Chapter 3
Students' Thinking About Integer Open Number Sentences

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Abstract We share a subset of the 41 underlying strategies that comprise five ways of reasoning about integer addition and subtraction: formal, order-based, analogy-based, computational, and emergent. The examples of the strategies are designed to provide clear comparisons and contrasts to support both teachers and researchers in understanding specific strategies within the ways of reasoning. The ability to categorize strategies into one of five ways of reasoning may enable teachers to organize knowledge of student thinking in ways that are useable and accessible for them and provide researchers with sufficient information about the strategies and ways of reasoning such that they can reliably build on this work.

Imagine how a student might solve the problem -3 + 6 = $\square$. Below we share several responses we heard from K–12 students who participated in our study.

Oscar: Oscar reaches for a provided number line and places his pen at -3. He moves his pen to the right one unit at a time while he counts, "One, two, three, four, five, six." His pen is now at 3 on the number line, and he answers, "Three" (Grade 7).

Alex: "It's like I owe my friend three dollars. And my mom gives me six dollars. I pay my friend three of the dollars that I got from my mom, and I still have three dollars, so my answer is three" (Grade 4).

Cole: "Three. The signs [for -3 and 6] are different, so I subtracted them and took the sign of the bigger number." When asked what he subtracted,

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L. Bofferding, N. M. Wessman-Enzinger (eds.), *Exploring the Integer Addition and Subtraction Landscape*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-90692-8_3

Cole continued, “I subtracted six minus three, which is three. Six is bigger than three, so I knew the answer had to be positive since six is positive.” When asked why it mattered which number was larger, Cole posed a related problem of -6 + 3. “Look, if it was, uh, like negative six plus three, you still subtract six minus three because they’re different signs. But if six *is negative* [with emphasis], then the answer is negative three” (Grade 7).

Fran-Olga: “I’ll just start by counting. [Fran-Olga moves her lips, presumably counting under her breath.] I don’t know. It’s either negative nine or three.” When asked to explain how she arrived at each answer, Fran-Olga replied, “Well if I go down into the negatives, it’s -4, -5, -6, -7, -8, -9. But if I go the other way, then [it’s] -2, -1, 0, 1, 2, 3. [Long pause] Maybe it’s switched. Wait. When I did three minus five, it [the operation in the problem] was minusing, and this one [this problem] is plussing. I’m thinking that since this one [points to 3 – 5 = □] was minus and I was going into the negatives, that this one [points to -3 + 6 = □] goes up. I think it’s three now” (Grade 2).

These responses are representative of the reasoning students across multiple grade levels used when solving open number sentences such as -3 + 6 = □, 5 – □ = 8, and □ + -2 = -10. When students solved these types of problems and shared their responses with us, we found that we could characterize their thinking about integer open number sentences into one of five broad ways of reasoning: *order-based*, *analogy-based*, *computational*, *formal*, and *emergent*. For us, a *way of reasoning (WoR)* about integer addition and subtraction involves a conceptualization of signed numbers in which the student draws on certain affordances or mathematical properties of the underlying conceptualization to engage in integer arithmetic. For example, in using an *order-based WoR*, one draws on the ordered and sequential nature of the set of integers and uses that property to reason about integer addition. We see this approach in both Oscar’s and Fran-Olga’s responses. In contrast, in a *computational WoR,* one treats numbers more abstractly and relies on rules and procedures to solve problems as we see in Cole’s response.

Although we briefly describe the five broad ways of reasoning (for a more detailed description, see Bishop et al., 2014), our goal in this chapter is to share the underlying strategies students used within each WoR about integer addition and subtraction. Our hope is that researchers and teachers will find both the more general ways of reasoning and the specific and detailed strategies useful to better understand students’ approaches to solving open number sentences and to guide future instruction.

Connections to Theory and Building From Existing Research

Our focus is on students’ mathematical thinking in the context of signed numbers, with a particular focus on how children think about integer addition and subtraction. Within mathematics education is a well-established tradition of studying students’ understanding of mathematical topics, including whole-number operations (Carpenter, Fennema, Franke, Levi, & Empson, 2014; Fuson, 1992), fractions (Empson & Levi, 2011; Hackenberg, 2010; Steffe & Olive, 2010), quantitative reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, 2010), limits and infinity (Swinyard & Larsen, 2012; Tall & Vinner, 1981; Williams, 1991), and integers (Bishop et al., 2014; Bofferding, 2014; Peled, 1991). Drawing from a Piagetian tradition, researchers working in this vein are generally interested in "… the way the child reasoned and the difficulties he encountered, the mistakes he made, his reasons for making them, and the methods he came up with in order to get to the right answers" (Piaget, as quoted in an interview with Bringuier, 1980, p. 9). Studies of students' mathematical thinking and cognition are grounded in constructivist theories of knowing and learning, and researchers within this tradition view students' mathematical thinking as important in its own right and distinct from established disciplinary views of a given topic as well as commonplace adult conceptions of mathematical topics.

Although a common constructivist heritage unites research in this tradition, scholars vary in their research designs and data sources (e.g., paired interviews in teaching experiment settings, individual, clinical interviews, or design experiments in classroom settings), their units of analysis (e.g., student reasoning about a particular task or evidence of construction of a particular mental scheme/structure), and the extent to which they incorporate Piagetian constructs such as operations, structures, and interiorization/internalized operations into analyses. In this chapter, we do not analyze students' mathematical thinking by looking for evidence of particular schemes, structures, or mental operations (e.g., levels of units). Instead we document strategies that students use when solving integer addition and subtraction problems. Through these more detailed strategies and their relationships to broader ways of reasoning, we seek to identify, describe, and categorize key features and patterns in students' problem-solving approaches that are general enough to provide a sense of coherence, yet are nuanced enough to sufficiently differentiate among students' solutions. We now turn to the literature base for a brief review of research related to students' conceptions of integers and the specific strategies they bring to bear when solving problems.

Students' struggles operating with negative numbers are well documented (Christou & Vosniadou, 2012; Gallardo, 1995; Kloosterman, 2012; Vlassis, 2002). Whereas Mora and Reck (2004) identified rules and procedures that students attempted to use when solving problems with negative numbers, others have found that children can make productive use of order, leveraging the sequential and ordered nature of numbers, to solve such problems, particularly with number lines (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Bofferding, 2014; Peled, 1991; Peled, Mukhopadhyay, & Resnick, 1989). Those using other lines of research have studied students' use of metaphors (see, e.g., Chaps. 5 and 6) and the efficacy of different contexts when engaging with integers and integer arithmetic (see, e.g., Chaps. 4 and 9). For example, Chiu (2001) identified categories of metaphors that students and experts used when solving integer problems, and Stephan and Akyuz (2012) developed an

Table 3.1 Literature-based problem-solving approaches for integer arithmetic

Problem-solving approach for integer arithmetic	Example/explanation	References
Rules and procedures	Operations with negative numbers are performed using rules, either correctly or incorrectly (e.g., applying rules for multiplication of signed numbers to addition and subtraction problems).	Chiu (2001) and Mora and Reck (2004)
Financial/ transactional context (debt, net worth, etc.)	Operations with negative numbers are related to money or other transactional contexts (giving/receiving) in which negatives are typically associated with debt or owing.	Chiu (2001), Peled and Carraher (2006), and Stephan and Akyuz (2012)
Other oppositional contexts and quantities	Negative numbers are used to represent a quantity of items with an unfavorable connotation (and in opposition to the positive quantity). For example, using two colors of chips or blocks to represent positive and negative numbers.	Chiu (2001) and Peled (1991)
Analogy to whole number	Negative numbers are related to whole numbers when solving integer arithmetic problems (e.g., using the known fact that 5 – 2 = 3 to evaluate the unknown expression -5 – -2).	Human and Murray (1987) and Murray (1985)
Number line, motion/movement	Imposing an ordering on signed numbers or using an existing ordering (as provided in a number line) and reasoning about addition/ subtraction as moving forward and backward.	Chiu (2001), Behrend and Mohs (2005–2006), Bofferding (2014), Murray (1985), Peled et al. (1989), Peled (1991), and Stephan and Akyuz (2012)
Logic	Comparing related problems such as 6 + -2 and 6 + 2 and using a fundamental mathematical property (e.g., inverse operations) to solve the related problem.	Human and Murray (1987)

instructional sequence about financial contexts (with a focus on net worth and incorporating the use of number lines) that positively supported students' understanding of integer addition and subtraction. Further, Murray (1985) and Bishop, Lamb, Philipp, Whitacre, and Schappelle (2016a, 2016b) found that some students could apply logical deductions based on the underlying structure of our number system to solve or explain their reasoning about integer open number sentences. Murray found that students used *logic* to solve problems by comparing a previously solved problem and a related new problem (e.g., 5 + -3 and 5 – -3) to aid in solving the new problem.

Because our focus in this chapter is on students' strategies for integer addition and subtraction, our above synthesis of existing research was also focused on different problem-solving approaches. In Table 3.1, we summarize problem-solving approaches documented in other scholar's work on integers and integer operations along with a brief example and relevant references for each.

Across the literature that documents differing conceptions of integers and integer arithmetic, we see a variety of productive problem-solving approaches. In this chapter we build on this research by describing students' integer strategies, including those in Table 3.1, and organizing those strategies within the broader ways of reasoning. We hope that this expanded framework, which combines both ways of reasoning and strategies, will support teachers to develop and use this knowledge in their instruction.

Conceptual Framework

Our goal within this chapter is to present a more nuanced and complete view of our ways -of -reasoning framework by defining and exemplifying many of the strategies within each WoR. To do so, we briefly describe the broader ways of reasoning into which the more detailed strategies are organized (see also Bishop et al., 2014; Bishop et al., 2016a for previous versions of the ways of reasoning). As mentioned earlier, a way of reasoning (WoR) is a general conceptualization and approach to solving integer addition and subtraction problems that is characterized on the bases of key features of students' solutions and the underlying views of number and operations at work. We identified five ways of reasoning that students across all participant groups in our study used when solving open number sentences: order-based, analogy-based, computational, formal, and emergent. (In earlier publications, we used different names for analogy-based and emergent reasoning, referring to analogy-based as magnitude and emergent as developmental or limited.) In Table 3.2 we define each WoR.

In the responses shared in the introduction to this chapter, four of the five ways of reasoning are represented. Alex's comparison of negative numbers to debt is an example of *analogy-based* reasoning, whereas Cole used *computational* reasoning when he invoked rules and properties in his solution. Oscar and Fran-Olga used *order-based* reasoning by ordering spoken number words and their corresponding written symbols to determine what was before and after a given number and then using these sequences to solve the problem. Fran-Olga's response also reflects a *formal* way of reasoning: In her explanation she compared the operations of addition and subtraction and used her informal understanding of inverses to argue that her answer was a necessary consequence of the relation between addition and subtraction and her assumptions from a previous problem. Within each *WoR* we wanted to identify specific and detailed strategies students brought to bear on each task (e.g., counting as a particular instantiation of the *order-based WoR* seen in Fran-Olga's response or the use of a number line as seen in Oscar's *order-based WoR*). A *strategy* is a subcategory of a particular *WoR* that further describes and differentiates student responses within the broader *WoR*. We view the five ways of reasoning as an organizing structure into which we can categorize more detailed strategies on the basis of the underlying views of number and operation leveraged in a given strategy's use. Given this view, the research questions guiding our study were the following: What strategies do students use when solving open number sentences with integers, and what is the relation among strategies and the broader ways of reasoning?

Table 3.2 Ways of reasoning

Ways of reasoning	Definition
Order-based	In this way of reasoning, one leverages the sequential and ordered nature of numbers to reason about a problem. Strategies include use of the number line with motion as well as counting forward or backward by 1s or another incrementing amount.
Analogy-based	This way of reasoning is characterized by relating numbers and, in particular, signed numbers, to another idea, concept, or object and reasoning about negative numbers on the basis of behaviors observed in this other concept. At times, signed numbers may be related to contexts (e.g., debt or digging holes). Analogy-based reasoning is often tied to ideas about cardinality and understanding a number as having magnitude.
Formal	In this way of reasoning, signed numbers are treated as formal objects that exist in a system and are subject to mathematical principles that govern behavior. Students may leverage the ideas of structural similarity, well-defined expressions, the structure of our number system, and fundamental principles (such as the field properties). This way of reasoning includes generalizing beyond a specific case by making a comparison to another, known, problem and appropriately adjusting one's heuristic so that the logic of the approach remains consistent, or generalizing beyond a specific case to apply properties of classes of numbers, such as generalizations about zero.
Computational	In this way of reasoning, one uses a procedure, rule, or calculation to arrive at an answer. For example, some students used a rule to change the operation of a given problem along with the corresponding sign of the subtrahend or second addend (i.e., changing 6 – -2 to 6 + 2 or 5 + -7 to 5 – 7). Students often explained these changes by referring to rules like "Keep Change Change" (keep the sign of the first quantity, change the operation, and change the sign of the second quantity). For a strategy to be placed into this category, the student may state a procedure or rule with or without sharing a justification.
Emergent	This category of reasoning often reflects preliminary attempts to compute with signed numbers. For many strategies in this category, the domain of possible solutions is locally restricted to nonnegatives. For example, a child may overgeneralize that addition always makes larger and, as a result, claim that a problem for which the sum is less than one of the addends ($6 + \square = 4$) has no answer. The domain of possible solutions appears to be restricted to natural numbers, and the effect (or possible effect) of adding a negative number is not considered.

Study Background and Methods

Participants

The data and findings we share are from a larger program of research wherein we investigated student's' conceptions of integers and integer operations across multiple grade levels. We interviewed 160 students at 11 schools in five districts in a large urban city in the Southwestern United States. Forty students at each of Grades 2, 4, 7, and 11 were randomly selected from students who returned consent forms. We chose these grades to provide a cross-sectional view of integer reasoning at different grade levels. The second and fourth graders provided insight into children's thinking before school instruction; the seventh graders reflected students' thinking immediately after school instruction on integers; and the 11th graders were chosen to represent the endpoint of students' integer reasoning in the K–12 setting.[1] During the interviews we noticed that some of our elementary-grade participants had knowledge of integers, whereas others did not. Consequently, we reorganized our second- and fourth-grade participant groups for analysis purposes. All students were placed into one of four groups: college-track ([CT], $n = 40$, eleventh-grade students), post instruction ([PI], $n = 40$, seventh-grade students who had recently completed instruction in integers), before instruction, with negatives ([BIN], $n = 39$, second and fourth graders with knowledge of negatives), and no evidence of negatives ([NEN], $n = 41$, second and fourth graders without knowledge of negatives). Group placements for second and fourth graders were made on the basis of responses to the introductory questions in the interview (see questions 1–4 in the Appendix).

Problem-Solving Interview

As part of the larger study, we developed, piloted, and revised a problem-solving interview over a period of 2 years. In addition to the 160 participants described in the previous section, we conducted pilot interviews with an additional 90 K–12 students across four interview cycles, each of which was focused on different grade levels of students (i.e., the first interview cycle targeted K–2 students, the second cycle targeted high school students, the third Grades 3–5, and the last cycle focused on middle school students). In each interview cycle, we tested new tasks and continued to refine the sequencing and phrasing of existing tasks to identify tasks likely to elicit students' integer reasoning.

Drawing from Piaget's method of clinical interviewing (Ginsburg, 1997), our initial goal with the interviews was to balance flexibility and standardization. Piaget described his approach as follows:

> You ask, you select, you fix the questions in advance. How can we, with our adult minds, know what will be interesting? If you follow the child wherever his answers lead spontaneously, instead of guiding him with preplanned questions, you can find out something new. … Of course there are three or four questions we always ask, but beyond that we can explore the whole area instead of sticking to fixed questions. (Piaget, as cited by Bringuier, 1980, p. 24)

Our pilot interviews were consistent with Piaget's description of his method. For example, it was not initially apparent to us that the open number sentences $-3 + 6 = \square$ and $6 + -3 = \square$ might encourage different reasoning; as adult experts,

[1] Note that we restricted our eleventh-grade participants to college-track students, that is, students who were enrolled in either calculus or precalculus during their eleventh-grade year. Our goal with the eleventh-grade students was to identify the best-case scenario for integer understanding when students finish their high school education.

we viewed -3 + 6 and 6 + -3 as equivalent because of the commutative property of addition. As we discovered in the pilot interviews, the location of -3 as the first or second addend influenced some students' approaches to the problem. Similarly, we were surprised when the seemingly similar problems of 6 – -2 = □ and -5 – -3 = □ (both involve subtracting a negative quantity) yielded widely differing responses from some children. One of our goals in conducting the pilot interviews was to pursue and uncover these differences in reasoning and the underlying conceptions from which they emerged. As a result, we routinely posed follow-up tasks to confirm or refute our working hypotheses about the ways students were reasoning and what features made problems more and less difficult. Because we customized follow-up questions on the basis of the specifics of the student's responses in the moment, they were not preplanned or standardized.

However, from the beginning of the project, our goal was to develop a standardized set of questions that would be posed to all students in the main study and from which we could compare students' reasoning within and across grade levels. In early 2011 we finalized the problem-solving interview and began conducting 160 interviews across the participant groups described earlier using a standardized set of questions. The one-on-one interview lasted 60–90 minutes and consisted of 56 total problems including introductory questions, open number sentences, context problems, comparison problems, and tasks involving variables and algebra (see Appendix for the complete interview). We found that solving open number sentences provided productive opportunities for students to reason about signed numbers; consequently, the analyses and findings we share in this chapter are based on the 25 open number sentences posed to students. (These open number sentences are questions 9–14, 16–28, and 30–35 in the interview shared in the Appendix.)

Analysis

For each open number sentence in each interview, we assigned a code for correctness and a code for the strategy (or strategies) the student used when solving the given problem. Each strategy was subsumed in one of the five ways of reasoning. For some problems, students used multiple ways of reasoning and, therefore, received more than one *WoR* code. Across all ways of reasoning, we identified a total of 41 strategies.[2] We developed our set of codes for the 41 strategies (and the 5 ways of reasoning) iteratively over a 3-year period. Moreover, this set of codes comprehensively captures the strategies students in our study used. Although some of the strategy codes we created are documented in existing literature (e.g., logic, use of number lines, converting to context), we did not use these codes a priori. Instead, we used a grounded theory approach to analysis so that our codes emerged from the student responses in the interviews (Corbin & Strauss, 2008).

[2] Two of these strategies, *unclear* (assigned when a strategy was not clear) and *other* (assigned when a student's response did not match an existing *WoR* or strategy), were used rarely and were not associated with one particular *WoR*. Although we include *unclear* and *other* as strategies, they are not subcategories of any single *WoR*.

Findings

In the text that follows, we expand the ways of reasoning framework by identifying and exemplifying the most common strategies within each *WoR*. We hope our categorizations will help readers to identify important differences and similarities among strategies and recognize the complexity and richness of students' thinking about integer addition and subtraction.

Common Strategies Within Ways of Reasoning

Across all ways of reasoning, we identified 41 total strategies. Table 3.3 identifies the most frequently invoked strategies within each *WoR* along with their overall percentage use. In general, we share the three most common strategies within each *WoR* in the following sections.[3]

Table 3.3 Examples of strategies within ways of reasoning and frequency of use

Way of reasoning	Strategy examples	Percentage use (of total problems posed)
Order-based	Number line	16.21%
	Jumping to zero	3.73%
	Counting by ones	3.26%
Analogy-based	Negatives like positives	6.82%
	Converts to context	3.32%
Computational	Keep change change	16.21%
	Negative sign subtractive	7.86%
	Changes order of terms	6.67%
	Equation	6.39%
	Same signs/different signs	3.73%
Formal	Infers sign	8.41%
	Generalization about zero/additive inverses	2.63%
	Logical necessity	<1%
Emergent	Addition makes larger/subtraction makes smaller	14.01%
	Ignores sign	10.78%
	Pascal	1.18%

[3] Because computational was the most frequently used *WoR*, we share more of the strategies within this *WoR*, and because the strategies other than *negatives like positives* and *converts to context* within *analogy-based WoR* were used so infrequently, we share only those two.

Order-Based *Order-based reasoning* was used on about one fourth of all problems posed. In this *WoR*, one leverages the sequential and ordered nature of numbers to reason about a problem. The most common strategies within this *WoR* were the *number line/motion* strategy, the *jumping to zero* strategy, and the *counting by ones* strategy. The *number line/motion* strategy was the most common within the *order-based WoR*, used on 16% of the problems posed. When using the *number line/motion* strategy, students treated the first addend and the sum (or the minuend and difference) as locations on the number line and the second addend (or subtrahend) as the number to move. The operations usually determined the direction of movement. To receive this code, students had to either explicitly use motion on a number line or share that they imagined moving on a number line when solving. For example, for the problem shared in the introduction, -3 + 6 = □, Oscar's strategy exemplifies *number line/motion*. His starting point is -3, the operation of addition indicates movement to the right, the second addend indicates the number to move, and the unknown is the ending location.

Another relatively common strategy demonstrating an *order-based WoR* that students used was *counting by ones*. Ellie, a second-grade student, counted up by ones to solve -3 + 6 = □ and from -2 to 4 to solve the problem -2 + □ = 4. She counted aloud saying, "Minus 2, minus 1 (raises one finger), zero (raises another finger)." She paused. "Wait, I lost count." Ellie then restarted her count, "Minus 2, minus 1 (raises one finger), zero (raises second finger), 1 (raises third finger), 2 (raises fourth finger), 3 (raises fifth finger), 4 (raises sixth finger)." Ellie's final answer was 6. We conjecture that Ellie's pause and restart ("I lost count") may indicate the additional cognitive demand required to begin her counting sequence with a negative number rather than a natural number. But her ability to successfully extend her counting sequence may be attributable to the fact that the direction of her counting was consistent with the addition of natural numbers (addition makes larger and thus one counts up toward the positive numbers to arrive at a sum). When, for example, Ellie solved the problem -5 + -1 = □, she adopted a different strategy and incorrectly answered -4; she may have abandoned a counting strategy for this problem because adding -1 to -5 would indicate a movement left on the number line for addition, or movement in the opposite direction than one would move with natural numbers.

The last strategy within the *order-based WoR* we share here is *jumping to zero,* which was used on just more than 3% of the problems posed. Opal's response to the problem -3 + 6 = □ exemplifies this strategy. Opal answered, "Three, "and then explained saying, "Half of 6 is 3, so then that would bring it [the running total] to the 0. And 3 more would bring it to the 3. And that would equal 6." Opal's strategy can be represented mathematically with the following series of equivalent expressions: -3 + 6 = -3 + (3 + 3) = (-3 + 3) + 3 = 0 + 3. By decomposing 6 into 3 plus 3, Opal was able to "jump to zero" by adding one of the 3s to -3. In general, this strategy involves strategically decomposing a number to obtain additive inverses so that the resultant partial sum is zero. However, students are unlikely to recognize either the underlying mathematical property they are implicitly using or its significance. We conjecture that Opal was treating zero like other decade numbers (e.g., 10 and 20) and using her knowledge of decomposition and incrementing to reach a *friendly* number as part of her computation. We believe that this type of order-based reasoning can be leveraged to formalize and explicitly name the concept of additive inverses that is at work in this strategy and encourage its continued use as appropriate.

Analogy-Based This *WoR* is characterized by relating signed numbers to another idea, concept, or object, often countable amounts or quantities, and reasoning about signed numbers on the basis of behaviors observed in this other concept. We named this *WoR analogy-based* because students created an analogy between signed numbers and some other concept. *Analogy-based reasoning* was used on about 13% of all problems posed.

Students compared negative numbers to positive numbers using a strategy we named *negatives like positives*, on about 7% of all problems. This strategy involves computing with negative numbers through explicit comparison to computing with positive numbers. This strategy was used productively across all grade levels. Consider Ricardo's (Grade 11) response to $-5 + -1 = \square$: "Negative five plus negative one equals negative six. I thought about this by changing this whole thing into a positive. So I just ignored the negatives for a little bit. So I knew five plus one equals six. But since it was negative, I added the negative after." When asked if changing the problem into a positive always worked, he replied, "So like this problem was applicable to change it to a positive since there were two negative numbers. But if you had like a negative and a positive, then that would be different." Ricardo was an 11th grader, but we also had many younger students who used this strategy. As an example, consider Jacob's (Grade 1) strategy for solving $-7 - \square = -5$. "Well for this one I need little cubes. … It would be like real numbers, but you just add the minus sign. You just do seven plus, well actually, seven minus two equals five. That's the answer for real numbers, so I just added a negative to all of them, and there is my answer." In these examples, we see that both Ricardo and Jacob compared the mathematical behavior of negative numbers to the behavior of positive numbers (or "real" numbers in Jacob's case) to solve problems involving the addition or subtraction of two negative integers.

Students also explicitly related signed numbers to contexts (e.g., debt or digging holes) on about 3% of all problems posed. Central to the *converts to context* strategy is that students used a context such as debt, digging holes, or bad guys that they deemed as related to negative numbers. As an example, consider Alex's solution to the problem in the introduction, $-3 + 6 = \square$, in which he interpreted -3 as representing a debt of \$3 and 6 as gaining \$6 from his mother. After taking \$3 from the money he was given to repay his debt, he had \$3 left. Another use of *converts to context* was relating signed numbers to digging and refilling holes. For example, Sawyer explained his answer of -3 to the problem $-5 + \square = -8$ by relating operations with negative numbers to digging and burying (his word for *refilling*) holes. For this problem he started with a hole five units deep: "Okay, if it [the unknown] would have been positive three, it would have canceled out; it would have buried some of the hole. [Instead] it's like we are digging a deeper hole and trying to get to negative eight." He applied the same context to think about the problem -2 + $\square$ = 4: "We start from negative two, and so it's like a hole and you need to fill it in." For Sawyer, the signs of the starting and ending numbers indicated whether he had a hole or a mound of dirt. He related the unknown in this problem to the action of filling in or burying the hole so that the result was a pile of dirt above ground.

Formal In a *formal WoR,* students treat negative numbers as formal objects that exist in a mathematical system and are subject to fundamental mathematical principles that govern their behavior. Students may generalize beyond a specific case to apply properties of classes of numbers or leverage underlying structures of our number system to make conjectures about which properties hold and do not hold upon successive extensions. *Formal reasoning* was used on just fewer than 12% of all problems posed.

The most common strategy within the *formal WoR*, *infers sign,* used on about 8% of all problems, involves examining the structural features of the problem—the operation in conjunction with the signs of the given numbers—to determine the sign of the answer prior to determining its magnitude. As an example of *infers sign*, consider Jane's thinking when solving the open number sentence $\square$ + 6 = 2. "Um, now we're trying to find, we know the number has to be negative. … The number that we're actually adding by [six], it's more than the actual, than our answer [two]. … So it has to be negative. So then if you know basic subtraction and addition, you know six minus what equal two. So it'd be four. … And it'd be negative four." Before she identified the magnitude of the unknown, Jane first determined the sign of the unknown by considering the operation, the signs of the given numbers, and their relative magnitudes. We considered this strategy to be a *formal WoR* because Jane is essentially making a claim about a class of problems—addition problems such that the sum is smaller than an addend (or, in other cases, subtraction problems such that the difference is greater than the minuend, like 5 – $\square$ = 8).

Sometimes students made generalizations that explicitly referenced the idea of additive inverses or the fact that the difference between any number and itself is zero. When a student invoked a general principle that $a - a = 0$ or $a + (-a) = 0$ (for $a \in Z$), we assigned the code *generalization about 0/additive inverses*. (Although we combine these strategies in our discussion here, we recognize important distinctions in them.) When using the *generalization about 0* strategy, students needed to indicate that the given problem was an instantiation of the generalization that any number minus itself is 0. One of our fourth-grade students, David, used this strategy when explaining how he thought about -5 – -5 = $\square$: "I know that any number subtract itself is zero." Because his language suggests that this is a general property and not true for just these particular numbers, we assigned the *generalization about 0* code to David's response.

Although the *additive inverses* strategy is related to *generalization about 0*, when using the *additive inverses* strategy, the student needed to explicitly mention three aspects we deemed critical to understanding additive inverses deeply: (a) the relation between *a* and *-a* (i.e., that they are inverses or opposites), (b) that the quantities are "canceling" (i.e., specify the importance of the operation of addition for additive inverses and the identity element of 0), and (c) that this claim is not specific to the numbers in the problem but is a generalization. For example, when solving the open number sentence $3 + \square = 0$, Belinda (an 11th grader) explained her answer of -3 saying, "I know that the opposite of three is negative three. And whenever you add things that are the same number but with different signs, positive or negative, it equals zero." Belinda identified the inverse relation between three and negative three describing them as opposites and also specified the operation and identity element involved (addition and zero). We interpreted her use of "whenever," the indexical noun "thing," and the second-person pronoun of "you" to indicate that Belinda was generalizing beyond the specific numbers given in the problem. Similarly, consider Kate's response to the problem $-8 + \square = 0$. She reasoned that, "If it [the sum] is going to equal zero, the way to cancel the eight out is to have the same number but have it in negative form." If Kate had stopped there with her explanation, she would not have received the *additive inverses* code. Although she alludes to the inverse relationship, identifies the importance of zero as the identity element, and seems to be moving toward a generalization with the phrase "same number in negative form," it's not clear how the canceling occurs. Critically, for us, the operation of addition had not yet been mentioned.[4] However, Kate did continue her explanation. "Because the same number on opposite sides of zero cancel each other out when you add them." In her last sentence, Kate indicated the importance of the operation of addition, and her language was more clearly generalized.

Another strategy within the *formal WoR*, *logical necessity,* was invoked infrequently but has promise for supporting powerful mathematical ideas. In the introduction, Fran-Olga used *logical necessity* in her response to $-3 + 6 = \square$. She was unsure which way to count (an *order-based WoR*) and considered answers of -9 and 3. After comparing the expressions -3 + 6 and 3 – 5, Fran-Olga settled on an answer of three. Because, on an earlier subtraction problem of 3 – 5, she had counted down "into the negatives," then for a problem that involved "plussing," Fran-Olga concluded she needed to count up. The key aspect of her reasoning was that "plussing" and "minusing" are inverse operations: If minusing goes down, then plussing goes up. Fran-Olga knew that addition and subtraction behaved oppositely in operating with whole numbers. She conjectured that the operations would still behave oppositely upon extension to the set of integers. In *logical necessity*, a student makes a comparison to another, known, problem and appropriately adjusts his or her reasoning so that the underlying logic of the system and the approach remain consistent; in this example, Fran-Olga maintained consistency with what she knew to be true for whole numbers. (We share an extensive examination of *logical necessity* in Bishop et al., 2016a, 2016b).

[4] Instead, Kate would have been assigned the strategy code, *magnitude*, which falls in the *analogy-based WoR* category. *Magnitude* strategies were used when students' responses indicated that they viewed a negative quantity as having magnitude, which enabled negative quantities to "cancel" an oppositional, positive quantity. Sometimes the "canceling" language was used when students used different colored chips to model and solve a problem. In these situations, another *analogy-based* strategy of *chips* was assigned as opposed to the *magnitude* code.

Computational A strategy coded as a *computational WoR* was based on a procedure, rule, or calculation. Because the most common WoR was computational, employed on about 40% of all problems posed, we share more strategies with this WoR to highlight the variety of computational strategies students in our study used. *KCC*, the most prevalent rule, is so named because many students shared the mnemonic Keep Change Change to indicate that they **K**eep the sign of the first number, **C**hange the operation, and **C**hange the sign of the second number. *KCC* was the most common strategy code across all ways of reasoning, used on about 16% of all problems.[5] The key feature of *KCC* is that the operation and second addend (or subtrahend) in the original expression are both changed to their opposites. In most instances, students referred to a mnemonic like KCC, boom boom, or the double stick trick when invoking this rule. But some students simply used the rule absent an accompanying memory aid, stating something like, "When a negative and a minus sign are together, they count as an addition." In both cases, the response was assigned the *KCC* strategy code. We exemplify this strategy in the following two responses and highlight the difficulty students typically had when asked to justify the validity of this rule. Gabriel, an 11th grader, invoked a mnemonic while solving the problem $5 - \square = 8$.

Gabriel: Negative three. Boom boom [writes -3 in the blank and when he states, "Boom boom," he draws two vertical line segments, one through the subtraction sign and one through the negative sign in -3].

Interviewer: Okay. So where does that boom boom come from? What was that?

Gabriel: It's magic.

Interviewer: Tell me a little bit more about the boom boom.

Gabriel: I remember learning in sixth grade or something, when you subtract a negative, you just do boom boom. And you add it I guess.

Interviewer: Okay, and why does that work?

Gabriel: Newton's third law—I don't know. Because you're taking away something that's negative? [Rising intonation]. Uh. [15-second pause] It just works.

Bea also gave an answer of -3 to the problem $5 - \square = 8$: "Just because, negative three, then I do the double stick trick. There is a minus [and a] negative so you add." When asked what the "double stick trick" was, Bea clarified, "Okay, when you have a subtraction sign [points to subtraction symbol in the expression 5 – -3 = 8] and then a negative number [points to negative sign for -3], they call it a double stick trick when you do this. [She draws two vertical lines, one through the subtraction sign the other through the negative sign in -3 so that the expression 5 – -3 = 8 is transformed into 5 + +3 = 8.] And so five plus three is eight."

The next two most common computational strategies—*negative sign subtractive* and *changes order of terms*—were used on roughly 8% and 7% of problems posed,

[5] The percentage use of 16% was driven by the CT students, who used *KCC* on 31.40% of all problems posed, sometimes in conjunction with another *WoR*.

respectively. Claire's response to the problem -3 + 6 = $\square$ reflects both of these strategies. "It's three. I know that six minus three is three. I just changed the order of the numbers and since three is negative, I subtracted." The interviewer pressed Claire, saying, "But the problem was negative three *plus* six. You *subtracted* and started with six instead of negative three." Claire again reiterated, "I just changed the order of the numbers and since three is negative, I subtracted." The interviewer continued, "Okay. When you changed the order of the numbers, I'm curious if you thought of the problem as six plus negative three, and then changed to subtraction? Or when you switched it, if you immediately thought of the problem as six minus three." Claire responded, "I immediately thought of it as a subtraction problem." In the strategy of *negative sign subtractive*, students indicate that the negative sign in the written symbolic form of a negative number, the - in -3, indicates the process of subtraction. Instead of being viewed as a quantity or mathematical object in its own right, -3 is understood as a quantity *to be subtracted*. Thus, Claire interpreted -3 to mean "subtract three." This strategy, which was used in all participant groups in our study, was one of the earlier historical conceptions mathematicians had for negative numbers (see Henley, 1999, for a discussion of "subtractive numbers"). In particular, almost two thirds of the college-track students used this strategy to solve the problem -3 + 6 = $\square$ by *subtracting* three from six.

Similar to most college-track students, Claire responded to the problem -3 + 6 = $\square$ by using *negative sign subtractive* simultaneously and in combination with *changes order of terms* to transform the original expression of -3 + 6 to the equivalent expression of 6 – 3. Claire was clear that she did not use the following sequence of transformations: -3 + 6 → 6 + -3 → 6 – 3, but instead went straight to the last expression. Her response exemplifies *changes order of terms* because she essentially applied the commutative property of addition to change the order of the addends, but she simultaneously changed what was an addition problem to a subtraction problem by interpreting -3 as subtractive, which is why her response was also assigned the *negative sign subtractive* strategy code. The college-track students were especially fluent, but almost always implicit, when changing the meaning of the minus sign from a negative number to subtraction.

Sometimes students added or subtracted a number to both sides of the open number sentence to "isolate the box." We named this strategy *equation* because students used *properties of equality* often associated with school-based instruction for solving one- and two-step equations. For example, Belinda's explanation for her solution to 6 + $\square$ = 4 was "I just subtracted six from both sides and got negative two." Many students explained that they had to "do the same thing to both sides," and some students insisted on rewriting the number sentences so that the box was replaced with a variable (i.e., 6 + $\square$ = 4 was rewritten as 6 + x = 4).

The last computational strategy we share was named the *same signs/different signs rule*, and it was used on just fewer than 4% of problem responses. This is a rule that applies only to addition problems, though we saw many students apply it incorrectly to subtraction problems. The *same signs/different signs rule* can be stated as follows: If the signs of the addends are the same, add their magnitudes, and keep the sign for the sum. If the signs are of the addends are different, find the difference of their magnitudes, and the difference should take the sign of the number with the larger magnitude. Cole's response to 6 + -3 = $\square$, shown in the introduction, is an example of this strategy. Because -3 and 6 had opposite signs, he subtracted three from six and assigned to that difference the sign of the addend with the larger magnitude, 6, which was positive. One student we interviewed recited a song to help her remember this rule (to the tune of *Row, Row, Row Your Boat*): "Same signs, add and keep. Different signs, subtract. Take the sign of the larger one, then you'll be exact."

Emergent The *emergent WoR* reflects students' initial attempts to compute with signed numbers. We chose the name *emergent* because many of the strategies students used in this *WoR* were not only sensible but with appropriate support could provide a strong foundation for integer reasoning from which more sophisticated strategies and ways of reasoning could emerge. Some students who had not yet heard of negative numbers ignored the negative sign or treated it as a subtraction symbol. Other students sometimes selectively restricted the domain of possible solutions to nonnegatives. Overall, *emergent* reasoning was used on about one third of all problems posed. The most common such strategy was *addition makes larger/ subtraction makes smaller (AML/SMS),* used on 14% of all problems posed.[6] The *AML/SMS* strategy stems from the overgeneralizations that addition always makes larger and subtraction always makes smaller and is related to conceptualizations of addition and subtraction as increasing and decreasing the cardinality of a set (Bishop et al., 2011; Bishop et al., 2014). For example, consider Oscar's response when solving 5 – $\square$ = 8. "Cuz, this [points to 8 in the written problem] is bigger than that [points to 5]. And if you minus three, if that [points to the minus sign] was a plus, um, it would be possible. … You couldn't take away, fff, fi, three out of five to equal eight 'cuz it would just equal two." Oscar then wrote "No" in the box. Ryan, too, used the *subtraction makes smaller* strategy for the same problem saying, "I wouldn't be able to do it because it would always be behind eight if it was minus something. Because if it was minus zero it would be five. It [the difference] would always be behind eight." Although both of these students had heard of negative numbers, they appeared to restrict the domain of possible solutions to whole numbers and did not consider the effect (or possible effect) of subtracting a negative number.

The second most common strategy within *emergent* reasoning was *ignores sign*. In this strategy students either ignore the negative sign throughout and treat it as though it does not exist or they *initially* ignore the negative sign and then account for it *after* finding a solution. The strategy *ignores sign* was used in just fewer than 11% of the problems posed and was mainly driven by second- and fourth-grade students in our study. Dahlia, a second grader, ignored the negative signs when solving -5 + -1 = $\square$ and treated -5 and -1 as if they were whole numbers. She read the problem aloud as "Five plus one" and immediately answered six. Dahlia then

[6]This percentage was driven by the BIN and NEN students, who used *AML/SMS* on 27.21% and 32.44% of all problems posed, respectively.

demonstrated the fact on her fingers saying, "Five (she held out five fingers on one hand) plus one (she held out her thumb on her other hand) would equal six." In contrast, Javier read the open number sentence $\square$ – 5 = -1 as "Box minus five equals negative one." He initially wrote 6 in the box and then revised his answer to -6. He explained, "Six minus five equals one. So I used negative six minus five so it could be negative one." Javier appeared to initially ignore the negative in -1 and solve instead the related number sentence of $\square$ – 5 = 1. When asked why the 6 was negative, he replied, "Because I, because if I don't have a negative and I subtract minus five, I won't be able to have negative one." Javier reasoned that for the difference to be negative one as opposed to one, the unknown needed to be negative. Thus, he assigned a negative sign to the unknown on the basis of the absence or presence of other negative numbers in the problem. Moreover, how Javier interpreted or made sense of signed numbers is unclear; he may have attended only to surface features embedded in the symbolization of these numbers.

Another strategy in the *emergent WoR* was used for open number sentences in which the magnitude of the subtrahend was larger than the magnitude of the minuend (e.g., 3 – 5 = $\square$, -2 – 7 = $\square$, -7 – -9 = $\square$). Students often declared that these problems were "not possible" to solve or gave an answer of zero. Consider Sam's response to the problem 3 – 5 = $\square$: "Three minus five is zero because you have three and you can't take away five. So take away the three, and it leaves you with zero." (When asked to solve 3 – 4 = $\square$ and 3 – 3 = $\square$, Sam answered 0 to both.) Similarly, Andrew was puzzled by the same task and said that solving 3 – 5 = $\square$ was "not possible." He shared his thinking, saying, "How come there's three and take away five? I don't have enough. 'Cuz look there's three (holding up three fingers) and I cannot take away five 'cuz there's not enough." We named this strategy *Pascal* for the mathematician and philosopher Blaise Pascal who gave a response not unlike Sam's. In his collection of unpublished philosophical and religious writings entitled *Pensées*, Pascal stated, "I know some who cannot understand that to take four from nothing leaves nothing" (1669/1941, p. 25).

Discussion and Implications

Because of their documented effectiveness in supporting students' learning, frameworks of students' mathematical thinking are deeply rooted in mathematics education research (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). In this chapter, we have contributed a descriptive framework for organizing and making sense of students' problem-solving strategies by relating them to the broader ways of reasoning about integer addition and subtraction. By combining strategies and ways of reasoning in our framework, we distinguish key details of student thinking in a way that provides organization and structure to student thinking in the realm of integers. Knowledge of specific strategies is beneficial because it can help teachers recognize and encourage the use of multiple, appropriate strategies and build toward more sophisticated strategies both within and across ways of reasoning (e.g., counting vs. jumping to zero vs. additive inverses). This knowledge also helps teachers to support students to select and use efficient strategies that are based on key features of problems. For example, we found that students often use *jumping to zero* (or another *order-based WoR strategy*) for problems like $-5 + \square = 3$ (an addition problem starting with a negative quantity, ending with a positive quantity, and with an unknown, positive, change value). (See Lamb, Bishop, Philipp, Whitacre, & Schappelle, 2017, for a discussion of integer addition and subtraction problem types and their relation to ways of reasoning.) And finally, the knowledge to differentiate multiple instantiations of a specific *WoR* (i.e., differentiating strategies within a *WoR*) can support the identification of common characteristics that unite those strategies within the *WoR*.

As discussed in the beginning of this chapter (see Table 3.1), researchers investigating the teaching, learning, and historical development of signed numbers have contributed studies and descriptions of students' thinking about integers that are consistent with both our broader ways of reasoning and many of the strategies we documented in this chapter (Bofferding, 2014; Chiu, 2001; Murray, 1985; Peled, 1991; Stephan & Akyuz, 2012). We extend this work by organizing key distinctions and patterns in children's solutions into a coherent framework that leverages the broader ways of reasoning as its central organizing feature.

Connecting Key Mathematics to Student Strategies

We believe the ways of reasoning framework holds promise for teachers because it can support their abilities to assess and interpret student thinking in the moment. Moreover, the strategies students use draw on important mathematical ideas. Therefore, knowing and recognizing differences among students' ways of reasoning and strategies as well as the underlying mathematical ideas embedded in specific strategies is important pedagogical content knowledge for teachers. However, the mathematical ideas in students' strategies are often unstated, unclear, or implicit, and teachers can experience difficulty in eliciting those ideas from students. We note that though we have selected the examples in this chapter for their clarity, student thinking is not always complete or clearly articulated; thus, in practice, asking probing questions to help elicit student thinking and connect student-generated strategies to underlying mathematical ideas is helpful. Further, at times, students may be unaware of the strategies they used. Providing students opportunities to regularly share their thinking may have multiple benefits. Students can become both more able to meaningfully communicate their mathematical ideas and more aware of the strategies that they actually used. Additionally, by making their own strategies more explicit to themselves, students may be able to use those strategies for problems with similar structure. In the following sections, we return to several strategies discussed earlier in the chapter, identify the key mathematical ideas embedded in these strategies, and offer suggestions for teachers to explore and make connections to those mathematical ideas.

Jumping to Zero and Additive Inverses We view the strategy *jumping to zero*, which is *order-based*, as significant for two reasons. First, students who jump to zero may recognize that decomposing numbers to get to a friendly number (in this case, 0) enables them to solve problems more efficiently than does counting by ones. Second, we suspect that the use of *jumping to zero* may be an important precursor to reasoning more formally about additive inverses—that is, using the *additive inverse* strategy in the *formal WoR*. For example, after sharing her strategy to $-3 + 6 = \square$, Opal and her classmates might be asked to consider the relation between 3 and -3 and what it means to be *opposites*. These types of conversations could support students to generalize the specific instantiation of the property $-3 + 3 = 0 = 3 + (-3)$ to all integers. In this case, we envision using the initial *order-based* reasoning to develop *formal* reasoning.

AML/SMS and Infers Sign In related work (Lamb et al., 2017), we shared how kernels of inferring the sign are present in *AML/SMS* strategies. We reiterate here that we believe that some strategies within the *emergent WoR* provide productive starting points for students' learning about negative numbers. For example, students who express *AML/SMS* strategies provide evidence that they have noticed features of the number system with which they have heretofore engaged, and thus they have recognized the underlying structure of addition and subtraction in the domain of natural numbers: Addition makes larger and subtraction makes smaller. After they have worked with negative values a, c, or both in problems with the form $a \pm b = c$, teachers and researchers can support students to develop a more nuanced assessment of their claims by having students consider what might happen to sums or differences when the b value is negative. This examination may support students in understanding the conditions under which *AML/SMS* holds and in recognizing that when AML does not hold, the sum may be less than or equal to a. When SMS does not hold, the difference may be greater than or equal to a. In this case, we envision using the initial *emergent* reasoning to develop *formal* reasoning.

Negative as Subtractive and Symbolic Flexibility In our research, we found that students often productively and appropriately treat the negative sign as a subtraction sign to efficiently solve open number sentences (i.e., the *negative sign subtractive* strategy in the *computational WoR*). We view the ability to seamlessly move between meanings of the minus sign and the operation as a desirable outcome of instruction (Arcavi, 1994; Lamb et al., 2012). For example, our college-track students successfully treated the subtraction sign as a negative number or treated a negative number as the operation of subtraction on almost one fourth of all problems they solved. However, the students who shared these strategies may have been so efficient and fluid when computing that they may not have recognized how or that they changed the problem. One goal may be to support students to be more explicit about when they are changing the meaning of the minus sign to aid their computations. See Lamb et al. (2012) for additional information and suggestions.

Negatives Like Positives and Ignores Sign We shared examples of two strategies that seem similar, *negatives like positives* and *ignores sign,* but were categorized as an *analogy-based WoR* and an *emergent WoR*, respectively. Despite the strategies' similarity, we provided evidence to support our claim that students were doing more than appending a sign when invoking *negatives like positives*. Rather, we determined that students had invoked *negatives like positives* only when they provided evidence of attending to more than surface features of the problem in their solutions. That is, had the students initially ignored signs, computed an answer, and appended a sign after computing, the responses would have been coded as *ignore signs*. We view *negatives like positives* as a productive strategy that teachers can leverage to discuss with students when the strategy is useful, to explore reasons the strategy makes sense mathematically, and to discuss important ideas including equivalent expressions and negation.

Final Thoughts

In this chapter, we shared five broad ways of reasoning about integer addition and subtraction and 16 (of the 41 identified) strategies that are subsumed under those ways of reasoning. Although we have shared the ways of reasoning in previous work, herein we sought to share some of the most common strategies with examples that provide clear comparisons and contrasts to support both teachers and researchers in understanding specific strategies within the ways of reasoning. The ability to categorize strategies into one of five ways of reasoning may enable teachers to organize knowledge of student thinking in ways that are useable and accessible for them and provide researchers with sufficient information about the strategies and ways of reasoning such that they can reliably build on this work.

Acknowledgments This manuscript is based on work supported by the National Science Foundation (NSF) under grant number DRL-0918780. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

Appendix

Problem-Solving Interview[7]

1. Name a big number. Can you name a bigger number?
2. Name a small number. Can you name a smaller number? If the child responds, "One," ask, "What if I gave that away? What number would you have then?" If the child responds, "Zero," ask, "Is there a number smaller than zero?"

[7] Students who provided no evidence of having knowledge of negative numbers (NENs) did not respond to items 16–22 or 30–35.

3. Can you count backward, starting at 5? If the child stops at 0 or 1, ask, "Can you keep counting back?" (If the child continues to count back, have the child stop counting at -5).

   Note. For Grades 2 and 4 students, the interviewer did not pose Question 4 unless the student had previously mentioned the term *negative*. The interviewer did not introduce the term *negative* or the notation for negative numbers unless the child mentioned them in responses to Questions 1–3.
4. *What can you tell me about negative numbers?*
5. $5 + 6 = \square$
6. $4 + \square = 9$
7. $\square - 4 = 6$
8. $8 - \square = 4$
9. $3 - 5 = \square$
10. $6 + \square = 4$
11. $5 - \square = 8$
12. $\square + 6 = 2$
13. $-3 + 6 = \square$
14. $-8 - 3 = \square$
15. *Yesterday you borrowed \$8 from your friend to buy a school t-shirt. Today you borrowed another \$5 from the same friend to buy lunch. What's the situation now?*
16. $-2 + \square = 4$
17. $\square - 5 = -1$
18. $-9 + \square = -4$
19. $-2 - \square = -8$
20. $-5 + \square = -8$
21. $-3 - \square = 2$
22. $-8 - \square = -2$
23. $-8 + \square = 0$
24. $-5 + -1 = \square$
25. $-5 - -3 = \square$
26. $6 - -2 = \square$
27. $6 + -3 = \square$
28. $3 + \square = 0$
29. *There is a bird flying 20 feet above the surface of the water and a fish swimming 5 feet below the surface of the water.* (Show picture of fish, bird, water surface.)

    How many feet higher is the bird than the fish?
30. $-5 - -5 = \square$
31. $-7 - -9 = \square$
32. $\square + -7 = -3$
33. $\square + -2 = -10$
34. $3 - \square = -6$
35. $-2 - 7 = \square$
36. -8 Point to -8. *Can you read this? What does it mean?*

For each pair of numbers, circle the larger, write "=" if they are equal, or write "?" if there is not enough information to tell which one is larger.

37.	3	7
38.	-7	3
39.	-5	-6
40.	+20	20
41.	-0	0
42.	-9	0
43.	- -4	-4
44.	-(-4)	-4
45.	-5	-100
46.	□	-□

47. *Is there anything you can write in the blank to make the following statement true*?
    5 = -____

For PI and CT students, we posed questions 48–54.

Circle the larger, write "=" if they are equal, or write "?" if there is not enough information to determine.

48.	x	x + x
49.	x	x + 1
50.	x + y	x – y
51.	-x	x
52.	7	x
53.	x	-7

54. If x < y, compare -x and -y.

For CT students, we posed questions 55 and 56.

55. What can you tell me about absolute value?

56a. Someone wrote this down as the definition of absolute value.

For any real number x, the **absolute value** of x is denoted by $|x|$ and is defined as

$$x = \begin{cases} x \text{ if } x \geq 0 \\ x \text{ if } x < 0 \end{cases}$$

Can you read this to me (point to definition of *absolute value*)*? What does this mean? Do you think this makes sense for the definition of absolute value? Why?*

56b. *According to this definition, explain what the absolute value of -2 is.*

Pose this follow-up question, if needed*: I am confused because negative 2 is less than zero. Doesn't this* (circling the –x in the definition for absolute value) *mean that my answer should be negative?*

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