$p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1})p(Tx_n, Tx_{n-1}) \leq \psi(K(x_n, x_{n-1})), \tag{1.13}$$

where

$$\begin{aligned} K(x_n, x_{n-1}) &= a_1 p(x_n, x_{n-1}) + a_2 p(x_n, Tx_n) + a_3 p(x_{n-1}, Tx_{n-1}) \\ &\quad + a_4[p(x_n, Tx_{n-1}) + p(x_{n-1}, Tx_n)] \\ &= a_1 p(x_n, x_{n-1}) + a_2 p(x_n, x_{n+1}) + a_3 p(x_{n-1}, x_n) \\ &\quad + a_4[p(x_n, x_n) + p(x_{n-1}, x_{n+1})]. \end{aligned}$$

Taking $(p_4)$ into account, we derive that

$$p(x_{n-1}, x_{n+1}) + p(x_n, x_n) \leq p(x_n, x_{n-1}) + p(x_n, x_{n+1})$$

Hence,

$$K(x_n, x_{n-1}) = (a_1 + a_3 + a_4)p(x_n, x_{n-1}) + (a_2 + a_4)p(x_n, x_{n+1}).$$

If for some $n \geq 1$, we have $p(x_n, x_{n-1}) \leq p(x_n, x_{n+1})$, then we get

$$K(x_n, x_{n-1}) \leq (a_1 + a_2 + a_3 + 2a_4)p(x_n, x_{n+1}) \leq p(x_n, x_{n+1}), \tag{1.14}$$

since $a_1 + a_2 + a_3 + a_4 \leq 1$. By taking (1.14) in consideration together with the fact that $\psi$ is a nondecreasing function, we obtain from the inequality (1.13) that

$$p(x_{n+1}, x_n) \leq \psi(K(x_n, x_{n-1})) \leq \psi(p(x_n, x_{n+1})) < p(x_n, x_{n+1}),$$

a contradiction. Thus, for all $n \geq 1$, we have

$$p(x_n, x_{n+1}) \leq p(x_n, x_{n-1}). \tag{1.15}$$

Using (1.13) and (1.15), we get that

$$p(x_{n+1}, x_n) \le \psi(p(x_n, x_{n-1})), \tag{1.16}$$

for all $n \ge 1$. By induction, we get

$$p(x_{n+1}, x_n) \le \psi^n(p(x_1, x_0)), \quad \text{for all} \;\; n \ge 1. \tag{1.17}$$

Due to Lemma (1.17) (i), we find that

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{1.18}$$

Again by keeping the expression (1.17) in the mind, and by using the triangular inequality ($p_4$), for all $k \ge 1$, we have

$$\begin{aligned} p(x_n, x_{n+k}) &\le p(x_n, x_{n+1}) + \cdots + p(x_{n+k-1}, x_{n+k}) - \sum_{j=1}^{k-1} \big(p(x_{n+j}, x_{n+j})\big) \\ &\le \sum_{j=n}^{n+k-1} \psi^j\,(p(x_1, x_0)) \\ &\le \sum_{j=n}^{+\infty} \psi^j\,(p(x_1, x_0)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} p(x_n, x_{n+k}) = 0,$$

and hence $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, p)$ is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} p(x_n, u) = 0 = \lim_{n\to\infty} p(x_n, x_{n+k}) = p(u, u). \tag{1.19}$$

Since $T$ is continuous, by Definition 4, we conclude from (1.19) that

$$\lim_{n\to\infty} p(x_{n+1}, Tu) = \lim_{n\to\infty} p(Tx_n, Tu) = 0. \tag{1.20}$$

On account of Lemma 2 together with (1.19) and (1.89), we find that $u$ is a fixed point of $T$, that is, $Tu = u$.

**Definition 9** We say that a non-empty set $X$ is regular if for each iterative sequence $\{x_n\}$ is in $X$ provides that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for all $k$.

**Theorem 4** *Let $(X, p)$ be a complete partial metric space and let $T : X \to X$ be an $(\alpha - \psi)$-type $K$-contraction. Suppose that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$.*

*Proof* As it is recognized easily, only the difference between Theorems 3 and 4 is the condition (iii). By following the lines in the proof of Theorem 3, we find an iterative sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. On account of (1.77) and the condition (iii) of Theorem 4, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all $k$. Applying (1.10), for all $k$, we get that

$$\begin{aligned} p(x_{n(k)+1}, Tu) = p(Tx_{n(k)}, Tu) &\leq \alpha(x_{n(k)}, u)p(Tx_{n(k)}, Tu) \\ &\leq \psi(K(x_{n(k)}, u)) < K(x_{n(k)}, u), \end{aligned} \tag{1.21}$$

where

$$\begin{aligned} K(x_{n(k)}, u) = a_1 p(x_{n(k)}, u) &+ a_2 p(x_{n(k)}, x_{n(k)+1}) \\ &+ a_3 p(u, Tu) + a_4[p(x_{n(k)}, Tu) + p(u, x_{n(k)+1})]. \end{aligned}$$

Letting $k \to \infty$ in the equality (1.21), we get that

$$p(u, Tu) \leq (a_3 + a_4)p(u, Tu), \tag{1.22}$$

which is a contradiction. Thus we have $p(u, Tu) = 0$, that is, $u = Tu$.

For the uniqueness of a fixed point derived in Theorems 3 and 4, we will consider the following hypothesis. (U) For all $u, v \in Fix(T)$, then $\alpha(u, v) \geq 1$.

**Theorem 5** *Putting condition (U) to the statements of Theorem 3 (resp. Theorem 4), we find that u is the unique fixed point of $T$.*

*Proof* Let $u, \nu$ be two distinct fixed point of $T$ and $p(u, \nu) > 0$. Note that in case $p(u, \nu) = 0$, there is nothing to prove. Due to the property of $\psi$, we have $\psi(pu, \nu)) > 0$.

On account of the condition (U) and the assumption of Theorem 3 (resp. Theorem 4)

$$\begin{aligned} p(u, \nu) &\leq \alpha(u, \nu)p(Tu, T\nu) \\ &\leq \psi(K(u, \nu)) = \psi(p(u, \nu)) \\ &< p(u, \nu), \end{aligned}$$

which is a contradiction. Thus, $u = \nu$.

**Definition 10** Let $T$ be a self-mapping defined on a partial metric space $(X, p)$. We say that $T$ is an $(\alpha - \psi)$-type *N-contraction* if there exist mappings $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)p(Tx, Ty) \leq \psi(N(x, y)) \text{ for all } x, y \in X, \tag{1.23}$$

where

$$N(x, y) := a_1 p(x, y) + a_2[p(x, Tx) + p(y, Ty)] + a_3[p(x, Ty) + p(y, Tx)], \tag{1.24}$$

where $0 \leq a_i \leq 1,\ i = 1, 2, 3$, and $a_1 + 2a_2 + 2a_3 \leq 1$.

**Theorem 6** *Let $(X, p)$ be a complete partial metric space and let $T : X \to X$ be an $(\alpha - \psi)$-type $N$-contraction. Suppose that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) either $T$ is continuous or $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$.*

Since $N(x, y) \leq K(x, y)$ for all $x, y \in X$, the proof of Theorem 6 can be derived as a consequence of Theorems 3 and 4.

On the other hand, for the uniqueness of a fixed point of the operator, defined in of Theorem 6, we shall propose a weaker condition:

$(H)$ For all $x, y \in \text{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$,

where $\text{Fix}(T)$ denotes the set of fixed points of $T$.

**Theorem 7** *Adding condition $(H)$ to the hypotheses of Theorem 6, we obtain that $u$ is the unique fixed point of $T$.*

*Proof* Suppose that $v$ is another fixed point of $T$. From (H), there exists $z \in X$ such that

$$\alpha(u, z) \geq 1 \text{ and } \alpha(v, z) \geq 1. \tag{1.25}$$

Since $T$ is $\alpha$-admissible and $u, v$ are the fixed point of $T$, the inequalities in (1.25) yield that

$$\alpha(u, T^n z) \geq 1 \text{ and } \alpha(v, T^n z) \geq 1, \quad \text{for all } n. \tag{1.26}$$

We construct an iterative sequence $\{z_n\}$ in $X$ by $z_{n+1} = Tz_n$ for all $n \geq 0$ and $z_0 = z$. From (1.26), for all $n$, we have

$$p(u, z_{n+1}) = p(Tu, Tz_n) \leq \alpha(u, z_n)p(Tu, Tz_n) \leq \psi(K(u, z_n)), \tag{1.27}$$

where

$$
\begin{aligned}
K(u, z_n) &= a_1 p(u, z_n) + a_2 p(u, Tu) + a_3 p(z_n, Tz_n) \\
&\quad + a_3[p(u, Tz_n) + p(z_n, Tu)] \\
&= a_1 p(u, z_n) + a_2[p(u, u) + p(z_n, z_{n+1})] + a_3[p(u, z_{n+1}) + p(z_n, u)] \\
&\leq a_1 p(u, z_n) + a_2[p(u, z_{n+1}) + p(u, z_{n+1}) - p(u, u)] \\
&\quad + a_4[p(u, z_{n+1}) + p(z_n, u)] \\
&= (a_1 + a_2 + a_2) p(u, z_n) + (a_2 + a_3 d)(u, z_{n+1}) + .
\end{aligned}
$$

Without loss of generality, we can suppose that $p(u, z_n) > 0$ for all $n$. If we have $p(u, z_n) \leq p(u, z_{n+1})$, then due to the monotone property of $\psi$, and the inequality (1.27), we find that

$$p(u, z_{n+1}) \leq \psi((a_1 + 2a_2 + 2a_3) p(u, z_{n+1})), \tag{1.28}$$

By keeping (1.27) and the monotone property of $\psi$ in the mind, if $\max\{p(u, z_n), p(u, z_{n+1})\} = p(u, z_{n+1})$, we get that, for all $n$,

$$p(u, z_{n+1}) \leq \psi((a_1 + 2a_2 + 2a_3) p(u, z_{n+1})) \leq \psi(p(u, z_{n+1})) < p(u, z_{n+1}),$$

which is a contradiction. Thus we have $\max\{p(u, z_n), p(u, z_{n+1})\} = p(u, z_n)$, and

$$p(u, z_{n+1}) \leq \psi(p(u, z_n)),$$

for all $n$. This implies that

$$p(u, z_n) \leq \psi^n(p(u, z_0)), \quad \text{for all} \;\; n \geq 1.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} p(z_n, u) = 0. \tag{1.29}$$

Similarly, one can show that

$$\lim_{n \to \infty} p(z_n, v) = 0. \tag{1.30}$$

From (1.29) and (1.30), it follows that $u = v$. Thus we proved that $u$ is the unique fixed point of $T$.

### 1.2.1 Some Immediate Consequences Due to Choice of Coefficients $a_1, a_2, a_3, a_4$

The following results can be derived from Theorems 3 to 7 by choosing the distinct coefficients $a_1, a_2, a_3, a_4$.

**Theorem 8** *Let $(X, p)$ be a complete partial metric space and $\alpha : X\times X \to [0,\infty)$ and $\psi \in \Psi$ Suppose that $T : X \to X$ fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq \psi(a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Ty)) \text{ for all } x, y \in X, \quad (1.31)$$

*where $0 \leq a_i \leq 1,\ i = 1, 2, 3,$ and $a_1 + a_2 + a_3 \leq 1$. Assume also that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) either $T$ is continuous or $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$. Additionally, if the condition $(U)$ is satisfied, then $u$ is the unique fixed point of $T$.*

*Proof* It is evident that

$$a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Ty) \leq K(x, y) \text{ for all } x, y \in X.$$

On account of the property $(\Psi_1)$, the desired results are observed from Theorem 5.

**Theorem 9** *Let $(X, p)$ be a complete partial metric space and $\alpha : X\times X \to [0,\infty)$ and $\psi \in \Psi$ Suppose that $T : X \to X$ fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq \psi(a_1 p(x, y) + a_2[p(x, Tx) + p(y, Ty)]) \text{ for all } x, y \in X, \quad (1.32)$$

*where $0 \leq a_i \leq 1,\ i = 1, 2,$ and $a_1 + 2a_2 \leq 1$. Suppose that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) either $T$ is continuous or $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$. Additionally, if the condition $(H)$ is satisfied, then $u$ is the unique fixed point of $T$.*

*Proof* It is clear that

$$a_1 p(x, y) + a_2[p(x, Tx) + p(y, Ty)] \leq N(x, y) \text{ for all } x, y \in X.$$

Taking the property $(\Psi_1)$ into account, the desired results are derived from Theorem 7.

**Theorem 10** *Let $(X, p)$ be a complete partial metric space and $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ Suppose that $T : X \to X$ fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq \psi(b_1 p(x, y) + b_2[p(x, Ty) + p(y, Tx)]) \quad \text{for all} \quad x, y \in X, \tag{1.33}$$

*where $0 \leq b_i \leq 1,\ i = 1, 2,$ and $b_1 + 2b_2 \leq 1$. Suppose that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) either $T$ is continuous or $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$. Additionally, if the condition $(H)$ is satisfied, then $u$ is the unique fixed point of $T$.*

*Proof* Keeping the property $(\Psi_1)$ in mind, together with the inequality below,

$$b_1 p(x, y) + b_2[p(x, Ty) + p(y, Tx)] \leq N(x, y) \text{ for all } x, y \in X,$$

we conclude the desired result from Theorem 7.

**Theorem 11** *Let $(X, p)$ be a complete partial metric space and $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ Suppose that $T : X \to X$ fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq \psi\left(\frac{[p(x, Tx) + p(y, Ty)]}{2}\right) \quad \text{for all} \quad x, y \in X. \tag{1.34}$$

*Suppose that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) either $T$ is continuous or $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$. Additionally, if the condition $(H)$ is satisfied, then $u$ is the unique fixed point of $T$.*

*Proof* The result follows from Theorem 7 due to the property $(\Psi_1)$ together with the inequality below:

$$\frac{[p(x, Tx) + p(y, Ty)]}{2} \leq N(x, y) \text{ for all } x, y \in X.$$

**Theorem 12** *Let $(X, p)$ be a complete partial metric space and $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ Suppose that $T : X \to X$ fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq \psi\left(\frac{p(x, Ty) + p(y, Tx)}{2}\right) \quad \text{for all} \quad x, y \in X. \tag{1.35}$$

*Suppose also that*

*(i)* $T$ *is* $\alpha$*-orbital admissible;*
*(ii)* *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
*(iii)* *either* $T$ *is continuous or* $X$ *is regular.*

*Then there exists* $u \in X$ *such that* $Tu = u$ *and* $p(u, u) = 0$*. Additionally, if the condition* $(H)$ *is satisfied, then* $u$ *is the unique fixed point of* $T$*.*

*Proof* The result is derived from Theorem 7 due to the property $(\Psi_1)$ and the inequality below:

$$\frac{p(x, Ty) + p(y, Tx)}{2} \leq N(x, y) \text{ for all } x, y \in X.$$

**Theorem 13** *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$ *and* $\psi \in \Psi$ *Suppose that* $T : X \to X$ *fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq \psi(p(x, y)) \text{ for all } x, y \in X. \tag{1.36}$$

*Suppose also that*

*(i)* $T$ *is* $\alpha$*-orbital admissible;*
*(ii)* *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
*(iii)* *either* $T$ *is continuous or* $X$ *is regular.*

*Then there exists* $u \in X$ *such that* $Tu = u$ *and* $p(u, u) = 0$*. Additionally, if the condition* $(H)$ *is satisfied, then* $u$ *is the unique fixed point of* $T$*.*

*Proof* Keeping Theorem 7 in mind, the inequality

$$p(x, y) \leq N(x, y) \text{ for all } x, y \in X,$$

and the property $(\Psi_1)$ yields the result.

Notice that, in this section, for the uniqueness of the fixed point, we use property $(U)$ only in Theorem 8. For the other theorems, we use property $(H)$ instead of $(U)$. It is clear that the condition $(U)$ is stronger than the condition $(H)$.

### 1.2.2 Some Consequences Due to Choice of $\psi$

In this part, we list some consequences of Theorems 3–7 by choosing $\psi(t) = kt$ for $k \in [0, 1)$. Notice that the class of $\Psi$ is very wide and it is possible to derive more consequences for different choice of $\psi$ apart from $\psi(t) = kt$.

**Theorem 14** *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$*. Suppose that* $T : X \to X$ *fulfills*

$$\alpha(x, y)p(Tx, Ty) \le a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Ty) + a_4[p(x, Ty) + p(y, Tx)] \text{ for all } x, y \in X. \tag{1.37}$$

*where* $0 \le a_i,\ i = 1, 2, 3, 4,$ *and* $a_1 + a_2 + a_3 + 2a_4 < 1$. *Suppose also that*

*(i)* $T$ *is* $\alpha$*-orbital admissible;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \ge 1$*;*
*(iii) either* $T$ *is continuous or* $X$ *is regular.*

*Then there exists* $u \in X$ *such that* $Tu = u$ *and* $p(u, u) = 0$. *Additionally, if the condition* $(U)$ *is satisfied, then* $u$ *is the unique fixed point of* $T$.

The results follow from Theorems 3–5, by choosing $\psi(t) = kt$ for $k \in [0, 1)$. Notice that the effect of $k \in [0, 1)$ can be seen from the revised criteria $a_1 + a_2 + a_3 + 2a_4 < 1$.

**Theorem 15** *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$. *Suppose that* $T : X \to X$ *fulfills*

$$\alpha(x, y)p(Tx, Ty) \le a_1 p(x, y) + a_2[p(x, Tx) + p(y, Ty)] + a_3[p(x, Ty) + p(y, Tx)] \text{ for all } x, y \in X. \tag{1.38}$$

*where* $0 \le a_i,\ i = 1, 2, 3,$ *and* $a_1 + 2a_2 + 2a_3 < 1$. *Suppose also that*

*(i)* $T$ *is* $\alpha$*-orbital admissible;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \ge 1$*;*
*(iii) either* $T$ *is continuous or* $X$ *is regular.*

*Then there exists* $u \in X$ *such that* $Tu = u$ *and* $p(u, u) = 0$. *Additionally, if the condition* $(H)$ *is satisfied, then* $u$ *is the unique fixed point of* $T$.

We skip the proof of this theorem since Theorems 6 and 7, by choosing $\psi(t) = kt$ for $k \in [0, 1)$. As in the consideration above, the criteria $k \in [0, 1)$ makes an effect on the conditions of $0 \le a_i,\ i = 1, 2, 3,$ as $a_1 + 2a_2 + 2a_3 < 1$.

**Theorem 16** *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$. *Suppose that* $T : X \to X$ *fulfills*

$$\alpha(x, y)p(Tx, Ty) \le a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Ty) \text{ for all } x, y \in X. \tag{1.39}$$

*where* $0 \le a_i,\ i = 1, 2, 3,$ *and* $a_1 + a_2 + a_3 < 1$. *Suppose also that*

*(i)* $T$ *is* $\alpha$*-orbital admissible;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \ge 1$*;*
*(iii) either* $T$ *is continuous or* $X$ *is regular.*

*Then there exists* $u \in X$ *such that* $Tu = u$ *and* $p(u, u) = 0$. *Additionally, if the condition* $(U)$ *is satisfied, then* $u$ *is the unique fixed point of* $T$.

**Theorem 17** *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$. *Suppose that* $T : X \to X$ *fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq a_1 p(x, y) + a_2[p(x, Tx) + p(y, Ty)] \text{ for all } x, y \in X. \tag{1.40}$$

*where* $0 \leq a_i,\ i = 1, 2,$ *and* $a_1 + 2a_2 < 1$. *Suppose also that*

*(i)* $T$ *is* $\alpha$*-orbital admissible;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
*(iii) either* $T$ *is continuous or* $X$ *is regular.*

*Then there exists* $u \in X$ *such that* $Tu = u$ *and* $p(u, u) = 0$. *Additionally, if the condition* $(H)$ *is satisfied, then* $u$ *is the unique fixed point of* $T$.

**Theorem 18** *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$. *Suppose that* $T : X \to X$ *fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq a_1 p(x, y) + a_3[p(x, Ty) + p(y, Tx)] \text{ for all } x, y \in X. \tag{1.41}$$

*where* $0 \leq a_i,\ i = 1, 3,$ *and* $a_1 + 2a_3 < 1$. *Suppose also that*

*(i)* $T$ *is* $\alpha$*-orbital admissible;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
*(iii) either* $T$ *is continuous or* $X$ *is regular.*

*Then there exists* $u \in X$ *such that* $Tu = u$ *and* $p(u, u) = 0$. *Additionally, if the condition* $(H)$ *is satisfied, then* $u$ *is the unique fixed point of* $T$.

**Theorem 19** *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$. *Suppose that* $T : X \to X$ *fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq a_2[p(x, Tx) + p(y, Ty)] \text{ for all } x, y \in X. \tag{1.42}$$

*where* $0 \leq 2a_2 < 1$. *Suppose also that*

*(i)* $T$ *is* $\alpha$*-orbital admissible;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
*(iii) either* $T$ *is continuous or* $X$ *is regular.*

*Then there exists* $u \in X$ *such that* $Tu = u$ *and* $p(u, u) = 0$. *Additionally, if the condition* $(H)$ *is satisfied, then* $u$ *is the unique fixed point of* $T$.

**Theorem 20** *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$. *Suppose that* $T : X \to X$ *fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq a_3[p(x, Ty) + p(y, Tx)] \text{ for all } x, y \in X. \tag{1.43}$$

*where* $0 \leq 2a_3 < 1$. *Suppose also that*

*(i)* *$T$ is $\alpha$-orbital admissible;*
*(ii)* *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii)* *either $T$ is continuous or $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$. Additionally, if the condition $(H)$ is satisfied, then $u$ is the unique fixed point of $T$.*

**Theorem 21** *Let $(X, p)$ be a complete partial metric space and $\alpha : X \times X \to [0, \infty)$. Suppose that $T : X \to X$ fulfills*

$$\alpha(x, y)p(Tx, Ty) \leq a_1 p(x, y) \quad \text{for all} \quad x, y \in X. \tag{1.44}$$

*where $0 \leq a_1 < 1$. Suppose also that*

*(i)* *$T$ is $\alpha$-orbital admissible;*
*(ii)* *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii)* *either $T$ is continuous or $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$. Additionally, if the condition $(H)$ is satisfied, then $u$ is the unique fixed point of $T$.*

### *1.2.3 Consequences in the Frame of Partial Metric Spaces with a Partial Order*

Existence of fixed point on metric spaces endowed with partial orders is one of the recent trends of metric fixed point theory that was initiated by Turinici [123] in 1986. Following this pioneer work, Ran and Reurings in [103] give more interesting results with an application to matrix equations.

**Definition 11** Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that $T$ is nondecreasing with respect to $\preceq$ if

$$x, y \in X, \ x \preceq y \Longrightarrow Tx \preceq Ty.$$

**Definition 12** Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all $n$.

**Definition 13** Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$. We say that $(X, \preceq, p)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k$.

We have the following result.

**Theorem 22** *Let $(X, \preceq)$ be a partially ordered set and $p$ be a partial metric on $X$ such that $(X, p)$ is complete. Let $T : X \to X$ be a nondecreasing mapping with* *respect to* $\preceq$*. Suppose that there exists a function* $\psi \in \Psi$ *such that*

$$p(Tx, Ty) \leq \psi(K(x, y)),$$

*for all* $x, y \in X$ *with* $x \succeq y$*, where* $K(x, y)$ *is defined as in (1.11). Suppose also that the following conditions hold:*

*(i) there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*;*
*(ii)* $T$ *is continuous or* $(X, \preceq, p)$ *is regular.*

*Then* $T$ *has a fixed point.*

*Proof* We define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

It is evident that $T$ is a $(\alpha - \psi)$-type $K$-contraction, that is,

$$\alpha(x, y)p(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in X$. On account of the condition (i), we have $\alpha(x_0, Tx_0) \geq 1$. Additionally, for all $x, y \in X$, from the monotone property of $T$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Hence, we find that $T$ is $\alpha$-admissible.

For the case, $T$ is continuous, the existence of a fixed point can be derived from Theorem 3.

Let us consider the other case. Assume that $(X, \preceq, p)$ is regular. Let $\{x_n\}$ be a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k$. Keeping the definition of $\alpha$ in mind, we find that $\alpha(x_{n(k)}, x) \geq 1$ for all $k$. Consequently, the existence of a fixed point follows from Theorem 4.

**Theorem 23** *Let* $(X, \preceq)$ *be a partially ordered set and* $p$ *be a partial metric on* $X$ *such that* $(X, p)$ *is complete. Let* $T : X \to X$ *be a nondecreasing mapping with respect to* $\preceq$*. Suppose that there exists a function* $\psi \in \Psi$ *such that*

$$p(Tx, Ty) \leq \psi(N(x, y)),$$

*for all* $x, y \in X$ *with* $x \succeq y$*, where* $K(x, y)$ *is defined as in (1.24). Suppose also that the following conditions hold:*

*(i) there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*;*
*(ii)* $T$ *is continuous or* $(X, \preceq, p)$ *is regular.*

*Then $T$ has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.*

*Proof* By following the lines in the proof of Theorem 22, we guarantee the existence of a fixed point of $T$.

Now we shall show the uniqueness of it. Let $x, y \in X$ be fixed points of $T$. By hypothesis, there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, which implies from the definition of $\alpha$ that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Thus we deduce the uniqueness of the fixed point by Theorem 7.

*Remark 3* Note that the techniques, used in this section, can be applied to all other results, Theorems 8–21. Regarding the analogy, we skip the state these results here.

### *1.2.4 Consequences in the Frame of Cyclic Contractive Mappings*

Existence and uniqueness of cyclic mappings have been investigated since the paper of Kirk et al. [80] appeared. In this part, we consider some consequences of our results in the setting of cyclic mappings.

**Corollary 1** *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space $(X, p)$ and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

*(I) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;*
*(II) there exists a function $\psi \in \Psi$ such that*

$$p(Tx, Ty) \leq \psi(K(x, y)), \text{ for all } (x, y) \in A_1 \times A_2,$$

*where $K(x, y)$ is defined as in (1.11).*

*Then $T$ has a fixed point that belongs to $A_1 \cap A_2$.*

*Proof* First, we observe that $(Y, p)$ is complete since $A_1$ and $A_2$ are closed subsets of the complete partial metric space $(X, p)$. Define the mapping $\alpha : Y \times Y \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 \text{ otherwise.} \end{cases}$$

Regarding (II) and the definition of $\alpha$, we find that

$$\alpha(x, y)p(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in Y$. Thus $T$ is a $(\alpha - \psi)$-type $K$-contraction.

Let $(x, y) \in Y \times Y$ such that $\alpha(x, y) \geq 1$. If $(x, y) \in A_1 \times A_2$, from (I), $(Tx, Ty) \in A_2 \times A_1$, which yields that $\alpha(Tx, Ty) \geq 1$. If $(x, y) \in A_2 \times A_1$, from (I), $(Tx, Ty) \in A_1 \times A_2$, which implies that $\alpha(Tx, Ty) \geq 1$. Consequently, we have $\alpha(Tx, Ty) \geq 1$. Hence, we conclude that $T$ is $\alpha$-admissible.

Moreover, from (I), for any $a \in A_1$, we have $(a, Ta) \in A_1 \times A_2$, which yields that $\alpha(a, Ta) \geq 1$.

Now, let $\{x_n\}$ be a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$. On account of the definition of $\alpha$ that

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \quad \text{for all } n.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we get that

$$(x, x) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which implies that $x \in A_1 \cap A_2$. Thus we get immediately from the definition of $\alpha$ that $\alpha(x_n, x) \geq 1$ for all $n$.

**Corollary 2** *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space $(X, d)$ and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

*(I) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;*
*(II) there exists a function $\psi \in \Psi$ such that*

$$p(Tx, Ty) \leq \psi(N(x, y)), \text{ for all} (x, y) \in A_1 \times A_2,$$

*where $K(x, y)$ is defined as in (1.24).*

*Then $T$ has a unique fixed point that belongs to $A_1 \cap A_2$.*

*Proof* By following the proof of Theorem 1, we conclude the existence of a fixed point of $T$. Now, we shall show the uniqueness of it. Let $x, y$ be distinct fixed point of $T$. Keeping (I) in mind, we find that $x, y \in A_1 \cap A_2$. Thus, for any $z \in Y$, we have $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Hence, the condition (H) is satisfied.

Now, all the hypotheses of Theorem 7 are satisfied. Consequently, we obtain that $T$ has a unique fixed point that belongs to $A_1 \cap A_2$ (from (I)).

*Remark 4* As in the previous section, used techniques in this section can be repeated for all other results, Theorems 8–21. We avoid to put all these results here, due to analogy.

### 1.2.5 More Consequences in the Frame of Standard Partial Metric Spaces

By letting $\alpha(x, y) = 1$ for all $x, y \in X$, in Theorems 3–7, we immediately get the following fixed point theorems.

**Theorem 24** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(K(x, y)),$$

*for all $x, y \in X$, where $K(x, y)$ is defined as in (1.11). Then $T$ has a unique fixed point.*

**Theorem 25** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(N(x, y)),$$

*for all $x, y \in X$, where $K(x, y)$ is defined as in (1.24). Then $T$ has a unique fixed point.*

**Theorem 26** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Ty)),$$

*for all $x, y \in X$, where $0 \leq a_i,\ i = 1, 2, 3$, and $a_1 + a_2 + a_3 < 1$. Then $T$ has a unique fixed point.*

**Theorem 27** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(a_1 p(x, y) + a_4[p(x, Ty) + p(y, Tx)]),$$

*for all $x, y \in X$, where $0 \leq a_i,\ i = 1, 4$, and $a_1 + 2a_4 < 1$. Then $T$ has a unique fixed point.*

**Theorem 28** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi\left(b_1 p(x, y) + b_2[p(x, Tx) + p(y, Ty)]\right),$$

*for all $x, y \in X$, where $0 \leq b_i,\ i = 1, 2$, and $b_1 + 2b_2 < 1$. Then $T$ has a unique fixed point.*

**Theorem 29** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi\left(\frac{[p(x, Tx) + p(y, Ty)]}{2}\right),$$

*for all $x, y \in X$. Then $T$ has a unique fixed point.*

**Theorem 30** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi\left(\frac{[p(x, Ty) + p(y, Tx)]}{2}\right),$$

*for all $x, y \in X$. Then $T$ has a unique fixed point.*

Observe that in the following theorems, we do not need to continuity of the mapping $T$. Indeed, the contraction criteria necessarily provides the continuity of $T$.

**Theorem 31** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(p(x, y)]),$$

*for all $x, y \in X$. Then $T$ has a unique fixed point.*

Letting $\psi(t) = kt$ with $k \in [0, 1)$, we derived the analog of Banach Contraction Mapping Principle in the setting of partial metric that was proved by Matthew [84, 85]

**Theorem 32** *Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists a $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kp(x, y),$$

*for all $x, y \in X$. Then $T$ has a unique fixed point.*

## 1.3 Fixed Point of Rational Type Contraction Mappings

**Definition 14** Let $T$ be a self-mapping defined on a partial metric space $(X, p)$. We say that $T$ is an $(\alpha - \psi)$-type *R-contraction* if there exist mappings $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)p(Tx, Ty) \leq \psi(R(x, y)) \text{ for all } x, y \in X, \tag{1.45}$$

where

$$R(x,y) := a_1 p(x,y) + a_2 p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)} + a_3 p(y,Tx)\frac{1+p(x,Ty)}{1+p(x,y)} \\ +a_4 p(x,Tx)\frac{1+p(y,Ty)}{1+p(x,y)} + a_5 p(x,Ty)\frac{1+p(y,Tx)}{1+p(x,y)} \tag{1.46}$$

where $0 \le a_i \le 1,\ i = 1,2,3,4,5$, and $a_1 + a_2 + 2a_3 + a_4 + 2a_5 \le 1$.

**Theorem 33** *Let $(X, p)$ be a complete partial metric space and let $T : X \to X$ be an $(\alpha - \psi)$-type $R$-contraction. Suppose that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
*(iii) $T$ is continuous.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u,u) = 0$.*

*Proof* As in Theorem 3, we construct an iterative sequence, by starting the given initial conditions at hypothesis $(ii)$ of theorem. More precisely, based on $(ii)$, we built an iterative sequence $\{x_n\}$ in $X$ by

$$x_{n+1} = Tx_n \text{ for all } n \ge 0.$$

Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n$. Indeed, if $x_{n_0} = x_{n_0+1}$ for some $n_0$, then $u = x_{n_0}$ is a fixed point of $T$ that terminates the proof.

Keeping $(ii)$ in mind again, one can derive the following implications

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \ge 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \ge 1,$$

since $T$ is $\alpha$-admissible. Recursively, we find that

$$\alpha(x_n, x_{n+1}) \ge 1, \quad \text{for all } n = 0, 1, \ldots \tag{1.47}$$

Combining (1.10) and (1.47), it follows, for all $n \ge 1$, that

$$p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1}) \le \alpha(x_n, x_{n-1}) p(Tx_n, Tx_{n-1}) \le \psi(R(x_n, x_{n-1})), \tag{1.48}$$

where

$$R(x_n, x_{n-1}) \\ = a_1 p(x_n, x_{n-1}) + a_2 p(x_{n-1}, Tx_{n-1})\frac{1+p(x_n, Tx_n)}{1+p(x_n, x_{n-1})} \\ +a_3 p(x_{n-1}, Tx_n)\frac{1+p(x_n, Tx_{n-1})}{1+p(x_n, x_{n-1})}$$

$$+a_4 p(x_n, Tx_n)\frac{1+p(x_{n-1}, Tx_{n-1})}{1+p(x_n, x_{n-1})} + a_5 p(x_n, Tx_{n-1})\frac{1+p(x_{n-1}, Tx_n)}{1+p(x_n, x_{n-1})}$$

$$= a_1 p(x_n, x_{n-1}) + a_2 p(x_{n-1}, x_n)\frac{1+p(x_n, x_{n+1})}{1+p(x_n, x_{n-1})}$$
$$+a_3 p(x_{n-1}, x_{n+1})\frac{1+p(x_n, x_n)}{1+p(x_n, x_{n-1})}$$

$$+a_4 p(x_n, x_{n+1})\frac{1+p(x_{n-1}, x_n)}{1+p(x_n, x_{n-1})} + a_5 p(x_n, x_n)\frac{1+p(x_{n-1}, x_{n+1})}{1+p(x_n, x_{n-1})}$$

$$\leq a_1 p(x_n, x_{n-1}) + a_2\frac{p(x_{n-1}, x_n) + p(x_{n-1}, x_n)p(x_n, x_{n+1})}{1+p(x_n, x_{n-1})}$$
$$+a_3[p(x_{n-1}, x_n) + p(x_n, x_{n+1})] + a_4 p(x_n, x_{n+1}) + a_5 p(x_n, x_{n+1})$$

Before further estimation of $R(x_n, x_{n-1})$, we clarify some steps on the above evaluation. It is clear that there are no changes on the terms of the coefficients $a_1$ and $a_2$. For the terms after the coefficient $a_3$, we apply both $(p_2)$ and modified triangle inequality $(p_4)$, that is

$$\begin{aligned}
&+a_5 p(x_n, x_n)\frac{1+p(x_{n-1}, x_{n+1})}{1+p(x_n, x_{n-1})}\\
&\leq p(x_{n-1}, x_{n+1}) \qquad \text{(by } (p_2)\text{, that is, } p(x,x) \leq p(x,y))\\
&\leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)\\
&\leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}).
\end{aligned}$$

Finally, we shall examine how to get the expression after the coefficient $a_5$:

$$\begin{aligned}
p(x_n, x_n)\frac{1+p(x_{n-1}, x_{n+1})}{1+p(x_n, x_{n-1})} &\leq p(x_n, x_n)\frac{1+p(x_{n-1}, x_n)+p(x_n, x_{n+1}) - p(x_n, x_n)}{1+p(x_n, x_{n-1})}\\
&\leq p(x_n, x_n)\frac{1+p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{1+p(x_n, x_{n-1})}\\
&\leq p(x_n, x_n)\frac{1+p(x_{n-1}, x_n)}{1+p(x_n, x_{n-1})}\frac{p(x_n, x_{n+1})}{1+p(x_n, x_{n-1})}\\
&\leq p(x_n, x_n)\frac{p(x_n, x_{n+1})}{1+p(x_n, x_{n-1})}\\
&= p(x_n, x_{n+1})\frac{p(x_n, x_n)}{1+p(x_n, x_{n-1})}\\
&\leq p(x_n, x_{n+1})\frac{p(x_n, x_{n-1})}{1+p(x_n, x_{n-1})}\\
&\leq p(x_n, x_{n+1})
\end{aligned}$$

To refine the estimation of $R(x_n, x_{n-1})$, we need to compare the terms $p(x_n, x_{n+1})$ and $p(x_n, x_{n-1})$. If for some $n \geq 1$, we have $p(x_n, x_{n-1}) \leq p(x_{n+1}, x_n)$, then we get

$$\begin{aligned}R(x_n, x_{n-1}) &\leq [a_1 + 2a_3 + a_4 + a_5]p(x_n, x_{n+1}) \\ &\quad + a_2 \frac{p(x_n, x_{n+1}) + p(x_{n-1}, x_n)p(x_n, x_{n+1})}{1 + p(x_n, x_{n-1})} \\ &= [a_1 + 2a_3 + a_4 + a_5]p(x_n, x_{n+1}) \\ &\quad + a_2 \frac{p(x_n, x_{n+1})[1 + p(x_{n-1}, x_n)]}{1 + p(x_n, x_{n-1})} \\ &\leq [a_1 + a_2 + 2a_3 + a_4 + a_5]p(x_n, x_{n+1}) \\ &\leq p(x_n, x_{n+1})\end{aligned} \tag{1.49}$$

since $a_1 + a_2 + a_3 + a_4 + a_5 \leq 1$.

On account of the inequalities (1.48), (1.49) together with the fact that $\psi$ is nondecreasing function, we find

$$p(x_{n+1}, x_n) \leq \psi(R(x_n, x_{n-1})) \leq \psi(p(x_n, x_{n+1})) < p(x_n, x_{n+1}), \tag{1.50}$$

a contradiction. Thus, for all $n \geq 1$, we have

$$p(x_n, x_{n+1}) \leq p(x_n, x_{n-1}). \tag{1.51}$$

Using (1.48) and (1.51), we get that

$$p(x_{n+1}, x_n) \leq \psi(p(x_n, x_{n-1})), \tag{1.52}$$

for all $n \geq 1$. By induction, we get

$$p(x_{n+1}, x_n) \leq \psi^n(p(x_1, x_0)), \quad \text{for all} \quad n \geq 1. \tag{1.53}$$

The rest of the proof is verbatim of the corresponding part in the proof of Theorem 3. Hence, we omit it the rest.

**Theorem 34** *Let $(X, p)$ be a complete partial metric space and let $T : X \to X$ be an $(\alpha - \psi)$-type $R$-contraction. Suppose that*

*(i) $T$ is $\alpha$-orbital admissible;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) $X$ is regular.*

*Then there exists $u \in X$ such that $Tu = u$ and $p(u, u) = 0$.*

*Proof* As in the lines in the proof of Theorem 33, we construct an iterative sequence, $\{x_n = Tx_{n-1}\}$ for all $n \geq 0$, converges for some $u \in X$ with $p(u, u) = 0$. Due to the condition (iii) of Theorem 34, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all $k$. Applying (1.45), for all $k$, we get that

$$\begin{aligned}p\left(x_{n(k)+1}, Tu\right) &= p\left(Tx_{n(k)}, Tu\right) \leq \alpha\left(x_{n(k)}, u\right) p\left(Tx_{n(k)}, Tu\right) \\ &\leq \psi\left(R(x_{n(k)}, u)\right) < R\left(x_{n(k)}, u\right),\end{aligned} \tag{1.54}$$

where

$$R(x_{n(k)},u)=a_1p(x_{n(k)},u)+a_2p(u,Tu)\frac{1+p\left(x_{n(k)},Tx_{n(k)}\right)}{1+p\left(x_{n(k)},u\right)}$$
$$+a_3p\left(u,Tx_{n(k)}\right)\frac{1+p(x_{n(k)},Tu)}{1+p(x_{n(k)},u)}$$
$$+a_4p\left(x_{n(k)},Tx_{n(k)}\right)\frac{1+p(u,Tu)}{1+p(x_{n(k)},u)}$$
$$+a_5p\left(x_{n(k)},Tu\right)\frac{1+p\left(u,Tx_{n(k)}\right)}{1+p\left(x_{n(k)},u\right)}$$

Letting $k\to\infty$ in the equality (1.54), we get that

$$p(u,Tu)\le(a_3+a_5)p(u,Tu), \tag{1.55}$$

a contradiction. Hence, we conclude $p(u,Tu)=0$, that is, $u=Tu$.

As it is expected, we shall consider an additional condition to guarantee the uniqueness of the fixed point.

**Theorem 35** *Putting condition (U) to the statements of Theorem 33 (resp. Theorem 34), we find that u is the unique fixed point of T.*

*Proof* Let $u,w$ be two distinct fixed point of $T$, that is, $p(u,w)>0$ and hence $\psi(pu,w))>0$. Keeping the condition (U) in the mind, we derive that

$$\begin{aligned}p(u,w)&\le\alpha(u,w)p(Tu,Tw)\\&\le\psi(K(u,w))=\psi(p(u,w))\\&<p(u,w),\end{aligned}$$

contradiction. Thus, $u=w$.

### 1.3.1 Remarks on the Immediate Consequences

As we demonstrate in Sect. 1.2, it is possible get a number of consequence of Theorems 33–35, by choosing the coefficients $a_i, i=1,2,3,4,5$, the functions $\psi$ and $\alpha$.

## 1.4 Common Fixed Point Results

**Definition 15** Let $(X,p)$ be a partial metric space and $S,T$ be two self-mappings on $(X,p)$. A point $z\in X$ is said to be common fixed point of $S$ and $T$, if $Sz=Tz=z$.

Let $(X, p)$ be a partial metric space and denote the closure of the set $\{p(x, y) : x, y \in X\}$ by $P$ and $P^3 = P \times P \times P$. A function $\phi : P^3 \to \mathbb{R}^+$ is right continuous if and only if

(S1) the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ decrease and converge to $a, b, c \in P$, respectively, then

$$\phi(a_n, b_n, c_n) \to \phi(a, b, c).$$

The function $\phi$ is called symmetric if and only if

$$\phi(a, b, c) = \phi(b, a, c), \text{ for all } (a, b, c) \in P^3.$$

In the spirit of Sehgal [118] we state the following definition for partial metric spaces.

**Definition 16** Let $(X, p)$ be a partial metric space and $S, T : X \to X$ be two mappings. The pair $(S, T)$ is said to satisfy Sehgal $k$-condition if and only if there are maps $I_S : S \times X \to \mathbb{Z}^+$ and $I_T : T \times X \to \mathbb{Z}^+$ such that if $r(x) = I_S(S, x)$ and $q(x) = I_T(T, x)$, then

$$p(S^{r(x)}x, T^{q(y)}y) \leq k\phi(p(S^{r(x)}x, x), p(y, T^{q(y)}y), p(x, y)) \tag{1.56}$$

for all $x, y \in X$, where $k \in \mathbb{R}$ and $\phi$ is a symmetric right continuous. If $0 \leq k < 1$, then we say that $(S, T)$ satisfy Sehgal contraction condition.

The following theorem extends the results of [118].

**Theorem 36** *Let $(X, p)$ be a complete partial metric space. Suppose $S, T : X \to X$ two mappings such that the pair $(S, T)$ satisfies Sehgal contraction.*

*(A) If $\phi(a, b, c) \leq \max\{a, b, c\}$, for $(a, b, c) \in P^3$, then $S$ and $T$ have a unique common fixed point in $X$, that is, $S^{r(z)}z = T^{q(z)}z = z$.*

*Proof* Let $x_0 \in X$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in a way that $x_2 = T^{q(x_1)}x_1$ and $x_1 = S^{r(x_0)}x_0$ and inductively

$$x_{2n+2} = T^{q(x_{2n+1})}x_{2n+1} \quad \text{and} \quad x_{2n+1} = S^{r(x_{2n})}x_{2n} \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

If $n$ is odd, due to (1.56), we have

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1}) \leq k\phi(p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1})) \tag{1.57}$$

Regarding the assumption of $(A)$,

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1}) \leq k \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \tag{1.58}$$

If $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_{n+1}, x_{n+2})$ then the expression (1.58) turns into

$$p(x_{n+1}, x_{n+2}) \leq kp(x_{n+1}, x_{n+2}).$$

Since $k < 1$, this is impossible. Thus, we have

$$p(x_{n+1}, x_{n+2}) \leq kp(x_n, x_{n+1}). \tag{1.59}$$

If $n$ is even, analogously we observe that $p(x_{n+1}, x_{n+2}) \leq kp(x_n, x_{n+1})$. Observe that $\{p(x_n, x_{n+1})\}$ is a non-negative, non-increasing sequence of reals. Regarding (1.59) one can observe that

$$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1), \; \forall n = 0, 1, 2, \ldots \tag{1.60}$$

Letting $n \to \infty$, the right-hand side of (1.60) tends to zero.

Consider now

$$\begin{aligned} d_p(x_{n+1}, x_{n+2}) &= 2p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) \\ &\leq 2p(x_{n+1}, x_{n+2}) \\ &\leq 2k^{n+1} p(x_0, x_1). \end{aligned} \tag{1.61}$$

Hence, regarding (1.60), we have $\lim_{n\to\infty} d_p(x_{n+1}, x_{n+2}) = 0$. Moreover,

$$\begin{aligned} d_p(x_{n+1}, x_{n+s}) &\leq d_p(x_{n+s-1}, x_{n+s}) + \cdots + d_p(x_{n+1}, x_{n+2}) \\ &\leq 2k^{n+s} p(x_0, x_1) + \cdots + 2k^{n+1} p(x_0, x_1) \end{aligned} \tag{1.62}$$

which implies that $\{x_n\}$ is a Cauchy sequence in $(X, d_p)$ that is, $d_p(x_n, x_m) \to 0$. Since $(X, p)$ is complete, by Lemma 1, $(X, d_p)$ is complete and the sequence $\{x_n\}$ is convergent in $(X, d_p)$, say $z \in X$.

By Lemma 1,

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) \tag{1.63}$$

Since $\{x_n\}$ is a Cauchy sequence in $(X, d_p)$, we have $\lim_{n,m\to\infty} d_p(x_n, x_m) = 0$. Since

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq p(x_n, x_{n+1}) \tag{1.64}$$

then by (1.60), it implies that

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq k^{n+1} p(x_0, x_1) \tag{1.65}$$

Thus from (1.60) and (1.65) the definition of $d_p$ we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Therefore from (1.63) we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{1.66}$$

We assert that $T^{q(z)}z = z$. Assume $T^{q(z)}z \neq z$, then $p(z, T^{q(z)}z) > 0$. Let $\{x_{2n(i)}\}$ be subsequence of $\{x_{2n}\}$ and hence $\{x_n\}$. Due to $(p_4)$, we have

$$\begin{aligned} p(Sx_{2n(i)}, T^{q(z)}z) &= p(Sx_{2n(i)}, T^{q(z)}z) \\ &\leq k\phi(p(x_{2n(i)}, x_{2n(i)+1}), p(T^{q(z)}z, z), p(x_{2n(i)}, z) \end{aligned} \tag{1.67}$$

Letting $n \to \infty$ and taking the assumption of $(A)$ and (1.66) into account, we get that

$$p(z, T^{q(z)}z) \leq k\phi(0, p(T^{q(z)}z, z), 0) \leq kp(T^{q(z)}z, z) \tag{1.68}$$

Since $k < 1$, then $p(T^{q(z)}z, z) = 0$. By Lemma 3, we get $T^{q(z)}z = z$. By considering the subsequence $\{x_{2n(i)+1}\}$ of $\{x_{2n+1}\}$, we obtain that $S^{r(z)}z = z$.

Assume now there exists $w \in X$ such that $S^{r(w)} = T^{q(w)}w = w$. By (PM3)

$$p(z, z) \leq p(z, w) \text{ and } p(w, w) \leq p(z, w) \tag{1.69}$$

Regarding that the function $\phi$ satisfies the condition of $(A)$ with (1.69), we get

$$\begin{aligned} p(z, w) = p(S^{r(z)}z, T^{q(w)}w) &\leq k\phi(p(z, S^{r(z)}z), p(T^{q(w)}w, w), p(z, w)) \\ &\leq k\phi(p(z, z), p(w, w), p(z, w)) \\ &\leq kp(z, w) \end{aligned}$$

Since $k < 1$, it yields a contradiction.

Thus, $p(z, w) = 0$ and by Lemma 3 we have $z = w$.

**Corollary 3** *Let $(X, p)$ be a complete partial metric space. Suppose $I_T$ and $I_S$ are defined as above. $S, T : X \to X$ two mappings such that the pair $(S, T)$ satisfies one of the following condition:*

*(A)* $p(S^{r(x)}x, T^{q(y)}y) \leq k \max\{p(S^{r(x)}x, x), p(y, T^{q(y)}y), p(x, y)\}$ *for some* $0 \leq k < 1$,

*(B)* $p(S^{r(x)}x, T^{q(y)}y) \leq \alpha p(S^{r(x)}x, x) + \beta p(y, T^{q(y)}y) + \gamma p(x, y)$ *for some non-negative reals* $\alpha, \beta, \gamma$ *with* $\alpha + \beta + \gamma < 1$.

*Then $S$ and $T$ have a unique common fixed point in $X$, that is, $S^{r(z)}z = T^{q(z)}z = z$.*

*Proof* For $(A)$, we choose a function $\phi(a, b, c) = \max\{a, b, c\}$ as in Theorem 36. In case of $(B)$, set $k = \alpha + \beta + \gamma$. Then $(A)$ implies $(B)$.

Notice that this corollary generalizes also some results in [31–34].

**Corollary 4** *Let $(X, p)$ be a complete partial metric space. $S, T : X \to X$ two mappings such that the pair $(S, T)$ satisfies the following condition:*

$$p(S^r x, T^q y) \leq k\phi(p(S^r x, x), p(y, T^q y), p(x, y)) \tag{1.70}$$

*for all $x, y \in X$ where $0 \leq k < 1$ and $\phi$ is symmetric right-continuous. If $\phi(a, b, c) \leq \max\{a, b, c\}$, then $S$ and $T$ have a unique common fixed point theorem.*

*Proof* By Theorem 36, by taking the maps $I_T$, $I_S$ as a constant, we get that $S^r$ and $T^q$ have a unique common fixed point, say $z \in X$. Now consider

$$S^q(Sz) = S^{q+1}z = S(S^q z) = Sz$$

which says that $Sz$ is a fixed point of $S^q$. Since $z$ is the unique fixed point of $S^q$, then $Sz = z$. Analogously, one can get $Tz = z$.

**Corollary 5** *Let $(X, p)$ be a complete partial metric space. $S, T : X \to X$ two mappings such that the pair $(S, T)$ satisfies the following condition:*

$$p(S^r x, T^q y) \leq k \max\{p(S^r x, x), p(y, T^q y), p(x, y)\} \tag{1.71}$$

*for all $x, y \in X$ where $0 \leq k < 1$ and $\phi$ is symmetric right-continuous. If $\phi(a, b, c) \leq \max\{a, b, c\}$, then $S$ and $T$ have a unique common fixed point theorem.*

**Corollary 6** *Let $(X, p)$ be a complete partial metric space. $S, T : X \to X$ two mappings such that the pair $(S, T)$ satisfies the following condition:*

$$p(S^r x, T^q y) \leq \alpha p(S^r x, x) + \beta p(y, T^q y) + \gamma p(x, y) \tag{1.72}$$

*for all $x, y \in X$ where for some non-negative reals $\alpha, \beta, \gamma$ with $\alpha + \beta + \gamma < 1$. $0 \leq k < 1$ and $\phi$ is symmetric right-continuous. If $\phi(a, b, c) \leq \max\{a, b, c\}$, then $S$ and $T$ have a unique common fixed point theorem.*

*Remark 5* Consider Corollary 6 and take $S = T$.

1. If we set $r = q$ in (1.72), then we get Reich type fixed point theorem (see, e.g., [12, 104]).
2. If we set $r = q = 1$ and $\gamma = 0$ in (1.72), we get Kannan type fixed point theorem (see, e.g., [12, 65])
3. If we set $r = q = 1$ and $\alpha = \beta = 0$ in (1.72), we get Banach type fixed point theorem (see, e.g., [12, 27, 85, 92] also [31–34])

*Example 4* Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$. It is clear that $(X, p)$ is a partial metric spaces but not a metric. Suppose $Sx = Tx = \frac{x}{2}$ and $I_S, I_T$ are constant mappings, such as $r(x) = 2 = q(y)$. Take $\phi(a, b, c) = \frac{1}{3}[a + b + c]$. Let

$p(x, y) = \max\{x, y\}$ for all $x, y \in X$. For $\frac{k}{3}$ the condition of Corollary 4 is satisfied. Clearly, 0 is the common fixed point of $S, T$.

*Example 5* Let $X = [1, 15]$ and $p(x, y) = \max\{x, y\}$. Here $(X, p)$ is a complete metric spaces. Define the self-mappings $S, T : X \to X$ as $Tx = \frac{x^2}{1+x}$ and $Sx = \begin{cases} \frac{x}{1+x} \text{ if } 1 < x \leq 15 \\ 0 \text{ if } 0 \leq x \leq 1 \end{cases}$. Set $\phi(a, b, c) = \frac{19}{20}\max\{x, y\}$. Without loss of generality, assume $y < x$. Thus, $p(Tx, x) = x$, $p(x, y) = x$, $p(Sy, y) = y$ and $p(Tx, Sy) = \frac{x^2}{1+x}$. Clearly, $p(Tx, Sy) = \frac{x^2}{1+x} \leq \phi(x, y, x) = \frac{19}{20}x$. Hence, it satisfies the conditions of Corollary 1.71 for $r = 1$ and $q = 1$, and 0 is the unique common fixed point of $S$ and $T$.

### 1.4.1 Further Common Fixed Point Results

Let $(X, p)$ be a partial metric space. Mappings $S, T : X \to X$ are called generalized $(\alpha - \psi)$-contractive pair, if there exist $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y)p(Tx, Sy) \leq \psi(C_{S,T}(x, y)) \tag{1.73}$$

for any $x, y \in X$, where

$$C_{S,T}(x, y) = a_1 p(x, y) + a_2 p(Tx, x) + a_3 p(Sy, y) + \frac{a_4}{2}[p(Tx, y) + p(Sy, x)] \tag{1.74}$$

with $0 \leq a_i \leq 1,\ i = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + 2a_4 \leq 1$.

**Theorem 37** *Let $(X, p)$ be a complete partial metric space and $S, T : X \to X$ be generalized $(\alpha - \psi)$-contractive pair. Suppose that*

*(i) $(S, T)$ is a generalized $\alpha$-admissible pair;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ ;*
*(iii) $\alpha(TSx, Sx) \geq 1$ for all $x \in X$;*
*(iv) $S$ and $T$ are continuous on $(X, p)$.*

*Then there exists $u \in X$ such that*

$$p(u, Tu) = p(Tu, Tu), \quad p(u, Su) = p(Su, Su) \quad \text{and} \quad p(u, u) = 0. \tag{1.75}$$

*Assume in addition that*

*(v) $\alpha(z, w) \geq 1$ for all $z, w \in \mathscr{C}(S, T)$, where $\mathscr{C}(S, T)$ denotes the set of common fixed points of $S$ and $T$.*

*Then, $u$ is a common fixed point of $S$ and $T$, that is, $u = Tu = Su$.*

*Proof* By assumption $(ii)$, there exists a point $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Take $x_1 = Sx_0$ and $x_2 = Tx_1$. By induction, we construct a sequence $(x_n)$ such that

$$x_{2n} = Tx_{2n-1} \quad \text{and} \quad x_{2n+1} = Sx_{2n} \quad \forall\ n = 1, 2, \ldots \tag{1.76}$$

We have $\alpha(x_0, x_1) \geq 1$ and since $(T, S)$ is a generalized $\alpha$-admissible pair, so

$$\begin{aligned} &\alpha(x_1, x_2) = \alpha(Sx_0, Tx_1) \geq 1 \quad \text{and} \\ &\alpha(x_2, x_3) = \alpha(Tx_1, Sx_2) = \alpha(TSx_0, STx_1) \geq 1. \end{aligned}$$

Similar to above, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all}\ \ n = 0, 1, \ldots. \tag{1.77}$$

On the other hand, by $(iii)$, we have

$$\alpha(x_2, x_1) = \alpha(TSx_0, Sx_0) \geq 1.$$

Applying again $(iii)$

$$\alpha(x_4, x_3) = \alpha(TSx_2, Sx_2) \geq 1.$$

Continuing the same process, we obtain

$$\alpha(x_{2n}, x_{2n-1}) \geq 1 \quad \text{for all}\ \ n = 1, 2, \ldots \tag{1.78}$$

We claim that $(x_n)$ is a Cauchy sequence in $(X, p)$. If $k$ is odd, due to (1.73), we have such that

$$p(x_{k+1}, x_{k+2}) \leq \alpha(x_{k+1}, x_{k+2})p(Tx_k, Sx_{k+1}) \leq \psi(C_{S,T}(x_k, x_{k+1})) \tag{1.79}$$

for any $x, y \in X$, where

$$\begin{aligned} C_{S,T}(x_k, x_{k+1}) = {} & a_1 p(x_k, x_{k+1}) + a_2 p(Tx_k, x_k) + a_3 p(Sx_{k+1}, x_{k+1}) \\ & + \frac{a_4}{2}[p(Tx_k, x_{k+1}) + p(Sx_{k+1}, x_k)] \end{aligned} \tag{1.80}$$

with $0 \leq a_i \leq 1,\ i = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + 2a_4 \leq 1$.

On account of modified triangle inequality $(p_4)$, we have

$$p(x_{k+1}, x_{k+1}) + p(x_{k+2}, x_k) \leq p(x_{k+2}, x_{k+1}) + p(x_{k+1}, x_k). \tag{1.81}$$

Thus, (1.80) turns into

$$C_{S,T}(x_k, x_{k+1}) = a_1 p(x_k, x_{k+1}) + a_2 p(x_{k+1}, x_k) + a_3 p(x_{k+2}, x_{k+1}), \\ \frac{a_4}{2}[p(x_{k+2}, x_{k+1}) + p(x_{k+1}, x_k)]\} \tag{1.82}$$

If $p(x_{k+1}, x_k) \leq p(x_{k+2}, x_{k+1})$, then (1.79) turns into

$$\begin{aligned} p(x_{k+1}, x_{k+2}) &= p(Tx_k, Sx_{k+1}) \leq \psi(C_{S,T}(x_k, x_{k+1})) \\ &\leq \psi([a_1 + a_2 + a_3 + 2a_4]p(x_{k+1}, x_{k+2})) \\ &\leq \psi(p(x_{k+1}, x_{k+2})) \quad \text{since } [a_1 + a_2 + a_3 + 2a_4] \leq 1 \\ &\quad < p(x_{k+1}, x_{k+2}) \text{ since } \phi(t) < t, \end{aligned} \tag{1.83}$$

which is a contradiction.

Hence, we derive, from the inequality (1.79), that

$$p(x_{k+1}, x_{k+2}) \leq \psi([a_1+a_2+a_3+2a_4]p(x_k, x_{k+1}))\psi(p(x_k, x_{k+1})) < p(x_k, x_{k+1}). \tag{1.84}$$

Analogously, for the case $k$ is even, we find that

$$\begin{aligned} p(x_{k+2}, x_{k+1}) &\leq \psi([a_1 + a_2 + a_3 + 2a_4]p(x_{k+1}, x_k)) \leq \psi(p(x_{k+1}, x_k)) \\ &< p(x_{k+1}, x_k). \end{aligned} \tag{1.85}$$

We get that $\{p(x_k, x_{k+1})\}$ is a non-negative, decreasing sequence of reals. Regarding (1.84) and (1.85) one gets

$$p(x_k, x_{k+1}) \leq \psi^k(p(x_0, x_1)), \ \forall k = 0, 1, 2, \ldots \tag{1.86}$$

Following the related steps in the proof of Theorem 3, we conclude that the sequence converges to $u \in X$ with

$$p(u, u) = \lim_{k\to\infty} p(x_k, u) = \lim_{k,m\to\infty} p(x_k, x_m) = 0. \tag{1.87}$$

We shall show that $Tu = u$. Since $T$ is continuous, by Definition 4, we conclude from (1.87) that

$$\lim_{k\to\infty} p(x_{2k}, Tu) = \lim_{n\to\infty} p(Tx_{2k-1}, Tu) = 0. \tag{1.88}$$

By Lemma 3, we get $Tu = u$.

In a similar way, due to the continuity of $S$, we conclude from (1.87) that

$$\lim_{k\to\infty} p(x_{2k+1}, Su) = \lim_{n\to\infty} p(Sx_{2k}, Su) = 0. \tag{1.89}$$

Hence, we conclude that $Tu = Su = u$.

As a last step, we shall show the uniqueness of the obtained common fixed point of $S$ and $T$. Suppose, on the contrary, that $S$ and $T$ have distinct common fixed points $u$ and $w$. On account of the condition $(v)$, we have

$$p(u,w) = p(Tuz, Sw) \leq \alpha(u,w)p(Tu, Sw) \leq \psi(C_{S,T}(u,w)) \tag{1.90}$$

where

$$\begin{aligned} C_{S,T}(u,w) &= a_1 p(u,w) + a_2 p(Tu,u) + a_3 p(Sw,w) \\ &\quad + \frac{a_4}{2}\left[p(Tu,w) + p(Sw,u)\right] \\ &= a_1 p(u,w) + \frac{a_4}{2}\left[p(u,w) + p(w,u)\right] \\ &= (a_1 + 2a_4)p(w,u). \end{aligned} \tag{1.91}$$

Hence, the inequality (1.90) yields that

$$\begin{aligned} p(u,w) = p(Tu,Sw) &\leq \alpha(u,w)p(Tu,Sw) \leq \psi(C_{S,T}(u,w)) \\ &\leq \psi((a_1+2a_4)p(w,u)) \leq \psi(p(u,w)) < p(u,w). \end{aligned} \tag{1.92}$$

Hence, by Lemma 3, we get $w = u$.

*Example 6* Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$ then $(X,p)$ is a complete partial metric space. Clearly, $p$ is not a metric. Suppose $S,T : X \to X$ such that $Sx = Tx = \frac{x}{3}$ and $\psi(t) = \frac{t}{2}$ . Without loss of generality assume $x \geq y$. Then

$$p(Tx,Sy) = \max\left\{\frac{x}{3}, \frac{y}{3}\right\} = \frac{x}{3} \tag{1.93}$$

$$\leq \frac{1}{2}C_{S,T}(x,y). \tag{1.94}$$

where

$$C_{S,T}(x,y) = a_1x + a_2x + a_3y + \frac{a_4}{2}\left[x + \max\left\{y, \frac{x}{3}\right\}\right] \leq [a_1+a_2+a_3+2a_4]x$$

Thus, the inequality (1.93) turns into

$$p(Tx,Sy) = \frac{x}{3} \leq \frac{1}{2}[a_1+a_2+a_3+2a_4]x \leq \frac{x}{2}$$

for all $x \in X$. Hence, all conditions of the Theorem 37 are satisfied. Indeed, 0 is the common fixed point of $S, T$.

**Proposition 1** *Let $(X,p)$ be a complete partial metric space and $S,T : X \to X$ be two self-mappings. Suppose there exists $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ such* *that*

$$\alpha(x, y)p(Tx, Sy) \leq \psi(C_{S^n,T^m}(x, y)) \tag{1.95}$$

*for any* $x, y \in X$ *and some positive integers* $m, n$*, where*

$$C_{S^n,T^m}(x, y) = a_1 p(x, y) + a_2 p(T^m x, x) + a_3 p(S^n y, y) \\ + \frac{a_4}{2}\left[p(T^m x, y) + p(S^n y, x)\right] \tag{1.96}$$

*with* $0 \leq a_i \leq 1,\ i = 1, 2, 3, 4,$ *and* $a_1 + a_2 + a_3 + 2a_4 \leq 1$. *Suppose also that*

*(i)* $(S, T)$ *is a generalized* $\alpha$*-admissible pair;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq 1$*;*
*(iii)* $\alpha(TSx, Sx) \geq 1$ *for all* $x \in X$*;*
*(iv)* $S$ *and* $T$ *are continuous on* $(X, p)$.

*Then there exists* $u \in X$ *such that*

$$p(u, Tu) = p(Tu, Tu), \quad p(u, Su) = p(Su, Su) \quad \text{and} \quad p(u, u) = 0. \tag{1.97}$$

*We, also, assume that*

*(v)* $\alpha(z, w) \geq 1$ *for all* $z, w \in \mathscr{C}(S^n, T^m)$*, where* $\mathscr{C}(S^n, T^m)$ *denotes the set of common fixed points of* $S$ *and* $T$.

*Then,* $u$ *is a common fixed point of* $S^n$ *and* $T^m$*, that is,* $u = T^m u = S^n u$.

*Proof* The proof is verbatim of the proof of Theorem 37 except the construction of the sequence. Here, we define a sequence $\{x_n\}_{n=1}^{\infty}$ in a way that $x_2 = T^m x_1$ and $x_1 = S^n x_0$ and inductively

$$x_{2k+2} = T^m x_{2k+1} \quad \text{and} \quad x_{2k+1} = S^n x_{2k} \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

The rest is verbatim.

The following theorem is a generalization of a common fixed point theorem that requires no commuting criteria (see, e.g., [26]).

**Theorem 38** *Assume in addition to Proposition 1*

*(v)* $\alpha(Tu, u) \geq 1$ *and* $\alpha(Su, u) \geq 1$ *for all* $u \in \mathscr{C}(S^n, T^m)$.

*Then,* $u$ *is a common fixed point of* $S$ *and* $T$*, that is,* $u = T^m u = S^n u$.

*Proof* Note that Proposition 1 yields that

$$T^m u = S^n u = u. \tag{1.98}$$

We shall indicate that $u$ is a common fixed point of $S$ and $T$. Suppose, on the contrary, that $p(Tu,u)>0$ and $p(Su,u)>0$. By (1.95) and (1.98),

$$\begin{aligned} p(Tu,u) &\le \alpha(Tu,u)p(Tu,u)=\alpha(Tu,u)p(TT^m u,S^n u) \\ &= p(T^m Tu,S^n u)\le \psi(C_{S^n,T^m}(Tu,u)) \end{aligned} \tag{1.99}$$

where

$$\begin{aligned} C_{S^n,T^m}(Tu,u) &= a_1 p(Tu,u)+a_2 p(T^m Tu,Tu)+a_3 p(S^n u,u) \\ &\quad + \frac{a_4}{2}\left[p(T^m Tu,u)+p(S^n u,Tu)\right] \\ &= a_1 p(Tu,u)+a_2 p(Tu,Tu)+a_3 p(u,u) \\ &\quad + \frac{a_4}{2}\left[p(Tu,u)+p(u,Tu)\right] \\ &\le [a_1+a_2+a_4]p(Tu,u),\ \text{dueto } (p_2). \end{aligned} \tag{1.100}$$

Hence, the expression (1.99) turns into

$$p(Tu,u)\le \psi([a_1+a_2+a_4]p(Tu,u))\le \psi(p(Tu,u))<p(Tu,u)$$

which is a contradiction. Hence, we have $p(Tu,u)=0$ and by Lemma 3, we get $Tu=u$. Analogously, one can obtain $Su=u$. Hence, $Tu=Su=u$.

For the uniqueness of the common fixed point $u$, assume the contrary. Suppose $w$ is another common fixed point of $S$ and $T$. Then,

$$p(u,w)\le \alpha(u,w)p(u,w)=\alpha(u,w)p(T^m u,S^n w)\le \psi(C_{S^n,T^m}(u,w)), \tag{1.101}$$

where, by the help of $(p_3)$,

$$\begin{aligned} C_{S^n,T^m}(u,w) &= a_1 p(u,w)+a_2 p(T^m u,u)+a_3 p(S^n w,w) \\ &\quad + \frac{a_4}{2}\left[p(T^m u,w)+p(S^n w,u)\right] \\ &= a_1 p(u,w)+a_2 p(u,u)+a_3 p(w,w)+a_4\frac{1}{2}\left[p(u,w)+p(w,u)\right] \\ &= [a_1+a_4]p(u,w) \end{aligned} \tag{1.102}$$

Therefore $p(u,w)\le \psi([a_1+a_4]p(u,w))\le \psi(p(u,w))<p(u,w)$, a contradiction. Hence, we have $p(u,w)=0$ which yields $u=w$ by Lemma 3. Hence, $u$ is a unique common fixed point of $S$ and $T$.

As we discussed in Sect. 1.2, we are able to derive further results from Theorems 37 and 38, by selecting the auxiliary functions $\alpha$, $\psi$ in a proper way, as well as the coefficients $a_1, a_2, a_3, a_4$.

### 1.4.2 Immediate Consequences

The results given in this section can be considered just immediate consequence of Theorem 37 by choosing the coefficients $a_i$, $i = 1, 2, 3, 4$ in a proper way.

**Theorem 39** *Let $(X, p)$ be a complete partial metric space and $S, T : X \to X$ be given mapping. Suppose that there exists $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x, y)p(Tx, Sy) \leq \psi(a_1 p(x, y) + a_2 p(Tx, x) + a_3 p(Sy, y)) \tag{1.103}$$

*for any $x, y \in X$, where $0 \leq a_i \leq 1$, $i = 1, 2, 3$, and $a_1 + a_2 + a_3 \leq 1$. Suppose also that*

*(i) $(S, T)$ is a generalized $\alpha$-admissible pair;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ ;*
*(iii) $\alpha(TSx, Sx) \geq 1$ for all $x \in X$;*
*(iv) $S$ and $T$ are continuous on $(X, p)$.*

*Then there exists $u \in X$ such that*

$$p(u, Tu) = p(Tu, Tu), \quad p(u, Su) = p(Su, Su) \quad \text{and} \quad p(u, u) = 0. \tag{1.104}$$

*Assume in addition that*

*(v) $\alpha(z, w) \geq 1$ for all $z, w \in \mathscr{C}(S, T)$,where $\mathscr{C}(S, T)$ denotes the set of common fixed points of $S$ and $T$.*

*Then, $u$ is a common fixed point of $S$ and $T$, that is, $u = Tu = Su$.*

**Theorem 40** *Let $(X, p)$ be a complete partial metric space and $S, T : X \to X$ be given mapping. Suppose that there exists $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x, y)p(Tx, Sy) \leq \psi\left(a_1 p(x, y) + \frac{a_4}{2}\left[p(Tx, y) + p(Sy, x)\right]\right) \tag{1.105}$$

*for any $x, y \in X$, where $0 \leq a_i \leq 1$, $i = 1, 4$, and $a_1 + 2a_4 \leq 1$. Suppose also that*

*(i) $(S, T)$ is a generalized $\alpha$-admissible pair;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*
*(iii) $\alpha(TSx, Sx) \geq 1$ for all $x \in X$;*
*(iv) $S$ and $T$ are continuous on $(X, p)$.*

*Then there exists* $u \in X$ *such that*

$$p(u, Tu) = p(Tu, Tu), \quad p(u, Su) = p(Su, Su) \quad \textit{and} \quad p(u, u) = 0. \tag{1.106}$$

*Assume in addition that*

*(v)* $\alpha(z, w) \geq 1$ *for all* $z, w \in \mathscr{C}(S, T)$*, where* $\mathscr{C}(S, T)$ *denotes the set of common fixed points of* $S$ *and* $T$.

*Then,* $u$ *is a common fixed point of* $S$ *and* $T$*, that is,* $u = Tu = Su$.

**Theorem 41** *Let* $(X, p)$ *be a complete partial metric space and* $S, T : X \to X$ *be given mapping. Suppose that there exists* $\psi \in \Psi$ *and* $\alpha : X \times X \to [0, \infty)$ *such that*

$$\alpha(x, y)p(Tx, Sy) \leq \psi(\frac{a_4}{2}[p(Tx, y) + p(Sy, x)]) \tag{1.107}$$

*for any* $x, y \in X$*, where* $0 \leq 2a_4 \leq 1$. *Suppose also that*

*(i)* $(S, T)$ *is a generalized* $\alpha$*-admissible pair;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq 1$*;*
*(iii)* $\alpha(TSx, Sx) \geq 1$ *for all* $x \in X$*;*
*(iv)* $S$ *and* $T$ *are continuous on* $(X, p)$.

*Then there exists* $u \in X$ *such that*

$$p(u, Tu) = p(Tu, Tu), \quad p(u, Su) = p(Su, Su) \quad \text{and} \quad p(u, u) = 0. \tag{1.108}$$

*Assume in addition that*

*(v)* $\alpha(z, w) \geq 1$ *for all* $z, w \in \mathscr{C}(S, T)$*,where* $\mathscr{C}(S, T)$ *denotes the set of common fixed points of* $S$ *and* $T$.

*Then,* $u$ *is a common fixed point of* $S$ *and* $T$*, that is,* $u = Tu = Su$.

**Theorem 42** *Let* $(X, p)$ *be a complete partial metric space and* $S, T : X \to X$ *be given mapping. Suppose that there exists* $\psi \in \Psi$ *and* $\alpha : X \times X \to [0, \infty)$ *such that*

$$\alpha(x, y)p(Tx, Sy) \leq \psi(C_{S,T}(x, y)) \tag{1.109}$$

*for any* $x, y \in X$*, where*

$$C_{S,T}(x, y) = a_1 p(x, y) + a_2 p(Tx, x) + a_3 p(Sy, y) + \frac{a_4}{2}[p(Tx, y) + p(Sy, x)] \tag{1.110}$$

*with* $0 \leq a_i \leq 1,\ i = 2, 3$ *and* $a_2 + a_3 \leq 1$. *Suppose also that*

*(i)* $(S, T)$ *is a generalized* $\alpha$*-admissible pair;*
*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq 1$*;*

*(iii)* $\alpha(TSx, Sx) \geq 1$ *for all* $x \in X$;
*(iv)* $S$ *and* $T$ *are continuous on* $(X, p)$.

*Then there exists* $u \in X$ *such that*

$$p(u, Tu) = p(Tu, Tu), \quad p(u, Su) = p(Su, Su) \quad \text{and} \quad p(u, u) = 0. \quad (1.111)$$

*Assume in addition that*

*(v)* $\alpha(z, w) \geq 1$ *for all* $z, w \in \mathscr{C}(S, T)$, *where* $\mathscr{C}(S, T)$ *denotes the set of common fixed points of* $S$ *and* $T$.

*Then,* $u$ *is a common fixed point of* $S$ *and* $T$, *that is,* $u = Tu = Su$.

### *1.4.3 Common Fixed Point Results for Four Mappings*

**Theorem 43** *Let* $(X, p)$ *be a complete partial metric space. Suppose that* $T$, $S$, $F$, *and* $G$ *are self-mappings on* $X$, *and each of* $F$ *and* $G$ *is continuous. Suppose also that* $T$, $F$ *and* $S$, $G$ *are commuting pairs and that*

$$T(X) \subset F(X), \quad S(X) \subset G(X). \quad (1.112)$$

*If there exists* $\psi \in \Psi$, *and* $m$, $n$ *in* $N$ *such that*

$$p(Tx, Sy) \leq \psi(M(x, y)) \quad (1.113)$$

*for any* $x$, $y$ *in* $X$ *where*

$$M(x, y) = a_1 p(Tx, Gx) + a_2 p(Sy, Fy) + 3a_p(Gx, Fy) + \frac{a_4}{2}\left[p(Tx, Fy) + p(Sy, Gx)\right], \quad (1.114)$$

*where* $0 \leq a_i \leq 1,\ i = 1, 2, 3, 4$, *and* $a_1 + a_2 + a_3 + 2a_4 \leq 1$. *Then,* $T$, $S$, $F$, *and* $G$ *have a unique common fixed point* $z$ *in* $X$.

*Proof* Fix $x_0 \in X$. Since $T(X) \subset F(X)$ and $S(X) \subset G(X)$, we can choose $x_1, x_2$ in $X$ such that $y_1 = Fx_1 = Tx_0$ and $y_2 = Gx_2 = Sx_1$. In general, we can choose $x_{2n-1}, x_{2n}$ in $X$ such that

$$y_{2n-1} = Fx_{2n-1} = Tx_{2n-2}, \quad y_{2n} = Gx_{2n} = Sx_{2n-1} \quad n = 1, 2, \ldots. \quad (1.115)$$

We claim that the constructive sequence $\{y_n\}$ is a Cauchy sequence. By (1.113) and (1.115),

$$d(y_{2n+1}, y_{2n+2}) = d(Fx_{2n+1}, Gx_{2n+2}) = d(Tx_{2n}, Sx_{2n+1}) \leq \psi(M(x_{2n}, x_{2n+1})), \tag{1.116}$$

where

$$\begin{aligned}
&M(x_{2n}, x_{2n+1}) \\
&\quad = a_1 p(Tx_{2n}, Gx_{2n}) + a_2 p(Sx_{2n+1} + a_3 Fx_{2n+1}), p(Gx_{2n}, Fx_{2n+1}), \\
&\quad\quad \frac{a_4}{2}\left[p(Tx_{2n}, Fx_{2n+1}) + p(Sx_{2n+1}, Gx_{2n})\right]\Big\} \\
&\quad = a_1 p(Tx_{2n}, Sx_{2n-1}) + a_2 p(Sx_{2n+1}, Tx_{2n}) + a_3 p(Sx_{2n-1}, Tx_{2n}), \\
&\quad\quad + \frac{a_4}{2}\left[p(Tx_{2n}, Tx_{2n}) + p(Sx_{2n+1}, Sx_{2n-1})\right]\Big\} \\
&\quad \leq a_1 p(Tx_{2n}, Sx_{2n-1}) + a_2 p(Sx_{2n+1}, Tx_{2n}) + a_3 p(Sx_{2n-1}, Tx_{2n}) \\
&\quad\quad + \frac{a_4}{2}[p(Sx_{2n-1}, Tx_{2n}) + p(Tx_{2n}, Sx_{2n+1})] \\
&\quad\quad \text{(derived from the modified triangle inequality } (p_4))
\end{aligned} \tag{1.117}$$

But if $p(Sx_{2n+1}, Tx_{2n}) \geq p(Sx_{2n-1}, Tx_{2n})$ then by (1.116)

$$\begin{aligned}
p(Sx_{2n+1}, Tx_{2n}) &\leq \psi(M(x_{2n}, x_{2n+1})) \\
&\quad \psi([a_1 + a_2 + a_3 + 2a_4]p(Sx_{2n+1}, Tx_{2n})) \\
&\leq \psi([a_1 + a_2 + a_3 + 2a_4]p(Sx_{2n+1}, Tx_{2n})) \\
&\leq \psi(p(Sx_{2n+1}, Tx_{2n})) \\
&\quad < [a_1 + a_2 + a_3 + 2a_4]p(Sx_{2n+1}, Tx_{2n}),
\end{aligned} \tag{1.118}$$

which is a contradiction. Hence, we have

$$p(Sx_{2n+1}, Tx_{2n}) \leq \psi(p(Sx_{2n-1}, Tx_{2n}) < p(Sx_{2n+1}, Tx_{2n}). \tag{1.119}$$

Analogously, we derive that

$$p(Sx_{2n+1}, Tx_{2n+2}) \leq \psi(p(Sx_{2n+1}, Tx_{2n})) < p(Sx_{2n+1}, Tx_{2n}). \tag{1.120}$$

Consequently, we get that

$$p(y_{n+1}, y_n) \leq \psi(p(y_n, y_{n-1})) < p(y_n, y_{n-1}) \text{ for all } n \in \mathbb{N}.$$

By routine calculation as in the proof of Theorem 3, we conclude that the constructed sequence $\{y_n\}$ is Cauchy. Since $X$ is complete, the sequence $\{y_n\}$ converges to a point $z \in X$. Consequently, the subsequences $\{T^m x_{2n}\}$, $\{S^n x_{2n-1}\}$, $\{Gx_{2n}\}$ and $\{Fx_{2n-1}\}$ converge to $z$.

Regarding that $T$, $F$ and $S$, $G$ are commuting pairs and the continuity of $G$ and $F$, the sequences $\{FFx_{2n-1}\}, \{SFx_{2n-1}\}$ tend to $Fz$, and the sequences $\{GGx_{2n}\}, \{TGx_{2n}\}$ tend to $Gz$, as $n \to \infty$.

Thus,

$$p(Gz, Fz) = \lim_{n\to\infty} p(TGx_{2n}, SFx_{2n-1}) \leq r \lim_{n\to\infty} M(Gx_{2n}, Fx_{2n-1}), \quad (1.121)$$

where

$$\begin{aligned} M(Gx_{2n}, Fx_{2n-1}) = a_1 p(TGx_{2n}, GGx_{2n}) + a_2 p(SFx_{2n-1}, FFx_{2n-1}) \\ + a_3 p(GGx_{2n}, FFx_{2n-1}), \\ \frac{a_4}{2}\left[p(TGx_{2n}, FFx_{2n-1}) + p(GGx_{2n}, S^n Fx_{2n-1})\right]. \end{aligned} \quad (1.122)$$

Since $\lim_{n\to\infty} M(Gx_{2n}, Fx_{2n-1}) = p(Gz, Fz)$, then $p(Gz, Fz) \leq \psi(p(Gz, Fz)) < p(Gz, Fz)$. It yields that $Gz = Fz$.

By repeating the same techniques, one can get

$$Tz = Sz = Fz = Gz = z. \quad (1.123)$$

We shall prove that $z$ is unique. Suppose, on the contrary that, there is another common fixed point $w \neq z$ of $S, T, F, G$. Hence, $p(z, w) = p(Tz, Sw) > 0$. Thus, we have

$$p(z, w) = p(Tz, Sw) \leq \psi(M(z, w)),$$

where

$$\begin{aligned} M(z, w) &= \max\{p(Tz, Gz), p(Sw, Fw), p(Gz, Fw), \\ &\qquad \frac{1}{2}[p(Tz, Fw) + p(Sw, Gz)]\Big\} \\ &= \max\left\{p(z, z), p(w, w), p(z, w), \frac{1}{2}[p(z, w) + p(w, z)]\right\} \\ &= p(z, w). \end{aligned} \quad (1.124)$$

Since $M(z, w) = p(z, w)$,

$$p(z, w) \leq \psi(p(z, w)) < p(z, w), \quad (1.125)$$

a contradiction. Therefore $p(z, w) = 0$ and by Lemma 3, we have $z = w$. Hence $z$ is the unique common fixed point of $S, T, F$, and $G$.

**Theorem 44** *Let $(X, p)$ be a complete partial metric space. Suppose that $T$, $S$, $F$, and $G$ are self-mappings on $X$, and each of $F$ and $G$ is continuous. Suppose also that $T$, $F$ and $S$, $G$ are commuting pairs and that*

$$T(X) \subset F(X), \quad S(X) \subset G(X). \tag{1.126}$$

*If there exists $k \in [0, 1)$, and $m$, $n$ in $N$ such that*

$$p(Tx, Sy) \leq kM(x, y) \tag{1.127}$$

*for any $x$, $y$ in $X$ where*

$$\begin{aligned} M(x, y) = a_1 p(Tx, Gx) + a_2 p(Sy, Fy) + 3a_p(Gx, Fy) \\ + \frac{a_4}{2}\left[p(Tx, Fy) + p(Sy, Gx)\right], \end{aligned} \tag{1.128}$$

*where $0 \leq a_i \leq 1$, $i = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + 2a_4 \leq 1$. Then, $T$, $S$, $F$, and $G$ have a unique common fixed point $z$ in $X$.*

*Proof* It is sufficient to take $\psi(t) = kt$, where $k \in [0, 1)$ in Theorem 43.

Regarding the relation between Theorem 43, one concludes that the following corollary from the previous theorem.

**Corollary 7** *Let $(X, p)$ be a complete partial metric space. Suppose that $A$, $B$, $F$, and $G$ are self-mappings on $X$, and each of $F$ and $G$ is continuous. Suppose also that $A$, $B$ and $S$, $G$ are commuting pairs and that*

$$A(X) \subset F(X), \quad B(X) \subset G(X). \tag{1.129}$$

*If there exists $r \in [0, 1)$, and $m$, $n$ in $N$ such that*

$$p(A^m x, B^n y) \leq rM(x, y) \tag{1.130}$$

*for any $x$, $y$ in $X$ where*

$$\begin{aligned} M(x, y) = \max\Big\{ p(A^m x, Gx), p(B^n y, Fy), p(Gx, Fy), \\ \frac{1}{2}\left[p(A^m x, Fy) + p(B^n y, Gx)\right]\Big\}, \end{aligned} \tag{1.131}$$

*then $A$, $B$, $F$, and $G$ have a unique common fixed point $z$ in $X$.*

*Proof* Due to Theorem 43

$$A^m z = B^n z = Fz = Gz = z. \tag{1.132}$$

Indeed, if we follows the proof of the corresponding theorem for $T = A^m$, $S = B^n$, we get (1.123) which is equivalent to (1.132). Thus, $A^m$, $B^n$, $F$, and $G$ have a unique common fixed point $z$ in $X$.

We claim that

$$Az = Bz = z. \tag{1.133}$$

By (1.130) and (1.132),

$$p(Az, z) = p(AA^m z, B^n z) = p(A^m Az, B^n z) \le r \lim_{n\to\infty} M(Az, z), \tag{1.134}$$

where

$$\begin{aligned} M(Az, z) = \max \Big\{ & p(A^m Az, GAz), p(B^n z, Fz), p(GAz, Fz), \\ & \frac{1}{2}\left[p(A^m Az, Fz) + p(B^n z, GAz)\right]\Big\} \\ = \max \Big\{ & p(AA^m z, AGz), p(z, z), p(AGz, z), \\ & \frac{1}{2}\left[p(AA^m z, z) + p(z, AGz)\right]\Big\} \\ = \max \Big\{ & p(Az, Az), 0, p(Az, z), \frac{1}{2}[p(Az, z) + p(z, Az)]\Big\} \\ = {} & p(Az, z). \end{aligned} \tag{1.135}$$

Hence, (1.134) is equivalent to $p(Az, z) \le rp(Az, z)$ which yields that $p(Az, z) = 0$, that is, $Az = z$. Analogously, one can get $Bz = z$. Thus, we observe

$$Az = Bz = z. \tag{1.136}$$

Combining (1.133) and (1.136), we obtain $Gz = Fz = Az = Bz = z$.

## 1.5 Ekeland Type Fixed Point Results

In this section, due to the relevance of Ekeland's principle in the literature over the last decades, the authors believe that extending this principle to the class of partial metric spaces could be useful for developing various applications (see, e.g., [39, 122]). As a consequence of our results, we obtain some fixed point theorems of

Caristi and Clarke types. These results are collected from the recent paper of Aydi et al. [24].

Now, we state and prove the following theorem.

**Theorem 45** *Let* $(X, p)$ *be a complete partial metric space and* $\phi : X \to \mathbb{R}^+$ *be a lower semi-continuous function. Let* $\varepsilon > 0$ *and* $x \in X$ *be such that*

$$\phi(x) \leq \inf_{t \in X} \phi(t) + \varepsilon \quad \textit{and} \quad \inf_{t \in X} p(x, t) < 1. \tag{1.137}$$

*Then there exists some point* $y \in X$ *such that*

$$\phi(y) \leq \phi(x), \tag{1.138}$$

$$p(x, y) \leq 1, \tag{1.139}$$

$$\forall z \in X \textit{ with } z \neq y, \;\; \phi(z) > \phi(y) - \varepsilon p(y, z). \tag{1.140}$$

*Proof* Let $x \in X$ be such that (1.137) holds. Define a sequence $\{x_n\}$ inductively, in the following way: for $n = 1$, take $x_1 := x$ so that $\phi(x_1) \leq \phi(x)$ and $p(x, x_1) = p(x, x) \leq 1$; for the other terms, assume that $x_n \in X$, with $\phi(x_n) \leq \phi(x)$ and $p(x, x_n) \leq 1$, is known and one of the following cases occurs:

(a) $\phi(x_n) - \phi(z) < \varepsilon p(x_n, z)$, for all $z \neq x_n$;
(b) there exists $z \neq x_n$ such that $\varepsilon p(x_n, z) \leq \phi(x_n) - \phi(z)$.

In case (a), if we take $y = x_n$, then (1.138)–(1.140) hold true trivially, since $\phi(y) = \phi(x_n) \leq \phi(x)$.

On the other hand, let $S_n$ be the set of all $z \in X$ such that case (b) holds. Then $x_{n+1} \in S_n$ is chosen in a way that

$$\phi(x_{n+1}) - \inf_{t \in S_n} \phi(t) \leq \frac{1}{2}\left[\phi(x_n) - \inf_{t \in S_n} \phi(t)\right]. \tag{1.141}$$

Consequently, one has

$$\varepsilon p(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1}), \quad \text{for all } \; n \in \mathbb{N} \tag{1.142}$$

and, by using the triangle inequality, one can obtain (for all $n \leq m$)

$$\begin{aligned} &\varepsilon p(x_n, x_m) \\ &\quad \leq \varepsilon[p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1})] \\ &\qquad + \varepsilon[p(x_{n+2}, x_{n+3}) + p(x_{n+3}, x_{n+4}) - p(x_{n+3}, x_{n+3})] + \cdots \end{aligned}$$

$$\begin{aligned}
&+\varepsilon[p(x_{m-2}, x_{m-1}) + p(x_{m-1}, x_m) - p(x_{m-1}, x_{m-1})] \\
&\le \varepsilon \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \\
&\le \sum_{k=n}^{m-1} (\phi(x_k) - \phi(x_{k+1})) \qquad (1.143) \\
&= \phi(x_n) - \phi(x_m).
\end{aligned}$$

By (1.142), the sequence $\{\phi(x_n)\}$ is non-increasing in $\mathbb{R}^+$ and bounded below by zero. Thus, the sequence $\{\phi(x_n)\}$ is convergent, which implies that the right-hand side of (1.143) tends to zero, that is, $p(x_n, x_m)$ tends to zero as $n, m \to +\infty$, so $\{x_n\}$ is a Cauchy sequence in the complete partial metric space $(X, p)$. By Lemma 1, $\{x_n\}$ is Cauchy in the metric space $(X, d_p)$ (also, it is complete). Then, there exists $y \in X$ such that $\{x_n\}$ is convergent to $y$ in $(X, d_p)$. Again by Lemma 1, we get

$$p(y, y) = \lim_{n\to+\infty} p(x_n, y) = \lim_{n,m\to+\infty} p(x_n, x_m). \qquad (1.144)$$

Since $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$, therefore by (1.144) we have $p(y, y) = 0$.
We claim that $y$ satisfies (1.138)–(1.140).

Due to (1.142), the sequence $\{\phi(x_n)\}$ is non-increasing, that is

$$\cdots \le \phi(x_{n+1}) \le \phi(x_n) \le \cdots \le \phi(x_1) \le \phi(x),$$

then (1.138) holds.

The inequality (1.139) is obtained by taking $n = 1$ in (1.143) and by using (1.137). Indeed, we have

$$\begin{aligned}
\varepsilon p(x, x_m) &= \varepsilon p(x_1, x_m) \\
&\le \phi(x) - \phi(x_m) \\
&\le \phi(x) - \inf_{t\in X} \phi(t) \le \varepsilon.
\end{aligned}$$

Hence, taking $m \to +\infty$ it follows that $p(x, y) \le 1$.

The inequality (1.140) is observed by the method of reductio ad absurdum. Assume (1.140) is not true, then there is $z \in X$ with $z \neq y$ such that

$$\phi(z) \le \phi(y) - \varepsilon p(y, z). \qquad (1.145)$$

Since $p(y, z) > 0$, we have

$$\phi(z) < \phi(y). \qquad (1.146)$$

By (1.143), we get

$$\phi(x_m) \leq \phi(x_n) - \varepsilon p(x_n, x_m), \quad \text{for all } n \leq m.$$

Then, taking $m \to +\infty$ in above inequality, one can obtain

$$\phi(y) \leq \liminf_{m\to+\infty} \phi(x_m) \leq \phi(x_n) - \varepsilon p(x_n, y). \tag{1.147}$$

From ($P_4$), we have

$$p(x_n, z) \leq p(x_n, y) + p(y, z) - p(y, y) = p(x_n, y) + p(y, z).$$

Next, using this inequality and (1.145), from (1.147) we get

$$\phi(z) \leq \phi(y) - \varepsilon p(y, z) \leq \phi(x_n) - \varepsilon p(x_n, z),$$

which implies that $z \in S_n$, for all $n \in \mathbb{N}$. Now, note that (1.141) can be written as

$$2\phi(x_{n+1}) - \phi(x_n) \leq \inf_{t\in S_n} \phi(t) \leq \phi(z). \tag{1.148}$$

Therefore, having in mind that $\{\phi(x_n)\}$ is a non-increasing sequence in $\mathbb{R}^+$, there exists $L \geq 0$ such that

$$\lim_{n\to+\infty} \phi(x_n) = L.$$

Letting $n \to +\infty$ in the previous inequality, then we get $L \leq \phi(z)$. On the other hand, since $\phi$ is l.s.c, then we have

$$\phi(y) \leq \liminf_{n\to+\infty} \phi(x_n) = L \tag{1.149}$$

and so we get $\phi(y) \leq \phi(z)$, that is a contradiction with respect to (1.146).

Notice that if in Theorem 45 we do not assume that $\inf_{t\in X} p(x, t) < 1$, then we can (only) deduce that there exists $y \in X$ such that (1.138) and (1.140) hold true.

Building on Theorem 45, we give the following result.

**Theorem 46** *Let $(X, p)$ be a complete partial metric space and $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function. Given $\varepsilon > 0$, then there exists $y \in X$ such that*

$$\phi(y) \leq \inf_{t\in X} \phi(t) + \varepsilon, \tag{1.150}$$

$$\forall z \in X, \;\; \phi(z) \geq \phi(y) - \varepsilon p(y, z). \tag{1.151}$$

*Proof* The proof is clear. Indeed, recalling the fact that there is always some point $x$ such that $\phi(x) \leq \inf_{t\in X} \phi(t) + \varepsilon$, then (1.150) and (1.151) follow from (1.138) and (1.140), respectively.

Notice that Theorem 45 is stronger than Theorem 46. Precisely, the main difference lies in inequality (1.137), which gives the whereabouts of point $x$ in $X$, and which has no counterpart in Theorem 46. Thus, Theorem 45 is said to be the strong statement, and Theorem 46 is said to be the weak statement.

### 1.5.1 Caristi's Fixed Point Theorem

The following theorem is an extension of the result of Caristi [42, Theorem 2.1]. We note that this theorem corresponds to [66, Theorem 5]. Here, we shorten the proof.

**Theorem 47** *Let $(X, p)$ be a complete partial metric space and let $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function. Then any mapping $T : X \to X$ satisfying*

$$p(x, Tx) \leq \phi(x) - \phi(Tx), \quad \textit{for each } x \in X \tag{1.152}$$

*has a fixed point in $X$.*

*Proof* We apply Theorem 46 (for $\varepsilon = \frac{1}{2}$) to the function $\phi$ satisfying (1.152) ($T$, verifying (1.152), is called a Caristi mapping on $(X, p)$). Then, there exists some point $y \in X$ such that

$$\forall t \in X, \quad \phi(t) \geq \phi(y) - \frac{1}{2} p(y, t).$$

This inequality holds also for $t = Ty$, therefore

$$\phi(y) - \phi(Ty) \leq \frac{1}{2} p(y, Ty).$$

Substituting $x = y$ in the inequality (1.152), one can get

$$p(y, Ty) \leq \phi(y) - \phi(Ty).$$

Comparing the last inequalities, we deduce that

$$p(y, Ty) \leq \frac{1}{2} p(y, Ty).$$

This holds unless $p(y, Ty) = 0$ and so by Lemma 3.1, we have $Ty = y$, that is, $T$ has a fixed point.

### 1.5.2 Clarke's Fixed Point Theorem

In 1976, Clarke [48] extended the Banach contraction principle for directional contractions (see condition (D) of Theorem 48) on closed convex subsets of Banach spaces.

**Theorem 48** *Let $X$ be a closed convex subset of a Banach space and let $T : X \to X$ be a continuous mapping satisfying the following condition:*

*(D) there exists $k \in (0, 1)$ such that corresponding to each $u \in X$, there exists $t \in (0, 1]$ for which $\|T(u_t) - T(u)\| \leq k\,\|u_t - u\|$, where $u_t = tT(u) + (1-t)u$ describes the line segment from $u$ to $T(u)$ as $t$ runs from 0 to 1.*

*Then, $T$ has a fixed point in $X$.*

*Proof* The main difference with the proof of Ekeland [50] is that here the proof is reposed on considering a partial metric (not a metric). First, we apply Theorem 46 to the functional $\varphi : X \to \mathbb{R}^+$ given by

$$\varphi(w) = \|w - T(w)\| + b,$$

for all $w \in X$, where $b > 0$ is arbitrary and $0 < \varepsilon < 1 - k$. Then, we define the partial metric $p : X \times X \to \mathbb{R}^+$ by

$$p(w, z) = \|w - z\| + b.$$

Clearly, $p$ is not a metric since $p(w, w) = b > 0$. Moreover,

$$d_p(w, z) = 2\|w - z\|$$

and so $(X, p)$ is a complete partial metric space.

Since $T : (X, \|\cdot\|) \to (X, \|\cdot\|)$ is continuous, then if $w_n \to w$ in $(X, \|\cdot\|)$, we have $T(w_n) \to T(w)$ in $(X, \|\cdot\|)$.

Note that $\varphi(w) = p(w, T(w))$. Now, let $w_n \to w$ in $(X, p)$, then

$$\lim_{n\to+\infty} p(w_n, w) = p(w, w).$$

By definition of the partial metric $p$, we get that

$$\lim_{n\to+\infty} \|w_n - w\| = 0.$$

Therefore $\lim_{n\to+\infty} \|T(w_n) - T(w)\| = 0$. As a consequence, we have

$$\lim_{n\to+\infty} \|w_n - T(w_n)\| = \|w - T(w)\|,$$

that is,

$$\lim_{n\to+\infty} \varphi(w_n) = \varphi(w).$$

We conclude that $\varphi$ is continuous and so is l.s.c in $X$. Due to Theorem 46, there exists some $y \in X$ such that

$$\forall\, w \in X,\ \ \varphi(w) \geq \varphi(y) - \varepsilon p(w, y)$$

that is,

$$\|w - T(w)\| \geq \|y - T(y)\| - \varepsilon(\|w - y\| + b). \tag{1.153}$$

By condition (D), there exist $k \in (0, 1)$ and $t \in (0, 1]$ such that

$$\|T(y_t) - T(y)\| \leq k\, \|y_t - y\| \leq k\, t\|y - T(y)\|.$$

Writing $w = y_t$ into the inequality (1.153), we get

$$\begin{aligned}
&\|y - T(y)\| \\
&\quad \leq \|y_t - T(y_t)\| + \varepsilon(\|y_t - y\| + b) \\
&\quad \leq \|y_t - T(y)\| + \|T(y) - T(y_t)\| + \varepsilon(t\|y - T(y)\| + b) \\
&\quad \leq \|y_t - T(y)\| + k\,t\|y - T(y)\| + \varepsilon(t\|y - T(y)\| + b).
\end{aligned}$$

Now, since $y_t$ belongs to the line segment $[y, T(y)]$, we have

$$\begin{aligned}
\|y - T(y)\| &= \|y - y_t\| + \|y_t - T(y)\| \\
&= t\|y - T(y)\| + \|y_t - T(y)\|.
\end{aligned}$$

It follows easily that

$$t\|y - T(y)\| \leq (k + \varepsilon)\, t\|y - T(y)\| + \varepsilon b,$$

for each $b > 0$. Consequently, letting $b \to 0$, we derive that

$$t\|y - T(y)\| \leq (k + \varepsilon)\, t\|y - T(y)\|.$$

Since $t > 0$, we divide by $t$ to obtain

$$\|y - T(y)\| \leq (k + \varepsilon)\|y - T(y)\|,$$

which holds unless that $y = Ty$, as $k + \varepsilon < 1$. Therefore, $y$ is a fixed point of $T$.

## 1.6 Nonunique Fixed Point Results

In this section, we prove some non-unique fixed point theorems for certain type of self-maps in the context of partial metric spaces. In fact, the fixed point theorems presented here can be considered as a continuation, in part, of the work of Ćirić [45], that is, the given theorems investigate conditions only for the existence of fixed points but not uniqueness. Our results generalize, enrich, and improve some earlier results on the topic in the literature (see, e.g., [3, 45, 69, 93]). We also give examples that show the advantages of using partial metric spaces instead of metric spaces in this context. The results of this section are mainly recollected from [73].

**Lemma 5 (See [73])** *Let* $(X, p)$ *be a partial metric space. A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $X$ *is a Cauchy sequence in* $(X, p)$ *if and only if it satisfies the following condition:*

$(*)$ *for each* $\varepsilon > 0$ *there is* $n_0 \in \mathbb{N}$ *such that* $p(x_n, x_m) - p(x_n, x_n) < \varepsilon$ *whenever* $n_0 \leq n \leq m$.

*Proof.* We first prove the "if" part. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $(X, p)$ satisfying $(*)$. We shall show that then the sequence $\{p(x_n, x_n)\}_{n\in\mathbb{N}}$ converges for the Euclidean metric on $\mathbb{R}^+$. Indeed, let $\varepsilon = 1$. Then, by $(*)$, there is $n_0 \in \mathbb{N}$ such that $p(x_n, x_n) \leq p(x_{n_0}, x_n) < 1 + p(x_{n_0}, x_{n_0})$ whenever $n \geq n_0$. Thus, the sequence $\{p(x_n, x_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^+$, so it has a subsequence $\{p(x_{n_k}, x_{n_k})\}_{k\in\mathbb{N}}$ that converges to an $a \in \mathbb{R}^+$ for the Euclidean metric. Now choose an $\varepsilon > 0$. Then, there is $k_0 \in \mathbb{N}$ such that condition $(*)$ is satisfied whenever $m \geq n \geq n_{k_0}$, and condition $\left|p(x_{n_k}, x_{n_k}) - a\right| < \varepsilon$ also holds for all $k \geq k_0$. Take any $n \geq n_{k_0}$. Then, we have

$$p(x_n, x_n) - a \leq p(x_n, x_{n_{k_0}}) - a < \varepsilon + p(x_{n_{k_0}}, x_{n_{k_0}}) - a < 2\varepsilon,$$

and for $k \in \mathbb{N}$ with $n_k \geq n$, we deduce that

$$a - p(x_n, x_n) < \varepsilon + p(x_{n_k}, x_{n_k}) - p(x_n, x_n) < 2\varepsilon.$$

Consequently $\lim_{n\to\infty} p(x_n, x_n) = a$. Then, by $(*)$, it immediately follows that $\lim_{n,m\to\infty} p(x_n, x_m) = a$. We conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, p)$.

The converse follows from Lemma 1.

For our purposes, we need to recall the following notion

**Definition 17 (See e.g. [73] cf. [45])** Let $(X, p)$ be a partial metric space and $T$ a self-map of $X$.

1. $T$ is called orbitally continuous if

$$\lim_{i,j\to\infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i\to\infty} p(T^{n_i}x, z) = p(z, z), \tag{1.154}$$

implies

$$\lim_{i,j\to\infty} p(TT^{n_i}x, TT^{n_j}x) = \lim_{i\to\infty} p(TT^{n_i}x, Tz) = p(Tz, Tz), \tag{1.155}$$

for each $x \in X$.

Equivalently, $T$ is orbitally continuous provided that if $T^{n_i}x \to z$ with respect to $\tau_{d_p}$, then $T^{n_i+1}x \to Tz$ with respect to $\tau_{d_p}$, for each $x \in X$.

2. $(X, p)$ is called orbitally complete if every Cauchy sequence of type $\{T^{n_i}x\}_{i\in\mathbb{N}}$ converges with respect to $\tau_{d_p}$, that is, if there is $z \in X$ such that

$$\lim_{i,j\to\infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i\to\infty} p(T^{n_i}x, z) = p(z, z). \tag{1.156}$$

In this section we give some non-unique fixed point theorems for partial metric spaces and present some examples illustrating our results.

**Theorem 49 (See [73])** *Let $T$ be an orbitally continuous self-map of a $T$-orbitally complete partial metric space $(X, p)$. If there is $k \in (0, 1)$ such that*

$$\begin{aligned} \min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\} \\ \le k(p(x, y) - p(x, x)) + p(y, y), \end{aligned} \tag{1.157}$$

*for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\omega}$ converges with respect to $\tau_{d_p}$ to a fixed point of $T$.*

*Proof.* Take an arbitrary point $x_0 \in X$. We define the iterative sequence $\{x_n\}_{n\in\omega}$ as follows:

$$x_{n+1} = Tx_n, \quad n \in \omega.$$

If there exists $n_0 \in \omega$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0}$ is a fixed point of $T$. Assume then that $x_n \neq x_{n+1}$ for each $n \in \omega$.

Substituting $x = x_n$ and $y = x_{n+1}$ in (1.157) we find the inequality

$$\begin{aligned} &\min\{p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \\ &\quad - \min\{d_m^p(x_n, x_{n+2}), d_m^p(x_{n+1}, x_{n+1})\} \\ &\quad\quad \le k(p(x_n, x_{n+1}) - p(x_n, x_n)) + p(x_{n+1}, x_{n+1}), \end{aligned}$$

Substituting now $x = x_{n+1}$ and $y = x_n$ in (1.157), we obtain

$$\begin{aligned} &\min\{p(x_{n+2}, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1})\} \\ &\quad - \min\{d_m^p(x_{n+1}, x_{n+1}), d_m^p(x_{n+2}, x_n)\} \\ &\quad\quad \le k(p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1})) + p(x_n, x_n), \end{aligned}$$

which imply that

$$\begin{aligned} &\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \\ &\quad \le k(p(x_n, x_{n+1}) - p(x_n, x_n)) + p(x_{n+1}, x_{n+1}), \end{aligned} \tag{1.158}$$

and

$$\begin{aligned} &\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \\ &\quad \le k(p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1})) + p(x_n, x_n). \end{aligned} \tag{1.159}$$

Suppose $p(x_{n_0}, x_{n_0+1}) \le p(x_{n_0+1}, x_{n_0+2})$ for some $n_0 \in \omega$. Then, from the preceding two inequalities we deduce that

$$\begin{aligned} (1-k)p(x_{n_0}, x_{n_0+1}) \le \min\{&p(x_{n_0+1}, x_{n_0+1}) - kp(x_{n_0}, x_{n_0}), \\ &p(x_{n_0}, x_{n_0}) - kp(x_{n_0+1}, x_{n_0+1})\}. \end{aligned}$$

If, for instance, $p(x_{n_0+1}, x_{n_0+1}) \le p(x_{n_0}, x_{n_0})$, we have

$$\begin{aligned} (1-k)p(x_{n_0}, x_{n_0+1}) &\le p(x_{n_0+1}, x_{n_0+1}) - kp(x_{n_0}, x_{n_0}) \\ &\le (1-k)p(x_{n_0+1}, x_{n_0+1}) \\ &\le (1-k)p(x_{n_0}, x_{n_0}), \end{aligned}$$

so, by using $(P2)$, $p(x_{n_0}, x_{n_0+1}) = p(x_{n_0}, x_{n_0}) = p(x_{n_0+1}, x_{n_0+1})$, and hence $x_{n_0} = x_{n_0+1}$, a contradiction.

Therefore $p(x_n, x_{n+1}) > p(x_{n+1}, x_{n+2})$ for all $n \in \omega$.

Hence, by (1.158) we get

$$\begin{aligned} p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) &\le k(p(x_n, x_{n+1}) - p(x_n, x_n)) \\ &\le k^2(p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})) \\ &\le \ldots \le k^{n+1}((p(x_0, x_1) - p(x_0, x_0)), \end{aligned} \tag{1.160}$$

for all $n \in \omega$.

We shall show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, p)$. Indeed, let $n, m \in \omega$ with $n < m$. Then, by using (1.160) and $(P4)$, we derive that

$$\begin{aligned} p(x_n, x_m) - p(x_n, x_n) &\le p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m) - \sum_{k=n}^{m-1} p(x_k, x_k) \\ &\le (k^n + \cdots + k^{m-1})p(x_0, x_1). \end{aligned}$$

Therefore, the sequence $\{x_n\}_{n\in\omega}$ satisfies condition $(*)$ of Lemma 5, so it is a Cauchy sequence in $(X, p)$. Since $x_n = T^n x_0$ for all $n$, and $(X, p)$ is $T$-orbitally complete, there is $z \in X$ such that $x_n \to z$ with respect to $\tau_{d_p}$. By the orbital continuity of $T$, we deduce that $x_n \to Tz$ with respect to $\tau_{d_p}$. Hence $z = Tz$ which concludes the proof.

**Corollary 8 ((See [73]) [45, Theorem 1])** *Let $T$ be an orbitally continuous self-map of a $T$-orbitally complete metric space $(X, d)$. If there is $k \in (0, 1)$ such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y), \tag{1.161}$$

*for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\omega}$ converges to a fixed point of $T$.*

The following are examples where Theorem 49 can be applied but not Corollary 8 for the metrics $d_p$ and $d_m^p$, and $p_0$, respectively.

*Example 7 (See [73])* Let $X = \{0, 1, 2\}$ and let $p$ be the partial metric on $X$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Define $T : X \to X$ by $T0 = T1 = 0$ and $T2 = 1$. Since $(X, p)$ is complete, then it is $T$-orbitally complete. Moreover, it is obvious that $T$ is orbitally continuous. An easy computation shows that

$$\begin{aligned}&\min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\}\\&\quad \leq \tfrac{1}{2}(p(x, y) - p(x, x)) + p(y, y),\end{aligned}$$

for all $x, y \in X$. So the conditions of Theorem 49 are satisfied. However,

$$\begin{aligned}&\min\{d_p(T1, T2), d_p(1, T1), d_p(2, T2)\} - \min\{d_p(1, T2), d_p(T1, 2)\}\\&\quad = 1 - 0 = 1 > k = kd_p(1, 2),\end{aligned}$$

for any $k \in (0, 1)$, so Corollary 8 cannot be applied to the complete metric space $(X, d_p)$. In fact, it cannot be applied to $(X, d_m^p)$, because $d_m^p = d_p$, in this case.

*Example 8 (See [73])* Let $X = [1, \infty)$ and let $p$ be the partial metric on $X$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Define $T : X \to X$ by $Tx = (x + 1)/2$ for all $x \in X$. Since $(X, p)$ is complete, then it is $T$-orbitally complete. Obviously $T$ is continuous with respect to $\tau_{d_p}$, so it is orbitally continuous.

Next we show that $T$ satisfies the contraction condition (1.157) for any $k \in (0, 1)$. We distinguish two cases for $x, y \in X$:
Case 1. $x = y$. Then

$$\begin{aligned}&\min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\}\\&= \min\left\{\frac{x+1}{2}, x, x\right\} - \left(x - \frac{x+1}{2}\right) = 1\\&\leq x = p(x, x) = k((p(x, y) - p(x, x)) + p(y, y).\end{aligned}$$

Case 2. $x \neq y$. We assume without loss of generality that $x > y$.

If $Tx \geq y$, we have

$$\begin{aligned}
&\min\{p(Tx,Ty), p(x,Tx), p(y,Ty)\} - \min\{d_m^p(x,Ty), d_m^p(Tx,y)\}\\
&= \min\left\{\frac{x+1}{2}, x, y\right\} - \min\left\{x - \frac{y+1}{2}, \frac{x+1}{2} - y\right\}\\
&= y - \left(\frac{x+1}{2} - y\right) = 2y - \frac{x+1}{2}\\
&\leq y = p(y,y) = k((p(x,y) - p(x,x)) + p(y,y).
\end{aligned}$$

If $Tx < y$, we have

$$\begin{aligned}
&\min\{p(Tx,Ty), p(x,Tx), p(y,Ty)\} - \min\{d_m^p(x,Ty), d_m^p(Tx,y)\}\\
&= \min\left\{\frac{x+1}{2}, x, y\right\} - \min\left\{x - \frac{y+1}{2}, y - \frac{x+1}{2}\right\}\\
&= \frac{x+1}{2} - \left(y - \frac{x+1}{2}\right) = x + 1 - y\\
&\quad < y = p(y,y) = k((p(x,y) - p(x,x)) + p(y,y).
\end{aligned}$$

Therefore, the conditions of Theorem 49 are satisfied. In fact $T$ has a (unique) fixed point, $x = 1$.

Finally, we show that Corollary 8 cannot be applied to the self-map $T$ and the complete metric space $(X, p_0)$. Indeed, given $k \in (0, 1)$, choose $x > 1$ such that $x + 1 > 2kx$, and let $y = Tx$. Then

$$\begin{aligned}
&\min\{p_0(Tx,Ty), p_0(x,Tx), p_0(y,Ty)\} - \min\{p_0(x,Ty), p_0(Tx,y)\}\\
&= \min\left\{\frac{x+1}{2}, x\right\} - \min\{x, 0\} = \frac{x+1}{2} > kx = kp_0(x,y).
\end{aligned}$$

Hence, the contraction condition (1.161) is not satisfied.

Our next result extends [45, Theorem 3] to partial metric spaces.

**Theorem 50 (See [73])** *Let $T$ be an orbitally continuous self-map of a partial metric space $(X, p)$. Suppose that $T$ satisfies the inequality*

$$\begin{aligned}
&\min\{p(Tx,Ty), p(x,Tx), p(y,Ty)\} - \min\{d_m^p(x,Ty), d_m^p(Tx,y)\}\\
&\quad < p(x,y) - p(x,x) + p(y,y),
\end{aligned} \tag{1.162}$$

*for all $x, y \in X$ with $x \neq y$. If for some $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\omega}$ has a cluster point $z \in X$ with respect to $\tau_{d_p}$, then $z$ is a fixed point of $T$.*

*Proof.* Let $x_0 \in X$ be such that the sequence $\{T^n x_0\}_{n\in\omega}$ has a cluster point $z \in X$ with respect to $\tau_{d_p}$. Define the iterative sequence $\{x_n\}_{n\in\omega}$ as $x_{n+1} = Tx_n, \quad n \in \omega$.

If there exists $n_0 \in \omega$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0}$ is a fixed point of $T$. Assume then that $x_n \neq x_{n+1}$ for each $n \in \omega$.

As in the proof of Theorem 49, substituting $x = x_n$ and $y = x_{n+1}$ in (1.162) we find the inequality

$$\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} < p(x_n, x_{n+1}) - p(x_n, x_n) + p(x_{n+1}, x_{n+1}),$$

and substituting $x = x_{n+1}$ and $y = x_n$ in (1.162), we obtain

$$\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} < p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1}) + p(x_n, x_n).$$

If $p(x_{n_0}, x_{n_0+1}) \leq p(x_{n_0+1}, x_{n_0+2})$ for some $n_0 \in \omega$, we deduce from the preceding two inequalities that $p(x_{n_0}, x_{n_0}) < p(x_{n_0+1}, x_{n_0+1})$ and $p(x_{n_0+1}, x_{n_0+1}) < p(x_{n_0}, x_{n_0})$, respectively, a contradiction.

Consequently $p(x_n, x_{n+1}) > p(x_{n+1}, x_{n+2})$ for all $n \in \omega$, and thus the sequence $\{p(T^n x_0, T^{n+1} x_0)\}_{n\in\omega}$ is convergent. Since $\{T^n x_0\}_{n\in\omega}$ has a cluster point $z \in X$ with respect to $\tau_{d_p}$, then there is a subsequence $\{T^{n_i} x_0\}_{i\in\omega}$ of $\{T^n x_0\}_{n\in\omega}$ which converges to $z$ with respect to $\tau_{d_p}$. By the orbital continuity of $T$ we have $T^{n_i+1} x_0 \to Tz$ with respect to $\tau_{d_p}$, so we have

$$\lim_{i\to\infty} p(T^{n_i} x_0, T^{n_i+1} x_0) = p(z, Tz). \tag{1.163}$$

Therefore

$$\lim_{n\to\infty} p(T^n x_0, T^{n+1} x_0) = p(z, Tz). \tag{1.164}$$

Again, by the orbital continuity of $T$ we have $T^{n_i+2} x_0 \to T^2 z$ with respect to $\tau_{d_p}$ and hence

$$\lim_{n\to\infty} p(T^{n+1} x_0, T^{n+2} x_0) = p(Tz, T^2 z),$$

so

$$p(Tz, T^2 z) = p(z, Tz). \tag{1.165}$$

Assume $Tz \neq z$, that is, $p(z, Tz) > 0$. So, one can replace $x$ and $y$ with $z$ and $Tz$, respectively, in (1.162) to deduce that

$$\min\{p(z, Tz), p(Tz, T^2 z)\} < p(z, Tz),$$

which yields that $p(Tz, T^2 z) < p(z, Tz)$. This contradicts the equality (1.165). Thus, $Tz = z$. The proof is complete.

Motivated by Ćirić's theorems [45], Pachpatte proved in [93, Theorem 1] that if $T$ is an orbitally continuous self-map of a $T$-orbitally complete metric space $(X, d)$ such that there is $k \in (0, 1)$ with

$$\begin{aligned}\min\{[d(Tx, Tx)]^2, d(x, y)d(Tx, Ty), [d(Ty, y)]^2\}\\ - \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\}\\ \le kd(x, Tx)d(Ty, y) \qquad (1.166)\end{aligned}$$

for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\omega}$ converges to a fixed point of $T$.

However, Pachpatte's theorem has a very limited field of application since under its conditions, if we denote by $z$ any fixed point of $T$, it follows that for each $y \in X$, $Ty = z$ or $Ty = y$. Indeed, let $y \neq z$ and suppose $Ty \neq z$. Then from

$$\begin{aligned}\min\{[d(Tz, Ty)]^2, d(z, y)d(Tz, Ty), [d(y, Ty)]^2\}\\ - \min\{d(z, Tz)d(y, Ty), d(z, Ty)d(y, Tz)\}\\ \le kd(z, Tz)d(y, Ty),\end{aligned}$$

it follows

$$\min\{[d(z, Ty)]^2, d(z, y)d(z, Ty), [d(y, Ty)]^2\} = 0.$$

Hence $d(y, Ty) = 0$, i.e., $y = Ty$.

In our next result we modify the contraction condition (1.166) and thus obtain a new fixed point theorem that avoids the inconvenience indicated above. In fact, this will be done in the more general setting of partial metric spaces and, to this end, the following notation will be used: If $p$ is a partial metric on a set $X$ we denote by $p'$ the function defined on $X \times X$ by $p'(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$. (Of course, $p' = p$ whenever $p$ is a metric on $X$.)

**Theorem 51 (See [73])** *Let $T$ be an orbitally continuous self-map of a $T$-orbitally complete partial metric space $(X, p)$. If there is $k \in (0, 1)$ such that*

$$\begin{aligned}\min\{[p'(x, Tx)]^2, p'(x, y)p'(Tx, Ty), [p'(y, Ty)]^2\}\\ - \min\{d_m^p(x, Tx)d_m^p(y, Ty), d_m^p(x, Ty)d_m^p(y, Tx)\}\\ \le k \min\{p'(x, Tx)p'(y, Ty), [p'(x, y)]^2\},\end{aligned} \qquad (1.167)$$

*for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\omega}$ converges with respect to $\tau_{d_p}$ to a fixed point of $T$.*

*Proof.* As in the proof of Theorem 49, take an arbitrary point $x_0 \in X$ and define the iterative sequence $\{x_n\}_{n\in\omega}$ as $x_{n+1} = Tx_n, \quad n \in \omega$.

If there exists $n_0 \in \omega$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0}$ is a fixed point of $T$. Assume then that $x_n \neq x_{n+1}$ for each $n \in \omega$.

Substituting $x = x_n$ and $y = x_{n+1}$ in (1.167) we find the inequality

$$\begin{aligned}&\min\{[p'(x_n, x_{n+1})]^2, p'(x_n, x_{n+1})p'(x_{n+1}, x_{n+2}), [p'(x_{n+1}, x_{n+2})]^2\}\\ &\quad \leq k\min\{p'(x_n, x_{n+1})p'(x_{n+1}, x_{n+2}), [p'(x_n, x_{n+1})]^2\}.\end{aligned} \tag{1.168}$$

By (1.168) we deduce that

$$\begin{aligned}&\min\{[p'(x_n, x_{n+1})]^2, p'(x_n, x_{n+1})p'(x_{n+1}, x_{n+2}), [p'(x_{n+1}, x_{n+2})]^2\}\\ &\quad = [p'(x_{n+1}, x_{n+2})]^2,\end{aligned}$$

and hence

$$p'(x_{n+1}, x_{n+2}) \leq kp'(x_n, x_{n+1}),$$

for all $n \in \omega$. Therefore

$$p(x_n, x_{n+1}) - p(x_n, x_n) \leq k^n(p(x_0, x_1) - p(x_0, x_0)),$$

for all $n \in \mathbb{N}$. As in the proof of Theorem 49, we deduce that the sequence $\{x_n\}_{n\in\omega}$ is Cauchy in $(X, p)$. Since $x_n = T^n x_0$ for all $n$, and $(X, p)$ is $T$-orbitally complete, there is $z \in X$ such that $x_n \to z$ with respect to $\tau_{d_p}$. By the orbital continuity of $T$, we deduce that $x_n \to Tz$ with respect to $\tau_{d_p}$. Hence $z = Tz$ which concludes the proof.

**Corollary 9 (See [73])** *Let $T$ be an orbitally continuous self-map of a $T$-orbitally complete metric space $(X, d)$. If there is $k \in (0, 1)$ such that*

$$\begin{aligned}&\min\{[d(x, Tx)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2\}\\ &\quad - \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\}\\ &\quad\quad \leq k\min\{d(x, Tx)d(y, Ty), [d(x, y)]^2\},\end{aligned} \tag{1.169}$$

*for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\omega}$ converges to a fixed point of $T$.*

*Remark 6* Note that if $(X, p)$ is the complete partial metric space of Example 1, then each orbitally continuous self-map $T$ of $X$ such that $Tx \leq x$ for all $x \in X$ has a fixed point. Indeed, for such a $T$ we have $p'(x, Tx) = 0$ for all $x \in X$, so condition (1.167) in Theorem 51 is trivially satisfied.

The following is an example where Theorem 51 can be applied but not Corollary 9 for any of the metrics $d_p$, $d_m^p$, and $p_0$.

*Example 9 (See [73])* Let $(X, p)$ be the partial metric space of Example 1. Define $T : X \to X$ by $Tx = x - 1$ if $x \geq 2$ and $Tx = 0$ if $x < 1$. Then $T$ is orbitally continuous because for each $x \in X$ one has $T^n x \to 0$ with respect to $\tau_{d_p}$, and $T0 = 0$. Moreover, by Remark 6 the contraction condition (1.167) is also satisfied, and thus all the conditions of Theorem 51 hold.

Now take $x \geq 3$ and $y = Tx$. Then $x - y = 1$, and $y \geq 2$. Hence

$$\begin{aligned}
&\min\{[d_p(x, Tx)]^2, d_p(x, y)d_p(Tx, Ty), [d_p(y, Ty)]^2\}\\
&\quad - \min\{d_p(x, Tx)d_p(y, Ty), d_p(x, Ty)d_p(y, Tx)\}\\
&\quad\quad = \min\{1, (x - y)^2, 1\} - 0 = 1\\
&\quad\quad = \min\{d_p(x, Tx)d_p(y, Ty), [d_p(x, y)]^2\}.
\end{aligned}$$

Therefore, condition (1.169) is not satisfied for any $k \in (0, 1)$, so we cannot apply Corollary 9 to $(X, d_p)$ (and thus to $(X, d_m^p)$ and the self-map $T$.

Finally, given $k \in (0, 1)$, choose $x \geq 3$ with $x > 1/(1 - k)$, and $y = Tx$. Then

$$\begin{aligned}
&\min\{[p_0(x, Tx)]^2, p_0(x, y)p_0(Tx, Ty), [p_0(y, Ty)]^2\}\\
&\quad - \min\{p_0(x, Tx)p_0(y, Ty), p_0(x, Ty)p_0(y, Tx)\}\\
&\quad\quad = \min\{x^2, x(x - 1), (x - 1)^2\} - 0 = (x - 1)^2 > kx(x - 1)\\
&\quad\quad = k \min\{p_0(x, Tx)p_0(y, Ty), [p_0(x, y)]^2\}.
\end{aligned}$$

Therefore, we cannot apply Corollary 9 to $(X, p_0)$ and the self-map $T$ (note that, in fact, $T$ is orbitally continuous for $(X, p_0)$).

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# Chapter 2
# Fixed Point Results for Mixed Multivalued Mappings of Feng-Liu Type on $M_b$-Metric Spaces

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## 2.1 Introduction and Preliminaries

Let $(X, d)$ be a metric space. We denote that $C(X)$ is the family of all nonempty closed subsets of $X$, $CB(X)$ is the family of all nonempty closed and bounded subsets of $X$. Pompeiu-Hausdorff metric is defined on $CB(X)$ as follows; for all $A, B \in CB(X)$

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}$$

(see [6]). Now, let $T : X \to CB(X)$. If there exists $\lambda \in (0, 1)$ such that $H(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$, then $T$ is called multivalued contraction. In 1969, Nadler [13] who is the first person to think fixed point result for multivalued mapping on metric space showed that if $(X, d)$ is a complete metric space and $T$ is a multivalued contraction mapping, then $T$ has a fixed point. That is, there exist $z \in X$ such that $z \in Tz$. After Nadler, many researchers have improved fixed point theory for multivalued mappings (see, for example, [3, 5, 7, 9, 14]). In a different way,

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Feng and Liu [10] generalized the Nadler's result without using Pompeiu-Hausdorff metric and by considering $C(X)$ instead of $CB(X)$ as follows:

**Theorem 1** *Let $(X, d)$ be a complete metric space and $T : X \to C(X)$ be a multivalued mapping. If for all $x \in X$ there exists $y \in I_b^x$ satisfying*

$$d(y, Ty) \leq cd(x, y),$$

*where*

$$I_b^x = \{y \in Tx : bd(x, y) \leq d(x, Tx)\}$$

*then $T$ has a fixed point in $X$ provided that $0 < c < b < 1$ and the function $f(x) = d(x, Tx)$ is lower semicontinuous.*

In 1994, Matthews [11] introduced the partial metric space as a generalization of ordinary metric space and then fixed point theory for both single valued and multivalued mappings on partial metric spaces has been studied by many authors [1, 8]. After that, to extend partial metric space, Asadi et al.[2] introduced the concept of $M$-metric space and obtained some fixed point theorems for single valued mappings on $M$-metric space. On the other hand, there is a different generalization of ordinary metric space in the literature known as $b$-metric space [4].

Recently, taking into account both of $M$-metric and $b$-metric on a nonempty set, Mlaiki et al. [12] generated the concept of $M_b$-metric to extend both of $M$-metric space and $b$-metric space as follows: Let $X$ be a nonempty set and $m_b : X \times X \to [0, \infty)$ be function. Then $m_b$ is called an $M_b$-metric on $X$ and $(X, m_b)$ is called an $M_b$-metric space, if the following conditions are satisfied: for all $x, y, z \in X$,

(m1) $m_b(x, x) = m_b(y, y) = m_b(x, y) \Leftrightarrow x = y$,
(m2) $m_{bxy} \leq m_b(x, y)$, where $m_{bxy} = \min\{m_b(x, x), m_b(y, y)\}$,
(m3) $m_b(x, y) = m_b(y, x)$,
(m4) there exists a real number $s \geq 1$ such that

$$m_b(x, y) - m_{bxy} \leq s[m_b(x, z) - m_{bxz} + m_b(z, y) - m_{bzy}] - m_b(z, z).$$

The number $s$ is called the coefficient of $M_b$-metric.

*Remark 1* Note that, as shown in [2], if $m$ is an $M$-metric on $X$, then $m(x, x)$ may not be zero for $x \in X$. However, if we take $x = y = z$ in the condition (m4) of the definition of $M_b$-metric, we have $0 \leq -m_b(x, x)$. Since $m_b$ is nonnegative valued, we obtain $m_b(x, x) = 0$ for all $x \in X$. To overcome this problem, we propose the following condition instead of (m4):

(m4)* for all $x, y, z \in X$, there exists a real number $s \geq 1$ such that

$$m_b(x, y) - m_{bxy} \leq s[m_b(x, z) - m_{bxz} + m_b(z, y) - m_{bzy}].$$

The rest of this paper, we will use the conditions (m1), (m2), (m3), and (m4)* for the concept of $M_b$-metric. Therefore, it is clear that every $M$-metric and every $b$-metric on a nonempty set $X$ are also $M_b$-metric.

*Example 1* Let $X = [0, \infty)$ and define a mapping by $m_b(x, y) = \min\{x^p, y^p\} + |x - y|^p$, where $p > 1$. Then $m_b$ is an $M_b$-metric (in our sense) with coefficient $s = 2^p$. Besides it is neither an $M$-metric nor $b$-metric on $X$. Also, since the condition (m4) is not satisfied for $x = y = z \neq 0$, then $m_b$ is not an $M_b$-metric in the sense of Mlaiki et al.

Let $(X, m_b)$ be an $M_b$-metric space, $x \in X$ and $r > 0$. The open ball with centered $x \in X$ and radius $r > 0$ is denoted by

$$B(m_b, x, r) = \{y \in X : m_b(x, y) < r + m_{bxy}\}.$$

Then, we call a subset $U$ of $X$ is open if and only if for all $x \in U$, there exists $r > 0$ such that $B(m_b, x, r) \subseteq U$. In this case, it can be shown that the family of all open subsets of $X$ is a topology on $X$, say $\tau_{m_b}$. Even though every partial metric $p$ is an $M_b$-metric on a nonempty set $X$ and every partial metric $p$ generates a $T_0$ topology on $X$, the topology $\tau_{m_b}$ may not be $T_0$ topology. The following example shows this fact.

*Example 2* Let $X = [0, 1]$ and $m_b(x, y) = \min\{x, y\}$, then $m_b$ is an $M_b$-metric on $X$ with coefficient $s = 2$. In this case for every $\varepsilon > 0$, we get

$$\begin{aligned} B(m_b, x, \varepsilon) &= \left\{y \in X : m_b(x, y) < m_{bxy} + \varepsilon\right\} \\ &= \{y \in X : 0 < \varepsilon\} \\ &= X \end{aligned}$$

for all $x \in X$. Therefore $\tau_{m_b} = \{\emptyset, X\}$ is not $T_0$ topology.

*Remark 2* Let $(X, m_b)$ be an $M_b$-metric space with the constant $s \geq 1$. Then the function defined by

$$b_m(x, y) = m_b(x, y) - 2m_{bxy} + M_{bxy}$$

is a $b$-metric on $X$ with the same constant $s$, where

$$M_{bxy} = \max\{m_b(x, x), m_b(y, y)\}.$$

*Remark 3* Let $(X, m_b)$ be an $M_b$-metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then the sequence $\{x_n\}$ converges to $x$ with respect to $\tau_{m_b}$ if and only if $\lim_{n\to\infty}(m_b(x_n, x) - m_{bx_nx}) = 0$. Indeed, let $x_n \overset{\tau_{m_b}}{\to} x$, and $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(m_b, x, \varepsilon)$ for all $n \geq n_0$. Therefore, for $n \geq n_0$ we have $m_b(x_n, x) < \varepsilon + m_{bx_nx}$ and so $\left|m_b(x_n, x) - m_{bx_nx}\right| < \varepsilon$. Thus we have $\lim_{n\to\infty}(m_b(x_n, x) - m_{bx_nx}) = 0$. Conversely, let $\lim_{n\to\infty}(m_b(x_n, x) - m_{bx_nx}) = 0$

and $U \in \tau_{m_b}$ such that $x \in U$. Then there exist $\varepsilon > 0$ such that $B(m_b, x, \varepsilon) \subseteq U$. Since $\lim_{n\to\infty}(m_b(x_n, x) - m_{bx_nx}) = 0$, there exist $n_0 \in \mathbb{N}$ such that $m_b(x_n, x) - m_{bx_nx} < \epsilon$ for all $n \geq n_0$. So that, $x_n \in B(m_b, x, \varepsilon) \subseteq U$ for all $n \geq n_0$. Hence, the sequence $\{x_n\}$ converges to $x$ with respect to $\tau_{m_b}$ on $X$. If the sequence $\{x_n\}$ converges to $x \in X$ with respect to $\tau_{m_b}$, we will call it as $M_b$-convergence.

**Definition 1** Let $(X, m_b)$ be an $M_b$-metric space and $\{x_n\}$ be a sequence in $X$.

1. $\{x_n\}$ is said to be $M_b$-Cauchy sequence if

$$\lim_{m,n\to\infty} m_b(x_n, x_m)$$

   exist and finite.
2. $(X, m_b)$ is said to be $M_b$-complete if every $M_b$-Cauchy sequence in $X$ is $M_b$-convergent to a point $x \in X$ such that

$$\lim_{n,m\to\infty} m_b(x_n, x_m) = m_b(x, x).$$

**Lemma 1** *Let $(X, m_b)$ be an $M_b$-metric space and $\{x_n\}$ be a sequence in $X$. Then:*

1. *$\{x_n\}$ is $M_b$-Cauchy sequence if and only if it is a Cauchy sequence in the b-metric space $(X, b_m)$.*
2. *$(X, m_b)$ is $M_b$-complete if and only if the b-metric space $(X, b_m)$ is complete. Moreover,*

$$\lim_{n\to\infty} b_m(x_n, x) = 0 \Leftrightarrow \left\{ \begin{array}{c} \lim_{n\to\infty}[m_b(x_n, x) - m_{bx_nx}] = 0 \\ \text{and} \\ \lim_{n,m\to\infty} m_b(x_n, x_m) = m_b(x, x) \end{array} \right\}$$

*Proof* Let $\{x_n\}$ be an $M_b$-Cauchy sequence in $(X, m_b)$. Then there exists $a \in \mathbb{R}$ such that $\lim_{n,k\to\infty} m_b(x_n, x_k) = a$. Thus, $\lim_{n\to\infty} m_b(x_n, x_n) = a$. Therefore we have

$$\lim_{n,k\to\infty} b_m(x_n, x_k) = \lim_{n,k\to\infty} \left[m_b(x_n, x_k) - 2m_{bx_nx_k} + M_{bx_nx_k}\right] = 0,$$

that is, $\{x_n\}$ is a Cauchy sequence in $(X, b_m)$.

Conversely, let $\{x_n\}$ be a Cauchy sequence in $(X, b_m)$. Thus for $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that $b_m(x_n, x_k) < 1$ for all $n, k \geq n_0$. Therefore we have

$$\begin{aligned} m_b(x_n, x_n) &= m_b(x_n, x_n) - m_b(x_{n_0}, x_{n_0}) + m_b(x_{n_0}, x_{n_0}) \\ &\leq \left|m_b(x_n, x_n) - m_b(x_{n_0}, x_{n_0})\right| + m_b(x_{n_0}, x_{n_0}) \\ &= M_{bx_nx_{n_0}} - m_{bx_nx_{n_0}} + m_b(x_{n_0}, x_{n_0}) \\ &\leq b_m(x_n, x_{n_0}) + m_b(x_{n_0}, x_{n_0}) \\ &< 1 + m_b(x_{n_0}, x_{n_0}). \end{aligned}$$

This shows that $\{m_b(x_n, x_n)\}$ is bounded sequence in $\mathbb{R}$ so there exists $a \in \mathbb{R}$ such that the subsequence $\{m_b(x_{n_k}, x_{n_k})\}$ converges to $a$. On the other hand, since $\{x_n\}$ is a Cauchy sequence in $(X, b_m)$, given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$|m_b(x_n, x_n) - m_b(x_k, x_k)| \leq b_m(x_n, x_k) < \varepsilon$$

for all $n, k \geq n_1$. Thus $\{m_b(x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$ and so we get

$$\lim_{n\to\infty} m_b(x_n, x_n) = a = \lim_{n\to\infty} m_{bx_nx_k} = \lim_{n\to\infty} M_{bx_nx_k}.$$

Now, since

$$\begin{aligned}|m_b(x_n, x_k) - a| &= \left|m_b(x_n, x_k) - 2m_{bx_nx_k} + M_{bx_nx_k} + 2m_{bx_nx_k} - M_{bx_nx_k} - a\right| \\ &\leq \left|m_b(x_n, x_k) - 2m_{bx_nx_k} + M_{bx_nx_k}\right| + \left|2m_{bx_nx_k} - M_{bx_nx_k} - a\right| \\ &= b_m(x_n, x_k) + \left|2m_{bx_nx_k} - M_{bx_nx_k} - a\right|\end{aligned}$$

we get $\lim_{n,k\to\infty} m_b(x_n, x_k) = a$. This shows that $\{x_n\}$ is an $M_b$-Cauchy sequence in $(X, m_b)$.

Now let $(X, m_b)$ be $M_b$-complete and $\{x_n\}$ be a Cauchy sequence in $(X, b_m)$. Then it is an $M_b$-Cauchy sequence in $(X, m_b)$ and so there exists $x \in X$ such that

$$\lim_{n\to\infty} \left[m_b(x_n, x) - m_{bx_nx}\right] = 0$$

and

$$\lim_{n,m\to\infty} m_b(x_n, x_m) = m_b(x, x).$$

Therefore we have

$$\begin{aligned}\lim_{n\to\infty} b_m(x_n, x) &= \lim_{n\to\infty} \left[m_b(x_n, x) - 2m_{bx_nx} + M_{bx_nx}\right] \\ &= \lim_{n\to\infty} \left[m_b(x_n, x) - m_{bx_nx} + M_{bx_nx} - m_{bx_nx}\right] \\ &= 0.\end{aligned}$$

Thus $(X, b_m)$ is complete $b$-metric space.

Conversely, let $(X, b_m)$ be a complete $b$-metric space and $\{x_n\}$ be an $M_b$-Cauchy sequence in $(X, m_b)$. Then it is a Cauchy sequence in $(X, b_m)$ and thus there exists $x \in X$ such that

$$\lim_{n\to\infty} b_m(x_n, x) = \lim_{n\to\infty} \left[m_b(x_n, x) - m_{bx_nx} + M_{bx_nx} - m_{bx_nx}\right] = 0.$$

Therefore since both $m_b(x_n, x) - m_{bx_nx}$ and $M_{bx_nx} - m_{bx_nx}$ are nonnegative, we get $\lim_{n\to\infty} \left[m_b(x_n, x) - m_{bx_nx}\right] = 0$ and $\lim_{n\to\infty} \left[M_{bx_nx} - m_{bx_nx}\right] = 0$. So that

$\{x_n\}$ is $M_b$-convergent to $x$. Since $\{x_n\}$ be an $M_b$-Cauchy sequence in $(X, m_b)$, it is sufficient to see that $\lim_{n\to\infty} m_b(x_n, x_n) = m_b(x, x)$. Furthermore, we get

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left[M_{bx_nx} - m_{bx_nx}\right] \\ &= \lim_{n\to\infty} |m_b(x_n, x_n) - m_b(x, x)| \end{aligned}$$

and so $\lim_{n\to\infty} m_b(x_n, x_n) = m_b(x, x)$.

In this article, we present Feng-Liu type fixed point results for multivalued mappings on $M_b$-metric space $(X, m_b)$. We denote by $C(X)$ the class of all nonempty closed subsets of $(X, m_b)$ and by $\overline{A}^b$ closure of $A \subseteq X$ with respect to $\tau_{m_b}$.

## 2.2 Main Result

We will begin this section with the following idea: Let $(X, d)$ be a metric space and $T : X \to C(X)$ be a multivalued mapping. By considering the Feng-Liu's fixed point theorem for $T$ on $X$, we can generate a fixed point result for single valued mapping. To do this, it is enough to take $Tx$ as singleton for all $x \in X$ because every singleton is closed set in metric space. But since the topology $\tau_{m_b}$ may not be $T_1$ topology (even it may not be $T_0$ topology as seen in Example 2), some single point sets may not be closed. Therefore we cannot generate a fixed point result for single valued mapping by the similar way. To overcome this problem, we will consider the mapping $T$ from an $M_b$-metric space $X$ to the class $X \cup C(X)$ as in [15]. For an $M_b$-metric space $(X, m_b)$, we will write $T : X \to X \cup C(X)$ if $Tx$ is singleton or $Tx \in C(X)$ for all $x \in X$. Since the mapping $T$ is both single-valued and multivalued, we will use mixed multivalued mapping for the mapping $T$ as in [15].

Now, let $(X, m_b)$ be an $M_b$-metric space. Let $T : X \to X \cup C(X)$ be a mixed multivalued mapping. For a positive constant $k \in (0, 1)$ and $x \in X$, define a set

$$T_k^x(m_b) = \{y \in Tx : km_b(x, y) \leq m_b(x, Tx)\},$$

where

$$m_b(x, Tx) = \inf\{m_b(x, y) : y \in Tx\}.$$

Now, if $|Tx| = 1$, then $T_k^x(m_b)$ is not empty. On the other hand, if $|Tx| > 1$ and $m_b(x, Tx) > 0$, then $T_k^x(m_b)$ is nonempty for all $k \in (0, 1)$. But, if $|Tx| > 1$ and $m_b(x, Tx) = 0$, then $T_k^x(m_b)$ may be empty.

*Example 3* Let $X = \left\{0, -1, -1 + \frac{1}{n} : n > 1, n \in \mathbb{N}\right\}$ and define an $M_b$-metric on $X$ with coefficient $s = 5$ as

$$m_b(x,y)=\begin{cases} 0 & , \quad x=y\in X\backslash\{-1\} \\ 2 & , \quad x=y=-1 \\ 1 & , \quad x\neq y\in\{0,-1\} \\ |x-y| & , \quad x\neq y\in\{-1,-1+\frac{1}{2n},n>1\} \\ 4 & , \quad \text{otherwise} \end{cases}.$$

Besides, the mapping $m_b$ is not $M$-metric. Indeed, if we take $x=0$, $y=-1+\frac{1}{2n}$, $z=-1$, the inequality $m_b(x,y)-m_{bxy}\leq[m_b(x,z)-m_{bxz}+m_b(z,y)-m_{bzy}]$ which is condition of $M$-metric is not satisfied. We define the mapping $T$ as $Tx=X$ for all $x\in X$. For $x=-1$, we get $m_b(x,Tx)=0$ and $m_b(x,y)>0$ for all $y\in Tx$. Hence $T_k^x(m_b)=\emptyset$ for all $k\in(0,1)$.

The following propositions are useful in the proof of main theorem.

**Proposition 1** *Let* $(X,m_b)$ *be an* $M_b$*-metric space,* $A\subseteq X$ *and* $x\in X$*. If* $m_b(x,A)=0$ *then* $x\in\overline{A}^b$.

*Proof* Let $m_b(x,A)=0$ and $U\in\tau_{m_b}$, $x\in U$. Then there exist $\epsilon>0$ such that $B(m_b,x,\epsilon)\subseteq U$. Since $m_b(x,A)=0$, there exists $x_\epsilon\in A$ such that $m_b(x,x_\epsilon)<\epsilon$. So, $m_b(x,x_\epsilon)<\epsilon+m_{bxx_\epsilon}$. In this case, since $x_\epsilon\in B(m_b,x,\epsilon)\subseteq U$ and $x_\epsilon\in A$, $A\cap U\neq\varnothing$. Thus, we get $x\in\overline{A}^b$.

**Proposition 2** *Let* $(X,m_b)$ *be an* $M_b$*-metric space,* $A\subseteq X$ *and* $x\in X$*. Then,* $\inf\{m_b(x,y)-m_{bxy}:y\in A\}=0$ *if and only if* $x\in\overline{A}^b$.

*Proof* Let $\inf\{m_b(x,y)-m_{bxy}:y\in A\}=0$ and $\epsilon>0$. Thus, there exist $y_\epsilon\in A$ such that $m_b(x,y_\epsilon)-m_{bxy_\epsilon}<\epsilon$. In this case, $y_\epsilon\in B(x,\epsilon)$ and thus $y_\epsilon\in A\cap B(x,\epsilon)$. So, $x\in\overline{A}^b$. Now, let $x\in\overline{A}^b$. Then, there exists $y_k\in A$ such that $m_b(x,y_k)-m_{bxy_k}<\frac{1}{k}$ for all $k\in\mathbb{N}$. Since $\inf\{m_b(x,y)-m_{bxy}:y\in A\}\leq m_b(x,y_k)-m_{bxy_k}<\frac{1}{k}$ for all $k\in\mathbb{N}$, we get that $\inf\{m_b(x,y)-m_{bxy}:y\in A\}=0$.

Before we prove our main result, we give the following new definition.

**Definition 2** A mixed multivalued mapping $T$ is called *x-lower orbitally continuous* if $|Tx|=1$ and a sequence $(x_n)$ in $X$ such that $x_{n+1}\in Tx_n$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}m_b(x_n,x)=m_b(x,x)$, then $m_b(Tx,Tx)\leq\liminf_{n\to\infty}m_b(x_n,Tx)$.

If $T$ is an *orbitally continuous* at a point $x$, then it is *lower orbitally continuous* at $x$. But reverse is not true. The following example shows this fact.

*Example 4* Let $X=\left\{0,-1,-1+\frac{1}{n}:n>1,n\in\mathbb{N}\right\}$ and define an $M_b$-metric on $X$ with coefficient $s=5$ as

$$m_b(x,y)=\begin{cases} 0 & , \quad x=y\in X \\ 1 & , \quad x\neq y\in\{0,-1\} \\ |x-y| & , \quad x\neq y\in\{-1,-1+\frac{1}{2n},n>1\} \\ 4 & , \quad \text{otherwise} \end{cases}.$$

Let $T:X\to X\cup C(X)$ satisfying

$$Tx = \begin{cases} X \ , x \in X \backslash \{-1\} \\ \{0\} \ , \quad x = -1 \end{cases}$$

and $(x_n) = (-1 + \frac{1}{2n})_{n\in\mathbb{N}}$. Clearly, $x_{n+1} \in Tx_n = X$ for all $n \in \mathbb{N}$ and

$\lim_{n\to\infty} m_b(x_n, -1) = 0 = m_b(-1, -1)$. On the other hand, it can be seen that $m_b(Tx, Tx) = m_b(0, 0) = 0 < \liminf_{n\to\infty} m_b(x_n, Tx) = \liminf_{n\to\infty} m_b(x_n, 0) = 4$ for $x = -1$.

**Theorem 2** *Let $(X, m_b)$ be an $M_b$-complete $M_b$-metric space with coefficient $s \geq 1$ and $T : X \to X \cup C(X)$ be a mixed multivalued map. If there exist a constant $c \in (0, 1)$ and $y \in T_k^x(m_b)$ such that for all any $x \in X$ with either $m_b(x, Tx) > 0$ and $|Tx| > 1$ or $|Tx| = 1$ satisfying*

$$m_b(y, Ty) \leq cm_b(x, y).$$

and

$$m_b(y, y) \leq m_b(x, y).$$

*Then there exists $z \in X$ such that $m_b(z, Tz) = 0$ provided that $sc < k$ and the function $f(x) = m_b(x, Tx)$ is lower semicontinuous. Further, if $|Tz| > 1$, then $z$ is a fixed point of $T$. If $|Tz| = 1$ and $T$ is $lower\ orbitally\ continuous$ at $z$, then $z$ is a fixed point of $T$.*

*Proof* We want to show that there exists a sequence $(x_n)$ in $X$ such that $x_{n+1} \in Tx_n$, $m_b(x_n, x_{n+1}) \leq \left(\frac{c}{k}\right)^n m_b(x_0, x_1)$ and $m_b(x_n, Tx_n) \leq \left(\frac{c}{k}\right)^n m_b(x_0, Tx_0)$ for all $n \in \mathbb{N}$. Let $x_0 \in X$ be an arbitrary point. Now, we consider two conditions:

$Case$ 1 : Let $|Tx_0| = 1$. In this case, there exist $x_1 \in X$ such that

$$km_b(x_0, x_1) \leq m_b(x_0, Tx_0)$$

and

$$m_b(x_1, Tx_1) \leq cm_b(x_0, x_1).$$

Again, there are two conditions. According to the first condition, it may be $|Tx_1| = 1$. From hypothesis, there exist $x_2 \in X$ such that

$$km_b(x_1, x_2) \leq m_b(x_1, Tx_1)$$

and

$$m_b(x_2, Tx_2) \leq cm_b(x_1, x_2).$$

Another situation say that it may be $|Tx_1| > 1$. If $m_b(x_1, Tx_1) = 0$, then $x_1 \in \overline{Tx_1}^b = Tx_1$. That is, $x_1$ is a fixed point of $T$. Let $m_b(x_1, Tx_1) > 0$. From hypothesis there exist $x_2 \in X$ such that

$$km_b(x_1, x_2) \leq m_b(x_1, Tx_1)$$

and

$$m_b(x_2, Tx_2) \leq cm_b(x_1, x_2).$$

*Case* 2 : Let $|Tx_0| > 1$. If $m_b(x_0, Tx_0) = 0$, then $x_0 \in \overline{Tx_0}^b = Tx_0$. That is, $x_0$ is a fixed point of $T$. So, the proof ends. If $m_b(x_0, Tx_0) > 0$, then from hypothesis, there exist $x_1 \in X$ such that

$$km_b(x_0, x_1) \leq m_b(x_0, Tx_0)$$

and

$$m_b(x_1, Tx_1) \leq cm_b(x_0, x_1).$$

Similar to *Case* 1, it may be $|Tx_1| = 1$ or $|Tx_1| > 1$. Let $|Tx_1| = 1$. From hypothesis, there exist $x_2 \in X$ such that

$$km_b(x_1, x_2) \leq m_b(x_1, Tx_1)$$

and

$$m_b(x_2, Tx_2) \leq cm_b(x_1, x_2).$$

Now, we assume $|Tx_1| > 1$. If $m_b(x_1, Tx_1) = 0$, then $x_1 \in \overline{Tx_1}^b = Tx_1$. That is, $x_1$ is a fixed point of $T$. If $m_b(x_1, Tx_1) > 0$, then there exist $x_2 \in X$ such that

$$km_b(x_1, x_2) \leq m_b(x_1, Tx_1)$$

and

$$m_b(x_2, Tx_2) \leq cm_b(x_1, x_2).$$

Keep going this process, we can generate a sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in Tx_n$,

$$km_b(x_n, x_{n+1}) \leq m_b(x_n, Tx_n) \tag{2.1}$$

and

$$m_b(x_{n+1}, Tx_{n+1}) \leq cm_b(x_n, x_{n+1}) \tag{2.2}$$

for all $n \in \mathbb{N}$. From (2.1) and (2.2), we can get

$$m_b(x_n, Tx_n) \leq \left(\frac{c}{k}\right)^n m_b(x_0, Tx_0) \tag{2.3}$$

and

$$m_b(x_n, x_{n+1}) \leq \left(\frac{c}{k}\right)^n m_b(x_0, x_1) \tag{2.4}$$

for all $n \in \mathbb{N}$. Moreover, from (2.3) and (2.4), we can get

$$\lim_{n\to\infty} m_b(x_n, Tx_n) = \lim_{n\to\infty} m_b(x_n, x_{n+1}) = 0.$$

Then for $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
m_b(x_n, x_m) - m_{bx_nx_m} &\leq s[(m_b(x_n, x_{n+1}) - m_{bx_nx_{n+1}}) \\
&\quad +(m_b(x_{n+1}, x_m) - m_{bx_{n+1}x_m})] \\
&\leq s(m_b(x_n, x_{n+1}) - m_{bx_nx_{n+1}}) \\
&\quad +s^2[(m_b(x_{n+1}, x_{n+2}) - m_{bx_{n+1}x_{n+2}}) \\
&\quad +(m_b(x_{n+2}, x_m) - m_{bx_{n+2}x_m})] \\
&\vdots \\
&\leq s(m_b(x_n, x_{n+1}) - m_{bx_nx_{n+1}}) + \cdots \\
&\quad +s^{m-n-1}(m_b(x_{m-1}, x_m) - m_{bx_{m-1}x_m}) \\
&\leq sm_b(x_n, x_{n+1}) + \cdots + s^{m-n-1}m_b(x_{m-1}, x_m) \\
&\leq \left(\frac{c}{k}\right)^n sm_b(x_0, x_1) + \cdots + \left(\frac{c}{k}\right)^{m-1} s^{m-n}m_b(x_0, x_1) \\
&\leq s\left(\frac{c}{k}\right)^n m_b(x_0, x_1)[1 + \cdots + \left(\frac{sc}{k}\right)^{m-n-1}] \\
&\leq \frac{s\left(\frac{c}{k}\right)^n}{1 - \frac{sc}{k}} m_b(x_0, x_1).
\end{aligned}$$

Since $c < k$, we get

$$\lim_{n,m\to\infty} \left(m_b(x_n, x_m) - m_{bx_nx_m}\right) = 0. \tag{2.5}$$

On the other hand,

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} m_b(x_{n+1}, x_{n+1}) \\
&\leq \lim_{n\to\infty} m_b(x_n, x_{n+1})
\end{aligned}$$

implies that

$$\lim_{n,m\to\infty} m_b(x_n, x_m) = 0 \tag{2.6}$$

From (2.5) and (2.6), we get that $\{x_n\}$ is an $M_b$-Cauchy sequence. Since $X$ is $M_b$-complete, there exist $z \in X$ such that

$$\lim_{n\to\infty} (m_b(x_n, z) - m_{bx_nz}) = 0,$$

that is, $\{x_n\}$ converges to $z$ and

$$\lim_{n,m\to\infty} m_b(x_n, x_m) = m_b(z, z). \tag{2.7}$$

Now, we show that $z$ is fixed point of $T$. From (2.3), we can say that the sequence $\{m_b(x_n, Tx_n)\}$ converges 0. Since $f(x) = m_b(x, Tx)$ is lower semicontinuous, we get

$$0 \le m_b(z, Tz) = f(z) \le \lim\inf_{n\to\infty} f(x_n) = \lim\inf_{n\to\infty} m_b(x_n, Tx_n) = 0,$$

that is, $m_b(z, Tz) = 0$.

If $|Tz| > 1$, then $z \in \overline{Tz}^b = Tz$. Now, assume $|Tz| = 1$ and $T$ is lower orbitally continuous at $z$. From (2.6) and (2.7), we can say $m_b(z, z) = 0$. Now, it is enough to show $m_b(Tz, Tz) = 0$. Since $\lim_{n\to\infty} m_b(x_n, z) = m_b(z, z)$ and $T$ is *lower orbitally continuous* at $z$, we can write

$$m_b(Tz, Tz) \le \lim\inf_{n\to\infty} m_b(x_n, Tz).$$

From the property of $(m2)$, we can write

$$m_b(x_n, Tz) - m_{bx_nTz} \le s\left\{m_b(x_n, z) - m_{bx_nz} + m_b(z, Tz) - m_{bzTz}\right\}$$

$$\lim_{n\to\infty} m(x_n, Tz) - m_{bx_nTz} \le s\left\{\lim_{n\to\infty} m_b(x_n, z) - m_{bx_nz} + \lim_{n\to\infty} m_b(z, Tz) - m_{bzTz}\right\}$$

$$0 \le m_b(Tz, Tz) \le \lim\inf_{n\to\infty} m_b(x_n, Tz) = 0.$$

From here, we can write

$$m_b(Tz, Tz) = 0.$$

So that, we get $z$ is fixed point of $T$.

Since every metric space, every partial metric space, every $M$-metric space are $M_b$-metric space, we can get the following corollaries from Theorem 2.

**Corollary 1 (Feng-Liu's Fixed Point Theorem)** *Let $(X, d)$ be a complete metric space and $T : X \to C(X)$ be a multivalued mapping. If there exist a constant $c \in (0, 1)$ and $y \in T_k^x(d)$ satisfying*

$$d(y, Ty) \le cd(x, y)$$

*for all $x \in X$. Then $T$ has a fixed point in $X$ provided that $c < k$ and the function $f(x) = d(x, Tx)$ is lower semicontinuous.*

*Proof* Since usual metric topology has $T_1$ property, we get $X \cup C(X) = C(X)$. In this case the *lower orbitally continuity* of $T$ is also satisfied. So, by Thereom 2, we have the conclusion.

**Corollary 2** *Let $(X, p)$ be a complete partial metric space and $T : X \to X \cup C(X)$ be a mixed multivalued mapping. If there exist a constant $c \in (0, 1)$ and $y \in T_k^x(p)$ such that for all $x \in X$ with either $p(x, Tx) > 0$ and $|Tx| > 1$ or $|Tx| = 1$ satisfying*

$$p(y, Ty) \leq cp(x, y)$$

*then $T$ has a fixed point in $X$ provided that $c < k$ and the function $f(x) = p(x, Tx)$ is lower semicontinuous.*

*Proof* Let $|Tx| = 1$ and consider a sequence $(x_n)$ in $X$ such that $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. Then, since $p(Tx, Tx) \leq p(x, Tx)$, we can get $p(Tx, Tx) \leq \liminf_{n\to\infty} p(x_n, Tx) = p(x, Tx)$. That is, $T$ is *lower orbitally continuous* at $x$. Then, by Thereom 2, we have the conclusion.

**Corollary 3** *Let $(X, m)$ be an $M$-complete $M$-metric space and $T : X \to X \cup C(X)$ be a mixed multivalued map. If there exist a constant $c \in (0, 1)$ and $y \in T_k^x(m)$ such that for all $x \in X$ with either $m(x, Tx) > 0$ and $|Tx| > 1$ or $|Tx| = 1$ satisfying*

$$m(y, Ty) \leq cm(x, y)$$

and

$$m(y, y) \leq m(x, y)$$

*then there exists $z \in X$ such that $m(z, Tz) = 0$ provided that $c < k$ and the function $f(x) = m(x, Tx)$ is lower semicontinuous. Further, if $|Tz| > 1$, then $z$ is a fixed point of $T$. If $|Tz| = 1$ and $T$ is lower orbitally continuous at $z$, then $z$ is a fixed point of $T$.*

Now, we give two examples convenient to results.

*Example 5* Let $X = \{0\} \cup [1, 2) \cup (2, \infty)$ and define a mapping $m_b$ by

$$m_b(x, y) = \begin{cases} 0 & , \quad x = y = 0 \\ 1 & , \quad x = y \in X \backslash \{0\} \\ \frac{x+y}{2} & , \quad x \neq y \in [1, 2) \\ 1 + |x - y| & , \quad x \neq y \in \{3, 3 + \frac{1}{n} : n > 1\} \\ 1.2 & , \quad \text{otherwise} \end{cases}$$

for all $x, y \in X$. Then, $m_b$ is an $M_b$-metric on $X$ with coefficient $s = 3$, but is not an $M$-metric space. Indeed, if we take $x = 1.5$, $y = 1.7$ and $z \in (2, \infty)$, the inequality $m_b(x, y) - m_{bxy} \leq [m_b(x, z) - m_{bxz} + m_b(z, y) - m_{bzy}]$ which is condition of $M$-metric is not satisfied. We can see that $(X, m_b)$ is $M_b$-complete $M_b$-metric space. Now define mapping $T : X \to X \cup C(X)$ as

$$Tx = \begin{cases} X & , x = 3 + \frac{1}{2} \\ \{0\} & , \text{ otherwise} \end{cases}.$$

It is clear that the function $f(x) = m_b(x, Tx)$ is lower semicontinuous with respect to $\tau_{m_b}$. On the other hand, for all $x \in X$ with either $m_b(x, Tx) > 0$ and $|Tx| > 1$ or $|Tx| = 1$ there exist $y \in T^x_{0,8}(m_b)$ such that

$$m_b(y, Ty) \leq c m_b(x, y) \text{ with } c = 0, 1.$$

So that, since $T$ is *lower orbitally continuous* at 0, then $T$ has a fixed point in $X$ from Theorem 2.

*Example 6* Let $X = \{0, 1\} \cup \left\{\frac{1}{n} : n > 2, n \in \mathbb{N}\right\}$ and define mapping $m_b$ as

$$m_b(x, y) = \begin{cases} 0 & , \quad x = y \in X \backslash \left\{1, \frac{1}{2n}\right\} \\ 1 & , \quad x = y = 1 \\ \frac{1}{2n} & , \quad x = y \in \left\{\frac{1}{2n} : n > 1, n \in \mathbb{N}\right\} \\ \frac{1}{2n} & , \quad \begin{array}{c} \left[x = 1 \text{and} y \in \left\{\frac{1}{2n} : n > 1, n \in \mathbb{N}\right\}\right] \\ \text{or} \\ \left[x \in \left\{\frac{1}{2n} : n > 1, n \in \mathbb{N}\right\} \text{and} y = 1\right] \end{array} \\ \min\{x, y\} & , \quad x \neq y \in \left\{\frac{1}{2n} : n > 1, n \in \mathbb{N}\right\} \\ \frac{1}{2} & , \quad \begin{array}{c} \left[x = \frac{1}{2m+1}, y = \frac{1}{2n} : n > 1, m \geq 1, n, m \in \mathbb{N}\right] \\ \text{or} \\ \left[x = \frac{1}{2n}, y = \frac{1}{2m+1} : n > 1, m \geq 1, n, m \in \mathbb{N}\right] \end{array} \\ 2 & , \quad \text{otherwise} \end{cases}.$$

Then $(X, m_b)$ is complete $M_b$-metric space with coefficient $s = 4$, but is not a $M$-metric space. Indeed, if we take $x = 1$, $y = \frac{1}{2n+1}$ and $z = \frac{1}{2m}$, the inequality

$$m_b(x,y) - m_{bxy} \leq [m_b(x,z) - m_{bxz} + m_b(z,y) - m_{bzy}]$$

which is condition of $M$-metric is not satisfied. Define a mapping $T : X \to X \cup C(X)$ by

$$Tx = \begin{cases} \{0\} \cup \{\frac{1}{2n+1} : n \geq 1, n \in \mathbb{N}\}, & x = 0 \\ \{0\} & , \text{otherwise} \end{cases}.$$

It is clear that the function $f(x) = m_b(x, Tx)$ is lower semicontinuous with respect to $\tau_{m_b}$. Besides, the mapping $T$ is *lower orbitally continuous*. On the other hand, for all $x \in X$ with either $m_b(x, Tx) > 0$ and $|Tx| > 1$ or $|Tx| = 1$ there exist $y \in T^x_{0,5}(m_b)$ such that

$$m_b(y, Ty) \leq c m_b(x, y) \text{with} c = 0, 1.$$

Hence, by applying Thereom 2, we can say that $T$ has a fixed point in $X$.

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# Chapter 3
# Hyers-Ulam and Hyers-Ulam-Rassias Stability for a Class of Integro-Differential Equations

**L. P. Castro and A. M. Simões**

## 3.1 Introduction

The concept of stability for functional, differential, integral and integro-differential equations has been studied in a quite extensive way during the last six decades and has earned particular interest due to their great number of applications (see [1, 3, 5, 6, 8–16, 18–23, 26] and the references therein). This occurs with particular emphasis in the case of Hyers-Ulam and Hyers-Ulam-Rassias stabilities. These stabilities were originated from a famous question raised by S. M. Ulam at the University of Wisconsin in 1940: "When a solution of an equation differing slightly from a given one must be somehow near to the solution of the given equation?" A first partial answer to this question was given by D. H. Hyers, for Banach spaces, in the case of an additive Cauchy equation. This is why the obtained result is nowadays called the Hyers-Ulam stability. Different generalizations of that initial answer of D. H. Hyers were obtained by T. Aoki [2], Z. Gajda [17] and Th.M. Rassias [25]. The interested reader can obtain a detailed description of these advances in [4]. Afterwards, new directions were introduced by Th.M. Rassias, see [24], introducing therefore the so-called Hyers-Ulam-Rassias stability.

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In this paper, we study the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability for the following class of Volterra integro-differential equation,

$$y'(x) = f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right), \quad y(a) = c \in \mathbb{R}, \quad (3.1)$$

with $y \in C^1([a, b])$, for $x \in [a, b]$ where, for starting, $a$ and $b$ are fixed real numbers, $f : [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous functions, and $\alpha : [a, b] \to [a, b]$ is a continuous delay function (i.e., fulfilling $\alpha(\tau) \leq \tau$ for all $\tau \in [a, b]$).

The formal definition of the above-mentioned stabilities are now introduced for our integro-differential equation (3.1).

If for each function $y$ satisfying

$$\left|y'(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right)\right| \leq \theta, \quad x \in [a, b], \quad (3.2)$$

where $\theta \geq 0$, there is a solution $y_0$ of the integro-differential equation and a constant $C > 0$ independent of $y$ and $y_0$ such that

$$|y(x) - y_0(x)| \leq C\,\theta, \quad (3.3)$$

for all $x \in [a, b]$, then we say that the integro-differential equation (3.1) has the Hyers-Ulam stability.

If for each function $y$ satisfying

$$\left|y'(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right)\right| \leq \sigma(x), \quad x \in [a, b], \quad (3.4)$$

where $\sigma$ is a non-negative function, there is a solution $y_0$ of the integro-differential equation and a constant $C > 0$ independent of $y$ and $y_0$ such that

$$|y(x) - y_0(x)| \leq C\,\sigma(x), \quad (3.5)$$

for all $x \in [a, b]$, then we say that the integro-differential equation (3.1) has the Hyers-Ulam-Rassias stability.

Some of the present techniques to study the stability of functional equations use a combination of fixed point results with a generalized metric in appropriate settings. In view of this, and just for the sake of completeness, let us recall the definition of a generalized metric and the corresponding Banach Fixed Point Theorem.

**Definition 1** Let $X$ be a nonempty set. We say that a function $d : X \times X \to [0, +\infty]$ is a generalized metric on $X$ if:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 1** *Let $(X, d)$ be a generalized complete metric space and $T : X \to X$ a strictly contractive operator with a Lipschitz constant $L < 1$. If there exists a non-negative integer $k$ such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following three propositions hold true:*

1. *the sequence $(T^n x)_{n\in\mathbb{N}}$ converges to a fixed point $x^*$ of $T$;*
2. *$x^*$ is the unique fixed point of $T$ in $X^* = \{y \in X : d(T^k x, y) < \infty\}$;*
3. *if $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(Ty, y). \tag{3.6}$$

## 3.2 Hyers-Ulam-Rassias Stability in the Finite Interval Case

In this section we will present sufficient conditions for the Hyers-Ulam-Rassias stability of the integro-differential equation (3.1), where $x \in [a, b]$, for some fixed real numbers $a$ and $b$.

We will consider the space of continuously differentiable functions $C^1([a, b])$ on $[a, b]$ endowed with a generalization of the Bielecki metric

$$d(u, v) = \sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{\sigma(x)}, \tag{3.7}$$

where $\sigma$ is a non-decreasing continuous function $\sigma : [a, b] \to (0, \infty)$. We recall that $\left(C^1([a, b]), d\right)$ is a complete metric spaces (cf., [7, 27]).

**Theorem 2** *Let $\alpha : [a, b] \to [a, b]$ be a continuous delay function with $\alpha(t) \leq t$ for all $t \in [a, b]$ and $\sigma : [a, b] \to (0, \infty)$ a non-decreasing continuous function. In addition, suppose that there is $\beta \in [0, 1)$ such that*

$$\int_a^x \sigma(\tau)d\tau \leq \beta\sigma(x), \tag{3.8}$$

*for all $x \in [a, b]$. Moreover, suppose that $f : [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq M\,(|u(x) - v(x)| + |g(x) - h(x)|) \tag{3.9}$$

*with $M > 0$ and the kernel $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|k(x, t, u(t), u(\alpha(t))) - k(x, t, v(t), v(\alpha(t)))| \leq L|u(\alpha(t)) - v(\alpha(t))| \tag{3.10}$$

*with $L > 0$.*

*If $y \in C^1([a,b])$ is such that*

$$\left| y'(x) - f\left(x, y(x), \int_a^x k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right)\right| \le \sigma(x),\ x \in [a,b], \quad (3.11)$$

*and $M\left(\beta + L\beta^2\right) < 1$, then there is a unique function $y_0 \in C^1([a,b])$ such that*

$$y_0'(x) = f\left(x, y_0(x), \int_a^x k(x,\tau,y_0(\tau),y_0(\alpha(\tau)))d\tau\right) \quad (3.12)$$

*and*

$$|y(x) - y_0(x)| \le \frac{\beta}{1 - M\left(\beta + L\beta^2\right)}\,\sigma(x) \quad (3.13)$$

*for all $x \in [a,b]$.*

*This means that under the above conditions, the integro-differential equation (3.1) has the Hyers-Ulam-Rassias stability.*

*Proof* By integration we have that

$$y'(x) = f\left(x, y(x), \int_a^x k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right) \quad (3.14)$$

is equivalent to

$$y(x) = c + \int_a^x f\left(s, y(s), \int_a^s k(s,\tau,y(\tau),y(\alpha(\tau)))d\tau\right)ds. \quad (3.15)$$

So, we will consider the operator $T : C^1([a,b]) \to C^1([a,b])$, defined by

$$(Tu)(x) = c + \int_a^x f\left(s, u(s), \int_a^s k(s,\tau,u(\tau),u(\alpha(\tau)))d\tau\right)ds, \quad (3.16)$$

for all $x \in [a,b]$ and $u \in C^1([a,b])$.

Note that for any continuous function $u$, $Tu$ is also continuous. Indeed,

$$\begin{aligned} &|(Tu)(x) - (Tu)(x_0)| \\ &= \left|\int_a^x f\left(s, u(s), \int_a^s k(s,\tau,u(\tau),u(\alpha(\tau)))d\tau\right)ds \right. \\ &\quad \left. - \int_a^{x_0} f\left(s, u(s), \int_a^s k(s,\tau,u(\tau),u(\alpha(\tau)))d\tau\right)ds\right| \\ &= \left|\int_x^{x_0} f\left(s, u(s), \int_a^s k(s,\tau,u(\tau),u(\alpha(\tau)))d\tau\right)ds\right| \longrightarrow 0 \quad (3.17) \end{aligned}$$

when $x \to x_0$.

Under the present conditions, we will deduce that the operator $T$ is strictly contractive with respect to the metric (3.7). Indeed, for all $u, v \in C^1([a, b])$, we have

$$
\begin{aligned}
d(Tu, Tv) &= \sup_{x\in[a,b]} \frac{|(Tu)(x) - (Tv)(x)|}{\sigma(x)} \\
&\leq M \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x |u(s) - v(s)|\, ds \\
&\quad + M \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^s |k(s, \tau, u(\tau), u(\alpha(\tau))) - k(s, \tau, v(\tau), v(\alpha(\tau)))|\, d\tau ds \\
&\leq M \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x |u(s) - v(s)|\, ds \\
&\quad + ML \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^s |u(\alpha(\tau)) - v(\alpha(\tau))|\, d\tau ds \\
&= M \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \sigma(s) \frac{|u(s) - v(s)|}{\sigma(s)} ds \\
&\quad + ML \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^s \sigma(\tau) \frac{|u(\alpha(\tau)) - v(\alpha(\tau))|}{\sigma(\tau)} d\tau ds \\
&\leq M \sup_{s\in[a,b]} \frac{|u(s) - v(s)|}{\sigma(s)} \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \sigma(s) ds \\
&\quad + ML \sup_{\tau\in[a,b]} \frac{|u(\tau) - v(\tau)|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^s \sigma(\tau) d\tau ds \\
&\leq M d(u, v)\beta + ML d(u, v) \sup_{x\in[a,b]} \frac{\beta^2 \sigma(x)}{\sigma(x)} \\
&= M \left(\beta + L\beta^2\right) d(u, v).
\end{aligned}
\tag{3.18}
$$

Due to the fact that $M\left(\beta + L\beta^2\right) < 1$ it follows that $T$ is strictly contractive. Thus, we can apply the above-mentioned Banach Fixed Point Theorem, which ensures that we have the Hyers-Ulam-Rassias stability for the integro-differential equation (3.1). Additionally, we can apply again the Banach Fixed Point Theorem, which guarantees us that

$$
d(y, y_0) \leq \frac{1}{1 - M\left(\beta + L\beta^2\right)} d(Ty, y). \tag{3.19}
$$

From the definition of the metric $d$, (3.8) and (3.11) follows that

$$\sup_{x\in[a,b]} \frac{|y(x)-y_0(x)|}{\sigma(x)} \le \frac{\beta}{1-M\left(\beta+L\beta^2\right)} \tag{3.20}$$

and consequently (3.13) holds. □

## 3.3 Hyers-Ulam Stability in the Finite Interval Case

In this section we will present sufficient conditions for the Hyers-Ulam stability of the integro-differential equation (3.1).

**Theorem 3** *Let $\alpha : [a,b] \to [a,b]$ be a continuous delay function with $\alpha(t) \le t$ for all $t \in [a,b]$ and $\sigma : [a,b] \to (0,\infty)$ a non-decreasing continuous function. In addition, suppose that there is $\beta \in [0,1)$ such that*

$$\int_a^x \sigma(\tau)d\tau \le \beta\sigma(x), \tag{3.21}$$

*for all $x \in [a,b]$. Moreover, suppose that $f : [a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),g(x)) - f(x,v(x),h(x))| \le M\left(|u(x)-v(x)| + |g(x)-h(x)|\right) \tag{3.22}$$

*with $M > 0$ and $k : [a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition*

$$|k(x,t,u(t),u(\alpha(t))) - k(x,t,v(t),v(\alpha(t)))| \le L|u(\alpha(t)) - v(\alpha(t))| \tag{3.23}$$

*with $L > 0$.*

*If $y \in C^1([a,b])$ is such that*

$$\left| y'(x) - f\left(x, y(x), \int_a^x k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right)\right| \le \theta, \quad x\in[a,b], \tag{3.24}$$

*where $\theta > 0$ and $M\left(\beta+L\beta^2\right) < 1$, then there is a unique function $y_0 \in C^1([a,b])$ such that*

$$y_0'(x) = f\left(x, y_0(x), \int_a^x k(x,t,y_0(t),y_0(\alpha(t)))dt\right) \tag{3.25}$$

*and*

$$|y(x) - y_0(x)| \leq \frac{(b-a)\sigma(b)}{\left[1 - M\left(\beta + L\beta^2\right)\right]\sigma(a)}\theta \tag{3.26}$$

*for all $x \in [a, b]$.*

*This means that under the above conditions, the integro-differential equation (3.1) has the Hyers-Ulam stability.*

This result can be obtained by using an analogous procedure as in the previous theorem (and so the full details of its proof are here omitted). In particular, we may consider the operator $T : C^1([a, b]) \to C^1([a, b])$, defined by

$$(Tu)(x) = c + \int_a^x f\left(s, u(s), \int_a^s k(s, \tau, u(\tau), u(\alpha(\tau)))d\tau\right) ds, \tag{3.27}$$

for all $x \in [a, b]$ and $u \in C^1([a, b])$, and conclude that $T$ is strictly contractive with respect to the metric (3.7), due to the fact that $M\left(\beta + L\beta^2\right) < 1$. Thus, we can again apply the Banach Fixed Point Theorem, which in this case leads us to the Hyers-Ulam stability for the integro-differential equation.

From (3.24) we have

$$-\theta \leq y'(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \leq \theta, \quad x \in [a, b]. \tag{3.28}$$

By integration in (3.28), we obtain

$$\left|y(x) - c - \int_a^x f\left(s, y(s), \int_a^s k(s, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) ds\right| \leq \int_a^x \theta d\tau \tag{3.29}$$

for all $x \in [a, b]$ and consequently

$$|y(x) - (Ty)(x)| \leq (b-a)\theta, \quad x \in [a, b]. \tag{3.30}$$

By (3.6), the definition of the metric $d$ and (3.30) it is easy to prove the inequality (3.26).

## 3.4 Hyers-Ulam-Rassias Stability in the Infinite Interval Case

In this section, we analyse the Hyers-Ulam-Rassias stability of the integro-differential equation (3.1) but, instead of considering a finite interval $[a, b]$ (with $a, b \in \mathbb{R}$), we will consider the infinite interval $[a, \infty)$, for some fixed $a \in \mathbb{R}$.

Thus, we will now be dealing with the integro-differential equation

$$y'(x) = f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right), \quad y(a) = c \in \mathbb{R}, \quad (3.31)$$

with $y \in C^1([a, \infty))$, $x \in [a, \infty)$ where $a$ is a fixed real number, $f : [a, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $k : [a, \infty) \times [a, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous functions, and $\alpha : [a, \infty) \to [a, \infty)$ is a continuous delay function which therefore fulfills $\alpha(\tau) \le \tau$ for all $\tau \in [a, \infty)$.

Our strategy will be based on a recurrence procedure due to the already obtained result for the corresponding finite interval case.

Let us consider a fixed non-decreasing continuous function $\sigma : [a, \infty) \to (\varepsilon, \omega)$, for some $\varepsilon, \omega > 0$ and the space $C_b^1([a, \infty))$ of bounded differentiable functions endowed with the metric

$$d_b(u, v) = \sup_{x \in [a,\infty)} \frac{|u(x) - v(x)|}{\sigma(x)}. \quad (3.32)$$

**Theorem 4** *Let $\alpha : [a, \infty) \to [a, \infty)$ be a continuous delay function with $\alpha(t) \le t$ for all $t \in [a, \infty)$ and $\sigma : [a, \infty) \to (\varepsilon, \omega)$, for some $\varepsilon, \omega > 0$, a non-decreasing continuous function. In addition, suppose that there is $\beta \in [0, 1)$ such that*

$$\int_a^x \sigma(\tau)d\tau \le \beta\sigma(x), \quad (3.33)$$

*for all $x \in [a, \infty)$. Moreover, suppose that $f : [a, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x, u(x), g(x)) - f(x, v(x), h(x))| \le M(|u(x) - v(x)| + |g(x) - h(x)|) \quad (3.34)$$

*with $M > 0$ and the kernel $k : [a, \infty) \times [a, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function so that $\int_a^x k(x, \tau, z(\tau), z(\alpha(\tau)))d\tau$ is a bounded continuous function for any bounded continuous function $z$. In addition, suppose that $k$ satisfies the Lipschitz condition*

$$|k(x, t, u(t), u(\alpha(t))) - k(x, t, v(t), v(\alpha(t)))| \le L|u(\alpha(t)) - v(\alpha(t))| \quad (3.35)$$

*with $L > 0$.*

*If $y \in C_b^1([a, \infty))$ is such that*

$$\left|y'(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right)\right| \le \sigma(x), \quad x \in [a, \infty), \quad (3.36)$$

*and* $M\left(\beta + L\beta^2\right) < 1$*, then there is a unique function* $y_0 \in C_b^1([a,\infty))$ *such that*

$$y_0'(x) = f\left(x, y_0(x), \int_a^x k(x, \tau, y_0(\tau), y_0(\alpha(\tau)))d\tau\right) \tag{3.37}$$

*and*

$$|y(x) - y_0(x)| \le \frac{\beta}{1 - M\left(\beta + L\beta^2\right)}\sigma(x) \tag{3.38}$$

*for all* $x \in [a,\infty)$*.*

*This means that under the above conditions, the integro-differential equation (3.31) has the Hyers-Ulam-Rassias stability.*

*Proof* For any $n \in \mathbb{N}$, we will define $I_n = [a, a+n]$. By Theorem 2, there exists a unique bounded differentiable function $y_{0,n} : I_n \to \mathbb{C}$ such that

$$y_{0,n}(x) = c + \int_a^x f\left(s, y_{0,n}(s), \int_a^s k(s, \tau, y_{0,n}(\tau), y_{0,n}(\alpha(\tau)))d\tau\right) ds \tag{3.39}$$

and

$$\left|y(x) - y_{0,n}(x)\right| \le \frac{\beta}{1 - M\left(\beta + L\beta^2\right)}\sigma(x) \tag{3.40}$$

for all $x \in I_n$. The uniqueness of $y_{0,n}$ implies that if $x \in I_n$ then

$$y_{0,n}(x) = y_{0,n+1}(x) = y_{0,n+2}(x) = \cdots . \tag{3.41}$$

For any $x \in [a,\infty)$, let us define $n(x) \in \mathbb{N}$ as $n(x) = \min\{n \in \mathbb{N} : \ x \in I_n\}$. We also define a function $y_0 : [a,\infty) \to \mathbb{C}$ by

$$y_0(x) = y_{0,n(x)}(x). \tag{3.42}$$

For any $x_1 \in [a,\infty)$, let $n_1 = n(x_1)$. Then $x_1 \in \operatorname{Int} I_{n_1+1}$ and there exists an $\epsilon > 0$ such that $y_0(x) = y_{0,n_1+1}(x)$ for all $x \in (x_1 - \epsilon, x_1 + \epsilon)$. By Theorem 2, $y_{0,n_1+1}$ is continuous at $x_1$, and so it is $y_0$.

Now, we will prove that $y_0$ satisfies

$$y_0(x) = c + \int_a^x f\left(s, y_0(s), \int_a^s k(s, \tau, y_0(\tau), y_0(\alpha(\tau)))d\tau\right) ds \tag{3.43}$$

and (3.38). For an arbitrary $x \in [a, \infty)$ we choose $n(x)$ such that $x \in I_{n(x)}$. By (3.39) and (3.42), we have

$$\begin{aligned} y_0(x) &= y_{0,n(x)}(x) \\ &= c + \int_a^x f\left(s, y_{0,n(x)}(s), \int_a^s k(s, \tau, y_{0,n(x)}(\tau), y_{0,n(x)}(\alpha(\tau)))d\tau\right) ds \\ &= c + \int_a^x f\left(s, y_0(s), \int_a^s k(s, \tau, y_0(\tau), y_0(\alpha(\tau)))d\tau\right) ds. \end{aligned} \tag{3.44}$$

Note that $n(\tau) \leq n(x)$, for any $\tau \in I_{n(x)}$, and it follows from (3.41) that $y_0(\tau) = y_{0,n(\tau)}(\tau) = y_{0,n(x)}(\tau)$, so, the last equality in (3.44) holds.

To prove (3.38), by (3.42) and (3.40), we have that for all $x \in [a, \infty)$,

$$|y(x) - y_0(x)| = \left|y(x) - y_{0,n(x)}(x)\right| \leq \frac{\beta}{1 - M(\beta - L\beta^2)}\sigma(x). \tag{3.45}$$

Finally, we will prove the uniqueness of $y_0$. Let us consider another bounded differentiable function $y_1$ which satisfies (3.37) and (3.38), for all $x \in [a, \infty)$. By the uniqueness of the solution on $I_{n(x)}$ for any $n(x) \in \mathbb{N}$ we have that $y_{0|I_{n(x)}} = y_{0,n(x)}$ and $y_{1|I_{n(x)}}$ satisfies (3.37) and (3.38) for all $x \in I_{n(x)}$, so

$$y_0(x) = y_0|_{I_{n(x)}}(x) = y_1|_{I_{n(x)}}(x) = y_1(x). \tag{3.46}$$

□

*Remark 1* With the necessary adaptations, Theorem 4 also holds true for infinite intervals of the type $(-\infty, b]$, with $b \in \mathbb{R}$, as well as for $(-\infty, \infty)$.

## 3.5 Examples

We will now present two concrete examples to illustrate the above presented results.

For a differentiable function $y : \left[0, \frac{2}{5}\right] \to \mathbb{R}$, let us consider the integro-differential equation

$$y'(x) = 1 + 2x - y(x) + \int_0^x \left(x(1+2x)y(\tau)e^{\tau(x-\tau)}\right) d\tau, \quad x \in \left[0, \frac{2}{5}\right], \tag{3.47}$$

as well as the continuous function $\sigma : \left[0, \frac{2}{5}\right] \to (0, \infty)$ defined by $\sigma(x) = 3e^x$ and the continuous delay function $\alpha : \left[0, \frac{2}{5}\right] \to \left[0, \frac{2}{5}\right]$ given by $\alpha(x) = x$.

We realize that all the conditions of Theorem 2 are here satisfied. In fact, $\alpha : \left[0, \frac{2}{5}\right] \to \left[0, \frac{2}{5}\right]$ defined by $\alpha(x) = x$ is a continuous function with $\alpha(x) \leq x$. For $\beta = 1/2$ we realize that $\sigma : \left[0, \frac{2}{5}\right] \to [0, \infty)$ defined by $\sigma(x) = 3e^x$, a continuous function, fulfills

$$\int_0^x 3e^\tau d\tau \leq \frac{3}{2}e^x = \beta\,\sigma(x), \quad x \in \left[0, \frac{2}{5}\right]. \tag{3.48}$$

Additionally $f : \left[0, \frac{2}{5}\right] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by

$$f(x, y(x), g(x)) = 1 + 2x - y(x) + g(x) \tag{3.49}$$

is a continuous function which satisfies

$$|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq |u(x) - v(x)| + |g(x) - h(x)| \tag{3.50}$$

for all $x \in \left[0, \frac{2}{5}\right]$, and so we may take the constant $M$ considered in Theorem 2 to be equal to 1.

The kernel $k : \left[0, \frac{2}{5}\right] \times \left[0, \frac{2}{5}\right] \times \mathbb{C} \to \mathbb{C}$ defined by

$$k(x, t, y(t), y(\alpha(t))) = x(1 + 2x)y(t)e^{t(x-t)} \tag{3.51}$$

is a continuous function which fulfils the condition

$$|k(x, t, u(t), u(\alpha(t))) - k(x, t, v(t), v(\alpha(t)))| \leq \frac{18}{25}e^{\frac{1}{25}}\,|u(\alpha(t)) - v(\alpha(t))| \tag{3.52}$$

for all $t \in [0, x]$ and $x \in \left[0, \frac{2}{5}\right]$, where we are using the constant $L = \frac{18}{25}e^{\frac{1}{25}}$. Thus, $M(\beta + L\beta^2) = \frac{1}{2} + \frac{9}{50}\,e^{1/25} < 1$.

If we choose $y(x) = \frac{e^{x^2}}{0.3}$, it follows

$$\left|y'(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right)\right| = \left|\frac{7}{3} + \frac{14}{3}x\right| \leq \sigma(x), \tag{3.53}$$

for all $x \in \left[0, \frac{2}{5}\right]$.

Therefore, this exhibits the Hyers-Ulam-Rassias stability of the integro-differential equation (3.47).

Moreover, by using the exact solution $y_0(x) = e^{x^2}$ we realize that

$$|y(x) - y_0(x)| = \left| \frac{e^{x^2}}{0.3} - e^{x^2} \right| \leq \frac{3\, e^x}{1 - \frac{9}{25}\, e^{1/25}} = \frac{\beta}{1 - M(\beta + L\beta^2)}\, \sigma(x) \quad (3.54)$$

for all $x \in \left[0, \frac{2}{5}\right]$.

Let us now turn to a second example in which the Hyers-Ulam stability is illustrated.

For differentiable functions $y : [0, 1] \to \mathbb{R}$, let us start by considering the integro-differential equation

$$y'(x) = (-2x - 4)e^{x/2} + 5y(x) + e^{x/2} \int_0^x ((\tau - x)y(\alpha(\tau)))\, d\tau, \quad (3.55)$$

for all $x \in [0, 1]$, as well as the continuous function $\sigma : [0, 1] \to (0, \infty)$ defined by $\sigma(x) = 1.1e^{10x}$ and the continuous delay function $\alpha : [0, 1] \to [0, 1/2]$ given by $\alpha(x) = x/2$.

We have all the conditions of Theorem 3 being satisfied. In fact, such $\alpha : [0, 1] \to [0, 1/2]$ defined by $\alpha(x) = x/2$ is a continuous function, and obviously $\alpha(x) \leq x$. Moreover, for $\beta = 1/10$ we have that $\sigma : [0, 1] \to (0, \infty)$ defined by $\sigma(x) = 1.1e^{10x}$ is a continuous function fulfilling

$$\int_0^x 1.1e^{10\tau} d\tau \leq \frac{1.1}{10} e^{10x} = \beta\, \sigma(x), \quad x \in [0, 1]. \quad (3.56)$$

Additionally $f : [0, 1] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by

$$f(x, y(x), g(x)) = (-2x - 4)e^{x/2} + 5y(x) + e^{x/2} g(x) \quad (3.57)$$

is a continuous function which fulfills

$$|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq 5\, (|u(x) - v(x)| + |g(x) - h(x)|) \quad (3.58)$$

for all $x \in [0, 1]$, and so the previous constant $M$ is here taking the value 5.

The kernel $k : [0, 1] \times [0, 1] \times \mathbb{C} \to \mathbb{C}$ defined by

$$k(x, t, y(t), y(\alpha(t))) = (t - x)y(\alpha(t)) \quad (3.59)$$

is a continuous function which fulfils the condition

$$|k(x, t, u(t), u(\alpha(t))) - k(x, t, v(t), v(\alpha(t)))| \leq \left| u\left(\frac{t}{2}\right) - v\left(\frac{t}{2}\right) \right| \quad (3.60)$$

for all $t \in [0, x]$ and $x \in [0, 1]$, where we may identify 1 as the constant previously denoted by $L$. Thus, $M(\beta + L\beta^2) = 11/20 < 1$.

If we choose $y(x) = 100\,e^x/99$, it follows

$$\left|y'(x)-f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right)\right|=\left|\left(-\frac{2}{99}x - \frac{4}{99}\right) e^{x/2}\right|\leq\theta \quad (3.61)$$

for all $x \in [0, 1]$ and where we consider $\theta := 0.1$.

Therefore, from Theorem 3, we have the Hyers-Ulam stability of the integro-differential equation (3.55).

In particular, having in mind the exact solution $y_0(x) = e^x$ of (3.55), it follows that

$$|y(x) - y_0(x)| = \left|\frac{100}{99}e^x - e^x\right| \leq \frac{2}{9}\,e^{10x} = \frac{(b-a)\,\sigma(b)}{[1 - M(\beta + L\beta^2)]\,\sigma(a)}\,\theta \quad (3.62)$$

for all $x \in [0, 1]$.

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# Part II
# New Mathematical Ideas

# Chapter 4
# Exact Solutions, Lie Symmetry Analysis and Conservation Laws of the Time Fractional Diffusion-Absorption Equation

**Mir Sajjad Hashemi, Zahra Balmeh, and Dumitru Baleanu**

## 4.1 Introduction

A pioneering investigation of localization-extinction phenomena nonlinear degenerate parabolic partial differential equations(PDEs) was firstly performed by Kersner in the 1960s–1970s. Key results about these PDEs, including equations from diffusion-absorption theory, are reflected by Kalashnikov in [1]. Among the discussed models of Kalashnikov, the diffusion-absorption equation with the critical absorption exponent:

$$v_t = \left(v^{\sigma} v_x\right)_x - v^{1-\sigma}, \tag{4.1}$$

is a famous one, where $\sigma$ is a positive parameter. It is well known in filtration theory that the terms $-v^{1-\sigma}$ show the absorption and describe the seepage on a permeable bed. Some explicit localized solutions of Eq. (4.1) were reported in [2]. From the filtration theory, by introducing $u = v^{\sigma}$, as pressure variable, we obtain a PDE with quadratic differential operator and constant sink

$$u_t = uu_{xx} + \frac{1}{\sigma}(u_x)^2 - \sigma. \tag{4.2}$$

In recent years, fractional calculus plays a very significant role in several branches of engineering and science [3, 4]. Many important phenomena, e.g. electro-

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magnetic, image processing, acoustics, electro-chemistry, and anomalous diffusion, are motivated to model by fractional derivatives. One of the benefits of fractional models is their better describing than integer ones and this motivates us to describe a significant and applicable model, i.e. TFDA equation in the fractional aspect. In general, it is difficult to find exact solutions of differential equations with fractional derivatives and we remind that investigation of some properties of fractional derivatives is very hard than the classical ones. Therefore, there is a huge motivation to find the exact solutions, conservation laws, and Lie symmetries of a famous equation like the TFDA equation. Time fractional version of Eq. (4.2) has the following form:

$$\partial_t^\alpha u = u u_{xx} + \frac{1}{\sigma}(u_x)^2 - \sigma, \quad \alpha \in (0,1), \tag{4.3}$$

where $\partial_t^\alpha u := D_t^\alpha u$ stands for Riemann-Liouville derivative defined by [5]

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t \frac{u(x,\xi)}{(t-\xi)^{\alpha-m+1}} d\xi, & m-1 < \alpha < m, \\ \frac{\partial^m u}{\partial t^m}, & \alpha = m \in \mathbb{N}. \end{cases} \tag{4.4}$$

This paper is organized as follows: Some basic properties of Lie symmetry analysis of the time fractional partial differential equations (FPDEs) are described in Sect. 4.2. Lie symmetries and invariant solution of TFDA equation are discussed in Sect. 4.3. Conservation laws of this equation by a generalized version of Ibragimov's method are obtained in Sect. 4.4. Exact solutions of Eq. (4.3) are discussed in Sect. 4.5 by using the invariant subspace method. Concluding remarks are given in the last section.

## 4.2 Lie Symmetry Analysis of FPDEs

Let us consider the Lie symmetry analysis of FPDEs [6–19]:

$$\Delta := \partial_t^\alpha u - F(x,t,u,u_x,u_{xx}) = 0, \quad 0 < \alpha < 1. \tag{4.5}$$

The infinitesimal operator of the local Lie group of point transformations which are admitted by Eq. (4.5) is:

$$X = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{4.6}$$

Applying Lie Theorem about an invariance condition concludes:

$$Pr^{(\alpha,2)} X(\Delta)|_{\Delta=0} = 0, \quad \Delta = \partial_t^\alpha u - F, \tag{4.7}$$

where

$$Pr^{(\alpha,2)}X = X + \phi_{\alpha}^{0}\partial_{\partial_t^{\alpha}u} + \phi^{x}\partial_{u_x} + \phi^{xx}\partial_{u_{xx}}, \tag{4.8}$$

and

$$\begin{aligned}
\phi^{x} &= D_x(\phi) - u_x D_x(\xi) - u_t D_x(\tau),\\
\phi^{xx} &= D_x(\phi^{x}) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi),\\
\phi_{\alpha}^{0} &= D_t^{\alpha}(\phi) + \xi D_t^{\alpha}(u_x) - D_t^{\alpha}(\xi u_x) + D_t^{\alpha}(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u).
\end{aligned}$$

Since the lower limit of integral in Riemann-Liouville fractional derivative is fixed, therefore it should be invariant with respect to the point transformations:

$$\tau(x,t,u)|_{t=0} = 0, \tag{4.9}$$

The $\alpha$th extended infinitesimal has the form:

$$\phi_{\alpha}^{0} = D_t^{\alpha}(\phi) + \xi D_t^{\alpha}(u_x) - D_t^{\alpha}(\xi u_x) + D_t^{\alpha}(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \tag{4.10}$$

where $D_t^{\alpha}$ denotes the total fractional derivative. Let us remind the fractional Leibniz rule:

$$D_t^{\alpha}\left[u(t)v(t)\right] = \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}u(t)D_t^{n}v(t), \tag{4.11}$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{4.12}$$

Thus from (4.11) we can rewrite (4.10) as follows:

$$\begin{aligned}
\phi_{\alpha}^{0} = -\alpha D_t(\tau)\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{n}(\xi)D_t^{\alpha-n}(u_x) + D_t^{\alpha}(\phi)\\
-\sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{n+1}(\tau)D_t^{\alpha-n}(u).
\end{aligned} \tag{4.13}$$

Also from the chain rule

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-g(t)]^{r}\frac{d^m}{dt^m}[g(t)^{k-r}]\frac{d^k f(g)}{dg^k} \tag{4.14}$$

and setting $f(t) = 1$, we get

$$D_t^\alpha(\phi) = \frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi_u \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n \phi_u}{\partial t^n} D_t^{\alpha-n}(u) + \chi \tag{4.15}$$

where

$$\chi = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} [-u]^r \frac{\partial^m}{\partial t^m} [u^{k-r}] \frac{\partial^{n-m+k} \phi}{\partial t^{n-m} \partial u^k}. \tag{4.16}$$

Therefore

$$\begin{aligned} \phi_\alpha^0 = {} & \frac{\partial^\alpha \phi}{\partial t^\alpha} + (\phi_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} + \chi \\ & + \sum_{n=1}^{\infty} \left[ \binom{\alpha}{n} \frac{\partial^\alpha \phi_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) \\ & - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x). \end{aligned}$$

## 4.3 Lie Symmetries and Invariant Solution of TFDA Equation

An over determined system of partial linear differential equations can be extracted by applying $pr^{(\alpha,2)}X$ to Eq. (4.3) as follows:

$$\begin{aligned} & \tau_u = \tau_x = \xi_t = \xi_u = \phi_{uu} = 0, \\ & -\phi + 2u\xi_x - \alpha u \tau_t = 0, \\ & 2\sigma u \phi_{xu} - \sigma u \xi_{xx} + 2\phi_x = 0, \\ & (1-\alpha)\tau_{tt} + 2\phi_{tu} = 0, \\ & (2-\alpha)\tau_{ttt} + 3\phi_{ttu} = 0, \\ & \phi_u + \alpha \tau_t - 2\xi_x = 0, \\ & -u\phi_{xx} - \sigma \phi_u + \alpha \sigma \tau_t - u \partial_t^\alpha \phi_u + \partial_t^\alpha \phi = 0, \end{aligned}$$

$$\sum_{k=3}^{\infty}\binom{\alpha}{k}\frac{\partial^{k+1}}{\partial t^k \partial u}\phi \times D_{t^{\alpha-k}}u$$
$$-\sum_{k=3}^{\infty}\frac{\binom{\alpha}{k}}{1+k}\left[(k-\alpha)\,D_{t^{\alpha-k}}u \times D_{t^{1+k}}\tau + (k+1)D_{t^{\alpha-k}}u_x \times D_{t^k}\xi\right] = 0. \tag{4.17}$$

Solving Eq. (4.17), we obtain the following infinitesimals:

$$\tau = c_1 + tc_2, \quad \xi = c_3 + xc_4, \quad \phi = 2uc_4 - \alpha uc_2, \tag{4.18}$$

where $c_1$, $c_2$, $c_3$, and $c_4$ are arbitrary constants and therefore:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}, \quad X_3 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{4.19}$$

Let us consider the invariant solution of Eq. (4.3) corresponding to the third vector field $X_3$. The similarity variable and similarity transformation corresponding to $X_3$ take the following form:

$$u(x,t) = x^2\mathfrak{F}(\zeta), \ \zeta = t, \tag{4.20}$$

where the function $\mathfrak{F}(\zeta)$ satisfies the following FODE:

$$x^2\left(\partial_t^{\alpha}\mathfrak{F}(\zeta) - 2\mathfrak{F}^2(\zeta) - \frac{4}{\sigma}\mathfrak{F}^2(\zeta)\right) = -\sigma. \tag{4.21}$$

Obviously, finding a general solution for Eq. (4.21) is not possible and we have to solve it in a restricted domain. To do this, we impose the following condition to Eq. (4.20):

$$\partial_t^{\alpha}\mathfrak{F}(\zeta) - \left(2 + \frac{4}{\sigma}\right)\mathfrak{F}^2(\zeta) = -\zeta^{-2\alpha}. \tag{4.22}$$

Corresponding exact solution of Eq. (4.22) has the following form:

$$\mathfrak{F}(\zeta) = \frac{\sigma\Gamma(1-\alpha) + \sqrt{\sigma^2\Gamma(1-\alpha)^2 + (8\sigma^2 + 16\sigma)\Gamma(1-2\alpha)^2}}{(\sigma+2)\Gamma(1-2\alpha)}\zeta^{-\alpha}. \tag{4.23}$$

Therefore, the invariant solution of Eq. (4.3) is

$$u(x,t) = \frac{\sigma\Gamma(1-\alpha) + \sqrt{\sigma^2\Gamma(1-\alpha)^2 + (8\sigma^2 + 16\sigma)\Gamma(1-2\alpha)^2}}{(\sigma+2)\Gamma(1-2\alpha)}t^{-\alpha}x^2, \tag{4.24}$$

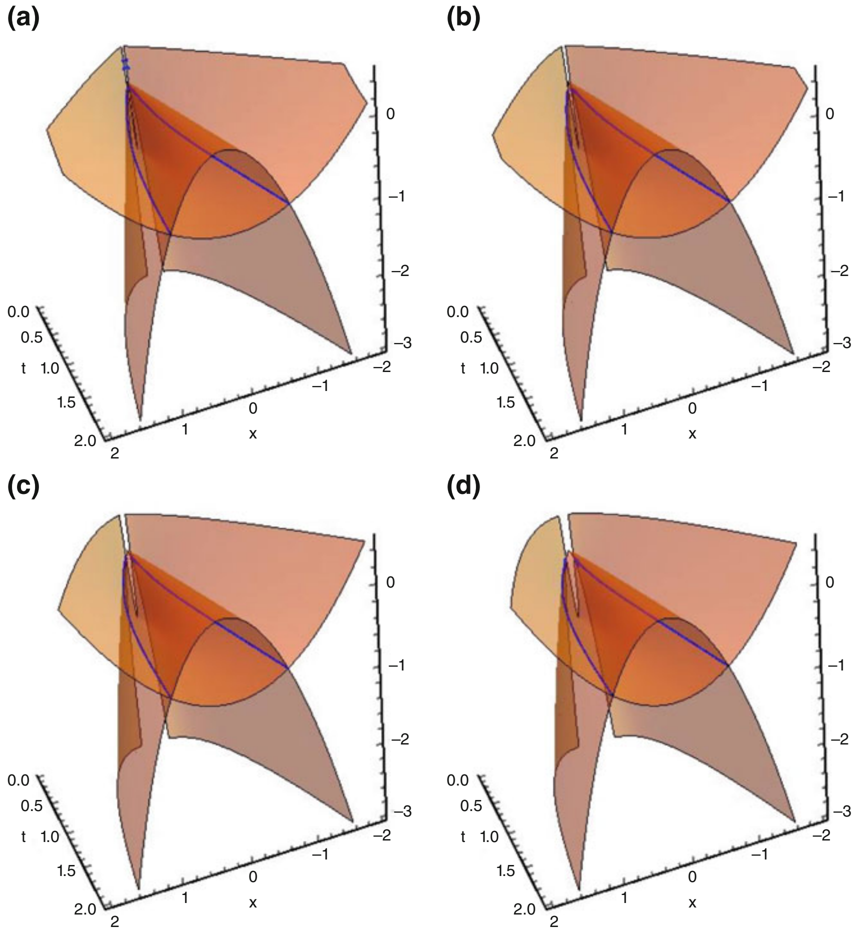


**Fig. 4.1** Exact solution of (4.24) with respect to $\sigma = 1$ and (**a**) $\alpha = 0.9$, (**b**) $\alpha = 0.8$, (**c**) $\alpha = 0.7$, (**d**) $\alpha = 0.6$

provided that

$$x^2 t^{-2\alpha} = \sigma. \tag{4.25}$$

Blue curves in Fig. 4.1a–d demonstrate the solution (4.24) in restricted domain (4.25) with various fractional orders and $\sigma = 1$.

## 4.4 Conservation Laws

In this section, we use Ibragimov method [20] for constructing the conservation laws of Eq. (4.3). To do this, we recall some preliminaries including the fractional derivatives and integrals that we use here. The Riemann-Liouville left-sided time-fractional derivative can be written in the form:

$$D_t^{\alpha} = D_t^m \left({}_0I_t^{m-\alpha} u\right), \tag{4.26}$$

where $D_t$ is differentiation operator with respect to $t$, $m = [\alpha] + 1$, and ${}_0I_t^{m-\alpha}u$ is the left-sided time-fractional integral of order $m - \alpha$ defined by

$$\left({}_0I_t^{m-\alpha} u\right)(x, t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u(x, \xi)}{(t-\xi)^{\alpha-m+1}} d\xi. \tag{4.27}$$

A vector field $\mathscr{C} = \left(\mathscr{C}^t, \mathscr{C}^x\right)$ is called a conserved vector for Eq. (4.3) if it satisfies the following conservation law:

$$D_t(\mathscr{C}^t) + D_x(\mathscr{C}^x) = 0. \tag{4.28}$$

Formal Lagrangian of the TFDA equation can be written as:

$$\mathscr{L} = v(x, t) \left[\partial_t^{\alpha} u - u u_{xx} - \frac{1}{\sigma} u_x^2 + \sigma\right], \tag{4.29}$$

where $v(x, t)$ denotes the dependent nonlocal variable. The Euler-Lagrange operator with respect to $u$ is as follows:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \left(D_t^{\alpha}\right)^* \frac{\partial}{\partial D_t^{\alpha} u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}}, \tag{4.30}$$

where $\left(D_t^{\alpha}\right)^*$ is the adjoint operator of $D_t^{\alpha}$ which one can find as:

$$\left(D_t^{\alpha}\right)^* = (-1)^m \, {}_tI_T^{m-\alpha} \left(D_t^m\right) \equiv {}_t^C D_T^{\alpha}. \tag{4.31}$$

Here ${}_t^C D_T^{\alpha}$ is the Caputo right-sided fractional differential operator of order $\alpha$ and

$$\left({}_tI_T^{m-\alpha} \psi\right)(x, t) = \frac{1}{\Gamma(m-\alpha)} \int_t^T \frac{\psi(x, \xi)}{(\xi - t)^{\alpha-m+1}} d\xi. \tag{4.32}$$

Now, we can construct the adjoint equation to the time-fractional diffusion-absorbtion equation as Euler-Lagrange equation:

$$\frac{\delta \mathscr{L}}{\delta u} = \left(D_t^{\alpha}\right)^* v - u v_{xx} - \left(\frac{2}{\sigma} - 2\right) (v u_x)_x = 0. \tag{4.33}$$

Taking into account the case of the variables $x$, $t$, and $u(x,t)$, we have

$$\overline{X} + D_t(\tau)I + D_x(\xi)I = \mathscr{W}\frac{\delta}{\delta u} + D_t\mathscr{N}^t + D_x\mathscr{N}^x, \tag{4.34}$$

where $I$ is the identity operator and $\mathscr{N}^t$, $\mathscr{N}^x$ are Noether operators and

$$\overline{X} = Pr^{(\alpha,2)}X,\ \mathscr{W} = \phi - \tau u_t - \xi u_x. \tag{4.35}$$

In Eq. (4.34), the operator $\mathscr{N}^t$ takes the form:

$$\mathscr{N}^t = \tau I + \sum_{k=0}^{m-1}(-1)^k D_t^{\alpha-1-k}(\mathscr{W})D_t^k\frac{\partial}{\partial D_t^\alpha u} - (-1)^m\mathfrak{J}\left(\mathscr{W}, D_t^m\frac{\partial}{\partial D_t^\alpha u}\right), \tag{4.36}$$

where

$$\mathfrak{J}(\Theta,\Psi) = \frac{1}{\Gamma(m-\alpha)}\int_0^t\int_t^T\frac{\Theta(x,\mu)\Psi(x,\eta)}{(\eta-\mu)^{\alpha+1-m}}d\eta d\mu. \tag{4.37}$$

Moreover, the operator $\mathscr{N}^x$ has the following form:

$$\mathscr{N}^x = \xi I + \mathscr{W}\left(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{xx}}\right) + D_x(\mathscr{W})\left(\frac{\partial}{\partial u_{xx}}\right). \tag{4.38}$$

When Eq. (4.3) is valid, for any its corresponding generator $X$ we have:

$$Pr^{(\alpha,2)}\mathscr{L} + D_t(\tau)\mathscr{L} + D_x(\xi)\mathscr{L} = 0, \tag{4.39}$$

which concludes the conservation law

$$D_t\left(\mathscr{N}^t\mathscr{L}\right) + D_x\left(\mathscr{N}^x\mathscr{L}\right) = 0. \tag{4.40}$$

Therefore, the components of the conserved vectors for Eq. (4.3) are

$$\mathscr{C}_i^t = vD_t^{\alpha-1}(\mathscr{W}_i) + \mathfrak{J}\left(\mathscr{W}_i, v_t\right), \tag{4.41}$$

$$\mathscr{C}_i^x = \mathscr{W}_i\left(v_x u + (1-\frac{2}{\sigma})vu_x\right) - vuD_x(\mathscr{W}_i),\quad i = 1,2,3, \tag{4.42}$$

where

$$\mathscr{W}_1 = -u_x,\quad \mathscr{W}_2 = -\alpha u - tu_t,\quad \mathscr{W}_3 = 2u - xu_x. \tag{4.43}$$

That is

$$\begin{aligned}
\mathscr{C}_1^t &= vD_t^{\alpha-1}(-u_x) + \mathfrak{J}\left(-u_x, v_t\right), \\
\mathscr{C}_1^x &= -u_x\left(v_x u + (1-\frac{2}{\sigma})vu_x\right) + vuu_{xx}, \\
\mathscr{C}_2^t &= -\alpha vD_t^{\alpha-1}(u) - tvD_t^{\alpha-1}(u_t) - (\alpha-1)vD_t^{\alpha-2}(u_t) - \alpha\mathfrak{J}\left(u, v_t\right) - \mathfrak{J}\left(tu_t, v_t\right), \\
\mathscr{C}_2^x &= -(\alpha u + tu_t)\left(v_x u + (1-\frac{2}{\sigma})vu_x\right) + vu(\alpha u_x - tu_{tx}),
\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}
\mathscr{C}_3^t &= 2vD_t^{\alpha-1}(u) - xvD_t^{\alpha-1}(u_x) + 2\mathfrak{J}\left(u, v_t\right) - \mathfrak{J}\left(xu_x, v_t\right), \\
\mathscr{C}_3^x &= (2u - xu_x)\left(v_x u + (1-\frac{2}{\sigma})vu_x\right) - vuu_x + xvuu_{xx}.
\end{aligned} \tag{4.45}$$

## 4.5 Exact Solutions by Invariant Subspace Method

In this section, we briefly describe the invariant subspace method applicable to the time FPDEs of the form [21–25]:

$$\partial_t^\alpha u = \Xi[u], \quad \alpha \in \mathbb{R}_+. \tag{4.46}$$

**Definition 1** A finite dimensional linear space $\mathfrak{W}_n = span\{\omega_1(x), \omega_2(x), \cdots, \omega_n(x)\}$ is said to be invariant subspace with respect to $\Xi$, if $\Xi[\mathfrak{W}_n] \subseteq \mathfrak{W}_n$.

Suppose that Eq. (4.46) admits an invariant subspace $\mathfrak{W}_n$. Then from the above definition, there exist the expansion coefficient functions $\psi_1, \psi_2, \cdots, \psi_n$ such that

$$\Xi\left[\sum_{i=1}^n \lambda_i\omega_i(x)\right] = \sum_{i=1}^n \psi_i(\lambda_1, \lambda_2, \cdots, \lambda_n)\omega_i(x), \quad \lambda_i \in \mathbb{R}. \tag{4.47}$$

Hence

$$u(x,t) = \sum_{i=1}^n \lambda_i(t)\omega_i(x), \tag{4.48}$$

is the solution of Eq. (4.46), if the expansion coefficients $\lambda_i(t),\ i = 1, \cdots, n$, satisfy a system of FODEs:

$$\begin{cases} \partial_t^\alpha \lambda_1(t) = \psi_1(\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)), \\ \vdots \\ \partial_t^\alpha \lambda_n(t) = \psi_n(\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)). \end{cases} \tag{4.49}$$

Now, in order to find the invariant subspace $\mathfrak{W}_n$ of a given fractional equation, one can use the following theorem [26].

**Theorem 1** *Let functions $\omega_i$, $i = 1, \cdots, n$ form the fundamental set of solutions of a linear nth order ODE*

$$L[y] \equiv y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0, \tag{4.50}$$

*and let $\Xi$ be a smooth enough function. Then the subspace $\mathfrak{W}_n = \mathrm{span}\{\omega_1(x), \omega_2(x), \cdots, \omega_n(x)\}$ is invariant with respect to the operator $\Xi$ of order $k \leq n-1$ if and only if*

$$L(\Xi[y])|_{L[y]=0} = 0. \tag{4.51}$$

*Moreover, dimension of the invariant subspace $\mathfrak{W}_n$ for the kth order nonlinear ODE operator $\Xi[y]$ satisfies $n \leq 2k+1$.*

Now, from Eqs. (4.3), (4.46) and Theorem 1 one can find that the dimension of invariant subspace $\mathfrak{W}_n$ for the operator $\Xi[u]$, corresponding to Eq. (4.3), satisfies $n \leq 2(2)+1 = 5$. After some calculations, we can find that $\mathfrak{W}_2 = \mathrm{span}\{1, x^2\}$ is the invariant subspace of $\Xi[u] = uu_{xx} + \frac{1}{\sigma}(u_x)^2 - \sigma$, because

$$\Xi[\lambda_1+\lambda_2 x^2] = \psi_1(\lambda_1, \lambda_2)+\psi_2(\lambda_1, \lambda_2)x^2 = 2\lambda_1\lambda_2-\sigma+2\left(1+\frac{2}{\sigma}\right)\lambda_2^2 x^2 \in \mathfrak{W}_2. \tag{4.52}$$

Therefore, in order to find the exact solution of the form:

$$u(x,t) = \lambda_1(t) + \lambda_2(t)x^2, \tag{4.53}$$

it is sufficient to solve the system of FODEs

$$\begin{cases} \partial_t^\alpha \lambda_1(t) = \psi_1(\lambda_1(t), \lambda_2(t)) = 2\lambda_1(t)\lambda_2(t) - \sigma, \\ \partial_t^\alpha \lambda_2(t) = \psi_2(\lambda_1(t), \lambda_2(t)) = 2\left(1+\frac{2}{\sigma}\right)\lambda_2^2(t). \end{cases} \tag{4.54}$$

Solving the second equation of (4.54) concludes

$$\lambda_2(t) = \frac{\Gamma(1-\alpha)t^{-\alpha}}{2\left(1+\frac{2}{\sigma}\right)\Gamma(1-2\alpha)}. \tag{4.55}$$

Substituting $\lambda_2(t)$ in the first equation of (4.54) and solving the obtained equation yields

$$\lambda_1(t) = \frac{\sigma(\sigma+2)\Gamma(1-2\alpha)t^{\alpha}}{\sigma\Gamma(1-\alpha)-(\sigma+2)\Gamma(\alpha+1)\Gamma(1-2\alpha)}. \tag{4.56}$$

Therefore, from Eq. (4.53) we obtain

$$u(x,t) = \frac{\sigma(\sigma+2)\Gamma(1-2\alpha)t^{\alpha}}{\sigma\Gamma(1-\alpha)-(\sigma+2)\Gamma(\alpha+1)\Gamma(1-2\alpha)} + \frac{\Gamma(1-\alpha)t^{-\alpha}}{2\left(1+\frac{2}{\sigma}\right)\Gamma(1-2\alpha)}x^2. \tag{4.57}$$

Figure 4.2 demonstrates the exact solution (4.57) in different time levels and various fractional orders.

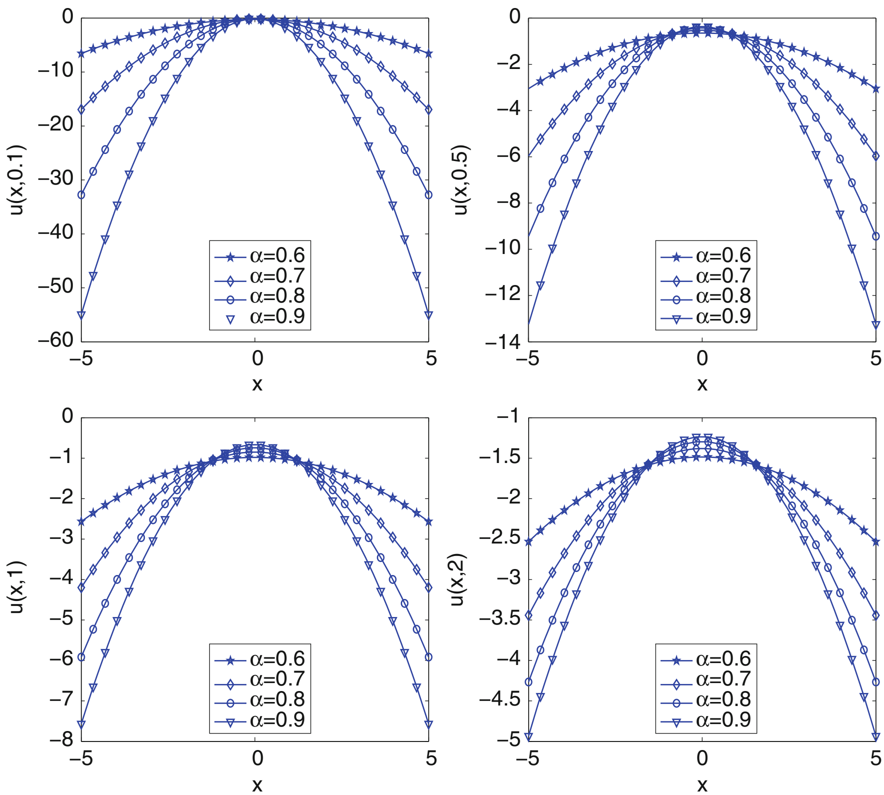


**Fig. 4.2** Exact solution of (4.57) with respect to $\sigma = 1$ and various $\alpha$ in different values of time

## 4.6 Conclusion

The symmetry properties of the TFDA equation and invariant solution in a restricted domain were investigated. Motivated by the Ibragimov's conservation laws theorem we obtain the conserved vectors. Exact solution of the TFDA equation is extracted by invariant subspace method.

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# Chapter 5
# Integral Balance Approach to 1-D Space-Fractional Diffusion Models

**Jordan Hristov**

## 5.1 Introduction

### *5.1.1 The Superdiffusion Model*

The description of anomalous transport processes often involves the use of fractional scaling. Fractional differential equations (FDEs) are suitable for modelling of anomalous diffusive processes such as subdiffusion (fractional in time) and superdiffusion (fractional in space) in non-homogeneous porous media [15], plasmas [3], and turbulent transports [2]. Since most of the analytical solutions for FDEs are difficult to obtain and most of them are approximate in nature [7, 15] resulting in forms unsuitable for post-solution applications and engineering analysis, numerical methods are most popular to solve FDEs [4, 9, 16, 29].

This chapter presents integral-balance solutions of the 1-D linear space-fractional equation (5.1) of order $1 < \beta < 2$ with three cases of diffusion coefficient: (1) Constant (space-independent) diffusion coefficient $D_\beta = D_{\beta 0}$, (2) power-law diffusion coefficient (5.2), and potential power-law diffusion coefficient (termed also as additive power-law relationship) (5.3):

$$\frac{\partial u(x,t)}{\partial t} =_{\mathrm{RL}} D_{\beta 0}(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta} \tag{5.1}$$

$$D_\beta(x) = D_{\beta 0} x^\alpha, \alpha \leq \beta, x \geq 0 \tag{5.2}$$

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$$D_\beta(x) = D_{\beta 0}x^\alpha + \gamma_x x^\alpha, \alpha \le \beta, \gamma_x > 0, x \ge 0, \tag{5.3}$$

The fractional derivative $\partial^\beta u(x,t)/\partial x^\beta$ in (5.1) is left-sided spatial derivative of either Riemann–Liouville (RL) (5.4) or Caputo type (5.5) of order $\beta$ $(1 < \beta < 2)$ [26, 27]

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} =_{\mathrm{RL}} D^x_\beta = \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_0^x \frac{u(x,t)}{(x-z)^{\beta-1}} dz \tag{5.4}$$

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} =_C D^x_\beta = \frac{1}{\Gamma(2-\beta)} \int_0^x \frac{1}{(x-z)^{\beta-1}} \frac{d^2 u(x,t)}{dx^2} dz \tag{5.5}$$

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{\partial^2 u(x,t)}{\partial x^2}, \beta = 2 \tag{5.6}$$

The common approach in solution of (5.1) is to apply numerical methods such as Galerking method [9, 29] and spectral-method [8], while analytical solutions are rare [1, 26, 27]. In this context, the solution to the space-fractional equation (5.1) with $D_\beta = D_{\beta 0}$ has been developed by Huang and Liu [26] by Green functions in terms of the similarity variable $\xi = x/(Dt)^{(1/\beta)}$. However, such solutions are not suitable for physical and engineering applications and this is the main reason to develop closed-form approximate solutions applying the integral-balance method.

### 5.1.2 The Coefficient $D_\beta(x)$ and Its Physically Correct Spatial Correlation

The power-law superdiffusivity $D_\beta(x) = D_{\beta 0}x^\alpha$, for instance, is commonly used in numerical examples demonstrating various solution approaches to (5.1) [1, 26, 27]. However, at $x = 0$ we have $D_\beta(x = 0) = 0$ and we see that Eq. (5.1) degenerates. Therefore, calculating the flux $q = -D_\beta(0)[\partial^\beta u(0,t)/\partial x^\beta]$ at the boundary $x = 0$ the physical inadequacy appears immediately since the transport coefficient $D_\beta(x)$ cannot be zero everywhere in the medium. To avoid this inadequacy the spatial approximation of $D_\beta(x)$ was suggested in [24] as (5.7)

$$D_\beta(x) = D_{\beta 0} + \gamma_x x^\alpha \Rightarrow D_\beta(x) = D_{\beta 0}(1 + k_x x^\alpha), \quad 0 \le x \le \infty \tag{5.7}$$

With (5.7) and $x = 0$ the model (5.1) is not yet a degenerate equation because $D_{\beta 0} \ne 0$. The dimension of $\gamma_x$ is $x^{\beta-\alpha}/s$ because the entire expression of $D_\beta(x)$ should have a dimension $m^\beta/s$; respectively the dimension of $k_x = \gamma_x/D_{\beta 0}$ is $1/m^\alpha$. The functional relationship (5.7) (see also (5.3)) was conceived in [24] as a correlation commonly used in integer-order models of diffusion and heat conduction [6, 12, 14].

The superdiffusivity in forms $D_{\beta 0}x^{\alpha}$ and $D_{\beta}(x) = D_{\beta 0} + \gamma_x x^{\alpha}$ with $\alpha > 0$ describes media with increasing space-dependent diffusivity as the diffusant penetrates into it. As a physical example, consider a underground water from a deep source flowing to the surface when the soil permeability increases (enhanced diffusion) in the direction from the source to the surface. In the opposite case, when $\alpha < 0$ the corresponding physical situation is a fluid infiltration into a soil with a compactness increasing in depth. In terms of superdiffusivity these cases as: *fast spatial superdiffusivity* ($\alpha > 0$), *slow spatial superdiffusivity* ($\alpha < 0$) and more comments are available elsewhere [23, 24].

The chapter encompasses the main steps and results of approximate analytical solutions of (5.1) with a constant diffusivity as well as the relationships presented by (5.2) and (5.3), developed by the integral-balance approach [13, 17]. The approach was successfully applied to models with time-fractional derivatives [18–21, 25]. The Dirichlet problem is used as an example demonstrating the method of solution.

## 5.2 The Integral Method

Consider the Dirichlet problem with initial and boundary condition

$$u(x,0) = 0, u(0,t) = 1, u(\infty,t) = 0, t \geq 0 \tag{5.8}$$

The integral-balance method [13, 17] uses the concept of a finite penetration depth $\delta(t)$. This assumption defines a sharp front of the solution, thus separating the medium of disturbed ($u(x,t) \neq 0$) and undisturbed part ($u(x,t) = 0$) separated by the position of $\delta(t)$ evolving in time. In a semi-infinite medium, this concept requires the boundary condition $u(\infty,t) = 0$ [see (5.8)] to be replaced by two new conditions (Goodman's conditions), namely: $u(\delta) = \partial u(\delta,t)/\partial x = 0$.

### *5.2.1 Principle Integration Techniques*

#### 5.2.1.1 Single-Integration Method

The integral over the penetration depth with respect to the space co-ordinate $x$ yields

$$\int_0^{\delta} \frac{\partial u(x,t)}{\partial t} dx = \int_0^{\delta} D_{\beta}(x) \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} dx \tag{5.9}$$

With the Lebnitz rule applied to left side of (5.9) we get

$$\frac{d}{dt} \int_0^{\delta} u(x,t) dx = \int_0^{\delta} D_{\beta}(x) \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} dx \tag{5.10}$$

This method is known (when $\beta = 2$) as the Heat-Balance integral Method (HBIM) of Goodman [13, 17].

#### 5.2.1.2 Double-Integration Method

The first step of DIM is integration from $x = 0$ to $\delta$ [20, 21]

$$\int_0^x \frac{\partial u(x,t)}{\partial t} dx = \int_0^x D_\beta(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta} dx \tag{5.11}$$

Since the integral in (5.9) can be presented as $\int_0^\delta u(x,t)dx = \int_0^x u(x,t)dx + \int_x^\delta u(x,t)dx$ [20, 21] we may subtract (5.11) from (5.9) that yields

$$\int_x^\delta \frac{\partial u(x,t)}{\partial t} dx = \int_x^\delta D_\beta(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta} dx \tag{5.12}$$

The second step of DIM is integration of (5.12) from 0 to $\delta$ [20, 21]. Then, applying the Leibniz rule we get

$$\frac{d}{dt} \int_0^\delta \left( \int_x^\delta u(x,t)dx \right) dx = \int_0^\delta \left( \int_x^\delta D_\beta(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta} dx \right) dx \tag{5.13}$$

Equation (5.13) is the principle integral relation of DIM when space-fractional derivatives are involved. Detailed analyses regarding the deficiencies of HBIM and the advantages of DIM are available elsewhere [12, 17–24].

#### 5.2.1.3 Assumed Profile

The integral-balance method, irrespective of the integration technique applied, suggests replacement of the function $u(x,t)$ by an assumed profile $u_a(x,\delta(t)$ expressed as function of the dimensionless space coordinate $x/\delta$ so that $0 < x/\delta < 1$. The profile may be of a polynomial type as in the classical application of HBIM [13, 17] or a parabolic profile with stipulated or unspecified exponent[17]. The approximate solutions encompassed by this chapter utilize a parabolic profile with unspecified exponent [17–21, 25] defined as

$$u_a(x,t) = \left(1 - \frac{x}{\delta}\right)^n, 0 < x < \delta, n > 0 \tag{5.14}$$

Therefore, the finite penetration depth concept (it means also a finite speed of the diffusant) transforms the problem defined in a semi-infinite area to a two-point problem by definition of the Goodman boundary conditions (5.9) at the front of the diffusion layer. The definition of the assumed profile (5.14) is general, without definition of the functional relationship of $\delta$ to $x, t$, and $n$. The replacement of $u(x,t)$ by $u_a(x,\delta(t))$ in the integral relations (5.10) or (5.13) results in an ordinary differential equation about $\delta(t)$, thus completing the definition of the assumed profile and the solution, too. However, if the exponent $n$ is not stipulated and cannot be defined by the boundary conditions, then by application of the least-square method the optimal exponent could be defined (see further in the text) [19, 22–25].

## 5.3 Space-Independent Diffusion Coefficient

### *5.3.1 Rescaling of the Diffusion Term*

We start with this simple case [22] since it is instructive about the initial rescaling of the diffusion equation. Changing the variable in diffusion term of (5.1) as $\eta = x/\delta$, where $0 < \eta < 1$, we get $u_a(x) \Rightarrow U_a(\eta) = (1-\eta)^n$ and the space-fractional derivative of the assumed profile can be rescaled as:

$$\frac{\partial^\beta u_a}{\partial x^\beta} = \left(\frac{1}{\delta}\right)^\beta \frac{\partial^\beta U_a(\eta)}{\partial x^\beta} \tag{5.15}$$

### *5.3.2 Single-Integration Method (HBIM)*

Further, changing the variable in the integral of RHS of (5.9) as $x \rightarrow \eta = x/\delta$ we get

$$D_{\beta 0}\int_0^\delta \frac{\partial^\beta u_a(x)}{\partial x^\beta}dx = D_{\beta 0}\delta^{1-\beta}\int_0^1 \frac{\partial^\beta U_a(\eta)}{\partial \eta^\beta}d\eta \tag{5.16}$$

The integration in the left side (5.10) from 0 to $\delta$, replacing $u(x,t)$ by $u_a(x,t)$ yields

$$\frac{d}{dt}\int_x^\delta u_a(x,t)dx = \frac{1}{n+1}\frac{d\delta}{dt} \tag{5.17}$$

Combining (5.17) with the result from the integration in (5.16) we get

$$\frac{1}{n+1}\frac{d\delta}{dt} = D_{\beta 0}\delta^{1-\beta}\Phi_1(n,\beta), \quad \Phi_1(n,\beta) = \int_0^1 \frac{\partial^\beta U_a(\eta)}{\partial \eta^\beta}d\eta \tag{5.18}$$

Therefore, the equation about $\delta$ [with time-independent term $\Phi_1(n,\beta)$] is

$$\frac{d\delta^\beta}{dt} = D_{\beta 0}(n+1)\Phi_1(n,\beta) \tag{5.19}$$

With $\delta(t = 0) = 0$ (no diffusion occurs at $t = 0$) the expression about $\delta(t)$ is [22]

$$\delta_1(t) = (D_{\beta 0} t)^{\frac{1}{\beta}} \left[\beta(n+1)\Phi_1(n, \beta)\right]^{\frac{1}{\beta}} \tag{5.20}$$

Here $\delta_1$ denotes the penetration depth developed by the single-integration technique of HBIM. Since $1 < \beta < 2$ when $\beta \to 2$ we have $\Phi_1(n, \beta) \to 1$ and consequently $\delta \to \sqrt{D_{\beta 0} t}$. Hence, the dimension of $D_{\beta 0}$ changes from $[m^\beta/s]$ to $[m^2/s]$ and $\delta_1$ reduces to the HBIM solution [13, 17] of the integer-order diffusion models.

### 5.3.3 Double-Integration Method (DIM)

With the re-scaled diffusion term (5.15) and changing the variables in the integrals of (5.13) we have [22]

$$\int_0^\delta \int_x^\delta \frac{\partial^\beta u_a(x,t)}{\partial x^\beta} dx dx = D_{\beta 0} \int_0^1 \int_\eta^1 \delta^2 \left(\frac{1}{\delta}\right)^\beta \frac{\partial^\beta U_a(\eta)}{\partial x^\beta} d\eta d\eta \tag{5.21}$$

Therefore, the equation of the penetration depth $\delta_2$ is [22]

$$\frac{1}{(n+1)(n+2} \frac{d(\delta^2)}{dt} = D_{\beta 0} \delta^{2-\beta} \Phi_2(n, \beta) \tag{5.22}$$

$$\Phi_2(n, \beta) = \int_0^1 \int_\eta^1 \frac{\partial^\beta U_a(\eta)}{\partial x^\beta} d\eta d\eta \tag{5.23}$$

The solution of (5.22) defines the penetration depth $\delta_2$

$$\delta_2(t) = \left(D_{\beta 0} t\right)^{\frac{1}{\beta}} \left[\frac{\beta}{2}(n+1)(n+2)\right]^{\frac{1}{\beta}} \tag{5.24}$$

For $\beta \to 2$ the scaling is $\delta_2 \to \sqrt{D_{\beta 0} t}$ and the dimension of $D_{\beta 0}$ changes from $[m^\beta/s]$ to $[m^2/s]$, while for $\beta \to 2$ the terms $\Phi_2(n, \beta) \to 1$ and (5.24) reduce to integer-order DIM solution classical diffusion equation [12, 20, 21].

### 5.3.4 Approximate Profiles

With developed functional relationships of $\delta_1$ and $\delta_2$ we may present the complete expressions of the approximate solutions, namely

#### 5.3.4.1 Single-Integration Method

$$u_a^1 = \left(1 - \frac{\xi}{N_1(n,\beta)}\right)^n, \qquad \xi = \frac{x}{(D_{\beta 0}t)^{1/\beta}} \tag{5.25}$$

$$N_1(n,b) = [\beta(n+1)\Phi_1(n,\beta)]^{1/\beta} \tag{5.26}$$

#### 5.3.4.2 Double-Integration Method

$$u_a^2 = \left(1 - \frac{\xi}{N_2(n,\beta)}\right)^n, \qquad \xi = \frac{x}{(D_{\beta 0}t)^{1/\beta}}, \tag{5.27}$$

$$N_2(n,b) = \left[\frac{\beta}{2}(n+1)(n+2)\Phi_2(n,\beta)\right]^{1/\beta} \tag{5.28}$$

It is noteworthy that the approximate profiles define the non-Boltzmann similarity variable $\xi = x/(D_{\beta 0}t)^{1/\beta}$ in a natural way. The position of the front corresponds to $u^a(\xi) = 0$ for $\xi = N_1(n\beta)$ or $\xi = N_2(n\beta)$, while its rate is proportional to $t^{1/\beta}$ and the characteristic diffusion length-scale is $(D_{\beta 0}t)^{1/\beta}$.

After the determination of $\delta_1(t)$ and $\delta_2(t)$ the only unresolved problems are the numerical evaluation of the factor $\Phi_1(n,\beta)$ and the approximate space derivative $\partial^\beta U_a(\eta)/\partial\eta^\beta$. In the sequel it is explained how this could be done.

### 5.3.5 *Evaluation of the Numerical Factors $\Phi_1(n,\beta)$ and $\Phi_1(n,\beta)$*

The evaluation of $\Phi(n,\beta)$ and $\Phi(n,\beta)$ can be done by expansion as a convergent series of $U_a(\eta) = (1-\eta)^n$ [21], namely

$$U_a(\eta) = (1-\eta)^n \approx \sum_{j=0}^{K} m_j \eta^j, m_j = \frac{U^{(j)}(0)}{\Gamma(j+1)}, 0 < \eta < 1 \tag{5.29}$$

The series $(1-\eta)^n \approx \sum_{j=0}^{K} m_j \eta^j$ converges rapidly (see [23] and the analysis therein). Now, we have to evaluate the approximate space derivative of the assumed profile and the space integral of it, depending on the integration technique applied.

#### 5.3.5.1 Single-Integration Method

Now, the fractional differentiation of the series (5.29) is

$$\frac{\partial U_a(\eta)}{\partial \eta} =_{\mathrm{RL}} D_x^{\beta} \approx \sum_{j=0}^{K} \frac{m_j}{\Gamma(j-\beta+1)} \eta^{j-\beta} = \eta^{-\beta} \sum_{j=0}^{K} \frac{m_j}{\Gamma(j-\beta+1)} \eta^{j} \quad (5.30)$$

If the Caputo space derivative is considered, for instance, then operation in (5.30) yields

$$\frac{\partial U_a(\eta)}{\partial \eta} =_{C} D_x^{\beta} \approx \sum_{j=1}^{K} \frac{m_j}{\Gamma(j-\beta+1)} \eta^{j-\beta} \quad (5.31)$$

with only difference in the first term of (5.30)

Therefore, by integration of the series (5.30), from 0 to 1, the approximation of $\Phi_1(n, \beta)$ can be obtained, namely

$$\Phi_1(n, \beta) \approx \int_0^1 \sum_{j=1}^{K} \frac{m_j}{\Gamma(j-\beta+1)} \eta^{j-\beta} d\eta \approx \sum_{j=0}^{K} \frac{m_j}{\Gamma(j-\beta+1)} \quad (5.32)$$

#### 5.3.5.2 Double-Integration Method

Applying the double integration to (5.30) the approximation of $\Phi_2(n, \beta)$ is [22]

$$\Phi_2(N, \beta) \approx \sum_{j=0}^{K} \left( \frac{1}{\Gamma(2+j-\beta)} - \frac{1}{\Gamma(3+j-\beta)} \right) \quad (5.33)$$

### 5.3.6 *The Number of Terms in the Truncated Series Expansion of $\Phi(n, \beta)$ and the Minimum Value of n*

Now, the reasonable question is: How many terms of the truncated series (5.32) and (5.33) are needed to assure the accuracy of the approximate integral-balance solution? It was established in [22] and [23] that 9 numbers of the series expansion (5.29) are enough (see more details and analysis in the original publications). Moreover, it was established that this condition permits to evaluate the minimum value of $n$ assuring $\Phi_1(n, \beta) > 0$ and $\Phi_2(n, \beta) > 0$, which is mandatory since $\delta$ is a physically defined variable with a dimension of length, decreases as the fractional order $\beta$ increases. The numerical simulations in Fig. 5.1 supports these comments.

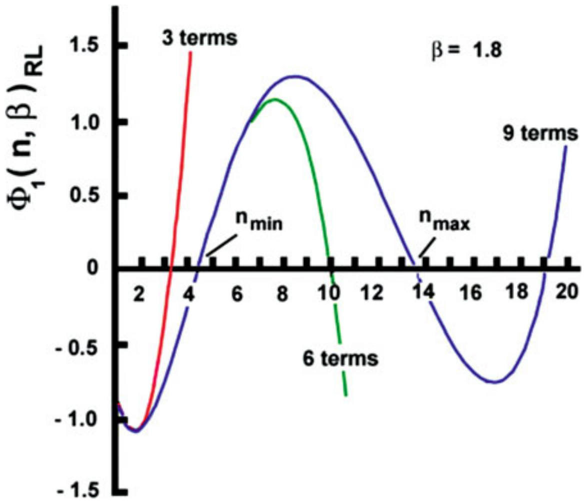


**Fig. 5.1** Effect of the exponent $n$ on $\Phi_1(n, \alpha, \beta)$ for $\alpha = 0$ and $\beta = 1.8$ and the number of terms of the truncated expansions of the assumed profile. Adapted from [22] by courtesy of *Thermal Science*

**Table 5.1** Optimal exponents of the approximate profile [22]

| $\beta$ | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 |
|---|---|---|---|---|---|---|---|---|---|
| $n_{opt}$ (HBIM) | 4.869 | 4.899 | 4.929 | 5.015 | 5.125 | 5.355 | 5.565 | 5.705 | 5.975 |
| $n_{opt}$ (DIM) | 9.697 | 9.414 | 9.125 | 8.410 | 8.575 | 8.315 | 8.101 | 7.765 | 7.850 |

### *5.3.7 Optimal Exponent of the Approximate Solution*

Applying the least-squares method to determine the undefined exponent $n$ we have to minimize the functional

$$E_L(n, \beta, t) = \int_0^{\delta} \left[ \frac{\partial u_a}{\partial t} - D_{\beta 0} \frac{\partial^{\beta} u_a}{\partial x^{\beta}} \right]^2 dx \rightarrow minimum \tag{5.34}$$

In the area where the integral-balance method is applied, this step is known as Langford criterion [12, 17, 25, 28]. It was successfully applied to subdiffusion equations [17, 19–21, 25]. In terms of the variable $\eta$ (5.34) can be expressed as [22]

$$E_L(n, \beta, t) = \frac{1}{\delta^{2\beta}} \int_0^1 \left[ n(1-\eta)^{n-1} \eta \delta^{\beta-1} \frac{d\delta}{dt} - D_{\beta 0} \frac{\partial^{\beta} U_a(\eta)}{\partial \eta^{\beta}} \right]^2 d\eta \tag{5.35}$$

The minimization procedure determining the optimal values of the exponent $n_{opt}$ is well described and we refer to [22–24] and [17, 25, 28] for details. It was successfully applied to subdiffusion equations [17, 19–21, 25]. The optimal exponents determined for the profiles (5.25) and (5.27) are summarized in Table 5.1.

### 5.3.8 Benchmarking to Exact Solutions

The exact solution of Huang and Liu [26] to the model (5.1) in terms of Caputo derivatives (here it is equivalent to the Riemann–Liouville derivative due to the zero initial conditions) expressed through the similarity variable $\xi = x/(D_{\beta 0}t)^{1/\beta}$ is:

$$u_{\mathrm{HL}}(x,t) = u(\xi) = \frac{L_{\beta}^{2-\beta}(x)}{L_{\beta}^{2-\beta}{}_{(x=0)}} = \frac{G_{1,b(x,t)}}{L_{\beta}^{2-\beta}{}_{(x=0)}} \tag{5.36}$$

where

$$L_{\beta}^{2-\beta}{}_{(x=0)} = \frac{1}{\pi\beta}\left(\frac{1}{\beta}\right)\cos\left[\frac{(2-\beta)\pi}{2\beta}\right] \tag{5.37}$$

and

$$G_{1,\beta}(x,t) = \frac{1}{\left(D_{\beta}t\right)^{\frac{1}{\beta}}} L_{\beta}^{2-\beta}\left(-\frac{x}{\left(D_{\beta}t\right)^{\frac{1}{\beta}}}\right), \quad 1 < \beta \leq 2 \tag{5.38}$$

$$L_{\beta}^{2-\beta}(x) = \frac{1}{\pi x}\sum_{k=1}^{\infty}(-x)^{k}\frac{\Gamma\left(1+\frac{k}{\beta}\right)}{n!}\sin\left[\frac{k\pi}{\beta}(1-\beta)\right], \quad x > 0. \tag{5.39}$$

Numerical simulations of these exact solutions are not easy tasks because the calculations are very sensitive to the number of terms used in the approximations (see detailed analysis in [22]). To avoid blow-ups the simulations used truncated series of 500 terms [22] indicating that such exact solutions can be handled only as textbook examples for simulations but, in fact,they are unpractical. The comparative plots in Fig. 5.2 [22] reveal that the maximum pointwise errors of approximations [compared to (5.36)] do not exceed 0.04, an error typical for integral-balance solutions [12, 17, 19–21, 25]

## 5.4 Power-Law Diffusion Coefficient $D_{\beta}(x) = D_{\beta 0}x^{\beta}$

### 5.4.1 Penetration Depth (HBIM)

With rescaled space-fractional derivative (5.15) and $D_{\beta}(x) = D_{\beta 0}x^{\alpha} = D_{\beta 0}\eta^{\alpha}\delta^{\alpha}$ the diffusion term of (5.1) takes the form

$$D_{\beta 0}x^{\alpha}\left(\frac{1}{\delta}\right)^{\beta}\frac{\partial^{\beta}U_{a}(x)}{\partial\eta^{\beta}} = D_{\beta 0}\eta^{\alpha}\delta^{\alpha-\beta}\frac{\partial^{\beta}U_{a}(x)}{\partial\eta^{\beta}} \tag{5.40}$$

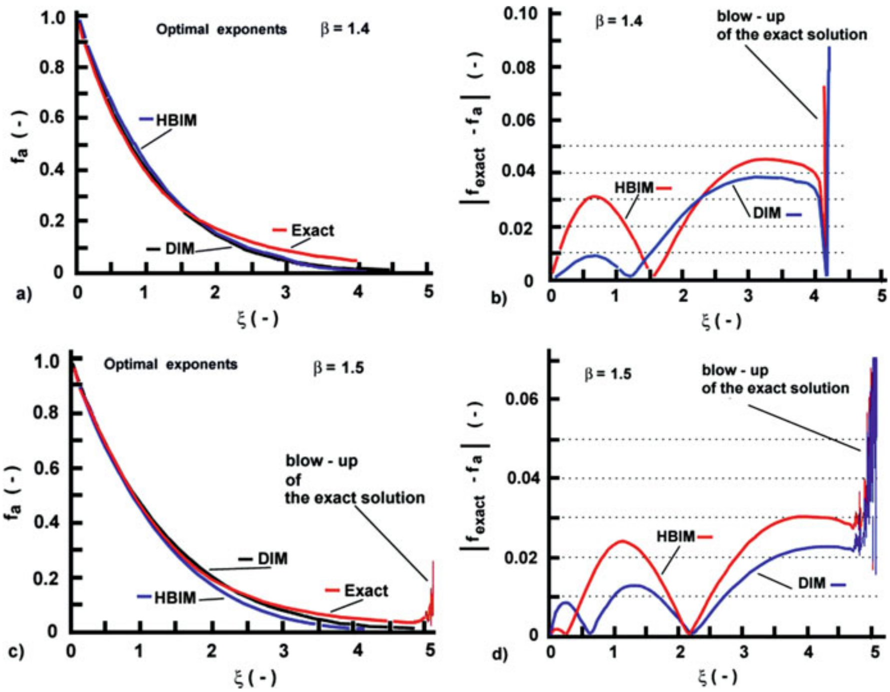


**Fig. 5.2** Approximate profiles compared to the exact ones (**a**, **c**, **e**) and pointwise errors (**b**, **d**, **f**) for $1.4 < \beta < 1.7$. Adapted from [22] by courtesy of *Thermal Science*

With the single integration of HBIM we get

$$\frac{1}{n+1}\frac{d\delta}{dt} = D_{\beta 0}\delta^{1+\alpha-\beta}\Phi_1^{\alpha}(n,\alpha,\beta) \tag{5.41}$$

$$\Phi_1^{\alpha}(n,\alpha,\beta) = \int_0^1 \eta^{\alpha}\frac{\partial^{\beta}U_a(\eta)}{\partial\eta^{\beta}}d\eta \tag{5.42}$$

where $\Phi_1^{\alpha}(n,\alpha,\beta)$ is time-independent.

Further, from (5.41) we get [23]

$$\frac{1}{\beta-\alpha}\frac{d\delta^{\beta-\alpha}}{dt} = D_{\beta 0}(n+1)\Phi_1^{\alpha}(n,\alpha,\beta) \tag{5.43}$$

$$\delta_p(t) = (D_{\beta 0}t)^{\frac{1}{\beta-\alpha}}[(\beta-\alpha)(n+1)\Phi_1^{\alpha}(n,\alpha,\beta)]^{\frac{1}{\beta-\alpha}} \tag{5.44}$$

where the index $p$ denotes $power - law$.

For $\alpha = 0$ this result reduces to $\delta_p = (D_{\beta 0} t)^{1/\beta}$ as it was developed for the case of $D_\beta(x) = D_{\beta 0} = const$. Further, since $1 < \beta < 2$, when $\beta \to 2$ we have $\delta_p(t) \to (D_{\beta 0} t)^{1/(2-\alpha)}$. Moreover, for $\alpha = 0$ and $\beta \to 2$ the penetration depth scales as $\delta_p \to \sqrt{Dt}$. Therefore, (5.44) reduces to all known versions of the penetration depth expressions by a simple change in the parameters, that actuality indicates its physical correctness, and the correctness of the solution, too. Besides, the scaling defined by (5.44) defines the conditions $\beta > \alpha$ and $1/(\beta - \alpha) > 0$, because, by definition, $\delta(t)$ should be *a growing in time penetration depth* .

Repeating the fractional differentiation of the approximated (by a series) space derivative, the final form of the approximation of $\Phi_1^\alpha(n, \alpha, \beta)$ is

$$\Phi_1(n, \alpha, \beta) = \sum_{j=0}^{K} \frac{m_j}{(j - \beta + \alpha + 1)\Gamma(j - \beta + 1)}, \varphi_1 = \eta^\alpha \frac{\partial^\beta U_a(\eta)}{\partial \eta_\beta} \tag{5.45}$$

Furthermore, the question about the number of term of the truncated series approximating the assumed profile has the same answer, that is nine terms are enough (see detailed discussion in [23]).

### 5.4.2 Approximate Profiles and Optimal Exponents

After determination of $\delta_p$ we may express the complete approximate solution as

$$u_{a-p} = \left(1 - \frac{x}{(D_{\beta 0} t)^{\frac{1}{\beta - \alpha}} N_p}\right)^n = \left(1 - \frac{\xi_p}{N_p}\right)^n \tag{5.46}$$

$$N_p = [(\beta - \alpha)(n + 1)\Phi_1(n, \alpha, \beta)]^{\frac{1}{\beta - \alpha}} \tag{5.47}$$

We stress again on the fact that due to the *construction* of the assumed profile, the approximate solution naturally defines the non-Boltzmann variable $\xi_p = x/(D_{\beta 0} t)^{1/(\beta - \alpha)}$.

The next step is determination of the optimal exponent that requires minimization of the following functional with respect to $n$

$$E_L(n, \alpha, \beta, n, t) = \int_0^1 \left(n(1 - \eta)^{n-1} \frac{\eta}{\delta} \frac{d\delta}{dt} - D_\beta(\eta^\alpha)\delta^{-\beta} \frac{\partial^\beta U_a(\eta)}{\partial \eta_\beta}\right) d\eta \tag{5.48}$$

The optimization procedure developed in [23] indicated that the optimal exponents are strongly dependent on $\beta$ but are practically unaffected by $\alpha$ (see Fig. 5.3). Numerical data are available elsewhere [23].

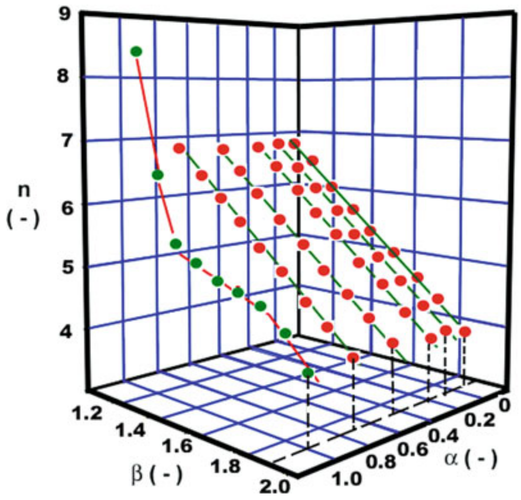


**Fig. 5.3** Three-dimensional scattered diagram $n_{\text{opt}} = f(\alpha, \beta)$. Adapted from [23]

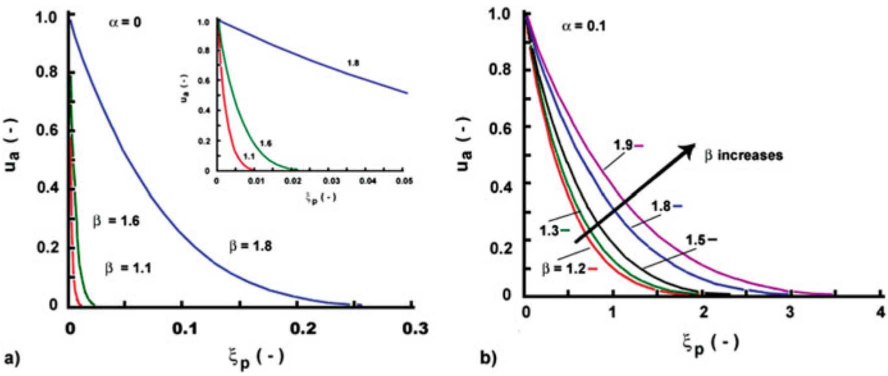


**Fig. 5.4** Approximate solutions. The order of arrangement of the profiles from left to right follows the increase in $\beta$. (**a**) $\alpha = 0$ (intermediate transport); (**b**) $\alpha = 0.1$ (intermediate transport). See comments on this regimes in Sect. 5.6

Approximate profiles as functions of the similarity variable $\xi_p = 1/(D_{\beta 0}t)^{1/(\beta-\alpha)}$ are shown in Figs. 5.4 and 5.5. These profiles belong to two groups. Precisely, for $0 \leq \alpha \leq 0.5$ roughly, the profiles with large $\beta$ propagate faster (Fig. 5.4a–c) while for $0.5 \leq \alpha \leq 1$ the behavior is just the opposite. The group formations is on the order of the arrangement of the profiles along the abscissa and the increase (or decrease) of $\beta$.

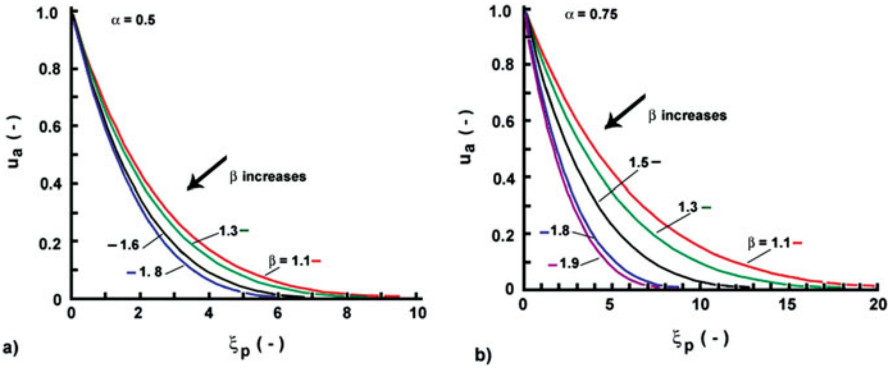


**Fig. 5.5** Approximate solution. The order of arrangement of the profiles from left to right follows the decrease in $\beta$. (**a**) $\alpha = 0.5$ (turbulent ($\beta < 1.5$) and intermediate ($\beta < 1.5$) transport). (**b**) $\alpha = 0.75$ (turbulent ($\beta < 1.8$) and intermediate ($\beta > 1.8$) transport)

## 5.5 Potential Power-Law (Additive) Diffusion Coefficient

### *5.5.1 Rescaling and Penetration Depth*

With $D_\beta(x) = D_{\beta 0} + \gamma_x x^\alpha$ (5.3) the diffusion equation (5.1) can be presented as [24]

$$\frac{\partial u(x,t)}{\partial t} = D_{\beta 0}\frac{\partial^\beta u(x,t)}{\partial x^\beta} + \gamma_x x^\alpha \frac{\partial^\beta u(x,t)}{\partial x^\beta} \tag{5.49}$$

With a change of the variable $x \longrightarrow \eta$ we get

$$\frac{\partial u(\eta)}{\partial t} = D_{\beta 0}\delta^{1-\beta}\frac{\partial^\beta F_a(\eta)}{\partial \eta^\beta} + \gamma_x \delta^{\alpha-\beta}\eta^\alpha \frac{\partial^\beta F_a(\eta)}{\partial \eta^\beta} \tag{5.50}$$

Now, applying HBIM (for the sake of simplicity) [24] we have

$$\frac{1}{\beta(n+1)}\frac{d\delta^\beta}{dt} = D_{\beta 0}\Phi_1(n,\alpha,\beta) + \delta^\alpha \gamma_x \Phi_2(n,\alpha,\beta) \tag{5.51}$$

$$\Phi_2(n,\alpha,\beta) = \int_0^1 \eta^\alpha \frac{\partial^\beta F_a(\eta)}{\partial \eta^\beta} d\eta \tag{5.52}$$

Now, with $y = \delta^\beta \Rightarrow \delta^\alpha = y^{\alpha/\beta}$ we get a reduced Bernoulli equation

$$\frac{dy}{dt} = a + by^p, \qquad p = \frac{\alpha}{\beta} \tag{5.53}$$

$$a = D_{\beta 0}\beta(n+1)\Phi_1(n,\alpha,\beta), \qquad b = \gamma_x \beta(n+1)\Phi_2(n,\alpha,\beta) \tag{5.54}$$

The solutions of (5.53) [24] use the substitution $y = \delta^{\beta} = \lambda t$ and the result is

$$\delta = \left(at + \frac{b\lambda^p}{(1+\alpha/\beta)}t^{1+\frac{\alpha}{\beta}}\right)^{\frac{1}{\beta}} \Longrightarrow \delta = (at)^{\frac{1}{\beta}}\left[1 + \frac{b}{a}\lambda^p \frac{1}{(1+\alpha/\beta)}t^{\frac{\alpha}{\beta}}\right]^{\frac{1}{\beta}} \tag{5.55}$$

The scaling analysis of (5.55) resulted in the fact that $\lambda = D_{\beta 0}$. For $b = 0$ we get the solution with the constant diffusion coefficient [22] presented earlier.

### 5.5.2 Time-Scales and Fractional Analogue of the Fourier Number

The penetration depth $\delta(t)$ (5.55) has a dimension of length. Thus, the second term in squared brackets of (5.55) should be dimensionless because the dimension of $(at)^{1/\beta}$ is length. In detail, the ratio $b/a = (\gamma_x/D_{\beta 0}^{1-\alpha/\beta})(\Phi_1/\Phi_0)$ is a scaled *characteristic time* (with a dimension $[s^{\alpha/\beta}]$) defined as [24]

$$t_1^{\alpha/\beta} = \left(D_{\beta 0}^{1-\alpha/\beta}/\gamma_s\right), \qquad t_1 = \left(D_{\beta 0}^{1-\alpha/\beta}/\gamma_s\right)^{\beta/\alpha} \tag{5.56}$$

and the ratio

$$t^{\alpha/\beta}/t_1^{\alpha/\beta} = (b/a)\lambda^{\alpha/\beta}t^{\alpha/\beta}/t_0 =_s H_r \tag{5.57}$$

is a scaled *fractional quasi-Fourier number* ${}_sH_r$ [24] which can be expressed as

$${}_sH_r = \left(\frac{\gamma_x}{D_{\beta 0}^{1-\alpha/\beta}}\right)^{-\beta/\alpha} \tag{5.58}$$

where the lower prefix $s$ means *space*

This result directly indicates that (5.3) and (5.7) are physically adequate definitions of the diffusion coefficients.

Alternatively, as it was demonstrated in [24], there is a second definition of characteristic diffusion time, namely

$$t_2 = \left(\frac{D_{\beta 0}}{\gamma_x^{1-k}}\right)^{\frac{1}{k}} = \frac{\gamma_x^{\beta/\alpha}}{D_{\beta 0}^{(\beta-\alpha)/\alpha}} = \left(\frac{\gamma_x^{\beta}}{D_{\beta 0}^{(\beta-\alpha)}}\right)^{\frac{1}{\alpha}} \tag{5.59}$$

As it was demonstrated in [24] that these are two identical definitions, because

$$\frac{{}_sHr_1}{{}_sHr_2} = \frac{t_2}{t_1} = \frac{\gamma^x_{\beta/\alpha}}{D^{\beta 0}_{\beta/\alpha} - 1} = 1 \tag{5.60}$$

#### 5.5.2.1 Approximate Profiles (Solutions)

The result (5.55) allows two forms of the approximate profiles [24], namely

$$u_{a1} = \left(1 - \frac{x}{(D_{\beta 0}t)^{\frac{1}{\beta}} N_1}\right)^n = \left(1 - \frac{\xi_1}{N_1}\right)^n \tag{5.61}$$

$$N_1 = [(\beta)(n+1)\Phi_0(n,\alpha,\beta)]^{\frac{1}{\beta}} \left[1 + ({}_sHr)^{\frac{\alpha}{\beta}} \frac{\Psi(n,\alpha,\beta)}{(1+\alpha/\beta)}\right]^{\frac{1}{\beta}} \tag{5.62}$$

$$\Psi(a,\alpha,\beta) = \frac{\Phi_1(n,\alpha,\beta)}{\Phi_0(n,\alpha,\beta)} \tag{5.63}$$

and

$$u_{a2} = \left(1 - \frac{x}{(\gamma_x t)^{\frac{1}{\beta-\alpha}} N_2}\right)^n \tag{5.64}$$

$$N_2 = \left[(\beta-\alpha)\Phi_1(n,\alpha,\beta)^{\frac{1}{\beta-\alpha}} + ({}_sHr)^{\frac{-\alpha}{\alpha-\beta}} \frac{(\beta-\alpha)\Phi_0(n,\alpha,\beta)}{[1-\alpha/(\beta-\alpha)]}\right]^{\frac{1}{\beta-\alpha}} \tag{5.65}$$

The evaluations of $\Phi_0(n,\alpha,\beta)$ and $\Phi_1(n,\alpha,\beta)$ by expansion of $U_a(\eta) = (1-\eta)^n$ as series is the same as in the previous sections. We will only give the evaluation of $\Psi(n,\alpha,\beta)$

$$\Psi(n,\alpha,\beta) = \sum_{j=0}^{K} \frac{m_j}{(j-\beta+\alpha+1)\Gamma(j-\beta+1)} \left(\sum_{j=0}^{K} \frac{m_j}{\Gamma(2+j-\beta)}\right)^{-1} \tag{5.66}$$

The determination of the optimal exponents of the profile requires minimization of the least-square error of approximation with a residual function defined as

$$R(u_a(x,t)) = n(1-\eta)^{n-1} \frac{n}{\delta}\frac{d\delta}{dt} - D_{\beta 0}\delta^{-\beta} \frac{\partial^\beta F_a(\eta)}{\partial \eta^\beta} - \gamma_x \delta^{\alpha-\beta} \frac{\partial^\beta F_a(\eta)}{\partial \eta^\beta} \tag{5.67}$$

The optimal exponents determined in [24] are practically independent of $\alpha$ but decrease with increase in $\beta$, as it is illustrated in Fig. 5.6 (the same was observed

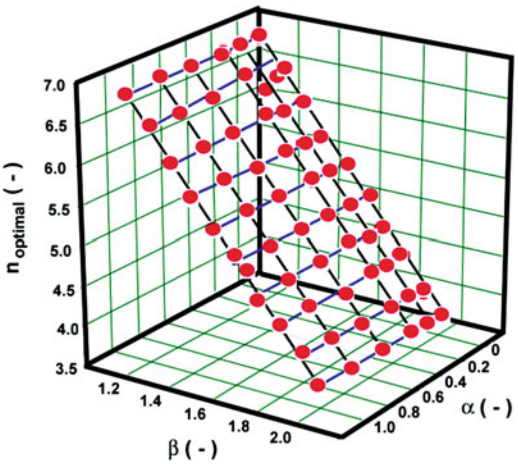


**Fig. 5.6** 3-D scattered diagram $n_{\mathrm{opt}} = f(\alpha, \beta)$. Adapted from [23]

with the diffusion coefficient $D_\beta(x) = D_{\beta 0}x^\alpha$). In this context, the relationship $n_{\mathrm{opt}} = f(\beta)$ is practically linear and be approximated as $n_{\mathrm{opt}} = 10 - 3.4\beta$ [24].

As with the integer-order models [17] and their time-fractional counterparts [18, 18–21, 25] the requirement to have a positive profile decaying in time impose the condition $n > 2$. All optimal exponents corresponding to the models discussed here satisfy this condition. Numerical data are available elsewhere [22–24].

## 5.6 Transport Regimes Modelled

### *5.6.1 Front Propagation and Effects of the Exponent α*

The values of $\alpha$ and $\beta$ control the diffusion process and as it was mentioned in Sect. 5.1.2 two principal superdiffusion regimes could be defined: *fast superdiffusion* and *slow superdiffusion*.

**Fast Superdiffusion** The results (5.44) and (5.64) where $\delta \equiv t^{\frac{1}{\beta-\alpha}}$ strictly require $\beta > \alpha$ because the penetration front is physically defined to be *a growing in time distance*. If the value of $\beta$ is stipulated, then with increase in $\alpha$ the exponent $1/(\beta - \alpha)$ will increase too and consequently faster front propagation will take place. Oppositely, when $\alpha$ is decreasing the speed controlling exponent $1/(\beta - \alpha)$ decreases and the diffusion decelerates, and *vice versa*.

Moreover, the speed of the front is $d\delta/dt \propto (1/(\beta - \alpha))t^{(1+\alpha-\beta)/(\beta-\alpha)}$. With $1+\alpha-\beta > 0$, the general condition $\beta > \alpha$ leads to an increase in time speed of the penetration front. In fact, this requires the condition $\alpha > \beta - 1$ to be obeyed. For example, if $\beta = 1.2$, this requirement defines a positive growth of the penetration depth when $0.2 < \alpha < 1$. Similarly for $\beta = 1.5$, we need $0.5 < \alpha < 1$ to be obeyed.

However, if $1+\alpha-\beta<0$, then the front propagates with a speed decaying in time. Precisely, the condition imposed in this case is $\alpha<\beta-1$. In addition, the ratio $(1-\beta)/\beta$ is always negative, and when $\alpha=0$ (constant diffusion coefficient), for instance, the speed of the front is $d\delta/dt \propto (1/\beta)t^{(1-\beta)/\beta}$, then the front will propagate with a speed decaying in time.

For the intermediate case $\alpha=\beta-1$ we have $\delta(t)=(D_{\beta 0}t)(n+1)\Phi_1(n,\alpha,\beta)$ and therefore the front propagates linearly in time and its speed does not change because $d\delta/dt = D_{\beta 0}(n+1)\Phi_1(n,\alpha,\beta) = \text{const}$.

**Slow Superdiffusion** The expression (5.44), without loss of generality, for $\alpha<0$ and $1<\beta<2$ can be written as

$$\delta_p(t) = (D_{\beta 0}t)^{\frac{1}{\beta+\alpha}}\left[(\beta+\alpha)(n+1)\Phi_1(n,\alpha,\beta)\right]^{\frac{1}{\beta+\alpha}} \tag{5.68}$$

For a given value of $\beta$, the exponent $1/(\beta+\alpha)$ decreases when the absolute value of $\alpha<0$ is increasing and therefore, the diffusion process decelerates. This is a *slow superdiffusion process*. Referring again to the case with $\alpha=0$, when the spatial damping effect depends only on the value of the fractional order $\beta$, we have the case of *slow spatial superdiffusion*. Subsequently, with $\alpha>0$ (*fast spatial superdiffusion*) the diffusant penetrates faster into the medium. Oppositely, for $\alpha<0$ (*slow spatial superdiffusion*) the diffusion will be slower than in the case with $\alpha=0$.

### 5.6.2 Mean Square Displacement and General Rules

The mean square displacement can be generalized as [5, 10, 11]

$$\langle x^2\rangle = \int_0^{\delta} x^2 u(x,t)dx \propto t^{\frac{2}{d_w}} \tag{5.69}$$

In general, the *diffusion on fractal structure* is defined with $d_w>2$, which means a dispersive transport with reduced diffusion, i.e. subdiffusion process because $\langle x^2\rangle \equiv t^{2/d_w}$ and $2/d_w<1$. For $d_w=2$ we have the Fickian diffusion. When $d_w<2$ we have *enhanced diffusion* with sub-cases: *intermediate transport* for $1<d_w<2$, a *ballistic transport* for $d_w=1$ and a *turbulent transport* for $d_w<1$. With a defined finite depth of the diffusion $\delta$ the mean squared displacement calculated with the profile (5.14) is

$$\langle x^2\rangle = \int_0^{\delta} x^2\left(1-\frac{x}{\delta}\right)^n dx = \frac{2\delta^2}{(n+1)(n+2)} \Longrightarrow \quad \langle x^2\rangle \equiv \delta^2 \propto t^{\frac{2}{(\beta-\alpha)}} \tag{5.70}$$

Hence, the difference $(\beta - \alpha)$ controls the transport regime, as in the slow and fast diffusion processes commented above. By definition we have $1 < \beta < 2$ and when $\alpha = 0$, i.e. the diffusion coefficient is space-independent with $\delta^2 \propto t^{1/\beta}$ and all regimes are subdiffusive. Further, when the diffusivity is represented by a power-law $D_\beta(x) = D_{\beta 0} x^\alpha$ or as the additive function (5.7) we have $\delta \propto t^{1/(\beta-\alpha)}$.

### 5.6.3 Regime Classification and Analysis of Two-Dimensional Profiles

The solutions presented against the similarity variable, the plots in Fig. 5.7, clearly demonstrate the effect of $\beta$ on the arrangement of the curves from left to right but the effect of $\alpha$ is not obvious. Hence, the question is: why the order of arrangements the profiles changes in both sides of the point $\beta - \alpha = 1$?

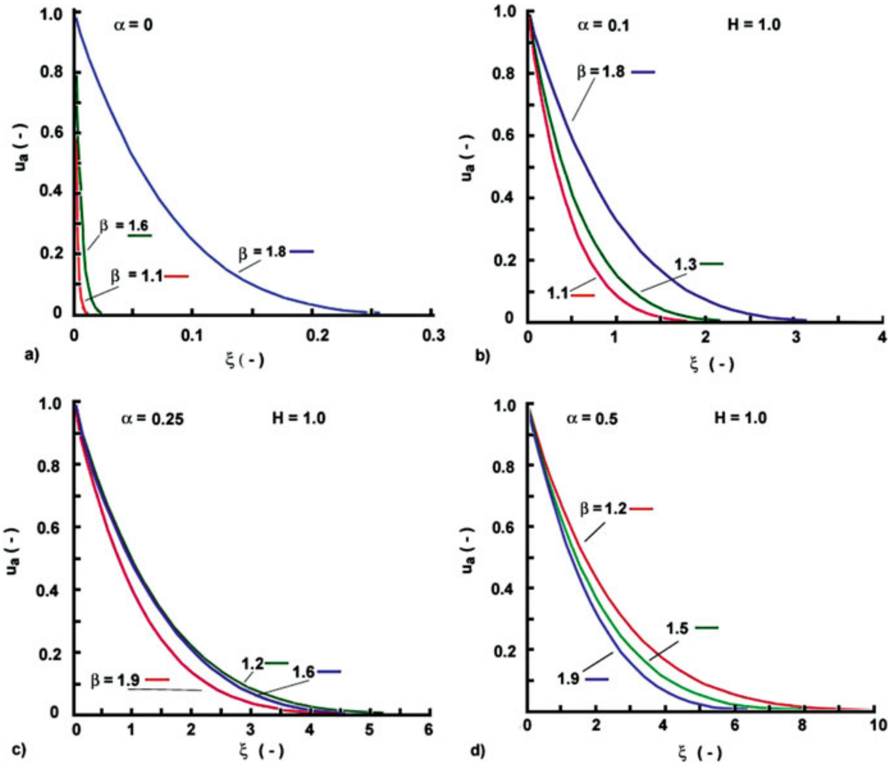


**Fig. 5.7** Approximate two-dimensional profiles. Situations when the order of arrangement of the profiles along the abscissa follows the decrease in the fractional order $\beta$ corresponding to the range $0 \leq \alpha \leq 0.5$. The case corresponds to intermediate and turbulent transport regimes (see the comments in the text). Case of ${}_sH_r = 1.0$. (**a**) $\alpha = 0$; (**b**) $\alpha = 0.1$; (**c**) $\alpha = 0.25$; (**d**) $\alpha = 0.5$. Adapted from [24]

The case $\beta - \alpha < 2$ encompasses all cases discussed here and therefore we have enhanced diffusion with the following sub-cases [5, 10, 11], namely:

For $(\beta - \alpha) < 2$ the transport regime is *intermediate*,
For $(\beta - \alpha) = 1$ there is a *ballistic transport* and
For $(\beta - \alpha) < 1$ a *turbulent transport* takes place.

The analysis in [24] (see the numerical data in Table 2 in this work) allows identifying the regimes as follows:

For $\alpha = 0$ and $\alpha = 0.1$ we have $0 < (\beta - \alpha) < 2$ and all profiles are propagating faster with increase in $\beta$ thus corresponding to the *intermediate transport* regime.

For $\alpha = 1$ in all cases $0 < (\beta - \alpha) < 1$ the profiles correspond to the *turbulent transport* regime.

For intermediate values of $\alpha$ the change in the transport mechanism depends on the value of $\beta$, namely

$$\alpha = 0.25, 1.3 < \beta < 1.9 \Longrightarrow 1 < (\beta - \alpha) < 2 \tag{5.71}$$

$$\alpha = 0.5, 1.5 < \beta < 1.9 \Longrightarrow 1.5 < (\beta - \alpha) < 2 \tag{5.72}$$

$$\alpha = 0.75, 1.8 < \beta < 1.8 \Longrightarrow 1 < (\beta - \alpha) < 2 \tag{5.73}$$

These estimations, developed in [24], demonstrate how with increase in $\alpha$ the transport behavior shifts from *intermediate* to *turbulent* regime, and *vice versa*. In addition, there exist two cases corresponding to $(\beta - \alpha) = 1$ where the fronts propagate with constant speeds [24]: for ($\alpha = 0.1$ and $\beta = 1.1$) as well as for ($\alpha = 0.5$ and $\beta = 1.5$).

We especially refer to the case for $\alpha = 0.5$, where the transport for $\beta < 1.5$ differs from that when $\beta > 1.5$, irrespective of the fact that the arrangement of the curves follows the same order as when $\beta - \alpha < 1$. In addition, for $\alpha = 0.75$ and $\alpha = 1$ all regimes are *turbulent*.

Three-dimensional profiles and regime classification, where the approximate solution $ua = f(\xi, \beta, \alpha = const.,_s H^r = const.)$ was simulated (with optimal exponent $n_{\text{opt}}$ approximated as $n_{\text{opt}} = 10 - 3.4\beta$, see the comments in Sect. 5.5.2.1) are analyzed in [24] and for details we refer to this work.

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# Chapter 6
# Fractional Order Filter Discretization by Particle Swarm Optimization Method

**Ozlem Imik, Baris Baykant Alagoz, Abdullah Ates, and Celaleddin Yeroglu**

## 6.1 Introduction

In general, filters are used to obtain desired frequency selectivity property by configuring amplitude response. They are mainly designed by shaping three types of characteristic regions that are pass bands, stop bands and transition bands. In general, stop bands are configured to reject frequency components of undesired signal from the original information signal. Stop band performance of filters is particularly important for filter applications that perform rejection of undesired signals such as noise, interference, harmonic distortions etc. After digital systems began to play a central role in daily life, digital filter design and implementation has turned into a central topic of signal processing studies. Discrete Linear Time Invariant (LTI) system models are employed in digital filter design and implementation. The design of digital filters is simplified by obtaining filter coefficients that provide a desired frequency selectivity in amplitude responses of discrete LTI systems.

Nowadays, due to their advantages to integer order counterparts, fractional-order LTI systems have gained importance for applied science and engineering problems [1–4]. Fractional-order continuous filter design includes the calculation of not only coefficients of LTI system transfer functions but also determination of real values of fractional orders. Adjustment of fractional orders gives more degree of freedom in shaping frequency response and thus make amplitude response of fractional-order filters more versatile compared to integer order counterparts that allow adjusting only filter coefficients by real numbers. In fact, fractional-order filter function is a more general form of filter function family, which includes the integer order filter functions [5]. A major advantage of fractional order filter function is that the slope of transition bands can be adjusted fractionally by using fractional orders

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[6]. Despite this advantages in filter response, digital realization of fractional-order filters are more complicated than the realization of integer order filters because of long memory effect of fractional order derivative operators. Fractional derivative of a function depends on all past values of functions. In other words, fractional order derivative is not localized to current value of function. For a fully approximation to the response of fractional-order filters by using an integer order transfer function, it may need infinite number of filter coefficients. For this reason, in practice, several approximation methods are developed for approximate implementation of fractional-order LTI systems by means of integer order functions in a certain degree of accuracy in operating ranges of applications [7–12]. In order to improve integer order approximation to fractional-order transfer functions, heuristic optimization methods such as genetic algorithm [4], PSO [13], discrete stochastic optimization [1] has been employed.

The current study focuses on stable IIR discrete filter designs to represent frequency selectivity properties of fractional-order filters, particularly at stop bands. Many analytical discretization methods do not ensure stability of resulting IIR filters, and unstable filter solutions are not useful in practice. To deal with filter stability problem and improve amplitude response fitting to fractional-order filter in desired frequency ranges, heuristic optimization can be utilized. The PSO algorithm is modified to obtain stable IIR filter coefficients that are approximating to amplitude responses of fractional-order continuous filter functions. Stability of resulting IIR discrete filters is ensured by setting very high cost values for particles resulting in unstable IIR discrete filter solutions. Thus, particles in the swarm are forced to move towards to search regions, where stable IIR filter solutions can exist. Illustrative design examples are shown to evaluate performance of proposed method and results are compared with the results of CFE approximation.

## 6.2 Theoretical Background

### *6.2.1 PSO Algorithm*

PSO algorithm is an effective heuristic search algorithm that emulates swarm intelligence. PSO algorithm imitates the collective behaviour of unsophisticated agents that can interact locally and form a global coherence [14]. Performance of PSO algorithm has been shown in diverse optimization problems such as control system implementation [11], filter design [15].

Particle motion in multidimensional search space is modelled by particle position $x_n[t]$ and particle velocity $v_n[t]$. This motion is calculated by the following formulas,

$$v_n\left[t+1\right] = wv_n\left[t\right] + c_1r_1\left(x_{L,n}\left[t\right] - x_n\left[t\right]\right) + c_2r_2\left(x_{G,n}\left[t\right] - x_n\left[t\right]\right) \tag{6.1}$$

$$x_n\left[t+1\right] = x_n\left[t\right] + v_n\left[t+1\right] \tag{6.2}$$

where, $x_{Ln}[t]$ is the personal best position, which is the best fitting solution in the current iteration, and $x_{Gn}[t]$ is the global best position, which is the best fitting solution of all iterations. Parameter $c_1$ and $c_2$ are personal learning coefficient and global learning coefficient, respectively. Parameter $w$ is the weight coefficient for particle inertias. In order to decrease inertia of particles during iterations, a damping rate ξ is applied as $w[t + 1] = w[t]\ \xi$ [16]. Equation (6.1) determines the new velocity $v_n[t + 1]$ of particles depending on particle inertia at the iteration time ($t$) and then new positions of particles $x_n[t + 1]$ are updated by Eq. (6.2). During the particle movements, the local best position and the global best position are updated according to an objective function. Particle positions represent solutions that are found during optimization process. Objective function evaluates fitness of each particle position as a solution of problem. Hence, devising objective function is important for success of optimization because objective functions determine the character of solutions, for which optimization algorithm is looking.

The following section introduces the proposed objective function that is devised for improvement of approximation performance at stop bands and stability of resulting filters.

### 6.2.2 Problem Formulation and Application of PSO Algorithm

Fractional-order continuous LTI filters can be written in general form as,

$$F_c(s) = \frac{\sum_{i=0}^{k} c_i s^{\beta_i}}{\sum_{i=0}^{m} d_i s^{\alpha_i}} \tag{6.3}$$

By using $s^\alpha = (j\omega)^\alpha = \omega^\alpha(\cos(\alpha\pi/2) + j\sin(\alpha\pi/2))$, amplitude response of fractional-order filters are expressed in general form as,

$$|F_c\,(j\omega)| = \frac{\left|\sum_{i=0}^{k} c_i \omega^{\beta_i}\left(\cos\left(\frac{\pi}{2}\beta_i\right) + j\,\sin\left(\frac{\pi}{2}\beta_i\right)\right)\right|}{\left|\sum_{i=0}^{m} d_i \omega^{\alpha_i}\left(\cos\left(\frac{\pi}{2}\alpha_i\right) + j\,\sin\left(\frac{\pi}{2}\alpha_i\right)\right)\right|} \tag{6.4}$$

Shaping amplitude responses of filter functions allows configuration of frequency selectivity properties of filters. By considering Eq. (6.4), we can see that filter coefficients $\{c_i, d_i\}$ and fractional orders $\{\beta_i, \alpha_i\}$ can be used to shape frequency response by assigning real numbers to them. This provides more option in frequency selectivity for filter functions compared to integer order counterparts that allow only integer numbers for the order parameters. As known, integer order LTI filter functions provide increments and decrements that can be multiples of 20 dB in amplitude response characteristics. However, fractional order LTI filter functions also allow fractions of 20 dB in slopes of amplitude response characteristics. For digital realization of the amplitude responses of filters, one needs the discretization of filter function. In the current study, the proposed algorithm optimize amplitude response of an initial randomly generated discrete IIR filter solution set to provide a better fitting to amplitude response of the continuous fractional-order filter function $F_c(s)$. The discrete IIR filter $F_d(s)$ can be expressed as,

$$F_d(z) = \frac{\sum_{i=0}^{l} a_i z^i}{\sum_{i=0}^{p} b_i z^i} \tag{6.5}$$

Particles of the swarm move in the search space of filter coefficients and positions of particles represent IIR filter solutions in this space. In other words, as particles move in directions that minimizes the objective function, they also modify the resulting discrete IIR filter functions according to the objective function, defined as:

$$f\,(a,b) = \frac{1}{L} \sum_{\omega_i \in (\omega_{\min},\omega_{\max})} \left(20\log_{10}|F_c\,(j\omega_i)| - 20\log_{10}\left|F_d\left(a,b,e^{j\omega_i T_s}\right)\right|\right)^2 \tag{6.6}$$

where, $a = [a_0\ a_1\ a_2\ a_3\ ...\ a_l]$ and $b = [b_0\ b_1\ b_2\ b_3\ ...\ b_\mathrm{p}]$ are IIR filter coefficients to be optimized in the sampled frequency range $\omega_i \in (\omega_{\min}, \omega_{\max})$, $i = 1, 2, 3, ...\ L$ [2]. To calculate Eq. (6.6), $z = e^{j\omega Ts}$ are used in Eq. (6.5). Parameter $T_s$ denotes the sampling period of discrete filter.

Positions of particles in the coefficient search space are expressed as,

$$x_n = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & \dots & b_p & a_0 & a_1 & a_2 & a_3 & \dots & a_l \end{bmatrix} \tag{6.7}$$

Decrease in values of objective functions allows approximation of amplitude response of $F_d(s)$ filter function to amplitude response of $F_c(s)$ filter function. Since phase response shaping does not require for filter applications, phase approximation of filters is not considered in the objective function. The following two assets are utilized for devising objective function for filter application.

1. Similarity of amplitude response is expressed by mean square of amplitude responses in logarithmic scale: The logarithmic scaling of magnitude in decibel results in compressing magnitudes and allows a finer optimization for very low magnitudes in amplitude response of filters, which indeed improves approximation performance at stop bands of filters. This effect can be clearly observed in frequency response data of example designs given in the following section.
2. Stability prevention is accomplished by assigning high value to objective function in case of unstable filter solution: In order to ensure stability of IIR filter design, we set a very high cost value ($f_{\max}$) to objective function for particles that

results in unstable IIR filter solutions. The objective function with this stability constraint can be expressed as,

$$f(a,b) = \begin{cases} \frac{1}{L} \sum_{\omega_i \in (\omega_{\min}, \omega_{\max})} \left(20\log_{10}|F_c(j\omega_i)| - 20\log_{10}\left|F_d\left(a, b, e^{j\omega_i T_s}\right)\right|\right)^2 & \text{stable} \\ f_{\max} & \text{unstable} \end{cases} \tag{6.8}$$

Here, very high value setting to $f_{\max}$ leads to keep other particles away from unstable filter solutions. This enforces particles to search for stable IIR filter solutions. Steps of the proposed PSO algorithm can be summarized as follows,

*Step 1*: Set an initial values to position and velocity of particles randomly within the search ranges.
*Step 2*: Find local best and global best positions according to objective function (Eq. 6.8).
*Step 3*: Update particle positions by using Eqs. (6.1) and (6.2), and determine local best and global best positions by calculating costs of particles according to objective function (Eq. 6.8).
*Step* 4: If the maximum number of iteration is reached, end optimization. Otherwise, go to Step 3.

## 6.3 Illustrative Examples

In this section, we present stable IIR filter design examples that can approximate to amplitude responses of fractional-order continuous filters. In PSO optimizations, we configured the search range of coefficients as $\alpha_i \in [-20, 20]$. Population size was set to 200 particles. Personal learning coefficients $c_1$ and $c_2$ were set to 1.5 and 2.0, respectively. The inertia weight ($w$) and damping rate ($\xi$) were set to 1 and 0.99, respectively. Sampling period for discrete filtering was assumed to 0.01 s, which allowed maximum filtering frequency ($\omega_{\max}$) up to 314.15 rad/s according to Nyquist sampling theorem.

*Example 1* Let us obtain a discrete IIR filter implementation of the fractional-order low pass filter given in [1] as,

$$F_{c1}(s) = \frac{1}{s^{0.5} + 1} \tag{6.9}$$

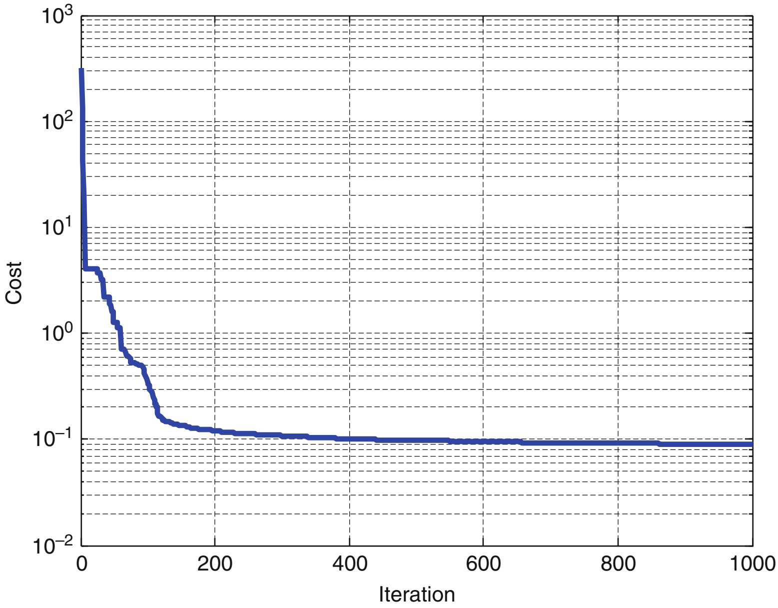


**Fig. 6.1** Change of cost values during optimization by PSO algorithm

After 1000 iteration, optimized IIR filter function was obtained as,

$$F_{d1}(s) = \frac{0.4887z^4 + 0.5235z^3 - 0.1822z^2 - 1.316z + 0.2847}{15.38z^4 - 10.99z^3 - 6.78z^2 - 1.131z + 3.8} \tag{6.10}$$

The IIR filter function, obtained by continuous CFE approximation with Tustin discretization, can be written as,

$$F_{cfe1}(s) = \frac{0.1116z^4 - 0.4119z^3 + 0.5651z^2 - 0.3432z + 0.07761}{z^4 - 3.884z^3 + 5.656z^2 - 3.658z + 0.8869} \tag{6.11}$$

Figure 6.1 shows the evolution of cost value during optimization of filter coefficients by PSO algorithm. Decrease of the cost values infers decrease of approximation errors of IIR filters during the optimization. The figure clearly indicates convergence of optimization process during iterations. Figure 6.2 compares the amplitude responses of continuous fractional-order filter function $F_{c1}(s)$, the obtained IIR filter function $F_{d1}(s)$, continuous integer order filter function approximation by CFE method and discrete IIR filter function approximation by CFE method with Tustin discretization (CFE + Tustin). As seen in figures, CFE method can provide better approximation at low frequency bands but it diverges at mid and high frequency ranges. Table 6.1 lists cost values of each design. The most of the magnitude error in CFE + Tustin discretization method comes from

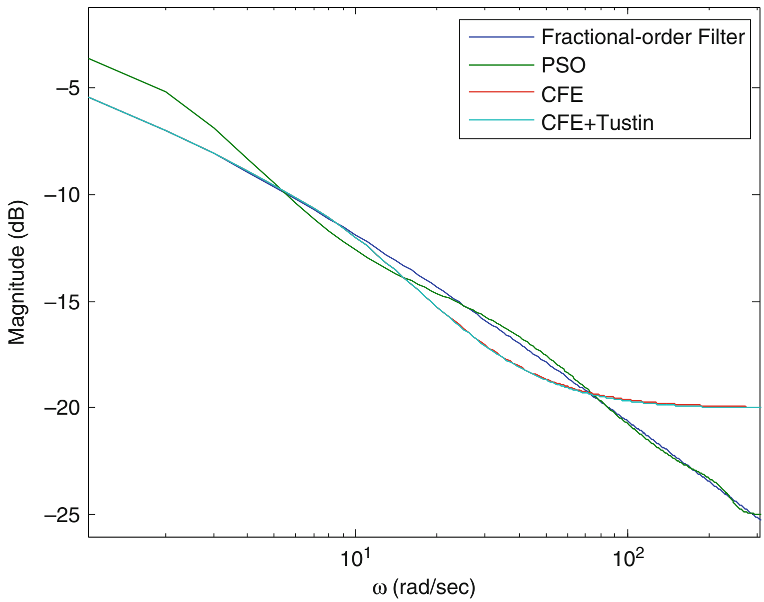


**Fig. 6.2** Comparison of amplitude responses of continuous fractional-order filter, IIR filter function by PSO, continuous integer order filter approximation by CFE and discrete IIR filter function approximation by CFE + Tustin methods

**Table 6.1** Mean squared errors for amplitude responses of IIR filter designs in Example 1

| IIR filter design methods | Mean squared errors |
|---|---|
| PSO | 0.0894 |
| CFE + Tustin discretization | 9.4034 |

high frequency parts of the spectrum. As known, stop band performance is more substantial for band reject filters, for instance, noise filtering. For low pass filters, stop band coincides to high frequency part of frequency spectrum. Since $F_{d1}(s)$ can provide better approximation at low magnitude values, which appear in high frequency parts of low pass filter spectrums, PSO can improve discrete realizations of low pass fractional-order filter functions in practice. Figure 6.3 shows time response of discrete $F_{d1}(s)$ and $F_{cfe1}(s)$ filter functions for sinusoidal input signal $(\sin(40t))$ and the results show that both IIR filters are stable filters.

*Example 2* Let us design a stable IIR filter approximating to the continuous fractional-order Chebyshev low pass filters given in the form of [6],

$$F_{c2}(s) = \frac{a_0}{a_1 s^{1+\alpha} + a_2 s^{\alpha} + 1} \tag{6.12}$$

for $\alpha = 0.2$, $a_0 = 3$, $a_1 = 3$ and $a_2 = 5$.

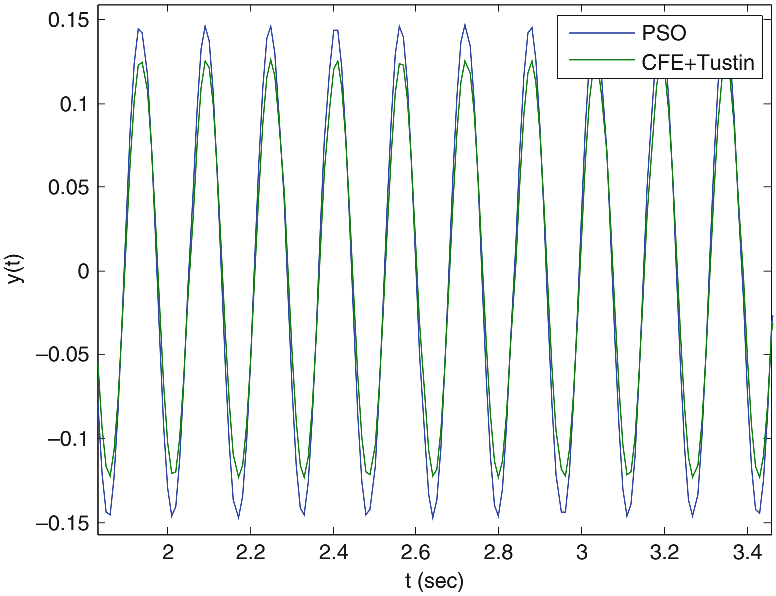


**Fig. 6.3** Time response of discrete filter functions for a sinusoidal input signal

After 1000 iteration, the optimized IIR filter function was obtained as,

$$F_{d2}(s) = \frac{-0.006286z^4 - 0.004119z^3 - 0.01841z^2 + 0.02865z - 0.009169}{19.95z^4 - 4.422z^3 - 15.4z^2 + 2.839z - 2.908} \tag{6.13}$$

The IIR filter function, obtained by CFE + Tustin method, can be written as,

$$F_{cfe2}(s) = \frac{\begin{array}{c}0.0007416z^9 - 0.004903z^8 - 0.01313z^7 - 0.0169z^6 + 0.006652z^5 \\ + 0.01021z^4 - 0.0169z^3 + 0.0111z^2 - 0.00363z + 0.0004868\end{array}}{\begin{array}{c}z^9 - 8.666z^8 + 33.36z^7 - 74.9z^6 + 108z^5 - 103.9z^4 \\ + 66.54z^3 - 27.39z^2 + 6.567z - 0.7009\end{array}} \tag{6.14}$$

Figure 6.4 shows the evolution of cost values. The figure confirms the convergence of optimization process during iterations. Figure 6.5 compares the amplitude responses of continuous fractional-order filter function $F_{c2}(s)$, the obtained IIR filter function $F_{d2}(s)$, continuous integer order filter function approximation by CFE method and discrete IIR filter function approximation by CFE with Tustin method. Figure 6.6 shows a close view of high frequency part in Fig. 6.5. As seen in figures, proposed PSO algorithm yields better approximation at stop band of the fractional-order filter function. However, CFE method can provide superior approximation

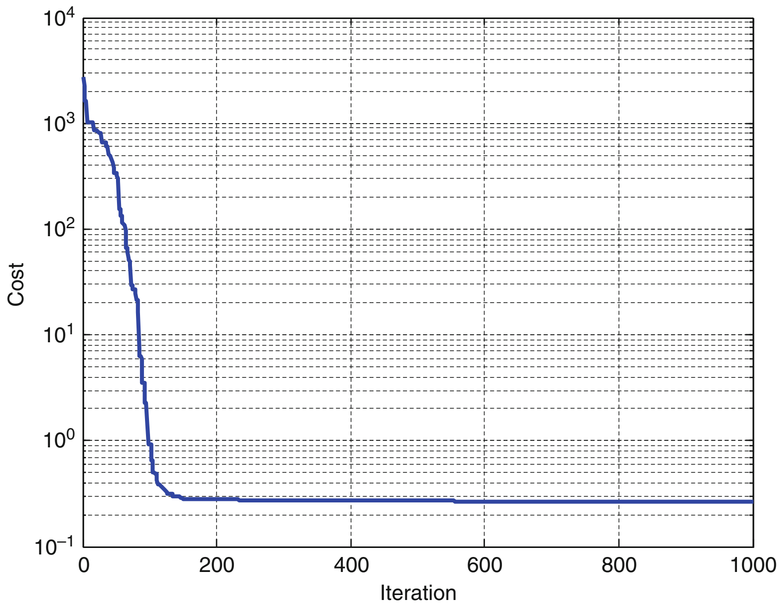


**Fig. 6.4** Change of cost values during optimization by PSO algorithm

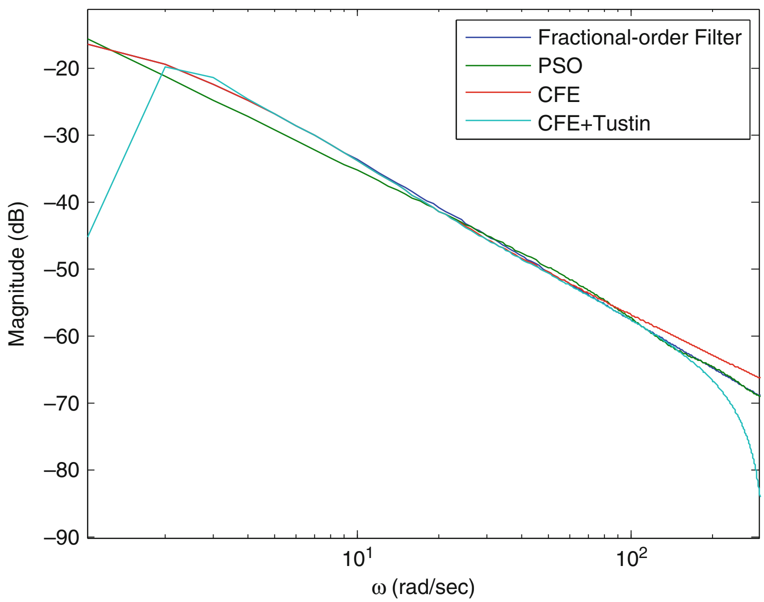


**Fig. 6.5** Comparison of amplitude responses of continuous fractional-order filter, IIR filter function by PSO, continuous integer order filter approximation by CFE and discrete IIR filter function approximation by CFE + Tustin methods

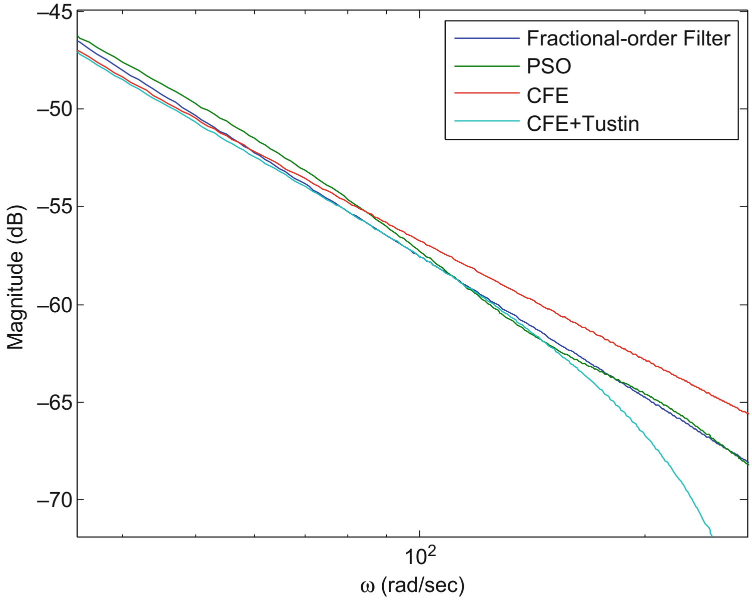


**Fig. 6.6** A close view from high frequency part of amplitude response of filters

**Table 6.2** Mean squared errors for amplitude responses of IIR filter designs in Example 2

| IIR filter design methods | Mean squared errors |
|---|---|
| PSO | 0.2627 |
| CFE + Tustin discretization | 76.2223 |

at low frequency region. Table 6.2 lists cost values of discrete filter designs by PSO and CFE + Tustin method. Figure 6.7 shows time response of discrete $F_{d2}(s)$ and $F_{cfe2}(s)$ filter functions for sinusoidal input signal ($\sin(40t)$). Results reveal that $F_{cfe2}(s)$ filter is a not stable because CFE method does not ensure stability of resulting filters. The proposed method enforces PSO to perform in the search space resulting in stable filters. This is an important advantage for discrete filter implementations.

## 6.4 Conclusions

This study presents application of PSO algorithm for fractional-order filter function discretization for digital signal processing applications. The proposed method contributes to digital filter design in two folds: (1) It enforces stability of optimized IIR filters by configuring very high cost values for particles that represent instable IIR filter solutions. This avoids movements of particle swarm towards the search space of unstable filter solutions. (2) Logarithmic scaling of amplitude responses in

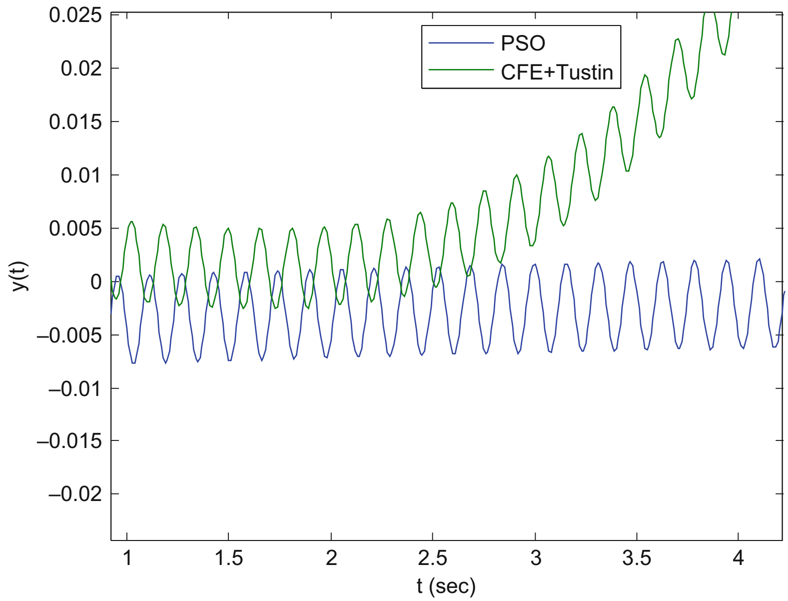


**Fig. 6.7** Time response of discrete filter functions for sinusoidal input signal

objective function increases optimization efforts of PSO algorithm at low magnitude part of filter spectrum. This leads better approximation at stop bands of filters, where amplitude response of filter takes very low values. Thus, it enables to better presentation of stop bands of fractional-order filters by stable IIR filters.

These two assets make the proposed PSO method useful for realization of fractional-order filter functions in digital system applications. Although, CFE method can provide superior approximation to frequency response of fractional-order transfer functions in low frequency region, it may not well represent the system responses at high frequency regions. Therefore, CFE approximation method is more suitable for low frequency applications, such as control system applications, which require a good phase and amplitude response matching in low frequency ranges. It should be noted that the proposed PSO method does not perform any phase response approximation.

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# Chapter 7
# On the Existence of Solution for a Sum Fractional Finite Difference Inclusion

**Vahid Ghorbanian, Shahram Rezapour, and Saeid Salehi**

## 7.1 Introduction

It has been published many works about the existence of solutions for some fractional finite difference equations by using different views (see, for example, [12–15, 18–21, 29, 30] and the references therein). One can find more details on elementary notions of fractional finite difference equations in [1, 9, 10, 16, 22–24, 28]. Also, there are many works on fractional differential inclusions (see for example, [2–11, 17, 25, 26] and the references therein). Recently, appeared a work on fractional finite difference inclusions [15]. We provide some preliminaries to investigate the existence of solution for a fractional finite difference inclusion.

As you know, the Gamma function has some known properties as $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. Define $t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for all $t, \nu \in \mathbb{R}$ whenever the right-hand side is defined. If $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then we define $t^{\underline{\nu}} = 0$. Also, one can verify that $\nu^{\underline{\nu}} = \nu^{\underline{\nu-1}} = \Gamma(\nu+1)$ and $t^{\underline{\nu+1}} = (t-\nu)t^{\underline{\nu}}$. We use the notations $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for all $a \in \mathbb{R}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ for all real numbers $a$ and $b$ whenever $b-a$ is a natural number. Let $\nu > 0$ be such that $m-1 < \nu \leq m$ for some natural number $m$. Then the $\nu$th fractional sum of $f$ based at $a$ is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} (t-\sigma(k))^{\underline{\nu-1}} f(k)$$

for all $t \in \mathbb{N}_{a+\nu}$. Similarly, we get $\Delta_a^{\nu} f(t) = \frac{1}{\Gamma(-\nu)} \sum_{k=a}^{t+\nu} (t-\sigma(k))^{\underline{-\nu-1}} f(k)$ for all $t \in \mathbb{N}_{a+m-\nu}$. We need the following result.

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**Lemma 7.1.1** *Let $\nu > 0$ with $m-1 < \nu \leq m$ for some natural number $m$. Then*

$$\Delta_a^\nu y(t) + \sum_{j=1}^{k} \Delta_a^{\nu-j} y(t-1+j) = \Delta_a^{\nu-k} y(t+k) + \sum_{i=1}^{k} \frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)} y(a-1+i) \quad (7.1)$$

*for all $k \in \mathbb{N}_1^{m-1}$.*

*Proof* Let $k=1$. Then, we have

$$\begin{aligned}
\Delta_a^\nu y(t) + \sum_{j=1}^{1} \Delta_a^{\nu-j} y(t-1+j) &= \Delta_a^\nu y(t) + \Delta_a^{\nu-1} y(t) \\
&= \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} y(s) + \frac{1}{\Gamma(-\nu+1)} \sum_{s=a}^{t+\nu-1} (t-\sigma(s))^{\underline{-\nu}} y(s) \\
&= \frac{1}{-\nu\Gamma(-\nu)} \sum_{s=a}^{t+\nu} \Delta(t-\sigma(s))^{\underline{-\nu}} y(s) + \frac{1}{\Gamma(-\nu+1)} \sum_{s=a}^{t+\nu-1} (t-\sigma(s))^{\underline{-\nu}} y(s) \\
&= \frac{1}{\Gamma(-\nu+1)} \sum_{s=a}^{t+\nu} \left[ (t+1-\sigma(s))^{\underline{-\nu}} y(s) - (t-\sigma(s))^{\underline{-\nu}} y(s) \right] \\
&\quad + \frac{1}{\Gamma(-\nu+1)} \sum_{s=a}^{t+\nu-1} (t-\sigma(s))^{\underline{-\nu}} y(s) \\
&= \frac{1}{\Gamma(-\nu+1)} \sum_{s=a}^{t+\nu} (t+1-\sigma(s))^{\underline{-\nu}} y(s) - \frac{1}{\Gamma(-\nu+1)} (t-\sigma(t+\nu))^{\underline{-\nu}} y(t+\nu) \\
&= \frac{1}{\Gamma(-\nu+1)} \sum_{s=a-1}^{t+\nu-1} (t+1-\sigma(s+1))^{\underline{-\nu}} y(s+1) - 0 \\
&= \frac{1}{\Gamma(-\nu+1)} \sum_{s=a-1}^{t+\nu-1} (t-\sigma(s))^{\underline{-\nu}} y(s+1) \\
&= \frac{1}{\Gamma(-\nu+1)} \sum_{s=a}^{t+\nu-1} (t-\sigma(s))^{\underline{-\nu}} y(s+1) + \frac{1}{\Gamma(-\nu+1)} (t-a)^{\underline{-\nu}} y(a) \\
&= \Delta_a^{\nu-1} y(t+1) + \frac{1}{\Gamma(-\nu+1)} (t-a)^{\underline{-\nu}} y(a).
\end{aligned}$$

Now, suppose that

$$\Delta_a^\nu y(t) + \sum_{j=1}^{n} \Delta_a^{\nu-j} y(t-1+j) = \Delta_a^{\nu-n} y(t+n) + \sum_{i=1}^{n} \frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)} y(a-1+i)$$

holds for $n \in \mathbb{N}_2^{m-2}$. Then, we have

$$\begin{aligned}
&\Delta_a^\nu y(t) + \sum_{j=1}^{n+1} \Delta_a^{\nu-j} y(t-1+j) \\
&= \Delta_a^\nu y(t) + \sum_{j=1}^{n} \Delta_a^{\nu-j} y(t-1+j) + \Delta_a^{\nu-n-1} y(t+n) \\
&= \Delta_a^{\nu-n} y(t+n) + \sum_{i=1}^{n} \frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)} y(a-1+i) + \Delta_a^{\nu-n-1} y(t+n) \\
&= \sum_{i=1}^{n} \frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)} y(a-1+i) + \frac{1}{\Gamma(-\nu+n)} \sum_{s=a}^{t+\nu-n} (t-\sigma(s))^{\underline{-\nu-1+n}} y(s+n) \\
&\quad + \frac{1}{\Gamma(-\nu+n+1)} \sum_{s=a}^{t+\nu-n-1} (t-\sigma(s))^{\underline{-\nu+n}} y(s+n) \\
&= \sum_{i=1}^{n} \frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)} y(a-1+i) \\
&\quad + \frac{1}{\Gamma(-\nu+n+1)} \sum_{s=a}^{t+\nu-n} \Delta(t-\sigma(s))^{\underline{-\nu+n}} y(s+n) \\
&\quad + \frac{1}{\Gamma(-\nu+n+1)} \sum_{s=a}^{t+\nu-n-1} (t-\sigma(s))^{\underline{-\nu+n}} y(s+n) \\
&= \sum_{i=1}^{n} \frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)} y(a-1+i) \\
&\quad + \frac{1}{\Gamma(-\nu+n+1)} \sum_{s=a}^{t+\nu-n} (t+1-\sigma(s))^{\underline{-\nu+n}} y(s+n) \\
&\quad - \frac{1}{\Gamma(-\nu+n+1)} (t-\sigma(t+\nu-n))^{\underline{-\nu+n}} y(s+n) \\
&= \sum_{i=1}^{n} \frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)} y(a-1+i)
\end{aligned}$$

$$+\frac{1}{\Gamma(-\nu+n+1)}\sum_{s=a-1}^{t+\nu-n-1}(t-\sigma(s))^{\underline{-\nu+n}}y(s+1+n)-0$$

$$=\sum_{i=1}^{n}\frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)}y(a-1+i)$$

$$+\frac{1}{\Gamma(-\nu+n+1)}\sum_{s=a}^{t+\nu-n-1}(t-\sigma(s))^{\underline{-\nu+n}}y(s+1+n)$$

$$+\frac{1}{\Gamma(-\nu+n+1)}(t-a)^{\underline{-\nu+n}}y(a+n)$$

$$=\Delta_a^{\nu-n-1}y(t+n+1)+\sum_{i=1}^{n+1}\frac{(t-a)^{\underline{-\nu-1+i}}}{\Gamma(-\nu+i)}y(a-1+i).$$

Thus, (7.1) holds for all $k \in \mathbb{N}_1^{m-1}$.

The following Lemma plays an important role in the fractional finite difference field (see [12]).

**Lemma 7.1.2** *Let $h : \mathbb{N}_0 \to \mathbb{R}$ be a mapping and $m-1 < \nu \le m$ for some natural number $m$. The general solution of the equation $\Delta_{\nu-m}^{\nu}x(t) = h(t)$ is given by*

$$x(t)=\sum_{i=1}^{m}c_i t^{\underline{\nu-i}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s), \tag{7.2}$$

*where $c_1, \cdots, c_m$ are arbitrary real constants.*

Let $(X, d)$ be a metric space. Denote by $P(X)$, $2^X$, $P_c(X)$, and $P_{cp}(X)$ the class of all subsets, the class of all nonempty subsets, the class of all closed subsets, and the class of all compact subsets of $X$, respectively. A mapping $Q : X \to 2^X$ is called a multifunction on $X$ and $u \in X$ is called a fixed point of $Q$ whenever $u \in Qu$. The (generalized) Pompeiu-Hausdorff metric $H_d$ on $P_c(X)$ is defined $H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B),\ \ \sup_{b\in B} d(A, b)\right\}$, where $d(A, b) = \inf_{a\in A} d(a, b)$ (see [8]). Let $\alpha : X \times X \to [0, \infty)$ be a map and $T : X \to 2^X$ a multifunction. We say that $X$ has the condition $(C_\alpha)$ whenever for each sequence $\{x_n\}$ in $X$ with $\alpha(x_n, x_{n+1}) \ge 1$ for all $n$ and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \ge 1$ for all $k$ [27]. Also the operator $T$ is called $\alpha$-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \ge 1$, we have $\alpha(y, z) \ge 1$ for all $z \in Ty$ [27]. Let $\Psi$ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ (for more details see [27]). In 2013, next result was proved by Mohammadi, Rezapour, and Shahzad (see [27]).

**Lemma 7.1.3** *Let $(X, d)$ be a complete metric space, $\alpha : X \times X \to [0, \infty)$ a map, $\psi \in \Psi$ a strictly increasing map, and $T : X \to CB(X)$ an $\alpha$-admissible multifunction such that $\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, and there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. If $X$ has the condition $(C_\alpha)$, then $T$ has a fixed point.*

## 7.2 Main Result

Now, we consider the fractional finite difference inclusion

$$\begin{aligned} &\Delta^{\nu}_{\nu-2}x(t) + \Delta^{\nu-1}_{\nu-2}x(t) + \Delta^{\nu-2}_{\nu-2}x(t+1) \\ &\quad \in F(t, x(t), \Delta x(t), \Delta^2 x(t), \Delta^{\mu} x(t), \Delta^{\gamma} x(t)) \end{aligned} \tag{7.3}$$

with the boundary conditions $x(\nu) = 0$ and $x(\nu + b + 2) = 0$, where $1 < \gamma \leq 2$, $0 < \mu \leq 1$, $3 < \nu \leq 4$, and $F : \mathbb{N}_{\nu}^{b+\nu+2} \times \mathbb{R}^5 \to 2^{\mathbb{R}}$ is a compact valued multifunction. First, we prove the following key result.

**Lemma 7.2.1** *Let $y : \mathbb{N}_0^b \to \mathbb{R}$ be a map and $3 < \nu \leq 4$. Then $x_0$ is a solution for the fractional finite difference equation*

$$\Delta^{\nu}_{\nu-2}x(t) + \Delta^{\nu-1}_{\nu-2}x(t) + \Delta^{\nu-2}_{\nu-2}x(t+1) = y(t) \tag{7.4}$$

*with the boundary conditions $x(\nu) = 0$ and $x(\nu + b + 2) = 0$ if and only if $x_0$ is a solution for the fractional sum equation*

$$x(t) = \sum_{s=0}^{b} G(t, s)\left[y(s) - \frac{(s-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}x(\nu-2) - \frac{(s-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}x(\nu-1)\right],$$

*where* $G(t, s) = \frac{1}{\Gamma(\nu)}\left[-\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}}(\nu+b-\sigma(s))^{\underline{\nu-1}} + (t-2-\sigma(s))^{\underline{\nu-1}}\right]$ *whenever* $0 \leq s \leq t - \nu - 2 \leq b$ *and* $G(t, s) = \frac{1}{\Gamma(\nu)}\left[-\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}}(\nu+b-\sigma(s))^{\underline{\nu-1}}\right]$ *whenever* $0 \leq t - \nu - 2 < s \leq b$.

*Proof* By using Lemma 7.1.1, the problem (7.4) is equivalent to the problem

$$\Delta^{\nu-2}_{\nu-2}x(t+2) + \frac{(t-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}x(\nu-2) + \frac{(t-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}x(\nu-1) = y(t)$$

and so is equivalent to the problem

$$\Delta^{\nu-2}_{\nu-2}x(t+2) = g(t), \tag{7.5}$$

where $g(t) = y(t) - \frac{(t-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}x(\nu-2) - \frac{(t-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}x(\nu-1)$. Thus, $x_0$ is a solution for the problem (7.4) if and only if $x_0$ is a solution for the problem (7.5). Let $x_0$ be a solution for the fractional finite difference equation $\Delta_{\nu-2}^{\nu-2}x(t+2) = g(t)$ with the boundary conditions $x(\nu) = 0$ and $x(\nu+b+2) = 0$. By using Lemma 7.1.2, we get

$$x_0(t+2) = c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} g(s).$$

By using the boundary condition $x_0(\nu) = 0$, we have

$$c_1(\nu-2)^{\underline{\nu-1}} + c_2(\nu-2)^{\underline{\nu-2}} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{-2} ((\nu-2) - \sigma(s))^{\underline{\nu-1}} g(s) = 0$$

and so $c_2 = 0$. Since $x_0(\nu+b+2) = 0$, we get

$$c_1 = -\frac{1}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-\sigma(s))^{\underline{\nu-1}} g(s).$$

Hence,

$$\begin{aligned} x_0(t) &= -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-\sigma(s))^{\underline{\nu-1}} g(s) \\ &\quad + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-2-\nu} (t-2-\sigma(s))^{\underline{\nu-1}} g(s) \\ &= \sum_{s=0}^{b} G(t,s) \Bigg[ y(s) - \frac{(s-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)} x(\nu-2) \\ &\quad - \frac{(s-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)} x(\nu-1) \Bigg]. \end{aligned}$$

Now let $x_0$ be a solution for the equation

$$x(t) = \sum_{s=0}^{b} G(t,s) \left[ y(s) - \frac{(s-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)} x(\nu-2) - \frac{(s-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)} x(\nu-1) \right].$$

Then,

$$x_0(t) = -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-\sigma(s))^{\underline{\nu-1}}g(s)$$

$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-2-\nu}(t-2-\sigma(s))^{\underline{\nu-1}}g(s).$$

Since $(\nu-2)^{\underline{\nu-1}} = 0$ and $\sum_{s=0}^{-2}(\nu-2-\sigma(s))^{\underline{\nu-1}}g(s) = 0$, we obtain $x_0(\nu) = 0$. By using a similar calculation, one can show that $x_0(\nu+b+2) = 0$. On the other hand, it is easy to see that

$$x_0(t+2) = c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}g(s)$$

is a solution for the equation $\Delta_{\nu-2}^{\nu-2}x(t+2) = g(t)$. This implies that

$$\Delta_{\nu-2}^{\nu}x_0(t) + \Delta_{\nu-2}^{\nu-1}x_0(t) + \Delta_{\nu-2}^{\nu-2}x_0(t+1) = y(t).$$

This completes the proof.

A function $x : \mathbb{N}_\nu^{b+\nu+2} \to \mathbb{R}$ is a solution of the problem (7.3) whenever it satisfies the boundary conditions and there exists a function $y : \mathbb{N}_0^b \to \mathbb{R}$ such that

$$y(t) \in F\Big(t, x(t), \Delta x(t), \Delta^2 x(t), \Delta^\mu x(t), \Delta^\gamma x(t)\Big)$$

for all $t \in \mathbb{N}_0^{b+1}$ and

$$x(t) = -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-\sigma(s))^{\underline{\nu-1}}g(s)$$

$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-2-\nu}(t-2-\sigma(s))^{\underline{\nu-1}}g(s).$$

Here, $g(s) = y(s) - \frac{(s-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}x(\nu-2) - \frac{(s-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}x(\nu-1)$ (see [15]). Let $\mathscr{X}$ be the set of all functions $x : \mathbb{N}_\nu^{b+\nu+2} \to \mathbb{R}$ endowed with the norm

$$\|x\| = \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|x(t)| + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|\Delta x(t)| + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|\Delta^2 x(t)|$$

$$+\max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|\Delta^\mu x(t)| + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|\Delta^\gamma x(t)|.$$

We show that $(\mathscr{X}, \|.\|)$ is a Banach space. Let $\{x_n\}$ be a Cauchy sequence in $\mathscr{X}$ and $\epsilon > 0$ be given. Choose a natural number $N$ such that $\|x_n - x_m\| < \epsilon$ for all $m, n > N$. Thus, we get $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |x_n(t)-x_m(t)| < \epsilon$, $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |\Delta x_n(t)-\Delta x_m(t)| < \epsilon$, $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |\Delta^2 x_n(t) - \Delta^2 x_m(t)| < \epsilon$, $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |\Delta^\mu x_n(t) - \Delta^\mu x_m(t)| < \epsilon$ and $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |\Delta^\gamma x_n(t) - \Delta^\gamma x_m(t)| < \epsilon$. Since $\mathbb{R}$ is complete, there are real numbers $x(t)$, $z(t)$, $w(t)$, $p(t)$ and $q(t)$ such that $x_n(t) \to x(t)$, $\Delta x_n(t) \to z(t)$, $\Delta^2 x_n(t) \to w(t)$, $\Delta^\mu x_n(t) \to p(t)$ and $\Delta^\gamma x_n(t) \to q(t)$ for all $t \in \mathbb{N}_\nu^{b+\nu+2}$. Note that $\Delta x_n(t) = x_n(t+1)-x_n(t)$ and so $\Delta x(t) = x(t+1)-x(t) = z(t)$. Similarly, we get $\Delta^2 x(t) = w(t)$. Also, we have $\Delta^\mu x_n(t) = \frac{1}{\Gamma(-\mu)} \sum_{s=0}^{t+\mu} (t - \sigma(s))^{\underline{-\mu-1}} x_n(s)$. Since $x_n(s) \to x(s)$, we get $\Delta^\mu x(t) = p(t)$. Similarly, we have $\Delta^\gamma x(t) = q(t)$. This implies that there exists a natural number $M$ such that $|x_n(t) - x(t)| < \frac{\epsilon}{5}$, $|\Delta x_n(t) - \Delta x(t)| < \frac{\epsilon}{5}$, $|\Delta^2 x_n(t) - \Delta^2 x(t)| < \frac{\epsilon}{5}$, $|\Delta^\mu x_n(t) - \Delta^\mu x(t)| < \frac{\epsilon}{5}$ and $|\Delta^\gamma x_n(t) - \Delta^\gamma x(t)| < \frac{\epsilon}{5}$ for all $t \in \mathbb{N}_\nu^{b+\nu+2}$ and $n > M$. Thus,

$$\begin{aligned} \|x_n - x\| = & \max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |x_n(t) - x(t)| + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |\Delta x_n(t) - \Delta x(t)| \\ & + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |\Delta^2 x_n(t) - \Delta^2 x(t)| + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |\Delta^\mu x_n(t) - \Delta^\mu x(t)| \\ & + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}} |\Delta^\gamma x_n(t) - \Delta^\gamma x(t)| < \epsilon \end{aligned}$$

for all $n > M$. This shows that $\mathscr{X}$ is a Banach space. Let $x \in \mathscr{X}$. Define the set of selections of $F$ by

$$\begin{aligned} S_{F,x} = \Big\{ & y : \mathbb{N}_0^b \to \mathbb{R} : \ y(t) \\ & \in F\Big(t, x(t), \Delta x(t), \Delta^2 x(t), \Delta^\mu x(t), \Delta^\gamma x(t)\Big) \text{ for all } t \in \mathbb{N}_0^b \Big\}. \end{aligned}$$

Since $F\Big(t, x(t), \Delta x(t), \Delta^2 x(t), \Delta^\mu x(t), \Delta^\gamma x(t)\Big) \neq \emptyset$, by using the selection axiom we get $S_{F,x}$ is nonempty.

**Theorem 7.2.2** *Suppose that $\psi \in \Psi$ and $F : \mathbb{N}_\nu^{b+\nu+2} \times \mathbb{R}^5 \to P_{cp}(\mathbb{R})$ is a multifunction such that $H_d\Big(F(t, x_1, x_2, x_3, x_4, x_5), F(t, z_1, z_2, z_3, z_4, z_5)\Big) \leq \psi\Big(\sum_{i=1}^5 |x_i - z_i|\Big)$ for all $t \in \mathbb{N}_\nu^{b+\nu+2}$ and $x_1, \cdots, x_5, z_1, \cdots, z_5 \in \mathbb{R}$. Then the fractional finite difference inclusion (7.3) has a solution.*

*Proof* We know that $S_{F,x}$ is nonempty for all $x \in \mathscr{X}$. Let $y \in S_{F,x}$ and

$$h(t) = -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-\sigma(s))^{\underline{\nu-1}} g(s)$$

$$+\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-2-\nu} (t-2-\sigma(s))^{\underline{\nu-1}} g(s)$$

for all $t \in \mathbb{N}_{\nu}^{\nu+b+2}$, where $g(s) = y(s) - \frac{(s-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)} x(\nu-2) - \frac{(s-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)} x(\nu-1)$. Thus, $h \in \mathscr{X}$ and so

$$\left\{h \in \mathscr{X} : \text{ there exists } y \in S_{F,x} \text{ such that } h(t) = w(t) \text{ for all } t \in \mathbb{N}_{\nu}^{\nu+b+2}\right\}$$

is nonempty, where

$$w(t) = -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-\sigma(s))^{\underline{\nu-1}} g(s)$$

$$+\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-2-\nu} (t-2-\sigma(s))^{\underline{\nu-1}} g(s).$$

Now, consider the operator $\mathscr{T} : \mathscr{X} \to 2^{\mathscr{X}}$ defined by

$$\mathscr{T}(x) = \left\{h \in \mathscr{X} : \text{ there exists } y \in S_{F,x} \text{ such that } h(t) = w(t) \text{ for all } t \in \mathbb{N}_{\nu}^{\nu+b+2}\right\}.$$

First, we show that $\mathscr{T}(x)$ is a closed subset of $\mathscr{X}$ for all $x \in \mathscr{X}$. Let $x \in \mathscr{X}$ and $\{u_n\}_{n\geq 1}$ be a sequence in $\mathscr{T}(x)$ with $u_n \to u$. For each $n$, choose $y_n \in S_{F,x}$ such that

$$u_n(t) = -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-\sigma(s))^{\underline{\nu-1}} g_n(s)$$

$$+\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-2-\nu} (t-2-\sigma(s))^{\underline{\nu-1}} g_n(s)$$

for all $t \in \mathbb{N}_{\nu}^{b+\nu+2}$ and $n \geq 1$, where

$$g_n(s) = y_n(s) - \frac{(s-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)} x(\nu-2) - \frac{(s-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)} x(\nu-1).$$

Since $F$ has compact values, $\{y_n\}_{n\geq 1}$ has a subsequence which converges to some $y : \mathbb{N}_0^b \to \mathbb{R}$. Denote the subsequence again by $\{y_n\}_{n\geq 1}$. It is easy to check that $y \in S_{F,x}$ and

$$u_n(t) \to u(t) = -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-\sigma(s))^{\underline{\nu-1}}g(s)$$
$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-2-\nu}(t-2-\sigma(s))^{\underline{\nu-1}}g(s).$$

This implies that $u \in \mathscr{T}(x)$ and so the multifunction $\mathscr{T}$ has closed values. Let $x, z \in \mathscr{X}$. Since $\mathscr{T}(x)$ is nonempty, for each $h_1 \in \mathscr{T}(x)$ there exists $y_1 \in S_{F,x}$ such that

$$h_1(t) = -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-\sigma(s))^{\underline{\nu-1}}g_1(s)$$
$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-2-\nu}(t-2-\sigma(s))^{\underline{\nu-1}}g_1(s)$$

for all $t \in \mathbb{N}_\nu^{\nu+b+2}$, where $g_1(s) = y_1(s) - \frac{(s-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}x(\nu-2) - \frac{(s-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}x(\nu-1)$. Similarly, for each $h_2 \in \mathscr{T}(z)$ there exists $y_2 \in S_{F,z}$ such that

$$h_2(t) = -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-\sigma(s))^{\underline{\nu-1}}g_2(s)$$
$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-2-\nu}(t-2-\sigma(s))^{\underline{\nu-1}}g_2(s)$$

for all $t \in \mathbb{N}_\nu^{\nu+b+2}$, where $g_2(s) = y_2(s) - \frac{(s-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}z(\nu-2) - \frac{(s-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}z(\nu-1)$. Since

$$H_d\Big(F\big(t, x(t), \Delta x(t), \Delta^2 x(t), \Delta^\mu x(t), \Delta^\gamma x(t)\big),$$
$$F\big(t, z(t), \Delta z(t), \Delta^2 z(t), \Delta^\mu z(t), \Delta^\gamma z(t)\big)\Big)$$
$$\leq \psi\Big(|x(t)-z(t)| + |\Delta x(t) - \Delta z(t)| + |\Delta^2 x(t) - \Delta^2 z(t)|$$
$$+ |\Delta^\mu x(t) - \Delta^\mu z(t)| + |\Delta^\gamma x(t) - \Delta^\gamma z(t)|\Big)$$

for all $x, z \in \mathscr{X}$, we get

$$|y_1(t) - y_2(t)| \le \psi\Big(|x(t) - z(t)| + |\Delta x(t) - \Delta z(t)| + |\Delta^2 x(t) - \Delta^2 z(t)| + |\Delta^{\mu} x(t) - \Delta^{\mu} z(t)| + |\Delta^{\gamma} x(t) - \Delta^{\gamma} z(t)|\Big).$$

for all $t \in \mathbb{N}_0^b$. Now, put

$$\Lambda_i = \max_{t \in \mathbb{N}_{\nu}^{b+\nu+2}} \left| \sum_{s=0}^{b} \left( \frac{(t-2-\sigma(s))^{\underline{\nu-i}}}{\Gamma(\nu+1-i)} - \frac{(t-2)^{\underline{\nu-i}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu+1-i)} \right) \right|$$

for $i=1, 2, 3$, $\Lambda_4 = \max_{t \in \mathbb{N}_{\nu}^{b+\nu+2}} \left| \sum_{s=0}^{b} \left( \frac{(t-2-\sigma(s))^{\underline{\nu-1-\mu}}}{\Gamma(\nu-\mu)} - \frac{(t-2)^{\underline{\nu-1-\mu}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu-\mu)} \right) \right|$,

$$\Lambda_5 = \max_{t \in \mathbb{N}_{\nu}^{b+\nu+2}} \left| \sum_{s=0}^{b} \left( \frac{(t-2-\sigma(s))^{\underline{\nu-1-\gamma}}}{\Gamma(\nu-\gamma)} - \frac{(t-2)^{\underline{\nu-1-\gamma}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu-\gamma)} \right) \right|$$

and

$$\Omega = \max_{t \in \mathbb{N}_0^b} \left| \frac{(t-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}(x(\nu-2) - z(\nu-2)) + \frac{(t-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}(x(\nu-1) - z(\nu-1)) \right|.$$

Thus,

$$\begin{aligned} &|h_1(t) - h_2(t)| \\ &= \Bigg| -\frac{(t-2)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-\sigma(s))^{\underline{\nu-1}}(g_1(s) - g_2(s)) \\ &\quad + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-2-\nu} (t-2-\sigma(s))^{\underline{\nu-1}}(g_1(s) - g_2(s)) \Bigg| \\ &\le \left| \sum_{s=0}^{b} \left( \frac{(t-2-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} - \frac{(t-2)^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \right)(g_1(s) - g_2(s)) \right| \\ &\le \max_{t \in \mathbb{N}_0^b} |g_1(t) - g_2(t)| \end{aligned}$$

$$\times \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}\left|\sum_{s=0}^{b}\left(\frac{(t-2-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}-\frac{(t-2)^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\right)\right|$$

$$=\max_{t\in\mathbb{N}_0^b}\left|(y_1(t)-y_2(t))-\frac{(t-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}(x(\nu-2)-z(\nu-2))\right.$$

$$\left.-\frac{(t-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}(x(\nu-1)-z(\nu-1))\right|\times\Lambda_1$$

$$\le\max_{t\in\mathbb{N}_0^b}\left|(y_1(t)-y_2(t))\right|\Lambda_1+\max_{t\in\mathbb{N}_0^b}\left|\frac{(t-\nu+2)^{\underline{-\nu}}}{\Gamma(-\nu+1)}(x(\nu-2)-z(\nu-2))\right.$$

$$\left.+\frac{(t-\nu+2)^{\underline{-\nu+1}}}{\Gamma(-\nu+2)}(x(\nu-1)-z(\nu-1))\right|\times\Lambda_1$$

$$=\left[\max_{t\in\mathbb{N}_0^b}\left|(y_1(t)-y_2(t))\right|+\Omega\right]\Lambda_1$$

$$\le\left[\psi\Big(|x(t)-z(t)|+|\Delta x(t)-\Delta z(t)|+|\Delta^2x(t)-\Delta^2z(t)|\right.$$

$$\left.+|\Delta^\mu x(t)-\Delta^\mu z(t)|+|\Delta^\gamma x(t)-\Delta^\gamma z(t)|\Big)+\Omega\right]\Lambda_1$$

for all $t\in\mathbb{N}_0^b$. Since

$$\Delta h_1(t)=-\frac{(t-2)^{\underline{\nu-2}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu-1)}\sum_{s=0}^{b}(\nu+b-\sigma(s))^{\underline{\nu-1}}g_1(s)$$

$$+\frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{t-3-\nu}(t-2-\sigma(s))^{\underline{\nu-2}}g_1(s),$$

by a similar calculation we get

$$|\Delta h_1(t)-\Delta h_2(t)|\le\left[\max_{t\in\mathbb{N}_0^b}\left|(y_1(t)-y_2(t))\right|+\Omega\right]\Lambda_2$$

$$\le\left[\psi\Big(|x(t)-z(t)|+|\Delta x(t)-\Delta z(t)|+|\Delta^2x(t)-\Delta^2z(t)|\right.$$

$$\left.+|\Delta^\mu x(t)-\Delta^\mu z(t)|+|\Delta^\gamma x(t)-\Delta^\gamma z(t)|\Big)+\Omega\right]\Lambda_2.$$

Also, we have

$$
\begin{aligned}
|\Delta^2 h_1(t) - \Delta^2 h_2(t)| &\leq \left[ \max_{t\in\mathbb{N}_0^b} \left|(y_1(t) - y_2(t))\right| + \Omega \right] \Lambda_3 \\
&\leq \Big[ \psi\Big( |x(t) - z(t)| + |\Delta x(t) - \Delta z(t)| + |\Delta^2 x(t) - \Delta^2 z(t)| \\
&\quad + |\Delta^\mu x(t) - \Delta^\mu z(t)| + |\Delta^\gamma x(t) - \Delta^\gamma z(t)| \Big) + \Omega \Big] \Lambda_3.
\end{aligned}
$$

Since

$$
\begin{aligned}
\Delta^\mu h_1(t) &= \frac{-(t-2)^{\underline{\nu-\mu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu-\mu)} \sum_{s=0}^{b} (\nu + b - \sigma(s))^{\underline{\nu-1}} g_1(s) \\
&\quad + \sum_{s=0}^{t-2-\nu-\mu} \frac{(t-2-\sigma(s))^{\underline{\nu-1-\mu}}}{\Gamma(\nu-\mu)} g_1(s),
\end{aligned}
$$

by using a simple calculation we obtain

$$
\begin{aligned}
|\Delta^\mu h_1(t) - \Delta^\mu h_2(t)| &\leq \left[ \max_{t\in\mathbb{N}_0^b} \left|(y_1(t) - y_2(t))\right| + \Omega \right] \Lambda_4 \\
&\leq \Big[ \psi\Big( |x(t) - z(t)| + |\Delta x(t) - \Delta z(t)| + |\Delta^2 x(t) - \Delta^2 z(t)| \\
&\quad + |\Delta^\mu x(t) - \Delta^\mu z(t)| + |\Delta^\gamma x(t) - \Delta^\gamma z(t)| \Big) + \Omega \Big] \Lambda_4
\end{aligned}
$$

and

$$
\begin{aligned}
|\Delta^\gamma h_1(t) - \Delta^\gamma h_2(t)| &\leq \left[ \max_{t\in\mathbb{N}_0^b} \left|(y_1(t) - y_2(t))\right| + \Omega \right] \Lambda_5 \\
&\leq \Big[ \psi\Big( |x(t) - z(t)| + |\Delta x(t) - \Delta z(t)| + |\Delta^2 x(t) - \Delta^2 z(t)| \\
&\quad + |\Delta^\mu x(t) - \Delta^\mu z(t)| + |\Delta^\gamma x(t) - \Delta^\gamma z(t)| \Big) + \Omega \Big] \Lambda_5.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|h_1 - h_2\| &= \max_{t\in\mathbb{N}_\nu^{b+2+\nu}} |h_1(t) - h_2(t)| + \max_{t\in\mathbb{N}_\nu^{b+2+\nu}} |\Delta h_1(t) - \Delta h_2(t)| \\
&\quad + \max_{t\in\mathbb{N}_\nu^{b+2+\nu}} |\Delta^2 h_1(t) - \Delta^2 h_2(t)|
\end{aligned}
$$

$$+\max_{t\in\mathbb{N}_\nu^{b+2+\nu}}|\Delta^\mu h_1(t)-\Delta^\mu h_2(t)|+\max_{t\in\mathbb{N}_\nu^{b+2+\nu}}|\Delta^\gamma h_1(t)-\Delta^\gamma h_2(t)|$$
$$\le\Big(\psi(|x(t)-z(t)|+|\Delta x(t)-\Delta z(t)|+|\Delta^2 x(t)-\Delta^2 z(t)|$$
$$+|\Delta^\mu x(t)-\Delta^\mu z(t)|+|\Delta^\gamma x(t)-\Delta^\gamma z(t)|)+\Omega\Big)$$
$$\times(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5)$$
$$\le(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5)(\psi(\|x-z\|)+\Omega),$$

for all $x, z\in\mathscr{X}$, $h_1\in\mathscr{T}(x)$ and $h_2\in\mathscr{T}(z)$. This implies that

$$H_d(\mathscr{T}(x),\mathscr{T}(z))\le(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5)(\psi(\|x-z\|)+\Omega)$$

for all $x, z\in\mathscr{X}$. Note that $\Omega$ is a non-negative real number. For each $x, z\in\mathscr{X}$, choose $\lambda_{x,z}\in[1,+\infty)$ such that $\Omega\le\lambda_{x,z}\psi(\|x-z\|)$. Thus,

$$H_d(\mathscr{T}(x),\mathscr{T}(z))\le(\lambda_{x,z}+1)(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5)\psi(\|x-z\|)$$

for all $x, z\in\mathscr{X}$. Now, define the map $\alpha:\mathscr{X}\times\mathscr{X}\to[0,\infty)$ by $\alpha(x,z)=1$ whenever $\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5\le\frac{1}{1+\lambda_{x,z}}$ and $\alpha(x,z)=\frac{1}{(\lambda_{x,z}+1)(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5)}$ otherwise. Thus, we get $\alpha(x,z)H_d(\mathscr{T}(x),\mathscr{T}(z))\le\psi(\|x-z\|)$ for all $x, z\in\mathscr{X}$. Since $\alpha(x,z)\le 1$ for all $x, z\in\mathscr{X}$, we get that $\mathscr{X}$ has the condition $(C_\alpha)$ and $\mathscr{T}$ is $\alpha$-admissible. By using Theorem 7.1.3, there exists $x^*\in\mathscr{X}$ such that $x^*\in\mathscr{T}(x^*)$. It is easy to see that $x^*$ is a solution for the problem (7.3).

*Example 7.2.1* Consider the fractional finite difference inclusion

$$\sum_{j=0}^{2}\Delta_{\sqrt{10}-2}^{\sqrt{10}-j}x(t-1+j!)$$
$$\in\left[0, 1+\frac{\sin x(t)}{e^t}+\frac{|\Delta x(t)|}{4t}+\frac{|\Delta^{0.5}x(t)|+\sqrt{3}|\Delta^{1.5}x(t)|}{t^4}+\frac{|\Delta^2 x(t)|}{\cosh(3t)}\right]$$

with the boundary value conditions $x(\sqrt{10})=0$ and $x(6+\sqrt{10})=0$. Put $\nu=\sqrt{10}$, $\mu=0.5$, $\gamma=1.5$, $b=4$ and $F(t,x_1,x_2,x_3,x_4,x_5)=\left[0\ ,\ 1+\frac{\sin x_1}{e^t}+\frac{|x_2|}{4t}+\frac{|x_4|+\sqrt{3}|x_5|}{t^4}+\frac{|x_3|}{\cosh(3t)}\right]$ for all $x_1,x_2,x_3,x_4,x_5\in\mathbb{R}$. Note that

$$1+\frac{\sin x_1}{e^t}+\frac{|x_2|}{4t}+\frac{|x_4|+\sqrt{3}|x_5|}{t^4}+\frac{|x_3|}{\cosh(3t)}>0,$$

$e^t \geq 10$, $4t \geq 10$, $\frac{t^4}{\sqrt{3}} \geq 10$ and $\cosh(3t) \geq 10$ for all $t \in \mathbb{N}_{\sqrt{10}}^{6+\sqrt{10}}$ and $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$. Now, put $\psi(z) = \frac{1}{10}z$ for all $z \geq 0$. Note that $\psi \in \Psi$ and

$$
\begin{aligned}
&H_d\Big(F(t, x_1, x_2, x_3, x_4, x_5), F(t, z_1, z_2, z_3, z_4, z_5)\Big) \\
&\quad \leq \Bigg| \frac{\sin x_1}{e^t} + \frac{|x_2|}{4t} + \frac{|x_4| + \sqrt{3}|x_5|}{t^4} + \frac{|x_3|}{\cosh(3t)} \\
&\quad\quad - \frac{\sin z_1}{e^t} - \frac{|z_2|}{4t} - \frac{|z_4| + \sqrt{3}|z_5|}{t^4} - \frac{|z_3|}{\cosh(3t)} \Bigg| \\
&\quad \leq \frac{|x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3| + |x_4 - z_4| + |x_5 - z_5|}{10} = \psi(\sum_{i=1}^{5} |x_i - z_i|)
\end{aligned}
$$

for all $t \in \mathbb{N}_{\sqrt{10}}^{6+\sqrt{10}}$ and $x_1, x_2, x_3, z_1, z_2, z_3 \in \mathbb{R}$. By using Theorem 7.2.2, this problem has at least one solution. Note that this problem is a special case for the problem (7.3).

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# Chapter 8
# Comparison on Solving a Class of Nonlinear Systems of Partial Differential Equations and Multiple Solutions of Second Order Differential Equations

**Ali Akgül, Esra Karatas Akgül, Yasir Khan, and Dumitru Baleanu**

## 8.1 Introduction

Many problems in science and engineering such as problems posed in solid state physics, fluid mechanics, chemical physics, plasma physics, optics, etc. are modelled as nonlinear partial differential equations (PDEs) or systems of nonlinear PDEs. Nonlinear systems of PDEs have taken much interest in working evolution equations. Many researchers have investigated the analytical and approximate solutions of nonlinear systems of PDEs by utilizing different techniques [7].

In this paper, a general technique is shown in the reproducing kernel space for searching the following class of nonlinear systems of PDEs:

$$
\begin{aligned}
A_1(f_1(\eta,\tau)) &= P_1(\eta,\tau,F(\eta,\tau)) + M_1(\eta,\tau),\\
&\dots \qquad\qquad\qquad\qquad\qquad\qquad ,\\
A_k(f_k(\eta,\tau)) &= P_k(\eta,\tau,F(\eta,\tau)) + M_k(\eta,\tau),\\
(\eta,\tau) \in \Omega &= [0,1]\times[0,1]
\end{aligned}
$$

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with the initial and boundary conditions

$$f_i(\eta,\tau)=0, \quad \text{for } \tau\le 0, \tag{8.1}$$

$$f_i(0,\tau)=0, \quad f_i(1,\tau)=0, \quad \text{for } \tau>0. \tag{8.2}$$

where $A_i$ and $P_i$ are linear and nonlinear differential operators for $i=1,2,\ldots,k$. $M_i(\eta,\tau)$ are given functions and $F(\eta,\tau)=[f_1(\eta,\tau), f_2(\eta,\tau),\ldots, f_k(\eta,\tau)]^T$ is an unknown vector function to be determined. Suppose this equation is of one-order derivative in $\tau$, and has a unique solution. We only take into consideration the homogeneous initial and boundary conditions, because the non-homogeneous initial and boundary conditions can be easily transformed to the homogeneous ones. The reproducing method has been implemented to several nonlinear problems [1]. For more details of this method, see [3, 6, 8].

We take into consideration the boundary value problems:

$$\begin{cases} u''(x)=\lambda\exp(\mu u(x)), \quad 0\le x\le 1,\\ u(0)=u(1)=0, \end{cases} \tag{8.3}$$

and

$$\big(\exp(x)v'(x)\big)'+|\ln x|=0, \forall x\in(0,\infty), x\ne 1, \Delta v'\,|_{x_1=1}=v^2(1),$$
$$v(0)=0, \quad v(\infty)=0. \tag{8.4}$$

The problem (8.3) shows up in implementations containing the diffusion of heat produced by positive temperature-dependent sources. If $\mu=1$, it springs in the analysis of Joule losses in electrically conducting solids, with $\lambda$ returning the square of constant current and $\exp(u)$ the temperature-dependent resistance, or frictional heating with $\lambda$ projecting the square of the constant shear stress and $\exp(u)$ the temperature-dependent fluidity. In particular if $\lambda=1$ and $\mu=-1$ the boundary value problem (8.3) has two solutions $u_1(x)$ and $u_2(x)$. Solution $u_1(x)$ drops below up to $-0.14050941\ldots$ and solution $u_2(x)$ up to $-4.0916146\ldots$.

Boundary value problem (8.4) has at least two positive solutions $v_1$, $v_2$ satisfying $0\le\|v_1\|\le\frac{1}{2}\le\|v_2\|$.

This work is ordered as follows. Section 8.2 presents some useful reproducing kernel functions. The representation of solutions and a related linear operator are given in Sect. 8.3. This section shows the main results. Examples are shown in Sect. 8.4. The final section contains some conclusions.

## 8.2 Preliminaries

**Definition 1** We present $G_2^1[0,1]$ by

$$G_2^1[0,1] = \{f \in AC[0,1] : \ f' \in L^2[0,1]\}.$$

The inner product and the norm in $G_2^1[0,1]$ are defined by

$$\langle f, g\rangle_{G_2^1} = f(0)g(0) + \int_0^1 f'(\eta)g'(\eta)\mathrm{d}\eta, \quad f, g \in G_2^1[0,1]$$

and

$$\|f\|_{G_2^1} = \sqrt{\langle f, f\rangle_{G_2^1}}, \quad f \in G_2^1[0,1].$$

**Theorem 1** *Reproducing kernel function* $\widetilde{Q}_\tau$ *of* $G_2^1[0,1]$ *is obtained as:*

$$\widetilde{Q}_\tau(\eta) = \sum_{i=1}^{2} c_i(\tau)\eta^{i-1}, \quad 0 \le \eta \le \tau \le 1, \sum_{i=1}^{2} d_i(\tau)\eta^{i-1}, \quad 0 \le \tau < \eta \le 1.$$

*Proof* By Definition 1, we have

$$\left\langle u, \widetilde{Q}_\tau \right\rangle_{G_2^1} = u(0)\widetilde{Q}_\tau(0) + \int_0^1 u'(\eta)\widetilde{Q}'_\tau(\eta)\mathrm{d}\eta, \tag{8.5}$$

We get

$$\left\langle u, \widetilde{Q}_\tau \right\rangle_{G_2^1} = u(0)\widetilde{Q}_\tau(0) + u(1)\widetilde{Q}'_\tau(1) - u(0)\widetilde{Q'}_\tau(0) - \int_0^1 u(\eta)\widetilde{Q}''_\tau(\eta)\mathrm{d}\eta,$$

by integrating by parts. Note that property of the reproducing kernel is

$$\langle u(\eta), \widetilde{Q}'_\tau(\eta)\rangle_{G_2^1} = u(\tau). \tag{8.6}$$

If

$$\begin{cases} \widetilde{Q}_\tau(0) - \widetilde{Q}'_\tau(0) = 0, \\ \widetilde{Q}'_\tau(1) = 0, \end{cases} \tag{8.7}$$

then, we get

$$-\widetilde{Q}''_\tau(\eta) = \delta(\eta - \tau).$$

If $\eta \neq \tau$, then we obtain

$$\widetilde{Q}''_\tau(\eta) = 0.$$

Thus, we have

$$\widetilde{Q}_\tau(\eta) = \begin{cases} c_1(\tau) + c_2(\tau)\eta, & 0 \leq \eta \leq \tau \leq 1, \\ \\ d_1(\tau) + d_2(\tau)\eta, & 0 \leq \tau < \eta \leq 1. \end{cases} \tag{8.8}$$

Since

$$-\widetilde{Q}''_\tau(\eta) = \delta(\eta - \tau),$$

we get

$$\widetilde{Q}_{\tau^+}(\tau) = \widetilde{Q}_{\tau^-}(\tau) \tag{8.9}$$

and

$$\widetilde{Q}'_{\tau^+}(\tau) - \widetilde{Q}'_{\tau^-}(\tau) = -1. \tag{8.10}$$

The unknown coefficients $c_i(\tau)$ and $d_i(\tau)$ $(i = 1, 2)$ can be obtained. Thus $\widetilde{Q}_\tau$ is acquired as

$$\widetilde{Q}_\tau(\eta) = 1 + \eta, \quad 0 \leq \eta \leq \tau \leq 1,\ 1 + \tau, \quad 0 \leq \tau < \eta \leq 1.$$

**Definition 2** We present the space $H_2^2[0, 1]$ as:

$$H_2^2[0, 1] = \{f \in AC[0, 1] : \ f' \in AC[0, 1],\ f'' \in L^2[0, 1],\ f(0) = 0\}.$$

The inner product and the norm in $H_2^2[0, 1]$ are presented as:

$$\langle f, g\rangle_{H_2^2} = f(0)g(0) + f'(0)g'(0) + \int_0^1 f''(\eta)g''(\eta)\mathrm{d}\eta, \quad f, g \in H_2^2[0, 1]$$

and

$$\|f\|_{H_2^2} = \sqrt{\langle f, f\rangle_{H_2^2}}, \quad f \in H_2^2[0, 1].$$

**Theorem 2** *Reproducing kernel function* $\widetilde{T}_\tau$ *of* $H_2^2[0,1]$ *is obtained by:*

$$\widetilde{T}_\tau(\eta) = \sum_{i=1}^{4} c_i(\tau)\eta^{i-1}, \quad 0 \le \eta \le \tau \le 1, \ \sum_{i=1}^{4} d_i(\tau)\eta^{i-1}, \quad 0 \le \tau < \eta \le 1.$$

*Proof* By Definition 2, we have

$$\langle f, \widetilde{T}_\tau \rangle_{H_2^2} = f(0)\widetilde{T}_\tau(0) + f'(0)\widetilde{T}'_\tau(0) + \int_0^1 f''(\eta)\widetilde{T}''_\tau(\eta)\mathrm{d}\tau. \tag{8.11}$$

Integrating this equation by parts two times, we get

$$\begin{aligned}\langle f, \widetilde{T}_\tau \rangle_{H_2^2} &= f(0)\widetilde{T}_\tau(0) + f'(0)\widetilde{T}'_\tau(0) + f'(1)\widetilde{T}''_\tau(1) - f'(0)\widetilde{T}''_\tau(0) \\ &\quad - f(1)\widetilde{T}'''_\tau(1) + f(0)\widetilde{T}'''_\tau(0) + \int_0^1 f(\eta)\widetilde{T}^{(4)}_\tau(\eta)\mathrm{d}\eta.\end{aligned}$$

We have

$$\langle f(\eta), \widetilde{T}_\tau(\eta) \rangle_{H_2^2} = f(\tau) \tag{8.12}$$

by reproducing property. Since $\widetilde{T}_\tau \in H_2^2[0,1]$, we have

$$\widetilde{T}_\tau(0) = 0. \tag{8.13}$$

If

$$\begin{cases} \widetilde{T}'_\tau(0) - \widetilde{T}''_\tau(0) = 0, \\ \widetilde{T}''_\tau(1) = 0, \\ \widetilde{T}'''_\tau(1) = 0, \end{cases} \tag{8.14}$$

then, we get

$$\widetilde{T}^{(4)}_\tau(\eta) = \delta(\eta - \tau).$$

When $\eta \ne \tau$, we get

$$\widetilde{T}^{(4)}_\tau(\eta) = 0.$$

Thus

$$\widetilde{Q}_\tau(\eta) = c_1(\tau) + c_2(\tau)\eta + c_3(\tau)\eta^2 + +c_4(\tau)\eta^3, \quad 0 \le \eta \le \tau \le 1,$$

$$d_1(\tau) + d_2(\tau)\eta + d_3(\tau)\eta^2 + d_4(\tau)\eta^3, \quad 0 \le \tau < \eta \le 1.$$

Since

$$\widetilde{T}_{\tau}^{(4)}(\eta) = \delta(\eta - \tau),$$

we obtain

$$\widetilde{T}_{\tau^+}^{(k)}(\tau) = \widetilde{T}_{\tau^-}^{(k)}(\tau), \quad k = 0, 1, 2 \tag{8.15}$$

and

$$\widetilde{T}_{\tau^+}'''(\tau) - \widetilde{T}_{\tau^-}'''(\tau) = 1. \tag{8.16}$$

The unknown coefficients $c_i(\tau)$ and $d_i(\tau)$ $(i = 1, 2, 3, 4)$ can be obtained. Thus $\widetilde{T}_{\tau}$ is achieved as

$$\begin{aligned}\widetilde{T}_{\tau}(\eta) = \eta\tau + \frac{(\eta)(\tau)^2}{2} + \frac{(\tau-\eta)^3}{6} - \frac{(\tau)^3}{6}, \quad 0 \le \eta \le \tau \le 1,\\ \eta\tau + \frac{(\tau)(\eta)^2}{2} + \frac{(\eta-\tau)^3}{6} - \frac{(\tau)^3}{6}, \quad 0 \le \tau < \eta \le 1.\end{aligned}$$

**Definition 3** We give $W_2^3[0, 1]$ as:

$$\begin{aligned}W_2^3[0, 1] = \{f \in AC[0, 1] : \ f', f'' \in AC[0, 1], \ f^{(3)} \in L^2[0, 1],\\ f(0) = f(1) = 0\}.\end{aligned}$$

The inner product and the norm in $W_2^3[0, 1]$ are defined by

$$\langle f, g\rangle_{W_2^3} = \sum_{i=0}^{2} f^{(i)}(0)g^{(i)}(0) + \int_0^1 f^{(3)}(\eta)g^{(3)}(\eta)\mathrm{d}\eta, \quad f, g \in W_2^3[0, 1]$$

and

$$\|f\|_{W_2^3} = \sqrt{\langle f, f\rangle_{W_2^3}}, \quad f \in W_2^3[0, 1].$$

**Theorem 3** *Reproducing kernel function* $R_{\tau}$ *of* $W_2^3[0, 1]$ *is obtained as:*

$$R_{\tau}(\eta) = \begin{cases} \sum_{i=1}^{5} c_i(\tau)\eta^i, & 0 \le \eta \le \tau \le 1,\\ \sum_{i=0}^{5} d_i(\tau)\eta^i, & 0 \le \tau < \eta \le 1,\end{cases} \tag{8.17}$$

*where*

$$
\begin{aligned}
c_1(\tau) &= -\frac{1}{156}\tau^5 + \frac{5}{156}\tau^4 - \frac{5}{78}\tau^3 - \frac{5}{26}\tau^2 + \frac{3}{13}\tau,\\
c_2(\tau) &= -\frac{1}{624}\tau^5 + \frac{5}{624}\tau^4 - \frac{5}{312}\tau^3 + \frac{21}{104}\tau^2 - \frac{5}{26}\tau,\\
c_3(\tau) &= -\frac{1}{1872}\tau^5 + \frac{5}{1872}\tau^4 - \frac{5}{936}\tau^3 + \frac{7}{104}\tau^2 - \frac{5}{78}\tau,\\
c_4(\tau) &= \frac{1}{3744}\tau^5 - \frac{5}{3744}\tau^4 + \frac{5}{1872}\tau^3 + \frac{5}{624}\tau^2 - \frac{1}{104}\tau,\\
c_5(\tau) &= -\frac{1}{18720}\tau^5 + \frac{1}{3744}\tau^4 - \frac{1}{1872}\tau^3 - \frac{1}{624}\tau^2 - \frac{1}{156}\tau + \frac{1}{120},\\
d_0(\tau) &= \frac{1}{120}\tau^5,\\
d_1(\tau) &= -\frac{1}{156}\tau^5 - \frac{1}{104}\tau^4 - \frac{5}{78}\tau^3 - \frac{5}{26}\tau^2 + \frac{3}{13}\tau,\\
d_2(\tau) &= -\frac{1}{624}\tau^5 + \frac{5}{624}\tau^4 + \frac{7}{104}\tau^3 + \frac{21}{104}\tau^2 - \frac{5}{26}\tau,\\
d_3(\tau) &= -\frac{1}{1872}\tau^5 + \frac{5}{1872}\tau^4 - \frac{5}{936}\tau^3 - \frac{5}{312}\tau^2 - \frac{5}{78}\tau,\\
d_4(\tau) &= \frac{1}{3744}\tau^5 - \frac{5}{3744}\tau^4 + \frac{5}{1872}\tau^3 + \frac{5}{624}\tau^2 + \frac{5}{156}\tau,\\
d_5(\tau) &= -\frac{1}{18720}\tau^5 + \frac{1}{3744}\tau^4 - \frac{1}{1872}\tau^3 - \frac{1}{624}\tau^2 - \frac{1}{156}\tau.
\end{aligned}
$$

*Proof* Let $f \in W_2^3[0, 1]$ and $0 \le \tau \le 1$. Note that

$$
R'_\tau(\eta) = \begin{cases} \sum_{i=0}^{4}(i+1)c_{i+1}(\tau)\eta^i, & 0 \le \eta < \tau \le 1,\\[2ex] \sum_{i=0}^{4}(i+1)d_{i+1}(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases}
$$

$$
R''_\tau(\eta) = \begin{cases} \sum_{i=0}^{3}(i+1)(i+2)c_{i+2}(\tau)\eta^i, & 0 \le \eta < \tau \le 1,\\[2ex] \sum_{i=0}^{3}(i+1)(i+2)d_{i+2}(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases}
$$

$$
R^{(3)}_\tau(\eta) = \begin{cases} \sum_{i=0}^{2}(i+1)(i+2)(i+3)c_{i+3}(\tau)\eta^i, & 0 \le \eta < \tau \le 1,\\[2ex] \sum_{i=0}^{2}(i+1)(i+2)(i+3)d_{i+3}(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases}
$$

$$R_\tau^{(4)}(\eta) = \begin{cases} \sum_{i=0}^{1}(i+1)(i+2)(i+3)(i+4)c_{i+4}(\tau)\eta^i, & 0 \le \eta < \tau \le 1, \\ \\ \sum_{i=0}^{1}(i+1)(i+2)(i+3)(i+4)d_{i+4}(\tau)\eta^i, & 0 \le \tau < \eta \le 1, \end{cases}$$

and

$$R_\tau^{(5)}(\eta) = \begin{cases} 120c_5(\tau), & 0 \le \eta < \tau \le 1, \\ \\ 120d_5(\tau), & 0 \le \tau < \eta \le 1. \end{cases}$$

By Definition 3 and integrating by parts, we obtain

$$\begin{aligned} \langle f, R_\tau \rangle_{W_2^3} &= \sum_{i=0}^{2} f^{(i)}(0) R_\tau^{(i)}(0) + \int_0^1 f^{(3)}(\eta) R_\tau^{(3)}(\eta) \mathrm{d}\eta \\ &= f'(0)R_\tau'(0) + f''(0)R_\tau''(0) + f''(1)R_\tau^{(3)}(1) - f''(0)R_\tau^{(3)}(0) \\ &\quad - f'(1)R_\tau^{(4)}(1) + f'(0)R_\tau^{(4)}(0) + \int_0^1 f'(\eta)R_\tau^{(5)}(\eta)\mathrm{d}\eta \\ &= c_1(\tau)f'(0) + 2c_2(y)f''(0) \\ &\quad + 6(d_3(\tau) + 4d_4(\tau) + 10d_5(\tau))f''(1) - 6c_3(\tau)f''(0) \\ &\quad - 24(d_4(\tau) + 5d_5(\tau))f'(1) + 24c_4(\tau)f'(0) \\ &\quad + \int_0^\tau 120c_5(\tau)f'(\eta)\mathrm{d}\eta + \int_\tau^1 120d_5(\tau)f'(\eta)\mathrm{d}\tau \\ &= (c_1(\tau) + 24c_4(\tau))f'(0) + 2(c_2(\tau) - 3c_3(\tau))f''(0) \\ &\quad + 6(d_3(\tau) + 4d_4(\tau) + 10d_5(\tau))f''(1) - 24(d_4(\tau) + 5d_5(\tau))f'(1) \\ &\quad + 120(c_5(\tau) - d_5(\tau))f(\tau) \\ &= f(\tau). \end{aligned}$$

**Definition 4** We give the binary space $W(\Omega)$ as:

$$W(\Omega) = \Big\{ f : \frac{\partial^3 f}{\partial \eta^2 \partial t} \in CC(\Omega), \quad \frac{\partial^5 f}{\partial \eta^3 \partial t^2} \in L^2(\Omega), \\ f(\eta, 0) = f(0, t) = f(1, t) = 0 \Big\},$$

where $CC$ denotes the space of completely continuous functions. The inner product and the norm in $W(\Omega)$ are obtained as:

$$\langle f, g\rangle_W = \sum_{i=0}^{1}\int_0^1\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial \eta^i}u(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial \eta^i}g(0,t)\right]\mathrm{d}t$$
$$+\sum_{j=0}^{1}\left\langle\frac{\partial^j}{\partial t^j}f(\cdot,0), \frac{\partial^j}{\partial t^j}g(\cdot,0)\right\rangle_{W_2^3}$$
$$+\int_0^1\int_0^1\left[\frac{\partial^3}{\partial \eta^3}\frac{\partial^2}{\partial t^2}f(\eta,t)\frac{\partial^3}{\partial \eta^3}\frac{\partial^2}{\partial t^2}g(\eta,t)\right]\mathrm{d}t\mathrm{d}\eta, \quad f,g \in W(\Omega)$$

and

$$\|g\|_W = \sqrt{\langle g, g\rangle_W}, \quad g \in W(\Omega).$$

**Lemma 1 (See [4, page 148])** *Reproducing kernel function $K_{(\tau,s)}$ of $W(\Omega)$ is given by:*

$$K_{(\tau,s)} = R_\tau r_s.$$

**Definition 5** We define the binary space $\widehat{W}(\Omega)$ by

$$\widehat{W}(\Omega) = \left\{f \in CC(\Omega) : \quad \frac{\partial^2 f}{\partial \eta \partial t} \in L^2(\Omega)\right\}.$$

The inner product and the norm in $\widehat{W}(\Omega)$ are obtained as:

$$\langle f, g\rangle_{\widehat{W}} = \int_0^1\left[\frac{\partial}{\partial t}f(0,t)\frac{\partial}{\partial t}g(0,t)\right]\mathrm{d}t + \langle f(\cdot,0), g(\cdot,0)\rangle_{G_2^1}$$
$$+\int_0^1\int_0^1\left[\frac{\partial}{\partial \eta}\frac{\partial}{\partial t}f(\eta,t)\frac{\partial}{\partial \eta}\frac{\partial}{\partial t}g(\eta,t)\right]\mathrm{d}t\mathrm{d}\eta, \quad f,g \in \widehat{W}(\Omega)$$

and

$$\|g\|_{\widehat{W}} = \sqrt{\langle g, g\rangle_{\widehat{W}}}, \quad g \in \widehat{W}(\Omega).$$

**Lemma 2 (See [4, page 23])** *Reproducing kernel function $G_{(\tau,s)}$ of $\widehat{W}(\Omega)$ is given as:*

$$G_{(\tau,s)} = (\widetilde{Q}_\tau)^2.$$

**Definition 6** We define the space $W_2^1[0,1]$ by

$$W_2^1[0,1] = \{u \in AC[0,1] : \ u' \in L^2[0,1]\}.$$

The inner product and the norm in $W_2^1[0, 1]$ are given as:

$$\langle u, g\rangle_{W_2^1} = \int_0^1 u(x)g(x) + u'(x)g'(x)\mathrm{d}x, \quad u, g \in G_2^1[0, 1] \tag{8.18}$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}, \quad u \in W_2^1[0, 1]. \tag{8.19}$$

The space $W_2^1[0, 1]$ is a reproducing kernel space, and its reproducing kernel function $T_x$ is obtained as [4]

$$T_x(y)= \frac{1}{2\sinh(1)}[\cosh(x + y - 1) + \cosh(|x - y| - 1)]. \tag{8.20}$$

## 8.3 Analytical and Approximate Solutions

We consider

$$A_1(f_1(\eta, t)) = P_1(\eta, t, F(\eta, t)) + M_1(\eta, t), \tag{8.21}$$

where $A_1 : W(\Omega) \to \widehat{W}(\Omega)$ is a bounded linear operator, $P_1$ is a nonlinear operator, $M_1(\eta, t)$ is an arbitrary function, and $F(\eta, t) = [f_1(\eta, t), f_2(\eta, t), \ldots, f_k(\eta, t)]^T$. The spaces $W(\Omega)$ and $\widehat{W}(\Omega)$ are reproducing kernel spaces which are defined according to the highest derivatives. We pick a countable dense subset $\{(\eta_j, t_j)\}_{j=1}^{\infty}$ in $\Omega$, and describe $\rho_j(\eta, t) = G_{(\eta_j,t_j)}(\eta, t)$, $\vartheta_{j_1}(\eta, t) = A_1^*\rho_j(\eta, t)$, where $A_1^*$ is the adjoint operator of $A_1$. It is simple to show that [2]

$$\vartheta_{j_1}(\eta, t) = A_1 K_{(\tau,s)}(\eta, t).$$

The solutions of (8.3) and (8.4) are considered in the reproducing kernel space $W_2^3[0, 1]$. On defining the linear operator $L : W_2^3[0, 1] \to W_2^1[0, 1]$ as

$$\mathrm{Lu}(x) = u''(x) \tag{8.22}$$

the problem changes the form:

$$\begin{cases} \mathrm{Lu} = f(x, u), & x \in [0, 1], \\ u(0) = u(1) = 0, \end{cases} \tag{8.23}$$

where $f(x, u) = \lambda \exp(\mu u(x))$.

In Eq. (8.23) since $u(x)$ is sufficiently smooth $L : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator. For model problem (8.4) similar things can be done.

**Theorem 4** *Assume that* $\{(\eta_j, t_j)\}_{j=1}^{\infty}$ *is dense in* $\Omega$, *then the solution of* (8.21) *can be shown as*

$$f_1(\eta, t) = \sum_{j=1}^{\infty} \sigma_{j_1} \vartheta_{j_1}(\eta, t), \tag{8.24}$$

*where the* $\sigma_{j_1}$ *are found by*

$$\begin{aligned} Z_1 \times \sigma_{j_1} = [&P_1(\eta_1, t_1, F(\eta_1, t_1)) + M_1(\eta_1, t_1), P_1(\eta_2, t_2, F(\eta_2, t_2)) \\ &+ M_1(\eta_2, t_2), \ldots]^T, \end{aligned} \tag{8.25}$$

$$Z_1 = [A_1 \vartheta_{j_1}(\eta, t)\ |_{(\eta,t)=(\eta_i,t_i)}]_{i,j=1,2,\ldots}, \quad \sigma_{j_1} = [\sigma_{11}, \sigma_{21}, \ldots]^T.$$

*Proof* $\{(\eta_j, t_j)\}_{j=1}^{\infty}$ is dense in $\Omega$. Therefore, $\vartheta_{j_1}(\eta, t)$ is complete system in $W(\Omega)$ [2]. We get

$$\begin{aligned} \langle \vartheta_{i_1}, \vartheta_{j_1} \rangle_{W(\Omega)} &= \langle A_1^* \rho_i(\eta, t), \vartheta_{j_1} \rangle_{W(\Omega)} = \langle \rho_i(\eta, t), A_1 \vartheta_{j_1} \rangle_{\widehat{W}(\Omega)} \\ &= A_1 \vartheta_{j_1}(\eta, t) = \langle f_1(\eta, t), \vartheta_{i_1} \rangle_{W(\Omega)} = \langle f_1(\eta, t), A_1^* \rho_{i_1} \rangle_{W(\Omega)} \\ &= \langle A_1 f_1(\eta, t), \rho_{i_1} \rangle_{\widehat{W}(\Omega)} = P_1(\eta_i, t_i, F(\eta_i, t_i)) + M_1(\eta_i, t_i). \end{aligned}$$

This completes the proof.

*Remark 1* If $P_\eta(\eta, t, F(\eta, t)) = 0$ for $\eta = 1, 2, \ldots, k$, then the analytical solution of each equation can be achieved and the approximate solution of each equation is the $m$-term intercept of the analytical solution which can be obtained by solving an $m \times m$ system of linear equations. If $P_\eta(\eta, t, F(\eta, t)) \neq 0$, then we need to construct an iterative method. We select the number of points $m$, the number of iterations $n$ and put the initial vector function $F_{0,m}(\eta, t) = [0, 0, \ldots, 0]^T$. Then the approximate solution is presented as:

$$F_{n,m}(\eta, t) = [f_{n,m,1}(\eta, t), f_{n,m,2}(\eta, t), \ldots, f_{n,m,k}(\eta, t)]^T,$$

where

$$\begin{cases} f_{n,m,1}(\eta, t) = \sum_{j=1}^{m} \sigma_{j_1} \vartheta_{j_1}(\eta, t), \\ \ldots \\ f_{n,m,k}(\eta, t) = \sum_{j=1}^{m} \sigma_{j_k} \vartheta_{j_k}(\eta, t). \end{cases}$$

**Theorem 5** *Suppose that* $\{(\eta_j, t_j)\}_{j=1}^{\infty}$ *is dense in* $\Omega$. *Then the approximate solution* $F_{n,m}(\eta, t)$ *converges to the analytical solution* $F(\eta, t)$.

*Proof* We have

$$A_\eta(f_{n,m,\eta}(\eta_j, t_j)) = P_\eta(\eta_j, t_j, F_{n-1,m}(\eta_j, t_j)) + M_\eta(\eta_j, t_j) \tag{8.26}$$

for

$$\eta = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, m, \quad \text{and } n = 1, 2, \ldots.$$

There exists a convergent subsequence $\{f_{n_\epsilon,m,\eta}(\eta, t)\}_{\epsilon=1}^{\infty}$ of $\{f_{n,m,\eta}(\eta, t)\}_{n=1}^{\infty}$ such that $f_{n_\epsilon,m,\eta}(\eta, t) \to u_\varpi(\eta, t)$ as $\epsilon \to \infty$, $m \to \infty$, for $\varpi = 1, 2, \ldots, k$. Then, we acquire

$$A_\varpi(f_{n_\epsilon,m,\varpi}(\eta_j, t_j)) = P_\varpi(\eta_j, t_j, F_{n_\epsilon-1,m}(\eta_j, t_j)) + M_\varpi(\eta_j, t_j). \tag{8.27}$$

The operators $A_\varpi$ and $P_\varpi$ are both continuous. Therefore it can be concluded that $F(\eta, t) = [f_1(\eta, t), \ldots, f_k(\eta, t)]^T$ is the analytical solution of (8.21) and $F_{n_\epsilon,m}(\eta, t) = [f_{n_\epsilon,m,1}(\eta, t), \ldots, f_{n_\epsilon,m,k}(\eta, t)]^T$ is the approximate solution of (8.21) after taking limit from both sides. This completes the proof.

It is obvious that $L : W_2^3[0, 1] \to W_2^1[0, 1]$ is a bounded linear operator. Put $\varphi_i(x) = T_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where $L^*$ is conjugate operator of $L$. The orthonormal system $\left\{\widehat{\Psi}_i(x)\right\}_{i=1}^{\infty}$ of $W_2^3[0, 1]$ can be obtained from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\widehat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots) \tag{8.28}$$

**Lemma 3 (See [5])** *Let* $\{x_i\}_{i=1}^{\infty}$ *be dense in* $[0, 1]$ *and* $\psi_i(x) = L_y R_x(y)\big|_{y=x_i}$. *Then the sequence* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is a complete system in* $W_2^3[0, 1]$.

**Theorem 6** *If* $u_1$ *and* $u_2$ *are the exact solutions of* (8.3), *then*

$$u_1(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k})\widehat{\Psi}_i(x) \tag{8.29}$$

*and*

$$u_2(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \gamma_{ij} f(x_j, u_{2_j})\widehat{\Psi}_i(x), \tag{8.30}$$

*where* $\{(x_i)\}_{i=1}^{\infty}$ *is dense in* $[0, 1]$.

*Proof* We have

$$\begin{aligned}
u_1(x) &= \sum_{i=1}^{\infty} \left\langle u_1(x), \widehat{\Psi}_i(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u_1(x), \Psi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u_1(x), L^* \varphi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle \mathrm{Lu}_1(x), \varphi_k(x) \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle f(x, u_1), T_{x_k} \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i(x).
\end{aligned}$$

Similar things can be done for $u_2$.

The approximate solutions $u_n(x)$ and $u_m(x)$ can be acquired from the $n$ and $m$ terms truncation of the exact solutions $u_1$ and $u_2$ as

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i(x), \tag{8.31}$$

and

$$u_m(x) = \sum_{i=1}^{m} \sum_{j=1}^{i} \gamma_{ij} f(x_k, u_{2_j}) \widehat{\Psi}_i(x). \tag{8.32}$$

**Theorem 7** *For any fixed* $u_{1_0}(x) \in W_2^3[0, 1]$ *assume that the following conditions are hold:*

*(i)*

$$u_n(x) = \sum_{i=1}^{n} A_i \widehat{\psi}_i(x), \tag{8.33}$$

$$A_i = \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_{k-1}}(x_k)), \tag{8.34}$$

*(ii)* $\|u_n\|_{W_2^3}$ *is bounded;*
*(iii)* $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[0, 1]$*;*
*(iv)* $f(x, u_1) \in W_2^1[0, 1]$ *for any* $u_1(x) \in W_2^3[0, 1]$.

*Then* $u_n(x)$ *converges to the exact solution of* (8.3) *in* $W_2^3[0, 1]$ *and*

$$u_1(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i(x),$$

*where* $A_i$ *is given by* (8.34)*.*

*Proof* We will show the convergence of $u_n(x)$. We get

$$u_{n+1}(x) = u_n(x) + A_{n+1}\widehat{\Psi}_{n+1}(x), \tag{8.35}$$

from the orthonormality of $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$, it follows that

$$\|u_{n+1}\|^2 = \|u_n\|^2 + A_{n+1}^2 = \|u_{n-1}\|^2 + A_n^2 + A_{n+1}^2 = \ldots = \sum_{i=1}^{n+1} A_i^2, \tag{8.36}$$

from boundedness of $\|u_n\|_{W_2^3}$, we obtain

$$\sum_{i=1}^{\infty} A_i^2 < \infty,$$

i.e.,

$$\{A_i\} \in l^2 \quad (i = 1, 2, \ldots).$$

Let $p > n$, in view of $\left(u_p - u_{p-1}\right) \perp \left(u_{p-1} - u_{p-2}\right) \perp \ldots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned} \left\|u_p - u_n\right\|_{W_2^3}^2 &= \left\|u_p - u_{p-1} + u_{p-1} - u_{p-2} + \ldots + u_{n+1} - u_n\right\|_{W_2^3}^2 \\ &\leq \left\|u_p - u_{p-1}\right\|_{W_2^3}^2 + \ldots + \|u_{n+1} - u_n\|_{W_2^3}^2 \\ &= \sum_{i=n+1}^{p} A_i^2 \to 0, \quad p, n \to \infty. \end{aligned}$$

Considering the completeness of $W_2^3[0, 1]$, there exists $u_1(x) \in W_2^3[0, 1]$, such that

$$u_n(x) \rightarrow u_1(x) \quad as\ n \rightarrow \infty.$$

(ii) Taking limits,

$$u_1(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i(x).$$

Since

$$\begin{aligned}(\mathrm{Lu}_1)\left(x_j\right) &= \sum_{i=1}^{\infty} A_i \left\langle L\widehat{\psi}_i(x), \varphi_j(x)\right\rangle_{W_2^1} = \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), L^*\varphi_j(x)\right\rangle_{W_2^3} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \psi_j(x)\right\rangle_{W_2^3},\end{aligned}$$

it follows that

$$\begin{aligned}\sum_{j=1}^{n} \beta_{nj}(\mathrm{Lu}_1)(x_j) &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \sum_{j=1}^{n} \beta_{nj}\psi_j(x)\right\rangle_{W_2^3} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \widehat{\psi}_n(x)\right\rangle_{W_2^3} = A_n.\end{aligned}$$

If $n = 1$, then

$$\mathrm{Lu}_1(x_1) = f(x_1, u_{1_0}(x_1)). \tag{8.37}$$

If $n = 2$, then

$$\beta_{21}(\mathrm{Lu}_1)(x_1) + \beta_{22}(\mathrm{Lu}_1)(x_2) = \beta_{21} f(x_1, u_{1_0}(x_1)) + \beta_{22} f(x_2, u_{1_1}(x_2)). \tag{8.38}$$

From (8.37) and (8.38), we have

$$(\mathrm{Lu}_1)(x_2) = f(x_2, u_{1_1}(x_2)).$$

We get

$$(\mathrm{Lu}_1)(x_j) = f(x_j, u_{1_{j-1}}(x_j)), \tag{8.39}$$

by induction. By the convergence of $u_n(x)$ we get

$$(\mathrm{Lu}_1)(y) = f(y, u_1(y)),$$

that is, $u_1(x)$ is the solution of (8.3) and

$$u_1(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i,$$

where $A_i$ are given by (8.34). It can be shown in a similar way that $u_2(x)$ is a solution of (8.4).

**Theorem 8** *If* $u_1 \in W_2^3[0,1]$*, then*

$$\|u_n - u_1\|_{W_2^3} \to 0, \quad n \to \infty.$$

*Moreover a sequence* $\|u_n - u_1\|_{W_2^3}$ *is monotonically decreasing in* $n$*.*

*Proof* We have

$$\|u_n - u_1\|_{W_2^3} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i \right\|_{W_2^3}.$$

Thus

$$\|u_n - u_1\|_{W_2^3} \to 0, \quad n \to \infty.$$

In addition

$$\|u_n - u_1\|_{W_2^3}^2 = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i \right\|_{W_2^3}^2$$

$$= \sum_{i=n+1}^{\infty} \left( \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{1_k}) \widehat{\Psi}_i \right)^2.$$

Clearly, $\|u_n - u_1\|_{W_2^3}$ is monotonically decreasing in $n$. In a similar way $\|u_m - u_2\|_{W_2^3}$ is monotonically decreasing in $m$. This completes the proof.

*Remark 2* Let us consider countable dense set $\{x_1, x_2, \ldots\} \in [0,1]$ and define

$$\varphi_i = T_{x_i}, \quad \Psi_i = L^* \varphi_i, \quad \widehat{\Psi}_i = \sum_{k=1}^{1} \beta_{ik} \Psi_k.$$

Then $\beta_{ik}$ coefficients can be found by

$$\beta_{11} = \frac{1}{\|\Psi_1\|}, \quad \beta_{ii} = \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}},$$

$$\beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad c_{ik} = \left\langle \Psi_i, \widehat{\Psi}_k \right\rangle.$$

In a similar way $\gamma_{ij}$ can be defined by using $Q_{x_i}$ .

## 8.4 Numerical Results

We conceive the following nonlinear system of partial differential equations by RKM:

$$\frac{\partial f}{\partial t} = \alpha + \frac{1}{Re}\frac{\partial}{\partial \tau}\left(\mu(T)\frac{\partial f}{\partial \tau}\right) - \frac{Ha^2}{Re} f - \frac{R}{Re}(f - f_p),$$

$$\frac{\partial f_p}{\partial t} = \frac{1}{ReG}(f - f_p),$$

$$\frac{\partial T}{\partial t} = \frac{1}{RePr}\frac{\partial}{\partial \tau}\left(K(T)\frac{\partial T}{\partial \tau}\right) + \frac{Ec}{Re}\mu(T)\left(\frac{\partial f}{\partial \tau}\right)^2 + \frac{Ec}{Re}Ha^2u^2 + \frac{2R}{3Pr}(T_p - T),$$

$$\frac{\partial T_p}{\partial t} = -L(T_p - T),$$

$$f(\tau,t) = f_p(\tau,t) = T(\tau,t) = T_p(\tau,t) = 0, \quad \text{for} \quad t \le 0,$$
$$f_p(0,t) = f_p(1,t) = T_p(0,t) = T(0,t) = 0, \quad \text{for} \quad t > 0,$$

$$\beta\frac{\partial f}{\partial \tau} = f, \quad \text{for } \tau = 0, 1,\ t > 0,$$

$$T_p(1,t) = T(1,t) = 1, \quad \text{for } t > 0.$$

where

$$\mu(T) = \exp(-aT), \quad K(T) = \exp(bT),$$

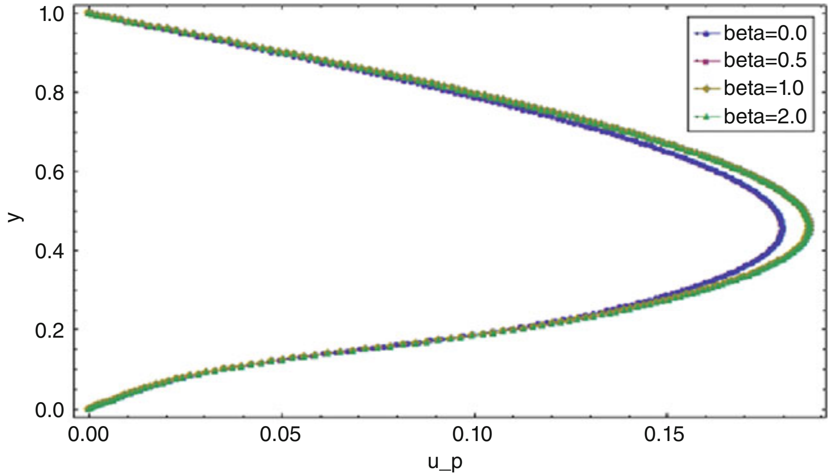


**Fig. 8.1** Approximate solutions of $u_p = f_p$ for various $\beta$

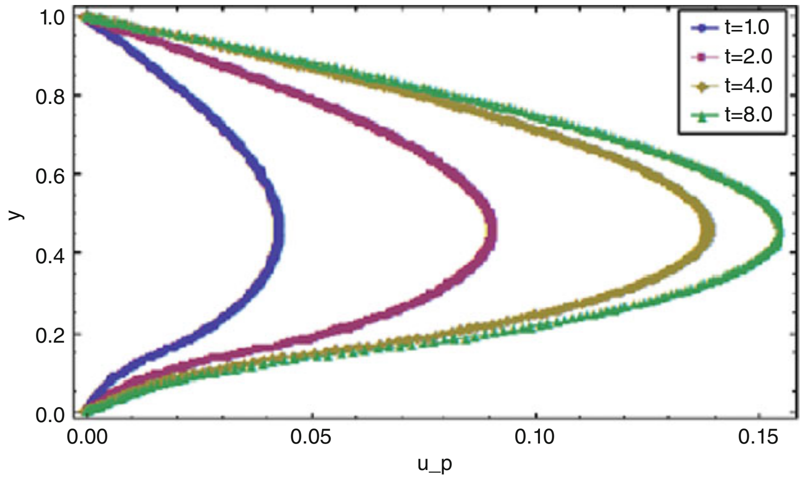


**Fig. 8.2** Approximate solutions of $u_p = f_p$ for various $t$

$$R = 0.5,\ G = 0.8,\ \alpha = 1,\ a = 1,\ b = 0.01,\ Pr = 7.1,$$
$$Ec = 0.2,\ Re = 1,\ L = 0.7,\ \beta = 1,\ t = 10.$$

We obtained the numerical results and demonstrated them in Figs. 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11, 8.12, 8.13, 8.14, 8.15, 8.16, 8.17, and 8.18.

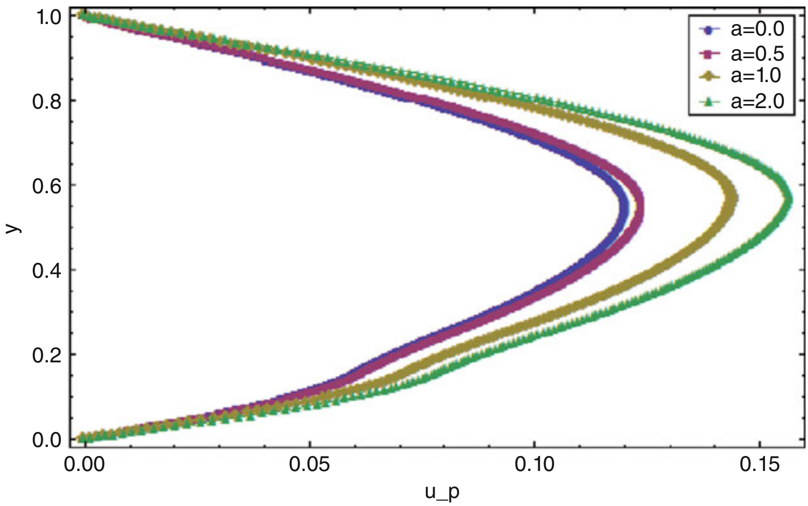


**Fig. 8.3** Approximate solutions of $u_p = f_p$ for various $a$

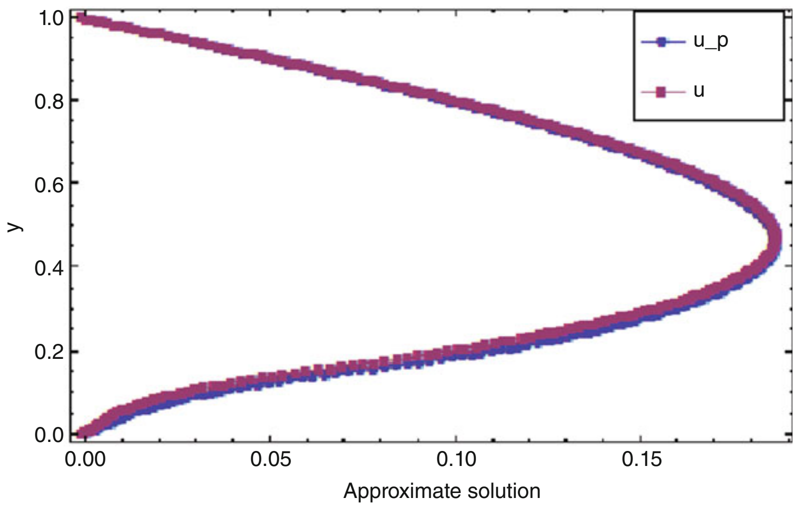


**Fig. 8.4** Comparison of approximate solutions of $u = f$ and $u_p = f_p$

*Example 1* We now consider (8.3). If $\mu\lambda < 0$, the problem (8.3) has as many solutions as the number of roots of the equation

$$\theta = \sqrt{2|\mu\lambda|} \cosh\left(\frac{\theta}{4}\right),$$

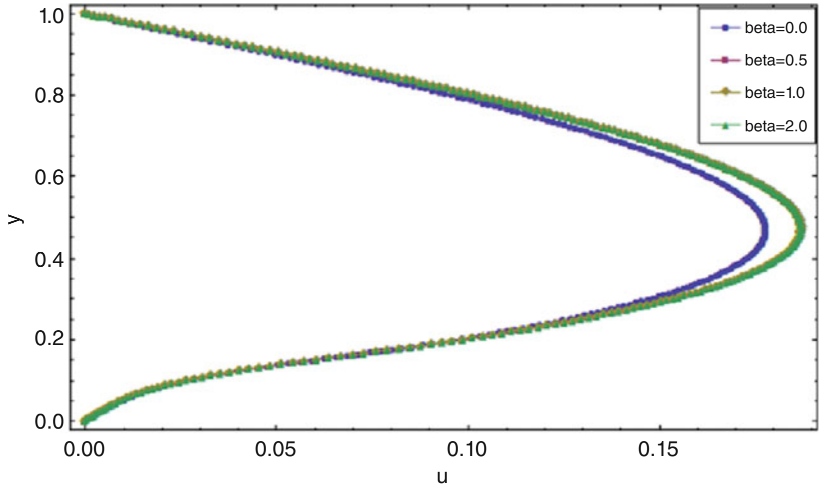


**Fig. 8.5** Approximate solutions of $u = f$ for different values of $\beta$

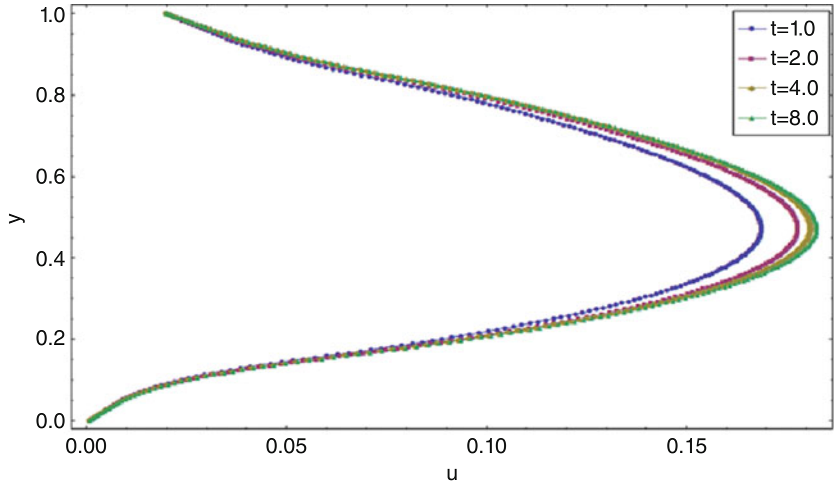


**Fig. 8.6** Approximate solutions of $u = f$ for various $t$

also for each such $\theta_i$

$$u_i(x) = -\frac{2}{\mu}\left\{\ln\left[\cosh\left(\frac{\theta_i}{2}\left(x-\frac{1}{2}\right)\right)\right] - \ln\left[\cosh\left(\frac{\theta_i}{4}\right)\right]\right\}.$$

We obtain Tables 8.1, 8.2, 8.3, 8.4, 8.5, and 8.6 by RKM.

*Example 2* We consider (8.4) for the second example. We obtain Tables 8.7, 8.8, and 8.9 by RKM.

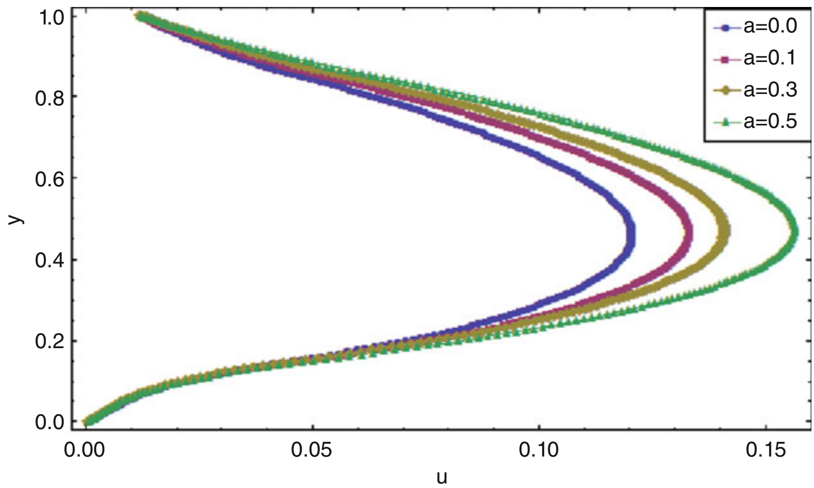


**Fig. 8.7** Approximate solutions of $u = f$ for various $a$

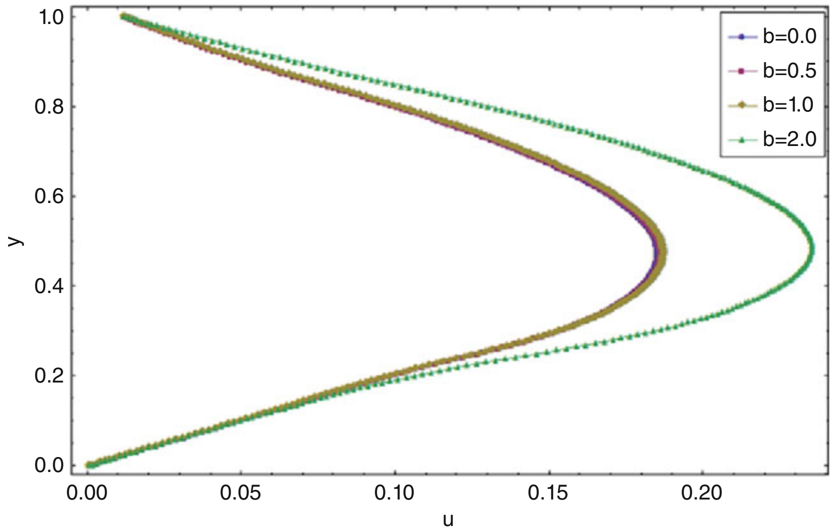


**Fig. 8.8** Approximate solutions of $u = f$ for various $b$

## 8.5 Conclusion

We studied approximate solutions of nonlinear systems of partial differential equations and multiple solutions of differential equations in the reproducing kernel space in this paper. We demonstrated our results with Tables 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, and 8.9 and Figs. 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11,

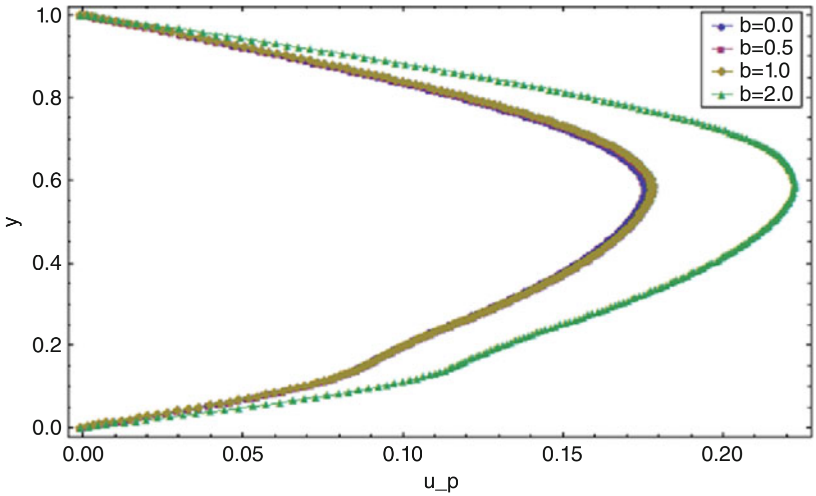


**Fig. 8.9** Approximate solutions of $u_p = f_p$ for various $b$

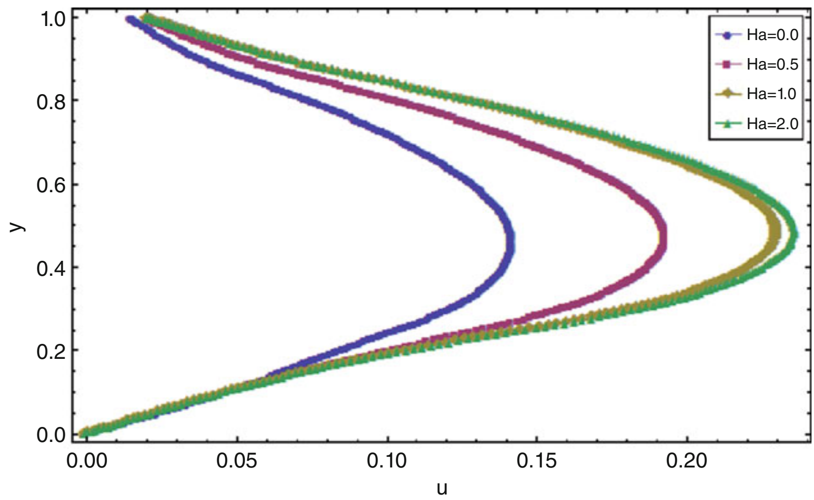


**Fig. 8.10** Approximate solutions of $u = f$ for various $Ha$

8.12, 8.13, 8.14, 8.15, 8.16, 8.17, and 8.18. We proved that the reproducing kernel method is an accurate technique for solving nonlinear systems of partial differential equations and second order differential equations.

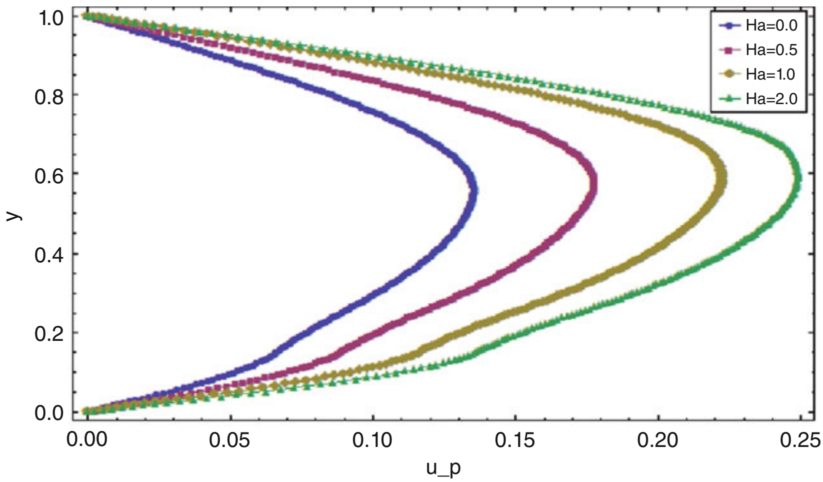


**Fig. 8.11** Approximate solutions of $u_p = f_p$ for various $Ha$

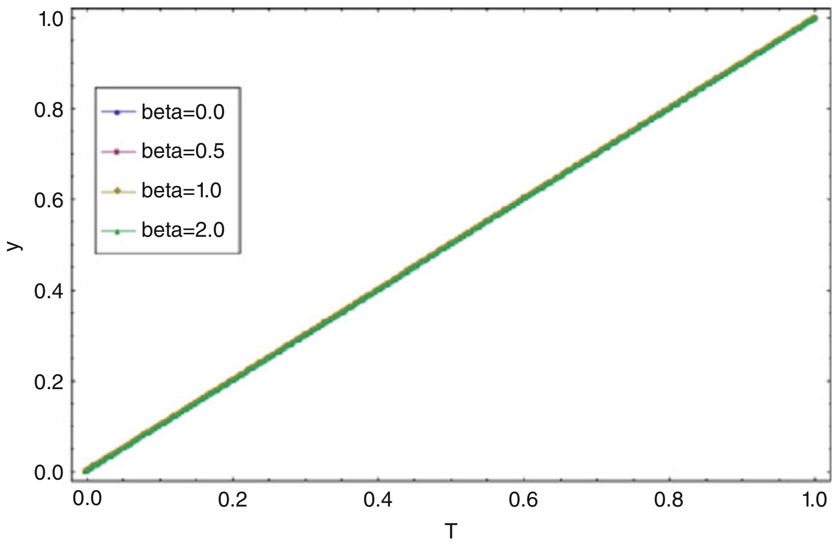


**Fig. 8.12** Approximate solutions of $T$ for various $\beta$

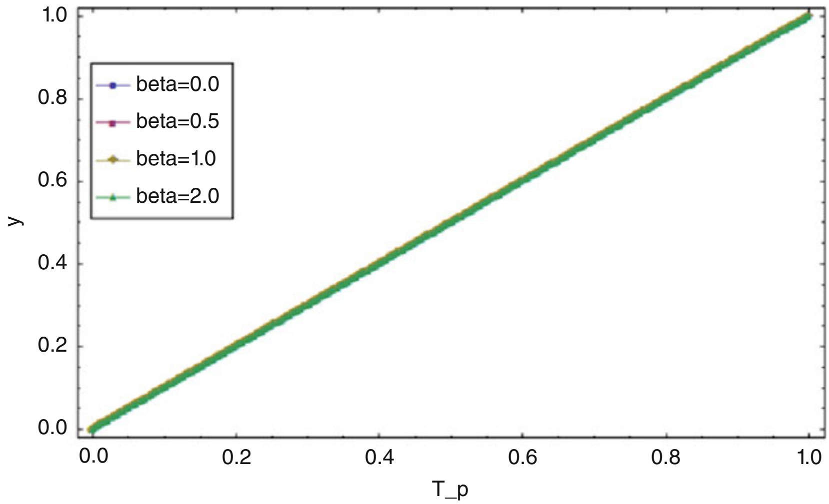


**Fig. 8.13** Approximate solutions of $T_p$ for various $\beta$

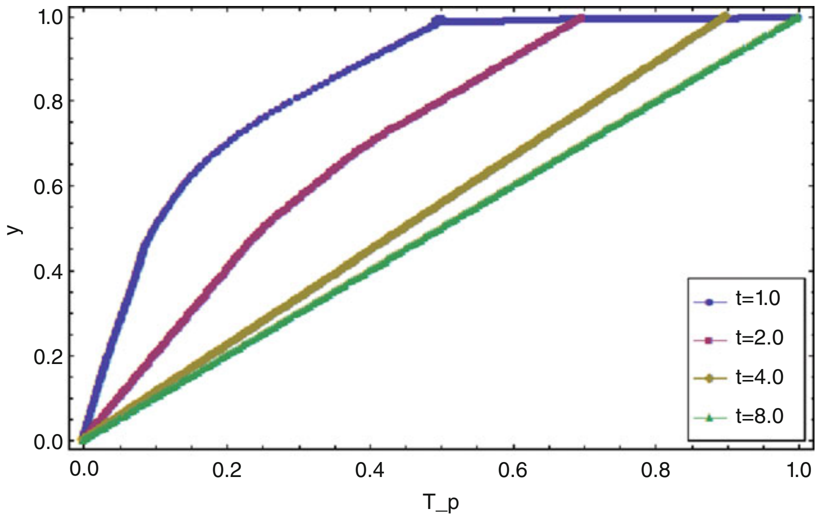


**Fig. 8.14** Approximate solutions of $T_p$ for various $t$

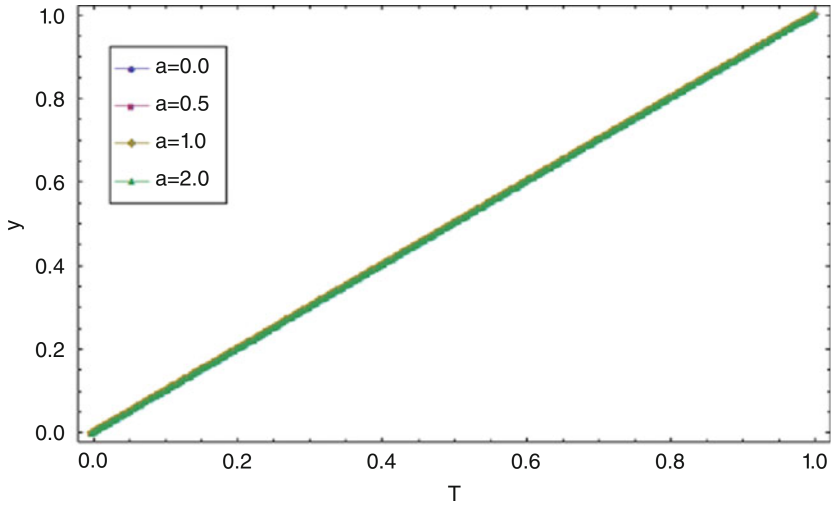


**Fig. 8.15** Approximate solutions of $T$ for various $a$

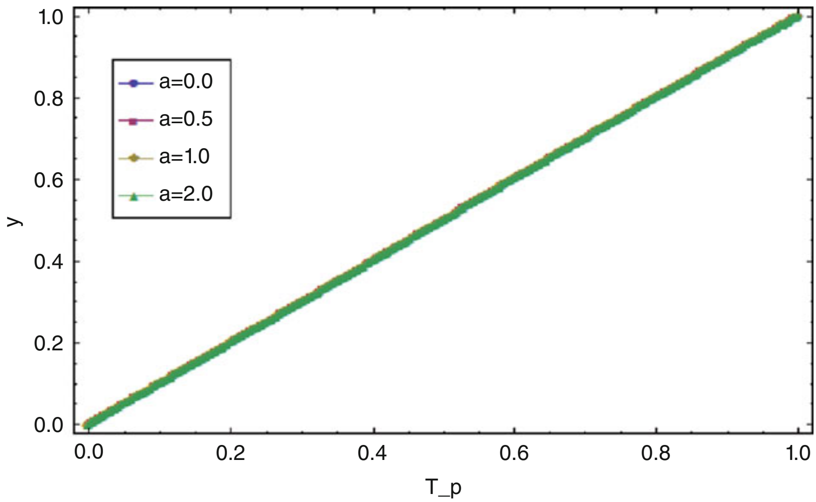


**Fig. 8.16** Approximate solutions of $T_p$ for various $a$

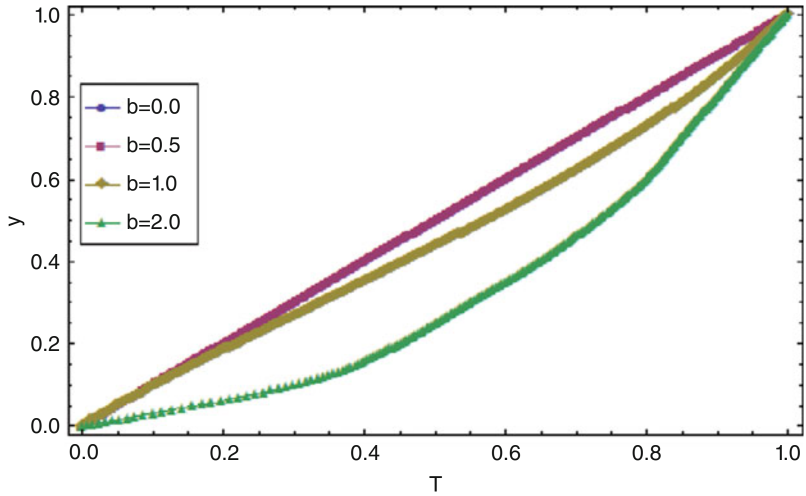


**Fig. 8.17** Approximate solutions of $T$ for various $b$

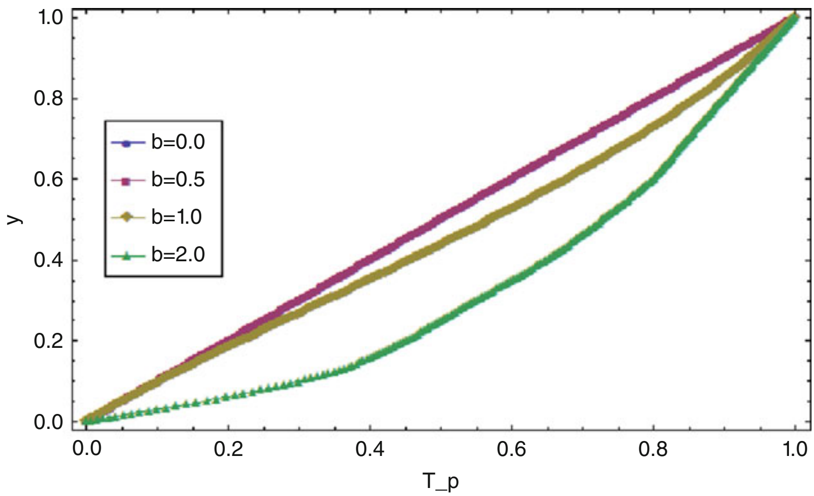


**Fig. 8.18** Approximate solutions of $T_p$ for various $b$

**Table 8.1** The numerical results of Example 1 for first solutions when $\theta = 1.51716$, $t_1 = 1.389$, $t_2 = 8.362$, $\lambda = 1$, and $\mu = -1$

| $x$ | Exact solution | Approximate solution ($m = 20$) |
|---|---|---|
| 0.1 | −0.0498467912422016 56122 | −0.04930059302385296 9510 |
| 0.2 | −0.0891899346230403 84704 | −0.08859095936942859 6421 |
| 0.3 | −0.11760909576028070931 | −0.11718659278945286749 |
| 0.4 | −0.13479025387538511030 | −0.13459000029373223235 |
| 0.5 | −0.14053921439128292991 | −0.14057905280854341332 |
| 0.6 | −0.13479025387538511030 | −0.13506729939580710291 |
| 0.7 | −0.1176090957602807093 | −0.11811210960560610009 |
| 0.8 | −0.08918993462304038 4704 | −0.08989439110906731 6779 |
| 0.9 | −0.04984679124220165 6122 | −0.05052085381355751 9252 |

| $x$ | Approximate solution ($m = 40$) |
|---|---|
| 0.1 | −0.04961547075493067 0101 |
| 0.2 | −0.08895586801468303 565 |
| 0.3 | −0.11742857480052264011 |
| 0.4 | −0.1346979968120595907 |
| 0.5 | −0.14054409043574415912 |
| 0.6 | −0.134882969385795952 |
| 0.7 | −0.11777439140332240699 |
| 0.8 | −0.08941583053876715 4698 |
| 0.9 | −0.05010823959491244 2539 |

**Table 8.2** Absolute errors of Example 1 for first solutions when $\theta = 1.51716$, $t_1 = 1.389$, $t_2 = 8.362$, $\lambda = 1$, and $\mu = -1$

| $x$ | Absolute error ($m = 20$) | Absolute error ($m = 40$) |
|---|---|---|
| 0.1 | 0.00054619821834868 6612 | 0.00023132048727098 6021 |
| 0.2 | 0.00059897525361178 8283 | 0.00023406660835734 9054 |
| 0.3 | 0.00042250297082784182 | 0.0001805209597580692 |
| 0.4 | 0.00020025358165287795 | 0.00009225706332551960 |
| 0.5 | 0.00003983841726048341 | 0.0000048760444612292 |
| 0.6 | 0.00027704552042199261 | 0.0000927155104108417 |
| 0.7 | 0.00050301384532539078 | 0.00016529564304169768 |
| 0.8 | 0.00070445648602693 2075 | 0.00022589591572676 9994 |
| 0.9 | 0.00067406257135586313 | 0.00026144835271078 6417 |

**Table 8.3** Relative errors of Example 1 for first solutions when $\theta = 1.51716$, $t_1 = 1.389$, $t_2 = 8.362$, $\lambda = 1$, and $\mu = -1$

| $x$ | Relative error ($m = 20$) | Relative error ($m = 40$) |
|---|---|---|
| 0.1 | 0.01095754018939257757 | 0.00464062944687889504З |
| 0.2 | 0.0067157270172172023205 | 0.0026243612504777277994 |
| 0.3 | 0.0035924344804845508337 | 0.0015349234563117461783 |
| 0.4 | 0.0014856681094911677931 | 0.00068444906566324999152 |
| 0.5 | 0.0002834683361013182336 | 0.000034695259129978821097 |
| 0.6 | 0.0020553824364640181301 | 0.00068785025434077810681 |
| 0.7 | 0.0042769978127428992563 | 0.0014054664902671738089 |
| 0.8 | 0.0078983854961246965039 | 0.0025327512199836777494 |
| 0.9 | 0.013522687309613288213 | 0.0052450387717120915581 |

**Table 8.4** The numerical results of Example 1 for second solutions when $\theta = 10.9387$, $t_1 = 2.247$, $t_2 = 8.034$, $\lambda = 1$, and $\mu = -1$

| $x$ | Exact solution | Approximate solution ($m = 20$) |
|---|---|---|
| 0.1 | −1.0772733167967386889 | −1.4723999999999999691 |
| 0.2 | −2.1223923410527448670 | −2.6175999999999998767 |
| 0.3 | −3.0773951005699656976 | −3.4355999999999997226 |
| 0.4 | −3.8061519589366269394 | −3.9263999999999995068 |
| 0.5 | −4.0914672451371118790 | −4.0899999999999992295 |
| 0.6 | −3.8061519589366269394 | −3.9263999999999988905 |
| 0.7 | −3.0773951005699656976 | −3.4355999999999984898 |
| 0.8 | −2.1223923410527448670 | −2.6175999999999980275 |
| 0.9 | −1.0772733167967386889 | −1.4723999999999975035 |

| $x$ | Approximate solution ($m = 40$) |
|---|---|
| 0.1 | −1.4705999999999999679 |
| 0.2 | −2.6143999999999998716 |
| 0.3 | −3.4313999999999997114 |
| 0.4 | −3.9215999999999994866 |
| 0.5 | −4.0849999999999991979 |
| 0.6 | −3.9215999999999988456 |
| 0.7 | −3.4313999999999984283 |
| 0.8 | −2.614399999999997947 |
| 0.9 | −1.4705999999999974016 |

**Table 8.5** Absolute error of Example 1 for second solutions when $\theta = 10.9387$, $t_1 = 2.247$, $t_2 = 8.034$, $\lambda = 1$, and $\mu = -1$

| $x$ | Absolute error ($m = 20$) | Absolute error ($m = 40$) |
|---|---|---|
| 0.1 | 0.3951266832032612802 | 0.393326683203261279 |
| 0.2 | 0.4952076589472550097 | 0.4920076589472550046 |
| 0.3 | 0.358204899430034025 | 0.3540048994300340138 |
| 0.4 | 0.1202480410633725674 | 0.1154480410633725472 |
| 0.5 | 0.0014672451371126495 | 0.0064672451371126811 |
| 0.6 | 0.1202480410633719511 | 0.1154480410633719062 |
| 0.7 | 0.3582048994300327922 | 0.3540048994300327307 |
| 0.8 | 0.4952076589472531605 | 0.49200765894725308 |
| 0.9 | 0.3951266832032588146 | 0.3933266832032587127 |

**Table 8.6** Relative error of Example 1 for second solutions when $\theta = 10.9387$, $t_1 = 2.247$, $t_2 = 8.034$, $\lambda = 1$, and $\mu = -1$

| $x$ | Relative error ($m = 20$) | Relative error ($m = 40$) |
|---|---|---|
| 0.1 | 0.36678406217111779588 | 0.36511317700953941112 |
| 0.2 | 0.23332521954994573834 | 0.23181748700770863839 |
| 0.3 | 0.11639873585412894767 | 0.11503394522350042442 |
| 0.4 | 0.031593074149611144625 | 0.030331957922044377369 |
| 0.5 | 0.00035861099434598545167 | 0.0015806664821279665282 |
| 0.6 | 0.031593074149610982703 | 0.030331957922044208957 |
| 0.7 | 0.11639873585412854707 | 0.11503394522350000748 |
| 0.8 | 0.23332521954994486706 | 0.23181748700770773158 |
| 0.9 | 0.36678406217111550714 | 0.3651131770095370289 |

**Table 8.7** The first approximate solutions of Example 2 when $t_1 = 3.026$, $t_2 = 17.629$

| $x$ | Approximate solution ($m = 20$) | Approximate solution ($m = 40$) |
|---|---|---|
| 0.10 | 0.046249255609840261107 | 0.047107959922349180118 |
| 0.20 | 0.067769610201103198051 | 0.069020525808629500991 |
| 0.25 | 0.072542992200350418849 | 0.073992569159451718608 |
| 0.30 | 0.074434206914517408745 | 0.076077916140538342689 |
| 0.40 | 0.071867699646151598001 | 0.07388068138764640163 |
| 0.50 | 0.063644001668100548478 | 0.065997331390535922972 |
| 0.60 | 0.052169172941129235241 | 0.054833380565893847229 |
| 0.70 | 0.039097646775136232426 | 0.042044528121172720677 |
| 0.75 | 0.03233520286288125028 | 0.035413465852768821882 |
| 0.80 | 0.02557036321912830349 | 0.02877375910539847071 |
| 0.90 | 0.012365702227148662923 | 0.015801844129696190555 |

**Table 8.8** The first approximate solutions of Example 2 when $t_1 = 3.026$, $t_2 = 17.629$

| $x$ | Approximate solution ($m = 20$ *and* $m = 40$) |
|---|---|
| $\frac{1}{8}$ | 0.053459133264424318687<br>0.054411793915706876268 |
| $\frac{3}{8}$ | 0.073154996223772261394<br>0.075078284156485197160 |
| $\frac{5}{8}$ | 0.048999016113865678553<br>0.051736471222444720848 |
| $\frac{7}{8}$ | 0.01560460687454551774<br>0.018984660238594126715 |

**Table 8.9** The second approximate solutions of Example 2 when $t_1 = 2.246$, $t_2 = 8.642$

| $x$ | Approximate solution ($m = 20$) | Approximate solution ($m = 40$) |
|---|---|---|
| 1.01 | 0.99882359294951381045 | 0.99882235929452273651 |
| 1.02 | 0.99765900933602205292 | 0.99765655449537351669 |
| 1.03 | 0.99650625041189210164 | 0.99650258720875611371 |
| 1.04 | 0.99536531943327912521 | 0.99536046161080033598 |
| 1.05 | 0.99423622266201998919 | 0.99423018573252504798 |
| 1.06 | 0.99311897036752715301 | 0.99311177274480118408 |
| 1.07 | 0.99201357782868256676 | 0.99200524224331477294 |
| 1.08 | 0.99092006633573157252 | 0.99091062153352994887 |
| 1.09 | 0.98983846419217680040 | 0.98982794691565197597 |
| 1.1 | 0.98876880771667206865 | 0.98875726496959026110 |

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# Chapter 9
# Effect of Edge Deletion and Addition on Zagreb Indices of Graphs

**Muge Togan, Aysun Yurttas, Ahmet Sinan Cevik, and Ismail Naci Cangul**

## 9.1 Introduction

Let $G = (V, E)$ be a simple graph with $| V(G) |= n$ vertices and $| E(G) |= m$ edges. That means, we do not allow loops nor multiple edges. For a vertex $v \in V(G)$, we denote the degree of $v$ by $d_G(v)$. A vertex with degree one is called a pendant vertex. Similarly, we shall use the term "pendant edge" for an edge having a pendant vertex. As usual, we denote by $P_n$, $C_n$, $S_n$, $K_n$, $K_{r,s}$, and $T_{r,s}$ the path, cycle, star, complete, complete bipartite, and tadpole graphs, respectively.

Topological graph indices are defined and used in many areas to study several properties of different objects such as atoms and molecules. Several topological graph indices have been defined and studied by many mathematicians and chemists as most graphs are generated from molecules by replacing atoms with vertices and bonds with edges. Two of the most important topological graph indices are called the first and second Zagreb indices denoted by

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v) \tag{9.1}$$

respectively.

They were first defined in 1972 by Gutman and Trinajstic [1], and are referred to due to their uses in QSAR and QSPR studies. In [2], some results on the first Zagreb index together with some other indices are given. In [3], the multiplicative versions

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of these indices are studied. For graph operations, these indices are calculated in [4]. Some relations between Zagreb indices and some other indices such as $ABC$, $GA$, and Randic indices are obtained in [5]. Zagreb indices of subdivision graphs were studied in [6] and these were calculated for the line graphs of the subdivision graphs in [7]. More generalized version of subdivision graphs is called $r$-subdivision graphs and Zagreb indices of $r$-subdivision graphs are calculated in [8]. These indices are calculated for several important graph classes in [9].

The authors recently studied the amount of change in the first and second Zagreb indices when a new edge is added to any simple graph:

**Theorem 1 ([10])** *Let $G$ be a simple graph. Let us add an edge $e$ to form a larger graph $G + \{e\}$.*

***i)*** *If the added edge $e$ is a pendant edge connecting the vertex $v_i$ of degree $d_i$ in $G$ with a new pendant vertex $v_{n+1}$ of degree $d_{n+1} = 1$, then*

$$M_1(G + \{e\}) = M_1(G) + 2(d_i + 1).$$

***ii)*** *If the added edge is not a pendant edge, in other words, any two vertices $v_i$, $v_j$ with degrees $d_i$, $d_j$ of the graph $G$, respectively, are connected by a new edge $e$, then*

$$M_1(G + \{e\}) = M_1(G) + 2(d_i + d_j + 1).$$

**Theorem 2 ([10])** *Let $G$ be a simple graph. Let us add the edge $e$ to form a larger graph $G + \{e\}$.*

***i)*** *If the added edge $e$ is a pendant edge connecting the vertex $v_i$ of degree $d_i$ in $G$ with a new vertex $v_{n+1}$ of degree $d_{n+1} = 1$, then*

$$M_2(G + \{e\}) = M_2(G) + d_1 + d_2 + \cdots + d_n + 1.$$

***ii)*** *Let $G$ have $m$ edges. If the added edge $e$ is not a pendant edge, in other words, if $e$ connects two vertices of the graph $G$, let us say $v_1$ and $v_2$ with degrees $d_1$ and $d_2$, respectively, then*

$$M_2(G + \{e\}) = M_2(G) + 2m + d_1 d_2 + 1.$$

As an application, the authors, in [10], considered six well-known graph types, namely path $P_n$, cycle $C_n$, star $S_n$, complete $K_n$, complete bipartite $K_{r,s}$, tadpole $T_{r,s}$, and calculate the increase in their first and second Zagreb indices after adding an arbitrary edge $e$. As in the general case, there are different types of edges with different vertex degrees. So the choice of the edge to be added is important and effects the change in the Zagreb indices. Thus the amount of increases in each case is given in an interval as in the following:

$$4 \leq M_1(P_n + \{e\}) - M_1(P_n) \leq 6\,, \quad 4 \leq M_2(P_n + \{e\}) - M_2(P_n) \leq 7\,,$$
$$6 \leq M_1(C_n + \{e\}) - M_1(C_n) \leq 10\,, \quad 7 \leq M_2(C_n + \{e\}) - M_2(C_n) \leq 17\,,$$

$$n^2 - n + 4 \leq M_1(S_n + \{e\}) - M_1(S_n) \leq n^2 + n\,,$$
$$n^2 - n + 2 \leq M_2(S_n + \{e\}) - M_2(S_n) \leq n^2\,,$$

$$M_1(K_n + \{e\}) - M_1(K_n) = 2n\,, \; M_2(K_n + \{e\}) - M_2(K_n) = n^2 - n + 1\,,$$

$$4(r + s) + 6 \leq M_1(T_{r,s} + \{e\}) - M_1(T_{r,s}) \leq 4(r + s) + 14\,,$$
$$4(r + s) + 8 \leq M_2(T_{r,s} + \{e\}) - M_2(T_{r,s}) \leq 4(r + s) + 30\,,$$

$$M_1(K_{r,s} + \{e\}) - M_1(K_{r,s}) = 2s + 2, \; M_2(K_{r,s} + \{e\}) - M_2(K_{r,s}) = rs + s + 1.$$

In this paper, our aim is to obtain the change of the first and second Zagreb indices of graphs when one or more edges are deleted. In this way, it is possible to obtain recursively the Zagreb indices of graphs in terms of smaller graphs, and knowing the Zagreb indices of some fundamental classes of graphs, it will be possible to calculate the Zagreb indices of all graphs.

## 9.2 Deleting One Edge from a Graph

In this section, we will determine the amount of change in the first and second Zagreb indices when one edge is deleted from any simple graph not necessarily connected. Later, we shall generalize this to the case where any $l$ edges are deleted and calculate the change in the first Zagreb index.

**Theorem 3** *Let $G$ be a simple graph. Let the edge $e \in E(G)$ connect two vertices $v_i$, $v_j$ with degrees $d_i$, $d_j$, respectively. Let us also denote by $G - \{e\}$ the graph $G$ with the edge $e$ is deleted. Then*

$$M_1(G) - M_1(G - \{e\}) = 2(d_i + d_j - 1)\,.$$

*Proof* Note that $M_1(G) = \sum_{k=1}^{n} d_k^2$ can be re-arranged as

$$M_1(G) = \sum_{\substack{k=1 \\ k \neq i,j}}^{n} d_k^2 + d_i^2 + d_j^2\,.$$

If we delete the edge $e$, then only the degrees of each vertices $v_i$ and $v_j$ decreases by 1. All other degrees remain the same. Therefore

$$M_1(G-\{e\}) = \sum_{\substack{k=1\\k\neq i,j}}^{n} d_k^2 + (d_i-1)^2 + (d_j-1)^2$$
$$= \sum_{\substack{k=1\\k\neq i,j}}^{n} d_k^2 - 2(d_i+d_j-1) = M_1(G) - 2(d_i+d_j-1)\,.$$

Hence the result follows.

A special situation occurs when we delete a pendant edge.

**Corollary 1** *Let $G$ be a simple graph. If $e$ is a pendant edge which connects two vertices $v_i$, $v_j$ with degrees $d_i$ and 1, respectively, then*

$$M_1(G) - M_1(G-\{e\}) = 2d_i\,.$$

*Proof* It easily follows from the equation $M_1(G) - M_1(G-\{e\}) = 2(d_i+d_j-1)$.

Secondly, let us consider the change in the second Zagreb index.

**Theorem 4** *Let $G$ be a simple graph. Let $e \in E(G)$ be an edge of $G$ to be deleted. As the labeling of the vertices is not important, let us call this edge as $e = v_1v_2$. Let the neighbors of $v_1$ be called $v_3, v_4, \ldots, v_k$, and the neighbors of $v_2$ be called $v_{k+1}, v_{k+2}, \ldots, v_t$. Note that some of these vertices may be the same. Then*

$$M_2(G) - M_2(G-\{e\}) = d_1d_2 + \sum_{j=3}^{t} d_j\,.$$

*Proof* We know that

$$M_2(G) = d_1d_2 + d_1(d_3+d_4+\cdots+d_k) + d_2(d_{k+1}+d_{k+2}+\cdots+d_t) + \sum_{\substack{r,s\in E(G)\\r,s\neq 1,2}} d_rd_s$$

and

$$M_2(G-\{e\}) = (d_1-1)(d_3+d_4+\cdots+d_k) + (d_2-1)(d_{k+1}+d_{k+2}+\cdots+d_t) + \sum_{\substack{r,s\in E(G)\\r,s\neq 1,2}} d_rd_s\,.$$

Hence the result follows.

As a special case, we have

**Corollary 2** *Let the deleted edge $e = v_1v_2$ be a pendant edge with degrees $d_1$ and $1$, respectively. Let $v_3, v_4, \ldots, v_k$ be other neighbors of $v_1$. Then*

$$M_2(G) - M_2(G - \{e\}) = d_1 + d_3 + d_4 + \cdots + d_k = \sum_{i=1}^{k} d_i - 1\,.$$

*Proof* Taking $d_2 = 1$ in Theorem 4, it follows.

## 9.3 Deleting Multiple Edges from a Graph

Now we give our interest to deletion of an arbitrary $l$ edges from a simple graph $G$. We shall denote the new graph by $G - \{e_1, e_2, \ldots, e_l\}$. Then we can formulate the change in the first Zagreb index as follows:

**Theorem 5** *Let $e_1, e_2, \ldots, e_m$ be the edges of a simple graph $G$. For $i = 1, 2, \ldots, m$, denote $e_i = v_{i_1}v_{i_2}$. If we delete $l$ edges from $G$, then*

$$M_1(G) - M_1(G - \{e_1, \ldots, e_l\}) \leq DM_1(e_1, e_2, \ldots, e_l)\,,$$

*where $l \leq m$, $DM_1(e_1, \ldots, e_l) = 2\left(\sum_{j=1}^{l}(d_{j_1} + d_{j_2}) - l\right)$.*

*That is when $l$ edges are deleted, the first Zagreb index decreases by at most two times the sum of the degrees of the vertices connected by the deleted edges minus $2l$. Nevertheless, if the deleted edges have no common vertex, then equality holds.*

*Proof* By Theorem 3, $M_1(G)$ decreases by $2\left(d_{1_1} + d_{1_2} - 1\right)$ when $e_1$ is deleted. Now we have the graph $G - e_1$. If we delete $e_2$ from $G - e_1$, $M_1(G - e_1)$ decreases further by $2\left(d_{2_1} + d_{2_2} - 1\right)$. At $l$-th step, the total decrease in $M_1(G)$ when $l$ edges are deleted would be

$$DM_1(e_1, \ldots, e_l) = 2\sum_{i=1}^{l}\left(d_{i_1} + d_{i_2} - 1\right)\,.$$

But, we must note that this decrease is only valid when the deleted edges have no common vertex. If there is at least one common vertex, then the decrease in $M_1(G)$ will be less than or equal to $DM_1$.

*Example 1* Let G be the graph as given in Fig. 9.1.

If we delete $e_4, e_6, e_8$ in any order, we have a new graph $G - \{e_4, e_6, e_8\}$ as in Fig. 9.2.

In here, while $M_1(G) = 6 \cdot 2^2 + 2 \cdot 3^2 = 42$ and $M_1(G - \{e_4, e_6, e_8\}) = 4 \cdot 1^2 + 4 \cdot 2^2 = 20$, the decrease is 22. Also $DM_1(e_4, e_6, e_8) = 2((2+2) + (3+3) + (2+2) - 3) = 22$ which is equal to the decrease in $M_1(G)$. Now let us delete three edges at least two of which have a common vertex. For example, let us delete $e_4$, $e_5$, and $e_8$ from $G$. Then we obtain the graph in Fig. 9.3.

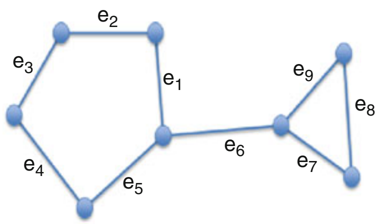

**Fig. 9.1** The graph $G$

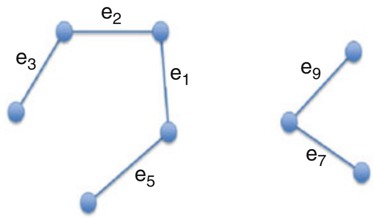

**Fig. 9.2** The graph with deleted edges $e_4$, $e_6$, and $e_8$

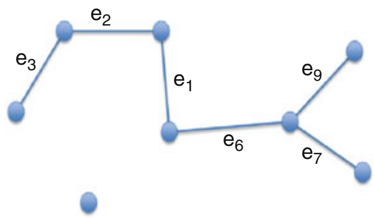

**Fig. 9.3** The graph with deleted edges $e_4$, $e_5$, and $e_8$

In this new graph, while $M_1(G - \{e_4, e_5, e_8\}) = 3 \cdot 1^2 + 3 \cdot 2^2 + 1 \cdot 3^2 = 24$, the decrease is 18. Also $DM_1(e_4, e_5, e_8) = 2((2+2)+(2+3)+(2+2)-3) = 20$ and

$$M_1(G) - M_1(G - \{e_4, e_5, e_8\}) < DM_1(e_4, e_5, e_8)\ .$$

## 9.4 Deleting One Edge from Some Well-Known Graphs

In this section, by considering six well-known graph types, we will calculate the change in their first and second Zagreb indices. Recall that if all degrees are the same, then the graph is called regular. Similarly if each edge in a graph $G$ has the vertex degrees $k$ and $l$, then we call the graph *biregular*. It is well known that all regular graphs are also biregular. For example, $C_n$, $S_n$, $K_n$, and $K_{r,s}$ are biregular graph types. On the other hand, if a graph is biregular, then the edge to be deleted is not important and the Zagreb indices will have the same change. Therefore, for biregular graphs, there is only one possible value for $M_i(G - \{e\})$, where $i = 1, 2$. If a graph is not biregular, then there are different types of edges with different vertex degrees. So the choice of the edge to be deleted is important and effects the change in the Zagreb indices.

Let us begin with path graphs $P_n$. There are two types of edges in $P_n$, two end edges with vertex degrees 1 and 2, and $n-3$ middle edges with both vertex degrees 2.

If we delete one of the end edges, say $e_1$, then

$$M_1(P_n - \{e_1\}) = 2 \cdot 1^2 + 1 \cdot 0^2 + (n-3) \cdot 2^2 = 4n - 10$$

and the decrease in the first Zagreb index is

$$M_1(P_n) - M_1(P_n - \{e_1\}) = (4n-6) - (4n-10) = 4 .$$

If we delete one of the middle edges, say $e_2$, then

$$M_1(P_n - \{e_2\}) = 4 \cdot 1^2 + (n-4) \cdot 2^2 = 4n - 12$$

and the decrease in the first Zagreb index is

$$M_1(P_n) - M_1(P_n - \{e_2\}) = (4n-6) - (4n-12) = 6 .$$

Therefore these arguments imply the following result.

**Corollary 3** *If any edge $e$ is deleted from $P_n$, then*

$$4n - 12 \leq M_1(P_n - \{e\}) \leq 4n - 10 .$$

Secondly, let us calculate the second Zagreb index for $P_n - \{e\}$. If $e_1$ is one of the end edges, then

$$M_2(P_n - \{e_1\}) = 2 \cdot (1 \cdot 2) + (n-4) \cdot (2 \cdot 2) = 4n - 12$$

and therefore $M_2(P_n) - M_2(P_n - \{e_1\}) = 4$. If $e_2$ is one of the middle edges, then

$$M_2(P_n - \{e_2\}) = 4 \cdot (1 \cdot 2) + (n-6) \cdot (2 \cdot 2) = 4n - 16$$

which implies that $M_2(P_n) - M_2(P_n - \{e_2\}) = 8$. Hence we proved

**Corollary 4** *If any edge is deleted from $P_n$, then*

$$4n - 16 \leq M_2(P_n - \{e\}) \leq 4n - 12 .$$

Now let us delete an arbitrary edge from $C_n$ and then calculate the change in two Zagreb indices. As $C_n$ is regular (and biregular), we obtain a path graph $P_n$ and after similar calculations, we get

**Corollary 5**

$$M_1(C_n - \{e\}) = 4n - 6 \quad \text{and} \quad M_2(C_n - \{e\}) = 4n - 8 .$$

*Hence $M_1$ decreases by* 6*, and $M_2$ decreases by* 8.

Now let us delete an arbitrary edge from a star graph $S_n$. As $S_n$ is biregular, all edges have vertex degrees 1 and $n-1$. Therefore

**Corollary 6**

$$M_1(S_n - \{e\}) = n^2 - 3n + 2 \quad \text{and} \quad M_2(S_n - \{e\}) = n^2 - 4n + 4\,.$$

*Hence, while the decrease in* $M_1$ *is* $2(n-1)$*, the decrease in* $M_2$ *is* $2n-3$.

For the complete graphs $K_n$ which are both regular and biregular, the changes in the Zagreb indices are given in the following result:

**Corollary 7**

$$M_1(K_n - \{e\}) = n^3 - 2n^2 - 3n + 6$$

*and*

$$M_2(K_n - \{e\}) = \frac{n^4 - 3n^3 - 3n^2 + 15n - 10}{2}\,.$$

*Hence the decreases in* $M_1$ *and* $M_2$ *are* $4n-6$ *and* $3n^2 - 8n + 5$*, respectively.*

*Proof* Recall that $M_1(K_n) = n(n-1)^2 = n^3 - 2n^2 + n$ and $M_2(K_n) = \binom{n}{2}(n-1)^2 = \frac{n^4 - 3n^3 + 3n^2 - n}{2}$. Now

$$M_1(K_n - \{e\}) = 2 \cdot (n-2)^2 + (n-2) \cdot (n-1)^2 = (n-2) \cdot (n^2 - 3)$$

and

$$M_2(K_n - \{e\}) = \binom{n-2}{2}(n-1) \cdot (n-1) + 2 \cdot (n-2) \cdot ((n-1) \cdot (n-2))$$
$$= \frac{(n-1) \cdot (n-2) \cdot (n^2 - 5)}{2}$$

and we proved the required results.

Now for the complete bipartite graph $K_{r,s}$, we have the following result:

**Corollary 8**

$$M_1(K_{r,s} - \{e\}) = (r+s)(rs-2) + 2$$

*and*

$$M_2(K_{r,s} - \{e\}) = rs(rs-3) + r + s\,.$$

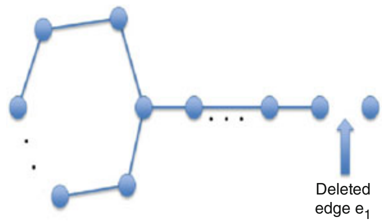


**Fig. 9.4** The graph with deleted edge $e_1$

*Hence the decreases in* $M_1$ *and* $M_2$ *when an edge is deleted are* $2(r+s-1)$ *and* $3rs-r-s$*, respectively.*

Finally, let us consider the tadpole graphs $T_{r,s}$. This time, the situation is a little bit more complicated. There are five types of edges that give different effects on the Zagreb indices when deleted. Let $n = r+s$. If we delete the unique pendant edge given in Fig. 9.4, then

$$\begin{aligned} M_1(T_{r,s} - \{e_1\}) &= 1 \cdot 1^2 + 1 \cdot 3^2 + (s-2+r-1) \cdot 2^2 \\ &= 4 \cdot (r+s) - 2 = 4n - 2 \end{aligned}$$

and

$$\begin{aligned} M_2(T_{r,s} - \{e_1\}) &= 1 \cdot (1 \cdot 2) + 3 \cdot (2 \cdot 3) + [s-3+r-2] \cdot (2 \cdot 2) \\ &= 4(r+s) = 4n \end{aligned}$$

which means that both the first and second Zagreb indices decrease by 4.

If we delete the edge $e_2$ next to the pendant edge, then we similarly get

$$\begin{aligned} M_1(T_{r,s} - \{e_2\}) &= 3 \cdot 1^2 + 1 \cdot 3^2 + (s-3+r-1) \cdot 2^2 \\ &= 4 \cdot (r+s) - 4 = 4n - 4 \end{aligned}$$

and

$$\begin{aligned} M_2(T_{r,s} - \{e_2\}) &= 1 \cdot (1 \cdot 1) + 1 \cdot (1 \cdot 2) + 3 \cdot (2 \cdot 3) + [s-4+r-2] \cdot (2 \cdot 2) \\ &= 4(r+s) - 3 = 4n - 3 \,. \end{aligned}$$

Hence the first and second Zagreb indices decrease by 6, and 7, respectively.

Thirdly, we delete one of the three edges, say $e_3$, incident to the vertex of degree 3. In this case

$$M_1(T_{r,s} - \{e_3\}) = 4n - 6 \quad \text{and} \quad M_2(T_{r,s} - \{e_3\}) = 4n - 8 \,.$$

Therefore the first and second Zagreb indices decrease by 8 and 12, respectively.

Fourthly, if we delete one of the three edges, say $e_4$, with vertex degrees 2 and 2, which are next to the edges deleted in the previous step, then

$$M_1(T_{r,s} - \{e_4\}) = 4(r + s) - 4 = 4n - 4$$

and

$$M_2(T_{r,s} - \{e_4\}) = 4(r + s) - 5 = 4n - 5$$

which means that the first and second Zagreb indices decrease by 6 and 9, respectively.

Finally, if we delete one of the remaining edges, say $e_5$, then

$$M_1(T_{r,s} - \{e_5\}) = M_2(T_{r,s} - \{e_5\}) = 4n - 4$$

which means that the first and second Zagreb indices decrease by 6 and 8, respectively.

All these give us the following result:

**Corollary 9**

$$4n - 6 \leq M_1(T_{r,s} - \{e\}) \leq 4n - 2 \quad \text{and} \quad 4n - 8 \leq M_2(T_{r,s} - \{e\}) \leq 4n\ .$$

*Example 2* Consider $T_{5,6}$. We know that $M_1(T_{5,6}) = 46$ and $M_2(T_{5,6}) = 48$. Let us delete an edge. As special cases, the upper bounds for $M_1$ and $M_2$ are obtained when the pendant edge is deleted, and the lower bounds are attained when one of the three edges incident to the vertex with degree three is deleted.

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# Chapter 10
# The Limit $q$-Bernstein Operators with Varying $q$

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## 10.1 Introduction

The Bernstein polynomials introduced in [2] represent—without any doubt—one of the greatest mathematical discoveries of the twentieth century. Their role and applications are not restricted to the approximation theory or even mathematics. These polynomials have been studied intensively and their connections with various branches of analysis such as convex and numerical analysis, total positivity, and the operator theory have been investigated. Due to the fact that the Bernstein polynomials of a continuous function $f$ on interval [0, 1] form an approximating sequence of shape-preserving operators, these polynomials are an indispensable tool of the computer-aided geometric design. It is a noteworthy fact that the DeCasteljeau algorithm and Bezier curves—both based on the Bernstein basis—were discovered in the research group of "Renault" concern. Currently, the bibliography on the Bernstein polynomials includes thousands of works, while new papers are constantly coming out, and new applications and generalizations are being discovered.

The aim of emerging generalizations is to create appropriate tools for various problems of analysis, differential equations, numerical analysis, and others. Due to the intensive development of the $q$-calculus, quite a few generalizations of Bernstein polynomials connected with the $q$-calculus have appeared. The most popular $q$-analogue of the Bernstein polynomials belongs to Philips [9], who proposed new polynomials known today as the *$q$-Bernstein polynomials*. This year we celebrate 20 years of these polynomials. During these two decades, the area attracted interest of many researches, produced a great number of interesting results, revealed new phenomena, and rich interrelations with other disciplines. We refer

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to [3, 4, 8, 10, 11] and the references therein. Although some properties of the $q$-Bernstein polynomials bear certain similarity to the properties of the classical ones, the involvement of parameter $q$ brings additional flexibility to their structure and requires entirely different approaches of investigation. In this work, we study problem which has no analogues in the classical case, namely the continuity of the limit $q$-Bernstein operator with respect to parameter $q$.

We use the following standard notation (cf. [1, Ch. 10, §10.2]):

$$(x;q)_0 := 1; \quad (x;q)_n := \prod_{k=0}^{n-1}(1-xq^k); \quad (x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k).$$

The entire function $(x;q)_\infty$, $0<q<1$, plays an important role in the reasoning of this paper. For the sake of convenience, we also denote it by:

$$\psi_q(x) := (x;q)_\infty. \tag{10.1}$$

The Taylor expansions, both of $\psi_q$ and $1/\psi_q$, were first obtained by Euler (cf. [1, Ch. 10, Cor. 10.2.2]):

$$\psi_q(x) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k} x^k \tag{10.2}$$

and

$$\frac{1}{\psi_q(x)} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}, \quad |x|<1. \tag{10.3}$$

The main object of this study is the limit $q$-Bernstein operator, which has appeared independently in several works from different approaches. See, for example, [7].

**Definition 1** Let $q \in [0,1]$. The *limit $q$-Bernstein operator* is defined by $f \mapsto B_q f$ as follows:

1. For $q \in (0,1)$,

$$(B_q f)(x) := \begin{cases} \psi_q(x)\cdot\sum_{k=0}^{\infty} \frac{f(1-q^k)x^k}{(q;q)_k} & \text{if } x\in[0,1), \\ f(1) & \text{if } x=1. \end{cases}$$

2. $B_0 := L$, where $(Lf)(x) := f(0)(1-x)+f(1)x$.
3. $B_1 := I$, the identity operator.

Whenever $q \in (0,1)$, it is convenient to introduce functions

$$p_k(q;x) := \frac{\psi_q(x)x^k}{(q;q)_k} \quad \text{for} \quad k=0,1,\ldots, \tag{10.4}$$

so that the limit $q$-Bernstein operator can be written in the form

$$B_q(f;x) = \sum_{k=0}^{\infty} f(1-q^k)p_k(q;x), \quad x \in [0,1).$$

By virtue of (10.3), it follows that

$$\sum_{k=0}^{\infty} p_k(q;x) = 1 \quad \text{for} \quad x \in [0,1). \tag{10.5}$$

The series in (10.5) converges uniformly on any compact subset of $[0, 1)$.

In the case $q \in (0, 1)$, the operator $B_q$ was introduced in [5] and its analytical and geometric properties were studied in a number of papers afterwards. A comprehensive review of the results on the limit $q$-Bernstein operator along with an ample bibliography can be found in [7]. In this paper, new results on the limit $q$-Bernstein operator are presented. To be more specific, the continuity of the mapping $q \mapsto B_q f$ with respect to $q$ has been investigated both in the strong and uniform operator topologies.

## 10.2 Statement of Results

Throughout the text, it is assumed that $q_n \in [0, 1]$, so that the corresponding operators $B_{q_n}$ are well-defined on $C[0, 1]$ equipped with the uniform norm. The following definitions are taken from [6, Section 4.9, Definition 4.9-1].

**Definition 2** A sequence $\{T_n\}$ of operators on a Banach space $X$ is said to be **strongly operator convergent** if $\{T_n x\}$ converges in the norm of $X$ for every $x \in X$.

**Definition 3** A sequence $\{T_n\}$ of operators on a Banach space $X$ is said to be **uniformly operator convergent** if $\{\|T_n - T\|\} \to 0$.

**Theorem 1** *Let* $q_n \to a \in [0, 1]$. *Then, for every* $f \in C[0, 1]$, *one has*

$$B_{q_n}(f;x) \to B_a(f;x) \quad as \quad n \to \infty$$

*uniformly on* $[0, 1]$. *In other words,* $B_a$ *is the strong operator limit of* $B_{q_n}$.

**Corollary 1** *The map* $q \mapsto B_q$ *is continuous in the strong operator topology on* $C[0, 1]$ *for* $q \in [0, 1]$.

The statement below demonstrates that the result of Theorem 1 cannot be extended for the uniform operator topology. What is more, the map $q \mapsto B_q$ is discontinuous at every point $q \in [0, 1]$.

**Theorem 2** *For every distinct* $r, q \in [0, 1]$,

$$\|B_q - B_r\| \geq \frac{1}{2}.$$

**Corollary 2** *With respect to the operator norm, the set* $\{B_q \,:\, q \in [0, 1]\}$ *consists of isolated points with the distance between any two distinct points being at least* $1/2$ *and at most 2.*

## 10.3 Auxiliary Results

**Assumption** From here on, whenever $a \in (0, 1)$, it will be assumed without any loss of generality that for some $a_1, a_2$ and all $n \in \mathbb{N}$,

$$0 < q_n < a_1 < a_2 < 1 \quad \text{and} \quad a < a_1 < a_2 < 1. \tag{10.6}$$

Before the proofs of Theorems 1 and 2 are presented, we give some auxiliary results. In what follows, by $C$ with some index we denote a positive constant whose value does not need to be specified. An index is used for numbering while the dependence on parameter(s) is indicated in parentheses.

**Lemma 1** *The following inequalities are true:*

*(i) if* $|u| \leq \frac{1}{2}$, *then* $|\ln(1+u)| \leq 2|u|$;
*(ii) if* $|\ln t| \leq 1$, *then* $|t-1| \leq 3|\ln(t)|$.

*Proof*

$(i)$ It is commonly known that $\ln(1+u) \leq u$ whenever $u \geq 0$. Let $u \in [-1/2, 0)$. Applying the Mean Value Theorem, we obtain:

$$\frac{\ln(1+u)}{u} = \frac{1}{1+c}, \quad c \in (-1/2, 0),$$

which yields $|\ln(1+u)| \leq 2|u|$ for $u \in [-1/2, 0)$.

$(ii)$ Applying the Mean Value Theorem to the function $f(x) = e^x$ one has:

$$\left|\frac{e^x - 1}{x}\right| \leq e < 3 \quad \text{for} \quad x \in [-1, 1].$$

Setting $x = \ln t$, one derives the statement. □

**Lemma 2** *The following estimate holds:*

$$\left|q_n^j - a^j\right| \leq C_1(a) a_2^j |q_n - a|,$$

*where* $C_1(a)$ *is independent from* $j$ *and* $n$.

*Proof* Since $q_n^j - a^j = (q_n - a)(q_n^{j-1} + q_n^{j-2}a + \cdots + a^{j-1})$, we have with the help of (10.6):

$$\left|q_n^j - a^j\right| \le |q_n - a| \left|a_1^{j-1} + a_1^{j-2}a_1 + \cdots + a_1^{j-1}\right|$$
$$= |q_n - a|\,(ja_1^{j-1}) \le |q_n - a|\, a_2^j \frac{ja_1^{j-1}}{a_2^j}.$$

Since the sequence $\left\{\frac{ja_1^{j-1}}{a_2^j}\right\}$ is bounded, say, by $C_1(a)$, the statement follows. □

**Lemma 3** *For all $j \in \mathbb{N}$, the following estimate holds:*

$$\left|\frac{q_n^j - a^j}{1 - a^j}\right| \le C_2(a)a_2^j|q_n - a|,$$

*where $C_2(a)$ is independent from $j$ and $n$.*

*Proof* With the help of Lemma 2,

$$\left|\frac{q_n^j - a^j}{1 - a^j}\right| \le \frac{\left|q_n^j - a^j\right|}{1-a} \le \frac{C_1(a)a_2^j|q_n - a|}{1-a} := C_2(a)a_2^j|q_n - a|.$$

□

**Corollary 3** *For all $j \in \mathbb{N}$ and $x \in [0, 1]$, the following inequality is valid:*

$$\left|\frac{(q_n^j - a^j)x}{1 - a^j x}\right| \le C_2(a)xa_2^j|q_n - a|.$$

**Lemma 4** *Let $\{q_n\} \to a < 1$. Then, for $n \in \mathbb{N}$, one has*

$$\left|\ln\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right)\right| \le C_3(a)a_2^j|q_n - a| \quad for \;\; all \quad j \in \mathbb{N}.$$

*Proof* Take $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|q_n - a| \le \frac{1}{2C_2(a)a_2}$. By using Lemma 1, we derive that for $n > N_1$

$$\left|\ln\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right)\right| \le 2C_2(a)a_2^j|q_n - a|,$$

and thence,

$$\left|\ln\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right)\right| \le C_3(a)a_2^j|q_n - a| \;\text{ for all }\; j \in \mathbb{N}.$$

□

**Lemma 5** *The following estimate holds for all $n, k \in \mathbb{N}$ :*

$$\left|\frac{1}{(q_n;q_n)_k}-\frac{1}{(a;a)_k}\right| \le C_4(a)|q_n-a|, \tag{10.7}$$

*where $C_4(a)$ is independent from $n$ and $k$.*

*Proof* First, notice that by virtue of (10.6), $(q_n;q_n)_k > (a_1;a_1)_k > (a_1;a_1)_\infty$, whence

$$0<\frac{1}{(q_n;q_n)_k}<\frac{1}{(a_1;a_1)_\infty}.$$

Therefore,

$$\begin{aligned}\left|\frac{1}{(q_n;q_n)_k}-\frac{1}{(a;a)_k}\right| &= \frac{1}{(q_n;q_n)_k}\left|1-\frac{(q_n;q_n)_k}{(a;a)_k}\right|\\ &\le \frac{1}{(a_1;a_1)_\infty}\left|\prod_{j=1}^{k}\frac{1-q_n^j}{1-a^j}-1\right|\\ &= \frac{1}{(a_1;a_1)_\infty}\left|\prod_{j=1}^{k}\left(1+\frac{a^j-q_n^j}{1-a^j}\right)-1\right|.\end{aligned}$$

Set $t:=\prod_{j=1}^{k}\left(1+\frac{a^j-q_n^j}{1-a^j}\right)$ and consider

$$|\ln t| = \left|\sum_{j=1}^{k}\ln\left(1+\frac{a^j-q_n^j}{1-a^j}\right)\right|.$$

By Lemma 4, one has, for $n$ large enough,

$$\begin{aligned}|\ln t| &\le \sum_{j=1}^{k}\left|\ln\left(1+\frac{a^j-q_n^j}{1-a^j}\right)\right| \le \sum_{j=1}^{k} C_3(a)a_2^j|q_n-a|\\ &\le C_3(a)|q_n-a|\sum_{j=0}^{\infty}a_2^j = \frac{C_3(a)}{1-a_2}|q_n-a|.\end{aligned}$$

Thence, $|\ln t| \le 1$ for $n$ large enough and Lemma 1 implies that, for those $n$,

$$|t-1| \le 3\frac{C_3(a)}{1-a_2}|q_n-a|.$$

This yields (10.7). □

**Lemma 6** *For all* $x \in [0, 1]$, *the following estimate holds:*

$$\left| \prod_{j=1}^{\infty} (1 - q_n^j x) - \prod_{j=1}^{\infty} (1 - a^j x) \right| \leq C_5(a) x |q_n - a|.$$

*Proof* First, notice that $(1 - q_n^j x) \geq (1 - a_1^j x) \geq (1 - a_1^j)$ for all $n$ and all $x \in [0, 1]$. Hence,

$$\left| \prod_{j=1}^{\infty} (1 - q_n^j x) - \prod_{j=1}^{\infty} (1 - a^j x) \right| \leq \frac{1}{(a_1; a_1)_\infty} \left| \prod_{j=1}^{\infty} \left( 1 + \frac{(a^j - q_n^j)x}{1 - a^j} \right) - 1 \right|.$$

As in the proof of Lemma 5, set:

$$t := \prod_{j=1}^{\infty} \left( 1 + \frac{(a^j - q_n^j)x}{1 - a^j} \right),$$

and use Corollary 3 along with Lemma 4 to derive

$$|\ln t| \leq \sum_{j=1}^{\infty} \left| \ln \left( 1 + \frac{(a^j - q_n^j)x}{1 - a^j} \right) \right| \leq C_3(a) x |q_n - a|$$

for $n$ large enough. Finally, one obtains:

$$\left| \prod_{j=1}^{\infty} (1 - q_n^j x) - \prod_{j=1}^{\infty} (1 - a^j x) \right| \leq \frac{1}{(a_1; a_1)_\infty} C_3(a) x |q_n - a| := C_5(a) x |q_n - a|.$$

□

**Corollary 4** *If* $\{q_n\} \to a \in (0, 1)$, *then one has:*

$$\prod_{j=1}^{\infty} (1 - q_n^j x) \to \prod_{j=1}^{\infty} (1 - a^j x) \quad \text{as} \quad n \to \infty$$

*uniformly on* $[0, 1]$.

**Lemma 7** *For every* $k \in \mathbb{N}$, *the following estimate holds:*

$$|p_k(q_n; x) - p_k(a; x)| \leq C_6(a) |q_n - a|,$$

*where* $p_k$ *are defined by (10.4). Therefore,* $p_k(q_n; x) \to p_k(a; x)$ *as* $n \to \infty$ *uniformly on* $[0, 1]$.

*Proof*

$$
\begin{aligned}
&|p_k(q_n; x) - p_k(a; x)| \\
&= x^k(1-x)\left|\frac{1}{(q_n; q_n)_k}\prod_{j=1}^{\infty}\left(1 - q_n^j x\right) - \frac{1}{(a; a)_k}\prod_{j=1}^{\infty}\left(1 - a^j x\right)\right| \\
&\leq \left|\frac{1}{(q_n; q_n)_k} - \frac{1}{(a; a)_k}\right|\prod_{j=1}^{\infty}(1 - q_n^j x) \\
&\quad + \frac{1}{(a; a)_k}\left|\prod_{j=1}^{\infty}(1 - q_n^j x) - \prod_{j=1}^{\infty}(1 - a^j x)\right|.
\end{aligned}
$$

Using the results of Lemmas 5 and 6, one obtains:

$$
\begin{aligned}
|p_k(q_n; x) - p_k(a; x)| &\leq C_4(a)|q_n - a| + \frac{1}{(a; a)_k} C_5(a)x|q_n - a| \\
&\leq \left(C_4(a) + \frac{C_5(a)}{(a; a)_\infty}\right)|q_n - a| := C_6(a)|q_n - a|.
\end{aligned}
$$

□

**Corollary 5** *For a fixed positive integer $N$, set:*

$$
S_N(q; x) = \sum_{k=0}^{N} f(1 - q_n^k)p_k(q_n; x). \tag{10.8}
$$

*Then, for any $f \in C[0, 1]$, $S_N(q_n; x) \to S_N(a; x)$ as $n \to \infty$ uniformly on* $[0, 1]$.

*Proof* The conclusion follows from the fact that $f(1 - q_n^k) \to f(1 - a^k)$ for each $k \in \mathbb{N}_0$ and Lemma 7. □

**Lemma 8** *Let $S_N$ be defined by (10.8). Then, the following estimate holds:*

$$
|S_N(q_n; x) - S_N(a; x)| \leq \omega_f(C_1(a)|q_n - a|) + \|f\|(N + 1)C_6(a)|q_n - a|,
$$

*where $\omega_f(\cdot)$ is the modulus of continuity on* $[0, 1]$ *and $\|\cdot\|$ is the uniform norm in* $C[0, 1]$.

*Proof* Consider

$$
|S_N(q_n; x) - S_N(a; x)| \leq \left|\sum_{k=0}^{N} f(1 - q_n^k)p_k(q_n; x) - \sum_{k=0}^{N} f(1 - a^k)p_k(q_n; x)\right|
$$

$$+\left|\sum_{k=0}^{N} f(1-a^k)p_k(q_n;x) - \sum_{k=0}^{N} f(1-a^k)p_k(a;x)\right|$$

$$\leq \sum_{k=0}^{N} |f(1-q_n^k) - f(1-a^k)|p_k(q_n;x)$$

$$+\|f\| \sum_{k=0}^{N} |p_k(q_n;x) - p_k(a;x))|.$$

Evidently, $|f(1-q_n^k) - f(1-a^k)| \leq \omega_f(|q_n^k - a^k|) \leq \omega_f(C_1(a)|q_n - a|)$ by Lemma 2. With the help of Eq. (10.5) and Lemma 7, one concludes that

$$|S_N(q_n;x) - S_N(a;x)| \leq \omega_f(C_1(a)|q_n - a|) + \|f\|(N+1)C_6(a)|q_n - a|.$$

□

*Remark 1* If $f$ satisfies the Lipschitz condition on [0, 1], then

$$|S_N(q_n;x) - S_N(a;x)| \leq C_7(a)|q_n - a|.$$

## 10.4 Proofs of Main Results

*Proof (Proof of Theorem 1)*

*Case 1* $a \in (0,1)$. Let $\varepsilon > 0$ be given. Since the limit $q$-Bernstein operator leaves linear functions invariant [7, Theorem 26, formula (53)], it can be assumed without loss of generality that $f(0) = f(1) = 0$. As $f$ is uniformly continuous on [0, 1], there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for all $x, y \in [0,1]$. Select $N \in \mathbb{N}$ in such a way that $a_1^k < \delta$ for $k > N$. Clearly, the selection of $N$ depends only on the value of $a_1$ and, therefore, on $a$. Then, for $k \geq N+1$, both $f(1-q_n^k) < \varepsilon$ and $f(1-a^k) < \varepsilon$. Consequently,

$$\left|B_{q_n}(f;x) - B_a(f;x)\right| \leq |S_N(q_n;x) - S_N(a;x)| + \left|\sum_{k=N+1}^{\infty} f(1-q_n^k)p_k(q_n;x)\right|$$

$$+\left|\sum_{k=N+1}^{\infty} f(1-a^k)p_k(a;x)\right|$$

$$:= J_1 + J_2 + J_3.$$

By Lemma 5, we conclude that $J_1 < \varepsilon$ for $n$ large enough. As for $J_2$, we have using (10.5):

$$J_2 \le \sum_{k=N+1}^{\infty} |f(1-q_n^k)| p_k(q_n; x) \le \varepsilon \sum_{k=N+1}^{\infty} p_k(q_n; x) < \varepsilon.$$

The term $J_3$ can be estimated in the same way. Finally, we obtain that, for $n$ large enough,

$$\left|B_{q_n}(f; x) - B_a(f; x)\right| < 3\varepsilon.$$

*Case 2* $a = 0$, that is, $\{q_n\} \to 0^+$. As before, it can be assumed without loss of generality that $f(0) = f(1) = 0$ implying $(B_0 f)(x) \equiv 0$. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x)| < \varepsilon$ when $x \in [1-\delta, 1]$. At the same time, there exists $N \in \mathbb{N}$ such that $1 - q_n^k \in [1-\delta, 1]$ for all $k \in \mathbb{N}$ and $n > N$. As a result, for $n > N$, one has:

$$|B_{q_n}(f; x)| \le \sum_{k=1}^{\infty} |f(1-q_n^k)| p_k(q_n; x) < \varepsilon.$$

The statement is proved.

*Case 3* $a = 1$. Here, we have to prove that, for every $f \in C[0, 1]$,

$$\left\{B_{q_n}(f; x)\right\} \to f(x) \ \text{ as } \ \{q_n\} \to 1^-. \tag{10.9}$$

Since $B_{q_n}$ is a positive linear operator on $C[0, 1]$ which reproduces linear functions, by Korovkin's Theorem it suffices to show that

$$\left\{B_{q_n}(x^2; x)\right\} \to x^2 \ \text{ as } \ \{q_n\} \to 1^-.$$

Plain calculations [7, formula (42)] reveal that:

$$B_{q_n}(x^2; x) = x^2 + (1-q_n)x(1-x),$$

yielding the needed conclusion.

The proof of the theorem is complete. □

*Proof (Proof of Theorem 2)* To begin with, we notice that, for every $q \in (0, 1)$ and every $\varepsilon > 0$, there exist $\delta = \delta(q, \varepsilon)$ and $N = N(q, \varepsilon)$ satisfying:

$$\sum_{k=1}^{N} p_k(q; x) > 1 - \varepsilon \ \text{ for } \ x \in [1-\delta, 1-\delta/2]. \tag{10.10}$$

Indeed, since $p_0(q;x) \to 0$ as $x \to 1^-$, there exists $\delta = \delta(q,\varepsilon) > 0$ such that $p_0(q;x) < \varepsilon/2$ when $x \in [1-\delta, 1]$. Apart from that, by (10.5), $\sum_{k=0}^{\infty} p_k(q;x) = 1$ and the convergence is uniform on each compact subset of $[0, 1)$. Hence, there exists $N = N(q,\varepsilon)$ such that

$$\sum_{k=0}^{N} p_k(q;x) > 1 - \varepsilon/2 \;\text{ for }\; x \in [1-\delta, 1-\delta/2],$$

which yields (10.10), because

$$\sum_{k=1}^{N} p_k(q;x) = \sum_{k=0}^{N} p_k(q;x) - p_0(q;x).$$

Next, assuming that $0 \le r < q \le 1$, we split the proof into several parts.

1. If $r = 0$ and $q = 1$, then $\|B_1 - B_0\| = 2$ and there is nothing to prove.
2. Let $r = 0$ and $q \in (0, 1)$ be fixed. Given $\varepsilon > 0$, choose $N = N(q,\varepsilon)$ to satisfy (10.10) and consider a function $f_N \in C[0,1]$, $\|f_N\| = 1$ satisfying the conditions:

$$f_N(1-q^k) = 1 \text{ for } k = 1, 2, \ldots, N,$$
$$f_N(1-q^k) = 0 \text{ for } k \neq 1, 2, \ldots, N.$$

   Consequently $\|B_q - B_0\| \ge 1 - \varepsilon$ for an arbitrary $\varepsilon > 0$, and thence $\|B_q - B_0\| \ge 1$.
3. Let $q = 1$, $r \in (0, 1)$, and $f \in C[0, 1]$ be chosen in such a way that $\|f\| = 1$ and $f(1-q^k) = 0$ for all $k \in \mathbb{N}_0$. Then $B_q f = 0$ and hence $\|B_q - B_1\| \ge \|f\| = 1$.
4. Finally, comes the case $0 < r < q < 1$. Here, exactly one of the following situations occurs:

   (i) $q^j \neq r^l$ for all $j, l \in \mathbb{N}$;
   (ii) $q^j = r^l$ for some $j, l \in \mathbb{N}$.

   We consider these cases separately.

   *Case (i)* As in previous arguments, given $\varepsilon > 0$, choose $\delta = \delta(r,\varepsilon) > 0$ and $N = N(r,\varepsilon)$ so that (10.10) is true. Now, let $f \in C[0, 1]$ be a function with $\|f\| = 1$, satisfying the conditions:

$$f(1-r^k) = 1 \text{ for } k = 1, 2, \ldots, N,$$
$$f(1-r^k) = 0 \text{ for } k \neq 1, 2, \ldots, N,$$
$$f(1-q^k) = 0 \text{ for all } k \in \mathbb{N}_0.$$

   Clearly,

$$|(B_q - B_r)(f;x)| = \sum_{k=1}^{N} p_k(r;x) > 1 - \varepsilon, \quad x \in [1-\delta, 1-\delta/2].$$

Since $\|f\| = 1$, the latter implies that $\|B_q - B_r\| \geq 1 - \varepsilon$ and, consequently, $\|B_q - B_r\| \geq 1$.

*Case (ii)* In this case, there exists an integer $m \geq 2$ such that

$$\{1 - r^k\}_{k=0}^{\infty} \cap \{1 - q^k\}_{k=0}^{\infty} = \{1 - r^{mk}\}_{k=0}^{\infty}.$$

Consider

$$\sum_{k=0}^{\infty} p_{mk}(r; x) = (x; r)_\infty \sum_{k=0}^{\infty} \frac{x^{mk}}{(r; r)_{mk}} < \frac{(x; r)_\infty}{(r; r)_\infty (1 - x^m)} \to \frac{1}{m} \text{ as } x \to 1^-.$$

Hence, for a given $\varepsilon > 0$, there exist $\delta_1 = \delta_1(r, \varepsilon) > 0$ such that

$$\sum_{k=0}^{\infty} p_{mk}(r; x) \leq \frac{1}{m} + \varepsilon/2 \text{ on } [1 - \delta_1, 1).$$

Using the last inequality and following the same line of reasoning as before, for an arbitrary $\varepsilon > 0$, one may select $\delta > 0$ and $N \in \mathbb{N}$ in such a way that

$$\begin{aligned} \sum_{k=1, m\nmid k}^{N} p_k(r; x) &= \sum_{k=0}^{N} p_k(r; x) - \sum_{k=0}^{\lfloor N/m \rfloor} p_{mk}(r; x) \\ &\geq 1 - \frac{1}{m} - \varepsilon \geq \frac{1}{2} - \varepsilon \text{ for } x \in [1 - \delta, 1 - \delta/2]. \end{aligned}$$

Now, let $f \in C[0, 1]$ be a function with $\|f\| = 1$, satisfying the conditions:

$$\begin{aligned} &f(1 - r^k) = 1 \text{ for } 1 \leq k \leq N, m \nmid k \\ &f(1 - r^k) = 0 \text{ for } 1 \leq k \leq N, m|k, \text{ or } k > N, \\ &f(1 - q^k) = 0 \text{ for } 1 - q^k \notin \{1 - r^j\}_{j=0}^{\infty}. \end{aligned}$$

Then,

$$\|(B_q - B_r)(f; x)\| = \max_{x \in [0,1]} \sum_{k=1, m\nmid k}^{N} p_k(r; x) \geq \frac{1}{2} - \varepsilon.$$

Thus, $\|B_q - B_r\| \geq 1/2$. The proof is complete. □

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# Chapter 11
# Localization of the Spectral Expansions Associated with the Partial Differential Operators

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## 11.1 Introduction

The solutions of the engineering problems can be obtained with the application of series or transformations depending on the domain, boundary, and initial conditions. For example, investigations of the various vibration processes in the real interval (finite, semi-finite or infinite) require application of the Fourier series and integrals. The problems of this type are the main reason for the importance of studying questions of convergence of the Fourier series and integrals in different topology.

Multidimensional case, in contrast to the one dimension, is involving various methods of summations of the Fourier series and integrals. Note that some of the summation methods are linked to the spectral theory of the partial differential operators. For example, a spherical partial sums of the multiple Fourier trigonometric series coincides with the spectral expansions associated with the Laplace operator on the torus.

The spectral expansions associated with the elliptic partial differential operators in the spaces of the smooth functions are well studied in many papers. But many phenomena in nature require for its description either "bad" functions or even they cannot be described by regular functions. Therefore, one has to deal with the distributions that describe only integral characteristics of phenomena.

Application of the spectral methods in the spaces of distributions leads to the study of convergence and/or summability problems of the spectral expansions of the linear continuous functional. We will study convergence and summability problems of the spectral expansions of distributions in the classical means in the domains where they coincide with the regular functions. We prove that the singularities of

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the distributions will effect negatively to the convergence and/or summability even at regular points. For example, the Fourier trigonometric series of the Dirac function diverges at a regular point due to the effect of the singularity at zero, although its arithmetic means converge.

## 11.2 Summability of the Spectral Expansions of Distributions

Let $\Omega$ - an arbitrary $N$ - dimensional domain. Consider a polynomial by $\xi,\ \xi \in R^N$, of order $2m$ with coefficients in $C^\infty(\Omega)$

$$A(x,\xi) = \sum_{|\alpha|\le 2m} a_\alpha(x)\xi^\alpha, \tag{11.1}$$

where $\alpha$ denotes a multi-index, i.e. $N$-dimensional vector with non-negative integer components $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ is called the length of the multi-index $\alpha$, $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha} \dots \xi_N^{\alpha}$, here $\xi_j$—component of the vector $\xi$.

Denote

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_i} \text{ and } D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_N^{\alpha_N}.$$

Differential operator

$$A(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x)D^\alpha, \tag{11.2}$$

is called an elliptic operator of the order $2m$, at the point $x$, if for any $\xi \in R^N$

$$A_0(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \ge c(x)|\xi|^{2m}, c(x) > 0.$$

$A(x,D)$ is called an elliptic operator in the domain $\Omega$ if it is elliptic in each point of the domain.

Denote by $C_0^\infty(\Omega)$ the space of all infinite differentiable functions in the domain $\Omega$ with the compact support in $\Omega$.

Let $A$ be an operator in the Hilbert space $L_2(\Omega)$ with the domain of definition $D(A) = C_0^\infty(\Omega)$, acting as $Au(x) = A(x,D)u(x), u(x) \in C_0^\infty(\Omega)$. Let $A$ be a symmetric operator, i.e. for any $u$ and $v$ from $C_0^\infty(\Omega)$

$$(u, v) = (u, Av).$$

Also suppose that $A$ is semi-bounded which means there is a constant $\mu$ such that for any $u \in C_0^\infty(\Omega)$

$$(u, u) \geq \mu(u, u).$$

From the Fredrix theorem [7] it follows that the operator $A$ has at least one self-adjoint extension $\hat{A}$ with the same lower bound $\mu$. By the John von Neumann spectral theorem (see, for instance, in [32]) the operator $\hat{A}$ can be represented as

$$\hat{A} = \int_\mu^\infty \lambda d E_\lambda.$$

where projectors $E_\lambda$ are integral operators

$$E_\lambda f = \int_\Omega \Theta(x, y, \lambda) f(y) dy, \ \ f \in L_2(\Omega). \tag{11.3}$$

The kernel $\Theta(x, y, \lambda)$ is called the spectral function of the operator $\hat{A}$ and the expression (11.3) is called the spectral expansions of $f$ associated with the self-adjoint operator $\hat{A}$.

Define the Riesz means of order $s \geq 0$, of the spectral expansions $E_\lambda f$ as follows:

$$E_\lambda^s f = \int_\mu^\lambda \left(1 - \frac{t}{\lambda}\right)^s d E_t.$$

The operators $E_\lambda^s$, as well as $E_\lambda$, are integral operators with the kernels

$$\Theta^s(x, y, \lambda) = \int_\mu^\lambda \left(1 - \frac{t}{\lambda}\right)^s d\Theta(x, y, t).$$

Note, from the Garding theorem [7] it follows that the function $\Theta(x, y, \lambda)$ belongs to $C^\infty(\Omega \times \Omega)$ for each $\lambda > 0$. This allows to define the spectral expansions of the distributions with the compact support.

Denote by $\mathscr{E}'(\Omega)$—the space of the linear continuous functionals on $C^\infty(\Omega)$. Then for any distribution $f \in \mathscr{E}'(\Omega)$ the Riesz means of its spectral expansions are defined as follows:

$$E_\lambda^s f(x) = < f, \Theta^s(x, y, \lambda) >, \tag{11.4}$$

where the functional $f$ is acting on $\Theta^s(x, y, \lambda)$ with respect to the second variable. Note that $E_\lambda^s f(x) \in C^\infty(\Omega)$ for any distribution $f$ from $\mathscr{E}'(\Omega)$, $s \geq 0$, and $\lambda > 0$.

The relation (11.4) can also be considered in the classical sense on the domains where $f$ coincides with the locally integrable function.

For any integer $\ell$ denote the Sobolev spaces $H^\ell(\Omega) = W_2^\ell(\Omega)$ [37].

**Theorem 1** *Let $f \in \mathscr{E}'(\Omega) \bigcap H^{-\ell}(\Omega)$, $\ell > 0$. If $s \geq (N-1)/2 + \ell$, then uniformly in any compact $K$ from $\Omega \setminus suppf$*

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = 0.$$

From the estimate of the spectral function $\Theta(x, y, \lambda)$ (see [13, theorem 6.1]) we get the following lemma.

**Lemma 1** *Let $\Omega_o$ be some sub-domain $\Omega_0 \subset \Omega$ and let $f \in \mathscr{E}'(\Omega) \bigcap H^{-\ell}(\Omega)$ be such that $suppf \subset \Omega_o$. Let $K$ be a compact set from $\Omega \setminus \Omega_o$ and $s = (N-1)/2 + \ell$. Then the estimate*

$$|E_\lambda^s f(x)| \leq C \| f \|_{-\ell}, \tag{11.5}$$

*is valid uniformly with respect to $x \in K$.*

The proof of the Theorem 1 follows from Lemma 1 and the fact that the space $C_0^\infty(\Omega)$ is dense in $\mathscr{E}'(\Omega) \bigcap H^{-\ell}(\Omega)$.

**Theorem 2** *Let $\ell > 0$ and let $x_0$ a point in the domain $\Omega$.*

*If $s < (N-1)/2 + \ell$, then there exists a distribution $f$ from $\mathscr{E}'(\Omega) \bigcap H^{-\ell}(\Omega)$, such that $x_0 \in \Omega \setminus suppf$ and*

$$\overline{\lim_{\lambda \longrightarrow \infty}} E_\lambda^s f(x_0) = +\infty.$$

Theorem 2 proves sharpness of the inequality $s \geq (N-1)/2 + \ell$, in the Theorem 1. It follows from the estimate of the spectral function from the bottom [13] and the Banach–Steinhaus theorem.

Using the Hermander theorem [12] Theorem 1 can be extended as follows:

**Theorem 3** *Let $\ell > 0$ and $\Omega_0$—a sub-domain of $\Omega$. If $s \geq (N-1)/2 + \ell$, then for any distribution $f \in C(\Omega) \cap \mathscr{E}'(\Omega) \cap H^{-\ell}(\Omega)$, then uniformly on each compact $K$ from $\Omega_0$*

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = f(x).$$

Case $p \neq 2$ is more complicated even for the spectral expansions of the smooth functions. In this case we have the following results [24].

For any real number $\ell$ by $W_p^{-\ell}(\Omega)$ denote the Sobolev spaces [37].

**Theorem 4** *Let $f \in \mathscr{E}'(\Omega) \bigcap W_p^{-\ell}(\Omega)$, , $\ell > 0$, $1 < p \leq 2$. If $s \geq (N-1)/p + \ell$, then uniformly in any compact $K$ from $\Omega \setminus suppf$*

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = 0.$$

A problem of the sharpness of the condition

$$s \geq (N-1)/p + \ell,$$

in Theorem 4 is complicated. This question is complicated even for the spectral expansions associated with the Laplace operator in the arbitrary domain.

## 11.3 Localization and Uniform Convergence of the Eigenfunction Expansions

Let $\Omega$ be a bounded domain in $R^N$ with the smooth boundary $\partial\Omega$. Let $\hat{A}$ be a self-adjoint extension of a positive formally elliptic differential operator of order $2m$ with the regular boundary conditions [7].

Denote by $\{u_n(x)\}$ a complete orthonormal in $L_2(\Omega)$ system of eigenfunctions of the operator $\hat{A}$ corresponding to the sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$. For any function $f \in L_2(\Omega)$ we introduce the Riesz means of order $s$ of the partial sums of the Fourier series as follows:

$$E_\lambda^s f(x) = \sum_{\lambda_n < \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x). \tag{11.6}$$

Here $\lambda > 0$, $f_n = (f, u_n)$ are the Fourier coefficients of the function $f$ with respect to the system $\{u_n(x)\}$.

Note that if $s = 0$, then (11.6) is just the partial sum of the Fourier series of the function $f$.

The precise conditions of the uniform convergence of $E_\lambda^s f(x)$ on the compact subsets of the domain $\Omega$ are established by Il'in (see in [15]).

**Theorem 5** *If*

$$\alpha \geq \frac{N-1}{2}, \ \alpha p > N, \ \ p \geq 1 \tag{11.7}$$

*then the Fourier series via the eigenfunctions of the Laplace operator of any function with compact support belonging to the Sobolev space* $W_p^\alpha(\Omega)$ *converges uniformly on any compact subset of the domain* $\Omega$.

Convergence of the Riesz means (11.6) of the smooth functions on the compact subsets of the domain $\Omega$ requires a modification of the condition (11.7) in Theorem 1 as follows:

$$\alpha + s \geq \frac{N-1}{2}, \ \alpha p > N, \ \ s \geq 0, \ \ p \geq 1. \tag{11.8}$$

The sharpness of the first inequality in (11.8) for the eigenfunction expansions associated with the Laplace operator is proved by Il'in (see in [15]). The preciseness of the second inequality in (1.3) follows from the fact that the condition $\alpha p \leq N$ implies the existence of an unbounded function with the compact support belonging to the appropriate Sobolev space for which its Fourier series cannot converge uniformly.

Moreover, the conditions (11.8) are sufficient for the functions in the Nikol'skii spaces $H_p^\alpha(\Omega)$. The last statement is proved in the case of expansions associated with the eigenfunctions of the Laplace operator by Il'in and Alimov [16], in the case of the expansions associated with the elliptic operators of the second order with the variable coefficients by Il'in and Moiseev [17]. Finally, for the general elliptic differential operators of order $2m$ Alimov is proved in [1] the following statement:

**Theorem 6** *If $f$ belongs to the space $\overset{\circ}{H}{}_p^\alpha(\Omega)$ and has the compact support in $\Omega$, then under the conditions (11.8) the Riesz means $E_\lambda^s f$ converge as $\lambda \to +\infty$ to $f$ uniformly on any compact $K \subset \Omega$.*

Here $\overset{\circ}{H}{}_p^\alpha(\Omega)$, $(\overset{\circ}{W}{}_p^\alpha(\Omega))$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm of the Nikol'skii (Sobolev) space $H_p^\alpha(\Omega)(W_p^\alpha(\Omega))$.

In the case in which the second condition in (11.8) is replaced by $\alpha p = N$, it is necessary to assume that the function $f$ is continuous ( see [3]).

**Theorem 7** *Let $\Omega_0$ be an arbitrary open subset of $\Omega$ and let*

$$\alpha + s > \frac{N-1}{2}, \quad \alpha p = N, \quad s \geq 0, \quad p \geq 1. \tag{11.9}$$

*Then for any function $f \in \overset{\circ}{W}{}_p^\alpha(\Omega)$ continuous on $\Omega_0$*

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = f(x) \tag{11.10}$$

*uniformly on any compact set $K \subset \Omega_0$.*

The first condition $\alpha + s > \frac{N-1}{2}$ in (11.9) is also precise [2].

**Theorem 8** *Let $x_0$ be an arbitrary point of the domain $\Omega$ and let*

$$\alpha + s = \frac{N-1}{2}, \quad \alpha p = N, \quad s \geq 0, \quad p \geq 1. \tag{11.11}$$

*Then there exists a function $f \in \overset{\circ}{W}{}_p^\alpha(\Omega)$, which is continuous in $\Omega$, and such that*

$$\overline{\lim_{\lambda \to \infty}} E_\lambda^s f(x_0) = +\infty. \tag{11.12}$$

These results are extended to the Nikol'skii spaces in [19].

## 11.4 Summability of the Multiple Fourier Series of the Periodic Distributions

We denote by $C^\infty(T^N)$ the space of $2\pi$ periodic in each variable, infinitely differentiable on $R^N$ functions, where $T^N = \{x \in R^N : -\pi < x_j \leq \pi\}$.

Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_N)$ denote a multi-index. The system of the semi-norms $Sup_{x\in T^N} |D^\gamma f(x)|$ produces a locally convex topology in $C^\infty(T^N)$, where $\gamma$ runs over the set of all multi-indexes. We denote by $\mathscr{E}(T^N)$ corresponding locally convex topological space. Let $\mathscr{E}'(T^N)$ be the space of the periodic distributions, i.e. the space of the continuous linear functionals on $\mathscr{E}(T^N)$.

For any distribution $f$ from $\mathscr{E}'(T^N)$ we define its Fourier coefficients $f_n$ as the action of the distribution $f$ on the test function $(2\pi)^{-\frac{N}{2}} \exp(-inx)$, where $x \in T^N$ and $n \in Z^N$ is $N$ dimensional vector with integer coordinates. Then $f$ can be represented by the Fourier series

$$f = (2\pi)^{-\frac{N}{2}} \sum_{n\in Z^N} f_n \exp(inx), \tag{11.13}$$

which always converges in the weak topology (see, for example, in [25]).

Consider the following polynomial:

$$P_m(n) = \left(\sum_{j=1}^{r+1} n_j^2\right)^{m+1} + \left(\sum_{j=r+2}^{N} n_j^2\right)^{m} \left(\sum_{j=1}^{N} n_j^2\right), \tag{11.14}$$

where $n = (n_1, n_2, \dots, n_N) \in Z^N$, $m$ is a positive integer number and $r = 0, 1, \dots, N-1$.

The polynomial $P_m(n)$ is a homogeneous of degree $2(m+1)$, i.e.

$$P_m(\lambda \cdot n) = \lambda^{2(m+1)} \cdot P_m(n)$$

and an elliptic, i.e.

$$P_m(n) > 0, \ n \neq 0.$$

Thus a family of bounded sets

$$\Lambda(\lambda) \ = \ \{n \in Z^N : P_m(n) < \lambda\}, \ \lambda \in R^+\}$$

enjoying the following properties:

(a) for any pairs $(\lambda_1, \lambda_2) \in R^+ \times R^+$ there is $\lambda \in R^+$, such that $\Lambda(\lambda_1) \cup \Lambda(\lambda_2) \subset \Lambda(\lambda)$.

(b) $\bigcup_{\lambda\in R^+} \Lambda(\lambda) = Z^N$.

Let $f \in \mathscr{E}'(T^N)$. Then $\Lambda$-partial sums of series (11.13) define by equality

$$E_\lambda f(x) = (2\pi)^{-\frac{N}{2}} \sum_{\Lambda(\lambda)} f_n \, \exp(in \cdot x). \tag{11.15}$$

For any real $s$, $s \geq 0$, we define the Riesz means of (11.15) by

$$E_\lambda^s f(x) = (2\pi)^{\frac{-N}{2}} \sum_{\Lambda(\lambda)} \left(1 - \frac{P_m(n)}{\lambda}\right)^s f_n \exp(inx). \tag{11.16}$$

At $s = 0$ we obtain the partial sums (11.15).

Summability of the series (11.13), as well as its regularization (11.16), depends on the power of singularity of $f$. In order to classify singularities of distributions, we apply the periodic Liouville spaces $L_p^\alpha(T^N)$, $1 < p \leq \infty$, $\alpha \in R$ [37].

**Theorem 9** *Let $f \in L_p^{-\alpha}(T^N) \bigcap \mathscr{E}'(T^N)$, $1 < p \leq 2$, $\alpha > 0$, and coincides with zero in $\Omega \subset T^N$. If*

$$s > \max\left\{\frac{(N-r-1)(1-\frac{1}{2m})}{p} + \frac{r}{2}, \frac{N-1}{2}\right\} + \alpha \tag{11.17}$$

*then uniformly on any compact set $K \subset \Omega$*

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = 0.$$

The Riesz means (11.16) can be written as

$$E_\lambda^s f(x) =< f, \; D_\lambda^s(x-y) >, \tag{11.18}$$

where $f$ acts on $D_\lambda^s(x-y)$ by $y$ and $D_\lambda^s(x)$ is the Riesz means of $\Lambda$-partial sums of the multiple Fourier series of the Delta function:

$$D_\lambda^s(x) = (2\pi)^{\frac{-N}{2}} \sum_{\Lambda(\lambda)} \left(1 - \frac{P_m(n)}{\lambda}\right)^s \exp(inx). \tag{11.19}$$

If $r = N-1$, then $D_\lambda^s(x)$ is exactly the Riesz means of the Dirichlet kernel [35].

First, we estimate (11.19) in the norm of the positive Liouville spaces. In this, we use the relation between the kernel (11.19) and the relevant kernel of Fourier integrals. Such a relation known as the Poisson summation formula. The kernel for the corresponding Fourier integrals can be described by the same polynomial $P_m$ replacing its argument range from $n \in Z^N$ to $\xi \in R^N$:

$$\Theta_\lambda^s(x) = (2\pi)^{\frac{-N}{2}} \int_{\Lambda(\lambda)} \left(1 - \frac{P_m(\xi)}{\lambda}\right)^s \exp(i\xi \cdot x) d\xi, \tag{11.20}$$

where in the definition of the domain $\Lambda(\lambda)$ its range must be changed accordingly.

The following asymptotic formula is valid for the kernel (11.20) [9]:

**Lemma 2** *Let* $x \in R^N, x = (x', x''), x' \in R^{r+1}, x'' \in R^{N-r-1}, 0 < \delta_0 < |x'|, \mu = \lambda^{\frac{1}{2(m+1)}}$. *Then for* $|x''| < \varepsilon\mu^{-(1-\frac{1}{2m})}, 0 < \varepsilon < \frac{1}{2}$, *and* $\mu \longrightarrow \infty$

$$\Theta_\lambda^s(x) = \frac{c\mu^N \cos(\mu|x'| + (\frac{r}{2} - s)\frac{\pi}{2})}{(\mu|x'|)^{\frac{r+2}{2}+s+\frac{N-1-r}{2m}}} \times \left(1 + O\left(\frac{1}{\mu|x|}\right) + O(|x''|\mu^{1-\frac{1}{2m}})\right) \tag{11.21}$$

Using the Poisson summation formula the following relation between two kernel can be established

$$D_\lambda^s(x) = \Theta_\lambda^s(x) + \Theta_{*,\lambda}^s(x), \tag{11.22}$$

where $\Theta_{*,\lambda}^s(x)$ defined as

$$\Theta_{*,\lambda}^s(x) = \sum_{n \in Z^N,\ n \neq 0} \Theta_\lambda^s(x + 2\pi n). \tag{11.23}$$

Then from Lemma 3 immediately obtain the following.

**Lemma 3** *Let* $\varepsilon > 0$ *an arbitrary small number and* $|x_i| \leq 2\pi - \varepsilon$, *for any* $i = 1, 2, 3, \ldots, N$. *If* $s$ *satisfies (11.17), then*

$$\Theta_{*,\lambda}^s(x) = O(\lambda^{\frac{1}{2(m+1)}})^{N-s-1-\frac{r}{2}-\frac{N-1-r}{2m}} \tag{11.24}$$

Lemma 4 provides an estimate of the second term in the right-hand side of (11.23). Moreover, if $0 < \delta_0 < |x'|$, then from (11.21) we obtain an estimation for the first term in (11.23). Thus, we proved the following.

**Lemma 4** *Let* $\varepsilon > 0$ *an arbitrary small number and* $|x'| > \varepsilon$. *If* $s$ *satisfies (11.17), then*

$$D_\lambda^s(x) = O(\lambda^{\frac{1}{2(m+1)}})^{N-s-1-\frac{r}{2}-\frac{N-1-r}{2m}} \tag{11.25}$$

We have the following estimation (see [12, 13]):
*Let* $K \subset\subset T^N$ *a compact set, then uniformly by* $x \in K$

$$\|D_\lambda^s(x-y)\|_{L_2(F)} = O(\lambda^{\frac{N-1-2s}{4(m+1)}}), \tag{11.26}$$

*where* $F$ *an arbitrary domain in* $T^N$ *such that* $\overline{F} \bigcap K = \emptyset$.

Then using the Stein interpolation theorem for the analytical family of the linear operators [34, 36] with $q = \infty$ (Lemma 5) and $q = 2$ (estimation (11.26)), obtain the following estimate the kernel $D_\lambda^s(x)$ in the norm of $L_q(T^N)$

**Lemma 5** *Let $s$ satisfy (11.17) and $K \subset\subset T^N$ an arbitrary compact set. Then uniformly by $x \in K$*

$$\|D_\lambda^s(x-y)\|_{L_q(F)} = O(\lambda^{\frac{1}{2(m+1)}})^{N-s-1-\frac{r}{2}-\frac{N-1-r}{2m}}, \tag{11.27}$$

*where $F$ an arbitrary domain in $T^N$ such that $\overline{F} \bigcap K = \emptyset$, $2 \leq q \leq \infty$.*

Let a distribution $f$ have a compact support and belong to the space $L_p^{-\alpha}(T^N)$, where $1 < p \leq 2, \alpha > 0$. Let $K$ be an arbitrary compact set from $T^N \backslash suppf$ and $s$ satisfy (11.17). Then from (11.18) we get the following

$$|E_\lambda^s f(x)| \leq \|f\|_{-\alpha,p} \|D_\lambda^s(x-y)\|_{\alpha,q,F}, \tag{11.28}$$

where $\| \; \|_{-\alpha,p}$ means a norm in the space $L_p^{-\alpha}(T^N)$ and $\| \; \|_{\alpha,q,F}$ means a norm in the space $L_q^\alpha(F)$, $\frac{1}{q} = 1 - \frac{1}{p}$ and $suppf \subset F$ such that $\overline{F} \bigcap K = \emptyset$.

Then from (11.28) and the Lemma 4.5 it follows

$$E_\lambda^s f(x) = O(1)\|f\|_{-\alpha,p} \tag{11.29}$$

uniformly by $x$ from $K$. Then from (11.29) we get the statement of Theorem 9.

## 11.5 Spherical Partial Sum of the Multiple Fourier Series and Equiconvergence with the Fourier Integral

For any distribution $f \in \mathscr{E}'(T^N)$ and any real number $s \geq 0$ the Riesz means of order $s$ of the spherical partial sums of the series (11.13) is defined by

$$\sigma_\lambda^s f(x) = (2\pi)^{\frac{-N}{2}} \sum_{|n|<\lambda} \left(1 - \frac{|n|^2}{\lambda^2}\right)^s f_n \exp(inx). \tag{11.30}$$

where $f_n$ is the value of the functional $f$ on a "test" function $(2\pi)^{-\frac{N}{2}} \exp(-iy\xi)$. If $s = 0$ and $f = \delta$, then from (11.31) we obtain the Direchlet kernel [8].

Let's extend the distribution $f$ to $\mathscr{R}^N$ as

$$g = \begin{cases} f \text{ in } T^N, \\ 0, \text{ in } \mathscr{R}^N \setminus T^N. \end{cases} \tag{11.31}$$

Note that the distribution $g$ belongs to the space $\mathscr{E}'(\mathscr{R}^N)$. Denote by $\hat{g}$ its Fourier transformation. For example, for the Delta function we have $\hat{\delta}(x) = 1$. Then the Bochner–Riesz means of order $s$ of the Fourier integral of the Delta function is

$$\Theta_\lambda^s(x) = (2\pi)^{\frac{-N}{2}} \int\limits_{|y|<\lambda} \left(1 - \frac{|y|^2}{\lambda^2}\right)^s \exp(iy \cdot x)dy. \tag{11.32}$$

Then we can define the Riesz means of the spherical partial sums of the Fourier integral for any distribution $g \in \mathscr{E}'(\mathscr{R}^N)$ as follows:

$$R_\lambda^s g(x) =< g,\ \Theta_\lambda^s(x-y) > . \tag{11.33}$$

where $g$ is acting on $\Theta_\lambda^s(x-y)$ with respect to the variable $y$.

In the critical index $s = \frac{N-1}{2}$, Bochner [10] proved that the localization for (11.33) holds and at the same time it fails for the partial sum (11.30) in the class $L_1$. He also proved that the localization in the critical index is valid for both expansions in $L_2$. This result for the expansions in eigenfunctions of the Laplace operator is proved by Levitan [20]. Below critical index $\frac{N-1}{2}$ the problem studied by Il'in (see in [15]).

Summability of the spectral expansions of distributions is studied in [3]. In [3] Alimov obtained precise conditions of the localization of the spectral expansions associated with the Laplace operator. These questions for the Fourier series were studied in [5] and for the Forier integral studied in [25].

**Theorem 10** *Let $\ell > 0$ and $s = \frac{N-1}{2} + \ell$. Then for any $f \in L_p^{-\ell}(T^N)$ with $1 < p \le 2$ and supp$f \subset \Omega \subset\subset T^N$*

$$\sigma_\lambda^s f(x) = R_\lambda^s F(x) + O(1)\|f\|_{-\ell,p},$$

*where $x \in T^N \setminus \overline{\Omega}$ and $\|\cdot\|_{-\ell,p}$ a norm in $L_p^{-\ell}(T^N)$ :*

$$\|f\|_{-\ell,p} = (2\pi)^{-\frac{N}{2}} \|(1+|n|^2)^{\ell/2} f_n \exp(inx)\|_p.$$

In case $s < \frac{N-1}{2} + \ell$ the statement of the Theorem 10 is not valid for any distribution [25]. In case $p = 2$ the Theorem 10 is proved in [27].

Let $\Theta_{*,\lambda}^s(x)$ denote the following.

$$\Theta_{*,\lambda}^s(x) = \sum_{n\in Z^N, n\neq 0} \Theta_\lambda^s(x + 2\pi n). \tag{11.34}$$

Then we get

**Lemma 6** *Let $\ell > 0$, $s = \frac{N-1}{2} + \ell$. Then uniformly on any compact set $K \subset T^N$*

$$|\Theta_{*,\lambda}^s(x)| = O(\lambda^{-\frac{\ell}{4}}). \tag{11.35}$$

As in the previous section using the Poisson formula of summation we get the following relation between expansions (11.30) and (11.33)

$$\sigma_\lambda^s f(x) - R_\lambda^s F(x) =< f, \Theta_{*,\lambda}^s(x-y) > . \tag{11.36}$$

Let $\Omega_0 \subset\subset \Omega$ and $suppf \subset \Omega_0$. Then from the Cauchy-Schwartz inequality, taking into account that $f \in L_p^{-\ell}(T^N)$, obtain the following.

$$| < f, \Theta_{*,\lambda}^s(x-y)| \le \|f\|_{-\ell,p} \|\Theta_{*,\lambda}^s(x-y)\|_{\ell,p,0} \tag{11.37}$$

where $\|\Theta_{*,\lambda}^s(x-y) > \|_{\ell,p,0}$ is a norm of $\Theta_{*,\lambda}^s(x-y)$ in $L_p^\ell(\Omega_0)$ via $y \in \Omega_0$.
From (11.37) and the Lemma 4.5 we get the following.

**Lemma 7** *Let* $s = \frac{N-1}{2} + \ell,\ \ell > 0,\ f \in L_p^{-\ell}(T^N) \cap \mathscr{E}'(T^N),\ 1 < p \le 2$ *and let* $suppf \subset \Omega \subset\subset T^N$.
*Then uniformly on any compact set* $K \subset T^N \setminus \overline{\Omega}$

$$< f, \Theta_{*,\lambda}^s(x-y) >= \mathscr{O}(1)\|f\|_{-\ell,p}$$

Then the statement of the Theorem 10 follows, in the standard way, from the Lemma 5.3. This statement is proved in [27] in case $p = 2$ and in [31] for any $p$.

## 11.6 Uniform Convergence on Closed Domains

The uniform convergence of the Fourier series on the closed domains $\overline{\Omega}$ was studied by Il'in (see [14]). In [14] for the eigenfunction expansions associated with the first, second, and third boundary conditions for the Laplace operator it was proved that if $f \in W_p^{\frac{N+2}{2}}$, $p > \frac{2N}{N-1}$ and the functions $f, \Delta f, \ldots, \Delta^\beta f$, up to a certain order $\beta$, satisfy the appropriate boundary conditions, then the Fourier series of $f$ converges uniformly on the closed domain $\overline{\Omega}$.

For the elliptic differential operator of order $2m$ with the regular boundary conditions Eskin (see [11]) proved that the eigenfunction expansion of a function in $\overset{\circ}{W}_p^{\frac{2N-1}{4}+\varepsilon}$ with any $\varepsilon > 0$ converges uniformly on the closed domains.

Moiseev studied the problem for the elliptic operators of second order for the first boundary value problem. In [21] it is proved that if $f$ is a function with compact support in the space $W_p^{\frac{N-1}{2}}$, $p > \frac{2N}{N-1}$, such that the series

$$\sum_{n=1}^{\infty} \lambda_n^{\frac{N-1}{2}} (\ln \lambda_n)^{2+\varepsilon} f_n^2$$

converges, then its expansion via eigenfunctions converges uniformly on the closed domain $\overline{\Omega}$.

Moreover, it was proved in [21] that the following estimate

$$\sum_{|\sqrt{\lambda_n}-\mu|\le 1} u_n^2(x) = O(\mu^{N-1}\ln^2\mu) \tag{11.38}$$

is valid uniformly on the closed domain $\overline{\Omega}$.

In [4] uniform convergence of expansions via eigenfunctions of the elliptic differential operator of order $2m$ with the Lopatinsky boundary condition was studied and the following result is proved.

**Theorem 11** *Let $f$ be an arbitrary continuous function with compact support in $\Omega$. Then the Riesz means $E_\lambda^s f(x)$ of order $s > \frac{N}{2}$ converge to $f$ uniformly on the closed domain $\overline{\Omega}$.*

In [22] by using estimate (11.1) the condition $s > \frac{N}{2}$ in the Theorem 11 was replaced by $s > \frac{N-1}{2}$.

We mention also the following result (see [23]).

**Theorem 12** *Let*

$$\alpha + s > \frac{N-1}{2}, \quad \alpha p \ge N, \quad s \ge 0, \quad p \ge 1. \tag{11.39}$$

*Then for any continuous function $f \in \overset{\circ}{H}{}_p^\alpha(\Omega)$ with the compact support in the domain $\Omega$ uniformly on $\overline{\Omega}$*

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = f(x). \tag{11.40}$$

Note, from Theorem 8 it follows that in the case $\alpha p = N$ the condition $\alpha + s > (N-1)/2$ is precise. In the case $\alpha p > N$ this problem is open. The question of the summability on the closed domain in the spaces of the distributions is studied in [6].

Now we consider the problem of convergence of expansions via eigenfunctions in the spaces with mixed norm.

The space of all measurable functions with finite norm

$$\|f\|_{L_{pq}(\mathscr{R}^N)} = \|\|f\|_{L_p(\mathscr{R}^k)}\|_{L_q(\mathscr{R}^{N-k})}$$

is called the space with mixed norm $L_{pq}(\mathscr{R}^N)$. If a function is defined in the domain $\Omega$ then the corresponding space can be defined by extending a function by zero outside of the domain $\Omega$.

By $H_{pq}^\alpha$ we denote the Banach space of all measurable functions with respect to the norm

$$\|f\|_{H_{pq}^\alpha(\Omega)} = \|f\|_{L_{pq}(\Omega)} + \sum_{|k|=\ell} \sup_z |z|^{-\kappa} \|\Delta_z^2 \partial^k f(y)\|_{L_{pq}(\Omega_{|z|})}.$$

Here $\alpha = \ell + \kappa$, $\ell$ is a nonnegative integer, $0 < \kappa \leq 1$, $p, q \geq 1$, $k = (k_1, k_2, \ldots, k_n)$ multiindex. $|k| = k_1 + k_2 + \cdots + k_n$ and $\partial^k f$ denotes the weak derivative

$$\partial^k f(y) = \frac{\partial^{|k|} f(y)}{\partial y_1^{k_1}, \partial y_2^{k_2}, \ldots, \partial y_n^{k_n}}.$$

The symbol $\Delta_z^2 \partial^k f(y)$ denotes the second difference of the function $\partial^k f(y)$ :

$$\Delta_z^2 \partial^k f(y) = \partial^k f(y+z) - 2\partial^k f(y) + \partial^k f(y).$$

$\|f\|_{L_{pq}(\Omega)}$ denotes the norm in the space $L_{pq}$ and, for $h > 0$, $\Omega_h = \{x \in \Omega : dist(x, \partial\Omega) > h\}$.

By $\overset{\circ}{H}{}^{\alpha}_{pq}(\Omega)$ denote the closure of $C_0^{\infty}(\Omega)$ with respect to the norm of the space $H^{\alpha}_{pq}(\Omega)$.

Using the methods of [1, 2] for functions in the spaces with the mixed norm appropriate theorems on convergence of the spectral expansions associated with the Laplace operator on compact subsets of the domain were obtained in [33].

**Theorem 13** *Let $f(x)$ be a continuous function with compact support in the domain $\Omega$ belonging to the space $\overset{\circ}{H}{}^{\alpha}_{pq}(\Omega)$ and*

$$\alpha > \frac{N-1}{2} - s, \; \alpha = \frac{N-k}{q} + \frac{k}{p}, \;\; 2 \leq p < q, \;\; 0 < k < N. \tag{11.41}$$

*Then uniformly on $\overline{\Omega}$*

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^s f(x) = f(x).$$

This theorem in the spaces of distributions from the Sobolev spaces with the mixed norm is proved in [28].

Note that Theorems 11 and 12 are obtained only for the eigenfunction expansions associated with the first boundary problem for the Laplace operator. Analogue of the theorem 6.2 in the generalized Sobolev spaces of distributions is proved in [6]. Recently in [29] analogue of Theorem 12 in the generalized Sobolev spaces of distributions is proved for the eigenfunction expansions associated with the Navie boundary conditions for the bi-harmonic operator.

The problems of the convergence/summability of the Fourier series (spectral expansions) on the closed domains remain open for any other boundary conditions than first boundary condition (including second and third type boundary conditions) even for the Laplace operator and other operators then the Laplace operator with any boundary conditions.

## 11.7 Spectral Expansions Associated with the Operators with Singular Coefficients

In this section we consider the Schrodinger operator $L = -\Delta + q(x)$ with the domain $C_0^\infty(R^N)$, where $q(x)$ is potential with singularity at 0 satisfies following conditions

$$\left| \frac{\partial^{|\alpha|} q(x)}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \dots \partial^{\alpha_N} x_N} \right| \le \text{const}\, |x|^{-1-\alpha},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index.

From the Kato-Rellich theorem (see in [32, p. 185]) it follows that operator $L$ is essentially self-adjoin and bounded from the bottom with some constant $\mu$.

Denote by $\hat{L}$ its only self-adjoin extension (closure) in $L_2(R^N)$. Let $\{E_\lambda\}$ be the corresponding spectral decomposition of unity. It is well known that the operators $E_\lambda$ are integral operators whose kernels $\Theta(x, y, \lambda)$ belong to the $C^\infty(R^N)$ with respect to the both variables $x$ and $y$ for any $\lambda$. The spectral decomposition of an arbitrary function $g \in L_2(R^N)$ is defined by the formula

$$E_\lambda g(x) = \int_{R^N} g(y)\Theta(x, y, \lambda) dy.$$

Let $f \in \mathscr{E}'(R^N)$. Since $\Theta(x, y, \lambda) \in C^\infty(R^N \times R^N)$, it follows that one can define the spectral decomposition of $f$ by the formula

$$E_\lambda f(x) = \langle f, \Theta(x, y, \lambda) \rangle,$$

where the functional $f$ acts on $\Theta(x, y, \lambda)$ with respect to the second argument.

For any $s \ge 0$, we introduce the Riesz means of the spectral decomposition of $f$ by the formula

$$E_\lambda^s f(x) = \langle f, \Theta^s(x, y, \lambda) \rangle,$$

where $\Theta^s(x, y, \lambda)$ is the Riesz mean of order $s$ of the spectral function,

$$\Theta^s(x, y, \lambda) = \int_\mu^\lambda \left( \frac{\lambda - t}{\lambda - \mu} \right)^s d_t \Theta(x, y, t).$$

**Theorem 14** *Let $\ell > 0$, $s \ge 0$, and $f \in \mathscr{E}'(R^N) \cap W_2^{-l}(R^N)$. If $s \ge (N - l)/2 + \ell$, then*

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = 0,$$

*uniformly with the respect to $x \in K$ for any compact subset $K \subset R^N \setminus supp(f)$..*

**Corollary 1** *Let* $f \in \mathscr{E}'(R^N) \cap W_2^{-\ell}(R^N), \ell > 0$, *and let the distribution* $f$ *coincide with a continuous function* $g(x)$ *in a domain* $D$. *If* $s \geq (N-l)/2 + \ell$, *then*

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = g(x)$$

*uniformly on each compact set* $K \subset D$.

Spectral expansions, associated with the Schrodinger operator are studied by Khalmukhamedov in various functional spaces (see in [18]). In [26] localization problem of expansions via eigenfunctions of the Schrodinger operator in the bounded domain in the spaces of distributions is studied and the sharp conditions are established. The summability problems of the eigenfunction expansions connected with one Schrodinger operator on the closed domain with the smooth boundary are studied in [30].

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# Chapter 12
# Energy Decay in a Quasilinear System with Finite and Infinite Memories

**Muhammad I. Mustafa**

## 12.1 Introduction

In this paper, we are concerned with the following coupled quasilinear system

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g_1(s)\Delta u(t-s)ds + f_1(u,v) = 0, & \text{in}\Omega \times (0,\infty) \\ |v_t|^\rho v_{tt} - \Delta v - \Delta v_{tt} + \int_0^\infty g_2(s)\Delta v(t-s)ds + f_2(u,v) = 0, & \text{in}\Omega \times (0,\infty) \\ u = v = 0, \quad \text{on}\partial\Omega \times (0,\infty) & \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega & \\ v(x,-t) = v_0(x,t),\ v_t(x,0) = v_1(x), \quad x \in \Omega, t \geq 0, & \end{cases} \tag{12.1}$$

where $u$ and $v$ denote the transverse displacements of waves, $\rho > 0$ and $(n-2)\rho \leq 2$, and $\Omega$ is an open bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The relaxation functions $g_1, g_2$ and the nonlinearities $f_1, f_2$ will be specified later.

System (12.1) arises in the theory of viscoelasticity and describes the interaction of two scalar fields. It can also be regarded as a generalization, in some sense, of the well-known Klein–Gordon system

$$\begin{cases} u_{tt} - \Delta u + m_1 u + k_1 u v^2 = 0 \\ v_{tt} - \Delta v + m_2 v + k_2 u^2 v = 0 \end{cases}$$

which arises in the study of quantum field theory [37]. See also [19] for a generalization of this system. A slightly more general system (on $\mathbb{R}^n$) has been also investigated by Zhang in [38].

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The problem with the viscoelastic effect described by finite-memory terms has been studied by several authors. In [2], Andrade and Mognon considered a problem with the nonlinearities

$$f_1(u, v) = |u|^{p-2} u\, |v|^p \text{ and } f_2(u, v) = |v|^{p-2} v\, |u|^p$$

where $p > 1$ if $n = 1, 2$ and $1 < p \leq \frac{n-1}{n-2}$ if $n \geq 3$. They proved the well-posedness for the problem under restrictive assumptions on the relaxation functions. In [36], Santos considered the system

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + f_1(u, v) = 0 \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + f_2(u, v) = 0 \end{aligned} \tag{12.2}$$

and the coupling

$$f_1(u, v) = a(u - v) \text{ and } f_2(u, v) = -a(u - v),$$

where $a$ is a positive constant, and assumed that

$$\begin{aligned} -a_1 g_i^p(t) \leq g_i'(t) \leq -a_2 g_i^p(t) \\ 0 \leq g_i''(t) \leq \gamma g_i^p(t), \qquad i = 1, 2, \end{aligned}$$

for some $1 \leq p < 2$. He proved that when the kernels decay exponentially (resp. polynomially) the first- and the second-order energy of the solution decays exponentially (resp. polynomially). In [26], Messaoudi and Tatar considered the following weaker conditions on the relaxation functions

$$g_i'(t) \leq -c_i g_i^{p_i}(t), \qquad i = 1, 2,$$

for some $1 \leq p_1, p_2 < 3/2$, and more general forms of nonlinearities. They proved an exponential decay for $(p_1, p_2) = (1, 1)$ and a polynomial decay for $(p_1, p_2) \neq (1, 1)$. Liu [16] used the same hypothesis for a quasilinear system with finite memories and established uniform decay results.

The problem, with a single viscoelastic equation, has been extensively discussed by many authors. An abstract equation with infinite memory

$$u_{tt} + Au - \int_0^\infty g(s)Au(t-s)ds = 0$$

was initially studied by Dafermos [9]. He showed that the energy tends asymptotically to zero, but no decay rate was given. Under the condition that g decays exponentially, the exponential decay of solutions of this system was obtained by Fabrizio and Lazzari [10], Giorgi et al. [11], Liu and Zheng [18], and Rivera and Naso [29] (in different contexts and using different approaches). Also, in the finite memory case, we refer to [4–6, 27, 28, 32] for subsequent results which proved that the energy decays exponentially if the relaxation function $g$ decays exponentially and polynomially if $g$ decays polynomially. Similarly, the uniform decay of solutions was obtained by Rivera et al. [30, 31] for localized viscoelastic damping.

The interaction between viscoelastic and frictional dampings was considered by Cavalcanti and Oquendo [7] who looked into wave equation of the form

$$u_{tt} - k_0 \Delta u + \int_0^t \operatorname{div}[a(x) g(t-\tau) \nabla u(\tau)] d\tau + b(x) h(u_t) + f(u) = 0,$$

and established exponential stability for $g$ decaying exponentially and $h$ linear and polynomial stability for $g$ decaying polynomially and $h$ having a polynomial growth near zero. We also refer to results in [8, 25] for viscoelastic equation with a frictional damping acting on a part of the boundary.

Then, a natural question was raised: how does the energy behave as the kernel function does not necessarily decay polynomially or exponentially? Messaoudi [20, 21] studied

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(\tau) d\tau = b \left| u \right|^{\gamma} u$$

for $b = 0$ and $b = 1$ and considered relaxation functions satisfying

$$g'(t) \le -\xi(t) g(t), \ t \ge 0 \tag{12.3}$$

where $\xi(t) : \mathbb{R}_+ \to \mathbb{R}_+$is a nonincreasing differentiable function with

$$\left| \frac{\xi'(t)}{\xi(t)} \right| \le k \tag{12.4}$$

for some constant $k$. He proved that the decay rate of the solution energy is similar to that of the relaxation function which is not necessarily of exponential or polynomial type. Han and Wang treated an abstract viscoelastic system in [13] using the conditions (12.3) and (12.4). After that, Messaoudi and Mustafa [24, 34] eliminated condition (12.4) and used only (12.3) to establish more general stability results of viscoelastic Timoshenko beams. Similarly, (12.3) was used by Messaoudi and Al-Gharabli [22] and Guesmia and Messaoudi [12] for nonlinear wave equation with infinite memory, and Liu [15] for the equation

$$\left| u_t \right|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g_1(t-\tau) \Delta u(\tau) d\tau + \alpha(t) h(u_t) = b \left| u \right|^{p-2} u$$

to get more general decay rates. Another step forward is the work of Alabau-Boussouira and Cannarsa [1] who considered wave equation with memory whose relaxation function is satisfying

$$g'(t) \leq -\chi\,(g(t)) \tag{12.5}$$

where $\chi$ is a nonnegative function, with $\chi(0) = \chi'(0) = 0$, and $\chi$ is strictly increasing and strictly convex on $(0, k_0]$, for some $k_0 > 0$.They also required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \qquad \int_0^{k_0} \frac{x\,dx}{\chi(x)} < 1, \qquad \lim\inf_{s\to 0^+} \frac{\chi(s)/s}{\chi'(s)} > \frac{1}{2}$$

and proved an energy decay result. In addition, if $\lim\sup_{s\to 0^+} \frac{\chi(s)/s}{\chi'(s)} < 1$ and $g'(t) = -\chi\,(g(t))$, then, in this case, an explicit rate of decay is given. Here, a new theorem was announced which was applied to some new examples giving optimal decay rates. These assumptions imposed on $\chi$ do not appear intrinsic to the result claimed, but rather to the method based on weighted energy inequalities with the use of convexity. Later on, Mustafa and Messaoudi [35] and Lasiecka et al. [14] similarly used (12.5) and provided another variant of that approach which was able to remove some of the constraints imposed in [1] and obtain explicit and general decay rate formulas.

Motivated by these works, Liu [17] imposed the conditions (12.3) and (12.4) on $g_1, g_2$ in the coupled system (12.2) and improved the earlier result in [26]. After that, Mustafa [33] treated (12.2) using only (12.3). Recently, the same was done by Messaoudi and Al-Gharabli [23] who studied a similar system but with infinite memories.

Our aim in this work is to investigate the asymptotic behavior of system (12.1) and establish an explicit energy decay formula. We use weaker conditions on the relaxation functions and provide more general decay rates for which the usual exponential and polynomial rates are only special cases. The paper is organized as follows. In Sect. 12.2, we present some notation and material needed for our work. Some technical lemmas and the proof of our main result will be given in Sect. 12.3.

## 12.2 Preliminaries

We use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms. Throughout this paper, $c$ is used to denote a generic positive constant. We consider the following hypotheses

(H1) $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ (for $i = 1, 2$) are differentiable functions such that

$$g_i(0) > 0, \qquad 1 - \int_0^\infty g_i(s)ds = l_i > 0, \tag{12.6}$$

and there exists a $C^1$ function $H : (0, \infty) \to (0, \infty)$ which is linear or it is strictly increasing and strictly convex $C^2$ function on $(0, r]$, $r \leq g(0)$, with $H(0) = H'(0) = 0$, such that

$$g_i'(t) \leq -H(g_i(t)), \quad (i = 1, 2) \ \forall t > 0.$$

(H2) $f_i : \mathbb{R}^2 \to \mathbb{R}$ (for $i = 1, 2$) are $C^1$ functions and there exists a function $F$ such that

$$f_1(x, y) = \frac{\partial F}{\partial x}, \ f_2(x, y) = \frac{\partial F}{\partial y},$$

$$F \geq 0, \qquad x f_1(x, y) + y f_2(x, y) - F(x, y) \geq 0,$$

and

$$\left|\frac{\partial f_i}{\partial x}(x, y)\right| + \left|\frac{\partial f_i}{\partial y}(x, y)\right| \leq q(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1}) \qquad \forall (x, y) \in \mathbb{R}^2 \tag{12.7}$$

for some constant $q > 0$ and $\beta_{ij} \geq 1$, $(n-2)\beta_{ij} \leq n$ for $i, j = 1, 2$.

(H3) The history function $v_0 \in L^2(\mathbb{R}_+; H_0^1(\Omega))$.

(H4) $\rho$ is a constant satisfying

$$\rho > 0, \qquad (n-2)\rho \leq 2$$

In the sequel we assume that system (12.1) has a unique solution

$$u, v \in L^\infty(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}_+; H_0^1(\Omega)) \cap W^{2,\infty}(\mathbb{R}_+; L^2(\Omega)).$$

This result can be proved, for initial data in suitable function spaces, using standard arguments such as the Galerkin method (see [33]).

Now, we introduce the energy functional

$$\begin{aligned} E(t) := & \frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}\,dx + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\int_\Omega |\nabla u|^2\,dx \\ & + \frac{1}{2}\int_\Omega |\nabla u_t|^2\,dx + \frac{1}{2}(g_1 \circ \nabla u)(t) + \frac{1}{\rho+2}\int_\Omega |v_t|^{\rho+2}\,dx \\ & + \frac{l_2}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2}\int_\Omega |\nabla v_t|^2\,dx + \frac{1}{2}(g_2 \circ \nabla v)(t) + \int_\Omega F(u, v)dx, \end{aligned}$$

where

$$(g_1 \circ w)(t) = \int_\Omega \int_0^t g_1(t-s)\,|w(t) - w(s)|^2\,dsdx$$

and

$$(g_2 \circ w)(t) = \int_\Omega \int_0^\infty g_2(s)\,|w(t) - w(t-s)|^2\, dsdx.$$

Our main stability result is the following.

**Theorem 12.2.1** *Assume that (H1)–(H4) are satisfied. Then there exist positive constants* $k_1, k_2, k_3$ *and* $\varepsilon_0$ *such that the solution of (12.1) satisfies*

$$E(t) \le k_3 H_1^{-1}(k_1 t + k_2) \qquad \forall t \ge 0, \tag{12.8}$$

*where*

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\varepsilon_0 s)} ds \qquad \text{and} \qquad H_0(t) = H(D(t)$$

*provided that* $D$ *is a positive* $C^1$ *function, with* $D(0) = 0$, *for which* $H_0$ *is strictly increasing and strictly convex* $C^2$ *function on* $(0, r]$ *and, for* $i = 1, 2$,

$$\int_0^{+\infty} \frac{g_i(s)}{H_0^{-1}(-g_i'(s))} ds < +\infty. \tag{12.9}$$

*Moreover, if* $\int_0^1 H_1(t)dt < +\infty$ *for some choice of* $D$, *then we have the improved estimate*

$$E(t) \le k_3 G^{-1}(k_1 t + k_2) \qquad \text{where} \qquad G(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)} ds. \tag{12.10}$$

*In particular, this last estimate is valid in the special case* $H(t) = ct^p$ *for* $1 \le p < 2$.

*Remarks*

1. Using the properties of $H$, one can show that the function $H_1$ is strictly decreasing and convex on $(0, 1]$, with $\lim\limits_{t\to 0} H_1(t) = +\infty$. Therefore, Theorem 12.2.1 ensures
$$\lim_{t\to+\infty} E(t) = 0.$$
2. The condition $g_i'(t) \le -cg_i^p(t)$, $1 \le p < 2$, assumes $g_i(t) \le \omega e^{-ct}$ when $p = 1$ and $g_i(t) \le \frac{\omega}{(t+1)^{\frac{1}{p-1}}}$ when $1 < p < 2$. Our result allows relaxation functions that are not necessarily of exponential or polynomial type decay. For instance, if
$$g_i(t) = a\exp(-t^q), \quad i = 1, 2$$

for $0 < q < 1$ and $a$ chosen so that $g_i$ satisfies (12.6), then $g_i'(t) = -H(g_i(t))$ where, for $t \in (0, r]$, $r < a$,

$$H(t) = \frac{qt}{[\ln(a/t)]^{\frac{1}{q}-1}}$$

which satisfies hypothesis (H1). Also, by taking $D(t) = t^{\alpha}$, (12.9) is satisfied for any $\alpha > 1$. Therefore, if (H2)–(H4) are also satisfied, then we can use Theorem 12.2.1 and do some calculations (see [35]) to deduce that the energy decays at the rate

$$E(t) \leq c\exp(-\omega t^{q}).$$

4. The well-known Jensen's inequality will be of essential use in establishing our main result. If $F$ is a convex function on $[a, b]$, $f : \Omega \to [a, b]$ and $h$ are integrable functions on $\Omega$, $h(x) \geq 0$, and $\int_{\Omega} h(x)dx = k > 0$, then Jensen's inequality states that

$$F\left[\frac{1}{k}\int_{\Omega} f(x)h(x)dx\right] \leq \frac{1}{k}\int_{\Omega} F[f(x)]h(x)dx.$$

5. By (H1), we easily deduce that $\lim_{t\to+\infty} g_i(t) = 0$. Similarly, assuming the existence of the limit, we find that $\lim_{t\to+\infty}(-g_i'(t)) = 0$. Hence, there is $t_1 > 0$ large enough such that $g_i(t_1) > 0$ and

$$\max\{g_i(t), -g_i'(t)\} < \min\{r, H(r), H_0(r)\}, \ (i = 1, 2) \qquad \forall t \geq t_1. \quad (12.11)$$

As $g_i$ is nonincreasing, $g_i(0) > 0$ and $g_i(t_1) > 0$, then $g_i(t) > 0$ for any $t \in [0, t_1]$ and

$$0 < g_i(t_1) \leq g_i(t) \leq g_i(0), \qquad (i = 1, 2)\ \forall t \in [0, t_1].$$

Therefore, since $H$ is a positive continuous function, then

$$a \leq H(g_i(t)) \leq b, \qquad (i = 1, 2)\ \forall t \in [0, t_1]$$

for some positive constants $a$ and $b$. Consequently, for all $t \in [0, t_1]$,

$$g_i'(t) \leq -H(g_i(t)) \leq -a = -\frac{a}{g_i(0)} g_i(0) \leq -\frac{a}{g_i(0)} g_i(t)$$

which gives, for some positive constant $\mu$,

$$g_i'(t) \leq -\mu g_i(t), \qquad (i = 1, 2)\ \forall t \in [0, t_1]. \quad (12.12)$$

6. If different functions $H_1$ and $H_2$ have the properties mentioned in (H1) such that $g_1'(t) \leq -H_1(g_1(t))$ and $g_2'(t) \leq -H_2(g_2(t))$, then there is $r < \min\{r_1, r_2\}$ small enough so that, say, $H_1(t) \leq H_2(t)$ on the interval $(0, r]$. Thus, the function $H(t) = H_1(t)$ satisfies (H1) for both functions $g_1$ and $g_2$, $\forall\, t \geq t_1$.
7. We observe that assumption (12.7) gives, for some positive constant $k$, that

$$|f_i(x, y)| \leq k(|x| + |y| + |x|^{\beta_{i1}} + |y|^{\beta_{i2}}) \tag{12.13}$$

   for all $(x, y) \in \mathbb{R}^2$ and $i = 1, 2$. Using (12.13), instead of (12.7), is sufficient to get the stability result obtained in Sect. 12.3, but (12.6) is necessary for the well-posedness of the system, see [33].
8. We will also be using the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for $2 \leq s \leq 2n/(n-2)$ if $n \geq 3$ or $s \geq 2$ if $n = 1, 2$; i.e., for any $\phi \in H_0^1(\Omega)$,

$$\|\phi\|_s \leq c\, \|\nabla\phi\|_2\,. \tag{12.14}$$

## 12.3 Proof of the Main Result

In this section we prove Theorem 12.2.1. For this purpose we establish several lemmas.

**Lemma 12.3.1** *Let $(u, v)$ be the solution of (12.1). Then the energy functional satisfies*

$$E'(t) = \frac{1}{2}(g_1' \circ \nabla u) - \frac{1}{2} g_1(t) \int_\Omega |\nabla u|^2\, dx + \frac{1}{2}(g_2' \circ \nabla v) \leq 0. \tag{12.15}$$

*Proof* By multiplying the first equation in (12.1) by $u_t$ and the second by $v_t$, integrating over $\Omega$, using integration by parts, hypotheses (H1)–(H4), and some manipulations, we obtain (12.15).■

Using Cauchy–Schwarz and Poincaré's inequalities, the proof of the following Lemma is immediate.

**Lemma 12.3.2** *There exist constants c,c′ > 0 such that*

$$\int_\Omega \left( \int_0^t g_1(s)(u(t) - u(t-s))ds \right)^2 dx \leq c(g_1 \circ u)(t) \leq c'(g_1 \circ \nabla u)(t)$$

$$\int_\Omega \left( \int_0^\infty g_2(s)(v(t) - v(t-s))ds \right)^2 dx \leq c(g_2 \circ v)(t) \leq c'(g_2 \circ \nabla v)(t)$$

Now we are going to construct a Lyapunov functional $\mathcal{L}$ equivalent to $E$. For this, we define several functionals which allow us to obtain the needed estimates.

**Lemma 12.3.3** *Under the assumptions (H1)–(H4), the functional $I$ defined by*

$$I(t) := \frac{1}{\rho+1}\int_\Omega u\,|u_t|^\rho\,u_t dx + \int_\Omega \nabla u\cdot\nabla u_t dx + \frac{1}{\rho+1}\int_\Omega v\,|v_t|^\rho\,v_t dx + \int_\Omega \nabla v\cdot\nabla v_t dx$$

*satisfies, along the solution, the estimate*

$$\begin{aligned} I'(t) \le & -\frac{l_1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla u_t|^2\,dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}\,dx + c(g_1\circ\nabla u)(t)\\ & -\frac{l_2}{2}\int_\Omega |\nabla v|^2\,dx + \int_\Omega |\nabla v_t|^2\,dx + \frac{1}{\rho+1}\int_\Omega |v_t|^{\rho+2}\,dx + c(g_2\circ\nabla v)(t)\\ & -\int_\Omega F(u,v)dx. \end{aligned} \tag{12.16}$$

*Proof* Direct computations, using (12.1), yield

$$\begin{aligned} I'(t) = & \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}\,dx - \int_\Omega |\nabla u|^2\,dx + \int_\Omega \nabla u\cdot\int_0^t g_1(s)\nabla u(t-s)dsdx\\ & + \int_\Omega |\nabla u_t|^2\,dx + \frac{1}{\rho+1}\int_\Omega |v_t|^{\rho+2}\,dx - \int_\Omega |\nabla v|^2\,dx + \int_\Omega |\nabla v_t|^2\,dx\\ & + \int_\Omega \nabla v\cdot\int_0^\infty g_2(s)\nabla v(t-s)dsdx - \int_\Omega [uf_1(u,v)+vf_2(u,v)]dx\\ \le & \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}\,dx - l_1\int_\Omega |\nabla u|^2\,dx + \int_\Omega \nabla u\\ & \cdot\int_0^t g_1(s)(\nabla u(t-s)-\nabla u(t))dsdx\\ & + \int_\Omega |\nabla u_t|^2\,dx + \frac{1}{\rho+1}\int_\Omega |v_t|^{\rho+2}\,dx - l_2\int_\Omega |\nabla v|^2\,dx + \int_\Omega |\nabla v_t|^2\,dx\\ & + \int_\Omega \nabla v\cdot\int_0^\infty g_2(s)(\nabla v(t-s)-\nabla v(t))dsdx - \int_\Omega F(u,v)dx. \end{aligned} \tag{12.17}$$

Now, using Young's inequality and Lemma 12.3.2, we obtain, for any $\delta > 0$,

$$\begin{aligned} & \int_\Omega \nabla u\cdot\int_0^t g_1(s)(\nabla u(t-s)-\nabla u(t))dsdx\\ & \le \delta\int_\Omega |\nabla u|^2\,dx + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1(s)\,|\nabla u(t-s)-\nabla u(t)|\,ds\right)^2 dx\\ & \le \delta\int_\Omega |\nabla u|^2\,dx + \frac{c}{\delta}(g_1\circ\nabla u)(t). \end{aligned} \tag{12.18}$$

Similar calculations also yield

$$\int_\Omega \nabla v \cdot \int_0^\infty g_2(s)(\nabla v(t-s) - \nabla v(t))dsdx \le \delta \int_\Omega |\nabla v|^2 \, dx + \frac{c}{\delta}(g_2 \circ \nabla v)(t). \tag{12.19}$$

Combining (12.17)–(12.19) and choosing $\delta$ small enough give (12.16).■

**Lemma 12.3.4** *Under the assumptions (H1)–(H4), the functional $K$ defined by*

$$K(t) = K_1(t) + K_2(t),$$

with

$$K_1(t) := \int_\Omega \left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right) \int_0^t g_1(s)(u(t) - u(t-s))dsdx$$
$$K_2(t) := \int_\Omega \left(\Delta v_t - \frac{|v_t|^\rho v_t}{\rho+1}\right) \int_0^\infty g_2(s)(v(t) - v(t-s))dsdx,$$

*satisfies, for any* $0 < \delta < 1$ *and for all* $t \ge t_1$*, the estimate*

$$\begin{aligned} K'(t) \le & -\frac{g_0}{\rho+1}\int_\Omega |u_t|^{\rho+2}\,dx - \frac{g_0}{2}\int_\Omega |\nabla u_t|^2\,dx + \delta c \int_\Omega |\nabla u|^2\,dx \\ & + \frac{c}{\delta}(g_1 \circ \nabla u)(t) - c(g_1' \circ \nabla u)(t) - \frac{g_0}{\rho+1}\int_\Omega |v_t|^{\rho+2}\,dx \\ & - \frac{g_0}{2}\int_\Omega |\nabla v_t|^2\,dx + \delta c \int_\Omega |\nabla v|^2\,dx + \frac{c}{\delta}(g_2 \circ \nabla v)(t) - c(g_2' \circ \nabla v)(t). \end{aligned} \tag{12.20}$$

*Here,* $g_0 = \min\{\int_0^{t_1} g_1(s)ds, 1 - l_2\}$ *where* $t_1 > 0$ *was introduced in (12.11).*

*Proof* By exploiting Eq. (12.1) and integrating by parts, we have

$$\begin{aligned} K_1'(t) = & \left(1 - \int_0^t g_1(s)ds\right)\int_\Omega \nabla u \cdot \int_0^t g_1(t-\tau)(\nabla u(t) - \nabla u(\tau))d\tau dx \\ & + \int_\Omega \left(\int_0^t g_1(t-\tau)\,|\nabla u(t) - \nabla u(\tau)|\,d\tau\right)^2 dx \\ & + \int_\Omega f_1(u,v)\int_0^t g_1(t-\tau)(u(t) - u(\tau))d\tau dx \\ & - \int_\Omega \nabla u_t \cdot \int_0^t g_1'(t-\tau)(\nabla u(t) - \nabla u(\tau))d\tau dx \\ & - \left(\int_0^t g_1(s)ds\right)\int_\Omega |\nabla u_t|^2\,dx \end{aligned}$$

$$-\int_\Omega \frac{|u_t|^\rho u_t}{\rho+1}\int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx$$
$$-\frac{\left(\int_0^t g_1(s)ds\right)}{\rho+1}\int_\Omega |u_t|^{\rho+2}dx. \tag{12.21}$$

For $K_2'(t)$, it will be $\left(1-\int_0^\infty g_2(s)ds\right)=l_2$ and $\left(\int_0^\infty g_2(s)ds\right)=1-l_2$. Using Young's inequality and Lemma 12.3.2, we obtain

$$\left(1-\int_0^t g_1(s)ds\right)\int_\Omega \nabla u\cdot\int_0^t g_1(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx$$
$$+\int_\Omega\left(\int_0^t g_1(t-\tau)\,|\nabla u(t)-\nabla u(\tau)|\,d\tau\right)^2 dx$$
$$\le \delta\int_\Omega |\nabla u|^2 dx+\frac{c}{\delta}(g_1\circ\nabla u)(t) \tag{12.22}$$

and

$$-\int_\Omega \nabla u_t\cdot\int_0^t g_1'(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx\le\delta_1\int_\Omega|\nabla u_t|^2dx-\frac{c}{\delta_1}(g_1'\circ\nabla u)(t). \tag{12.23}$$

To estimate the third and sixth terms in the right-hand side of (12.21), we use (12.13), (12.14), Lemma 12.3.2, and the fact that

$$l_1\|\nabla u\|_2^2+\|\nabla u_t\|_2^2+l_2\|\nabla v\|_2^2+\|\nabla v_t\|_2^2\le 2E(t)\le 2E(0)$$

to get

$$\int_\Omega f_1(u,v)\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx$$
$$\le\delta c\int_\Omega(|u|^2+|v|^2+|u|^{2\beta_{11}}+|v|^{2\beta_{12}})dx$$
$$+\frac{c}{\delta}\int_\Omega\left(\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau\right)^2dx$$
$$\le\delta c(\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\nabla u\|_2^{2\beta_{11}}+\|\nabla v\|_2^{2\beta_{12}})+\frac{c}{\delta}(g_1\circ\nabla u)(t)$$
$$=\delta c(\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\nabla u\|_2^{2(\beta_{11}-1)}\|\nabla u\|_2^2+\|\nabla v\|_2^{2(\beta_{12}-1)}\|\nabla v\|_2^2)$$
$$+\frac{c}{\delta}(g_1\circ\nabla u)(t)$$
$$\le\delta c(\|\nabla u\|_2^2+\|\nabla v\|_2^2+[2E(0)]^{(\beta_{11}-1)}\|\nabla u\|_2^2+[2E(0)]^{(\beta_{12}-1)}\|\nabla v\|_2^2)$$

$$
\begin{aligned}
&+\frac{c}{\delta}(g_1 \circ \nabla u)(t) \\
&\leq \delta c \left\|\nabla u\right\|_2^2 + \delta c \left\|\nabla v\right\|_2^2 + \frac{c}{\delta}(g_1 \circ \nabla u)(t). \qquad (12.24)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_\Omega \frac{|u_t|^\rho u_t}{\rho+1} \int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx \\
&\leq \delta_1 \int_\Omega |u_t|^{2(\rho+1)} dx - \frac{c}{\delta_1}(g_1' \circ \nabla u)(t) \\
&\leq c\delta_1 \left\|\nabla u_t\right\|_2^{2(\rho+1)} - \frac{c}{\delta_1}(g_1' \circ \nabla u)(t) \\
&\leq c\delta_1 [2E(0)]^\rho \left\|\nabla u_t\right\|_2^2 - \frac{c}{\delta_1}(g_1' \circ \nabla u)(t). \qquad (12.25)
\end{aligned}
$$

By combining (12.21)–(12.25), using $-\left(\int_0^t g_1(s)ds\right) \leq -g_0$ for all $t \geq t_1$, and taking $\delta_1$ small enough, we obtain

$$
\begin{aligned}
K_1'(t) \leq &-\frac{g_0}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx - \frac{g_0}{2}\int_\Omega |\nabla u_t|^2 dx + \delta c \int_\Omega |\nabla u|^2 dx \\
&+ \frac{c}{\delta}(g_1 \circ \nabla u)(t) - c(g_1' \circ \nabla u)(t)
\end{aligned}
$$

Since $K_2$ can be dealt with similarly, (12.20) is established.■

*Proof of Theorem 12.2.1* For $N_1 > 0$, let

$$
\mathcal{L}(t) := N_1 E(t) + \frac{4}{g_0} K(t) + I(t)
$$

By combining (12.15), (12.16), (12.20), and taking $\delta = \frac{lg_0}{16c}$ in (12.20), where $l = \min\{l_1, l_2\}$, we obtain, for all $t \geq t_1$,

$$
\begin{aligned}
\mathcal{L}'(t) \leq &-\frac{l}{4}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \frac{3}{\rho+1}\int_\Omega (|u_t|^{\rho+2} + |v_t|^{\rho+2})dx \\
&- \int_\Omega |\nabla u_t|^2 dx - \int_\Omega |\nabla v_t|^2 dx - \int_\Omega F(u,v)dx \\
&+ \left(\frac{64c^2}{lg_0^2} + c\right)[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] \\
&+ \left(\frac{1}{2}N_1 - \frac{4c}{g_0}\right)[(g_1' \circ \nabla u)(t) + (g_2' \circ \nabla v)(t)].
\end{aligned}
$$

At this point, we choose $N_1$ large enough so that

$$\left(\frac{1}{2}N_1 - \frac{4c}{g_0}\right) > 0.$$

So, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -\frac{l}{4}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \frac{3}{\rho+1}\int_\Omega (|u_t|^{\rho+2} + |v_t|^{\rho+2})dx \\ & -\int_\Omega |\nabla u_t|^2\, dx - \int_\Omega |\nabla v_t|^2\, dx - \int_\Omega F(u,v)dx \\ & + c(g_1 \circ \nabla u)(t) + c(g_2 \circ \nabla v)(t),\end{aligned}$$

which yields

$$\mathcal{L}'(t) \leq -mE(t) + c(g_1 \circ \nabla u)(t) + c(g_2 \circ \nabla v)(t), \quad \text{for all } t \geq t_1. \tag{12.26}$$

On the other hand, we can choose $N_1$ even larger so that

$$\mathcal{L}(t) \sim E(t)$$

which means that, for some constants $a_1, a_2 > 0$,

$$a_1 E(t) \leq \mathcal{L}(t) \leq a_2 E(t).$$

Now, we use (12.12) and (12.15) to conclude that, for any $t \geq t_1$,

$$\begin{aligned}&\int_0^{t_1} g_1(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\, dxds \\ &\quad + \int_0^{t_1} g_2(s)\int_\Omega |\nabla v(t) - \nabla v(t-s)|^2\, dxds \\ &\quad\leq -\frac{1}{\mu}\int_0^{t_1} g_1'(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\, dxds \\ &\qquad -\frac{1}{\mu}\int_0^{t_1} g_2'(s)\int_\Omega |\nabla v(t) - \nabla v(t-s)|^2\, dxds \\ &\quad\leq -cE'(t).\end{aligned} \tag{12.27}$$

Next, we take $F(t) = \mathcal{L}(t) + cE(t)$, which is clearly equivalent to $E(t)$, and use (12.26) and (12.27), to get

$$F'(t) \leq -mE(t) + c\int_{t_1}^{t} g_1(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\, dxds$$

$$+c\int_{t_1}^{\infty} g_2(s)\int_{\Omega}|\nabla v(t)-\nabla v(t-s)|^2\,dxds. \tag{12.28}$$

**(I)** $H(t)$ **is linear:** In this case, estimate (12.28) yields

$$F'(t)\le -mE(t)-c(g_1'\circ\nabla u)(t)-c(g_2'\circ\nabla v)(t)\le -mE(t)-cE'(t),\ \ \forall t\ge t_1$$

which gives

$$(F+cE)'(t)\le -mE(t).$$

Hence, using the fact that $F+cE\sim E$, we easily obtain

$$E(t)\le k_1e^{-k_2t}.$$

**(II)** $H(t)$ **is nonlinear:** We define $I(t)$ by

$$I(t):=\int_{t_1}^{t}\frac{g_1(s)}{H_0^{-1}(-g_1'(s))}\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds$$

where $H_0$ is such that (12.9) is satisfied. Using (12.9), (12.15), and choosing $t_1$ even larger if needed, we deduce that, for all $t\ge t_1$,

$$\begin{aligned}I(t)&\le 2\int_{t_1}^{t}\frac{g_1(s)}{H_0^{-1}(-g_1'(s))}\int_{\Omega}(|\nabla u(t)|^2+|\nabla u(t-s)|^2)dxds\\&\le cE(0)\int_{t_1}^{t}\frac{g_1(s)}{H_0^{-1}(-g_1'(s))}ds<1\end{aligned}\tag{12.29}$$

We also assume, without loss of generality, that $I(t)>0$; otherwise, (12.28) yields an exponential decay. In addition, we define $\xi(t)$ by

$$\xi(t):=-\int_{t_1}^{t}g_1'(s)\frac{g_1(s)}{H_0^{-1}(-g_1'(s))}\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds$$

and infer from (H1) and the properties of $H_0$ and $D$ that

$$\frac{g_i(s)}{H_0^{-1}(-g_i'(s))}\le\frac{g_i(s)}{H_0^{-1}(H(g_i(s)))}=\frac{g_i(s)}{D^{-1}(g_i(s))}\le k_0\quad\forall i=1,2,$$

for some positive constant $k_0$. Then, using (12.15) and choosing $t_1$ even larger (if needed), one can easily see that $\xi(t)$ satisfies, for all $t\ge t_1$,

$$\begin{aligned}
\xi(t) &\leq -k_0 \int_{t_1}^{t} g_1'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\
&\leq -cE(0) \int_{t_1}^{t} g_1'(s) \leq c g_1(t_1) \\
&< \frac{1}{2} \min\{r, H(r), H_0(r)\}. \qquad (12.30)
\end{aligned}$$

Since $H_0$ is strictly convex on $(0, r]$ and $H_0(0) = 0$, then

$$H_0(\theta x) \leq \theta H_0(x)$$

provided $0 \leq \theta \leq 1$ and $x \in (0, r]$. The use of this fact, hypothesis (H1), (12.11), (12.29), (12.30), and Jensen's inequality leads to

$$\begin{aligned}
\xi(t) &= \frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0[H_0^{-1}(-g_1'(s))] \frac{g_1(s)}{H_0^{-1}(-g_1'(s))} \\
&\quad \times \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\
&\geq \frac{1}{I(t)} \int_{t_1}^{t} H_0[I(t) H_0^{-1}(-g_1'(s))] \frac{g_1(s)}{H_0^{-1}(-g_1'(s))} \\
&\quad \times \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\
&\geq H_0\left( \frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0^{-1}(-g_1'(s)) \frac{g_1(s)}{H_0^{-1}(-g_1'(s))} \right. \\
&\quad \left. \times \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \right) \\
&= H_0\left( \int_{t_1}^{t} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \right)
\end{aligned}$$

This implies that

$$\int_{t_1}^{t} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \leq H_0^{-1}(\xi(t)). \qquad (12.31)$$

We also define

$$\phi(t) := \int_{t_1}^{\infty} \frac{g_2(s)}{H_0^{-1}(-g_2'(s))} \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds$$

$$\chi(t) := -\int_{t_1}^{\infty} g_2'(s) \frac{g_2(s)}{H_0^{-1}(-g_2'(s))} \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds.$$

We similarly deduce that

$$\phi(t) < 1$$

and

$$\chi(t) < \frac{1}{2}\min\{r, H(r), H_0(r)\}. \tag{12.32}$$

Repeating the above steps, we arrive at

$$\int_{t_1}^{\infty} g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2 \, dx ds \leq H_0^{-1}(\chi(t)). \tag{12.33}$$

Then, inserting the estimates (12.31) and (12.33) into (12.28) and using the properties of $H_0$, we obtain

$$\begin{aligned} F'(t) &\leq -mE(t) + cH_0^{-1}(\xi(t)) + H_0^{-1}(\chi(t)) \\ &\leq -mE(t) + cH_0^{-1}(\xi(t) + \chi(t)), \qquad \forall t \geq t_1. \end{aligned} \tag{12.34}$$

Now, for $\varepsilon_0 < r$ and $c_0 > 0$, using (12.34), and the fact that $E' \leq 0$, $H_0' > 0$, $H_0'' > 0$ on $(0, r]$, we find that the functional $F_1$, defined by

$$F_1(t) := H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F(t) + c_0 E(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 F_1(t) \leq E(t) \leq \alpha_2 F_1(t) \tag{12.35}$$

and

$$\begin{aligned} F_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H_0''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F(t) + H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F'(t) + c_0 E'(t) \\ &\leq -mE(t) H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cH_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) H_0^{-1}(\xi(t) + \chi(t)) + c_0 E'(t). \end{aligned} \tag{12.36}$$

Let $H_0^*$ be the convex conjugate of $H_0$ in the sense of Young (see [3, pp. 61–64]), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \qquad \text{if} s \in (0, H_0'(r)] \tag{12.37}$$

and $H_0^*$ satisfies the following Young's inequality

$$AB \le H_0^*(A) + H_0(B), \qquad \text{if} A \in (0, H_0'(r)], B \in (0, r] \tag{12.38}$$

With $A = H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = H_0^{-1}(\xi(t) + \chi(t))$, using (12.15), (12.30), (12.32), and (12.36)–(12.38), we arrive at

$$\begin{aligned} F_1'(t) &\le -mE(t)H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cH_1^*\left(H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) \\ &\quad + c\xi(t) + c\chi(t) + c_0 E'(t) \\ &\le -mE(t)H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)} H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) + c_0 E'(t). \end{aligned}$$

Consequently, with a suitable choice of $\varepsilon_0$ and $c_0$, we obtain

$$F_1'(t) \le -\tau \left(\frac{E(t)}{E(0)}\right) H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -\tau H_2\left(\frac{E(t)}{E(0)}\right), \tag{12.39}$$

where $H_2(t) = tH_0'(\varepsilon_0 t)$.

Since $H_2'(t) = H_0'(\varepsilon_0 t) + \varepsilon_0 t H_0''(\varepsilon_0 t)$, then, using the strict convexity of $H_0$ on $(0, r]$, we find that $H_2'(t)$, $H_2(t) > 0$ on $(0, 1]$. Thus, with

$$R(t) = \epsilon \frac{\alpha_1 F_1(t)}{E(0)}, 0 < \epsilon < 1$$

taking into account (12.35) and (12.39), we have

$$R(t) \sim E(t) \tag{12.40}$$

and, for some $k_1 > 0$,

$$R'(t) \le -\epsilon k_1 H_2(R(t)).$$

Then, a simple integration and a suitable choice of $\epsilon$ yield, for some $k_2 > 0$,

$$R(t) \le H_1^{-1}(k_1 t + k_2). \tag{12.41}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds$.

Here, we used, based on the properties of $H_2$, the fact that $H_1$ is strictly decreasing on $(0, 1]$ and $\lim_{t \to 0} H_1(t) = +\infty$. Using (12.40)–(12.41) and by virtue of continuity and boundedness of $E$, we obtain (12.8).

Moreover, if $\int_0^1 H_1(t)dt < +\infty$, then $\int_0^{+\infty} H_1^{-1}(t)dt < +\infty \underset{\text{by (12.8)}}{\Longrightarrow}$ $\int_0^{+\infty} E(t)dt < +\infty$. Then, using (H3), we have

$$\begin{aligned}
&\int_0^t \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\, dxds + \int_0^\infty \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2\, dxds \\
&\quad \le 2 \int_0^t \int_\Omega |\nabla u(t)|^2\, dxds + 2 \int_0^t \int_\Omega |\nabla u(t-s)|^2\, dxds \\
&\qquad +2 \int_0^\infty \int_\Omega |\nabla v(t)|^2\, dxds + 2 \int_0^t \int_\Omega |\nabla v(t-s)|^2\, dxds \\
&\qquad +2 \int_0^\infty \int_\Omega |\nabla v_0(s)|^2\, dxds \\
&\quad \le c \int_0^\infty E(s)ds + 2 \int_0^\infty \int_\Omega |\nabla v_0(s)|^2\, dxds \\
&\quad < +\infty.
\end{aligned}$$

Therefore, we can repeat the same procedures with

$$\begin{aligned}
I(t) &:= \int_{t_1}^t \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\, dxds, \\
\phi(t) &:= \int_{t_1}^\infty \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2\, dxds
\end{aligned}$$

and

$$\begin{aligned}
\xi(t) &:= -\int_{t_1}^t g_1'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\, dxds, \\
\chi(t) &:= -\int_{t_1}^\infty g_2'(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2\, dxds
\end{aligned}$$

to establish (12.10).

**(III)** $H(t) = ct^p$ where $1 < p < 2$: This means that, for $i = 1, 2$,

$$g_i(t) \le \frac{\omega}{(t+1)^{\frac{1}{p-1}}}$$

which we use with Holder's inequality, for parameter $q_1 > p$, and (H3) and the fact that $g_2(t)$ and $E(t)$ are decreasing to get

$$(g_2 \circ \nabla v)(t)$$

$$= \int_0^\infty g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dx ds$$

$$= \int_0^\infty g_2^{\frac{p}{q_1}}(s) g_2^{1-\frac{p}{q_1}}(s)$$

$$\times \left(\|\nabla v(t) - \nabla v(t-s)\|^2\right)^{\frac{1}{q_1}} \left(\|\nabla v(t) - \nabla v(t-s)\|^2\right)^{1-\frac{1}{q_1}} ds$$

$$\leq \left[\int_0^\infty g_2^p(s) \, \|\nabla v(t) - \nabla v(t-s)\|^2 \, ds\right]^{\frac{1}{q_1}}$$

$$\times \left[\int_0^\infty g_2^{\frac{q_1-p}{q_1-1}}(s) \, \|\nabla v(t) - \nabla v(t-s)\|^2 \, ds\right]^{\frac{q_1-1}{q_1}}$$

$$\leq \left[-c \int_0^\infty g_2'(s) \, \|\nabla v(t) - \nabla v(t-s)\|^2 \, ds\right]^{\frac{1}{q_1}}$$

$$\times \left[2 \int_0^\infty g_2^{\frac{q_1-p}{q_1-1}}(s) \left(\|\nabla v(t)\|^2 + \|\nabla v(t-s)\|^2\right) ds\right]^{\frac{q_1-1}{q_1}}$$

$$\leq \left[-cE'(t)\right]^{\frac{1}{q_1}} \left[\begin{array}{l} 2 \int_0^\infty g_2^{\frac{q_1-p}{q_1-1}}(s) \, \|\nabla v(t)\|^2 \, ds \\ \quad + 2 \int_0^t g_2^{\frac{q_1-p}{q_1-1}}(s) \, \|\nabla v(t-s)\|^2 \, dx ds \\ \quad + 2 g_2^{\frac{q_1-p}{q_1-1}}(0) \int_0^\infty \|\nabla v_0(s)\|^2 \, ds \end{array}\right]^{\frac{q_1-1}{q_1}}$$

$$\leq \left[-cE'(t)\right]^{\frac{1}{q_1}} \left[cE(0) \int_0^\infty \left(\frac{\omega}{(s+1)^{\frac{1}{p-1}}}\right)^{\frac{q_1-p}{q_1-1}} ds + c\right]^{\frac{q_1-1}{q_1}}.$$

Therefore, under the condition $\left[1 - \frac{q_1-p}{(q_1-1)(p-1)} < 0\right]$ which requires

$$\frac{1}{q_1 - 1} < \frac{2-p}{p-1},$$

we guarantee that

$$(g_2 \circ \nabla v)(t) \leq c \left[-E'(t)\right]^{\frac{1}{q_1}} = c H_{01}^{-1}(-E'(t))$$

where $H_{01}(t) = t^{q_1}$. Similarly, we get

$$(g_1 \circ \nabla u)(t) \leq c H_{01}^{-1}(-E'(t))$$

which lead, as in case II above, to (12.8) giving the following energy decay rates

$$E(t) \leq \frac{c}{(t+1)^{\frac{1}{q_1-1}}}.$$

Making use of this result gives, when $s < t$,

$$\|\nabla v(t)\|^2 + \|\nabla v(t-s)\|^2 \leq \frac{c}{(t+1)^{\frac{1}{q_1-1}}} + \frac{c}{(t-s+1)^{\frac{1}{q_1-1}}}$$

and repeating the process with another parameter $q_2$, where $q_1 > q_2 \geq p$, we similarly find

$$\begin{aligned}
&(g_2 \circ \nabla v)(t) \\
&\leq \left[-cE'(t)\right]^{\frac{1}{q_2}} \begin{bmatrix} 2\int_0^\infty g_2^{\frac{q_2-p}{q_2-1}}(s)\, \|\nabla v(t)\|^2\, ds \\ \quad +2\int_0^t g_2^{\frac{q_2-p}{q_2-1}}(s)\, \|\nabla v(t-s)\|^2\, dxds \\ \quad +2g_2^{\frac{q_2-p}{q_2-1}}(0)\int_0^\infty \|\nabla v_0(s)\|^2\, ds \end{bmatrix}^{\frac{q_2-1}{q_2}} \\
&\leq \left[-cE'(t)\right]^{\frac{1}{q_2}} \left[ c\int_0^\infty \left(\frac{\omega}{(s+1)^{\frac{1}{p-1}}}\right)^{\frac{q_2-p}{q_2-1}} \right. \\
&\quad \left. \times \left(\frac{1}{(t+1)^{\frac{1}{q_1-1}}} + \frac{1}{(t-s+1)^{\frac{1}{q_1-1}}}\right) ds + c \right]^{\frac{q_2-1}{q_2}} \\
&\leq c\left[-E'(t)\right]^{\frac{1}{q_2}} = cH_{02}^{-1}(-E'(t))
\end{aligned}$$

where $H_{02}(t) = t^{q_2}$ provided that $\left[1 - \frac{q_2-p}{(q_2-1)(p-1)} - \frac{1}{q_1-1} < 0\right]$ which requires

$$\frac{1}{q_2-1} < \frac{1}{q_1-1} + \frac{2-p}{p-1} < 2\left(\frac{2-p}{p-1}\right).$$

This yields the following improved energy decay rate

$$E(t) \leq \frac{c}{(t+1)^{\frac{1}{q_2-1}}}.$$

We continue this process and get a sequence $(q_n)$ with the requirement

$$\frac{1}{q_n - 1} < \frac{1}{q_{n-1} - 1} + \frac{2-p}{p-1} < \frac{1}{q_{n-2} - 1} + 2\left(\frac{2-p}{p-1}\right) < \cdots < n\left(\frac{2-p}{p-1}\right)$$

which allows to reach the optimal value $q_n = p$ in finitely many steps $n$ provided $n > \frac{1}{2-p}$. In this case, we obtain, for $p \in (1, 2)$, the optimal decay rate

$$E(t) \leq \frac{c}{(t+1)^{\frac{1}{p-1}}}. \quad \blacksquare$$

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# Chapter 13
# A Modified and Enhanced Ant Colony Optimization Algorithm for Traveling Salesman Problem

**Leila Eskandari, Ahmad Jafarian, Parastoo Rahimloo, and Dumitru Baleanu**

## 13.1 Introduction

The traveling salesman problem is one of the famous problems which has been proposed in 1800 by W.R. Hamilton and T. Kirkman and then the common form of this problem has been studied by the mathematicians like K. Menger from Harvard and H. Whitney from Princeton University. This problem is explained as follows: We have some cities and we know the cost of traveling between the cities. It needs to be found the path with minimum cost which only visit the each city's one time and get back to the starting point. All the paths that could find for each problem in TSP are calculated by this $\frac{1}{2}(n-1)!$ for $n > 2$ where n is the number of cities. Indeed, this formula calculates the number of Hamiltonian cycles in a complete graph [10].

The TSP problem could be solved with various methods such as neural network [1], mimetic computing, simulating annealing [6], Ga algorithm [3], PSO algorithm [11], ACO algorithm [15], and other evolutionary algorithms. Among these methods, ACO is mostly used for problems in which it needs to find the path and minimize the cost for it, like the TSP problem. Recently proposed methods based on the evolutionary algorithms are PSO-ACO [4], PSO-ACO-3Opt [9], and ACOBOA [8]. In PSO-ACO, a novel combination has been introduced by using a PSO, which is improved by the ACO algorithm. The PSO-ACO-3Opt is the another new hybrid algorithm based on the particle swarm optimization, ant colony optimization, and

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k-opt ($k = 3$) method [2]. The PSO algorithm is used for finding proper values of parameters $\alpha$ and $\beta$ which are used for city selection operations in the ACO algorithm and specifies significance of inter-city pheromone and distances. The 3-Opt algorithm is used for the purpose of improving city selection operations, which could not be improved due to falling in local minimums by the ACO algorithm the performance. The novel hybrid algorithm ACOBOA finds the balance between exploiting the optimal solution and enlarging the search space. This algorithm is based on the ACO and Bean optimization algorithm (BOA) [14]. The results of the experiments show that ACOBOA has better optimization performance and efficiency than the general ant colony optimization algorithm and genetic algorithm. In this study, the ACO algorithm has been used, but the modification has been applied on updating the matrix of pheromones $\tau$ which helps in converging to the best solution and not trap in local.

The rest of the paper is organized as follows: Sect. 13.2 illustrates the TSP problem, Sect. 13.3 defines the ACO algorithm, Sect. 13.4 discusses the MEACO algorithm, and Sect. 13.5 presents six benchmark TSP problems applied for the experiments. Finally, the last section presents the concluding remarks.

## 13.2 Traveling Salesman Problem (TSP)

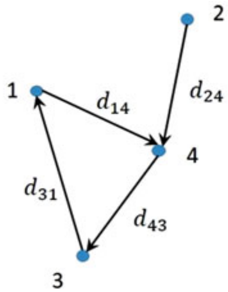
For solving traveling salesman problem each city has an integer number and the real value coordination *(x,y)* where $x$ and $y \in R$. The integer numbers permutation shows the visiting order of city in traveling salesman problem and the coordination is used for calculating the distance between city $i$ and $j$ by the following Euclidean distance formula:

$$d_{i,j} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)2} \tag{13.1}$$

It needs to compute the distance between cities and stores it in the Matrix with name of Distance matrix and uses it in calculating the sum of distance for specific permutation which needs to be minimized by algorithm. For example, if we have 4 cities, then the permutation 2431 is the one probable solution for TSP problem. This permutation means that the starting city is city 2, then goes to city 4, and at last city 1, then will come back to the starting point, city 2. The tour for this probable solution is $2 \to 4, 4 \to 3, 3 \to 1$, and $1 \to 2$. The distance for this tour is computed from distance matrix which has been computed before starting algorithm main iteration. Figure 13.1 shows the example for TSP problem with 4 cities and the following numerical example for distance matrix illustrates this problem:

$$D = \begin{bmatrix} 0 & d_{12} & d_{13} & d_{14} \\ d_{21} & 0 & d_{23} & d_{24} \\ d_{31} & d_{32} & 0 & d_{34} \\ d_{41} & d_{42} & d_{43} & 0 \end{bmatrix}.$$

**Fig. 13.1** Simple graph for TSP problem and the distance between cities




Sum of distance for route $2 \rightarrow 4, 4 \rightarrow 3, 3 \rightarrow 1$, and $1 \rightarrow 2$ is calculated as follows: $= d_{24} + d_{43} + d_{31} + d_{14}$. Algorithm run for permutation of tour 59 (like 1342, 1234 and etc). until, it finds the tour which has minimum Sum of distance.

## 13.3 Ant Colony Optimization Algorithm

The ACO algorithm was one of the nature inspired algorithms which for the first time was developed by Dorigo et al. [15] and it has been mimicked actual ant colony behaviors. The research on the behavior of ants in real life shows that the ants have the ability to find the shortest path between their nest and food source. The most major feature in seeking the shortest path is the evaporation rate, chemical matter of pheromone that ants drop on the route which they have chosen. Ants in a colony mostly select a path where pheromone rate is high. Selection of city $j$, to which an ant in the city $i$ in iteration $t$ will go, is made according to Eq. (13.2).

$$P_{ij}^{k} = \begin{cases} \frac{[\tau_{ij}(t)]^{\alpha}[\eta_{ij}(t)]^{\beta}}{\sum[\tau_{ij}(t)]^{\alpha}[\eta_{ij}(t)]^{\beta}} & \text{if } j \text{ is the allowed city} \\ 0 & \text{otherwise} \end{cases} \tag{13.2}$$

In Eq. (13.2), $\tau_{ij}$ indicates the amount of pheromones between $i$ and $j$ cities, $\eta_{ij}$ indicates information $(1/d_{ij})$ pertaining to distance between $i$ and $j$ cities, and $j$ displays cities where $k$th ant can go. An ant chooses the city with the highest ratio of $P_{ij}$ by making a greedy selection. Parameters $\alpha$ and $\beta$ are used for determining the significance between the amount of pheromones and distance inter-city. $k$th ant completes one total tour by using Eq. (13.2). The above-mentioned operation is repeated in $t$ iteration for all ants that are present in the colony. The amount of pheromones left by an ant on a route that it has used is determined according to Eq. (13.3).

$$\Delta_{ij}^{k}(t, t+1) = \begin{cases} \frac{Q}{L^{k}} & \text{if } (i, j) \in r \text{ out performed by the } k\text{th ant} \\ 0 & \text{otherwise} \end{cases}, \tag{13.3}$$

where $L^k$ represents a distance of the tour, $Q$ represents a constant number, and k represents *kth* ant in the colony. The total amount of pheromones that ants, which are present in the colony and use the route between cities $i$ and $j$ have left, is calculated by using Eq. (13.4).

$$\Delta_{ij}^{k}(t, t+1) = \sum_{k=1}^{n} \Delta_{ij}^{k}(t, t+1) \tag{13.4}$$

Amount of pheromones, which will be found in inter-city routes in iteration (t + 1), is determined as in Eq. (13.5) depending on the impact of evaporation as well.

$$\tau_{ij}(t+1) = (1-\rho)\tau_{ij}(t) + \Delta_{ij}^{k}(t, t+1) \tag{13.5}$$

In Eq. (13.5), $\rho$ is the coefficient of evaporation and receives a value at intervals [0 1]. When the maximum number of iterations is reached, the shortest tour length obtained is regarded as the solution of the problem.

## 13.4 Proposed Method for TSP Problem

In solving the TSP problem with ACO algorithm, increasing the number of cities in TSP problem causes to increase the complexity of the problem and ACO parameters $\alpha$ and $\beta$ need to adjust properly for reaching the optimum path. Like the ACO algorithm, proposed method starts the procedure by the initialization of the parameters, $\tau$ (pheromone matrix) and $\eta$ (distance matrix) for the specific TSP problem. After the initialization, algorithm enters to the main iteration and starts to select a path base on the rate of pheromone on each path and same as the original ACO, a new path has been created for $\text{ant}_k$ and if the new path better than the previous personal best path for that ant ($\text{ant}_k^{\text{pbest}}$) then $\text{ant}_k^{\text{pbest}}$ will update with new path. By updating the personal best the algorithm find the better path base on the previous personal experiences. The personal best of ant ($\text{ant}_k^{\text{pbest}}$) is mutated by mutation function (swap, insertion and reversion) [7] and the output ($newsol$) compares with previous personal best. The mutation operators used in this method are swap, insertion, and reversion which are randomly chosen and executed by calling the mutation function. Figure 13.2 shows these operators and their effects on probable solution. These operators are defined as follows:

- Swap: The position of selected positions 2 and position 5 in Fig. 13.2 are exchanged (position refers to the array index).
- Reversion: In this operator, besides conducting swap, the positions located between the swapped positions are reversed, too.
- Insertion: Select one position and insert it after another one. Other positions are shifted to the right.

**Fig. 13.2** Mutation operators in detailed view

| | | | | | | |
|---|---|---|---|---|---|---|
| Given solution | 3 | 2 | 6 | 5 | 1 | 4 |
| Swap | 3 | 1 | 6 | 5 | 2 | 4 |
| Insertion | 3 | 2 | 1 | 6 | 5 | 4 |
| Reversion | 3 | 1 | 5 | 6 | 2 | 4 |

If the mutation output is better than the previous best then it will update, otherwise the algorithm ends this ant procedure and goes to the next one. After finishing the each ant process, it is time to select Gbest from the population. The Gbest is also entered as input to the mutation function and the output is compared with the previous Gbest. If the new solution is better than Gbest, then it will set as Gbest for this iteration. After the process of Gbest, algorithm reaches stopping criteria and if it is satisfied it will end, otherwise it is going to start another iteration for ants of the population. The algorithm for MEACO is as follows:

**MEACO algorithm**

Initialization (n (population size), $\tau$, $\alpha$, $\beta$, $\rho$, Q )
**While** ( criterion )
  **for** k=1,2,…,n **do** (k ant)
   Determine the route for $ant_k$ by Eq. (1)
   Calculate the sum of distance (fitness) for route $ant_k$
   Calculate $\beta$ using Eq. (4).
   Update the $\tau$ pheromone for route $ant_k$.
   **If** (fitness($ant_k$) < fitness($ant_k^{pbest}$)) **then**
    $ant_k^{pbest}$= $ant_k$
   **end if**
   Apply mutation to $ant_k^{pbest}$, newsol=mutate($ant_k^{pbest}$)
   **If** (fitness(newsol) < fitness($ant_k^{pbest}$)) **then**
    $ant_k^{pbest}$= newsol
    **If** (fitness($ant_k^{pbest}$) < fitness(Gbest)) **then**
    Gbest= $ant_k^{pbest}$
    Update the $\tau$ pheromone for route Gbest.
    **end if**
   **end if**
  **end for k**
  Sort the population based on fitness and determine the Gbest
  Apply mutation to Gbest, newsol=mutate(Gbest)
  **If** (fitness(newsol) < fitness(Gbest)) **then**
   Gbest=newsol
   Update the $\tau$ pheromone for route Gbest.
  **end if**
**end while**
Output the optimum route.

**Table 13.1** MEACO results for TSP benchmark problems

| Problem | BKS | Best | Mean | Worst | Std | Time (s) |
|---|---|---|---|---|---|---|
| Bayas29 | 2020 | 2020 | 2541 | 2632 | 0.61 | 302.3 |
| Berlin52 | 7542 | 7544 | 7598 | 7603 | 0.2 | 201.14 |
| Dantzig42 | 699 | 679.2019 | 683.542 | 685.443 | 15.27 | 409.5 |
| Eil51 | 426 | 432.5456 | 468.4586 | 472.4745 | 52.13 | 207.2 |
| Kroa100 | 21,282 | 21,295 | 21,333 | 21,398 | 17.12 | 508.9 |
| St70 | 675 | 693.4521 | 695.4538 | 696.1258 | 2.35 | 320.7 |

## 13.5 Experimental Result Ant Testing Setup

The proposed method has been tested on six TSP benchmark problems which are accessible from this source [12]. The parameter setting for MEACO algorithm is as follows: $\alpha = 2$, $\beta = 2$, $\rho = 0.02$, Q=100 and the amount of pheromone at the starting was set to 1 for each path from one city to another. The testing is performed on the computer with the following features: CPU 2.1 GHZ, Ram 8 GB and Matlab 2013 running on a computer with windows 10. The stopping criteria was set to reach a Maximum iteration number max-iter=20,000. Table 13.1 shows the result of benchmark TSP problems Bayas29, Berlin52, Dantzig42, Eil51, Kroa100 and St70 based on the Mean (average), Std (standard deviation), Worst, Best, and Time. In this table optimum route cost for each problem has been presented besides the MEACO results and it can be seen that MEACO could find proper results for problems especially for Dantzig42 which has been shown in Fig. 13.3. The results for other TSP problems are greatly minimized and MEACO could achieve best performance in solving these problems. In Table 13.2, MEACO has been compared with other famous algorithms PSO, GA, SA, ACO for problems Dantzig42 and St70. The Friedman non-parametric test results [5] for these two problems show that MEACO has better performance than the other algorithms in solving them and scores the minimum rank among the other algorithms. The $p$-value shows the difference between the mean results. The results for other algorithms in Table 13.2, have been taken from reference [13]. Figure 13.4 depicts the optimum solution for these TSP problems. The Dantzig42 TSP problem optimum tour is as follows:

$11 \rightarrow 12 \rightarrow 23 \rightarrow 22 \rightarrow 17 \rightarrow 16 \rightarrow 13 \rightarrow 14 \rightarrow 15 \rightarrow 18 \rightarrow 19 \rightarrow 20 \rightarrow 21 \rightarrow 28 \rightarrow 29 \rightarrow 30 \rightarrow 31 \rightarrow 32 \rightarrow 33 \rightarrow 34 \rightarrow 35 \rightarrow 36 \rightarrow 37 \rightarrow 38 \rightarrow 39 \rightarrow 40 \rightarrow 41 \rightarrow 42 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10 \rightarrow 25 \rightarrow 26 \rightarrow 27 \rightarrow 24 \rightarrow 11$.

## 13.6 Conclusion

The ACO is one of the efficient nature inspired metaheuristic algorithms, which has outperformed most of the algorithms in solving the various optimizing discreet problems. In this paper, we have modified and enhanced the ACO algorithm to

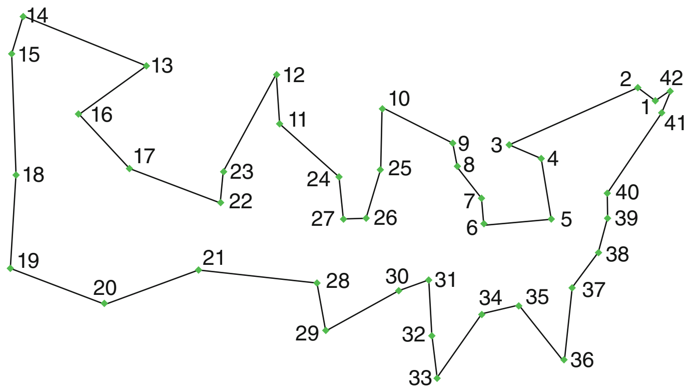



**Fig. 13.3** Optimum Tour for TSP problem Dantzig42

**Table 13.2** MEACO comparison with other famous algorithms for TSP benchmark problems Dantzig 42 and St70

| Problem | | PSO | GA | SA | ACO | MEACO |
|---|---|---|---|---|---|---|
| Dantzig42 | Best | 679.2019 | 679.2019 | 686.2082 | 703.8294 | 679.2019 |
| | Avg | 699.8715 | 715.8312 | 722.8525 | 724.8758 | 683.542 |
| | Std | 13.1413 | 22.0551 | 22.6387 | 10.5018 | 15.27 |
| St70 | Best | 677.1945 | 692.4504 | 709.3605 | 699.2357 | 693.4521 |
| | Avg | 717.7294 | 732.0563 | 781.5419 | 710.3917 | 695.4538 |
| | Std | 22.7000 | 102.0129 | 33.9260 | 4.7699 | 2.35 |
| Friedman test | Rank | 2.5 | 3.5 | 4.5 | 3.5 | 1 |
| | $p$-value | 0.231 | | | | |
| | Statistic | 5.6 | | | | |

design a new method for seeking the optimum path in TSP problem. The proposed algorithm begins to search from the ACO algorithm and applies the mutation to the personal best and global best, which are used in updating the ants' tours. The experiment based on benchmark TSP problems showed that proposed hybrid is dominant than other famous algorithms and could find proper tours for each problem and these results are better than ACO.

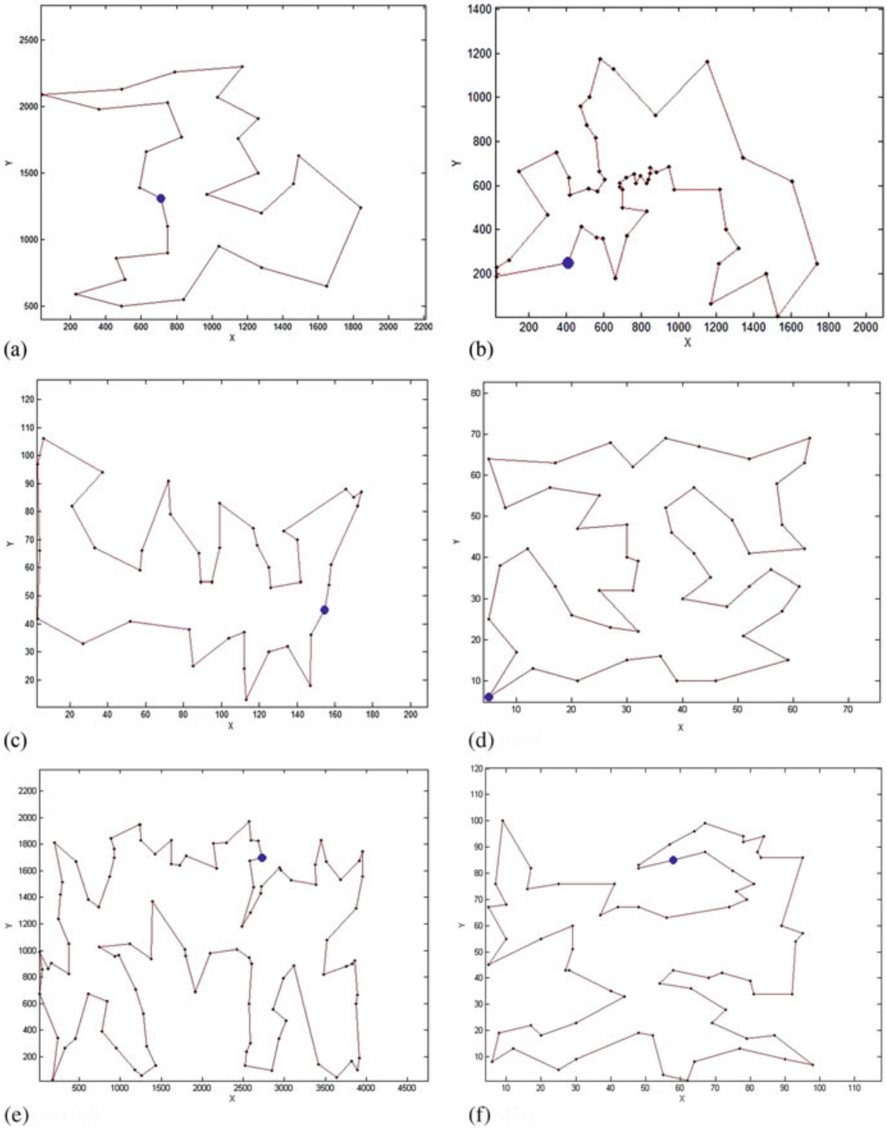


**Fig. 13.4** Optimum routes found by proposed method for benchmark TSP problems. (**a**) Bayas29. (**b**) Berlin52. (**c**) Dantzig42. (**d**) Eil51. (**e**) Kroa100. (**f**) St70

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# Chapter 14
# A Note on the Upper Bound of Average Distance via Irregularity Index

**Nihat Akgunes, Ismail Naci Cangul, and Ahmet Sinan Cevik**

## 14.1 Introduction and Preliminaries

Let $G = (V, E)$ be a simple, finite connected graph with the vertex set $V = V(G)$ of order $n$ and the edge set $E = E(G)$. The *average distance* $\mu(G)$ of $G$ is defined as

$$\mu(G) = \binom{n}{2}^{-1} \sum_{\{u,v\} \subseteq V(G)} d_G(u, v),$$

where $d_G(u, v)$ denotes the *distance* (length of the shortest path) between two vertices $u, v$ of $G$. The *diameter* $diam(G)$ of $G$ is defined as the maximum distance $d_G(u, v)$ over all pairs of vertices $u$ and $v$ in $G$. The *degree* $deg_G(v)$ of a vertex $v$ of $G$ is the number of vertices adjacent to $v$. Among all degrees, the *minimum degree* of vertices is denoted by $\delta$ in graph $G$. We also have the *degree sequence* $DS(G)$ which is a sequence of degrees of vertices of $G$. It has been defined a new parameter for graphs by Mukwembi [10], namely the *irregularity index* of $G$ and denoted by $t = t(G)$. In fact $t$ is the number of distinct terms in $DS(G)$. Although there exist

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very huge number of studies on the degree sequence of graphs [8, 9], there is very limited number on the irregularity index (see, for instance, in [1, 10]).

In this paper we also need the concept average distance (or, equivalently, mean distance) which was first introduced by Doyle and Graver in the paper [3]. This important graph parameter is the measure of the *compactness* of a graph $G$. The average distance had been found in so many application areas in nature science such as in (telecommunications) network, architecture, inter-computer connections, molecular structure, etc. We can refer [5, 7, 11, 12] for some of these depicted applications and examples. However, another application of average distance was given by Fajtlowicz and Waller [4] as a computer program GRAFFITI. This program made the attractive conjecture that, for every $\delta$-regular connected graph $G$ (i.e. connected graphs in which all vertices have the same degree $\delta$) of order $n$,

$$\mu(G) \leq \frac{n}{\delta}. \tag{14.1}$$

By considering any connected graph of order $n$ with the minimum degree $\delta$, an asymptotically slightly stronger form of this conjecture was proved by Kouider and Winkler [6] as

$$\mu(G) \leq \frac{n}{\delta + 1} + 2. \tag{14.2}$$

After that, by considering special *trees*, Dankelmann and Entringer [2] presented the following bounds:

1. If $T$ is a spanning tree, then $\mu(T) \leq \frac{n}{\delta} + 5$.
2. If $G$ is triangle-free, then for a spanning tree $T$ of $G$ we have

$$\mu(T) \leq \frac{2}{3}\frac{n}{\delta} + \frac{25}{3}.$$

3. If $G$ is the $C_4$-free graph, then again for a spanning tree $T$ of $C_4$,

$$\mu(T) \leq \frac{5}{3}\frac{n}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} + \frac{29}{3},$$

where $C_4$ is the cycle graph of length 4.

In the light of the results in papers [2, 4, 6], by considering the parameters $t$, $n$, and $\delta$, we will state and prove an upper bound (see Theorem 14.1 in below) for the average distance of any simple connected graph in this paper. Furthermore we will note that this bound is more stronger than the bounds given in (14.1), (14.2) and (1)–(3) above (see Remark 14.1 in below).

## 14.2 The Main Result

To prove our theorem, we need the following notation which can also be found, for instance, in [5].

The *neighbourhood* $N_G(x)$ of a vertex $x \in V(G)$ is the set of all vertices adjacent to $x$. The *closed neighbourhood* $N_G[x]$ of a vertex $x \in V(G)$ contains $N_G(x)$ and the vertex $x$ itself. For a subset $S \subseteq V$, let us assume that $G[S]$ denotes the subgraph induced by $S$ in $G$. Then the *distance* between a vertex $x$ and $S$, denoted by $d_G(x, S)$, is defined as $\min_{v \in S} d_G(x, v)$. The *closed neighbourhood* of $S$ is the set $\cup_{x \in S} N_G[x]$ and denoted by $N_G[S]$. The *k-th power* of $G$, denoted by $G^k$, is the graph with the same vertex set as $G$ in which two vertices $u \neq v \in V(G)$ are adjacent if $d_G(u, v) \leq k$. For a subset $A \subseteq V(G)$, the *subgraph* of $G^k$ induced by $A$ is denoted by $G^k[A]$. For a positive integer $k$, a *k-packing* of $G$ is a subset $A \subseteq V(G)$ with $d_G(a, b) > k$ for all $a, b \in A$.

The following lemma plays a central role in the proof of our main result.

**Lemma 14.1 ([10])** *Let $G$ be a connected graph of order $n$, minimum degree $\delta$ and diameter $d$, where $d \neq 3, 4$. If $A$ is maximal 2-packing set, then*

$$|A| \leq \frac{n - t + 1}{\delta + 1}.$$

**Theorem 14.1** *Let $G$ be a simple, connected graph of order $n$ with the minimum degree $\delta$ and diameter $d$, where $d \neq 3, 4$. Suppose that $t$ is the irregularity index of $G$. Then we have an upper bound*

$$\mu(G) \leq \frac{n - t + 1}{\delta + 1} + 1. \tag{14.3}$$

*Moreover, this inequality is essentially tight.*

*Proof* Let $DS(G)$ be the degree sequence of $G$ having $t$ distinct terms. In other words, the irregularity index is $t$.

Firstly we find a maximal 2-packing $A \subseteq V(G)$ of $G$ using the following procedure: For a chosen vertex $v$ of $V(G)$, let $A = \{v\}$. If there exists another vertex $u$ in $V(G)$ having the condition $d_G(u, A) = 3$, add $u$ to $A$. After that add all such these vertices $u'$ having the same condition $d_G(u', A) = 3$ to $A$ until each of the vertices not in $A$ is within distance two of $A$.

As the next step, let $T_1 \leq G$ be the forest with vertex set $N_G[A]$ and whose edge set consists of all edges incident with a vertex in $A$. By our construction on $A$, there exist $|A| - 1$ edges in $G$, each of them joining two neighbors of distinct elements of $A$, whose addition to $T_1$ yields a tree $T_2 \leq G$. Now every vertex $u$ not in $T_2$ is adjacent to some vertex $u''$ in $T_2$.

Let $T[A]$ be a spanning tree of $G$ with the edge set $E(T_2) \cup \left\{uu'' \middle| u \in V(G) - V(T_2)\right\}$. By our construction, $T^3[A]$ is connected, and we have

$$\mu(T[A]) \leq 3\mu(T^3[A]) \tag{14.4}$$

and

$$\mu(T^3[A]) \leq \frac{|A|+1}{3}. \tag{14.5}$$

We note that the equality holds in both (14.4) and (14.5) if and only if $T^3[A]$ is path graph.

Now, by (14.4), (14.5) and Lemma 14.1, we clearly have

$$\mu(T[A]) \leq |A| + 1$$

which implies that

$$\mu(T[A]) \leq \frac{n-t+1}{\delta+1} + 1.$$

However, since $T[A]$ is a spanning tree of $G$, we finally get

$$\mu(G) \leq \mu(T[A]) \leq \frac{n-t+1}{\delta+1} + 1.$$

Hence the result.

*Remark 14.1* The bound obtained in (14.3) is the tightest bound among all other bounds depicted in (14.1), (14.2) and (1)–(3) as in the previous section. By using the definition of irregularity index and the fact of $\delta + 1$ is a positive integer, we can show it very basically as in the following:

$$t \geq 1 \Rightarrow n \geq 1 - t + n \Rightarrow \frac{n}{\delta+1} \geq \frac{n-t+1}{\delta+1}$$
$$\Rightarrow \frac{n}{\delta+1} + 2 \geq \frac{n-t+1}{\delta+1} + 2 \geq \frac{n-t+1}{\delta+1} + 1.$$

For an application of Theorem 14.1, we can present regular graphs. First of all, since complete graphs $K_n$ are the specific cases of the regular graphs, it is easy to see that Theorem 14.1 satisfies for $K_n$. Moreover, for any regular graph $G$, since $t$ is always equal to 1 by the meaning of regular graphs, we have the following corollary as a consequence of Theorem 14.1.

**Corollary 14.1** *Let G be a $\delta$-regular graph. Then the inequality*

$$\mu(G) \leq \frac{n}{\delta+1} + 1$$

*holds.*

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