Chapter 13
Future Teachers' Use of Multiplication and Fractions When Expressing Proportional Relationships

İbrahim Burak Ölmez

Abstract The purpose of this study was to investigate how six future middle grades mathematics teachers used explicit, quantitative definitions for multiplication and for fractions when reasoning about proportional relationships. The future teachers were recruited from a preparation program in the United States based on their performance on a fractions survey. The data collection consisted of 1-hour semi-structured interviews with each future teacher. An explanatory case study was used to make comparisons across the future teachers. Results revealed that explicit use of the quantitative definition of multiplication is a helpful organizing tool for future teachers to generate and explain equations for proportional relationships.

Keywords Definition of multiplication · Definition for fractions
Future middle grades teachers · Proportional relationships · Algebraic equations

13.1 Introduction

One of the most central and difficult concepts of elementary and secondary mathematics education is ratios and proportional relationships (e.g., Kilpatrick, Swafford, & Findell, 2001; Lamon, 2007; National Council of Teachers of Mathematics, 2000). Lesh, Post, and Behr (1988) considered ratios and proportional relationships to be the capstone of elementary mathematics and the cornerstone of high school mathematics. Vergnaud (1983, 1988) described these relationships as part of the multiplicative conceptual field—a web of interrelated ideas including multiplication, division, fractions, and more. The National Mathematics Advisory Panel (2008) stated that the interrelated ideas of ratios, proportional relationships, and fractions are foundational for algebra.

İ. B. Ölmez (✉)
Department of Mathematics and Science Education,
University of Georgia, 105 Aderhold Hall, Athens,
GA 30602, USA
e-mail: i.burakolmez@hotmail.com

M. E. Strutchens et al. (eds.), *Educating Prospective Secondary Mathematics Teachers*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-91059-8_13

Past research on teachers' understandings of the multiplicative conceptual field has reported that in-service and future teachers have trouble when explaining the product of two fractions or decimals embedded in problem situations despite their correct computation of the algorithms (e.g., Ball, Lubienski, & Mewborn, 2001; Izsák, 2008; Tirosh & Graeber, 1990). Past research has also acknowledged that many teachers in the U.S. struggle explaining division (i.e., multiplication with an unknown factor) when it is embedded in problem situations although they can determine the quotient of two fractions or decimals (e.g., Armstrong & Bezuk, 1995; Jansen & Hohensee, 2016).

The relatively few studies that have examined how in-service and future teachers reason about ratios and proportional relationships have demonstrated that teachers have many of the same difficulties reported in the much larger literature on students' reasoning about proportional relationships. In one of these studies, middle grades teachers were found to show poor performance on test items that their students are expected to solve (Post, Harel, Behr, & Lesh, 1991). In another study, Harel and Behr (1995) reported that many teachers were not able to solve the problems involving proportional relationships correctly. Rather, these teachers guessed at operations by performing each of these operations on the quantities until reaching out a reasonable solution or they searched for particular words (i.e., key words) in the problems to decide which operation to use. In addition to these findings, past research has documented that teachers can have a hard time distinguishing missing-value problems that describe directly proportional relationships from ones that do not (e.g., Cramer, Post, & Currier, 1993; Fisher, 1988; Izsák & Jacobson, 2017) and can struggle conceiving a ratio as a measure of a physical attribute, such as steepness (e.g., Simon & Blume, 1994).

The purpose of this study was to examine how six future middle grades mathematics teachers used explicit, quantitative definitions for multiplication and for fractions to develop equations that relate quantities in a proportional relationship. Both definitions were introduced in content courses that the future teachers were completing as part of a preparation program in the United States. The following research question was addressed in this study:

- How do future middle grades mathematics teachers reason with quantitative definitions for multiplication and for fractions when solving proportion problems?

The study makes two contributions. First, existing studies have consistently acknowledged that solving proportions involving whole-number multiples is easier than solving proportions involving fraction multiples (e.g., Kaput & West, 1994; Karplus, Pulos, & Stage, 1983), but no studies have examined how explicit, quantitative definitions for multiplication and for fractions can support and constrain reasoning about proportional relationships. Second, the present study demonstrates that the quantitative definition of multiplication is accessible to future teachers in terms of generating and explaining appropriate equations that involve proportional relationships.

13.2 Theoretical Framework

The framework for this study is based on the quantitative definition of multiplication explicated by Beckmann and Izsák (2015) as in Fig. 13.1. In the equation M · N = P, the multiplier, M, is interpreted as the number of groups; the multiplicand, N, is the number of units in one group; and the product, P, is the number of units in M groups.

This study is also based on a definition for fractions that is consistent with the one found in the Common Core State Standards in the United States (CCSS) (Common Core State Standards Initiative, 2010). A two-part definition for a fraction is as follows:

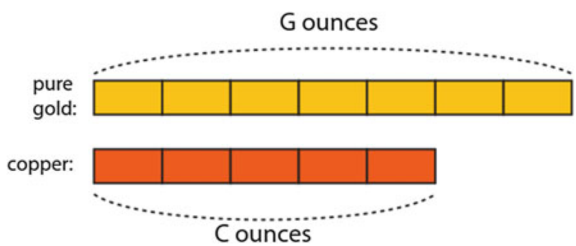
(a) $1/b$ is the quantity formed by one part when a unit amount (or whole) is divided into b equal parts; each part is $1/b$ of the unit amount.
(b) a/b is the quantity formed by a parts of size $1/b$ of the unit amount.

Beckmann and Izsák (2015) explained how maintaining distinct roles played by the multiplier and multiplicand in Fig. 13.1 leads to two distinct perspectives on proportional relationships, one termed the multiple batches perspective, which has been studied widely, and the other termed the variable parts perspective, which has been largely overlooked in mathematics education research.

For the present study, we focused on the variable parts perspective, which we illustrate with the Jewelry Problem. The variable parts perspective, combined with a drawn model called a strip diagram (Fig. 13.2) supports at least two different solutions (Beckmann, Izsák, & Ölmez, 2015).

Fig. 13.1 A quantitative definition of multiplication

Fig. 13.2 A strip diagram for the Jewelry Problem



Jewelry Problem: A company makes jewelry using gold and copper. The company uses different weights of gold and copper on different days, but always in the same 7–5 ratio. Let G and C be some unspecified number of ounces of gold and copper the company will use that same day. Please use a strip diagram to help you explain the relationship between G and C.

One method future teachers can use to develop equations relating G ounces of gold and C ounces of copper is the "how much in one part" method (Fig. 13.3). With this method, future teachers can view the total amount of gold as one group consisting of seven parts and the total amount of copper as one group consisting of five parts. Here the number of parts is fixed and, by convention, all parts contain the same number of ounces. At the same time, the number of ounces in each of the 12 parts can vary with the total amount of jewelry gold being made. Future teachers can solve the problem by determining the number of ounces in one part, $C/5$ oz, and by using the quantitative definition of multiplication to generate the following equation:

$$7\,(\text{groups}) \cdot C/5\,(\text{ounces in one group}) = G(\text{ounces in 7 groups})$$

A second method future teachers can use is the "how many total amounts" method (Fig. 13.4). With this method, future teachers can treat the copper strip as 1 group of C ounces. By asking how many groups of five parts are in seven parts and applying the definition for fractions to the copper strip as unit amount or whole, future teachers can see that the gold strip consists of 7 parts each containing the same number of ounces as 1/5 of the copper strip, and the five parts copper strip fits into the seven parts gold strip 7/5 times. Thus, future teachers can conclude that the amount of gold, the G ounces, is 7/5 groups, and use the definition of multiplication to generate the following equation:

$$7/5\,(\text{groups}) \cdot C/5(\text{ounces in one group}) = G(\text{ounces in 7/5 groups})$$

Past research on understanding teachers' solutions to problems similar to the Jewelry Problem has found that teachers tend to resort to cross-multiplication as a rote computation algorithm without reasoning about the quantities and guess at

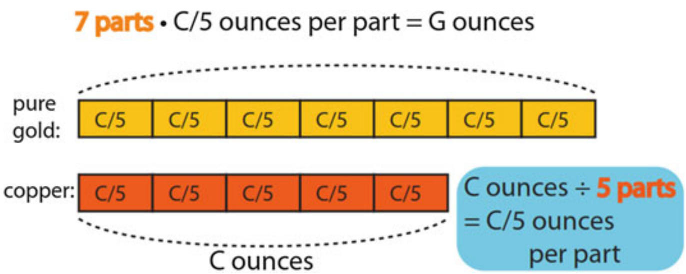


Fig. 13.3 "How much in one part" method (Variable Parts Perspective)

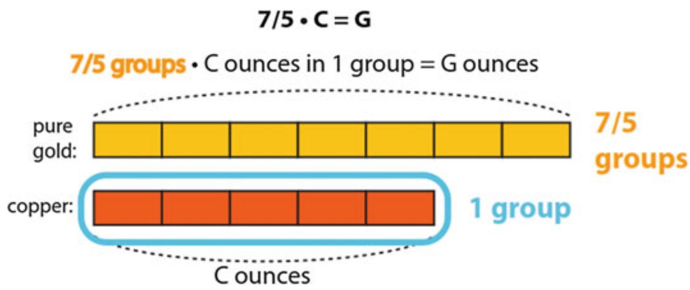


Fig. 13.4 "How many total amounts" method (Variable Parts Perspective)

operations (e.g., Fisher, 1988; Harel & Behr, 1995); have difficulty coordinating two proportionally related quantities (e.g., Orrill & Brown, 2012); and rely on additive relationships rather than multiplicative ones when reasoning about these quantities (e.g., Ölmez, 2016). Moreover, current recommendations on middle grades students suggest developing their conceptual understanding for solving these problems before teaching cross-multiplication as an algorithm (Siegler et al., 2010) and exposing these students to solve such problems using math drawings like strip diagrams and double number lines (CCSS, 2010). To apply the current recommendations for middle grades students, future middle grades teachers, as prospective teachers of these students, should have capacities to reason about proportionally-related quantities and connect their capacities within the multiplicative conceptual field. Therefore, the theoretical framework of this study is consistent with the recommendations of *The Mathematical Education of Teaching II* for middle grades teachers (Conference Board of the Mathematical Sciences, 2012, p. 39).

13.3 Methods

Data for this paper come from a larger on-going study of future middle grades (grades 4–8) mathematics teachers' multiplicative reasoning. At the time of the study, the six future middle grades teachers were enrolled in a teacher education program at a large university in the Southern United States. The future teachers had already taken a first semester calculus course required by the program. As part of the program, they also took a content course on number and operations in Fall 2014 and a content course on algebra in Spring 2015. The sequence of instruction in the number and operations course were as follows: numbers, the base-ten system, the definition of fractions, equivalent fractions, comparing fractions, fraction addition and subtraction, the definition of multiplication, properties of multiplication, applying properties of multiplication, fraction multiplication, division of whole numbers, fraction division, and connecting division with fractions. The main focus in this course was to develop a foundation for future teachers to use the quantitative definitions for multiplication and for fractions. In addition, the sequence of instruction in the algebra course followed identification of proportional relationships, the multiple batches perspective, the variable parts perspective, developing equations in two variables and developing equations of lines that come through the origin. The main focus in this course was to introduce the two perspectives on proportional relationships and strip diagrams.

A project team member was the instructor for both courses. Both content courses were taught from Mathematics for Elementary Teachers with Activities (Beckmann, 2014). Each course met for three 50-min sessions each week for 16 weeks. Most class sessions started with about 30 min of small group work on a set of problems focused on a particular topic. During whole-class discussion, the future teachers shared their solutions to problems that involved proportional relationships and asked each other questions. The instructor wrapped up the discussion by highlighting the key ideas that emerged during the conversation.

The project team selected six future teachers from a class of 22 future middle grades mathematics teachers who took the number and operations course in Fall 2014. The project team recruited future teachers who were mathematically diverse based on their performance on a survey (Bradshaw, Izsák, Templin, & Jacobson, 2014) that assessed facility with multiplication and division of fractions in terms of measured quantities. An explanatory case study was used because a case study is appropriate when the aim is to investigate causal relationships (Yin, 1993). Another project team member conducted six semi-structured hour-long cognitive interviews with each future teacher during the two semesters in 2014–2015. During the interviews, future teachers solved paper-and-pencil tasks similar to, but not the same as, those used in their course work. They were each interviewed twice during the numbers and operations course in Fall 2014 and four times during the algebra course in Spring 2015. All interviews were videotaped and transcribed verbatim for analysis.

Data for the present study consisted of future teachers' interview videos, audio transcripts, and a scanned copy of each future teacher's written work for two tasks that were given during the fourth interview, which was conducted through the middle of the algebra course. Main goals of the previous three interviews were to examine the extent to which future teachers identify groups, number of units in each group, and product amount in their definitions of multiplication, and distinguish distinct types of division in their definitions. The two interview tasks of the fourth interview were as follows:

Task 1 "How do you interpret the meaning of $1/6 \cdot X$?"
Task 2 Jewelry Problem (the same Jewelry Problem discussed above)

At the time of the fourth interview, future teachers had studied the quantitative definitions for multiplication and for fractions as described above during the numbers and operations course and had received initial instruction on ratios and proportional relationships during the algebra course, but they had not yet had instruction in developing equations in two variables, such as $5/7 \cdot G = C$. Thus, the interview was designed to probe students' initial capacities to reason about the quantitative definitions for multiplication and fractions when developing equations for proportional relationships before instruction.

The project team reviewed the data multiple times by placing interview transcripts side-by-side with the videos, and examined future teachers' words, gestures, and inscriptions for evidence of their thinking processes. The future teachers' initial responses to the tasks, not follow-up questions, were analyzed to obtain their ideas and ways of reasoning that they felt most comfortable with and confident in. To analyze their responses, detailed summaries describing each future teacher's reasoning on each task were written. The team then analyzed each detailed summary to identify emerging themes such as ideas, concepts, and ways of reasoning that the future teacher demonstrated as they worked on a task. Specifically, the team focused on the future teachers' use of key resources such as the definitions for multiplication and for fractions, the use of the variable parts perspective, and the use of strip diagrams. As more passes were taken through the data, it became increasingly clear that there was a substantial diversity between future teachers' use of the definition of multiplication, and their ability to develop appropriate equations for the Jewelry Problem.

13.4 Results

Table 13.1 provides a summary of the performance of all six future teachers in terms of their use of the quantitative definitions for multiplication and for fractions while working on Task 1 and Task 2, and their ability to generate correct equations in Task 2. According to the table, the future teachers who consistently used the definition of multiplication during Task 1 and Task 2 (i.e., consistent with instruction), also generated correct equations for the proportional relationship in Task 2. On the other hand, the future teachers who did not make explicit use of the definition of multiplication appeared to have a hard time developing correct equations.

Table 13.1 A summary table presenting the future teachers' performance

Names[a]	Definition of multiplication in Task 1	Definition of fractions in Task 1 or Task 2	Definition of multiplication equations in Task 2	Generating correct in Task 2
Alice	No	Not enough evidence	No	No
Jeff	No	Yes	No	No
Linda	Yes	Not enough evidence	Yes	Yes
Claire	Yes	Yes	Yes	Yes
Diana	No	Yes	No	No
Kelly	Yes	Yes	Yes	Yes

[a]All names are pseudonyms

13.4.1 Future Teachers' Performance on Task 1

In response to Task 1, half of the future teachers (3 out of 6) did not use the quantitative definition of multiplication consistently (see Table 13.1). Although Alice used the definition of multiplication, her use was not appropriate and consistent across the interview because her referent units lacked the precision with which units were discussed in class. Instead of using the word "number" and an explicit referent unit for each term in her equation, she used the phrase "how many" and left out referent units for *X* and for the product (Fig. 13.5a). To express her thinking in this task, she drew the strip diagram in Fig. 13.5b, but she interpreted the 1 group as 1 part of the whole strip instead of 6 parts of the whole strip.

Alice: So, 1/6 is our number of groups, and *X* would be the size of each group. So, we're pretty much… *X would be like contained in the 1/6* when you… if that makes sense. So, just multiplying 1/6 · *X*, which is how many we have in 1 group, would equal how many we get in all of the groups of 1/6 of *X* size.

Interviewer: Could you give a context or a drawing to sort of also communicate that thinking?

Alice: I guess I would start with this, and we could cut it into 6 parts. So, one of them would be 1/6, and that'll be the size of the group (draws Fig. 13.5b).

Interviewer: Maybe I didn't hear it. This whole strip from here to here (points to each end of the strip in Fig. 13.5b), what does this represent?

Alice: It's just 6. I guess 6 parts. So, this is like one whole, 6 outta 6 (writes 6/6), so one of these would be 1/6, and that's what this is, too (writes 1/6 in the 1-part strip in Fig. 13.5b). And so, *this is 1 group* (points to the 1-part strip). And there's like… I guess *X* is… there's like *X is a number of some sort*. So, say *it's like 2*, there's 2 parts in each of these little things (writes 2 in five parts of the 6-part strip in Fig. 13.5b). *It's hard to like come up with a drawing. So, we start with our how many groups, and then the size of the group would be like 2*. So, we have… would have 1/6 · 2 equals how many would be in all of the groups, so how many would be in this (points to the 1-part strip in Fig. 13.5b).

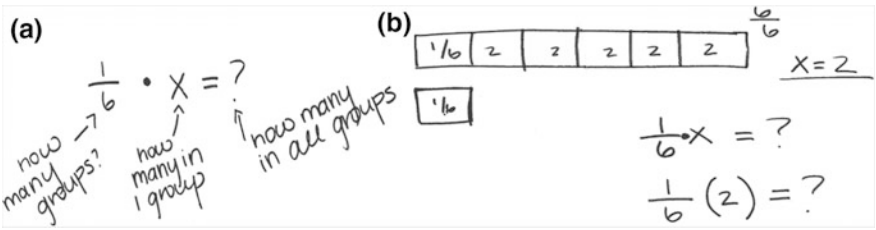


Fig. 13.5 **a** Alice's equation in Task 1. **b** Alice's representation of her equation

The data demonstrate that Alice labeled some parts of her strip diagram with 1/6 and others with 2 as a specific value for *X*. After completing her strip diagram drawing, she also acknowledged that for her to explain the definition of the expression with a drawing was challenging (even though drawings were used regularly in the content courses). Alice's association of 2 with each part rather than the entire strip indicated her incorrect coordination between units and groups.

Moreover, Jeff and Diana did not interpret 1/6 · *X* using the definition of multiplication in an appropriate way because they could not interpret 1/6 as the number of groups. Jeff switched the order of the given multiplication, turning 1/6 from the multiplier position into the multiplicand, as indicated in the strip diagram he drew (Fig. 13.6a, b):

Jeff:	So right here (points to Fig. 13.6a), I drew a strip or a bar, and divided it into 6 equal parts, which gave me 1/6. And so, I see *1/6 times X as 1/6 times a certain amount of groups of 1/6. So, if it was X equals 2, I would have 2 groups of 1/6. Or if X was 6, I would have 6 groups of 1/6*, which would give us the 1.
Interviewer:	If you had, say, 7 groups?
Jeff:	I would just do the 1/6 seven times, and see that it was 7/6 (draws Fig. 13.6b).
Interviewer:	Where do you see the *X* in your diagram?
Jeff:	Each part. So, 1, 2, 3, 4, 5, 6, 7 (counts each circle in Fig. 13.6b).
Interviewer:	Each one of those circles is an *X*? Am I understanding you correctly?
Jeff:	Okay. So, I see the *X* as, I shouldn't have drawn all the circles (points to Fig. 13.6b), I see the *X* as the whole 7 pieces of 1/6, because if we're saying *X* equals 7, so we have *7 groups of 1/6*.
Interviewer:	I think you said it, but can you tell me again… when you look at this expression (points to Task 1), 1/6 · *X*, and can you just interpret it in terms of the meaning of multiplication from class?
Jeff:	Okay. So, I see it as 1/6 times a certain number of groups of 1/6. So here (points to Fig. 13.6b) *X* is 7, so it is 7 groups of 1/6.

When Jeff discussed "do[ing] 1/6 seven times," he used 7 as the number of groups or multiplier, not 1/6, indicating his switch of the order of the multiplier and

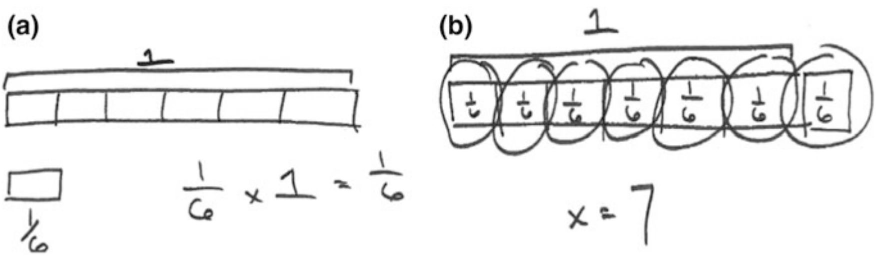


Fig. 13.6 **a** Jeff's view of the whole strip. **b** Jeff switches the order of 1/6 · *X*

multiplicand. Although Jeff could not use the definition of multiplication consistently as discussed above, his consideration of 7/6 as "7 groups of 1/6" indicated his use of the definition for fractions. In contrast to Jeff and Diana's misinterpretation of $1/6 \cdot X$, they used the definition of multiplication as "4 groups with X in each group" when they were asked to interpret $4 \cdot X$. When the interviewer asked Jeff whether his interpretations of $4 \cdot X$ and $1/6 \cdot X$ are same or not, Jeff seemed to be aware of his different interpretations as follows:

Interviewer: Do you think that you're using one interpretation of multiplication for the original $1/6 \cdot X$ problem and the new problem $4 \cdot X$, or do you think that you're using sort of different ways of thinking about multiplication for each example?

Jeff: This one (points to $4 \cdot X$), I'm using the definition of multiplication that I'm most common to in the class. This one (points to $1/6 \cdot X$), I didn't use that.

Interviewer: Why?

Jeff: I think just initially I switched the order, because it was hard for me to see 1/6 as a number of groups… if that makes sense.

Jeff acknowledged that for him it was difficult to imagine 1/6 as the number of groups in his drawing. That it is difficult for him to think of 1/6 as the number of groups indicated his struggle of coordinating units and groups.

Of the future teachers who were able to use the quantitative definition of multiplication consistently (3 out of 6), all three maintained the distinct roles for the multiplier and for the multiplicand (see Table 13.1). Although Linda initially seemed to interpret $1/6 \cdot X$ in a way different from class instruction by saying "1/6 is the size of the group and X is the whole in the group," later in the interview she clarified that she meant "number of groups" when she said "size of the group."

Interviewer: How do you interpret the meaning of $1/6 \cdot X$?

Linda: When I see $1/6 \cdot X$, I think of X being the whole and then dividing that into 6 parts, and then taking one of those parts. And so, that would be $1/6 \cdot X$ where this is X (points to the whole drawing in Fig. 13.7).

Interviewer: How would you use the meaning of multiplication that you've been discussing in class to interpret that statement?

Linda: So, *1/6 would be like the size of the group and X would be like a… the whole in the group*. So, the whole would be X or 6/6, and then we're taking 1/6 of that whole.

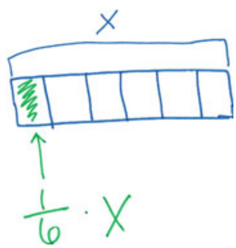


Fig. 13.7 Linda's interpretation of 1/6

The data make clear that she considered the unit for 1/6 (i.e., multiplier) as *X* and she saw that *X* (i.e., multiplicand) as 1 whole group, indicating her coordination of units and groups.

Furthermore, Claire used the definition of multiplication in an explicit and consistent way by coordinating units and groups appropriately in the expression of $1/6 \cdot X$ (Fig. 13.8).

Claire: The way I interpret it (points to $1/6 \cdot X$) is $1/6 \cdot X$ is 1/6 groups of *X*. Should I show you an example of what… like with a drawing?

Interviewer: That was my next question.

Claire: All right. Let's say this drawing (draws Fig. 13.8) is *X*, we see that there's 1, 2, 3, 4, 5, 6 groups that make the whole. So, this is our *X* (points to the whole drawing in Fig. 13.8). I'll use a different color. And, this (circles one part of the whole drawing with six parts in Fig. 13.8) is one of 6 parts of *X* or 1/6 of *X*. So, this (points to the circled part in Fig. 13.8) is *1/6 of group of the whole X*.

Similarly, Kelly identified the referent units for each term, coordinated units and groups in her use of the definition of multiplication for $1/6 \cdot X$, and also used the definition for fractions (Fig. 13.9).

Kelly: We have *1/6th of a group*. And *in that group, it's size X*. And then the product will be the total size of 1/6th of a group.

Interviewer: Could you give like a context and or drawing to explain that?

Kelly: Okay (draws Fig. 13.9). One group, *the size of one group is X*, so there is *X* in one group but we want to know how much or how many of X is in one 1/6th of a group. So, if I take this one group, divide it into 6, each part is one part of 6 total parts, *each part of 1/6th in size*. And if I just want to look at this 1/6th part of my one group, there will be $1/6^{th}$ of *X* inside it.

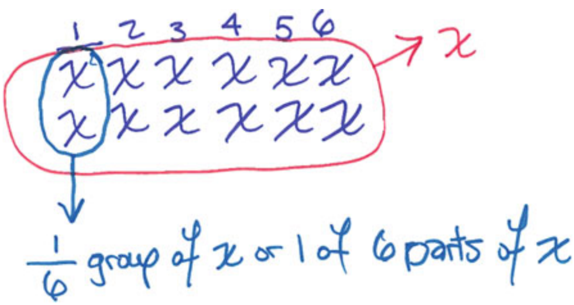


Fig. 13.8 Claire's interpretation of 1/6

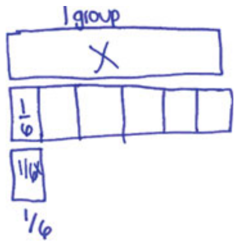


Fig. 13.9 Kelly's interpretation of the whole strip

The data indicate that Kelly and Claire could coordinate the whole strip as the multiplicand and the unit amount for the fraction 1/6 as multiplier. Their identification of the units in the expression $1/6 \cdot X$ and their association of the entire strip as 1 group in their drawings indicated that they could coordinate units and groups appropriately, at least for this task.

13.4.2 Future Teachers' Performance on Task 2

In response to Task 2, which asks future teachers to use a strip diagram to explain the relationship between the amounts of gold and copper that are in 7–5 ratio, Alice, Jeff, and Diana did not generate an appropriate equation involving G and C (see Table 13.1). In several cases, these future teachers' interpretation of the equal sign was inconsistent with normative usage, which in this task would equate the number of ounces of gold and of copper. From the data, it is not clear if the future teachers might have employed more normative usage of the equal sign in another situation or if their interpretation of the equal sign caused problems when they tried to apply the definition of multiplication in this task.

Alice did not form an equation that related the ounces of gold and copper appropriately, despite writing separate equations with whole number multipliers $5 \cdot C/5 = ?$ for copper and $7 \cdot G/7 = ?$ for gold. When she was reminded to produce an equation that involves the ounces of gold and copper, she wrote $5\ C = 7\ G$ as follows:

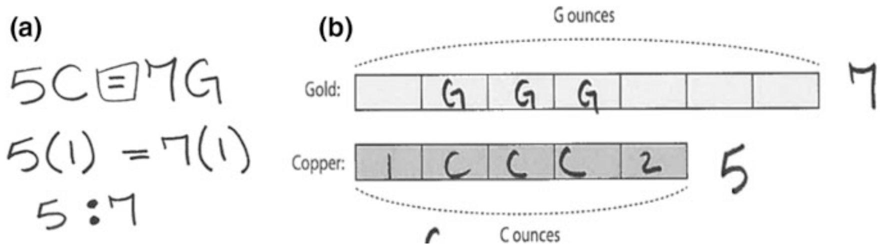


Fig. 13.10 **a** Alice's equation in Task 2. **b** Alice's interpretation of the strip diagram

Interviewer: Could you produce an equation that had *G* and *C* in it?

Alice: My equation, I guess, would be *5 parts copper equals 7 parts gold* (writes the equation in Fig. 13.10a).

Interviewer: How are you thinking about that?

Alice: I guess for *C* would be how many parts are in each of these? So, if like 5… if our size of the parts is 1, we would still be within that 5 to 7 ratio. So *not necessarily that these are equal, but they're proportionate*.

Interviewer: Yeah, I was going to ask you how you were thinking about *the equal sign*. So, you're saying it's indicating that they're proportionate.

Alice: Yes, *because they aren't equal*. Because 5 and 7 aren't equal, but … *they have this relationship of 5 to 7*. Then, I go back to think that *C* ounces and *G* ounces aren't going to be the same. But when I think about … each of these as like a *G* and each of these as *C*, I was like … thinking about those as size of the parts then your proportional relationship would be the same (puts *G* and *C* into each part in Fig. 13.10b). But I don't know how to think of it as like how *G* and *C* are related.

Alice appeared to understand that *G* and *C* are in a 7 to 5 ratio, but she used the equal sign to indicate certain amounts of gold and copper were associated with one another, not to indicate that numbers of ounces of gold and copper would be the same. In addition, she did not use the definition of multiplication to generate an appropriate equation.

Moreover, Jeff did not succeed in generating an appropriate equation either. He interpreted *G* as 7 oz and *C* as 5 oz in the given task, and he developed an incorrect equation as 7 + 5 = 12 oz (Fig. 13.11).

Jeff: I'm thinking about setting up another equation (other than his current Eq. 7 + 5 = 12 oz), but I'm having a hard time figuring out the parts of the equation.

Interviewer: How are you interpreting the diagram that we provided (refers to the given strip diagram in Task 2)? Walk me through kind of how you're interpreting it.

Jeff: I'm seeing the gold (points to *G* in the given diagram) as 7 oz. And then this *C* ounces as 5.

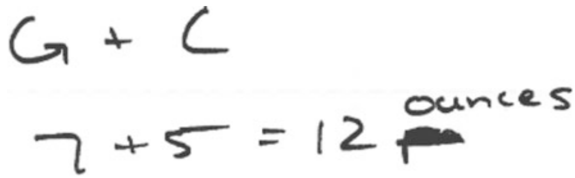


Fig. 13.11 Jeff's equation in Task 2

Although Jeff did not feel comfortable with his Eq. 7 + 5 = 12 oz and searched for another equation, he acknowledged that he could not figure out the specific terms that would form that equation. Jeff's interpretation of *G* ounces and *C* ounces as numeric values instead of variables (i.e., conflation of ounces and parts) indicated that he was not coordinating units and groups appropriately. In addition, he did not use the definition of multiplication to generate an appropriate equation that relate ounces of gold and copper.

Diana produced incorrect equations as $C = 5 \cdot 1/7$ and $G = 7 \cdot 1/5$ with whole number multipliers, and she interpreted the equations in terms of the definition for fractions (i.e., "5 parts each of size 1/7 of *G* ounces" and "7 parts each of size 1/5 of *C* ounces"), even though she was consistently reminded to use the definition of multiplication.

Diana: Okay. So, I guess the relationship would be for every 7 oz of gold there are 5 oz of copper or for every 7, I guess, *G* ounces there are 5 *C* ounces. Or, you could say the *G ounces is 7 parts, each a size 1/5 of the C ounces* or vice versa would be *C* ounces is equal to *5 parts, each of size 1/7 of the G ounces*.

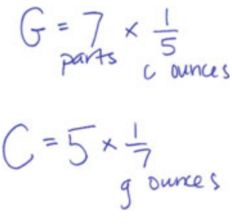
Interviewer: So, what would you write for that? For either one of those possibilities?

Diana: So, I guess *G* would be equal to *7 parts (I'd) say times 1/5 of C ounces* and then the other one would be *C* is equal to *5 parts each of size 1/7 of G ounces* (writes $C = 5 \cdot 1/7$ and $G = 7 \cdot 1/5$ in Fig. 13.12).

Interviewer: Why are you writing the *C* ounces and the *G* ounces sort of down below (points to her equations in Fig. 13.12)?

Diana: I guess just showing *what 1/5 and 1/7 represents*. So, like you see like the number problem, but you see what each number means below it.

Fig. 13.12 Diana's equations in Task 2



Diana's only use of the definition for fractions and her lack of reliance on use of the definition of multiplication did not lead to generation of an appropriate equation in this task. Moreover, her notations of "C ounces" and "G ounces" underneath her equations $C = 5 \cdot 1/7$ and $G = 7 \cdot 1/5$ in Fig. 13.12 and her explanation for her use of such notations above provided evidence that she might interpret the equal sign as association between the ounces of gold and copper instead of equality of these ounces.

On the other hand, Linda, Claire, and Kelly, who demonstrated solid understanding of the definition of multiplication in Task 1, continued using this definition and succeeded in generating a correct equation in Task 2 (see Table 13.1). They all produced an equation with a fraction multiplier, such as $C = 5/7 \cdot G$, by perceiving 5/7 as "the number of groups" in their strip diagram drawings. Linda, immediately, indicated the relationship between the ounces of gold and copper as "copper is 5/7 of gold" and "gold is 7/5 of copper," and she generated correctly $C = 5/7 \cdot G$ as follows:

Linda: So, copper is 5/7 of gold or gold is 7/5 of copper (points to the given strip diagram drawing in Task 2). Copper equals $5/7 \cdot G$ where *G is the whole and 5/7 is the size of the group* (writes $C = 5/7 \cdot G$).

Interviewer: How do you interpret the equal sign? You wrote an equal sign right there (refers to her equation $C = 5/7 \cdot G$). When you read that, what are you thinking?

Linda: Copper equals 5/7 of gold… The number of ounces copper has. Yeah, copper has 5/7 the number of ounces as gold.

The data show that Linda interpreted the equal sign as having the same number of ounces of gold and copper rather than an association between them. Her use of the definition of multiplication as "5/7 is the size of the group" (i.e., "the number of groups" for her) and "G is the whole" appeared to regulate her thinking in generating the appropriate equation.

Furthermore, Kelly generated two correct equations of $C = 5 \cdot G/7$ and $C = 5/7 \cdot G$ and presented them in one equation as $5 \cdot G/7 = 5/7 \cdot G$. She used the definition of multiplication in both of her equations as "5 groups" and "G/7 in each group" for $C = 5 \cdot G/7$, and "5/7 groups" and "G in one group" for $C = 5/7 \cdot G$. Her use of the definition of multiplication seemed to help her develop appropriate equations.

Kelly: 5 groups, G/7 in each group (refers to the left hand side of her equation in Fig. 13.13), so it's like 5/7ths G of like, in terms of the gold is in copper.

Interviewer: The left side and the right side are a little different, the way you've written them. (refers to her equation in Fig. 13.13)

Kelly: I just combined them (refers to both sides of her equation in Fig. 13.13).

Interviewer: Could you explain the meaning of multiplication in each of those cases?

Kelly: *There are 5 groups and each of the groups has G/7 in them* (points to the left hand side in Fig. 13.13) … For this 5/7th G (points to the right hand side in Fig. 13.13), *5/7ths is my group* because 7/7th would be one group of gold. And the copper is 5/7th group, so we're looking at 5/7th of G, which is 7/7th or a whole.

Fig. 13.13 Kelly's equation in Task 2

Finally, Claire kept the language used in definitions for multiplication and for fractions distinct. She interpreted $5/7 \cdot G = C$ from both definition of fractions (i.e., "5 parts each size 1/7 of the whole amount of gold") and the definition of multiplication (i.e., "5/7 groups times the amount of ounces in 1 group is equal to the amount of copper in 5/7 groups of gold").

Interviewer: How would you explain the meaning of the equal sign (refers to her Eq. $5/7\ G = C$ in Fig. 13.14b)?

Claire: That means they're equivalent or the same amount of ounces, because the gold has like 7 parts versus the copper has 5 parts. But *if we take 5/7 of that gold, we're going to have 5 of the 7 parts, which is equivalent to the parts of copper* (points to the strip diagram accompanying Task 2). So, 5/7 of the amount of gold is equivalent or the same amount of ounces as the total amount of copper we have.

Interviewer: Could you interpret the left hand side of this (points to her Eq. $5/7\ G = C$ in Fig. 13.14b) for me using the meaning of multiplication?

Claire: I know two meanings, so… like I know… *the fraction meaning*, and I also know *the meaning of the multiplication*. And you're talking about multiplication, right?

Interviewer: Tell me about both.

Claire: Well, with the fraction meaning 5/7 that means we have… the whole unit is 7 parts and the amount of parts we have is 5 of that… those 7 parts or 5 of the whole unit. So it's *5 parts each size 1/7*. And with the multiplication… So, 5/7 groups times the amount of gold that we have is the amount of copper we have. So, it's like *5/7 groups times gold, which is the amount of ounces in 1 group or the amount of gold, is equal to copper, which is total for the 5/7 groups of gold*.

Claire exhibited solid performance based on an appropriate interpretation of the equal sign, the definitions for multiplication and for fractions in addition to her generation of single, correct equation. The given strip diagram appeared to support her performance for perceiving the relationships between the ounces of gold and copper.

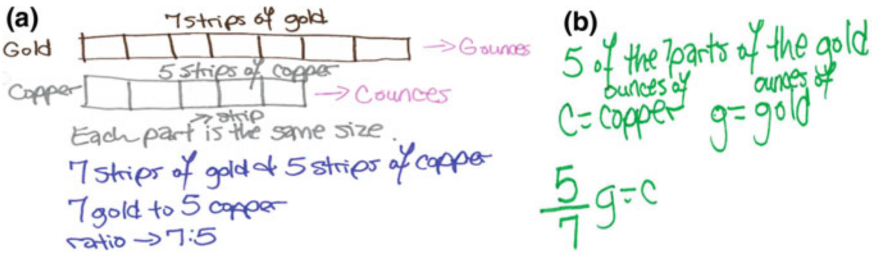


Fig. 13.14 **a** Claire's strip diagram in Task 2. **b** Claire's equation in Task 2

13.5 Conclusion and Discussion

The purpose of the present study was to examine how six future middle grades mathematics teachers used quantitative definitions for multiplication and for fractions when developing and explaining equations for proportional relationships. The results revealed that future teachers' use of explicit, quantitative definition of multiplication in a consistent way co-occurred with their success when generating and explaining equations that involve proportional relationships. In other words, future teachers' explicit use of the definition of multiplication facilitated their ability to develop equations and to reason about proportional relationships. All three future teachers (i.e., Linda, Claire, and Kelly), who demonstrated an explicit and consistent use of the definition of multiplication, seemed to use this definition as an organizing tool to regulate their reasoning process while working on the given tasks. Their use of the definition as an organizing tool apparently enabled them to coordinate units and groups appropriately, to keep distinct wording between the definitions for multiplication and for fractions, and to generate and explain appropriate equations for proportional relationships. In addition, their interpretation of the equal sign was consistent with normative usage, which is to equate the number of ounces of gold and of copper. Furthermore, when multipliers were placed as fractions in the tasks, these future teachers were able to identify the multiplier as the number of groups by viewing how many groups (or parts) of one strip were in another strip in their drawings.

On the other hand, the remaining three future teachers (i.e., Alice, Jeff, and Diana), who did not explicitly express the quantitative definition of multiplication, had difficulties in generating equations and in reasoning about proportional relationships. These future teachers experienced consistent difficulties distinguishing the different roles played by the multiplier and multiplicand, and coordinating units and groups appropriately. Specifically, they had trouble identifying the multiplier as the number of groups when multipliers were fractions, and this even caused some of them to switch the order of multiplier and multiplicand so that the multipliers were whole numbers. Moreover, the placement of fractions as multipliers in the tasks caused these teachers to misuse phrasing from the definition for fractions when applying the definition of multiplication. Thus, lack of the explicit use of the definition of multiplication in a consistent way apparently contributed to these future teachers' blended wording from both definitions rather than keeping them distinct, and to struggle with reasoning about proportional relationships. Moreover, these future teachers' interpretations of the equal sign were inconsistent with normative usage, at least for Task 2.

Given that developing algebraic equations is difficult and this study took place before instruction on equations, it is important that three of the six future teachers generated and explained appropriate equations that involve a proportional relationship by using the definition of multiplication. Therefore, the results of this study suggest that explicit use of the quantitative definition of multiplication is helpful organizing tool for future teachers in terms of developing equations and reasoning about proportional relationships. Furthermore, past research has documented teachers' difficulties with particular topics such as fraction multiplication and multiplication with an unknown factor (e.g., Armstrong & Bezuk, 1995; Izsák, 2008). Regarding that past research has not examined the extent to which teachers' facilities with multiplication and fractions can support and constrain their generation of equations and reasoning about proportional relationships, the present study concludes that using the quantitative definition of multiplication to construct viable arguments related to proportional relationships and equations is accessible to future middle grades teachers. Robust arguments about proportional relationships and equations depend in turn on the ability to keep distinct the definitions for multiplication and for fractions. In addition to stating the quantitative definition of multiplication verbally, future teachers also need to visualize how many parts of one strip are nested in another strip by viewing the relationships between the multiplier and multiplicand in the strip diagrams.

As the main implication of this study, future teachers in middle grades programs should be given opportunities to develop capacities for reasoning with quantitative definitions for multiplication and for fractions across problem situations. Future studies should continue, with larger samples and tasks, to evaluate the effects of using explicit, quantitative definitions for multiplication and for fractions on both teachers and students' generation of equations and their reasoning about proportional relationships. Additional research is also needed to compare the effects of using these definitions for future middle grades mathematics teachers in developing and explaining algebraic equations before and after class instruction.

Acknowledgements I would like to thank Andrew Izsák and Sybilla Beckmann for helpful comments on earlier drafts of this monograph. This research was supported by the National Science Foundation under Grant No. DRL-1420307. The opinions expressed are those of the author and do not necessarily reflect the views of the NSF.

References

Armstrong, B., & Bezuk, N. (1995). Multiplication and division of fractions: The search for meaning. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 85–119). Albany, NY: State University of New York Press.

Ball, D., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433–456). Washington, D.C.: American Educational Research Association.

Beckmann, S. (2014). *Mathematics for elementary teachers with activities*. Boston: Pearson.

Beckmann, S., & Izsák, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education, 46*(1), 17–38.

Beckmann, S., Izsák, A., & Ölmez, İ. B. (2015). From multiplication to proportional reasoning. In X. Sun, B. Kaur, & J. Novotna (Eds.), *Conference proceedings of ICMI Study 23: Primary mathematics study on whole numbers* (pp. 518–525). Macau: University of Macau.

Bradshaw, L., Izsák, A., Templin, J., & Jacobson, E. (2014). Diagnosing teachers' understandings of rational number: Building a multidimensional test within the diagnostic classification framework. *Educational Measurement: Issues and Practice, 33*(1), 2–14.

Common Core State Standards Initiative. (2010). *The common core state standards for mathematics*. Washington, DC: Author.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. Washington, DC: Author.

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159–178). New York: Macmillan.

Fisher, L. (1988). Strategies used by secondary mathematics teachers to solve proportion problems. *Journal for Research in Mathematics Education, 19*(2), 157–168.

Harel, G., & Behr, M. (1995). Teachers' solutions for multiplicative problems. *Hiroshima Journal of Mathematics Education, 3,* 31–51.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction, 26*(1), 95–143. https://doi.org/10.1080/07370000701798529.

Izsák, A., & Jacobson, E. (2017). Preservice teachers' reasoning about relationships that are and are not proportional: A knowledge-in-pieces account. *Journal for Research in Mathematics Education, 48*(3), 300–339.

Jansen, A., & Hohensee, C. (2016). Examining and elaborating upon the nature of elementary prospective teachers' conceptions of partitive division with fractions. *Journal of Mathematics Teacher Education, 19*(6), 503–522. https://doi.org/10.1007/s10857-015-9312-0.

Kaput, J., & West, M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235–287). Albany: State University of New York Press.

Karplus, R., Pulos, S., & Stage, E. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*. New York: Academic Press.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 629–667). Charlotte, NC: Information Age Publishing.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education. Retrieved from http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf.

Orrill, C. H., & Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education, 15*(5), 381–403.

Ölmez, İ. B. (2016). Two distinct perspectives on ratios: Additive and multiplicative relationships between quantities. *Elementary Education Online, 15*(1), 186–203.

Post, T., Harel, G., Behr, M., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. Carpenter, & S. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177–198). Albany, NY: State University of New York Press.

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., … Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE #2010-4039). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from whatworks.ed.gov/publications/practiceguides.

Simon, M. A., & Blume, G. W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 13*(2), 183–197.

Tirosh, D., & Graeber, A. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21*(2), 98–108. https://doi.org/10.2307/749137.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). Orlando, FL: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (pp. 141–161). Reston, VA: National Council of Teachers of Mathematics.

Yin, R. K. (1993). *Applied social research methods series: Vol. 34. Applications of Case Study Research*. London: SAGE publishing.