Research in Mathematics Education
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Exploring the Integer Addition and Subtraction Landscape

Perspectives on Integer Thinking

Springer

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To our children

Preface

Connecting Pathways Across the Integer Addition and Subtraction Landscape

> The only way to solve negative problems is you have to, like you probably have to just go to try and do it…you gotta pretend. (2nd grader)

One of the delights of childhood is having the freedom to pretend and consider "what if's" free from pressure. Perhaps that is why we find ourselves drawn to children's conceptions around negatives. Negative numbers are playful things within children's imaginations, and as they consider numbers they cannot see, we gain insight into children's reasoning about numbers and operations. Interestingly, we find evidence of similar integer conceptions and reasoning across grade levels (both aligned with conventional meanings and not). Therefore, students' understanding of integers seems less connected with how old they are and more dependent on their prior knowledge. This makes the topic of negative integers an ideal example for highlighting the importance of eliciting student thinking and building on their current understanding in instruction.

Why Focus on Integers Now?

The emerging foundation of research on thinking about integers highlights students' sophisticated reasoning about integers, demonstrated even by young children, and critical nonconventional conceptions that they can develop if integers are ignored. Understanding children's thinking about integers is pivotal for planning better instructional experiences in elementary and middle school, preparing prospective elementary and middle school teachers, and supporting inservice teachers. An increase in research into integer thinking and learning in recent years highlighted the need to bring together multiple viewpoints to establish our current understanding of the integer landscape.

Children's thinking is the soil of the landscape because it is foundational to our building of knowledge on thinking about integers and can influence what knowledge arises from it. Across the decades, as a field we have recognized that children have difficulties with making sense of integers and integer operations: What is the composition of the soil? (What knowledge do students bring to the topic of integers?) How can we navigate across unstable ground? (How do students make sense of integers in light of their prior understanding?) Scholarly conversations around integer addition and subtraction also highlight and classify the sophisticated reasoning that children are capable of when solving integer addition and subtraction problems: How do we distinguish one type of soil from another? (How can we classify students' strategies and ways of reasoning?) We need to focus in depth on how play, instructional contexts, or different number sentence types may contribute to children's thinking about integers and integer addition and subtraction.

In the landscape of integers, specific instructional contexts (e.g., elevator models, balloon models), models (e.g., number line, cancellation), and metaphors (e.g, moving along a path, collecting objects) are the mountains behind the clouds. Important, but hard to pin down. We see the prominence of contexts, models, and metaphors, but they are distant. How far away are the mountains? (How well do the models or contexts align with the operations?) What do the mountains look like? (How do students use contexts, models, and metaphors when presented in different ways?) How is one mountain different than its neighbor? (What distinguishes an integer model from an instructional context or even one number line model from another?) How easy is the mountain to scale? (What shortcuts do the contexts, models, and metaphors support? How easy are they for students to understand?) A substantial portion of the integer literature focuses on instructional models and metaphors for supporting integer thinking around addition and subtraction. In some cases, there are subtle (or not so subtle) arguments for one versus another. We need to continue gaining insight into when and how these models support students' learning and when they break down.

As we traverse any landscape, there are leaders and signposts that guide the way along existing (or new!) paths. Both teachers and prospective teachers (PTs) are the tour guides through the integer landscape (magnitude, value, operations) for the new tourists (students). Even experts can find some landscapes difficult to cross; we know that secondary and university students find thinking and learning about integers challenging. What characterizes their challenges? Sometimes signposts are mislabeled or tour guides stumble; PTs often have nonconventional conceptions about integer addition and subtraction or rely on procedures that they have difficulty explaining, which make it difficult for them to lead. How can we capitalize on their strengths? As tour guides through this landscape, teachers and PTs need to develop deep conceptual understandings in order that they may make instructional choices, evaluate curricula, and support student thinking in their own classrooms. We need more investigations into this type of deep knowledge that PTs have and need about integer addition and subtraction.

Let's Begin!

This book represents the collaborative work of researchers from different perspectives about integer addition and subtraction with a common goal of exploring the integer landscape. Although the commentary chapters provide connections among various landmarks in the landscape, they also point to additional paths for future research that need to be explored. We hope the work within these chapters inspires you on your journey, as it did for us.

West Lafayette, IN, USA — Laura Bofferding
Newberg, OR, USA — Nicole M. Wessman-Enzinger

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Part I

Beginning the Journey: Children's Thinking About Integer Addition and Subtraction

Chapter 1
Playing with Integer Concepts: A Quest for Structure

Laura Bofferding, Mahtob Aqazade, and Sherri Farmer

Abstract How children play around with new numerical concepts can provide important information about the structure and patterns they notice in number systems. In this chapter, we report on data from 243 second graders who were asked to fill in missing numbers on a number path (encouraging them to play around with numbers less than zero) and to solve integer arithmetic problems (encouraging them to play around with the concepts of addition and subtraction involving negative integers). When playing around with the number paths, students made patterns, continued the number sequence in interesting ways, and used invented notation. When playing around with the operations, they interpreted negative signs as subtraction signs or added negative signs to their answers. Their play with the number path often connected to their play with operations, revealing that although some students were attuned to the pattern in the order of numbers and operations as movement in a particular direction, others focused more on numerical values and operations as changes in amount. The various ways in which children played with integers provide insight into their conceptual change process and can provide guidance for ways teachers could help students build on their logic.

> Put simply, play is not so much an activity as it is an acceptance of uncertainty and a willingness to move...But it is not an abandonment of our quest for structure, order, pattern, and comprehensibility. Quite the opposite, these are the ends of play.
>
> But these ends are revealed only in the playing, for play is not simply random activity. Rather, by opening the door to the as yet unexperienced, to the possible, play reveals what is not yet known as it simultaneously offers space to support learning. (Davis, 1996, p. 222)

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As expressed in the opening quote, children's play helps them make sense of the world. An early introduction to negative numbers provides a rich context for exploring children's willingness to play with numbers (Featherstone, 2000). Further, their play can illuminate which aspects of structure and pattern stand out in their minds and can provide a space for them to wonder about new observations. The integer literature illuminates several instances where young children were given the opportunity to play around with the idea of numbers before zero with delightful results, which we describe next.

Encouraging playful thinking can begin with a simple question. Behrend and Mohs (2005–2006) discussed such a situation, where first graders were challenged to determine if numbers go on forever. After reasoning that they could count in the positive direction endlessly, the teacher asked if they could in the other direction as well. Having heard about them from a sibling, one student introduced the idea of negative numbers, and over the course of the year, students began using negatives in calculations, such as −2 + 12 = 10.

Having students consider numbers to the left of zero is a common theme in playful integer activities for young students. Aze (1989) introduced a game with 7- and 8-year-olds involving movements on a physical number line. One child moved on the number line as the rest of the class told the child how much to add or subtract in order to move. The number line was labeled 0–9 with space on either end, and students inevitably suggested a movement that required them to determine what to do to the left of 0. Even if they had not heard of negative numbers, many students suggested moving to a new tick mark, and one class playfully named each number they needed to the left of zero with a classmate's name. They were even able to solve problems such as 2 – 5 based on their system.

Similarly, Wilcox (2008) described a game, in which she had her first-grade daughter move along on a number line with unlabeled spaces for negatives by drawing + and – cards followed by a numeral. Zero, as a reference point, played a crucial role in understanding the numbers less than zero. The child originally called -3 "zero" but when pushed to distinguish it from the other zero called it "zero cousin spot" and later "zero cousin minus three spot." Similar to students described by Aze (1989), when using the number line, the girl could add 1 to the -3 spot to get to -2. However, her reasoning was limited when working with objects; with blocks, she did not yet extend the structure of her new system to deal with situations such as 3 – 4, in which case there were not enough blocks to take away.

The number line games just described focus on exploring integers as locations as a result of translations. Games that focus more directly on quantity tend to encourage counterbalance conceptual models (Wessman-Enzinger & Mooney, 2014) and often involve positive and negative chips or double-sided abaci. For example, a group of third and fourth graders played a game where they had to keep track of scores (black counters) and forfeits (red counters) as well as their running totals after each turn (Liebeck, 1990). Through the process of game play, they were able to articulate that four scores and two forfeits would result in a score of 2. More impressively, when they ran out of red counters, they decided they could take away a black counter instead. Similarly, fifth graders played a dice game where they either scored for a yellow team or a blue team by moving beads on a double abacus. They were able to reason that removing a bead from one team corresponded to gaining a bead on the other team (Williams, Linchevski, & Kutscher, 2008). The fifth graders eventually applied this compensation rule to addition and subtraction in order to solve addition and subtraction integer problems. These games helped students consider integers as physical quantities rather than as locations.

The above examples highlight open-ended number line games or object-based games where students sought to find structure and comprehensibility in their use of numbers. In the study described in this chapter, we took inspiration from the integer number line examples above. We used a fill-in-the-blanks number path as an example of an open-ended or unexpected task that can provide teachers with insight into what children see as important, such as the role of zero (Wilcox, 2008). We also went a step further by posing unexpected integer problems to children. By analyzing how students play around with numbers when working with these types of tasks, we can gain insight into the structure, order, or patterns they see and to what extent their play aligns with our formal understanding of integers.

In this chapter, we present two situations where children had the opportunity to play with numbers and symbols and illuminate their meaning making of integers. Looking at their answers through a lens of play, we give legitimacy to children's interactions with the numbers. Further, we identify instances where play reveals important insights that teachers and researchers can leverage for encouraging students' developing understanding of integers.

Conceptual Change

The lens of play as articulated by Davis (1996) in the opening quote explains play as "an acceptance of uncertainty and a willingness to move...play reveals what is not yet known as it simultaneously offers space to support learning" (p. 222). This perspective is consistent with a framework theory approach to conceptual change, which posits that children form their initial conceptions based on experiences in the world and modify them as they encounter new experiences (Vosniadou & Brewer, 1992). In particular, the process children go through to modify their conceptions involves a willingness to play with new ideas. Children experience conceptual change on a daily basis, interacting in a world where much is yet unknown to them, but they are willing to forge ahead and try to find order in all that they encounter.

More specifically, in the realm of integers, children's *initial integer mental models* are based on their experiences with whole numbers; therefore, they may ignore negative signs completely or order negatives correctly but treat them as equivalent to positives (Bofferding, 2014). Some may make their answers negative because they notice the negative sign and think it is important (symmetric meaning of the minus sign) (Bofferding, 2010; Vlassis, 2008); others will interpret the negative sign as a subtraction sign (binary meaning of the minus sign) (Bofferding, 2010; Vlassis, 2008) based on their experiences with whole number subtraction. These students will often consider negatives to be amounts taken away, worth zero. Therefore, they will either order them next to their positive counterparts (as amounts that will be taken away) or next to zero (Bofferding, 2014; Peled, Mukhopadhyay, & Resnick, 1989). All of these responses indicate a "quest for structure, order, pattern, and comprehensibility" (Davis, 1996, p. 222).

Students who have more exposure to negatives may indicate that negatives are less than zero but continue to play around with the meaning of integer values in relation to the numerals used to express them. Classified as exhibiting *synthetic mental models*, these students interpret negatives with larger absolute value (further from 0) to be more than negatives with smaller absolute value (e.g., -6 > -2) (Bofferding, 2014). Therefore, when solving problems such as -5 + 3, they might add 5 + 3 and make the answer negative (symmetric meaning of the minus sign), or they might start at -5 (treating the negative sign as designating a negative number, the unary meaning) and count -6, -7, -8. Students who order negatives correctly and treat negatives closer to 0 as larger (not those with larger absolute value) exhibit *formal mental models* (Bofferding, 2014).

Although the formal mental models are based on an adult perspective, the conceptual change approach values children's alternative ways of thinking, grounded in the theory that there is structure and order in children's thinking, based on their prior experiences. In particular, how students who exhibit initial mental models play with numbers and respond provides insight into what aspects of number are salient to them. Likewise, the responses of students who exhibit synthetic models are valuable because they highlight children's efforts to find structure with new numbers and indicate where additional instruction could be fruitful. Therefore, this chapter addresses two main research questions:

1. What does second graders' play with numbers when filling in an integer number path suggest about their understanding of numbers?
2. Based on their integer number path play, what does their subsequent play with integer operations suggest about their understanding of numbers and operations?

Our Data

As part of a larger 5-year study, we worked with second graders on a series of interventions involving negative numbers. The data described come from a total of 243 second graders across 12 classrooms in a rural school district in the Midwest (110 students from year 1 and 133 students from year 2). In this district, there were approximately 40% English-language learners and 69% receiving free or reduced lunch. Several classrooms (but not all) had number lines in their classrooms that included negative numbers, but negative numbers were not part of their curriculum.

Fig. 1.1 Integer number path

At the beginning of each year, we administered to students a written pretest composed of a variety of integer tasks. We asked them to complete the tasks as best they could, answering whatever they thought. We emphasized that we were interested in how and what they thought about and even told them they could make up answers if they got stuck. In this sense, we had playful tasks as part of our assessment. Out of the six main tasks, we focus our analysis here on two that were the most open-ended: a number path task (one item) and an integer arithmetic task (several items). A day or two after students took the pretest, we individually interviewed 20% of students in order to learn more about their reasoning on the pretest.

Number Path Task

For the number path task, students saw a number path with the numbers 3, 4, and 5 labeled (see Fig. 1.1), and we asked them to fill in the missing numbers as if they were counting backward. There were enough empty spaces that students could label to -11. If they stopped at 1 or 0 and inquired about the remaining spaces, we asked them if there was anything else they could fill in and encouraged them to do "whatever you think."

Integer Arithmetic Task

The integer arithmetic task differed across the 2 years, with the first year focusing on 14 addition problems involving integers and the second year focusing on 17 subtraction problems involving integers (with four addition problems also included). The problems included adding or subtracting with two negative integers or with both a positive and negative integer. In the case of subtraction, there were also problems where students had to subtract two positive numbers in order to get a negative answer. See the appendix for the list of integer arithmetic questions each year. In this discussion of the results, we group students based on whether they used any negatives on their number path as well as whether they had any negative number answers on the arithmetic problems that reasonably could be obtained using the numbers given. Therefore, for the sake of grouping students in the results section, if a student included only one negative answer without meaningful reasoning (e.g., -7 + -1 = -100), we did not count this as using negatives.

What Is an Instance of Play?

Using a liberal interpretation of Davis's (1996) words, "Play is not so much an activity as it is an acceptance of uncertainty and a willingness to move" (p. 222), we interpreted all of students' answers as a version of play, as they showed a willingness to move their thinking even if uncertain about negative integer tasks. Including negatives on a counting backward task is not standard at this age level. Not including negatives on the number path demonstrates the order and structure that fit their understanding when given essentially more space than they need. On the other hand, their willingness to include negatives (or something else) when we did not specifically ask suggests playfulness and provides a different insight into their structuring of number. Similarly, whether or not students use negatives or pay attention to the negative signs on the arithmetic problems provides insight into patterns or structure they notice. Nonetheless, we organize the presentation of the results for the two tasks according to students who play with whole numbers only and students who play with negative numbers.

Number Path Results

Based on how they played around with completing the number paths, the 243 second graders' responses fell into one of several categories. We describe these sets of responses under the larger categories of students who play with whole numbers versus students who play with negative numbers, and we follow each section with a discussion of what we can learn about these students' playfulness.

Playing Around with Whole Numbers

The majority of students (50%) only played with positive or whole numbers on the number path, leaving several of the spaces blank. Therefore, they left 11 spaces blank and then filled in numerals 0–2 (see Fig. 1.2).

Otherwise, they left 12 spaces blank and only wrote numerals 1–5. For example, when explaining the number path, one student (3.A12) counted back to zero, and another student (3.A05) counted backward, "Five, four, three, two, one," and, when asked if there were any other numbers, responded, "No, that's it, I think." A few students (4%) used this stem to create a repeating pattern (see Fig. 1.3).

Fig. 1.2 Number path with numerals 0–2 filled in

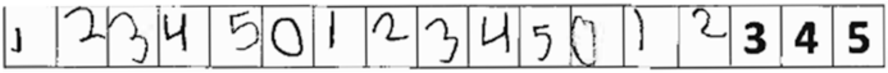

Fig. 1.3 Number path with repeating pattern

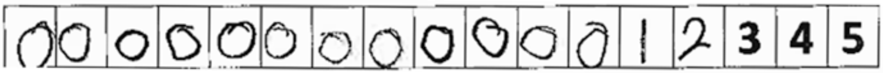

Fig. 1.4 Number path with repeated zeroes

Fig. 1.5 Number path with two counts

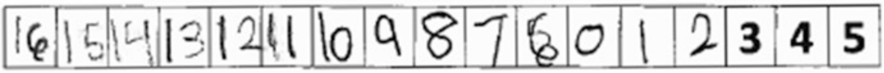

Fig. 1.6 Number path with wrapped numbers

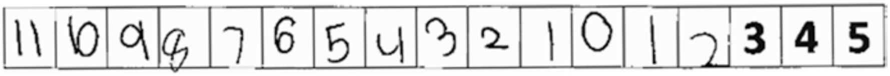

Fig. 1.7 Number path with symmetric numbers

Two of these students extended the number path, repeating the sequence from 1 to 10 twice.

Another trend for students who primarily used positive or whole numbers was to continue the number sequence in some way (9%). Out of those who answered in this way, six of the students continued the number sequence with repeated zeroes (see Fig. 1.4).

Of the rest, some counted up from the left end of the number path and counted down from the right (see Fig. 1.5).

Others continued the number sequence after 5 by wrapping numbers around to the left of zero (see Fig. 1.6).

Student 3.E09 explained, “I saw five, then four, then three...so I went two, one, zero, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen.” When asked how the student knew to put the six to the left of zero, the student responded, “Because there was no room for it [after five]. I just put it—shoved it in there.”

Aside from continuing the positive number sequence, a few students (7%) created symmetric number paths. The most common way they did this was to start at 5 on the right and write the decreasing number sequence to zero and then write the increasing number sequence on the left side of zero (see Fig. 1.7).

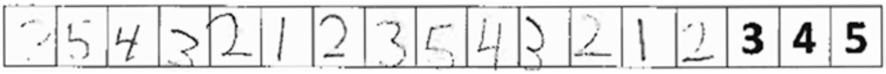

Fig. 1.8 Number path with symmetric numbers and no zero

One student added in an extra zero, while a few left out zero. One student who left out zero also did a symmetric pattern (see Fig. 1.8).

Discussion of Whole Number Play

Interestingly, half of the second graders did not play around with the "extra" spaces on the number path and only filled in the positive integers (1–5) or whole numbers (0–5). A few of these students questioned why there were extra spaces while filling in the number path. Many others stopped filling in the numbers, perhaps due to confidence that there are no more numbers below zero, similar to the above examples. Another reason for leaving blanks could be reluctance to deviate from teachers' "normal" expectations. Having confidence in an answer when the problem situations are unfamiliar is an important trait and speaks to the strength of these students' whole number understanding. Those students who continued the number sequence on the left side of zero by wrapping the numbers around exhibited knowledge that the positive number sequence continues beyond five (something they clearly wanted to make sure we knew!), and their willingness to place these numbers on the other side of zero suggests that they do not necessarily visualize numbers in a linear manner. We constrained them by providing a set number of boxes, which likely influenced where they chose to put the numbers. In terms of integer exposure, these students would benefit from number path games similar to those described by Aze (1989) and Wilcox (2008) to encourage students to further play with numbers. Teachers could encourage students to consider what numbers go on the other side of zero and to think about how to notate them and to provide them with opportunities to think about numbers continuing indefinitely in both directions.

Other students, who used the whole numbers to make a pattern, demonstrated a focus on the repetitive nature of numbers. Patterning with number is important in making sense of place value and can be leveraged to help students focus on similarities between positive and negative numbers, such as through a focus on symmetry (Tsang, Blair, Bofferding, & Schwartz, 2015). Both students who leveraged patterns and symmetry in their answers could benefit from experiences that involve comparing patterns in increasing magnitudes from zero (e.g., 1, 2, 3 versus -1, -2, -3) and exploring how the patterns are similar yet different in terms of how they are ordered (i.e., -3, -2, -1 versus 1, 2, 3).

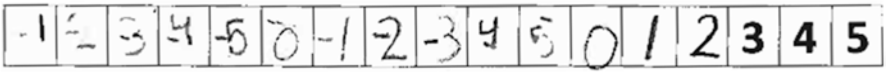

Fig. 1.9 Number path pattern with negatives

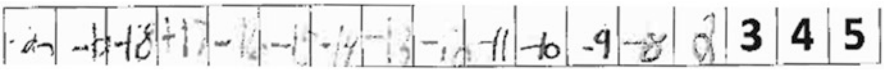

Fig. 1.10 Number path with a negative and positive sequence

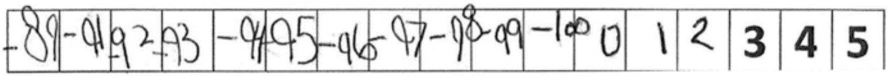

Fig. 1.11 Number path with a reversed negative sequence

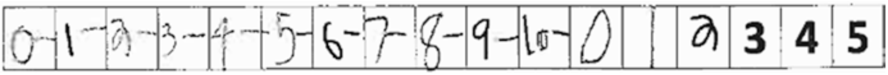

Fig. 1.12 Number path with a reversed negative sequence and two zeroes

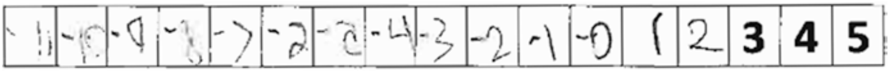

Fig. 1.13 Number path with negative zero

Playing Around with Negatives

A small group of students (7%) included negatives on their number paths in ways that are not considered formal. One student even added negatives onto their pattern (see Fig. 1.9).

As with the patterns, one student added negatives when writing a sequence (see Fig. 1.10).

Further, two students reversed the order of the negatives and wrote -89 to -100 to the left of zero (although both skipped one number in the sequence).

The student who completed the number path shown in Fig. 1.11 said that she knew “minus a hundred” went next to zero “because it’s a low one. It’s a low number.” Others either used negative zero instead of zero or used both versions of zero, sometimes putting the negative sign on the right side of the number (see Fig. 1.12).

Similarly, some of the students ordered the negatives correctly but left out zero, used negative zero (see Fig. 1.13), or used both positive zero and negative zero.

One of the students who left out zero included it later on a number line task. Finally, 23% of students correctly filled in the number path to show the integer sequence, with five of these students creating their own notation to designate negatives. These notations included putting an apostrophe, which the student called

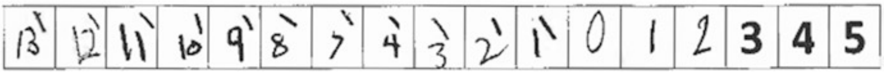

Fig. 1.14 Number path with apostrophe notation for negatives

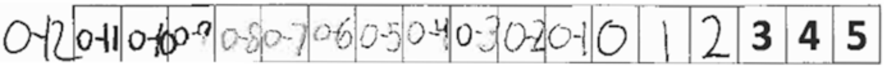

Fig. 1.15 Number path with invented notation for negatives

"negative" or "N" after each number to indicate a negative (see Fig. 1.14), putting an "n" above each number, placing a zero before the number (e.g., 02, 01), or putting a "0-" before each number (e.g., 0-2, 0-1) (see Fig. 1.15). This latter student (2.Q02) called 0-1 "negative one" and indicated, "It means zero, and the line means negative, and then the number."

Discussion: Negative Number Play

The small group of students who included negatives in informal ways provides interesting insight into students' beginning conceptions of negatives. The inclusion of negative zero suggests that students are attuned to symmetry and the similarity in notation between positive and negative numbers and, based on that pattern, feel the pattern should extend to zero. Although the notion of negative zero exists in computing, students need to understand that we do not write both zero and negative zero on the number path because their values are the same (i.e., -0 = 0; this might make more sense to students who are exploring multiplication with integers: -1 × 0 = 0). Students who reversed the order of the negatives on the number path also tried to find structure in the numbers but did so by focusing on the numerals and copying the order of the positive number sequence or by writing them in order of increasing magnitudes within each set of numbers. In this way, these students demonstrated a "separate negative number ray" conception (Widjaja, Stacey, & Steinle, 2011, p. 86). These students need additional experience thinking about the relation between integer order and values through exploring them on a continuum from most negative to most positive (e.g., Bofferding, 2014) and through contexts such as elevation (e.g., Swanson, 2010), temperature (e.g., Pratt & Simpson, 2004), or net worth (e.g., Stephan & Akyuz, 2012). Some of these students indicated that they knew about negatives because they "just saw it somewhere" (Y2.R07) or because a sibling had told them (Y3.D06).

Given that some of the students had previously heard about negatives, it is not surprising that some of them wrote them correctly. Further, some of them may have copied the negatives from number lines posted in their classrooms, suggesting that some students rely on external cues to support their playfulness – it is perhaps equally enlightening that many of them ignored the number lines in their rooms.

Those who made up their own notation, although a small number, may have similarly heard about negatives but did not know the formal notation; yet, their willingness to create consistent notation suggests a high degree of playfulness. Further, their use of an alternate notation suggests they understand the importance of distinguishing types of numbers through notation. When possible, teachers should help these students connect their playful notations to the formal notations.

Arithmetic Results

As seen by the number path results, students had a variety of ways of playing with whole numbers and negative numbers that illuminated how they were structuring and finding patterns in number. However, their responses on the number path did not directly relate to their interpretations of the arithmetic problems in most cases. That is, most children did not count on their number paths in order to solve the arithmetic problems. Yet, depending on students' prior experiences with negative numbers (as illustrated through their number path responses), they played around with the problems in different ways. Our presentation of the arithmetic results revolves around two main sections: one focused on the students who only included whole numbers on their number paths and one focused on the students who included negative numbers on their number paths. Within each section, we further break apart the results based on students who gave only positive answers on the arithmetic problems and those who included negative answers on the arithmetic problems. We follow these two subsections with a discussion of the students' playfulness. Recall that based on the play perspective highlighted by Davis (1996), we interpret all responses as playful because they provide us with insight into how students find structure in the problems.

Students Who Only Included Whole Numbers on the Number Path

Whole Number Answers on Arithmetic As mentioned earlier, the majority of students completed the number path with positives or the whole number sequence. Unsurprisingly, only including whole numbers on the number path was significantly correlated with only giving whole number answers on the arithmetic problems ($r = 0.519$ and $r = 0.475$ for addition problems in years 1 and 2, respectively, and 0.585 for subtraction problems in year 2).

Among the 169 students who did not include negatives on their number paths, 151 did not include negative integers in their arithmetic responses. For those who solved subtraction problems in the second year, some avoided negatives on problems such as $1 - 4$ by reversing the order of the numbers and solving $4 - 1 = 3$;

others ignored all of the negative signs in the problems. When solving -6 + -4, Student 2.P07 drew six dots and then four dots and answered ten. Being more explicit about ignoring the negatives, when solving -4 + -3 = 7, Student 3.B14 crossed out the negative signs and said, “I did four plus three equals seven.” When asked why she crossed out the negatives, she responded, “Because I didn’t wanted them.” In fact, she felt free to change anything that did not work for her. On 1 – 4 she “switched these numbers” and solved it as 4 – 1. Further, on -7 – -9, she changed the problem to 10 – 6 because, in her words, “Seven wasn’t big enough. So I just changed it into, seven to a bigger number; nine to a smaller number.” Similarly, Student 3.B09 ignored all of the negatives. If the absolute value of the second number was larger or equal to the absolute value of the first number in the subtraction problems, she generally answered with the absolute value of the first number, essentially ignoring the second numbers.

Even though these students did not include negatives, many of them indicated through their strategies that they paid attention to the negative signs in the problems. One way they did this was by interpreting negatives as worth zero; several students articulated reasoning that the numbers were taken away. For example, when solving -9 + 2 = 2, Student 3.F02 stated, “I thought that it said like, minus. So, I already had a nine, and I minused – like I took nine away, and then I had a zero, and I get two, and it’s like plus two.” Therefore, this student interpreted the problem as 9 – 9 + 2, leveraging the binary meaning of the minus sign to subtract the number from itself.

However, this was not the only use of the binary meaning. At the extreme, Student 3.J05 answered 0 for almost every problem because she used subtraction in multiple ways. Typically, she subtracted negatives from themselves (i.e., -8 or “eight minus” meant 8 – 8). This somehow led her to an answer for 0 on -9 + 2. She said, “Because nine minus plus two equals zero.” Further, on problems such as 4 – 5, she also answered 0 because, “If you have four and minus five, it equals zero.” She consistently used a binary meaning of the minus sign. Placement in this group was associated with providing significantly more answers of zero on the subtraction problems (correlation of 0.283), possibly because they were more likely to ignore the negatives on the arithmetic problems and answer 0 for problems such as -3 – 3, -8 – -8, and -5 – -5 or because they reasoned that they couldn’t take any more away on problems such as 4 – 5 or -2 – 3 (interpreted as 2 – 3).

Another student who consistently used the binary meaning was Student 2.Y02, who solved integer addition problems by subtracting the smaller absolute value from the larger absolute value. For example, when solving -6 + -4 and answering 2, he stated, “Minus six plus minus four and then, um, I had so I did, um, I drew six and I marked out four and then, um, two was my answer.” Therefore, although he read that “-6” had a “minus,” he used it as positive but then interpreted the second negative sign as a subtraction sign. Further, for 6 + -8 he said, “Six minus eight, so I drew eight since it’s a little bigger. I took away six.” Along the same lines, when solving problems where students had to subtract a negative, some students saw the symbols as two subtraction signs. Student 3.D08 when solving 9 – -2 answered 5 and said, “Because I thought it was two minuses, so I just had put five,” essentially solving 9 – 2 – 2. Student 3.F02, similar to 3.D08, took away the second number twice when subtracting a negative number in solving 3 – -1 and explained, “Because three minus one and then like—this one I thought it minused two—like one more, and then I put one.”

Negative Answers on Arithmetic Even though many students did not include negative integers on their number paths, 18 of these 169 students provided meaningful negative answers for at least one arithmetic problem. Beyond a focus on the binary meaning of the minus sign, students in this category also frequently applied the symmetric meaning of the minus sign. Student 2.P02 was rather unique in his treatment of the numbers. He played around with the relations among numbers based on fact families or familiar sequences while also attending to the negative. For example, when solving -4 + 6, he answered -8: “I knew two plus two equals four and then I added two and then six, then I went six and then eight and then got that [-8].” The student saw 2, 4, 6, and 8 as related, but although he talked about them as positive in his solution process, he provided a negative answer. Other students added the absolute value of the integers and added the negative sign to the answer; whereas, some subtracted first and added the negative sign to their answer. Student 2.O08 in solving -6 + -4 said, “Negative six plus negative four equals two...Negative two**.** ‘Cause both of the numbers have negative and there’s two.” When asked how he knew it would be negative two, he replied, “Because it’s subtracting.” This response suggests that he either interpreted the negative sign in front of the four with both a binary (subtraction) meaning as well as a symmetric meaning or interpreted the negative sign in front of the six with a symmetric meaning and the negative in front of the four with a binary meaning. Student 3.L01 displayed a strong symmetric meaning of the minus sign. All of his answers were negative with over 80% of them consistent with subtracting the smaller absolute value from the larger absolute value and making the answer negative, regardless of the signs of the numbers. For example, on -7 – -9, he answered -2; on -3 – 3, he answered -0; and on -5 – -5, he answered -0.

Similar to students in the previous part, some of these students also treated negative numbers as worth zero in their arithmetic responses. Student 3.H06 interpreted the negative numbers as zero in some problems. For example, when solving 9 – -2, she indicated that -2 “equal zero [so] you get nine.” Yet, her interpretation of negatives as worth zero was more complex. When solving subtraction problems with two negatives, she provided negative answers (e.g., solving -5 – -5 as -10 and -7 – -9 as -16). When describing her solution to -2 – -6 as -8, she said, “I’m in the zeroes on the number line.” If subtraction is interpreted as moving to the left on a number line, her description would match and suggests that she may have some unary conception of negatives as locations on a number line “in the zeroes.” There was clear evidence that at least one other student treated negatives as attached to a numeral (unary meaning). Student 3.B13 started at a negative number and counted toward the negative direction, indicating a unary understanding of the minus sign. He solved -4 + -3 by counting, “Negative four, negative five, negative six, negative seven.” However, as with the previous student, Student 3.H06, Student 3.B13 solved 9 – -2 and answered 9 by indicating, “Negative two doesn’t make anything,” interpreting it as worth zero.

Discussion Many of the students exhibited initial mental models for integer order and value in these tasks. Their strategies (ignoring of the negative signs and changing problems so that the larger absolute value was first) highlight the strength of their mental models for whole number operations and the structure they expect to find in arithmetic problems. As seen with previous negative integer research, students’ play with the problems where they had to subtract a number of larger magnitude from a number of smaller magnitude revealed what they know. Those who reversed the order of the numbers tried to apply the commutative property to subtraction, whereas others subtracted and got zero because they could not take any more away (Bofferding, 2011). The expectation that they could commute numbers with subtraction is further evidence that even young students look for structure in mathematics. By restricting the numbers they expose children to (i.e., only presenting whole numbers in early years), adults inadvertently provide situations where students are forced to try to find structure in limited situations.

On the other hand, a handful of students played around with the negative signs as they tried to make sense of them. Based on their mental models of whole numbers, where minus signs indicate subtraction, it is logical that they would interpret negatives as a binary sign as well. Students who interpret negative numbers as numbers subtracted from themselves are not far off the mark (and exhibit more of a transition I mental model; see Bofferding, 2014). Indeed, instead of interpreting -5 as 5 – 5, one could interpret it as the result of 0 – 5. Student 3.H06 provided some indication that these “zero numbers” can be ordered (as on a number path); however, she only answered with negative solutions if both minuend and subtrahend were negative. This suggests that she was operating with a divided number line mental model (Peled et al., 1989) and had not developed special rules to cross between negative and positive sides. Acknowledgment of negatives as their own type of number (located separately from positives on a number path) is a key step. Students who solely rely on the binary meaning of the minus sign and subtract twice for problems such as 9 – -2 would benefit from exploring both negatives as locations as well as the distance meaning of subtraction. By considering the distance between two numbers, students could reason that the distance from -2 to 9 is 11 (Whitacre, Schoen, Champagne, & Goddard, 2016–2017).

Finally, Student 3.B13 presents an interesting case because he did not include negatives on his number path but was able to count into the negatives when solving some of the problems. It is possible he forgot about them until he saw them on the test, or the number path question itself may have been the issue. The question asked students to fill in the missing numbers, but students may have thought this only referred to the whole numbers or “normal numbers” as many described them. Another possibility is that due to their school context, they believed they were expected to only fill in whole numbers; in fact, some students who knew about negative numbers specifically asked us if they could fill them in on their number paths. This suggests that the rigidness of school-based tasks could limit students' willingness to play and thus find or demonstrate additional order and structure in mathematics. If we take the opening quote by Davis to heart, this could lead to a lack of play and students concluding that mathematics is a set of rules to memorize, a belief held by many students (National Research Council, 2001).

Students Who Included Negatives on the Number Path

Whole Number Answers on Arithmetic Interestingly, 25 students among 74 students, who had negative numbers on their number path, did not include negative numbers in their arithmetic problems, or their use of negative answers did not convey a meaningful strategy. Not all these students placed the negative integers on the number path in the correct order. Also, some of these students used their own notation of the negative sign when filling in the numbers below zero (e.g., 11N, 10N, 9N, 8N, 7N, 6N, 5N, 4N, 3N, 2N, 1N, 0, 1, 2, 3, 4, 5) but did not include them on the arithmetic problems. The arithmetic results of those students who used their own notation for negatives on the number path highlight the importance of connecting their notation to the formal notation. Three of these students ignored all of the negative signs and added the integers based on their absolute values (e.g., -9 + 2 = 11, 5 + -2 = 7). Therefore, the arithmetic items potentially underestimated these students' ability to reason about integer operations compared to if they had been given problems using their own notation. Even some who correctly completed the number path using formal negative notation ignored the negative sign in the problems and added the integers with their absolute value.

Some students paid attention to the negatives in the problems and played around with the meaning of negatives such that their responses did not include negative integers. Student 2.R04 when solving -6 + -4 said, "Since it's both negatives, so it would equal zero. Because [if] it was six plus four it would be ten, but it's negative six plus negative four so it's zero." Unlike students who put whole numbers on their number line, Student 2.R04 did not talk about negatives as taking away themselves but talked about them as being "nothing," suggesting he saw the negative as attached to the number, a unary meaning of the minus sign, but attributed a value of zero to negatives.

On the other hand, Student 2.V06 described the negatives as "taking away," playing with the binary meaning of the minus sign as seen with other children. In solving -2 + 0, she said, "Take away two plus zero, and it equals zero." For addition problems starting with the smaller absolute value number, she answered 0. For example, when solving 6 + -8, she explained, "You can't take away eight so I just took away all of them." She even did this if the first number was the one that was negative, as in -4 + 6 which she answered with 0, consistently using the binary meaning of the negative sign. Others, such as Student 2.R01, only subtracted the integers when the second number was negative; otherwise she either treated the negatives as worth zero or ignored the negative signs, answering with nonnegative numbers. Treating the negative sign as the subtraction function was more prevalent among students who worked on subtraction problems.

Negative Answers on Arithmetic As mentioned, 74 students among 243 second graders had negative numbers on their number path. More than half of these 74 students (49 students) provided negative numbers in their arithmetic responses, and often their treatment of the negative sign varied. Student 3.J04 was an anomaly in this group. To form her answers, she took the numerals from the problem and put them together to make a two-digit number, which she made negative in one instance (e.g., -5 – 9 = -59). On the other hand, many of these 49 students ordered the negative integers on the number path in nonstandard ways, which for a few corresponded to their arithmetic strategies and responses. For example, Student 3.J02 had -0 instead of 0 on the number path; likewise, he answered -0 in arithmetic problems like -8 – -8. He also answered -0 for the majority of the other problems, such as -3 – 3 as well as 4 – 5. Student 3.J02 was not the only student who responded with -0 for some problems. Student 3.C13 had both 0 and -0 on the number path and also answered -0 for -3 – 3 and -5 – -5; similarly, she typically subtracted the smaller absolute value from the larger and made the answer negative. This suggests a focus on the symmetric meaning of the minus sign. Other students expressed the symmetric meaning more clearly, such as Student 2. O09 who solved -6 + -4 = -10 and explained, “Because I knew that six plus four equals ten, so I thought that it would be the same in negatives.”

Similar to the previous groups of students, one of the frequent uses of the minus sign for these students was applying the binary meaning as subtracting two integers. Student 2.Y09 in solving -6 + -4 answered 2 and said, “Minus six plus minus four, I had six and then I went back four and I got two” and for -9 + 2 answered 7 and explained, “Nine minus two plus two equals seven. I just went back two.” Student 2.O09 sometimes treated the negative sign as subtraction and explained his response for -5 + 5, “Because negative five plus five would be like five minus five.” Unlike the other groups who had students subtracting numbers from themselves, making negatives result in zero values (i.e., negatives became worth zero only after being subtracted from themselves), students in this group tended to acknowledge negatives as a separate class of numbers, to which they ascribed the value of zero – treating the negative sign with its unary meaning but worth 0. For example, Student 2. R10 in solving -5 + 5 answered 5 and said, “Because that’s a negative number. It means that it’s one of the zeros.” Also, Student 2.R09 explained her answer for 4 + -5 as 4, “Because five, because negative five is um, is worth none.”

A couple of students applied the unary meaning to the negative signs but also interpreted them with an additional meaning. Student 2.O09, for example, described his answer for -3 + 1 = -2, “I knew that it’d go up except it was subtraction...’cause I knew that three minus one would equal two...because I knew that it was negative number instead of an actual number.” He talked about going up from -3, treating the negative with its unary meaning, talked about the problem related to subtraction (binary meaning), and also used the symmetric meaning, relating his answer of -2 to 2. Likewise, Student 2.U01, who applied the unary meaning of the minus sign in some problems, solved -1 + -7 = -8 and explained, "Negative one plus negative seven is kinda one plus seven. But I did, but I counted one and got negative eight." Therefore, the student both counted within the negatives (unary meaning) and related the problem to its positive counterpart (symmetric meaning). Although he talked about the symmetric meaning, he had a strong unary meaning of the minus sign and answered almost every arithmetic problem correctly. He knew that adding a negative resulted in a more negative answer, and for -9 + 2 = -7, he explained, "I started at negative nine and then counted backwards, then it's negative eight then negative seven."

Students' treatment of negative numbers as points on the number line or in the counting list (the use of the unary meaning of the minus sign) sets this group apart from students in the other groups. Their use of negatives did not always align with formal procedures; however, they exhibited a system of rules that likely reflected their positive number understanding. Student 3.H05 started to assign movements to numbers. She described her answer for -9 + 2 as -11, "Because you're in the negatives, and you want to go up to two more. So, I found out it was eleven." Her meaning of going up corresponds to a positive number notion of addition. Note that she also called her answer 11, even though she wrote -11. Further, she explained her response for -7 – -9, "Negative seven minus negative nine equals negative two... I started at negative nine, and I went back seven, and I landed on negative two." Even when subtracting two positive numbers, she always started with the larger number (e.g., solving 1 – 4 as 3), and the direction she counted was consistent with a positive notion of subtraction where the magnitude of the answer is smaller than the initial number.

Student 2.O07 had a different, yet consistent, way of reasoning about integer operations. She consistently started at the initial number and described the directional movements, as if using a mental number line, on that initial number. If the initial number in the addition problem was positive, she clarified, "If you wanted to go down, you could be like seven plus negative three equals four." She solved all problems of the form positive plus negative correctly, counting down from the positive number. All addition problems with the initial numbers as negative corresponded to an upward movement to her, regardless of the sign of the second number. In solving -5 + 5 = 0, she reasoned, "Negative five and if you add five, you would get negative four, negative three, negative two, negative one, zero." When asked how she knew when to go up, she provided another example, "Like when it's negative nine plus two equals negative seven." Consequently, she correctly answered all problems of the form negative plus positive. However, she also used an upward movement to solve negative plus negative problems. Therefore, she explained her answer for -6 + -4, "I started with negative six, negative five, negative four, negative three, negative two." Further she responded 6 for -1 + -7 by applying the same strategy (starting at -1 and moving more positive 7).

Discussion Although it is less surprising that students who represented negatives on the number path with their own notation would ignore them on the arithmetic problems, it is more surprising that those who included negatives would ignore them. However, this makes sense in terms of conceptual change and integer order and value mental models. Students may have an understanding of the order of the negative numbers before attaching value meaning to them (Bofferding, 2014). In fact, some of the students may have copied the numbers from the classroom number lines. Therefore, they found other ways to make sense of the symbols that fit into their existing mental models of whole number values.

When we look beyond answers as being correct or incorrect and look at how students play around with numbers, we learn what elements of numerical structure stand out to students. For those who did not provide negative answers but interpreted negative signs as subtraction signs, their answers suggest patterns in students' thinking. For some, the minus sign meant subtraction regardless of its position, and they always subtracted from left to right, so -4 + 6 meant 4 - 6 just as 6 + -8 meant 6 – 8. For others, negatives at the beginning of a number sentence meant the number was subtracted from itself, in which case -4 + 6 meant 4 – 4 + 6. Distinguishing between these two situations might be an initial step toward interpreting negative signs as different from subtraction signs.

One of the striking differences between students who used negatives on their number paths and those who did not was the increasing number of students who treated negatives as numbers (or locations on a number path). Several of these students were able to count in the negatives and through zero, similar to young students in other studies (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011). Rather than playing around with the meaning of the negative signs, their play centered on the meaning of the operations in relation to negative numbers. Acknowledging this play as legitimate sensemaking, regardless of correct or incorrect answers, is important because their play highlights powerful reasoning on their part as they try to coordinate concepts related to magnitude, order, addition, and subtraction. Analyzing these attempts can help teachers and researchers identify potential areas to focus on in instruction and continued research. For example, Student 2.O07 differentiated between the direction of movement when adding a positive number to a positive number and a negative number to a positive number. Unlike with others who focus on the magnitude of numbers, this focus on direction helped her make sense of adding a positive to a negative as going up (movement to the right on a number line). Likewise, in a comparison between second and fifth graders, Aqazade (2017) found that second graders who compared worked examples of the problem comparisons 5 + 3 versus -5 + 3 (highlighting the consistent direction when adding a positive number) followed by -2 + -5 versus 2 + -5 (highlighting the consistent direction when adding a negative number) made larger gains and performed higher than fifth graders by posttest. One hypothesis for why the fifth graders made fewer gains was that they were more set in their initial strategies and less open to playing around and changing the way they thought.

Making Space for Learning

By shifting to a lens of play, introducing concepts that may seem “too abstract” for children based on an adult perspective suddenly gives children the freedom to play around and ultimately provides us with insight into how they are trying to make sense of the world. The results presented here highlight some important moments in children’s conceptual change journey from whole number to integer understanding. One key concept revolves around students’ interpretation of the negative sign and how it relates to integer values. On the one hand, some children interpret the sign as disconnected and representing a subtraction sign. These students interpret negatives as subtracted from themselves and worth zero (Bofferding, 2014). Subtly, but strikingly different, another subset of students interprets the negative as attached to numeral to form a negative number, which they also concluded had a value of zero. Some of these students treated the negatives as locations on a number path while simultaneously considering them to all have zero values; often they preferred one aspect over another in their arithmetic answers.

Another key concept involves students’ interpretation of the operations. For many students, addition (even with negatives) involves a process of “going up.” However, their interpretation of what “going up” means differed. In the case of Student 3.H05, going up meant getting a higher absolute value either in the positives or negatives, depending on the initial number in the problem. Yet for Student 2.O07, going up corresponded to moving to the right on a number path; although this allowed her to add positive numbers to negative ones, it did not lead to correct answers when adding two negatives. This points to the importance of focusing on the meaning of operating with directed magnitudes and helping students reason about the differences between operating with a positive and negative number in conjunction with addressing their overall meanings of the operations.

The wealth of information we can learn from students’ opportunities to play with new concepts has implications on the types of tasks we give to students. Providing students with problems that challenge their existing mental models can illuminate their current mental models as well as “what is not yet known as it simultaneously offers space to support learning” (Davis, 1996, p. 222) and encourage mathematical wonder. In this view, assessment is less about determining students’ mastery and more about understanding their efforts to find structure in mathematics.

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Appendix

Year 1: addition arithmetic problems

-5 + 5 =	-2 + 0 =	-4 + 6 =
-9 + 2 =	-3 + 1 =	-1 + 8 =
5 + -2 =	4 + -5 =	9 + -9 =
6 + -8 =	7 + -3 =	0 + -9 =
-6 + -4 =	-1 + -7 =	

Year 2: addition and subtraction arithmetic problems

-4 + -3 =	-9 + 2 =	4 + -6 =	-8 + 8 =
1 – 4 =	4 – 5 =	0 – 9 =	9 – -2 =
3 – -1 =	1 – -6 =	6 – -7 =	-7 – 3 =
-3 – 3 =	-5 – 9 =	-2 – 3 =	-7 – -9 =
-2 – - 6 =	-8 – -5 =	-2 – -1 =	-8 – -8 =
-5 – -5 =			

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Chapter 2
Integer Play and Playing with Integers

Nicole M. Wessman-Enzinger

Abstract This chapter describes instances of play within a teaching episode on integer addition and subtraction. Specifically, this chapter makes the theoretical distinction between integer play and playing with integers. Describing instances of integer play and playing with integers is important for facilitating this type of intellectual play in the future. The playful curiosities arising out of integer addition and subtraction tended to be concepts that we think of prerequisite knowledge (e.g., magnitude or order, sign of zero) or knowledge that is more nuanced for integer addition and subtraction (e.g., how negative and positive integers can "balance" each other). Instances of integer play and playing with integers are connected to the work of mathematicians, highlighting the importance of play in school mathematics.

Embracing the identity of a mathematician or participating in the work of a mathematician may seem like a foreign idea, especially to elementary school students. Yet, children are more capable of approaching mathematics similar to research mathematicians than they realize:

> Young children develop mathematical strategies, grapple with important mathematical ideas, use mathematics in their play, and play with mathematics. Young children often enjoy their mathematical work and play. Indeed, despite its immaturity, young children's mathematics bears some resemblance to research mathematicians' activity. Both young children and mathematicians ask and think about deep questions, invent solutions, apply mathematics to solve real problems, and play with mathematics. (Ginsburg, 2006, p. 158)

A key idea expressed by Ginsburg is the idea of play. He posits that through play students deeply engage in mathematics, reminiscent of mathematicians. The idea of fusing play with mathematics comes at a pivotal time in education and society. Increased educational testing (Ravitch, 2010), demands to meet expectations of standards (e.g., National Governors Association Center for Best Practices and

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Council of Chief State School Officers [NGA and CCSSO], 2010), and increased needs for children to pursue STEM careers in the future (Ellis, Fosdick, & Rasmussen, 2016; Olson & Riordan, 2012) are just some of the contemporary pressures. As stress continues to build around the increase in testing and expectations in standards, there is also a push to extend play throughout elementary school (Parks, 2015). Including play in mathematics may reduce stressful mathematical experiences. Engaging children in playful experiences of mathematicians may also have the potential to provide increased opportunities for access to more complex mathematical concepts. Although there are calls for mathematical play (Ginsburg, 2006) and prolonged play in school (Parks, 2015), most of these play experiences are described with young children. But, are children in late elementary school able to learn advanced mathematical concepts through play? This chapter illuminates the potential of play for supporting children's mathematical thinking and learning about integers and integer operations in Grade 5.

Elements of Play

Children, like research mathematicians, engage in mathematical play and playful mathematics (Ginsburg, 2006). Ginsburg classified mathematical play as engaging in mathematics embedded in play. For instance, when building block towers, children may count their blocks or compare the heights of block towers as they play. Ginsburg also classified playful mathematics as play centered on mathematics. This may happen when students engage in play that is purposefully mathematical—like playing a walking game on a number line.

These types of play, mathematical play and playful mathematics, should not be reserved for just young children (Parks, 2015) or just mathematical topics typically advocated at their grade level (Featherstone, 2000). Play can help them investigate new concepts as well. Parks (2015) lamented about the need for play throughout elementary school, "as children move through the primary grades and have fewer and fewer opportunities for play, finding ways to bring choice, excitement, movement, imagination, and curiosity into formal lessons becomes more and more important" (p. 112). We know that children are capable of sophisticated reasoning about integers (Bofferding, 2014) and integer addition and subtraction (Bishop et al., 2014). Mathematical play and playful mathematics may be a space for children to engage in topics, like integers, at later elementary grades and before age levels recommended in standards (NGA & CCSSO, 2010).

Identifying elements of mathematical play and playful mathematics (see Table 2.1), even with older children, can help distinguish the intellectual, but playful, experiences that children engage in as they play with integers (Featherstone, 2000; Parks, 2015). Burghardt (2011) described essential criteria for play: spontaneous or pleasurable, not fully functional, different from similar serious behaviors, repeated, and initiated in the absence of stress. First, play must be spontaneous and pleasurable—it is a necessary requirement that play is fun and enjoyable for children.

Table 2.1 Elements of play

Criteria for play (Burghardt, 2011)	Additional criteria for play (Parks, 2015)
Spontaneous or pleasurable	Opportunities for social engagement
Not fully functional	Creative thinking
Different from similar serious behaviors	Appealing materials
Repeated	Physical movement
Initiated in the absence of stress	Imagination

Second, play is not fully functional because it is not necessary for survival but has some functional aspect. Play may be functional, like building a castle out of blocks for a doll. In this way, play may serve some sort of function and have delayed benefits. Third, play must also include some qualities that differentiate it from serious behaviors. Children dancing or pretending to be an animal, for example, are different than typical behaviors in the surrounding environment. Fourth, play also includes elements of repetition because children will often repeatedly play until a skill is mastered. For example, children may try to build a tall block tower. As they build this tower, it may topple over, but they will continue to repeatedly build this tower until it stands. Last, play must be initiated in the absence of stress—play is voluntary and takes place in a safe, relaxed environment. Burghardt noted that all of these criteria must be met in some capacity for true play. However, play includes additional criteria, such as social engagement, creative thinking, appealing materials, physical movement, and imagination (Parks, 2015).

Insight into how these elements of play are present as children engage in integer play and play with integers is needed. Identifying elements of play and describing instances of them provides insight into the opportunities and spaces for deep, intellectual, and mathematical thought. Describing instances of intellectual play may also offer insight into how play may be supported in school mathematics throughout elementary school.

Imaginative Play Supports Thinking and Learning About Integers

One of the prevalent themes in the literature across history is that the thinking and learning about integer addition and subtraction is notoriously challenging (e.g., Bishop et al., 2014; Piaget, 1948; Thomaidis, 1993). Yet, we are gaining deeper insights into the ways that children think about integers (Bofferding, 2014) and integer addition and subtraction (Bishop et al., 2014; Bofferding, 2010; see Chap. 3). One reason the negative integers may be so challenging is the lack of physical embodiment of them (Martínez, 2006; Peled & Carraher, 2008). That is, the negative integers cannot be used as objects that physically exist (e.g., -2 fish) without opposites and an abstract one-to-one mapping of an integer to an object (e.g., stating that a red chip represents -1). Because of the physical constraints of the negative integers, the integers are not as naturally accessible in play as the whole numbers or natural numbers.

Even so, Featherstone (2000) illustrated that play can be built around the imaginative world of integers. She presented an illustration of a Grade 3 student journaling about additive inverses in a playful way: "-pat + pat = 0" (p. 14). Educators and researchers also utilize games for the teaching and learning of integer addition and subtraction (e.g., Bofferding & Hoffman, 2014; Wessman-Enzinger & Bofferding, 2014). Games may be the space to encourage the imaginative mathematical play that Featherstone discussed. Play often centers on mathematics (Ginsburg, 2006; Parks, 2015), such as playing on a linear board game (e.g., Bofferding & Hoffman, 2014; Siegler & Ramani, 2009).

Yet, mathematical play is not just playing with a game but is *play with numbers* (Featherstone, 2000; Ginsburg, 2006; Steffe & Wiegel, 1994). Boaler (2016) supports play with numbers for all students: "The best and most important start we can give our students is to encourage them to play with numbers and shapes, thinking about what patterns and ideas they can see" (p. 34). Featherstone (2000) argued that as children engage with integers, this may be a "territory for mathematically imaginative play" (p. 20). She also connected some features of play to exploring integers in elementary school. For example, one of the defined attributes of play is that play exists in a separate, outside world (Huizinga, 1955). That is, the child is able to step outside of reality into this other world. Featherstone (2000) proposed that the integers themselves are this imaginative world. She wrote, "The territory below zero is a separate world for elementary students. It is an outside the 'real' world of natural numbers - numbers that are in daily use both inside and outside of school" (p. 20). This type of imaginative play may be a way to share integers sooner and prolong play in schools.

We need more descriptions and insight into this imaginary world of play with integers that the children often step into. This chapter highlights how these different types of play, playing a game and engaging in mathematically imaginative play, work together to support thinking and learning about integers and integer addition and subtraction. Specially, this chapter illustrates specific instances of integer play and playing with integers and connects these instances to the elements of play described by Burghardt (2011) and Parks (2015). Then, these instances of play (integer play and playing with integers) are connected to the work of research mathematicians to show the potential for play in upper elementary grades.

Context of the Study

The data reported on in this chapter comes from a 12-week teaching experiment (Steffe & Thompson, 2000) with three Grade 5 students designed to examine the teaching and learning of integers, specifically negative integers. The teaching experiment was comprised of nine group sessions and eight individual sessions for each child. During these sessions, the students were introduced to four conceptual models for integer addition and subtraction ([CMIAS], Wessman-Enzinger & Mooney, 2014)—bookkeeping, counterbalance, translation, and relativity—through the use of various contextualized problems and activities. Although these CMIAS were introduced throughout the teaching experiment, it was not expected that the students would use only these models; there were opportunities for students to think about the addition and subtraction of integers freely as they engaged in activities during the group sessions.

The mathematics that the students discussed and the misconceptions they held influenced the content and development of the group sessions of the teaching experiment. I served as the teacher-researcher for this teaching experiment. A second researcher was the witness for most of the group sessions. He took field notes during the group sessions. In addition to taking field notes, he also periodically asked questions of the participants during the sessions. After each group session with the students, the witness and I debriefed about the students' thinking and learning that appeared to be emerging during the sessions. We also discussed plans for the next group session, and I considered his observations and suggestions for the next instructional moves, based on the students' responses in that session. After each individual and group session, I wrote reflections about what I noticed as the teacher-researcher, what I thought the next instructional moves should be, and why I thought that move should be made.

The focus of this chapter is on the fourth group session because it serves as an example of the playful mathematics imbedded in mathematical play (Ginsburg, 2006). This group session incorporated playing an integer-focused card game, during which the students engaged with mathematics in ways that we had not planned. I present this case to illustrate the power of mathematical play for creating opportunities for play with mathematics and to show how such play can support mathematical thinking.

Integer Play

The mathematical goals of the integer play in the group session constituted adding integers, subtracting integers, developing a counterbalance conceptual model (Wessman-Enzinger & Mooney, 2014), and distinguishing the minus symbol from the negative symbol.

In the card game, *Integers: Draw or Discard*, drawing cards aligned with integer addition and discarding cards aligned with integer subtraction (Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014); therefore, the game fostered discussion of both addition and subtraction during this session. After including the drawn cards to their hand, the children determined their total points of their cards by adding. Discarding a card was similar to subtraction—as the point value of the card was taken away from the total hand. Thus, if students discarded a negative integer card, they considered the effects of subtracting a negative integer.

Developing thinking about integers with a counterbalance conceptual model (Wessman-Enzinger & Mooney, 2014) constituted another mathematical goal of the integer play. Because students often develop their conceptions of number with discrete, countable objects, developing thinking that supports this is important. In contrast to movements on a number line, thinking with a counterbalance conceptual model provides the opportunity to think about integers as "tangible." Within a counterbalance conceptual model, integers are conceptualized as two distinct quantities that neutralize. Ideas of neutralization are also important in mathematics from contexts like electron charges to areas beneath curves in calculus. However, as the students began this group session in the teaching experiment, the students did not appear to see the "neutralization" in the quantities. To emphasize this "neutralization" with a context, I decided to use a card game that uses integer cards from -8 to 8 (Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014). I selected this card game because the cards are integer quantities that would remain present in their hands of cards, giving the students opportunities to experience neutralization and, consequently, the potential to develop the counterbalance conceptual model. For example, if a student had a hand of 2, -2, and 7, it was worth the same in this game as a hand of 3, -3, and 7.

Central to the notion of counterbalance, another goal included that children begin to make distinctions between the subtraction symbol and negative symbol (Gallardo & Rojano, 1994). For those reasons, after game play, the children were asked to make sense of fictitious children's hands of cards and write number sentences modeling the drawing or discarding of cards. This was done to help promote thinking and learning about both integer subtraction and the differentiation between the negative symbol (e.g., used when writing an integer in a number sentence) and the subtraction symbol (e.g., used when writing a number sentence for discarding). Some of the ways that these students engaged in these types of mathematical ideas during the integer play will be discussed in the following section.

Integer play with this card game satisfied Burghardt's (2011) essential criteria for play. This game provided the *opportunity for pleasurable experiences* because the children demonstrated excitement about playing the game and generally enjoyed playing with cards. This integer card game included *not fully functional* behaviors as the game was not necessary for survival but had the potential to satisfy the mathematical goals highlighted above. The game served as an activity *different from similar serious behavior*—the game included negative integers, which the children did not use during regular school instruction, and the children played the game outside of math class during their free time. The children played several rounds of the game and asked to keep playing after the game concluded—this illustrates *repetition* in play. *Initiated in the absence of stress*, the children volunteered to participate in this game play, which took place separate from formal instruction in a room outside of their classroom. The following excerpts will illustrate some of the mathematical goals achieved through this integer play and highlight the additional criteria of play achieved (Parks, 2015).

Integer Play: Addition of Integers

From the inauguration of the teaching experiment, the children illustrated an ability to add integers with success. Consequently, as the children engaged in the game play, they naturally added their cards with ease and did not initiate discussion about addition. In the first move of the game, Kim drew two cards:

Kim: Negative seven and eight.
Me: She had negative seven and eight. What do you think her point total is?
Alice: One.
Jace: One.
Me: Why do you all think it's one?
Jace: Because eight minus seven equals one.

All three students performed calculations repeatedly for the function of determining their scores. However, the students did not reference addition. Even when Jace needed to add two cards in this excerpt, -7 + 8, he interpreted this as 8 – 7, suggesting an interpretation of the negative as a subtraction sign (Bofferding, 2010). Rather than discussion about addition per se, the children's discussion focused on the discard of cards or how to get the largest point total, which often included making decisions between drawing a card (adding) or discarding a card (subtracting). This consequently resulted in children talking about situations where they were confronted with initial ideas of subtraction; they considered situations where it was better to discard larger negative integers from their card hands (subtraction) rather than to draw smaller negative integers to their card hands (addition), which was a mathematical expectation of this game (see, e.g., Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014). It was expected that the main opportunities for thinking and learning would be centered on the subtraction of integers and developing a counterbalance conceptual model (Wessman-Enzinger & Mooney, 2014), which are described next.

Integer Play: Subtraction of Integers

All three of the children discarded negative cards (e.g., -2) throughout the game and recognized that this increased the total points of their hands. The children successfully played ten rounds of the game, where each child confronted the option of discarding a card with a negative integer. Each child did this action and increased their point total; yet, the children did not necessarily explicitly recognize this physical action in the game play as subtraction. For this reason, at the end of the game, the children were asked to write number sentences representing some of their hands and fictitious children's hands of cards in order to see if they conceptualized discarding cards as subtraction. However, the children had difficulty writing number sentences. When the children began writing number sentences, they did not use subtraction for discarding cards; they would, instead, write the addition of the positive score. I prompted the children to think about how they could write number sentences that preserved the negativity of their cards. In the last minutes of the group's session, they began writing number sentences that involved the subtraction of integers. Jace and Alice worked together on writing a number sentence on a whiteboard together, while Kim observed, for discarding a -5 card from a fictitious student's hand of cards. The fictitious student's hand of cards included -3, -5, and 8, with a -2 card as an option to draw; therefore, they had decided it was best to take the current hand of cards (worth 0 points) and discard the -5.

Alice (Writes 0 + 5 while whispering)
Jace: (Whispers)
Alice: Can you speak a little louder?
Jace: Sure. I did zero (points at the 0 in 0 – -5 = 5) because that's what he had after the first problem. And then I did minus negative points (points at the "–" and then "-5") because he discarded the negative five and now he has five because there's not longer a negative five in the problem. In the first problem that he did. So that just adds five to it. Technically (gestures with fingers and makes "air quotes").
Alice: (Looks at me) Well, I don't get how he got his answer of five.
Kim: I don't get it.
Jace: Alice, you're just doing what I did here (points at Alice's writing: 0 + 5).
Kim: (Gets up out of seat and walks to the board where Jace and Alice are.)
Alice: Yeah, but I don't get how he get got five.
Kim: This was his first problem (circles Alice's number sentence before she simplified to find the initial point total: 8 + -3 + -5). And then this is the second problem (circles Jace's number sentence 0 – -5 = 5).
Alice: Yeah, but I don't get how he got this answer (points at 5).

Kim wrote a number sentence with addition for discarding a -5 card. Although Jace was able to write a number sentence with subtraction and potentially make this connection at the end of this session, as Kim and Alice questioned him, he stated that he was confused too. In this excerpt, the children have generalized that their point total will go up by the absolute value of the negative they discard, a sophisticated observation. What remained was a matter of facilitating the children in connecting this generalization to subtracting a negative, which is an idea that may be developed later.

As the students shared this type of thinking about integers, they engaged in an *opportunity for social engagement* (Parks, 2015). Alice and Kim communicated to Jace their confusion, and Jace explained his thinking while they listened. This excerpt also illustrates *physical movement* and *use of materials that are appealing* (Parks, 2015). At first, Alice and Jace moved from the table to the whiteboard to discuss writing a number sentence, and then Kim followed. During the teaching experiment, the children often left the site where the cameras were to go write on the whiteboard. The whiteboard was an appealing department from their position at the table with paper and pencil. In general, when the students engaged in deep thinking together, they would move to this space, much like mathematicians around a chalkboard.

Given the challenges of writing a number sentence for the moves made in integer game play, subsequent sessions were developed to address subtracting negative integers. The challenges associated with the subtraction of integers lingered throughout the weeks of the teaching experience. The children's difficulty writing subtraction number sentences, but ease with discarding a negative and adding the amount to the deck, supports discussion about the difficulty of subtracting integers (e.g., Bofferding & Wessman-Enzinger, 2017) and supports research that demonstrates children's thinking is often different than adult's thinking (e.g., Bishop et al., 2014; Bofferding, 2014). However, this excerpt was included in demonstration of the initial thinking about integer subtraction that can happen during game play. Examining children's discussions during play experiences provides insight into their thinking, which may be supported later.

Integer Play: Counterbalance

From the beginning of the teaching experiment and throughout this group session, the children did not have difficulty with adding integers. However, despite their abilities to successfully add integers, the children did not all appear to draw on a counterbalance conceptual model. The counterbalance conceptual model involves children conceptualizing the addition of integers as integers that neutralize or balance each other out (Wessman-Enzinger & Mooney, 2014). In the following excerpt, Kim is faced with a decision to either discard a -7 card or to draw a +7 card. These moves have the same effect on her total points in her hand, and the children confront and reflect on this in the following excerpt.

Kim: It's the same, I think.
Alice: You could have … never mind.
Jace: No, because it would be zero, too.
Alice: I know something she [Kim] could do and it would make her score even higher, but… I'm not going to say it.
Kim: I don't think it could have.
Me: What do you think would make her score even higher?
Alice: If she picked this up (points at the 7 card).
Kim: I don't care really.
Jace: No, because she would still have the same amount.
Alice: Because she would have, then she would have, oh yeah… she would still have seven.
Jace: Yeah, because negative seven plus seven equal zero. So, should have still have…
Kim: Boom. Now I have eight points. Yay.

In this excerpt, Kim initially thought that discarding a -7 (subtracting -7) might be the same as drawing a +7 (adding 7). Alice thought that drawing a +7 card would make the score higher, than discarding a -7. The children discussed this. As part of that discussion, Jace provided the justification that 7 + -7 = 0—utilizing additive inverses is an important component when beginning to make sense of the counterbalance conceptual model. Jace reflected on this more than once, later during game play, stating:

> Alright, so. I have eight even though I have two eights in here. Actually, I have three if I count the negative eight. So… (writes on paper). Yeah, so I had an eight. I got a negative eight, so it's zero. So just got another eight and now it's eight.

Jace verbally recognized that $a + -a = 0$ in two instances during this group session. Although Kim and Alice did not verbally make those observations, they participated in the discussions where Jace shared this with them. Developing ideas about the additive inverses of integers is an important component to developing the use of the counterbalance conceptual model (Wessman-Enzinger, 2015; Wessman-Enzinger & Mooney, 2014). This excerpt highlights *creative thinking* (Parks, 2015) from Jace. Jace, without prompting from his peers or me, shared what he noticed about inverses. In this sense, Jace created this mathematics and shared his thinking about this observation. Although his peers did not ask him questions about his observations about inverses, his openness exposed Alice and Kim to this idea.

Integer Play: Minus Sign Versus Negative Sign

As the children engaged with integers through the game play, Jace highlighted that the role of a minus sign and negative sign is distinct (e.g., Bofferding, 2014)—a learning goal of the game with inclusion of the negatives on cards (use of negative sign) and discarding cards (use of minus sign when writing a number sentence). As the children wrote number sentences for representative hands of cards, Jace stated, "When you have a subtraction symbol (points at the 'minus' symbol) and a negative symbol (points at the negative number) you are just adding," referring to the number sentence 0 – -5 = 5. Kim, not convinced, stated, "Well, you are actually at zero." Jace responded, "If you take away a negative number that means that the negative number is no longer there. So like (starts writing on the board) five minus negative three would equal eight."

In this excerpt, Jace was trying to develop a rule for subtracting negative integers. For example, when Jace solved -7 + 8 in the previous section, he utilized 8–7 without discussion about this procedure. In this excerpt, Jace focused on the nuances of the sign and explicitly verbalized his procedure, but Kim and Alice were not convinced. Although it is noteworthy that Jace was trying to develop a rule or procedure for himself, through this discourse, he distinguished the negative symbol from the subtraction symbol. In this excerpt, the students focused on the minus symbol and the negative symbol. As they focused on the signs, treating the negative integer with its sign different than a minus sign represents the *use of materials that are appealing* (Parks, 2015). The students, prior to and during the teaching experiment, did not experience negative integers during their typical school day. In fact, according to *Common Core State Standards for Mathematics* recommendations (NGA & CCSSO, 2010), these students would not encounter subtraction of integers until 2 years later, and, as participants in this study, the students were attuned to this because they mentioned how they did not work with negative numbers during their typical school day and only within this teaching experiment. And, again, this negotiation on the differentiation of the role of the subtraction symbol and the negative symbol illustrates *opportunity for social engagement between the students* (Parks, 2015). This social engagement element of play was pivotal for addressing the mathematical goals of the integer play. In the past two excerpts, Jace verbalized a good understanding of concepts such as integer subtraction and symbol use. Through integer play, the students all engaged with these mathematical goals as they asked questions, discussed, and listened based on their understandings.

Although making sense of integer addition and subtraction, the counterbalance conceptual model, and differentiating the negative sign from the minus sign constituted the intended mathematical goals of the integer play, it was not important that the children mastered these ideas. Through conversation with each other, they were exposed to other ideas, like the ones that Jace presented in the past two excerpts that they had not played around with yet. Play is an ongoing activity that children use to help make sense of situations, and we cannot expect mastery immediately—especially with difficult ideas of integer subtraction. Providing opportunities for engagement with integer play is the point, because through play the children have the opportunity to work through different ideas and try new concepts out. Furthermore, the students thought about and engaged in other mathematics as they were playful with the integers in ways that were not planned by myself and the witness to the teaching experiment. The subsequent section highlights the robust mathematical ideas that may immerge when children play with integers.

Playing with Integers

The students engaged in integer play as they interacted in the game, *Integers: Draw or Discard*. Although immersed in integer play, the students played with integers in ways that occurred outside of the mathematical objectives of the game—*playing with the integers*. As the students created, wondered, imagined, and questioned with integers, they played with the integers. Three cases illustrating how the children played with the integers in this group session will be presented next. Two of cases illustrate the robust thinking and wondering they engaged in directly tied to integer addition and subtraction. Although the third illustration of playing with the integers does not connect to integer addition and subtraction, it connects to other advanced mathematical ideas. Each of these playing with integer cases will be linked to the work of mathematicians.

Playing with Integers: Order Versus Magnitude

Before the students began engaging in integer play, I explained the directions of the card game. I then asked the children who should go first. The following transcript illustrates the children playing with integers in this setting.

Me: So I was thinking... How do we decide who goes first though?
Kim: Rock, paper, scissors.
Alice: Or, who draws the highest card?
Kim: Yeah, draw highest card.
Jace: Yeah.
Me: Ok, so everyone takes...
Jace: Everyone takes one card and whoever has the highest.

(Alice, Jace, and Kim draw cards. Alice draws a -4 card, Jace draws a -8 card, and Kim draws a -7 card.)

Kim: I totally lost.
Alice: I did too.
Jace: I got negative eight.
Alice: I got negative four.
Me: Ok. And, you got what?
Kim: Negative seven...
Jace: So she goes first (points at Alice with -4).
Kim: (points at Jace with -8) So Jace's is the highest actually.
Alice: No, I am.
Jace: No, well...
Me: So, who is the highest?
Alice: (raises card in the air) Me!
Kim: Jace because his is the biggest in the negatives. Because we all have negatives, so.
Alice: Well, mine would be the biggest.
Jace: Well, she's the closest to one (pointing to Alice).
Me: So somebody said that they think Jace's is the biggest because it's negative eight.
Alice: (Shakes head no.)
Me: And, then Alice says no. So why did you think that Jace's is the biggest?
Kim: I don't know. They're all negative numbers and just like find out which one is bigger.
Jace: (Gasps.) I was wondering why you would want to discard cards. I'm like if they are all whatever why would you want to put one down. Ok, now I see.
Kim: Now I know why (holds the -8 card up in the air).
Me: And what's yours?
Alice: Negative four (holds up card).
Me: So which one do you think is bigger?

Alice: Mine.
Me: Why do you think yours is bigger?
Alice: It's closest to one. It's highest out of all of them.
Kim: Well, yeah.
Jace: Mmm-hmm.
Kim: So I'm second. I'm second (waves hands and card in the air) .

After this, I suggested that the children draw two new cards and start the game play. Although they never explicitly verbalized who should go first, Alice played first.

This excerpt highlights the elements of play: function, creativity, social engagement, and absence of stress. The children's suggestion of how to decide who should go first illustrated a *functional* component of playfulness (Burghardt, 2011); the students wanted to play the game and needed to decide who should go first, resulting in this mathematical discussion. This excerpt is playful because the children illustrated *creative thinking* (Parks, 2015); they created the ideas of order and magnitude when comparing integers. This excerpt is also playful because the children participated in *social engagement* (Parks, 2015); although the students did not verbalize a conclusive agreement on which card was "biggest," they decided to let Alice go first and played without conflict. The children freely had this discussion in the *absence of stress* (Burghardt, 2011); the children decided how they would determine who would go first in excitement to begin game play. During this freely chosen activity, the cards unexpectedly, and serendipitously, revealed all negative integers.

Distinguishing between order and magnitude of the integers is an important component of what it means to understand the integers and represents perquisite knowledge for integer addition and subtraction (Bofferding, 2014). Through deciding who should play the game first, the children played with the integers as they initiated a discussion about order and magnitude. Alice drew a -4 card; Jace drew a -8 card; and Kim drew a -7 card. The children found themselves in a situation grappling with order versus magnitude during the comparison of three negative integers: -4, -7, -8. Kim stated that -8 was "bigger" than the other numbers because -8 is "more negative"—employing magnitude-based reasoning (Bofferding, 2014). Alice and Jace reasoned that -4 is "highest" and "biggest" because it is closer to 1—employing order-based reasoning (Bofferding, 2014). Language issues of "bigger" and "higher" are also important tenants of the prerequisite knowledge that children need to make sense of as they begin to learning addition and subtraction (Bofferding & Hoffman, 2015).

As a society, we culturally emphasize order over magnitude with integer comparisons. That is, when comparing numbers like -4, -7, and -8, -4 > -8 is expected because of order, -4 is close to zero on the number line or -4 is more to the right on the number line than -8. However, often the work of mathematicians is magnitude based. That is, there are times when -8 is "bigger" than -4. For example, consider two velocity vectors, one with magnitude -8 and another with magnitude -4. The vector with a magnitude -8 would be considered "bigger." Also, this excerpt illustrates the children engaging in play that became an unresolved mathematical problem for them around order and magnitude. Sometimes mathematicians work on problems that are not resolved right away. This is the expected and normative work of mathematicians.

Playing with Integers: Permutations

Throughout the entire session, as the children played the integer game, they determined their total points in the game with the sum of the cards in their hand. Each of the children successfully wrote his or her total points on the recording sheet. However, throughout the entire session, the children would make jokes about having a point total that was different from what they were recording. The children physically moved their cards around on the table in different positions, using only cards with positive integers represented on them, to make "pretend" point totals. The excerpt of transcript below is from the first instance of this type of play in the session.

Alice: I have forty points. (Arranges cards 4 and 0 next to each other to look like 40.)

(Kim continues with game and draws a card.)

Kim: I will just take this one. (Takes cards and writes on recording sheet.)
Alice: Kim has like one hundred.
Kim: Nine.
Alice. Or, eighteen points. (Reaches over and touches Kim's cards, moving the 1 and 8 card next to each other.)

The children continued engaging in the integer play with the stated rules of the game; however, several times during this integer play, the children continued to arrange their positive cards, and notably not their negative cards, into different, "pretend" point totals. Although initiated by Alice, Kim did this later in the integer game play. Kim stated, "I made up thirty-eight and you guys are up in the eight hundreds"—referring to ordering the positive integer cards and notably not writing these point totals down. Alice and Jace participated in making permutations of their cards repeatedly as well. Looking at her hand that consisted of both positive and negative integers, she pulled the cards 0, 4, and 8 out of the hand. Discussing her actual point total, Alice whispered to Jace, "I have twelve. You have two more than me" and continued playfully, "I have eight hundred and four." Jace replied, "I'm going to lose. She has eight hundred and forty"—helping Alice make a larger valued number out of her current permutation.

This excerpt highlights elements of play: spontaneity, different from similar serious behaviors, repeated, creativity, and imagination. Without prompting the children engaged in extra, unplanned mathematics. The children played with the integers by making permutations with their positive integers *spontaneously*—an element of playfulness (Burghardt, 2011). This excerpt is also *different from similar serious behaviors* (Burghardt, 2011); in fact, the children attuned to this difference and did not record these "pretend," permutated scores on their recording sheet. This excerpt is playful because it illustrates the children engaging in an act that was *pleasurable and lighthearted* to them (Burghardt, 2011); the children treated these permutations as pretend scores as they continued with the expected directions of the game and recorded different point scores than they verbally stated with the permutations. The children *repeated* this type of play throughout the session (Burghardt, 2011). This example is also playful because it illustrates the *creative thinking* and *imaginations* of the children (Parks, 2015); they created this play with permutations and imagined larger scores than they actually had based on the rules of the integer game.

The children constructed permutations with the positive integers only. They ordered their positive integer cards, utilized the place value system, and made new point totals from the permutation that would give the largest positive number. The children implicitly recognized that the base-10 system utilizes positive digits in the place value system, rather than negative digits. That is, if you have -1 and -8 cards, they were more than likely not permutated because -1 and -8 are not utilized as digits to make numbers. In order to use the negative cards, the students would have needed to take a negative card and place it first, like -1 and 8 to make -18 or 8 – 1. However, they did not do this. In addition to constructing permutations with the positive integer cards, they reasoned about what permutation provided the largest positive number. In this sense, as the children played with the integers, they also played with the idea of permutations. Although permutations are an important mathematical concept, it is not explicitly needed prerequisite knowledge for the teaching and learning of integer addition and subtraction. This is a consequence of the freedom of play; without prompting, the children engaged in extra mathematics. Although not a mathematical goal of original integer play, the children fearlessly played with integers in a mathematically productive way.

The ways that the children played with the integers in this excerpt mirrors the ways that mathematicians play with numbers as well. Similar to the work of the children in this excerpt, mathematicians engage in recreational mathematics (see, e.g., *Journal of Recreational Mathematics*). Some mathematics is simply for the joy and interest of doing mathematics (e.g., logic puzzles, happy numbers, star tangrams). In fact, often within the domain of recreational mathematics, permutations or combinations with integers are necessary. For example, pentominoes are common puzzles accessible to children but are also the basis for some interesting recreational mathematics (see, e.g., Golomb, 1994; Wessman-Enzinger, 2013). A pentomino is created by permutations of the five unit squares in such a way that each square touches another square on at least one side—creating 12 pentominoes. Some recreational mathematics topics have included creating twin pentomino towers (e.g., stacking pentominoes vertically, creating the same-shaped towers with different pieces). Although the children's play did not directly relate to integer addition and subtraction, the children did play with integers through permutations—a mathematically substantial way linked to the work of mathematicians (see, e.g., Knuth, 2000).

Playing with Integers: Zero

After the children played ten rounds of the game, they were shown various hands of cards from fictitious children. Alice, Jace, and Kim considered these hands of cards, played with their physical cards, and decided what move the fictitious children should make. The children also wrote number sentences for the point totals of the various hands when drawing or discarding cards. As the children attempted to write a number sentence for a hand of cards, Jace posed a question.

Jace: I have a question. Would zero count as a negative number?
Me: Do you think that zero would count as a negative number?
Alice: No.
Kim: Hmm... No.
Jace: Well, it's not a whole number.
Kim: I think it would actually equal both.
Me: You think it would equal both?
Kim: I mean it would be both. (Shakes hand side to side).
Alice: I think it's kind of in the middle.
Jace: Because zero is nothing.
Me: Hmmm.
Jace: And, negative numbers are nothing. But, it doesn't have a negative symbol in front of it.
Alice: Zero's like not a number because it's nothing.
Jace: Well, so is negative numbers.
Kim: (Laughs.)
Alice: Yeah, but they're something.
Jace: My mind is blown.
Alice: (Laughs.)
Kim: Zero is sort of important. It's like the line below the whole numbers to let you know when you are starting the negatives.
Alice: I think the answer for this one (points at the sheet of paper, returning to the trying to write a number sentence for a hand of cards) is five, but I don't get my number sentence.

The children grappled the nature of zero in this excerpt. They initiated a discussion about whether zero is negative or not. In addition to discussing whether zero is negative or not, Alice wondered if zero is not even a number, which then prompted Jace to reflect on the physical embodiment of the integers, stating that "negative numbers are nothing" also. Children often have misconceptions about zero (e.g., Bofferding & Alexander, 2011; Gallardo & Hernández, 2006; Seidelmann, 2004), and making sense of zero as neither a positive nor negative number is important. Recognizing that zero is neither positive nor negative is a component of highlighting the symmetry of the negatives with zero as the center.

This excerpt highlights elements of play: spontaneity, imagination, social engagement, creativity, and stress-free initiation. This excerpt is playful because Jace *spontaneously* asked a question about whether zero is negative, also highlighting his *imaginative thinking* about the integers (Burghardt, 2011). Also illustrating playfulness, the children engaged in *social engagement*, considered Jace's question, and shared their opinions (Parks, 2015). This excerpt is also playful because the children illustrated *creative thinking* (Parks, 2015); they thought that maybe zero was not a number, maybe zero was both positive and negative, or maybe zero was just a number in the middle. Illustrating an *initiation in a stress-free environment*, in a freely chosen discussion, Alice decided to transition from this conversation back to the task of writing a number sentence (Burghardt, 2011).

The ways that the children contemplated the nature of zero in this excerpt mimics the historical struggles mathematicians faced as they made sense of zero as well. Gallardo and Hernández (2006) wrote about this, "Piaget (1960) states that one of the great discoveries in the history of mathematics was the fact that the zero and negatives were converted into numbers" (p. 153). Historically, mathematicians have also grappled with similar ideas about the nature of zero (Kaplan, 1999), and these children did as well through their wonderings of the positivity and negativity of zero.

Discussion

This chapter described both instances of integer play and playing with integers within a specific group session of a teaching experiment on integer addition and subtraction. Describing instances of integer play (e.g., a game with integers) and playing with integers (e.g., contemplating the negativity of zero) that children and students engage in is important in order to facilitate these types of play in the future. Although the descriptions of integer play and playing with integers in this chapter come from a specific instructional experience designed for integer addition and subtraction for Grade 5 students, these instances specify the rich creativity and meaningful mathematics that children play with. Not only do these instances of play highlight robust mathematics of children connected to the work of research mathematics, but integer play is a way to share integer instruction earlier than recommendations, and playing with integers is a way to prolong play in school and can also serve as a way to provide equitable instruction for children.

Integer Play as a Way to Bring Integers to Curriculum Sooner

We are situated in an era where research illustrates that young children are capable of reasoning about integers (e.g., Bofferding, 2014); yet, standards do not suggest instruction with integers until later grades (NGA & CSSSO, 2010), and most curriculum in the USA supports this as well (Whitacre et al., 2011). Illustrating instances of integer play and playing with integers may provide an outlet for bringing thinking and learning with integers to earlier grades. Although it is not novel to suggest integer instruction earlier (see, e.g., Bofferding, 2014), current recommendations currently maintain integer operations in Grade 7. Yet, Bofferding and Hoffman (2015) illustrated that children are capable of engaging with integers, as young as kindergarten, in game play, and this type of game play is productive in developing conceptions of numbers.

Why Integers? Although Grade 5 is not much sooner than recommendations in standards (e.g., NGA & CCSSO, 2010), even playing with integer operations 2 years prior to formal instruction will be beneficial to break generalizations formed by whole numbers (e.g., adding always makes larger, Bofferding & Wessman-Enzinger, 2017). Children are capable of many things, but there should be a focus on integers in elementary school to confront misconceptions of working with only positive integers. As illustrated in both this chapter and entire book, by working with integers, children confront the ideas that:

- Addition does not always make the sum "larger."
- Subtraction does not always make the difference "smaller."
- "Larger" and "smaller" have different meanings with order-based and magnitude-based reasoning when extending beyond positive numbers.
- The number line does not just extend infinitely in only one direction.

Because the physical embodiment of the negative integers is not as natural as the counting numbers (e.g., 1, 2, 3, …) or positive real numbers (e.g., 1/2, 0.4), there is something inherently playful with the integers that is due to its challenging nature compared to other numbers. By engaging in work with integers, children potentially gain a deeper understanding of the number systems they are, by standard recommendations, supposed to learn. As illustrated in this chapter, the children also gain more than that when working with integers—they gain experiences of thinking like a mathematician as they create uses of integer operations, make sense of magnitude- and order-based reasoning, or even make permutations of positive integers.

Yes, we need to teach operations with whole numbers and positive integers and positive rational numbers as the standards recommend. But, is that truly possible when we are potentially generating and establishing deep misconceptions (e.g., subtraction always makes smaller)? Not only do we need to utilize integer play and utilize it sooner than recommendations, but we also need to allow for children to play with integers and examine the ways that children play with integers as they engage in this type of play.

Integer Play and Playing with Integers as a Way to Prolong Play in Schools

Parks (2015) shared the importance of incorporating play beyond early childhood—suggesting that even children in Grades 2 and 3 should have time set aside for play. Featherstone (2000) illustrated in a Grade 3 classroom that the use of negative integers opened a space for imaginative mathematical play in the classroom. The instances presented in this chapter of children *playing with integers* illustrated more elements of play than even in the *integer play* section. As the children played with integers, they enjoyed their creative mathematics, which included extra mathematics than the planned mathematical goals of the game. For example, as the children made their permutations of positive integers, they were joking with each other. They laughed, spoke in silly voices, and did not take their permutated score seriously. As they discarded negative integers and made sense of zero scores, they laughed and teased each other around a ficitious game with pretend scores.

Alice: I have eight thousand and ... (Alice making a joke as she permutated her positive cards.)
Kim: (Takes the card from the center pile and writes on the paper). Oh my god, I will just have to add it. Now I have negative fifteen. Sad day.
Jace: (J flips the center card.) Oh my god!

(laughter)

Jace: (Discards his final card.) I hope you guys are happy. I have nothing. Wait no, I should take that one. Now, Kim you are in second place.
Kim: (Claps hands together) Woot!

When the game ended, the children expressed continued joy about engaging in this play by asking to continue to play.

Jace: We are the champions.
Kim: Do you have another one (holding up a recording sheet)?

This points to a twofold implication centered on prolonged play in school. First, utilizing games in later elementary grades, when typical conventions of play may not have as a prominent of a role, is one way to prolong play in schools. While the use of game play does not necessarily dictate play (e.g., a game on multiplication facts will likely not have the same results), integer play and playing with integers offer enough imagination and challenge to support authentic play. Second, playing with integers effectively engages students in mathematics at a time when many children seem scared of it—providing a space for children to be fearless and creative in mathematics.

Playing with Integers as an Equity Tool

With integer play, teachers determine the play and set the mathematical goals. I, for instance, planned to use an integer game (Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014) and started the group session with predetermined mathematical goals. In the selected excerpts highlighted in this chapter, Jace appeared to conceptualize the integers in these intended ways and explained this reasoning to his peers, Alice and Kim. However, when playing with integers, the students set the agenda and determined what mathematics would be explored. In the integer play, Jace seemed to shine: noticing inverses and differentiating the use of the minus symbol from negative symbol. But, when playing with the integers, other students brought their mathematics to the table. Alice and Kim questioned the role of zero and compared the nuances in order and magnitude, a goal I did not plan.

Playing with integers not only provided opportunity for earlier integer instruction and prolonged play in school but also provided an equitable opportunity for all students to be successful mathematically. Because of the freedom of playing with integers, rather than just integer play, the children entered the play and mathematics in their own way, freely sharing their creative, playful, and valuable ideas—like permutations. Providing space for playing with integers is a pedagogical tool for equitable practices in school mathematics.

Integer Play and Playing with Integers as a Space for Future Research

The children created, invented, and played with the integers—this is the beauty of games. With integer play and playing with integers, there are opportunities for unlimited mathematical experiences—the children in these excerpts played with more mathematics than planned in the intended mathematical goals of the game. As researchers and educators, we want to pick games where this potential for playing with integers is large, and the only way we can know that for sure is by studying them. Then, if additional opportunities for mathematics arise, we can modify the games to encourage it more. For example, a revised version of the game could require that whoever draws the largest card has to go first to encourage more debates about order and magnitude like Alice, Jace, and Kim engaged in.

Conclusion

These instances of integer play highlight that children are capable of thinking about integer addition and subtraction. Through integer play, children encountered opportunities for playing with integers in novel ways. The excerpts of playing with integers illustrate the playful curiosities arising out of integer addition and subtraction that tended to be concepts that we think of as “prerequisite knowledge” (e.g., magnitude or order, sign of zero). Yet, students also began developing integer knowledge that is more nuanced for integer addition and subtraction (e.g., how negatives and positives can “balance” each other) during integer play. Because the children demonstrated capability in solving some integer addition and subtraction problems in this session and throughout the teaching experiment, these examples of integer play and playing with integers highlights that learning about typical prerequisite knowledge (e.g., order, magnitude, use of minus sign) may be developed in tandem with integer addition and subtraction. Furthermore, not only did the children engage in thinking about addition and subtraction of integers, as well as other integer concepts, the children engaged in the work of mathematicians. As children played with the integers and engaged in the work of young mathematicians, they did the thinking and learning most important to integers: imaginative and creative play.

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Chapter 3

Students' Thinking About Integer Open Number Sentences

Jessica P. Bishop, Lisa L. Lamb, Randolph A. Philipp, Ian Whitacre, and Bonnie P. Schappelle

Abstract We share a subset of the 41 underlying strategies that comprise five ways of reasoning about integer addition and subtraction: formal, order-based, analogy-based, computational, and emergent. The examples of the strategies are designed to provide clear comparisons and contrasts to support both teachers and researchers in understanding specific strategies within the ways of reasoning. The ability to categorize strategies into one of five ways of reasoning may enable teachers to organize knowledge of student thinking in ways that are useable and accessible for them and provide researchers with sufficient information about the strategies and ways of reasoning such that they can reliably build on this work.

Imagine how a student might solve the problem -3 + 6 = □. Below we share several responses we heard from K–12 students who participated in our study.

Oscar: Oscar reaches for a provided number line and places his pen at -3. He moves his pen to the right one unit at a time while he counts, "One, two, three, four, five, six." His pen is now at 3 on the number line, and he answers, "Three" (Grade 7).

Alex: "It's like I owe my friend three dollars. And my mom gives me six dollars. I pay my friend three of the dollars that I got from my mom, and I still have three dollars, so my answer is three" (Grade 4).

Cole: "Three. The signs [for -3 and 6] are different, so I subtracted them and took the sign of the bigger number." When asked what he subtracted,

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Cole continued, “I subtracted six minus three, which is three. Six is bigger than three, so I knew the answer had to be positive since six is positive.” When asked why it mattered which number was larger, Cole posed a related problem of -6 + 3. “Look, if it was, uh, like negative six plus three, you still subtract six minus three because they’re different signs. But if six *is negative* [with emphasis], then the answer is negative three” (Grade 7).

Fran-Olga: “I’ll just start by counting. [Fran-Olga moves her lips, presumably counting under her breath.] I don’t know. It’s either negative nine or three.” When asked to explain how she arrived at each answer, Fran-Olga replied, “Well if I go down into the negatives, it’s -4, -5, -6, -7, -8, -9. But if I go the other way, then [it’s] -2, -1, 0, 1, 2, 3. [Long pause] Maybe it’s switched. Wait. When I did three minus five, it [the operation in the problem] was minusing, and this one [this problem] is plussing. I’m thinking that since this one [points to $3 - 5 = \square$] was minus and I was going into the negatives, that this one [points to $-3 + 6 = \square$] goes up. I think it’s three now” (Grade 2).

These responses are representative of the reasoning students across multiple grade levels used when solving open number sentences such as $-3 + 6 = \square$, $5 - \square = 8$, and $\square + -2 = -10$. When students solved these types of problems and shared their responses with us, we found that we could characterize their thinking about integer open number sentences into one of five broad ways of reasoning: *order-based*, *analogy-based*, *computational*, *formal*, and *emergent*. For us, a *way of reasoning (WoR)* about integer addition and subtraction involves a conceptualization of signed numbers in which the student draws on certain affordances or mathematical properties of the underlying conceptualization to engage in integer arithmetic. For example, in using an *order-based WoR*, one draws on the ordered and sequential nature of the set of integers and uses that property to reason about integer addition. We see this approach in both Oscar’s and Fran-Olga’s responses. In contrast, in a *computational WoR,* one treats numbers more abstractly and relies on rules and procedures to solve problems as we see in Cole’s response.

Although we briefly describe the five broad ways of reasoning (for a more detailed description, see Bishop et al., 2014), our goal in this chapter is to share the underlying strategies students used within each WoR about integer addition and subtraction. Our hope is that researchers and teachers will find both the more general ways of reasoning and the specific and detailed strategies useful to better understand students’ approaches to solving open number sentences and to guide future instruction.

Connections to Theory and Building From Existing Research

Our focus is on students’ mathematical thinking in the context of signed numbers, with a particular focus on how children think about integer addition and subtraction. Within mathematics education is a well-established tradition of studying students’ understanding of mathematical topics, including whole-number operations (Carpenter, Fennema, Franke, Levi, & Empson, 2014; Fuson, 1992), fractions (Empson & Levi, 2011; Hackenberg, 2010; Steffe & Olive, 2010), quantitative reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, 2010), limits and infinity (Swinyard & Larsen, 2012; Tall & Vinner, 1981; Williams, 1991), and integers (Bishop et al., 2014; Bofferding, 2014; Peled, 1991). Drawing from a Piagetian tradition, researchers working in this vein are generally interested in "… the way the child reasoned and the difficulties he encountered, the mistakes he made, his reasons for making them, and the methods he came up with in order to get to the right answers" (Piaget, as quoted in an interview with Bringuier, 1980, p. 9). Studies of students' mathematical thinking and cognition are grounded in constructivist theories of knowing and learning, and researchers within this tradition view students' mathematical thinking as important in its own right and distinct from established disciplinary views of a given topic as well as commonplace adult conceptions of mathematical topics.

Although a common constructivist heritage unites research in this tradition, scholars vary in their research designs and data sources (e.g., paired interviews in teaching experiment settings, individual, clinical interviews, or design experiments in classroom settings), their units of analysis (e.g., student reasoning about a particular task or evidence of construction of a particular mental scheme/structure), and the extent to which they incorporate Piagetian constructs such as operations, structures, and interiorization/internalized operations into analyses. In this chapter, we do not analyze students' mathematical thinking by looking for evidence of particular schemes, structures, or mental operations (e.g., levels of units). Instead we document strategies that students use when solving integer addition and subtraction problems. Through these more detailed strategies and their relationships to broader ways of reasoning, we seek to identify, describe, and categorize key features and patterns in students' problem-solving approaches that are general enough to provide a sense of coherence, yet are nuanced enough to sufficiently differentiate among students' solutions. We now turn to the literature base for a brief review of research related to students' conceptions of integers and the specific strategies they bring to bear when solving problems.

Students' struggles operating with negative numbers are well documented (Christou & Vosniadou, 2012; Gallardo, 1995; Kloosterman, 2012; Vlassis, 2002). Whereas Mora and Reck (2004) identified rules and procedures that students attempted to use when solving problems with negative numbers, others have found that children can make productive use of order, leveraging the sequential and ordered nature of numbers, to solve such problems, particularly with number lines (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Bofferding, 2014; Peled, 1991; Peled, Mukhopadhyay, & Resnick, 1989). Those using other lines of research have studied students' use of metaphors (see, e.g., Chaps. 5 and 6) and the efficacy of different contexts when engaging with integers and integer arithmetic (see, e.g., Chaps. 4 and 9). For example, Chiu (2001) identified categories of metaphors that students and experts used when solving integer problems, and Stephan and Akyuz (2012) developed an

Table 3.1 Literature-based problem-solving approaches for integer arithmetic

Problem-solving approach for integer arithmetic	Example/explanation	References
Rules and procedures	Operations with negative numbers are performed using rules, either correctly or incorrectly (e.g., applying rules for multiplication of signed numbers to addition and subtraction problems).	Chiu (2001) and Mora and Reck (2004)
Financial/ transactional context (debt, net worth, etc.)	Operations with negative numbers are related to money or other transactional contexts (giving/receiving) in which negatives are typically associated with debt or owing.	Chiu (2001), Peled and Carraher (2006), and Stephan and Akyuz (2012)
Other oppositional contexts and quantities	Negative numbers are used to represent a quantity of items with an unfavorable connotation (and in opposition to the positive quantity). For example, using two colors of chips or blocks to represent positive and negative numbers.	Chiu (2001) and Peled (1991)
Analogy to whole number	Negative numbers are related to whole numbers when solving integer arithmetic problems (e.g., using the known fact that 5 – 2 = 3 to evaluate the unknown expression -5 – -2).	Human and Murray (1987) and Murray (1985)
Number line, motion/movement	Imposing an ordering on signed numbers or using an existing ordering (as provided in a number line) and reasoning about addition/ subtraction as moving forward and backward.	Chiu (2001), Behrend and Mohs (2005–2006), Bofferding (2014), Murray (1985), Peled et al. (1989), Peled (1991), and Stephan and Akyuz (2012)
Logic	Comparing related problems such as 6 + -2 and 6 + 2 and using a fundamental mathematical property (e.g., inverse operations) to solve the related problem.	Human and Murray (1987)

instructional sequence about financial contexts (with a focus on net worth and incorporating the use of number lines) that positively supported students' understanding of integer addition and subtraction. Further, Murray (1985) and Bishop, Lamb, Philipp, Whitacre, and Schappelle (2016a, 2016b) found that some students could apply logical deductions based on the underlying structure of our number system to solve or explain their reasoning about integer open number sentences. Murray found that students used *logic* to solve problems by comparing a previously solved problem and a related new problem (e.g., 5 + -3 and 5 – -3) to aid in solving the new problem.

Because our focus in this chapter is on students' strategies for integer addition and subtraction, our above synthesis of existing research was also focused on different problem-solving approaches. In Table 3.1, we summarize problem-solving approaches documented in other scholar's work on integers and integer operations along with a brief example and relevant references for each.

Across the literature that documents differing conceptions of integers and integer arithmetic, we see a variety of productive problem-solving approaches. In this chapter we build on this research by describing students' integer strategies, including those in Table 3.1, and organizing those strategies within the broader ways of reasoning. We hope that this expanded framework, which combines both ways of reasoning and strategies, will support teachers to develop and use this knowledge in their instruction.

Conceptual Framework

Our goal within this chapter is to present a more nuanced and complete view of our ways -of -reasoning framework by defining and exemplifying many of the strategies within each WoR. To do so, we briefly describe the broader ways of reasoning into which the more detailed strategies are organized (see also Bishop et al., 2014; Bishop et al., 2016a for previous versions of the ways of reasoning). As mentioned earlier, a way of reasoning (WoR) is a general conceptualization and approach to solving integer addition and subtraction problems that is characterized on the bases of key features of students' solutions and the underlying views of number and operations at work. We identified five ways of reasoning that students across all participant groups in our study used when solving open number sentences: order-based, analogy-based, computational, formal, and emergent. (In earlier publications, we used different names for analogy-based and emergent reasoning, referring to analogy-based as magnitude and emergent as developmental or limited.) In Table 3.2 we define each WoR.

In the responses shared in the introduction to this chapter, four of the five ways of reasoning are represented. Alex's comparison of negative numbers to debt is an example of *analogy-based* reasoning, whereas Cole used *computational* reasoning when he invoked rules and properties in his solution. Oscar and Fran-Olga used *order-based* reasoning by ordering spoken number words and their corresponding written symbols to determine what was before and after a given number and then using these sequences to solve the problem. Fran-Olga's response also reflects a *formal* way of reasoning: In her explanation she compared the operations of addition and subtraction and used her informal understanding of inverses to argue that her answer was a necessary consequence of the relation between addition and subtraction and her assumptions from a previous problem. Within each *WoR* we wanted to identify specific and detailed strategies students brought to bear on each task (e.g., counting as a particular instantiation of the *order-based WoR* seen in Fran-Olga's response or the use of a number line as seen in Oscar's *order-based WoR*). A *strategy* is a subcategory of a particular *WoR* that further describes and differentiates student responses within the broader *WoR*. We view the five ways of reasoning as an organizing structure into which we can categorize more detailed strategies on the basis of the underlying views of number and operation leveraged in a given strategy's use. Given this view, the research questions guiding our study were the following: What strategies do students use when solving open number sentences with integers, and what is the relation among strategies and the broader ways of reasoning?

Table 3.2 Ways of reasoning

Ways of reasoning	Definition
Order-based	In this way of reasoning, one leverages the sequential and ordered nature of numbers to reason about a problem. Strategies include use of the number line with motion as well as counting forward or backward by 1s or another incrementing amount.
Analogy-based	This way of reasoning is characterized by relating numbers and, in particular, signed numbers, to another idea, concept, or object and reasoning about negative numbers on the basis of behaviors observed in this other concept. At times, signed numbers may be related to contexts (e.g., debt or digging holes). Analogy-based reasoning is often tied to ideas about cardinality and understanding a number as having magnitude.
Formal	In this way of reasoning, signed numbers are treated as formal objects that exist in a system and are subject to mathematical principles that govern behavior. Students may leverage the ideas of structural similarity, well-defined expressions, the structure of our number system, and fundamental principles (such as the field properties). This way of reasoning includes generalizing beyond a specific case by making a comparison to another, known, problem and appropriately adjusting one's heuristic so that the logic of the approach remains consistent, or generalizing beyond a specific case to apply properties of classes of numbers, such as generalizations about zero.
Computational	In this way of reasoning, one uses a procedure, rule, or calculation to arrive at an answer. For example, some students used a rule to change the operation of a given problem along with the corresponding sign of the subtrahend or second addend (i.e., changing 6 – -2 to 6 + 2 or 5 + -7 to 5 – 7). Students often explained these changes by referring to rules like "Keep Change Change" (keep the sign of the first quantity, change the operation, and change the sign of the second quantity). For a strategy to be placed into this category, the student may state a procedure or rule with or without sharing a justification.
Emergent	This category of reasoning often reflects preliminary attempts to compute with signed numbers. For many strategies in this category, the domain of possible solutions is locally restricted to nonnegatives. For example, a child may overgeneralize that addition always makes larger and, as a result, claim that a problem for which the sum is less than one of the addends (6 + □ = 4) has no answer. The domain of possible solutions appears to be restricted to natural numbers, and the effect (or possible effect) of adding a negative number is not considered.

Study Background and Methods

Participants

The data and findings we share are from a larger program of research wherein we investigated student's' conceptions of integers and integer operations across multiple grade levels. We interviewed 160 students at 11 schools in five districts in a large urban city in the Southwestern United States. Forty students at each of Grades 2, 4, 7, and 11 were randomly selected from students who returned consent forms. We chose these grades to provide a cross-sectional view of integer reasoning at different grade levels. The second and fourth graders provided insight into children's thinking before school instruction; the seventh graders reflected students' thinking immediately after school instruction on integers; and the 11th graders were chosen to represent the endpoint of students' integer reasoning in the K–12 setting.[1] During the interviews we noticed that some of our elementary-grade participants had knowledge of integers, whereas others did not. Consequently, we reorganized our second- and fourth-grade participant groups for analysis purposes. All students were placed into one of four groups: college-track ([CT], $n = 40$, eleventh-grade students), post instruction ([PI], $n = 40$, seventh-grade students who had recently completed instruction in integers), before instruction, with negatives ([BIN], $n = 39$, second and fourth graders with knowledge of negatives), and no evidence of negatives ([NEN], $n = 41$, second and fourth graders without knowledge of negatives). Group placements for second and fourth graders were made on the basis of responses to the introductory questions in the interview (see questions 1–4 in the Appendix).

Problem-Solving Interview

As part of the larger study, we developed, piloted, and revised a problem-solving interview over a period of 2 years. In addition to the 160 participants described in the previous section, we conducted pilot interviews with an additional 90 K–12 students across four interview cycles, each of which was focused on different grade levels of students (i.e., the first interview cycle targeted K–2 students, the second cycle targeted high school students, the third Grades 3–5, and the last cycle focused on middle school students). In each interview cycle, we tested new tasks and continued to refine the sequencing and phrasing of existing tasks to identify tasks likely to elicit students' integer reasoning.

Drawing from Piaget's method of clinical interviewing (Ginsburg, 1997), our initial goal with the interviews was to balance flexibility and standardization. Piaget described his approach as follows:

> You ask, you select, you fix the questions in advance. How can we, with our adult minds, know what will be interesting? If you follow the child wherever his answers lead spontaneously, instead of guiding him with preplanned questions, you can find out something new. … Of course there are three or four questions we always ask, but beyond that we can explore the whole area instead of sticking to fixed questions. (Piaget, as cited by Bringuier, 1980, p. 24)

Our pilot interviews were consistent with Piaget's description of his method. For example, it was not initially apparent to us that the open number sentences $-3 + 6 = \square$ and $6 + -3 = \square$ might encourage different reasoning; as adult experts,

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[1] Note that we restricted our eleventh-grade participants to college-track students, that is, students who were enrolled in either calculus or precalculus during their eleventh-grade year. Our goal with the eleventh-grade students was to identify the best-case scenario for integer understanding when students finish their high school education.

we viewed -3 + 6 and 6 + -3 as equivalent because of the commutative property of addition. As we discovered in the pilot interviews, the location of -3 as the first or second addend influenced some students' approaches to the problem. Similarly, we were surprised when the seemingly similar problems of 6 – -2 = ☐ and -5 – -3 = ☐ (both involve subtracting a negative quantity) yielded widely differing responses from some children. One of our goals in conducting the pilot interviews was to pursue and uncover these differences in reasoning and the underlying conceptions from which they emerged. As a result, we routinely posed follow-up tasks to confirm or refute our working hypotheses about the ways students were reasoning and what features made problems more and less difficult. Because we customized follow-up questions on the basis of the specifics of the student's responses in the moment, they were not preplanned or standardized.

However, from the beginning of the project, our goal was to develop a standardized set of questions that would be posed to all students in the main study and from which we could compare students' reasoning within and across grade levels. In early 2011 we finalized the problem-solving interview and began conducting 160 interviews across the participant groups described earlier using a standardized set of questions. The one-on-one interview lasted 60–90 minutes and consisted of 56 total problems including introductory questions, open number sentences, context problems, comparison problems, and tasks involving variables and algebra (see Appendix for the complete interview). We found that solving open number sentences provided productive opportunities for students to reason about signed numbers; consequently, the analyses and findings we share in this chapter are based on the 25 open number sentences posed to students. (These open number sentences are questions 9–14, 16–28, and 30–35 in the interview shared in the Appendix.)

Analysis

For each open number sentence in each interview, we assigned a code for correctness and a code for the strategy (or strategies) the student used when solving the given problem. Each strategy was subsumed in one of the five ways of reasoning. For some problems, students used multiple ways of reasoning and, therefore, received more than one *WoR* code. Across all ways of reasoning, we identified a total of 41 strategies.[2] We developed our set of codes for the 41 strategies (and the 5 ways of reasoning) iteratively over a 3-year period. Moreover, this set of codes comprehensively captures the strategies students in our study used. Although some of the strategy codes we created are documented in existing literature (e.g., logic, use of number lines, converting to context), we did not use these codes a priori. Instead, we used a grounded theory approach to analysis so that our codes emerged from the student responses in the interviews (Corbin & Strauss, 2008).

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[2] Two of these strategies, *unclear* (assigned when a strategy was not clear) and *other* (assigned when a student's response did not match an existing *WoR* or strategy), were used rarely and were not associated with one particular *WoR*. Although we include *unclear* and *other* as strategies, they are not subcategories of any single *WoR*.

Findings

In the text that follows, we expand the ways of reasoning framework by identifying and exemplifying the most common strategies within each *WoR*. We hope our categorizations will help readers to identify important differences and similarities among strategies and recognize the complexity and richness of students' thinking about integer addition and subtraction.

Common Strategies Within Ways of Reasoning

Across all ways of reasoning, we identified 41 total strategies. Table 3.3 identifies the most frequently invoked strategies within each *WoR* along with their overall percentage use. In general, we share the three most common strategies within each *WoR* in the following sections.[3]

Table 3.3 Examples of strategies within ways of reasoning and frequency of use

Way of reasoning	Strategy examples	Percentage use (of total problems posed)
Order-based	Number line	16.21%
	Jumping to zero	3.73%
	Counting by ones	3.26%
Analogy-based	Negatives like positives	6.82%
	Converts to context	3.32%
Computational	Keep change change	16.21%
	Negative sign subtractive	7.86%
	Changes order of terms	6.67%
	Equation	6.39%
	Same signs/different signs	3.73%
Formal	Infers sign	8.41%
	Generalization about zero/additive inverses	2.63%
	Logical necessity	<1%
Emergent	Addition makes larger/subtraction makes smaller	14.01%
	Ignores sign	10.78%
	Pascal	1.18%

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[3] Because computational was the most frequently used *WoR*, we share more of the strategies within this *WoR*, and because the strategies other than *negatives like positives* and *converts to context* within *analogy-based WoR* were used so infrequently, we share only those two.

Order-Based *Order-based reasoning* was used on about one fourth of all problems posed. In this *WoR*, one leverages the sequential and ordered nature of numbers to reason about a problem. The most common strategies within this *WoR* were the *number line/motion* strategy, the *jumping to zero* strategy, and the *counting by ones* strategy. The *number line/motion* strategy was the most common within the *order-based WoR*, used on 16% of the problems posed. When using the *number line/motion* strategy, students treated the first addend and the sum (or the minuend and difference) as locations on the number line and the second addend (or subtrahend) as the number to move. The operations usually determined the direction of movement. To receive this code, students had to either explicitly use motion on a number line or share that they imagined moving on a number line when solving. For example, for the problem shared in the introduction, $-3 + 6 = \square$, Oscar's strategy exemplifies *number line/motion*. His starting point is -3, the operation of addition indicates movement to the right, the second addend indicates the number to move, and the unknown is the ending location.

Another relatively common strategy demonstrating an *order-based WoR* that students used was *counting by ones*. Ellie, a second-grade student, counted up by ones to solve $-3 + 6 = \square$ and from -2 to 4 to solve the problem $-2 + \square = 4$. She counted aloud saying, “Minus 2, minus 1 (raises one finger), zero (raises another finger).” She paused. “Wait, I lost count.” Ellie then restarted her count, “Minus 2, minus 1 (raises one finger), zero (raises second finger), 1 (raises third finger), 2 (raises fourth finger), 3 (raises fifth finger), 4 (raises sixth finger).” Ellie's final answer was 6. We conjecture that Ellie's pause and restart (“I lost count”) may indicate the additional cognitive demand required to begin her counting sequence with a negative number rather than a natural number. But her ability to successfully extend her counting sequence may be attributable to the fact that the direction of her counting was consistent with the addition of natural numbers (addition makes larger and thus one counts up toward the positive numbers to arrive at a sum). When, for example, Ellie solved the problem $-5 + -1 = \square$, she adopted a different strategy and incorrectly answered -4; she may have abandoned a counting strategy for this problem because adding -1 to -5 would indicate a movement left on the number line for addition, or movement in the opposite direction than one would move with natural numbers.

The last strategy within the *order-based WoR* we share here is *jumping to zero,* which was used on just more than 3% of the problems posed. Opal's response to the problem $-3 + 6 = \square$ exemplifies this strategy. Opal answered, “Three, “and then explained saying, “Half of 6 is 3, so then that would bring it [the running total] to the 0. And 3 more would bring it to the 3. And that would equal 6.” Opal's strategy can be represented mathematically with the following series of equivalent expressions: $-3 + 6 = -3 + (3 + 3) = (-3 + 3) + 3 = 0 + 3$. By decomposing 6 into 3 plus 3, Opal was able to “jump to zero” by adding one of the 3s to -3. In general, this strategy involves strategically decomposing a number to obtain additive inverses so that the resultant partial sum is zero. However, students are unlikely to recognize either the underlying mathematical property they are implicitly using or its significance. We conjecture that Opal was treating zero like other decade numbers (e.g., 10 and 20) and using her knowledge of decomposition and incrementing to reach a *friendly* number as part of her computation. We believe that this type of order-based reasoning can be leveraged to formalize and explicitly name the concept of additive inverses that is at work in this strategy and encourage its continued use as appropriate.

Analogy-Based This *WoR* is characterized by relating signed numbers to another idea, concept, or object, often countable amounts or quantities, and reasoning about signed numbers on the basis of behaviors observed in this other concept. We named this *WoR analogy-based* because students created an analogy between signed numbers and some other concept. *Analogy-based reasoning* was used on about 13% of all problems posed.

Students compared negative numbers to positive numbers using a strategy we named *negatives like positives*, on about 7% of all problems. This strategy involves computing with negative numbers through explicit comparison to computing with positive numbers. This strategy was used productively across all grade levels. Consider Ricardo's (Grade 11) response to -5 + -1 = □: "Negative five plus negative one equals negative six. I thought about this by changing this whole thing into a positive. So I just ignored the negatives for a little bit. So I knew five plus one equals six. But since it was negative, I added the negative after." When asked if changing the problem into a positive always worked, he replied, "So like this problem was applicable to change it to a positive since there were two negative numbers. But if you had like a negative and a positive, then that would be different." Ricardo was an 11th grader, but we also had many younger students who used this strategy. As an example, consider Jacob's (Grade 1) strategy for solving -7 – □ = -5. "Well for this one I need little cubes. … It would be like real numbers, but you just add the minus sign. You just do seven plus, well actually, seven minus two equals five. That's the answer for real numbers, so I just added a negative to all of them, and there is my answer." In these examples, we see that both Ricardo and Jacob compared the mathematical behavior of negative numbers to the behavior of positive numbers (or "real" numbers in Jacob's case) to solve problems involving the addition or subtraction of two negative integers.

Students also explicitly related signed numbers to contexts (e.g., debt or digging holes) on about 3% of all problems posed. Central to the *converts to context* strategy is that students used a context such as debt, digging holes, or bad guys that they deemed as related to negative numbers. As an example, consider Alex's solution to the problem in the introduction, -3 + 6 = □, in which he interpreted -3 as representing a debt of $3 and 6 as gaining $6 from his mother. After taking $3 from the money he was given to repay his debt, he had $3 left. Another use of *converts to context* was relating signed numbers to digging and refilling holes. For example, Sawyer explained his answer of -3 to the problem -5 + □ = -8 by relating operations with negative numbers to digging and burying (his word for *refilling*) holes. For this problem he started with a hole five units deep: "Okay, if it [the unknown] would have been positive three, it would have canceled out; it would have buried some of the hole. [Instead] it's like we are digging a deeper hole and trying to get to negative eight.” He applied the same context to think about the problem -2 + □ = 4: “We start from negative two, and so it’s like a hole and you need to fill it in.” For Sawyer, the signs of the starting and ending numbers indicated whether he had a hole or a mound of dirt. He related the unknown in this problem to the action of filling in or burying the hole so that the result was a pile of dirt above ground.

Formal In a *formal WoR,* students treat negative numbers as formal objects that exist in a mathematical system and are subject to fundamental mathematical principles that govern their behavior. Students may generalize beyond a specific case to apply properties of classes of numbers or leverage underlying structures of our number system to make conjectures about which properties hold and do not hold upon successive extensions. *Formal reasoning* was used on just fewer than 12% of all problems posed.

The most common strategy within the *formal WoR*, *infers sign,* used on about 8% of all problems, involves examining the structural features of the problem—the operation in conjunction with the signs of the given numbers—to determine the sign of the answer prior to determining its magnitude. As an example of *infers sign*, consider Jane’s thinking when solving the open number sentence □ + 6 = 2. “Um, now we’re trying to find, we know the number has to be negative. … The number that we’re actually adding by [six], it’s more than the actual, than our answer [two]. … So it has to be negative. So then if you know basic subtraction and addition, you know six minus what equal two. So it’d be four. … And it’d be negative four.” Before she identified the magnitude of the unknown, Jane first determined the sign of the unknown by considering the operation, the signs of the given numbers, and their relative magnitudes. We considered this strategy to be a *formal WoR* because Jane is essentially making a claim about a class of problems—addition problems such that the sum is smaller than an addend (or, in other cases, subtraction problems such that the difference is greater than the minuend, like 5 – □ = 8).

Sometimes students made generalizations that explicitly referenced the idea of additive inverses or the fact that the difference between any number and itself is zero. When a student invoked a general principle that $a - a = 0$ or $a + (-a) = 0$ (for $a \in Z$), we assigned the code *generalization about 0/additive inverses*. (Although we combine these strategies in our discussion here, we recognize important distinctions in them.) When using the *generalization about 0* strategy, students needed to indicate that the given problem was an instantiation of the generalization that any number minus itself is 0. One of our fourth-grade students, David, used this strategy when explaining how he thought about -5 – -5 = □: “I know that any number subtract itself is zero.” Because his language suggests that this is a general property and not true for just these particular numbers, we assigned the *generalization about 0* code to David’s response.

Although the *additive inverses* strategy is related to *generalization about 0*, when using the *additive inverses* strategy, the student needed to explicitly mention three aspects we deemed critical to understanding additive inverses deeply: (a) the relation between *a* and -*a* (i.e., that they are inverses or opposites), (b) that the quantities are “canceling” (i.e., specify the importance of the operation of addition for additive inverses and the identity element of 0), and (c) that this claim is not specific to the numbers in the problem but is a generalization. For example, when solving the open number sentence $3 + \square = 0$, Belinda (an 11th grader) explained her answer of -3 saying, “I know that the opposite of three is negative three. And whenever you add things that are the same number but with different signs, positive or negative, it equals zero.” Belinda identified the inverse relation between three and negative three describing them as opposites and also specified the operation and identity element involved (addition and zero). We interpreted her use of “whenever,” the indexical noun “thing,” and the second-person pronoun of “you” to indicate that Belinda was generalizing beyond the specific numbers given in the problem. Similarly, consider Kate’s response to the problem $-8 + \square = 0$. She reasoned that, “If it [the sum] is going to equal zero, the way to cancel the eight out is to have the same number but have it in negative form.” If Kate had stopped there with her explanation, she would not have received the *additive inverses* code. Although she alludes to the inverse relationship, identifies the importance of zero as the identity element, and seems to be moving toward a generalization with the phrase “same number in negative form,” it’s not clear how the canceling occurs. Critically, for us, the operation of addition had not yet been mentioned.[4] However, Kate did continue her explanation. “Because the same number on opposite sides of zero cancel each other out when you add them.” In her last sentence, Kate indicated the importance of the operation of addition, and her language was more clearly generalized.

Another strategy within the *formal WoR*, *logical necessity,* was invoked infrequently but has promise for supporting powerful mathematical ideas. In the introduction, Fran-Olga used *logical necessity* in her response to $-3 + 6 = \square$. She was unsure which way to count (an *order-based WoR*) and considered answers of -9 and 3. After comparing the expressions -3 + 6 and 3 – 5, Fran-Olga settled on an answer of three. Because, on an earlier subtraction problem of 3 – 5, she had counted down “into the negatives,” then for a problem that involved “plussing,” Fran-Olga concluded she needed to count up. The key aspect of her reasoning was that “plussing” and “minusing” are inverse operations: If minusing goes down, then plussing goes up. Fran-Olga knew that addition and subtraction behaved oppositely in operating with whole numbers. She conjectured that the operations would still behave oppositely upon extension to the set of integers. In *logical necessity*, a student makes a comparison to another, known, problem and appropriately adjusts his or her reasoning so that the underlying logic of the system and the approach remain consistent; in this example, Fran-Olga maintained consistency with what she knew to be true for whole numbers. (We share an extensive examination of *logical necessity* in Bishop et al., 2016a, 2016b).

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[4] Instead, Kate would have been assigned the strategy code, *magnitude*, which falls in the *analogy-based WoR* category. *Magnitude* strategies were used when students’ responses indicated that they viewed a negative quantity as having magnitude, which enabled negative quantities to “cancel” an oppositional, positive quantity. Sometimes the “canceling” language was used when students used different colored chips to model and solve a problem. In these situations, another *analogy-based* strategy of *chips* was assigned as opposed to the *magnitude* code.

Computational A strategy coded as a *computational WoR* was based on a procedure, rule, or calculation. Because the most common WoR was computational, employed on about 40% of all problems posed, we share more strategies with this WoR to highlight the variety of computational strategies students in our study used. *KCC*, the most prevalent rule, is so named because many students shared the mnemonic Keep Change Change to indicate that they **K**eep the sign of the first number, **C**hange the operation, and **C**hange the sign of the second number. *KCC* was the most common strategy code across all ways of reasoning, used on about 16% of all problems.[5] The key feature of *KCC* is that the operation and second addend (or subtrahend) in the original expression are both changed to their opposites. In most instances, students referred to a mnemonic like KCC, boom boom, or the double stick trick when invoking this rule. But some students simply used the rule absent an accompanying memory aid, stating something like, "When a negative and a minus sign are together, they count as an addition." In both cases, the response was assigned the *KCC* strategy code. We exemplify this strategy in the following two responses and highlight the difficulty students typically had when asked to justify the validity of this rule. Gabriel, an 11th grader, invoked a mnemonic while solving the problem 5 – □ = 8.

Gabriel: Negative three. Boom boom [writes -3 in the blank and when he states, "Boom boom," he draws two vertical line segments, one through the subtraction sign and one through the negative sign in -3].
Interviewer: Okay. So where does that boom boom come from? What was that?
Gabriel: It's magic.
Interviewer: Tell me a little bit more about the boom boom.
Gabriel: I remember learning in sixth grade or something, when you subtract a negative, you just do boom boom. And you add it I guess.
Interviewer: Okay, and why does that work?
Gabriel: Newton's third law—I don't know. Because you're taking away something that's negative? [Rising intonation]. Uh. [15-second pause] It just works.

Bea also gave an answer of -3 to the problem 5 – □ = 8: "Just because, negative three, then I do the double stick trick. There is a minus [and a] negative so you add." When asked what the "double stick trick" was, Bea clarified, "Okay, when you have a subtraction sign [points to subtraction symbol in the expression 5 – -3 = 8] and then a negative number [points to negative sign for -3], they call it a double stick trick when you do this. [She draws two vertical lines, one through the subtraction sign the other through the negative sign in -3 so that the expression 5 – -3 = 8 is transformed into 5 + +3 = 8.] And so five plus three is eight."

The next two most common computational strategies—*negative sign subtractive* and *changes order of terms*—were used on roughly 8% and 7% of problems posed,

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[5] The percentage use of 16% was driven by the CT students, who used *KCC* on 31.40% of all problems posed, sometimes in conjunction with another *WoR*.

respectively. Claire's response to the problem -3 + 6 = □ reflects both of these strategies. "It's three. I know that six minus three is three. I just changed the order of the numbers and since three is negative, I subtracted." The interviewer pressed Claire, saying, "But the problem was negative three *plus* six. You *subtracted* and started with six instead of negative three." Claire again reiterated, "I just changed the order of the numbers and since three is negative, I subtracted." The interviewer continued, "Okay. When you changed the order of the numbers, I'm curious if you thought of the problem as six plus negative three, and then changed to subtraction? Or when you switched it, if you immediately thought of the problem as six minus three." Claire responded, "I immediately thought of it as a subtraction problem." In the strategy of *negative sign subtractive*, students indicate that the negative sign in the written symbolic form of a negative number, the - in -3, indicates the process of subtraction. Instead of being viewed as a quantity or mathematical object in its own right, -3 is understood as a quantity *to be subtracted*. Thus, Claire interpreted -3 to mean "subtract three." This strategy, which was used in all participant groups in our study, was one of the earlier historical conceptions mathematicians had for negative numbers (see Henley, 1999, for a discussion of "subtractive numbers"). In particular, almost two thirds of the college-track students used this strategy to solve the problem -3 + 6 = □ by *subtracting* three from six.

Similar to most college-track students, Claire responded to the problem -3 + 6 = □ by using *negative sign subtractive* simultaneously and in combination with *changes order of terms* to transform the original expression of -3 + 6 to the equivalent expression of 6 – 3. Claire was clear that she did not use the following sequence of transformations: -3 + 6 → 6 + -3 → 6 – 3, but instead went straight to the last expression. Her response exemplifies *changes order of terms* because she essentially applied the commutative property of addition to change the order of the addends, but she simultaneously changed what was an addition problem to a subtraction problem by interpreting -3 as subtractive, which is why her response was also assigned the *negative sign subtractive* strategy code. The college-track students were especially fluent, but almost always implicit, when changing the meaning of the minus sign from a negative number to subtraction.

Sometimes students added or subtracted a number to both sides of the open number sentence to "isolate the box." We named this strategy *equation* because students used *properties of equality* often associated with school-based instruction for solving one- and two-step equations. For example, Belinda's explanation for her solution to 6 + □ = 4 was "I just subtracted six from both sides and got negative two." Many students explained that they had to "do the same thing to both sides," and some students insisted on rewriting the number sentences so that the box was replaced with a variable (i.e., 6 + □ = 4 was rewritten as 6 + x = 4).

The last computational strategy we share was named the *same signs/different signs rule*, and it was used on just fewer than 4% of problem responses. This is a rule that applies only to addition problems, though we saw many students apply it incorrectly to subtraction problems. The *same signs/different signs rule* can be stated as follows: If the signs of the addends are the same, add their magnitudes, and keep the sign for the sum. If the signs are of the addends are different, find the difference of their magnitudes, and the difference should take the sign of the number with the larger magnitude. Cole's response to 6 + -3 = □, shown in the introduction, is an example of this strategy. Because -3 and 6 had opposite signs, he subtracted three from six and assigned to that difference the sign of the addend with the larger magnitude, 6, which was positive. One student we interviewed recited a song to help her remember this rule (to the tune of *Row, Row, Row Your Boat*): "Same signs, add and keep. Different signs, subtract. Take the sign of the larger one, then you'll be exact."

Emergent The *emergent WoR* reflects students' initial attempts to compute with signed numbers. We chose the name *emergent* because many of the strategies students used in this *WoR* were not only sensible but with appropriate support could provide a strong foundation for integer reasoning from which more sophisticated strategies and ways of reasoning could emerge. Some students who had not yet heard of negative numbers ignored the negative sign or treated it as a subtraction symbol. Other students sometimes selectively restricted the domain of possible solutions to nonnegatives. Overall, *emergent* reasoning was used on about one third of all problems posed. The most common such strategy was *addition makes larger/subtraction makes smaller (AML/SMS),* used on 14% of all problems posed.[6] The *AML/SMS* strategy stems from the overgeneralizations that addition always makes larger and subtraction always makes smaller and is related to conceptualizations of addition and subtraction as increasing and decreasing the cardinality of a set (Bishop et al., 2011; Bishop et al., 2014). For example, consider Oscar's response when solving 5 – □ = 8. "Cuz, this [points to 8 in the written problem] is bigger than that [points to 5]. And if you minus three, if that [points to the minus sign] was a plus, um, it would be possible. … You couldn't take away, fff, fi, three out of five to equal eight 'cuz it would just equal two." Oscar then wrote "No" in the box. Ryan, too, used the *subtraction makes smaller* strategy for the same problem saying, "I wouldn't be able to do it because it would always be behind eight if it was minus something. Because if it was minus zero it would be five. It [the difference] would always be behind eight." Although both of these students had heard of negative numbers, they appeared to restrict the domain of possible solutions to whole numbers and did not consider the effect (or possible effect) of subtracting a negative number.

The second most common strategy within *emergent* reasoning was *ignores sign*. In this strategy students either ignore the negative sign throughout and treat it as though it does not exist or they *initially* ignore the negative sign and then account for it *after* finding a solution. The strategy *ignores sign* was used in just fewer than 11% of the problems posed and was mainly driven by second- and fourth-grade students in our study. Dahlia, a second grader, ignored the negative signs when solving -5 + -1 = □ and treated -5 and -1 as if they were whole numbers. She read the problem aloud as "Five plus one" and immediately answered six. Dahlia then

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[6] This percentage was driven by the BIN and NEN students, who used *AML/SMS* on 27.21% and 32.44% of all problems posed, respectively.

demonstrated the fact on her fingers saying, "Five (she held out five fingers on one hand) plus one (she held out her thumb on her other hand) would equal six." In contrast, Javier read the open number sentence □ – 5 = -1 as "Box minus five equals negative one." He initially wrote 6 in the box and then revised his answer to -6. He explained, "Six minus five equals one. So I used negative six minus five so it could be negative one." Javier appeared to initially ignore the negative in -1 and solve instead the related number sentence of □ – 5 = 1. When asked why the 6 was negative, he replied, "Because I, because if I don't have a negative and I subtract minus five, I won't be able to have negative one." Javier reasoned that for the difference to be negative one as opposed to one, the unknown needed to be negative. Thus, he assigned a negative sign to the unknown on the basis of the absence or presence of other negative numbers in the problem. Moreover, how Javier interpreted or made sense of signed numbers is unclear; he may have attended only to surface features embedded in the symbolization of these numbers.

Another strategy in the *emergent WoR* was used for open number sentences in which the magnitude of the subtrahend was larger than the magnitude of the minuend (e.g., 3 – 5 = □, -2 – 7 = □, -7 – -9 = □). Students often declared that these problems were "not possible" to solve or gave an answer of zero. Consider Sam's response to the problem 3 – 5 = □: "Three minus five is zero because you have three and you can't take away five. So take away the three, and it leaves you with zero." (When asked to solve 3 – 4 = □ and 3 – 3 = □, Sam answered 0 to both.) Similarly, Andrew was puzzled by the same task and said that solving 3 – 5 = □ was "not possible." He shared his thinking, saying, "How come there's three and take away five? I don't have enough. 'Cuz look there's three (holding up three fingers) and I cannot take away five 'cuz there's not enough." We named this strategy *Pascal* for the mathematician and philosopher Blaise Pascal who gave a response not unlike Sam's. In his collection of unpublished philosophical and religious writings entitled *Pensées*, Pascal stated, "I know some who cannot understand that to take four from nothing leaves nothing" (1669/1941, p. 25).

Discussion and Implications

Because of their documented effectiveness in supporting students' learning, frameworks of students' mathematical thinking are deeply rooted in mathematics education research (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). In this chapter, we have contributed a descriptive framework for organizing and making sense of students' problem-solving strategies by relating them to the broader ways of reasoning about integer addition and subtraction. By combining strategies and ways of reasoning in our framework, we distinguish key details of student thinking in a way that provides organization and structure to student thinking in the realm of integers. Knowledge of specific strategies is beneficial because it can help teachers recognize and encourage the use of multiple, appropriate strategies and build toward more sophisticated strategies both within and across ways of reasoning (e.g., counting vs. jumping to zero vs. additive inverses). This knowledge also helps teachers to support students to select and use efficient strategies that are based on key features of problems. For example, we found that students often use *jumping to zero* (or another *order-based WoR strategy*) for problems like -5 + ☐ = 3 (an addition problem starting with a negative quantity, ending with a positive quantity, and with an unknown, positive, change value). (See Lamb, Bishop, Philipp, Whitacre, & Schappelle, 2017, for a discussion of integer addition and subtraction problem types and their relation to ways of reasoning.) And finally, the knowledge to differentiate multiple instantiations of a specific *WoR* (i.e., differentiating strategies within a *WoR*) can support the identification of common characteristics that unite those strategies within the *WoR*.

As discussed in the beginning of this chapter (see Table 3.1), researchers investigating the teaching, learning, and historical development of signed numbers have contributed studies and descriptions of students' thinking about integers that are consistent with both our broader ways of reasoning and many of the strategies we documented in this chapter (Bofferding, 2014; Chiu, 2001; Murray, 1985; Peled, 1991; Stephan & Akyuz, 2012). We extend this work by organizing key distinctions and patterns in children's solutions into a coherent framework that leverages the broader ways of reasoning as its central organizing feature.

Connecting Key Mathematics to Student Strategies

We believe the ways of reasoning framework holds promise for teachers because it can support their abilities to assess and interpret student thinking in the moment. Moreover, the strategies students use draw on important mathematical ideas. Therefore, knowing and recognizing differences among students' ways of reasoning and strategies as well as the underlying mathematical ideas embedded in specific strategies is important pedagogical content knowledge for teachers. However, the mathematical ideas in students' strategies are often unstated, unclear, or implicit, and teachers can experience difficulty in eliciting those ideas from students. We note that though we have selected the examples in this chapter for their clarity, student thinking is not always complete or clearly articulated; thus, in practice, asking probing questions to help elicit student thinking and connect student-generated strategies to underlying mathematical ideas is helpful. Further, at times, students may be unaware of the strategies they used. Providing students opportunities to regularly share their thinking may have multiple benefits. Students can become both more able to meaningfully communicate their mathematical ideas and more aware of the strategies that they actually used. Additionally, by making their own strategies more explicit to themselves, students may be able to use those strategies for problems with similar structure. In the following sections, we return to several strategies discussed earlier in the chapter, identify the key mathematical ideas embedded in these strategies, and offer suggestions for teachers to explore and make connections to those mathematical ideas.

Jumping to Zero and Additive Inverses We view the strategy *jumping to zero*, which is *order-based*, as significant for two reasons. First, students who jump to zero may recognize that decomposing numbers to get to a friendly number (in this case, 0) enables them to solve problems more efficiently than does counting by ones. Second, we suspect that the use of *jumping to zero* may be an important precursor to reasoning more formally about additive inverses—that is, using the *additive inverse* strategy in the *formal WoR*. For example, after sharing her strategy to $-3 + 6 = \square$, Opal and her classmates might be asked to consider the relation between 3 and -3 and what it means to be *opposites*. These types of conversations could support students to generalize the specific instantiation of the property $-3 + 3 = 0 = 3 + (-3)$ to all integers. In this case, we envision using the initial *order-based* reasoning to develop *formal* reasoning.

AML/SMS and Infers Sign In related work (Lamb et al., 2017), we shared how kernels of inferring the sign are present in *AML/SMS* strategies. We reiterate here that we believe that some strategies within the *emergent WoR* provide productive starting points for students' learning about negative numbers. For example, students who express *AML/SMS* strategies provide evidence that they have noticed features of the number system with which they have heretofore engaged, and thus they have recognized the underlying structure of addition and subtraction in the domain of natural numbers: Addition makes larger and subtraction makes smaller. After they have worked with negative values a, c, or both in problems with the form $a \pm b = c$, teachers and researchers can support students to develop a more nuanced assessment of their claims by having students consider what might happen to sums or differences when the b value is negative. This examination may support students in understanding the conditions under which *AML/SMS* holds and in recognizing that when AML does not hold, the sum may be less than or equal to a. When SMS does not hold, the difference may be greater than or equal to a. In this case, we envision using the initial *emergent* reasoning to develop *formal* reasoning.

Negative as Subtractive and Symbolic Flexibility In our research, we found that students often productively and appropriately treat the negative sign as a subtraction sign to efficiently solve open number sentences (i.e., the *negative sign subtractive* strategy in the *computational WoR*). We view the ability to seamlessly move between meanings of the minus sign and the operation as a desirable outcome of instruction (Arcavi, 1994; Lamb et al., 2012). For example, our college-track students successfully treated the subtraction sign as a negative number or treated a negative number as the operation of subtraction on almost one fourth of all problems they solved. However, the students who shared these strategies may have been so efficient and fluid when computing that they may not have recognized how or that they changed the problem. One goal may be to support students to be more explicit about when they are changing the meaning of the minus sign to aid their computations. See Lamb et al. (2012) for additional information and suggestions.

Negatives Like Positives and Ignores Sign We shared examples of two strategies that seem similar, *negatives like positives* and *ignores sign,* but were categorized as an *analogy-based WoR* and an *emergent WoR*, respectively. Despite the strategies' similarity, we provided evidence to support our claim that students were doing more than appending a sign when invoking *negatives like positives*. Rather, we determined that students had invoked *negatives like positives* only when they provided evidence of attending to more than surface features of the problem in their solutions. That is, had the students initially ignored signs, computed an answer, and appended a sign after computing, the responses would have been coded as *ignore signs*. We view *negatives like positives* as a productive strategy that teachers can leverage to discuss with students when the strategy is useful, to explore reasons the strategy makes sense mathematically, and to discuss important ideas including equivalent expressions and negation.

Final Thoughts

In this chapter, we shared five broad ways of reasoning about integer addition and subtraction and 16 (of the 41 identified) strategies that are subsumed under those ways of reasoning. Although we have shared the ways of reasoning in previous work, herein we sought to share some of the most common strategies with examples that provide clear comparisons and contrasts to support both teachers and researchers in understanding specific strategies within the ways of reasoning. The ability to categorize strategies into one of five ways of reasoning may enable teachers to organize knowledge of student thinking in ways that are useable and accessible for them and provide researchers with sufficient information about the strategies and ways of reasoning such that they can reliably build on this work.

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Appendix

Problem-Solving Interview[7]

1. Name a big number. Can you name a bigger number?
2. Name a small number. Can you name a smaller number? If the child responds, "One," ask, "What if I gave that away? What number would you have then?" If the child responds, "Zero," ask, "Is there a number smaller than zero?"

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[7] Students who provided no evidence of having knowledge of negative numbers (NENs) did not respond to items 16–22 or 30–35.

3. Can you count backward, starting at 5? If the child stops at 0 or 1, ask, "Can you keep counting back?" (If the child continues to count back, have the child stop counting at −5).

   Note. For Grades 2 and 4 students, the interviewer did not pose Question 4 unless the student had previously mentioned the term *negative*. The interviewer did not introduce the term *negative* or the notation for negative numbers unless the child mentioned them in responses to Questions 1–3.
4. *What can you tell me about negative numbers?*
5. $5 + 6 = \square$
6. $4 + \square = 9$
7. $\square - 4 = 6$
8. $8 - \square = 4$
9. $3 - 5 = \square$
10. $6 + \square = 4$
11. $5 - \square = 8$
12. $\square + 6 = 2$
13. $-3 + 6 = \square$
14. $-8 - 3 = \square$
15. *Yesterday you borrowed \$8 from your friend to buy a school t-shirt. Today you borrowed another \$5 from the same friend to buy lunch. What's the situation now?*
16. $-2 + \square = 4$
17. $\square - 5 = -1$
18. $-9 + \square = -4$
19. $-2 - \square = -8$
20. $-5 + \square = -8$
21. $-3 - \square = 2$
22. $-8 - \square = -2$
23. $-8 + \square = 0$
24. $-5 + -1 = \square$
25. $-5 - -3 = \square$
26. $6 - -2 = \square$
27. $6 + -3 = \square$
28. $3 + \square = 0$
29. *There is a bird flying 20 feet above the surface of the water and a fish swimming 5 feet below the surface of the water.* (Show picture of fish, bird, water surface.)

   How many feet higher is the bird than the fish?
30. $-5 - -5 = \square$
31. $-7 - -9 = \square$
32. $\square + -7 = -3$
33. $\square + -2 = -10$
34. $3 - \square = -6$
35. $-2 - 7 = \square$
36. -8 Point to -8. *Can you read this? What does it mean?*

For each pair of numbers, circle the larger, write "=" if they are equal, or write "?" if there is not enough information to tell which one is larger.

37. 3 7
38. -7 3
39. -5 -6
40. +20 20
41. -0 0
42. -9 0
43. - -4 -4
44. -(-4) -4
45. -5 -100
46. □ -□
47. *Is there anything you can write in the blank to make the following statement true*?
    5 = -____

For PI and CT students, we posed questions 48–54.

Circle the larger, write "=" if they are equal, or write "?" if there is not enough information to determine.

48. x x + x
49. x x + 1
50. x + y x – y
51. -x x
52. 7 x
53. x -7
54. If x < y, compare -x and -y.

For CT students, we posed questions 55 and 56.

55. What can you tell me about absolute value?

56a. Someone wrote this down as the definition of absolute value.

For any real number x, the **absolute value** of x is denoted by $|x|$ and is defined as

$$x = \{ \begin{matrix} x \text{ if } x \geq 0 \\ x \text{if } x < 0 \end{matrix}$$

Can you read this to me (point to definition of *absolute value*)*? What does this mean? Do you think this makes sense for the definition of absolute value? Why?*

56b. *According to this definition, explain what the absolute value of -2 is.*

Pose this follow-up question, if needed*: I am confused because negative 2 is less than zero. Doesn't this* (circling the –x in the definition for absolute value) *mean that my answer should be negative?*

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Part II
Multiple Pathways: Models and Metaphors for Integer Addition and Subtraction

Chapter 4
Teaching Integers to Students with Disabilities: Three Case Studies

Michelle Stephan and Didem Akyuz

Abstract The main purpose of this chapter is to document the reasoning of three students, two with disabilities and one with mathematical difficulties, as they participated in and contributed to the classroom mathematical practices established by a seventh-grade class during integer instruction. The integer instructional sequence was designed to support students' increasingly sophisticated reasoning about integers and meaning making for integer addition and subtraction. This chapter builds on our prior work that identified the classroom mathematical practices established by the teacher and students during implementation of the integer sequence (Stephan M, Akyuz D, J Res Math Edu 43:428–464, 2012). In particular, because the integer sequence was implemented in an inclusive setting, we have the unique opportunity to document the learning of two students with disabilities and one with difficulties as they participated in an inquiry environment, contributing to the development of the classroom mathematical practices. These three students participated in a classroom teaching experiment held in a co-taught classroom in which students with disabilities were included with regular education students. Therefore, the three case studies we present illustrate a rare analysis of the integer learning of students with disabilities. The primary research question that we seek to answer through this chapter is, how do students with learning disabilities make meaningful contributions to the development of classroom mathematical practices and gain intellectual autonomy in the process of learning integer concepts and operations?

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The main purpose of this chapter is to document the reasoning of three students, two with disabilities and one with mathematical difficulties, as they participated in and contributed to the classroom mathematical practices established by a seventh-grade class during integer instruction. The integer instructional sequence was designed to support students' increasingly sophisticated reasoning about integers and meaning making for integer addition and subtraction. This chapter builds on our prior work that identified the classroom mathematical practices established by the teacher and students during implementation of the integer sequence (Stephan & Akyuz, 2012). In particular, because the integer sequence was implemented in an inclusive setting, we have the unique opportunity to document the learning of two students with disabilities and one with difficulties as they participated in an inquiry environment, contributing to the development of the classroom mathematical practices. These three students participated in a classroom teaching experiment held in a co-taught classroom in which students with disabilities were included with regular education students. Therefore, the three case studies we present illustrate a rare analysis of the integer learning of students with disabilities. The primary research question that we seek to answer through this chapter is, how do students with learning disabilities make meaningful contributions to the development of classroom mathematical practices and gain intellectual autonomy in the process of learning integer concepts and operations?

Research on Teaching Integers

Learning integers is considered by some researchers to be the first time students encounter algebraic situations, particularly due to the abstract nature of negative numbers (Gallardo, 2002). Students have difficulty conceptualizing numbers less than zero, creating negative numbers as mathematical objects, and formalizing rules for integer arithmetic (Gallardo, 2002; Hefendehl-Hebeker, 1991; Liebeck, 1990). Understanding that the opposite of a negative number is a positive number is particularly challenging (Lytle, 1994; Smith, 1995). This is not surprising as historically negative numbers were considered "absurd" early in their conception because mathematicians had not developed a way to understand numbers less than zero (e.g., see Bishop et al., 2014). Even though many contexts and instructional strategies exist to support students' understanding of negative numbers, there is no consensus about how to teach negative numbers. Most studies focus on modeling negative numbers with physical objects, while grounding the mathematical activity in experientially real contexts (De Bock, Deprez, Van Dooren, Roelens, & Verschaffel, 2011; Kaminski, Sloutsky, & Heckler, 2008; Linchevski & Williams, 1999; Lytle, 1994; Smith, 1995; Streefland, 1996). For example, Battista (1983) suggested using two-colored chips to represent positively and negatively charged particles. Linchevski and Williams (1999), for their part, designed a disco scenario in which students keep track of the number of attendees entering and leaving a disco stage using a double-wired abacus. Many researchers also emphasize that negative numbers cannot be modeled with physical objects because you cannot have fewer than zero number of objects (see, e.g., Bishop et al., 2014). Five minus two, for example, can be modeled using five counters and removing two. However, when young students are asked problems such as 2−5, they struggle with taking five away when there are only two counters initially. Students often change the problem to 5−2 and subtract to get 3 as they do not have a full comprehension of negative numbers (e.g., Murray, 1985).

In addition to using different contexts, one of the important tools used in teaching integers in school is the number line (e.g., Heeffer, 2011). It can help students visualize the position of numbers relative to each other and to construct unary, binary, and symmetric meanings of the minus sign (Vlassis, 2004). Here, unary refers to the quantity as a negative object; in other words, the sign is "attached to the number" (Vlassis, 2008, p. 561). Binary refers to actions such as taking away, completing (as in how much more is needed to have 25, if you have 10) and finding difference between numbers (Gallardo & Rojano, 1994). Finally, symmetric refers to the symbol that signifies taking the opposite of a number. For the expression -(-10), the first negative sign would signify the operation of taking the opposite of -10. Researchers state that the number line can support the unary and symmetrical meaning of signs and can support representations where operations between integers are displayed on the number line with the binary nature of signs (Vlassis, 2004).

For integers, a combination of the number line model and real-life contexts can make the number line more meaningful for elementary and middle school students (Beswick, 2011; Stephan & Akyuz, 2012). For example, vertical number lines can be used meaningfully when paired with a financial context. Students can explain assets, debts, and net worth concepts flexibly on a vertical number line while giving meaning to negative integers and learning operations of addition and subtraction of integers (Akyuz, Stephan, & Dixon, 2012). However, in classrooms, number lines are usually represented horizontally so teachers can easily place them on the wall or the board, and mathematics textbooks support using horizontal models in solutions or representations (Beswick, 2011). One of the important points here is to support the number line with a context where students can make sense of it. For instance, if a context involving going "up" and "down" in net worth is used, employing a vertical number line instead of a horizontal one may be more meaningful for students.

As a consequence of the results from prior research, we designed an integer instructional sequence that uses finance as a context and a vertical number line as a potential model (see Stephan and Akyuz (2012) for more information). Our goal was to support seventh-graders' development of meaning for integer addition and subtraction in a more inquiry, autonomous fashion rather than by direct instruction. Because our research site was an inclusive stetting that involved both regular and special education students, we hoped that our context, model, and inquiry instruction would be supportive of students with disabilities as well.

Teaching Integers to Students with Mathematical Disabilities and Difficulties

Several students in the classroom had been diagnosed with some type of disability; therefore, it is important to define what we mean by the term mathematical disability. Mathematical disabilities (MD) are neurologically based disorders that impede the learning of mathematics; they are cognitive impairments in mathematical ability that do not stem from poor instruction or other factors (Mazzocco, 2007). In fact, Mazzocco distinguishes MD from mathematics *difficulties*, the latter referring to students who do poorly in mathematics because of poor instruction or other disorders like ADHD or mathematics anxiety (Shalev, 2007).

Although direct instruction is promoted as the primary method of instruction with students with special needs, it is becoming increasingly common for researchers of mathematics disabilities to advocate for a blended approach to instruction, which incorporates techniques from both direct and inquiry instruction (Hudson, Miller, & Butler, 2006; Scheuermann, Deshler, & Schumaker, 2009). For example, to teach integers, special education researchers suggest using (1) manipulatives such as two-colored chips that signify positive or negative numbers (Lytle, 1994) or (2) online algebra tiles with temperature or elevation contexts (Maccini & Ruhl, 2000). The teacher then models how to use the manipulatives correctly and provides multiple opportunities for the students to learn her method. Using manipulatives or visual models to teach is typically acknowledged as the contribution from the inquiry approach, while explicit teaching of the way to use the manipulatives incorporates direct instruction.

Related to the blended approach, special education researchers also advocate the use of a technique known as concrete-representational-abstract, abbreviated as CRA (Witzel, Mercer, & Miller, 2003). In this technique, students with disabilities are first allowed to experiment with physical manipulatives to develop intuition for the underlying arithmetic and algebraic operations. These concrete experiences are then augmented with representational activities, in which students are shown how to represent the same information using pictures, figures, and drawings. It is only after the students have mastered these concrete and representational approaches that the instruction transitions to more abstract concepts that involve the use of variables and equations. This approach has shown to improve the mathematics performance of students with disabilities under paired instruction experiments (Witzel et al., 2003).

In this chapter, we offer an approach to teaching integers for understanding that, on the surface, looks like the blended instructional model elaborated above. However, our view is that the blended model is a superficial conjoining of instructional strategies that are used in both explicit and implicit models. Rather, we contend that inquiry instruction has been watered down to superficial techniques and no longer preserves the fundamental belief that distinguishes it from traditional, direct instruction: developing students' intellectual autonomy (Kamii, 1982; Piaget, 1948/1973).

Intellectual autonomy refers to the belief that one is responsible for making sense of problematic situations and does not rely on the authority of others to govern their reasoning. Heteronomy refers to the belief that one's thinking is governed by outside authorities. For example, students who are heteronomous would rely on the teacher to show them how to solve a problem, while autonomous students believe that it is their responsibility to explore the situation. Students who are heteronomous can be heard saying, "Can you tell me how to do this?" or "the calculator said the answer was 29, so I must be wrong." Autonomous students might say things such as, "Wait, don't tell me, let me figure it out first" or "the calculator said the answer is 29. I must have punched something in wrong because that doesn't make sense." Individuals who teach integers for heteronomy might use manipulatives, but they would show students how to use them and have them practice for proficiency. Teachers teaching for autonomy would focus on problem solving first and let students choose manipulatives as reasoning devices, not show a set of steps that must be used with them.

In this chapter, we present three case studies that illustrate the development of integer reasoning from a seventh-grade classroom design experiment in which the teachers used a genuine inquiry approach, *teaching integers for autonomy*. Two of the students had an Individualized Education Plan (IEP[1]) that contained their diagnoses (one with a mathematics disability and the other with a language disability) and education plans with goals for improvement. Since the third student had not been formally tested for disability services, at best we can conclude that he had a mathematical difficulty because he had scored in the lowest 25% of students on the state achievement test. For ease of reading, we will continue to refer to this group of students as a group with special needs, recognizing that the third student who has not been formally diagnosed yet is in the lowest 25% in mathematics achievement. In the sections that follow, we describe the teachers and class context involved in the study and the integer instructional sequence that was implemented during the classroom design experiment. We follow with our methodology for conducting the study and analyzing the learning of the classroom.

Context

The integer instructional sequence was implemented in a public middle school (grades 6–8, ages 11–13) in Central Florida, United States. There were approximately 1500 students enrolled in the school, which served students from a variety of backgrounds, primarily middle class. When the experiment was conducted, the first author had 3 years full-time teaching experience. Prior to that, she had been a mathematics education professor and specialized in designing instruction to support

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[1] In the United States, by law, a student who is diagnosed with a disability must have an Individualized Education Plan (IEP) created that lists the services to be provided as well as ways to measure the student's progress.

inquiry-teaching approaches consistent with *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000). The class consisted of 13 boys, including the 3 identified with mathematical difficulties, and 7 girls. Five of the students (two of which appear in this chapter) had been formally diagnosed as having learning disabilities, and over half the class was performing below grade level (one of which is the third student in this case study). There was also a 10-year veteran co-teacher who was certified in special education. Stephan and her co-teacher had co-taught for 3 years. The classroom teaching experiment was conducted in the third quarter of the school year. Therefore, norms consistent with establishing a standards-based environment (NCTM, 2000) had already been set and were relatively stable during the instruction reported here (Akyuz, 2010).

Integers Instructional Sequence

To develop the instructional sequence, we followed the tenets of Realistic Mathematics Education (RME) (Freudenthal, 1973; Gravemeijer, 1994). Although the origins of RME are in mathematics, the heuristics align well with the concrete-representational-abstract (CRA) design approach recently developed in the special education field (Maccini & Gagnon, 2005). First, RME suggests that instruction start with an *experientially real context* for students, meaning that students do not have to have actually experienced the situation but have to be able to imagine themselves in it. The experientially real context underlying the integer sequence involved determining a person's financial net worth. After students discussed that assets and debts comprise a net worth, the teacher introduced a financial worth statement (for a full description of the instructional sequence, see Stephan & Akyuz, 2012). Working with the financial statement was a way to ground students in a concrete situation that provided the semantic grounding for future abstract mathematical activity. This is similar to the concrete phase mentioned in the CRA special education approach. The first set of activities asked students to compare the net worth of two people. Although they were encouraged to solve the problem any way they wished, students typically added the assets, added the debts, and subtracted one total from the other total. Students began to understand that debts reduce net worth, whereas assets increase net worth. These net worth statements presented both assets and debts as positive whole numbers; situations and activities later on introduced positive *and* negative signs.

A second heuristic of RME suggests that the mathematical activities should build students' reasoning gradually from the concrete to the abstract (see Miller & Hudson, 2007; Witzel et al., 2003). Teachers should use manipulatives, pictures, tools, and other items to reinforce students' reasoning with imagery. Instruction should be intentionally designed so that students reorganize their thinking progressively toward more abstract ideas. In line with this heuristic, the previous activities encouraged students to build meaningful imagery and understanding for net worth as a combination of assets and debts or quantities that are opposites. The next set of tasks continued this idea by progressively presenting assets and debts in a more integer-like format. At this point, introducing assets and debts with signs was enough to move students from the concrete to slightly more abstract thinking. In the preceding task, students were able to organize their mathematical activities within an experientially real context. They were able to develop the relation between assets and debts as affecting a net worth in opposite ways.

The next tasks focused on *transactions* or operations. Instead of figuring out a person's net worth, transaction tasks encouraged students to take an already determined net worth and alter it with a *transaction*. Students were asked to judge the effect that various transactions have on net worth, such as when making good or bad decisions. For example, students were asked to determine whether Ann, who had taken away a $200 asset, had made a decision that was good or bad for her net worth. During the whole class discussion, the teacher introduced a way of symbolizing each transaction. In Ann's case of -(+200), the first sign symbolized the action (add or subtract), and the second symbolized an asset or debt. Students wrote each transaction on the page using symbols.

The third heuristic of RME is that students should be encouraged to create models of their concrete activity. These models should become reasoning devices for more abstract thinking. The next set of tasks, also categorized as transaction tasks, consisted of word problems such as this: *Nancy has a net worth of $5000. A debt of $3000 is taken away. Is this good or bad? What is her net worth now?* Students were also asked to write a situation in a number sentence, such as +$5000 − (-$3000) = $8000. The progression involved presenting problems with words and then presenting problems with symbols only. Students had to find the new net worth each time. Additionally, we introduced the empty, vertical number line (VNL) as a way to help students structure their operations. For the Nancy problem, students might have inscribed their reasoning as follows (see Fig. 4.1):

The vertical number line begins as a model of performing transactions on net worth; it later becomes a model for reasoning with integer quantities. By the time students encountered these last few examples, they were making calculation mistakes only, not conceptual ones. Although the three heuristics do not correlate perfectly with the CRA approach from the special education literature, the instructional sequence is consistent with the notion of moving from concrete to more abstract activity.

Methodology

In this section we detail the research methodology that guided the classroom-based teaching experiment (Cobb & Yackel, 1996) on integers. We first explain the theoretical perspective and associated framework used to guide our analysis. Then, we turn to the data collection and analysis technique.

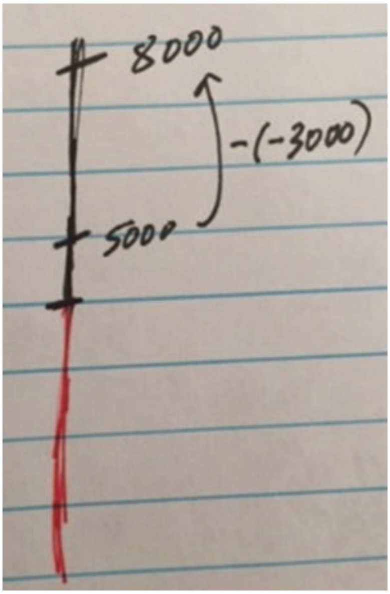


Fig. 4.1 Inscription of 5000 – (-3000)

Theoretical Perspective

The theory we draw on to make sense of students' learning, while we are in a classroom or when we are conducting analyses at the completion of an experiment, is a version of social constructivism called the *emergent perspective* (see Cobb & Yackel, 1996; Stephan, Bowers, Cobb & Gravemeijer, 2003; Stephan & Cobb, 2003). Briefly, this theory draws from constructivist theories that specify learning as an organic, autoregulated series of cognitive reorganizations (Steffe, von Glasersfeld, Richards, & Cobb, 1983; von Glasersfeld, 1995) and interactionist theories that emphasize learning as a social accomplishment (Bauersfeld, 1992; Blumer, 1969). The emergent perspective is one attempt to transcend the individual versus social dichotomy by taking learning to be both simultaneously. In other words, learning is characterized as both an individual and a social process with neither taking primacy over the other. Students are viewed as reorganizing their learning as they both participate in *and* contribute to the social (and mathematical) context of which they are a part. Motivated by this theoretical perspective, Cobb and Yackel (1996) constructed an interpretive framework useful for detailing the learning of a classroom community and its participants.

The Interpretative Framework Cobb and Yackel (1996) developed the framework that guides our analyses of student learning. This interpretive framework emerged out of an attempt to conduct analyses that coordinate individual students' mathematical development with the social context of the classroom (see Table 4.1).

Table 4.1 The interpretive framework for analyzing classrooms

Social perspective	Individual perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

The left side of the framework draws on an interactionist (i.e., social) view of communal or collective classroom processes (Bauersfeld, Krummheuer, & Voigt, 1988; Blumer, 1969). The individual perspective draws on psychological constructivist views of students' activity as they participate in the development of these communal processes (von Glasersfeld, 1995). The relation between the two sides of the framework is said to be reflexive.

A mathematical practice can be described as the taken-as-shared ways of reasoning and arguing mathematically (Cobb, Stephan, McClain, & Gravemeijer, 2001). Classroom mathematical practices (CMPs) evolve as the teacher and students discuss situations, problems, and solution methods and often include aspects of symbolizing and notating (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). We have already documented the classroom mathematical practices of the seventh-grade classroom as the integers instructional sequence was implemented:

Practice 1: Interpreting net worth as a positive or negative quantity
Practice 2: Using zero as a point of reference for calculations
Practice 3: Comparing zero as a point of reference for calculations
Practice 4: Reasoning with a vertical number line to determine the results of addition and subtraction operations
Practice 5: Determining the meaning of positive/negative signs (for more details, see Stephan & Akyuz, 2012)

In this chapter we offer an analysis using a sociological lens that complements the documentation of the mathematical practices. The ways in which three students with disabilities participated in and contributed to the constitution of the classroom mathematical practices above complete the picture of the learning of the classroom community that was begun with the 2012 analysis. This analysis also provides insight into the ways in which students, particularly those with disabilities, make meaning for integer operations.

Data Collection and Analysis

The teacher (Stephan) and researcher (Akyuz) conducted pre- and post-interviews with all 20 students to assess their understanding of integers. In addition, pretests and posttests were administered to each student to document learning from a quantitative point of view. All pretest and posttest questions were drawn from Smith's

(1995) study and included both procedural and conceptual problems. Results from both pre-interviews and pretests indicated that students had some previous knowledge of integer operations, and students were far more successful with addition of integers than with subtraction.

Once the integer instruction began (in March, 2009), data collection continued for 5 weeks and included the class observations recorded through video and audio recordings, field notes taken by the second author, daily interviews of the teacher (first author) by the second author, and the weekly meetings consisting of three other seventh-grade teachers and both authors. At the time of the experiment, the school district had adopted *Connected Mathematics Project* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), a reform textbook, which advocated a student-centered approach to teaching. Units within this textbook take at minimum 4 weeks to teach; thus, 5 weeks was a realistic and acceptable time period for our instruction.

The data utilized for the analysis in this chapter came from the pretests, posttests, and corresponding interviews with each student as well as small group and whole class discussions. All pre- and post-interviews with the three students were transcribed. Additionally, all whole class discussions and small group sessions with the three students were transcribed. We color-coded all instances of talk from any of the three case study students (Nathan, red; Stuart, green; Seth, purple). Then, we cut and pasted each student's sentences into a spreadsheet, along with the day on which the speech occurred, and coded each instance according to the specific classroom mathematical practice (CMP) that the student's statement referred to. For example, Table 4.2 shows a portion of our coding scheme in which we listed every contribution that a student made in whole class or small group discussion.

We then used the coding from the spreadsheet as data to analyze each student's way of participating in or contributing to the classroom mathematical practices by looking at, say, Nathan's growth from the beginning of the table to the end. We also analyzed each student's pre- and post-interviews to document each student's integer reasoning before and after the instructional intervention. In the forthcoming analysis, we begin with results from each student's interviews and then present case studies that show their growth across instruction. First, we introduce the participants in the classroom teaching experiment and the instructional sequence for integers.

Pre-interviews

For the current study, three students, Nathan, Stuart, and Seth, were chosen for the current analysis based upon three facts: (1) each student can be categorized as having either a cognitive or mathematical disability with an IEP or a mathematical difficulty, (2) the three of them worked in a small group together almost the entire school year, and (3) each student participated verbally in both whole class and small group for a significant amount of time, thus allowing for a fruitful analysis.

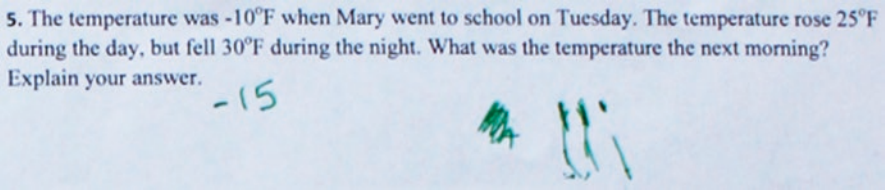
Table 4.2 A sample of the spreadsheet used in coding students' reasoning as related to a classroom mathematical practice (CMP)

Student Contribution	CMP	Day
Teacher: Nathan says, how did you get negative 400?	CMP 1b	19
Nathan: Yes, I did not understand that part.		
Nathan: Yes. Since he was negative, 8,400 is negative so and she subtracted by the positive 8,000 since it is more than zero he is still in negative.	CMP1b	19
Teacher: He pays off…say it again Stuart; real loud.	CMP 1a	19
Stuart: He pays off debts by using assets.		
Teacher: So he uses his assets to pay off his debts.		
Teacher: All right. What is his pay off amount? How much he pays off?	CMP 1a	19
Nathan: 8,400 and he still has to pay 400 dollars.		
Seth: So he goes negative 400. She called this as pay off (T writes - 400 next to 8,000). That is gonna take 8,000 dollars to pay off and then he still in debt.	CMP2a	19
Teacher: Seth, do you think Carl will be in a bad spot or in a good spot?	CMP1b	20
Seth: In a bad spot.		
Seth: First of all…the debit is how much he owes, this is how much he owns.		
Student: It is not debit, it is debt.		
Teacher: Whatever, you all know what he means.		
Seth: He owes more than he owns.		
Stuart: Do not use any words, just say own and owes.		

Introducing Nathan

Nathan had been diagnosed with a learning disability prior to the start of the school year and was assigned to Ms. Smith, the special education co-teacher. On the state achievement test for the sixth grade, Nathan had earned a level 1 qualification, the lowest achievement score possible (which ranges from 1 to 5, 3 and higher being proficient). His IEP included extra time on homework assignments, quizzes, and tests as well as specific mathematics calculation proficiency goals. His mathematical confidence was very low coming into the seventh grade, and Nathan struggled with basic computations. In addition to being enrolled in the co-taught regular mathematics session, he was assigned to the same regular education teacher for another mathematics class session, which was designed to support students who were underperforming on the state test. Thus, Nathan had two class periods of mathematics with the same regular education teacher: one, in a co-taught setting with both a regular and a special education teacher and the other only with the regular education teacher and students who were also underperforming.

Rather than explain how Nathan solved each interview task separately, we draw some general conclusions about his integer reasoning and computation. Nathan had



5. The temperature was -10°F when Mary went to school on Tuesday. The temperature rose 25°F during the day, but fell 30°F during the night. What was the temperature the next morning? Explain your answer.

-15

Fig. 4.2 Nathan's reasoning on the temperature problem

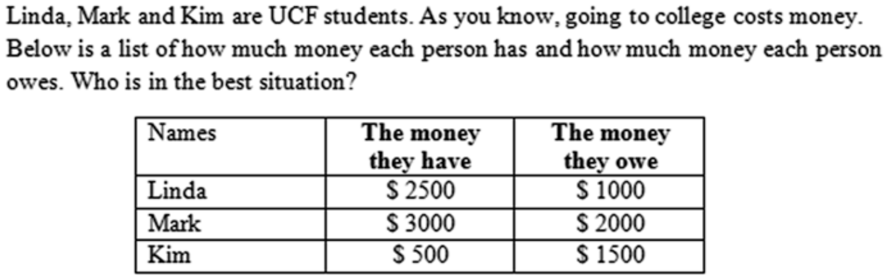
difficulty interpreting both realistic problems as well as computing number sentences. For example, on the temperature problem (Fig. 4.2), Nathan drew two and one-half tally marks to stand for 25 degrees. He drew three full tallies directly underneath those and matched up the two long tallies in each group, saying,

> A tally mark stands for ten…10, 20, and that would be the 5. Then, 30 would be the 10, 20, 30. Then, you match these up (pointing to the two tally marks from the 25 and the two tally marks from the 30) and then you have 15 left.

Nathan simply looked at the remaining tally from the bottom group and the one-half tally from the top group and put them together to conclude that the temperature must be 15 degrees. He then added that the 15 must be negative because "you start with less and subtract by more, [so] you would get a lower number." The answer -15 is coincidentally correct but with erroneous reasoning. Nathan's calculations did not take into account that the temperature started with -10 degrees. Consider if the starting temperature were -30 °F; Nathan's strategy would still have given him -15 as his answer, but the correct answer is actually -35 °F. He used tally marks to represent only *two* of the quantities in the problem (rising 25 °F and falling 30 °F) and lost track of their meaning, combining leftover amounts after deleting the two tallies that matched. This is a typical strategy of someone with a mathematics difficulty; in the absence of meaning for their problem solving, the student sometimes manipulates the quantities in what appears to be random combinations.

When it came to questions about assets and debts (see Fig. 4.3), Nathan first analyzed each person's situation, saying that Kim was in a bad situation because she had a lot of debt, Mark is doing well, and so is Linda. Hence, Nathan seemed to have an intuition that debts and assets are connected, but he could not combine those in a meaningful way. When the researcher pushed him to determine who was in the best financial state, he attempted to find the difference between assets and debts by dividing, say 1500 by 500. When asked what the 3 in his calculation stood for, he replied, skeptically, "The difference…I don't know" and abandoned division. Eventually, he argued that Mark was on top because he had the most money ($3000 versus $2500 and $500), focusing only on the positive value for each student and not the debt.

For the bare number problems such as 9 + (-12), -6 + (-3), 5 – (-7), and -4 – (+8), Nathan only constructed a correct answer for the second one, but it was a coincidence (his answers in order: -21, -9, -2, +4). Nathan looked at the sign in the middle of the problem and performed that operation no matter what the sign of the second



Linda, Mark and Kim are UCF students. As you know, going to college costs money. Below is a list of how much money each person has and how much money each person owes. Who is in the best situation?

Names	**The money they have**	**The money they owe**
Linda	$ 2500	$ 1000
Mark	$ 3000	$ 2000
Kim	$ 500	$ 1500

Fig. 4.3 The college student question

addend was and adjusted the sign by giving his answer the sign of the largest addend. For example, 9 + (-12) was -21 because the plus sign in the middle led him to add 9 and 12 to get 21. Since the 12 had a minus sign and was larger than 9, his answer was -21.

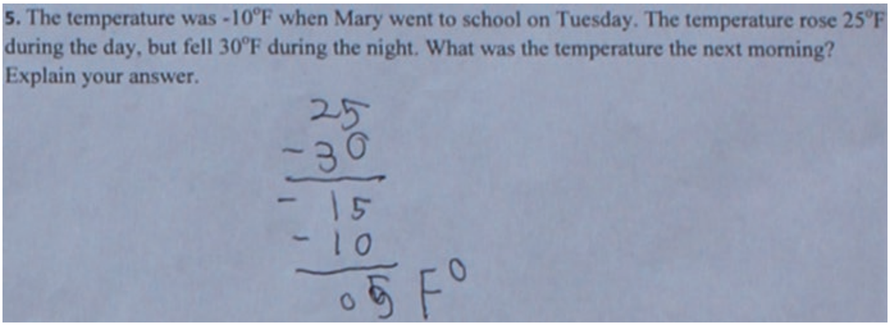
In conclusion, Nathan's pre-instruction interview indicated that he had difficulty making meaning out of the quantities in word problems related to negative amounts and that, on bare number problems, he was able to make sense out of adding two negatives but did not make sense of adding or subtracting negatives.

Introducing Seth

Seth had an IEP and was listed on the special educator's roster because of a language processing difficulty as opposed to a mathematics difficulty. However, Seth had difficulty articulating his thinking both out loud and in written form, often leaving out words, not writing complete sentences and using words that were nonsensical or incorrect for the argument he was making. He had difficulty putting into words the mathematical connections that he had formed. Because he was proficient on the state test, he did not have an extra mathematics class. He also had high confidence in mathematics reasoning but low confidence in articulating an argument both in oral and written form.

Like Stuart, Seth had a strong sense of how debts and assets affect each other to form a net result. For problems involving money, such as that in Fig. 4.3, Seth also analyzed each situation and concluded fairly quickly and confidently that Linda would be in the best financial situation. To make this decision, he found the difference in the two positive numbers (asset and debt) for each person and argued that Kim would be in the worst situation because he [sic] had 500 minus 1500 and would be in the negative 1000. Additionally, Seth argued that Kim only has $500 and has $1500 debt and needs "1000 more dollars to get back to zero. Zero is the diameter around here."

For the temperature problem (see Fig. 4.4), Seth said that he kept the -10 temperature alone for the morning and subtracted 30 from 25.



5. The temperature was -10°F when Mary went to school on Tuesday. The temperature rose 25°F during the day, but fell 30°F during the night. What was the temperature the next morning? Explain your answer.

25
-30
-15
-10
05 F°

Fig. 4.4 Seth's solution to the temperature problem

> Seth: This is hard, but I think I can do it…I subtract both of them 25 by 30 and I get -15. Negative 15 minus negative 10…we're taking away from the negatives and since they're both negatives, they gonna go up to the positives since negative is the exact opposite of positive. You've got 5 bigger Fahrenheit right there.

In this excerpt, Seth illustrated that he knew that rising and falling temperatures had opposite effects on each other but was unable to manage the symbolic meaning to calculate a correct difference. When attempting to take into account the morning temperature of -10, again he indicated that he remembered a rule that two negatives make a positive, but he did not make meaning of that rule in this context.

Finally, for all number sentences, Seth erroneously used the strategy of subtracting the smaller number from the bigger and keeping the sign of the bigger number. This produced incorrect answers to all but the first problem, 9 + (-12). When asked to explain his solution to the first two again, Seth spontaneously drew a horizontal number line to show left and right movements, depending on the problem. This led him to change his answer to the second problem -6 + (-3) to get -9.

In conclusion, Seth had a strong sense of the opposite effects that positive and negative quantities have on one another. However, like Stuart, he had difficulty interpreting his actions in purely symbolic situations and at times in context.

Introducing Stuart

Stuart was considered a student with a mathematical difficulty as he performed in the lowest 25 percentile on his mathematics achievement test with a non-proficient score of 2 on the state test. He was on the regular education teacher's roster and did not have an IEP like Nathan and Seth. Since Stuart was underperforming on the state test, he, like Nathan, was assigned an extra support mathematics class with the same regular education teacher.

Stuart had a strong sense of how debts and assets affect each other to form a net result. For problems involving money, such as that in Fig. 4.3 (the college student question), Stuart analyzed each situation and concluded fairly quickly and confidently that Linda would be in the best financial situation. To make this decision, Stuart found the difference in the two positive numbers (asset and debt) for each person and argued that Linda would be in the best situation because she has "$1500 left" after she rids herself of the $1000 debt.

For the temperature problem (see Fig. 4.4), Stuart started by finding what he termed a range by subtracting 25° from 30° to get -5°. When asked about the -10°, Stuart said that this number did not really matter, and he did not take it into account. Like Nathan, Stuart worked only with the temperature change and did not consider the original temperature.

For the bare number problems such as 9 + (-12), -6 + (-3), 5 – (-7), and -4 – (8), Stuart recalled integer addition and subtraction rules he had been taught the previous year and attempted to implement them for the problems. For the first two problems, he used them correctly and, when asked why they worked, simply repeated the rules he said he was taught. For 5 – (-7), he found -2 because the subtraction sign prompted him to subtract the 7 and 5 to get 2, but he was not clear why it would be a -2. For the last problem, he claimed that he added 8 to the -4 to get a 4.

In conclusion, Stuart had some sense of the effect that negatives (debts) and positives (assets) have on each other and applied this in other contextual situations, such as temperature. However, when it came to bare symbolic questions, he had difficulty interpreting the meaning of the positive and negative signs and their implications for action. Stuart fell back on integer operation rules that he had been taught previously.

Implications of Pre-interview Results for Instruction

The analysis of these three students' reasoning, when added to our analysis of their classmates' interviews, led us to make the following conclusions about our instructional starting and ending points:

1. Students could reason with positive and negative quantities in a context, knowing that positive and negative quantities had an opposite effect on each other. Even students reasoning like Nathan understood this, despite not counting the debt in their final assessment.
2. The first conclusion led us to believe that we had chosen a realistic starting point for instruction that would build on students' notions of the relations between assets and debts (+ and – quantities). All three case study students could reason meaningfully with positive and negative quantities having opposite effects on each other. One of our learning goals initially, therefore, was to help students make meaning for net worth as a combination of positive and negative quantities.
3. Finally, given the spontaneous use of horizontal number lines by Seth and three other students, we were encouraged that students' reasoning with the empty, vertical number line might serve as a model of their transactions with net worths and

shift to a model for reasoning about integer quantities. We also conjectured that the vertical direction could evoke more meaningful and contextual imagery for students who had otherwise been using a horizontal number line mechanically.

In the next section, we describe how each case study student participated in and contributed to the development of the CMPs that emerged during the class sessions. We then present an analysis of each student's post-interview integer tasks, comparing their reasoning with pre-interview results.

Case Studies

The following analysis is organized by students' participation in and contribution to the five classroom mathematical practices (CMPs). Rather than present each case separately, we merge them together within each CMP, which has the advantage of showing the diversity in reasoning as well as how the three students negotiated meaning, at times, together in their small group.

Classroom Mathematical Practice One

The first classroom mathematical practice that was established involved making meaning for net worth as a combination of assets and debts (or in integer terms, signed quantities; Fig. 4.5). The first day of the instructional sequence, the teacher introduced the context through a story about Oprah Winfrey and asked students to name assets and debts that may contribute to her net worth. After discussions about a net worth statement, students were given an activity sheet that had Cindy's and Bobby's fictional net worth statements on it, and they were to determine which person was worth more (see Fig. 4.6).

Students were not told how to determine a person's net worth; they were merely asked to establish who was worth more and were left to small group exploration to come to a conclusion. The following small group discussion signifies how Seth and Stuart attempted to make sense of net worth.

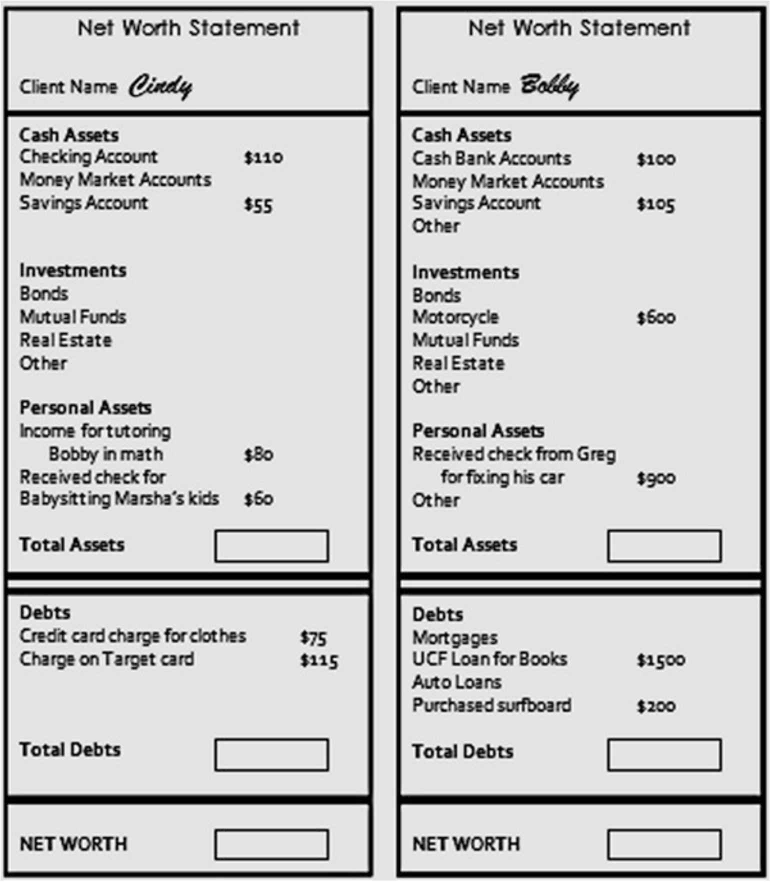
Stuart: What did you get for total debts? Cindy?
Seth: 190.
Stuart: How did you use the total assets and total debts to get the net worth?
Seth: 305 minus 190.
Stuart: What'd you do to get them?
Seth: Debt means you need money. Asset means you gain money.
Stuart: Ah! I get it! How would you…1700 [referring to Bobby's net worth]. We need a calculator.

This excerpt shows that Stuart was initially unsure of the way assets and debts interact to create a net worth, as seen when he asked Seth how he combined the total assets and debts to get net worth. This might be surprising given Stuart's interview

Practice 1: Interpreting net worth as a positive or negative quantity

a) Net worth is a combination of a positive and a negative value [when the assets and debts are both nonzero].
b) When a negative value is greater than [in absolute value] a positive, the combination is negative.

Fig. 4.5 Classroom mathematical practice one (Stephan & Akyuz, 2012, p. 442)



Net Worth Statement

Client Name *Cindy*

Cash Assets	
Checking Account	$110
Money Market Accounts	
Savings Account	$55
Investments	
Bonds	
Mutual Funds	
Real Estate	
Other	
Personal Assets	
Income for tutoring Bobby in math	$80
Received check for Babysitting Marsha's kids	$60
Total Assets	
Debts	
Credit card charge for clothes	$75
Charge on Target card	$115
Total Debts	
NET WORTH	

Net Worth Statement

Client Name *Bobby*

Cash Assets	
Cash Bank Accounts	$100
Money Market Accounts	
Savings Account	$105
Other	
Investments	
Bonds	
Motorcycle	$600
Mutual Funds	
Real Estate	
Other	
Personal Assets	
Received check from Greg for fixing his car	$900
Other	
Total Assets	
Debts	
Mortgages	
UCF Loan for Books	$1500
Auto Loans	
Purchased surfboard	$200
Total Debts	
NET WORTH	

Fig. 4.6 Cindy and Bobby activity sheet

solutions in which he found the difference between debts and assets quite readily. However, net worth was a new term for him, and it may not have been readily apparent that net worth is the combination of assets and debts. As soon as Seth reminded Stuart that debt is money that you need, and assets are money you gain, Stuart was able to create a meaningful strategy for calculating net worth, which he began to implement for Bobby. In other words, a simple reminder that assets and debts have opposite meaning prompted Stuart to combine the two by finding their difference. For Nathan's part, he interpreted net worth as the difference between total assets and debts and even argued correctly in a whole class discussion that a –$190,000 net worth is not a debt.

Dusty: The negative 190,000 is how much Brad is in debt.
Teacher: (She writes the board) Dusty says it is how much Brad in debt. You all agree with that?
Students: Yes.
Teacher: Who doesn't agree with that? Nathan?
Nathan: I think it is how much the net worth Brad has.
Teacher: [writes on the board: *Nathan: It is net worth.*] Why do not you think it is his debt? [other students try to answer]. Nathan? Hold on, hold on [other students trying to talk out of turn]!
Nathan: Because, I do not know. I just do not get it [Dusty's interpretation].

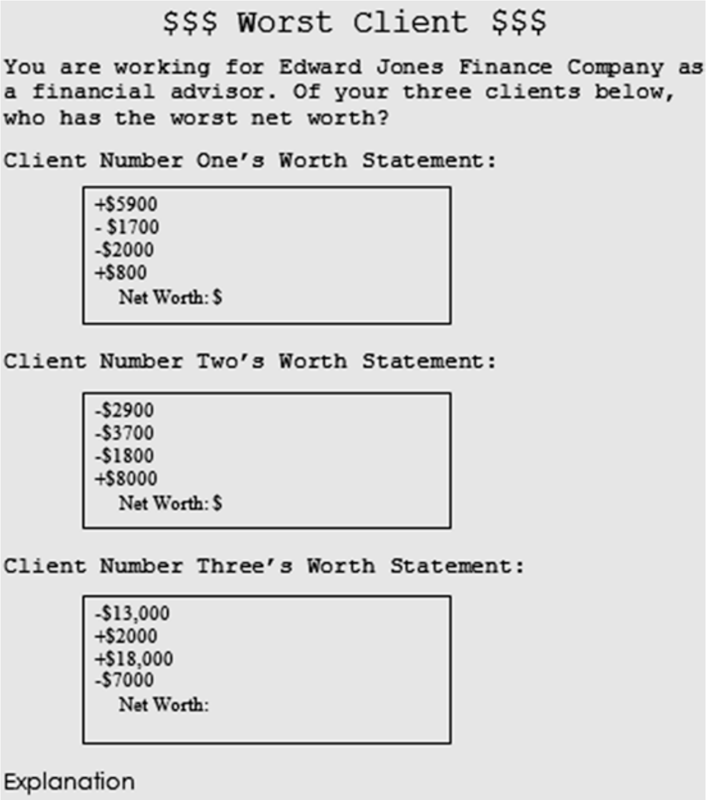
Nathan rejected Dusty's argument that Brad's final amount signified his debt, implying that Nathan understood net worth to stand for the difference between debts and assets, not just debt itself. Nathan's reasoning about net worth can be seen simultaneously as his mathematical understanding and as a contribution to the emerging classroom mathematical practice.

By Day 3 of the instructional sequence, all three students were interpreting net worth as the difference between total assets and debts and understanding that when the total debt is bigger (in absolute value) than the total assets, then the resulting net worth is negative. In a small group discussion of the Brad and Angelina task, Nathan corrected Stuart who claimed incorrectly that Brad's net worth was +190,000.

Stuart: Brad's NW is 190,000.
Nathan: I don't get that. −190,000. He's a LOT less.
Stuart: We picked Angelina [T visits their desks].
Teacher: Because?
Seth: [Angelina is] 90,000 POSITIVE. Negative means less than the positive.
Nathan: What they said!

Nathan's rebuttal suggests that he understood that when your total debts outweigh your assets, the net worth should be called negative. Additionally, when comparing Brad's −190,000 net worth to Angelina's net worth of +90,000, Nathan understood that negative values are less than positive ones.

As the instructional sequence progressed, students developed two main ways to determine net worth. For example, consider the activity sheet that was given to students on Day 3 of the experiment (Fig. 4.7). Most students developed at least one of three ways to find each client's net worth. The first method was to work through the assets and debts linearly, such as 5900 − 1700, then use that total of 4200 − 2000 to get 2200, and then add 800 to get a total of +$3000 for Client One. The second method resembled students' interpretation of net worth as the difference between total assets and debts: 5900 + 800 = 6700 total assets; 1700 + 2000 = 3700 total debts; 6700 − 3700 = 3000 net worth. For his part, Nathan typically used the linear method to calculate a person's net worth, while Seth and Stuart used the total assets, total debts, and difference method fairly consistently.



$$$ Worst Client $$$

You are working for Edward Jones Finance Company as a financial advisor. Of your three clients below, who has the worst net worth?

Client Number One's Worth Statement:

+$5900
- $1700
-$2000
+$800
Net Worth: $

Client Number Two's Worth Statement:

-$2900
-$3700
-$1800
+$8000
Net Worth: $

Client Number Three's Worth Statement:

-$13,000
+$2000
+$18,000
-$7000
Net Worth:

Explanation

Fig. 4.7 Worst client activity sheet

Table 4.3 Summary of students' participation in and contribution to CMP1

Student	Participation in/contribution to CMP1
Nathan	Immediate sense that net worth is combination of total assets and debts (participation in CMP1)
	The higher quantity determines the sign of the net worth (contribution to CMP1)
Seth	Immediate sense that net worth is combination of total assets and debts (participation in CMP1; aided Stuart in his development)
Stuart	No immediate sense that net worth is combination of total assets and debts but stable by Day 3 (participation in CMP1)

In summary (Table 4.3), all three students reached a point by Day 3 of the instructional sequence in which they understood that net worth is a difference of assets and debts and that the higher value determines the sign of the net worth. Their interactions with each other in small group provided the opportunity for Stuart to reorganize his thinking, with Seth and Nathan creating that understanding almost immediately.

Practice 2: Using zero as a point of reference for calculations

a) Referencing zero to determine net worth
b) Referencing zero to compare two net worths
c) Referencing zero to add or subtract integers
d) Cancelling equal positive and negative quantities

Practice 3: Comparing integers using a vertical number line

a) Higher [in absolute value] negative numbers are farther away from zero.
b) Structuring the gap between two integers to find the difference

Fig. 4.8 Classroom mathematical practices two and three

Classroom Mathematical Practices Two and Three

We discuss the three students' participation in and contribution to classroom mathematical practices two and three together because several of the mathematical ideas from both practices emerged simultaneously and became taken-as-shared at the same time (see Fig. 4.8). Therefore, in writing the analysis, it became impossible to separate the two and is easier to read together than separately. In other classroom design experiments, we have also found that the establishment of CMPs is not necessarily linear in time but can overlap (Rasmussen & Stephan, 2008).

All three students used the term "payoff" in their reasoning when finding net worth. For example, when referring to Client Two's net worth (Fig. 4.7), Nathan said, "[he has] 8,400 [of debt], and he still has to pay 400 dollars." Consider Seth's whole class contribution, "So he goes negative 400 [net worth]. She called this a payoff [Teacher writes the word, "payoff" next to 8000 on the board]. That is gonna take Norman 8000 dollars to pay off and then he is still in debt." The metaphor of *paying off* became a consistent communication and imagery device to help students both make sense of and justify to others their actions with the quantities. In this way, both Nathan and Seth's whole class explanations can be considered as an act of contributing to the constitution of the first taken-as-shared idea under classroom mathematical practice two (referencing zero to determine net worth). While their statements did not explicitly mention the term "zero," the action of *paying off* means using your assets' value to get to zero assets, and if there is any amount left to pay off, it means there is a negative net worth.

Paying off continued to be strong imagery for all three students, especially as the tasks changed to comparing net worths rather than computing net worths. On Day 5, the teacher posed a task that prompted students to compare the net worths of two individuals (Fig. 4.9). In small group, Nathan, Seth, and Stuart each quickly determined that Gilligan is worth more than Mary Ann by $1000 (i.e., 3000 – 2000), and when the teacher visited their group to report that some other students claimed the answer was 5000, Nathan said, "Trust me, 5000 is wrong!" and Seth demanded, "Can I defend 1000!?" The ensuing whole class discussion was pivotal for all students' development—Nathan, Stuart, and Seth in particular.

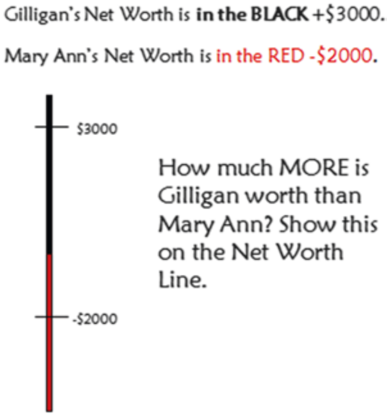


Fig. 4.9 First task involving comparing, not computing, net worths

Seth: …So we want to find out how much Gilligan [places 3000 appropriately on the vertical number line]. This is Gilligan, right?

Teacher: Yes, 3000 was Gilligan's.

Seth: So, I found out how much more money that Gilligan has than whoever is [Seth writes -2000 in the red area of the VNL, and Teacher writes Mary Ann next to it]. So basically what you do is 3000 plus 2000 but it is not like this (draws arrow from zero to 3000), it is here in the negatives, but it does not make it all the way since it starts from the zero and it cut short 1000, since the 1000 between them.

Teacher: You guys need to put your hands up. Wait, your hand is not up to show your point of view, your hands are up to question Seth's. All right, there is a big difference.

Dusty: Why did you add 3000 to negative 2000?

Seth: Because we try to find out how much more Gilligan has than Mary Ann. So what we did was 3000 plus 2000 since it is negative I believe we do minus. It just goes down or -5000 [Seth is imagining starting with Gilligan and moving down toward Mary Ann's net worth, thus a negative 5000].

Teacher: Did it clear it up? It did not for me. Okay you got more questions.

Bradley: See what I do not understand is you are in negative 2000 but Gilligan is 3000 more, it will take Mary Ann 2000 just to go back to zero [Seth writes $3000 - 2000 = 1000$].

Stuart: *How much more*, more is even in bold letters, *is Gilligan worth more than Mary Ann*? *Show on a net worth line* [paraphrasing the directions]. This is the net worth line, correct? *More,* which means basically you have 2000 in debts and you got 3000 here, we are trying to find how many…wait a second now I understand [why it's 5000]. Never mind!

Bradley: Yes. Mary Ann is in negative 2000, she just paid 2000 to just to get up to zero [He writes on the Dusty's solution]. I put plus 2000 because she just paid 2000 and then she has to pay 3000 to get up to here [to Gilligan's net worth].

Stuart: You are in the red.

Bradley: She needs to pay 3000.

Teacher: So I think Bradley what you are saying is the gap between here and here [between -2000 and 0], 2000 both Gage and Bradley have said that she had 2000 dollars to get to black and then 3000 to get up Gilligan, so total 5000.

Fig. 4.10 How much more is 8000 than −10,000

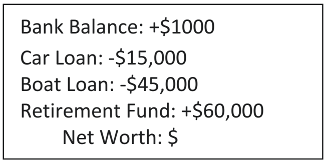
This whole class discussion was a pivotal exchange in many ways. First, Seth and Stuart's incorrect solution was critical for making meaning of the gap between two different-signed integers. Both Seth and Stuart argued that the difference between 3000 and -2000 was 1000 until Stuart reasoned with the VNL to make sense of the distance between the two. Second, reasoning with the VNL provided the visual support for mathematizing the gap between the two integers. The imagery of *paying off* was evoked by students to justify why the gap was 5000 rather than 1000, and the color-coding of the VNL made it possible to show exactly where the transition between paying off (i.e., getting to zero) and needing more money was located. Participation in this conversation was significant for Stuart in two ways. He drew on *going through zero* strategies and *paying off* imagery for the remainder of the experiment. Additionally, when Stuart was in doubt of a solution, he backed up his answer by drawing a VNL for support. Seth, too, reasoned with the VNL to structure the gap between two integer quantities, particularly when the strategy necessitated going through zero. Their reasoning in this episode constitutes both evidence of their current integer understanding as well as contributions to the second taken-as-shared mathematical idea in CMP2 and CMP3.

At the beginning of class the next day, the teacher posed the problem in Fig. 4.10. All three students were structuring the gap between two integers in very meaningful ways using the VNL:

Seth and Stuart: I got it!
Nathan: I got the answer. Do you guys? 18,000? I did the line for the zero. I put -10,000 and drew an arrow going up. 10,000 going up and then added 8000 to go the rest of the way up.

This small group excerpt shows that Stuart and Seth both solved the new problem very quickly with Nathan explaining by structuring the space between the two integers in two chunks that go through zero. This way of reasoning became stable on this day when comparing net worths but also recurred at a later part of the instructional sequence when performing operations. For example, to solve a problem like 125 + (-225), Nathan organized his reasoning on the VNL to go through zero as a strategy, as illustrated in our prior report (Stephan & Akyuz, 2012):

Nathan: It is 225 how much he owes minus how much he has right now. We subtract from -100.
Teacher: What is your number line about?



Bank Balance: +$1000
Car Loan: -$15,000
Boat Loan: -$45,000
Retirement Fund: +$60,000
Net Worth: $

Fig. 4.11 Client One's net worth statement

Nathan: I went 125 down [from 125 to 0] and 100 more [from 0 to -100] and added up them and gave me 225.
Teacher: Gage, what do you want to add?
Gage: 125 is the original and when you go to zero, 100 is left.
Dusty: What is the bottom?
Nathan: I went 125 to zero and then zero to 100 and 100 to zero and added them up (p. 455).

In this example, Nathan, like many students, used a going through zero strategy to solve operations with the VNL. Participation in the previous taken-as-shared activity of structuring the space between two integer amounts evolved now to structuring the number line through zero to solve number sentences (CMP2 idea 3, CMP3 idea 2). Seth and Stuart also used this strategy and way of using zero as a reference in their calculations.

One final mathematical idea became taken-as-shared in classroom mathematical practice #2: the invention of Seth's cancellation method. During the small group work on the worst client problem, Seth invented an efficient way to determine the net worth of Client One (Fig. 4.11) by canceling the +60,000 with the combination of -15,000 and -45,000.

In the whole class discussion on Day 3, Seth shared this strategy with classmates:

Seth: What I did was, I added it [−15,000 and −45,000] and this turned out to be 60, and there is another 60,000. So basically what I did was cross this off [car loan, boat loan, and retirement fund] since these two cancel each other off so it is equal to zero.
Mark: There is also 1000 there.
Teacher: He is getting there.
Seth: So since there is only 1000, I skipped the adding zero to 1000.
Teacher: How about Carl? I want everyone to explain Seth's steps.
Carl: He added 15,000 to 45,000. He got 60,000.
Teacher: And what is this stand for Carl?
Carl: Car loan and boat loan.
Teacher: What are the names for these guys?
Students: Debts.
Teacher: Okay debts. Keep going Carl.
Carl: And then he saw the retirement fund and just said that 60,000 minus 60,000 is zero. And there was a bank balance which was 1000, and that was his net worth.
Teacher: That was pretty cool, wasn't it? How many people added the assets and got 61,000? You added all these assets and got 61,000, and then how many people subtracted 60,000 from it? That was another way I saw around the room.
Tisha: An easy way.
Teacher: What is easier?

Table 4.4 Students' reasoning as they participated in and contributed to CMP2 and CMP3

Student	Participation in/contribution to CMP2 and CMP3
Nathan	Initially, Nathan had difficulty interpreting and structuring the distance between two integer amounts on the VNL; after a major contribution in class, Nathan developed strong imagery for going through zero to structure the gap between two numbers using the VNL. The *payoff* imagery was also strong, supportive imagery for Nathan (contribution to CMP2 and participation in CMP3). Going through zero became a very powerful reasoning tool even when performing calculations on bare number problems (participation in CMP2)
Seth	Seth's reasoning involved simple, mechanical operations until his lack of structuring the distance between 3000 and −2000 was questioned in whole class. It is during his argument that he began to see the virtue of using the number line to structure the gap between two integer quantities, and he reasoned with zero as a referent point from here on (contribution to CMP2 and CMP3 and participation in CMP3). Additionally, Seth contributed to the establishment of CMP2 when he offered the cancellation method as another zero-reference strategy (contribution to CMP2)
Stuart	Stuart's integer reasoning was also crucial in the process of constituting CMP2 and CMP3 when he used the VNL to structure the distance between 3000 and −2000 (contribution to CMP2 and CMP3 and participation in CMP3). We contend that reasoning with the VNL to correct his thinking made such an impact on Stuart that he continued to use the VNL in future reasoning but particularly when checking his answers for correctness (participation in CMP2 and CMP3)

Tisha: That [referring to totaling assets and totaling debts, then finding the difference].

Teacher: Easy for you but this was one easier for Seth. He said he cancels these out. He used this word "cancel out" [Teacher writes the word on the board]. So I am gonna put this Seth's Cancel Out Method [writes on the board].

This excerpt marks the first time that Seth's canceling strategy is made public in the discourse. Most students had still been using one of two strategies (difference between total assets and total debts or adding and subtracting linearly). However, Seth's verbalization on Day 3 contributed to the establishment of the cancellation reasoning as a taken-as-shared mathematical idea in CMP2. In fact, days later, other students used and referred to Seth's cancellation method, indicating that it was taken-as-shared.

Table 4.4 summarizes the ongoing mathematical development of Seth, Nathan, and Stuart up to this point in CMP2 and CMP3. One theme was clear in analyzing these three students' reasoning as they participated in CMP2 and CMP3: the emergence of the VNL and payoff metaphor as powerful reasoning devices for structuring their mathematical activity around zero as a reference point.

Classroom Mathematical Practice Four

As the instructional activities progressed, we introduced tasks that we hoped would help students develop meaning for operations with integers. We knew from the literature (e.g., Liebeck, 1990) that students tend to perform well when adding integers but show a decline in scores on problems that involve subtracting integers.

Practice 4: Reasoning with a vertical number line to determine the results of addition and subtraction operations

a) Transactions can have a positive or negative effect on a quantity.
b) A vertical number line can be used to find the results of integer operations.
c) Subtraction of integers is not commutative.

Fig. 4.12 Classroom mathematical practice four

Therefore, our class spent considerable time and mathematical activity trying to make meaning for both adding and subtracting an amount from a beginning net worth. Our approach was to place operations in the context of performing a variety of transactions on a net worth, support students' meaningful interpretation of these operations, and gradually symbolize these transactions with two integer signs. As students engaged in instructional activities with these goals, two classroom-level mathematical practices emerged. In this section we focus on Nathan, Seth, and Stuart's mathematical development as they participated in and contributed to the fourth practice, which had three taken-as-shared ideas associated with it (Fig. 4.12).

We first introduced the idea of transactions by posing the problem called *Don't Cry Over Spilled Milk*: The activity sheet showed a fictional net worth statement with assets and debts labeled on it, but Abigail had spilled milk on it and all that was showing were a couple of assets and debts and her final net worth. The text below this ruined net worth statement read, *Abigail lost an asset (a valuable coin) worth $8000. She wanted to figure out what she was worth now that the asset was taken away from her net worth statement. BUT the only copy of her net worth statement she could find has milk stains on it. Can you help her figure out her net worth now?* (Stephan, 2009, p. 20). Students could see that her original net worth was −$10,000.

In small group work, Seth started the discussion by revealing his thinking to Stuart and Nathan: "That's gonna make her net worth go lower. So I did 10,000 + -8000. The key word I'm focusing on is negative." Seth immediately saw that the transaction of losing a coin would make Abigail's net worth go lower. Although he referred to 10,000 without a negative (possibly due to his language disability), Seth's conclusion was -18,000 as a final net worth. However, Nathan and Stuart did not respond to Seth's claim during the small group work; yet, in the follow-up whole class discussion, they appear to have the same interpretation, as seen when Nathan argued that Abigail's new net worth would be -18,000. For another problem in which a fictional character named James added a $500 asset, Nathan created a realistic scenario to explain, "If he won a lottery, 500 dollars, he is adding to his net worth."

The next task was designed to focus students' attention only on the transactions, not the computational result of them. For example, students were asked to determine if the following transactions by fictional characters were good or bad: **Ann** took away an asset of (+$200) from her net worth statement, **Bradley** added an asset of (+$3000), **Christian** took away an asset of (+$50), and **Ernie** took away a debt of (-$5400). Nathan exclaimed that this task was easy and got all of them correct. Seth and Stuart got all but one correct due to misreading the problem. This illustrates that, when put into context, even subtracting a debt, "−(-)," made sense to students; they argued that it is a good thing to take away debt. None of these students had difficulty

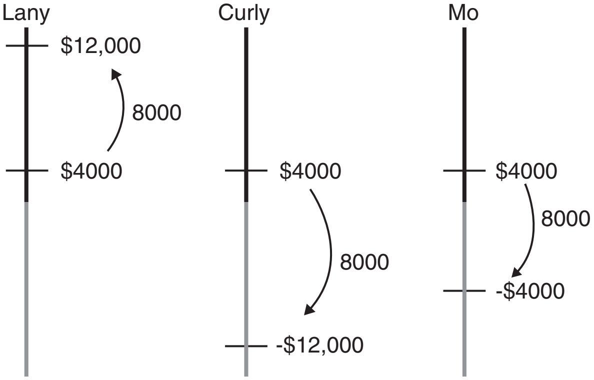


Fig. 4.13 Which VNL represents 4000 + (−8000)

writing the transactions in symbols either (e.g., Ann's transaction was symbolized as – (+200); similarly, Bradley, + (+3000); Christian, – (+50); and Ernie, – (-5400)). However, the trouble began when attempting to coordinate the symbols and context to determine the new net worth after a particular transaction had taken place. For example on Day 9, Nathan and Stuart discussed the problem of finding Maria's new net worth if the following happened: 50 – (-100).

Nathan: How do you know [it's a positive net worth]? She has a negative number.
Stuart: You're taking away a debt so that gives you a boost.

Nathan tended to focus mainly on the sign between the two numbers, as in his pre-interview work, and perform that operation. Stuart and Seth, for their part, had much more success coordinating transactions with a starting number posed in symbols. These excerpts illustrate the reasoning of each student as they participated in and contributed to the taken-as-shared idea that transactions can have a positive or negative effect on a quantity (CMP4).

While Seth and Stuart were successful by Day 9 (3 days after introducing transactions with symbols), Nathan did not create a stable meaning and strategy until Day 20, 15 class periods after its first introduction. One of the tools that was crucial to Nathan's reorganization was the VNL. Consider the whole class discussion about the following problem (Fig. 4.13) that asked which VNL properly signifies the number sentence 4000 + (-8000).

Seth: Curly seems impossible.
Nathan: I know what we did wrong. You start at 4000, you're adding some more [debt]. You have to kind of subtract 4000 from 8000 [to get -4000, so Mo's solution].
Teacher: Nathan says he started with 4000 and you are adding on [debt], so you end up [going] 4000 down to zero.

Seth also used the VNL for solving number problems. In a whole class discussion on Day 15, Seth explained his answer to the problem -426 + (29).

Seth: I only like number lines when I'm explaining. You add asset of 29. There is no negative sign there. Mathematicians do not put a sign there. So it is -426 to go up 29 so it is -397 (Fig. 4.14).

Fig. 4.14 Seth's drawing for the problem -426 + (29)

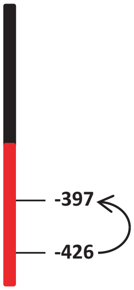
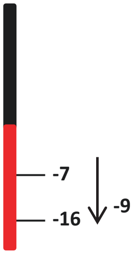


Fig. 4.15 Stuart's drawing for his solution to -7 – 9



Seth's explanation with his drawing can be cast as both his mathematical reasoning and a contribution to CMP4, particularly the idea that the VNL can be used to find the results of integer operations. Stuart also used the VNL as a reasoning device exemplified at the beginning of the whole class discussion on Day 17 for -7 – 9 (Fig. 4.15).

Stuart: I am going to make a number line. We are already down here -7 and what happened is that Norman rented a car he has to pay for it, so it is minus 9. That is -16.

The summary of each student's mathematical development can be seen in Table 4.5. One common theme is that the vertical number line was essential in supporting each student's mathematical reorganizations as they participated in and contributed to the establishment of the fourth mathematical practice. Also, none of the students contributed much to the establishment of the third taken-as-shared idea that subtraction of integers is not commutative, and there is no data to suggest the meanings they held for this idea.

Classroom Mathematical Practice Five

The remainder of the instructional sequence focused on the dual nature of the negative sign as both a characteristic of an object and as a transformation (Thompson & Dreyfus, 1988) or a state versus an operator (Glaeser, 1981; Streefland, 1996). In other words, a negative sign can be interpreted dynamically as "taking away" or statically as "negative" (Vlassis, 2004). Additionally, tasks were designed to support students exploring the "multiplication rules" (i.e., two of the same signs make

Table 4.5 Each student's participation in and contribution to CMP4

Student	Participation in/contribution to CMP4
Nathan	While Nathan had no difficulty interpreting the meaning of transactions in the context of adding or subtracting assets and debts, when it came to coordinating these interpretations with symbols in order to solve a transaction problem, it took him a total of 15 class periods to solidify his reasoning (participation in and contribution to CMP4)
Seth	Seth had no difficulty interpreting the meaning of transactions in context either and seemed to develop a stable understanding and strategy for coordinating transactions with net worths to calculate new net worths. This stability was rather quick in that it took three class periods, even though he had slight numerical (not integer) errors (participation in CMP4)
Stuart	Stuart also had no difficulty interpreting the meaning of transactions in context and seemed to develop a stable understanding and strategy for coordinating transactions with net worths to calculate new net worths. This stability was fast in that it took three class periods, even though there were slight numerical (not integer) errors (participation in CMP4)

Practice 5: Determining the meaning of positive/negative signs

a) Different operations (transactions) can have the same effect on a quantity.
b) A minus sign is different from a negative sign.

Fig. 4.16 Classroom mathematical practice five

a positive, two different signs make a negative). The final classroom mathematical practice that emerged involved the establishment of the following two mathematical ideas (Fig. 4.16).

Seth and Stuart developed a strong sense that two positives or negatives had the same effect on net worth. In fact, Stuart was credited in the whole class discussion as the first person to offer that idea as a conjecture and had it named after him. On Day 8, students solved the problem, *Maria has a net worth of 50 and – (−100) happens to her. What is her new net worth?* In the course of the whole class conversation, Stuart exclaimed:

Stuart: Wait a second, now I have a conjecture. If you write the symbol like that.
Teacher: Like what?
Stuart: You do – (-) and if you do + (+) you get the same answer if the numbers are same.
Teacher: Same thing is happening, I think we already have mentioned that [Adam mentioned it].
Teacher: It is smart decision, same thing happens, doesn't it? [Mark writes a conjecture and gives it to her, she reads it]. Actually, that is exactly what Stuart and Adam said.

Although Adam noticed the same idea in the discussion previously, for Stuart, this was a brand new conjecture. He excitedly presented it to the class, and the teacher acknowledged both he and Adam for their unique contribution. As it turned

Table 4.6 Each student's participation in and contribution to CMP5

Student	Participation in/contribution to CMP5
Nathan	Nathan did not contribute to the establishment of CMP5
Seth	Seth did not contribute to the establishment of CMP5
Stuart	Stuart contributed mostly to the establishment of the first mathematical idea in CMP5. He offered a conjecture that two of the same signs would produce a good result for net worth and later that opposite signs would have a bad effect (contribution to CMP5)

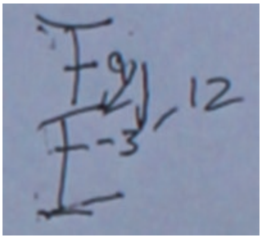


Fig. 4.17 Stuart's solution to the first number problem

out, in small group work in previous days, both Seth and Stuart noticed the same idea but did not generalize it to all integer pairs the way that Stuart did in class. In this way, Stuart's reasoning shows his mathematical abstraction and counts as a contribution to the establishment of CMP5. Later, the teacher asked Stuart to help her write his conjecture on the board (see Stephan & Akyuz, 2012, p. 455) and that is where opposite signs were said to have a bad effect on net worth (in addition to two of the same signs having a good effect). Nathan, on the other hand, recognized that a – (-) and a + (+) would individually make an improvement on his net worth, but never appeared to generalize these and opposite-signed pairs for all integers. A summary of the students' reasoning in the fifth classroom mathematical practice is summarized in Table 4.6.

Post-Interviews

As the analysis indicates, the students made incredible progress in their integer development. All three worked together in a small group, developing rapport and integer reasoning with mutual support. Their post-interview results suggest that each student became proficient at solving the bare number problems that they had such difficulty with before. In fact, Nathan, Seth, and Stuart correctly solved all of the bare number sentences, the temperature problem, and the money situation that was described in the pre-interview section. Nathan and Seth both used the language of assets, debts, and net worths in order to explain their solutions to each bare number problem. Stuart, on the other hand, solved the first one, 9 + (-12), by simply adding to get -21. When asked to explain, he paused and said that he wanted to check his work on a number line, drew it, and changed his answer to -3 (see Fig. 4.17).

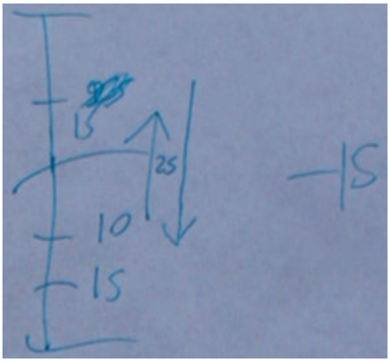


Fig. 4.18 Stuart's solution to the temperature problem

Table 4.7 Case study of students' test score results (30 is the highest score possible)

	Pretest raw score correct		Posttest raw score correct	
	Addition	Subtraction	Addition	Subtraction
Nathan	24	3	26	25
Seth	24	8	26	21
Stuart	16	2	20	14

For all future bare number problems, Stuart reasoned with a VNL and produced accurate interpretations and results. Additionally, he used a VNL for the temperature problem and stated that it helped him organize so many numbers (see Fig. 4.18).

Seth and Nathan both solved the temperature problem correctly this time by starting with -10, moving up 25 to +15° and down 30 more to -15° with no drawn number line. However, when pressed for an explanation, Seth could be seen gesturing up and down an invisible vertical number line on the table as he explained going to zero and then 15 degrees lower to his final answer.

The most significant findings involve the comparison between their scores on the integer pretest and posttest (see Table 4.7). All three students' scores for integer addition were already high on the pretest, but they made significant increases. Most notably, all three students made statistically significant gains on integer subtraction questions, a result that had eluded most research programs in the past (for more details see Stephan & Akyuz, 2012).

Conclusion

Our goal in this study was to examine how three students who have disabilities or math difficulties participated in and contributed to the development of the mathematical practices identified by Stephan and Akyuz (2012). To do so, we provided the mathematical reasoning of three students and how their reasoning supported the development of mathematical practices. Since we had rich data sources, such as small group audiotapes, videotapes of classroom sessions, pre-post interviews, test scores of students, and students' artifacts from each class session, we could examine their reasoning and determine how it contributed to each mathematical practice.

The pre-interviews and pretests provided us with useful insights about the three students' understanding of integers before the beginning of the instruction. We first found that although students could do reasoning and calculations with positive and negative integers, they mostly did not understand the meaning of their operations. Second, we observed that students felt more comfortable in some contexts than others. For instance, while the students appeared to struggle more with a temperature context, they seemed to be more successful with an "owing" and "owning" context. This gave us confidence about the net worth-based instructional sequence that was planned. Finally, we observed that students spontaneously attempted to use the horizontal number line during the solutions of the interview questions (see, e.g., Mukhopadhyay, Resnick, & Schauble, 1990). This also supported our intuition that a vertical number line, which matches better with the concept of "going up" and "going down" in finance contexts, could serve as a helpful tool during reasoning with integer problems.

After the instruction started, we observed how these three students contributed to the development of mathematical practices. From the analyses, we can conclude that the three students actively participated in the class and had a chance to develop intellectual autonomy (Kamii, 1982; Piaget, 1948/1973), answering our primary research question. This suggests that, if a suitable environment is created, even students who have disabilities and difficulties can make contributions to and participate in the development of classroom mathematical practices. This finding contributes to the existing literature in that most of the earlier studies advocate the use of a concrete-representational-abstract (CRA) approach to teach mathematics to students with disabilities and disorders (Miller & Hudson, 2007; Witzel et al., 2003). The current work shows that under a genuine inquiry approach (as opposed to blended inquiry Hudson et al., 2006; Scheuermann et al., 2009) that is empowered by a realistic instructional context, students with disabilities can also develop autonomy without resorting to direct instruction that limits their intellectual autonomy. In other words, the struggling students in this study were not given a manipulative and shown what to do with it. Rather, the students created meaning for their activity with integer quantities as they made sense of situations on their own or with partners. The VNL inscription was not imposed in a top-down manner, with certain steps to be memorized, but rather in a way that allowed students to use it in ways that made sense with their current reasoning.

Another important result we found was that the vertical number line served as a valuable tool in helping with the students' reasoning. In the CRA approach advocated by special education, instruction typically begins with a concrete, physical manipulative and proceeds toward more abstract representations and then symbols. However, because negative numbers cannot be genuinely represented with physical objects, using a concrete manipulative is not a viable starting point for integer operations, in our view. Rather, we found that grounding instruction in a realistic context of finance, coupled with the inscriptional device (VNL) and student imagery (e.g., *paying off*), was a more supportive instructional approach. We conjecture that the vertical number line worked well as it directly matched to the concept of going up and down in finance. It seems therefore appropriate to suggest that the tool that is used as an aid in reasoning should be compatible with the image and the words used within the context. A future comparison study could provide more evidence regarding how vertical and horizontal number lines affect the understanding of students within different contexts.

Finally, when we analyzed students' post-interviews and posttests, we found that the three students had a significant improvement in understanding integers, especially operations with negative numbers. Additionally, state test scores showed us that these students not only became successful in negative numbers but also with the other topics taught throughout the year. Nathan's test score increased from level 1 to level 3 (low non-proficiency to proficient), Stuart's from level 2 to level 3 (non-proficient to proficient), and Seth's from level 3 to level 5 (low proficient to the highest level of proficiency). We argue that students' growth is due in large part to the increased opportunity to explore and discuss mathmeatical ideas with their peers rather than be shown one method by the teacher. This underscores the importance of giving more opportunities to students with disabilities in which they can utilize their own cognitive resources to solve problems in conceptually meaningful ways.

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Chapter 5
Take It Away or Walk the Other Way? Finding Positive Solutions for Integer Subtraction

Julie Nurnberger-Haag

Abstract Practicing teachers as well as researchers, mathematicians, and teacher educators have offered opinions and theoretical critiques of the multiple models used to teach integer arithmetic. Few studies, however, have investigated what students learn with models or empirically compared affordances and constraints of integer models. This led me to investigate how 160 fifth- and sixth-grade students who were learning integer arithmetic for the first time could benefit from a particular model. Each integer model encouraged students to conceive of numbers using distinct conceptual metaphors and move in certain ways to represent integer subtraction. Thus, I used embodied cognition to illuminate ways a manipulative-based cancellation model (chip model) and a physically enacted number line model (walk-it-off model) differentially impacted students' subtraction knowledge. Integer subtraction, particularly the idea that subtracting a negative number could create a positive solution is especially difficult for students regardless of age, so assessment of this construct deserved a special focus in the test design of the larger study. This chapter reports students' accuracy and reasoning on this difficult subtraction type 5 weeks after instruction with their assigned model. Findings for practice suggest the walk-it-off model was more effective as the first model students used and more research is needed.

Internationally, middle-grade students find integer arithmetic difficult, and some of the issues students have with subsequent mathematics are due to these difficulties (Altiparmak & Özdoğan, 2010; Bishop et al., 2014; Gallardo, 2002; Ryan & Williams, 2007). About three decades ago, Thompson and Dreyfus (1988) identified the need for research to compare how different instructional models impact students' learning of integer addition and subtraction. Although this research is still in its infancy (e.g., Liebeck, 1990; Nurnberger-Haag, 2015; Tsang, Blair, Bofferding, & Schwartz, 2015), in practice, multiple instructional models are promoted in the methods textbooks from which prospective teachers learn how to teach integer

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arithmetic (e.g., Reys, Lindquist, Lambin, & Smith, 2014; van de Walle, Karp, & Bay-Williams, 2010). These models are then reinforced by many of the mathematics textbooks that schools expect in-service teachers to use with students. Although students physically move (or imagine moving) to use some integer models, embodied cognition perspectives are just beginning to be used to design studies that compare and analyze these models. Thus, to inform practice as well as research, I use perspectives and research from embodied cognition (Barsalou, 2008; Goldin-Meadow, Cook, & Mitchell, 2009; Lakoff & Núñez, 2000) to illuminate the ways a manipulative-based cancellation model and a physically enacted number line model differentially impacted students' learning of integer subtraction.

Integer Arithmetic and Instructional Model Research

The literature review focuses first on prior research about integer addition and subtraction relevant to the current study, followed by investigations of integer models.

Integer Arithmetic Knowledge

Robust integer knowledge encompasses many ways of thinking, ordering numbers, arithmetic proficiency with the four primary operations, multiple meanings of the negative sign, and applications to various contexts (e.g., Bofferding, 2014; Chiu, 2001; Lakoff & Núñez, 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010; Thompson & Dreyfus, 1988; Vlassis, 2008). The *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010) used in the United States suggests that sixth-grade students should learn what negative numbers are and notation for the opposite of positive numbers. It is not until seventh grade when students are scheduled to learn the primary operations of rational numbers, which include integers (NGA & CCSSO, 2010).

Elsewhere I analyze student data of all four primary operations together as well as other integer constructs (Nurnberger-Haag, 2015) because students' reasoning among the operations is related. For example, after learning addition and subtraction, some students mistakenly treat the negative signs in products as though they are subtraction signs (Ryan & Williams, 2007; Vlassis, 2008). Given the difficulty arising from integer subtraction, it deserves a closer look. In particular, the idea that subtraction of negative numbers could yield positive solutions is challenging (Ryan & Williams, 2007). With nonnegative integers, subtraction is more difficult than addition (Fuson, 1990), so this is one reason subtraction of negative numbers is also difficult. Periasamy and Zaman (2009) who created a 24-item measure of integer subtraction found 14-year-olds in Malaysia who had studied integer arithmetic still found many subtraction problems involving negative numbers troublesome.

Depending on the problem types, accuracy of items ranged from about 38% to 63% on single-digit (SD) subtraction tasks and from about 36% to 66% on double-digit (DD) subtraction tasks. Only about half of the students answered -5 – 2 correctly (Periasamy & Zaman, 2009, p. 366), but a broader sample of the same age students in the United Kingdom fared worse on a similar problem, with about one-third correctly answering the problem -6 – 3 (Ryan & Williams, 2007, p. 218). In Periasamy and Zaman's (2009) study, students had the most difficulty on all types in which the minuend was a negative number (both SD and DD subtraction problems).

Another reason that subtraction of integers is problematic is that students typically generalize from whole-number operations and often have been explicitly told that adding makes a number have a greater solution, whereas subtracting makes a number smaller (Karp, Bush, & Dougherty, 2014). Although the items just described are consistent with this generalization, some integer subtraction problem-structures violate this generalization and pose additional difficulties for students. Regardless of whether the problems were SD or DD, only about 40% of students could accurately answer problems with the structure $N_1 - N_2$ where $N < 0$ and N_2 has a greater absolute value (Periasamy & Zaman, 2009, p. 376–77). Such problems not only contradict the generalization from whole numbers by having solutions greater than the minuends, but students must accept that what began as a negative number became a positive solution.

Instruction of Integer Subtraction and Addition Using Models

Three categories of instructional methods for teaching integers are typical: (a) cancellation models in which two objects cancel, (b) number line models, or (c) abstract methods (Küchemann, 1981). However, educators, including researchers, disagree about the efficacy of these methods. Since this report focuses on integer models, cancellation and number line models are discussed in more detail.

Cancellation Models Models for "which the integers are regarded as discrete entities or objects, constructed in such a way that the positive integers cancel out the negative integers" are cancellation models (Küchemann, 1981, p. 87). Cancellation models may be rooted in contexts such as hot and cold cubes or charged particles, may involve games, or may be acontextual, such as color-coded objects (Cotter, 1969; Goldin & Shteingold, 2001; Jencks & Peck, 1977; Liebeck, 1990; Linchevski & Williams, 1999; Ponce, 2007). Such models may foster students' learning of integer subtraction because students can literally remove or take away objects that represent positives and negatives analogous to how they used to represent subtraction of whole numbers (Küchemann, 1981; Liebeck, 1990). Others see additional mathematical value of representing integer operations with things that cancel because the models could allow students to explore that taking away positive numbers or adding negative numbers corresponds to subtraction as adding the additive inverse (French, 2001; Linchevski & Williams, 1999; Semadeni, 1984).

Of the many cancellation models portrayed in practitioner journals, online, in mathematics textbooks, as well as in textbooks for how to teach mathematics, chip models are prevalent. These models typically represent opposite numbers with different-colored chips (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006; van de Walle, Karp, & Bay-Williams, 2010). Some researchers share the perspective that chip models and some other cancellation approaches do not reflect real mathematics but are "formed artificially" for instruction in school mathematics (Roussat, 2010; Schwarz, Kohn, & Resnick, 1993–1994; Umland, 2011; Vig, Murray, & Star, 2014). In a theoretical analysis, Vig, Murray, and Star (2014) explained that rules to use models that are not mathematical rules are where the models "break" (Vig et al., p. 74). For example, some integer subtraction problem-structures break or violate mathematical rules because students need to add more chips in order to have enough chips to subtract (Vig, Murray, & Star, 2014). Although Liebeck (1990) promoted a cancellation model, she also noted that students found these problem types, in which they needed to add more chips before subtracting, the most difficult. Thompson and Dreyfus (1988) also noted that cancellation models fail to represent the "negation" or opposite operation meaning of the negative sign (p.131). These analyses have not considered embodied cognition. In the Theoretical Influences section, I will further analyze these issues with chip models in terms of embodied cognition.

Number Line Models Students' learning of integer addition and subtraction with number line models has been studied in various contexts such as temperature, elevation, and animals or humans on a number line or elevator (e.g., Sfard, 2007; Thompson & Dreyfus, 1988). Proponents claim using number lines for integer instruction is better than cancellation models precisely because they avoid treating numbers as things, and number lines are also disciplinary representations (Freudenthal, 1973; Thompson & Dreyfus, 1988). With little empirical evidence, some argue that students find number lines intuitive for addition and subtraction (Freudenthal, 1973), whereas others claim number lines are not helpful for integer subtraction (Küchemann, 1981). These publications refer to "the" number line model as though students and educators use the representation of a number line in a uniform way (e.g., Küchemann, 1981; Liebeck, 1990). I use the term typical number line model to refer to this model that authors have typically referred to as "the" number line. I will discuss variations of number line models as part of the theoretical perspectives section.

Typical number line models have been theoretically critiqued about the ways they represent addition and subtraction (Bofferding, 2014; Nurnberger-Haag, 2007; Stephan & Akyuz, 2012). Difficulties and benefits students have using a number line to add and subtract have been investigated with students from at least kindergarten to sixth grade and beyond (e.g., Bofferding, 2014; Bruno & Martinon, 1999; Ernest, 1985; Thompson & Dreyfus, 1988). For example, Thompson and Dreyfus (1988) found that after instruction on addition, the two sixth-grade students still had difficulty with problem-structures that did not have solutions that were the sum of the magnitudes (e.g., -30 + 20 was more difficult than -30 + -20). The aspects of the study that are most salient for the analysis here are the instruction group size, time, and model used. The students experienced more specialized instruction than possible in typical classrooms, because each student worked with a researcher for what Thompson and Dreyfus (1988) called "11 highly individualized lessons" over the course of 6 weeks (p. 130). After using a number line model, students still had difficulty with integer addition. Although such individualized research informs the field about how students think, the fact that students still found addition so difficult after individualized instruction, which is impractical in real classrooms, is troublesome (Thompson & Dreyfus, 1988). This also opens the question of how well certain number line models support students' learning of integer subtraction, given that subtraction is more difficult.

Research Needed to Compare Models How, if at all, can models support building procedural fluency and conceptual understanding of integer subtraction? At least one study used traditional cognitive perspectives to experimentally compare a number line model to no model (Moreno & Mayer, 1999). From mathematics education perspectives, research of addition and subtraction using cancellation or number line models has investigated how students learn with a single, researcher-developed model embedded in a context (Linchevski & Williams, 1999; Pettis & Glancy, 2015; Stephan & Akyus, 2012; see also, Chaps. 4 and 10). For the single operation of addition, Tsang, Blair, Bofferding, and Schwartz (2015) investigated a hybrid model that integrated cancellation on a number line (stacking and folding conditions) compared to what seemed to have been a typical number line model (jumping condition). This study found no calculation accuracy differences between conditions for addition, but the students in the folding condition more frequently expressed a symmetric conception of integers, which is an important aspect of integer knowledge (Tsang, Blair, Bofferding, & Schwartz, 2015).

To my knowledge the only existing study to compare a cancellation model (not a hybrid model) to a number line model is an often-cited study, which compared students' addition and subtraction performance after students were assigned to learn with a manipulative chip model embedded in a context or an acontextual number line model that students enacted by walking (Liebeck, 1990). That report concluded the chip model with a scoring context was better than the number line model tested (which was a typical number line model). Although that study is frequently cited, there were many limitations that invalidate the conclusions. For example, the instructional approach in each condition was quite different, different teachers instructed each condition, and only ten students participated in each group. Moreover, no pretest was used to verify similarity of groups prior to instruction or to make it possible to identify whether instruction was the reason for student knowledge found at the time of testing, not to mention that all of the problems on the delayed posttest (the only test given) had numbers with an absolute value less than or equal to 3.

Theoretical Influences

In spite of the dearth of research about the benefits and problems with individual models, the same students may be exposed to multiple instructional models. Minimally, curricular and methods textbooks often promote one model from each class of cancellation and number line models. On the one hand, some textbooks state differences exist and argue the merits of either a cancellation model or a number line over the other as previously described (e.g., Küchemann, 1981; Thompson & Dreyfus, 1988). On the other hand, a popular methods text informed future teachers there are no differences: "Although the two models appear quite different [chip and number line], they are alike mathematically" (van de Walle, Karp, & Bay-Williams, 2010, p. 499). This perspective that two models are the same because the targeted performance is the same limits the field's understanding of how different models foster different ways of thinking, such as the conceptual metaphors students enact with such models and how this influences student learning.

Conceptual Metaphor Theory

Given emerging research about how cognition is embodied (Barsalou, 2008; Gibbs, 2011), it behooves us to investigate how the ways students are encouraged to move and think during mathematics learning influences their achievement and understanding. Conceptual metaphor theory (CMT) and empirical research showing that how humans move is part of cognition would seem to be crucial perspectives to shed light on ways integer instructional models may differentially impact cognition and learning.

The "fallacy …that metaphor is only about the ways we *talk* and not about conceptualization and reasoning" (Lakoff & Johnson, 2003/1980, p. 245), likely, has been a barrier to educational researchers considering conceptual metaphor theory as a theoretical frame. So let's consider an example of a conceptual metaphor unrelated to mathematics to illustrate how humans pervasively use conceptual metaphors to reason about concepts. For example, to conceive of the abstract concept of what an "idea" is, humans across many cultures conceive of "ideas" in terms of various conceptual metaphors that afford different inferences or meanings about that concept (Lakoff & Johnson, 2003/1980). Some examples are IDEAS ARE BUILDINGS, IDEAS ARE PRODUCTS, IDEAS ARE RESOURCES, or IDEAS ARE LIGHT-SOURCES. None of these ways of conceiving of abstract "ideas" fit literal meanings because ideas do not really have foundations, are not concrete things that can be made in factories or resources obtained from the world, nor are ideas visible to the human eye. Yet, humans make sense of such an abstract concept by thinking about them in terms of their prior physical experiences with foundations, things that can be tangibly bought or produced, or seen. CMT has been extensively debated and critiqued, especially in terms of its applicability to mathematics (Gibbs, 2011; Lakoff & Núñez, 2000; Sinclair & Schiralli, 2003; Wood, 2010). Yet, when Gibbs (2011) reviewed these critiques and empirical evidence, he concluded that "CMT …has great explanatory power,…as well as for broader theories of human cognition" (p. 556).

Drawing on the theoretical work of Lakoff and Núñez (2000), Chiu (2001) empirically found evidence that facility with multiple conceptual metaphors may characterize expert understanding of integer arithmetic. For example, conceiving of negatives as objects that cancel with positives may be necessary for ideas of chemical reactions, but it distinctly differs from the ways people think about numbers as lengths or positions on a number line. To rely exclusively on either conception could interfere with students' developing understanding in varied contexts that draw on other metaphors, such as thermometers or elevation. Research of procedural and conceptual use of integer arithmetic via conceptual metaphors is in its infancy. Thus far, investigations have assessed what metaphors people used while thinking (Nurnberger-Haag, 2013; Chiu, 2001; Kilhamn, 2011), rather than how specific metaphor-based physical motions that integer models promote impact students' learning.

In this chapter I use CMT to analyze integer arithmetic learning with models because it affords a theoretical lens based on broader theories of human cognition to distinguish the mathematical representations from the distinct ways humans think about these representations. The ARITHMETIC AS COLLECTING OBJECTS,[1] ARITHMETIC AS MEASURING, and ARITHMETIC AS MOVING ALONG A PATH metaphors are conceptual metaphors relevant to cancellation and number line models (Kilhamn, 2011; Lakoff & Núñez, 2000; Nurnberger-Haag, 2013, 2015). Chip models in particular encourage an object-based conception of what a negative quantity is and that to calculate with negative numbers, a person collects and regroups these positive and negative objects in various ways (COLLECTING OBJECTS metaphor). Although all number line models use a commonly accepted representation of a number line, there at least two ways to think with a number line by drawing on different conceptual metaphors. As Descartes did, a number line representation can be thought of using a MEASURING metaphor (Berlinghoff & Gouvêa, 2002; Lakoff & Núñez, 2000) in which numbers are thought of as simply the end of a length. Number line representations can also be thought of with a MOVING ALONG A PATH metaphor (Kilhamn, 2011; Lakoff & Núñez, 2000; Nurnberger-Haag, 2007), in which numbers can be conceived of as points on a line found by moving to that point and operations can be conceived of as motion along that line.

Integer Model Movements

How people physically move impacts their cognition (Antle, Corness, & Bevans, 2013; Barsalou, 2008; Glenberg & Kaschak, 2002; Kontra, Fischer, Lyons, & Beilock, 2015). When adults and children physically move in ways that are

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[1] I use the verb forms as opposed to the noun forms of the metaphors to better reflect the patterns of interacting with the world as part of an ongoing dynamic system, in other words "enactive metaphors" (Gallagher, & Lindgren, 2015; Nurnberger-Haag, 2014; Smith, 2005).

consistent with intended ideas, this aids cognition, whereas when prompted to move in ways that are inconsistent with intended ideas, this interferes with cognition (Anelli, Lugli, Baroni, Borghi, & Nicoletti, 2014; Day & Goldstone, 2011; Glenberg & Kaschak, 2002; Goldin-Meadow, Cook, & Mitchell, 2009; Kontra, Fischer, Lyons, & Beilock, 2015). Just as humans' cognition is influenced outside of the classroom by their physical body movements, students' cognition may be just as influenced or potentially more so when learning something difficult and effortful in classrooms. Evidence has shown that when students' physical motions are consistent with both whole-number and integer arithmetic operations, students learn more than when their physical motions are inconsistent with the intended operations (Goldin-Meadow, Cook, & Mitchell, 2009; Nurnberger-Haag, 2015). Research about the difficult topic of integer subtraction would benefit from understanding how students' explicit or implicit physical motions with such models and the conceptual metaphors models encourage students to enact may influence learning. I define model movements as the patterns of physical motions found when different students and teachers enact particular models, not the idiosyncratic motions of individuals. Different patterns of model movements with chips constitute different models as do the unique sets of ways students can move on number lines.

Number Line Model-Movement Consistency with Subtraction A typical number line model often found in schools and online resources (e.g., Math Forum, 2001) has students enact a MOVING ON A PATH metaphor by facing the positive or negative direction depending on the sign of a number (negative or positive number, respectively). To represent an addition or subtraction operation, students move forward (addition) or backward (subtraction). In other words, typical number line models inform students to use the signs of the problem to follow a fixed system of *which* direction to face and *which* direction to move. Typical number line models do not include an opposite meaning of the "-" symbols, because they represent "-" with backwards and forwards motion or positive or negative facing directions. Although children in gym class are sometimes asked to run drills backwards and forwards, walking backwards is not typical of how humans move.

In contrast, students always walk forward with the walk-it-off number line model (referred to subsequently as walk-it-off; Nurnberger-Haag, 2007), which encourages students to enact the same metaphor with movements that differ from a typical number line model and different ways of thinking about how those movements represent the mathematical symbols. The meaning of the "-" symbol as the opposite was designed into the walk-it-off model to promote the opposite operator meaning of the negative sign important in algebra that was missing from typical number line models and chip models (Nurnberger-Haag, 2007; Thompson & Dreyfus, 1988). From an embodied perspective, since humans typically walk forward in the world or specifically when walking on a path, this may feel more intuitive to enact a MOVING ON A PATH metaphor using forward motion. Students begin facing the whole numbers with which they are most familiar and read the signs (negative signs, positive signs, or addition and subtraction signs) to decide in what direction to move relative to their current position. In other words, the walk-it-off model has students consider whether to *change direction* or go in the *opposite* direction (Nurnberger-Haag, 2007). To subtract with the walk-it-off model, the minuend is a location on a number line. Students always begin facing toward the default positive numbers, rather than determined by the sign since the minuend is a location not a direction. A subtraction sign means turn the opposite direction (an addition sign means to maintain direction). The subtrahend is a directed magnitude, so this signals whether to turn the opposite direction and also the distance to move. That is, if the subtrahend is positive, a student maintains the direction and walks the given magnitude, but if the number is negative, then the student turns the opposite direction and walks the given distance. Figure 5.1 illustrates an example problem.

To summarize then the key difference between typical number line models and the walk-it-off model is that typical models tell students *which* direction to face, whereas "+" and "-" symbols in the walk-it-off model mean *whether to change* direction (Nurnberger-Haag, 2007). When the symbol "-" represents subtraction or a directed subtrahend in the walk-it-off model, the symbol means turn the *opposite direction*, and the sign "+" means continue in the same direction (Nurnberger-Haag, 2007). For this reason, the walk-it-off model was the MOVING ON A PATH metaphor-based model used in this study.

Chip Model-Movement Consistency with Subtraction Just as there are multiple ways to move on number lines that constitute different models, I use embodied perspectives on learning to point out there are also multiple chip models (see, e.g., Chap. 10). In this chapter I refer to the common chip model, shown in Figure 5.1, as extra-zero-

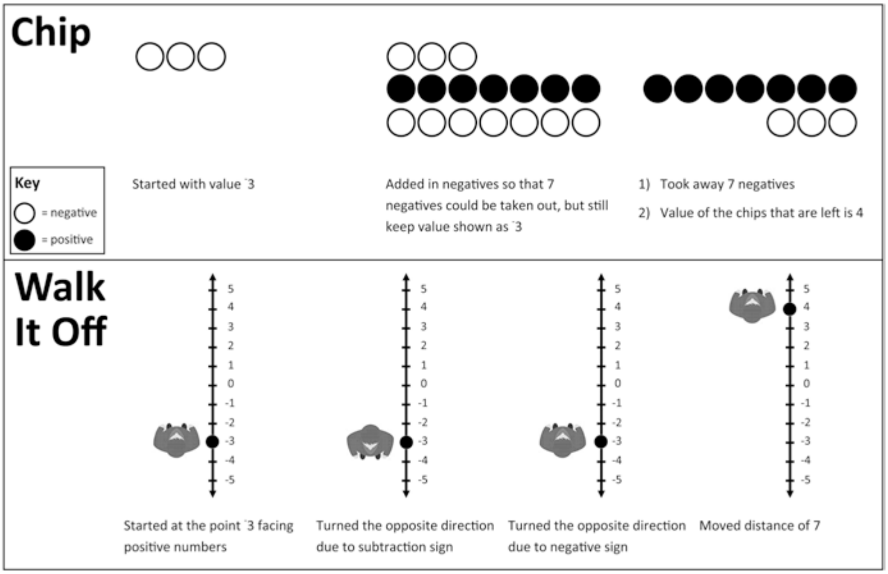


Fig. 5.1 An example of how a group from each integer model condition solved the problem -3 – -7 during instruction

value-when-needed chip model and the other chip models as extra-zero-value-at-setup (see Chapter 10 this volume). Although the timing of movements differ among these chip models, the language students can use to describe their physical movements is the same for both models, such as “put in,” “take out,” or “take away.” With the extra-zero-value-at-setup chip model, at the onset of any problem, students put in several chips, which represents a zero value (e.g., several white chips and the same number of black chips). Regardless of the target problem, students who enact an extra-zero-value-at-setup model use multiple chips to represent zero at the onset. However, students who use the more common chip model only put in this inconsistent zero value when the problem-structure creates this need (extra-zero-value-when-needed). For example, to do the problem -5 – -3 with the common chip model, a student can begin with five negative chips and remove three negatives to model the subtraction operation without adding any quantities of chips that are inconsistent with the mathematical problem. In contrast, the problem -5 – 2 would require putting in or collecting enough objects to remove two positives (i.e., two positives and two negatives). From an embodied perspective, it is only because the model expects students to think of negatives as physical things (i.e., a COLLECTING OBJECTS metaphor) that to do the problem -3 – -7 in Fig. 5.1, students actually move in ways that complete the problem -3 + (-7 +7) – -7. Although the problem could be accomplished by putting in various zero value quantities (e.g., see Chap. 10), the chip model necessitates that students move chips in these certain ways due to the nature of the model in which actual objects represent negative quantities. Therefore, as shown in these examples, the structure of each subtraction problem requires different sequences of model movements and different model-movement consistencies.

Although someone might argue that the extra-zero-value-at-setup is simpler than the extra-zero-value-when-needed chip model, each is more simple or complex in different ways. The first requires extra work for all problem types, and students must anticipate how large of a field of objects they need to represent additive inverses. This makes this model more complex in one way, yet simpler in that all extra motions occur at setup. In contrast, with an extra-zero-value-when-needed model, students enact extra motions to add zero values before performing an operation, which may be simpler because students only think of unnecessary quantities if needed. Consequently, for this first study comparing a COLLECTING OBJECTS-based integer model to a MOVING ON A PATH integer model, the models compared were walk-it-off and extra-zero-value-when-needed, which, in the rest of the chapter, I will simply refer to as chip model.

Focus of Study

The purpose of the larger study from which this data was drawn was to investigate benefits and issues with using either this chip model or the walk-it-off model as the first model students use in formal instruction of all four primary operations with negative numbers. An original goal was to better understand which conceptual metaphor might better support initial procedural and conceptual learning of integer arithmetic, and the focus on metaphor-based models was, in part, to provide theoretical and practical insights educators could use in their own classrooms. Many metaphor-based model pairs could have been compared, but in light of cognitive alignment between theorized model movements and integer arithmetic, to begin this research of comparing metaphor-based models, I selected the potentially most-aligned model for each metaphor.

Given that the larger-scale analyses of integer arithmetic with all operations found that longer-term results 5 weeks after instruction can differ from performance on day-after tests (Nurnberger-Haag, 2015), the analysis here focuses on this longer-term understanding by analyzing delayed posttests. This chapter focuses on how students reasoned after instruction of a single integer model that promoted either the COLLECTING OBJECTS metaphor or the MOVING ON A PATH metaphor with respect to subtraction of negative numbers. Earlier analyses demonstrated that the walk-it-off model better supported overall integer arithmetic performance than the studied chip model (Nurnberger-Haag, 2015), but this analysis focuses in-depth on how students reasoned in terms of conceptual metaphors on counterintuitive integer subtraction problems. Similar to how another study focused on a very specific aspect of integer knowledge, such as the property of additive inverses (Tsang, Blair, Bofferding, & Schwartz, 2015), this analysis focuses on whether and in what ways students who used either model were able in the longer term (5 weeks after instruction) to produce positive solutions when subtraction problem-structures warranted it (i.e., solutions were greater than zero) and if they could extend their understanding to more difficult problems of the same type. The specific research questions reported here are as follows:

- Which, if either, model better supports students to accept solutions greater than zero (positive solutions) for appropriate integer subtraction problems?
- Which, if either, integer model better supports students to extend their knowledge to larger magnitudes not experienced during subtraction instruction?
- In terms of conceptual metaphor theory, how did students reason on single-digit subtraction compared to double-digit subtraction? What similarities and differences occurred due to integer model?

Methods

The data reported here were collected as part of a larger pre-post-delayed posttest instructional study, in which I randomly assigned classes in two rural districts to instruction with the COLLECTING OBJECTS-based (extra-zero-value-when-needed chip model) integer model or the MOVING ON A PATH-based (walk-it-off number line) integer model. I taught both models in both districts during the year prior to when negative number arithmetic is taught in each district. The state website indicated that the students at this grade level at both districts (first semester sixth grade at District A and second semester fifth grade at District B) were primarily European American, and 45% of the students were eligible for free or reduced lunch. I taught four classes with the chip model, and the other four classes experienced the same lessons with the walk-it-off number line model. Overall, the eight-lesson unit addressed ordering numbers, addition, subtraction, multiplication, and division. Analyses of pretest integer operations and whole-number fact tests confirmed that the students assigned to a chip model or walk-it-off model instructional conditions were not significantly different (Nurnberger-Haag, 2015). Rather than posttests, to assess longer-term learning, the focus of this chapter was on the delayed posttests of students assigned to the chip model ($n = 80$) and walk-it off model ($n = 80$); their pretests were considered to look for confirmatory or contradictory evidence that instruction supported learning.

Data Sources

The seven-item Explain and Draw Test (EDT) used open-response items intended to elicit explanations using words and drawings. The EDT assessed all the constructs students experienced during instruction (ordering numbers, addition, subtraction, multiplication, division) and an additional construct not taught regarding the notation for opposites of expressions. The EDT included two subtraction problems: (1) a SD subtraction problem that assessed students' ability to provide a positive answer when subtracting two negatives (e.g., "Show the students what -3 – -5 means.") and (2) a DD subtraction problem that assessed if either integer model used during instruction better supported students' ability to extend their reasoning to larger magnitudes than experienced during instruction (e.g., "Show the students what -52 – -85 means."). The wording of the items was informed by Chiu (2001), and EDT directions prompted students to "explain in words and by drawing" so that another student their age who had not already learned this would understand.

Instruction

When I first learned about chip models in 1991, I wished my prior teachers had taught me with the chip model, because as someone who understood integer operations, I thought it made so much sense and continued to think so throughout my teacher training and later teaching. In fact, as a student teacher, I found the typical number line model problematic in terms of how it represented the numbers and operations, which is why I set out to design a number line model that incorporated the opposite meaning of a negative sign (Nurnberger-Haag, 2007). It was my own students who helped me understand that what I thought was intuitive about chip models may not be so for students. This prompted my desire to better understand how students learn integer arithmetic with chip and number line models.

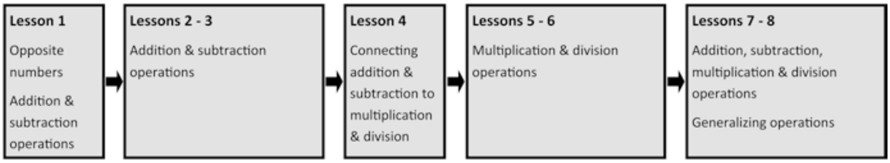


Fig. 5.2 Primary topic focus for each of the eight lessons

For this first study comparing chip and number line models, I assigned students to trios that reflected a range of pretest scores on a 46-item skill-based integer operation test and a whole-number single-digit fact test of the 4 primary operations (used as a proxy for prior student achievement in arithmetic). Students worked on tasks with their assigned trio (or duo if class size required it) during the entire unit. After trios worked on sets of related tasks or other learning opportunities, I led whole-class discussions about their processes.

The instructional unit consisted of eight lessons. Figure 5.2 illustrates the focus of each lesson's instruction. Since this chapter focuses on particular types of subtraction problems, details relevant to subtraction are discussed further. During instruction, students primarily had opportunities to subtract single-digit numbers and sometimes double-digit numbers ranging from -20 to 20. Further, student groups explored tasks such as writing addition and subtraction equations to make 0 and -4. These task prompts had additional constraints to ensure that students grappled with at least one equation involving subtraction of a negative number. The students also played a dice game to provoke students to discover that subtraction can make a number larger, smaller, or maintain the same solution depending on the quantities subtracted. I never taught the students rules typical of algebra textbooks. At the end of the unit, however, students were prompted to individually make generalizing conjectures, some of which were selected for small group debate as *never true*, *sometimes true*, or *always true* with follow-up class discussion (Heck & DeFord, 2012).

Students had several opportunities to explore the range of subtraction problem types of negative numbers with their respective model. Instruction began by building on students' understanding of concepts and procedures of addition and subtraction with nonnegative integers to enact the assigned metaphor and then extending the metaphor to work with negative numbers. In this way, even though the final efficient uses of each model involved particular procedures, these procedures were built from and connected to students' prior understanding. In the chip model classes, students were first encouraged to build on their whole-number understanding of addition and subtraction with a COLLECTING OBJECTS metaphor by using black chips to act out these operations, and they identified that white was the opposite of black, so they agreed to treat white chips as negative numbers. I then encouraged them to grapple with how to move chips when there were insufficient chips of the right kind to remove. In the classes where students were assigned to use the walk-it-off model, students were given 10-foot open-number lines (i.e., students had to draw any tick marks or numbers) to place on the floor then try whole-number addition and subtraction by physically enacting a MOVING ON A PATH metaphor. I then had them consider how their movements must differ for subtracting a whole number versus a negative number. For each model, we explicitly discussed the meanings of each symbol within bare-number problems (operations and signs composing numerals) .

Data Analysis

Two coders qualitatively coded the 160 EDT delayed posttests. I was one coder and at the time of this analysis had 20 years teaching experience. The second coder was a doctoral student with a master's degree in mathematics and 11 years teaching experience. Since pseudonyms can evoke unintentional bias, I use students' randomly assigned identification numbers. Gender pronouns are used solely to aide readability.

Single-Digit Problem with a Positive Solution For this analysis of 160 students' delayed posttest solutions to the SD item, success was defined as a solution greater than zero with supporting reasoning (i.e., explanations that did not contradict the solution provided, even if the students made a calculation error). I removed 19 students whose explanations supported their positive solution but actually explained a different problem (such as when attempting to solve -3 – -5, the solution given really explained 0 – -3 = 3; or when the student used a rule such as "subtracting two negative numbers always has a positive answer" that was mathematically valid for this problem-structure but would be inaccurate if a student used the rule to solve -5 – -3). On the SD item, 56 students (35%) provided positive solutions with reasoning that did not contradict their solution or the original problem. Note that although coders allowed for positive solutions with calculation errors, 54 of the 56 students' positive solutions on the EDT SD subtraction problem were accurate.

Extension to Double-Digit Subtraction Analyses of extension from SD to DD subtraction included accuracy, consistency or sameness of reasoning from one problem-structure to the other, and conceptual metaphor expressed.

Extension The goal of the extension analysis was to determine successful or unsuccessful extension from SD to DD subtraction, where successful extension involved students providing accurate solutions with reasoning that could support the solution on both SD and DD problems (we allowed for calculation errors if the processes could lead to accurate solutions). To develop these final categories, both coders conducted multiple passes of the data, determining ways that students extended their understanding. In a single pass, the coders independently coded each student's delayed posttest in order of randomly assigned identification numbers. The coders then discussed the codes and, to prevent coding drift in a later week, conducted additional passes in the same way to confirm and refine existing codes. A final pass of the data was conducted beginning with the assigned code (i.e., evidence of extension, potential extension, no evidence of extension, or evidence did not extend) to confirm that a student with this code in comparison with the other students with the same code was still validly coded. Two of 56 codes were changed (96.4% agreement from prior coding pass) because both coders agreed with each other that the prior code was not accurate. Inter-coder agreement of the final data was 100%.

Sameness of Reasoning from SD to DD While blind to students' assigned integer model, each coder documented whether the reasoning the student used on the DD problem was the same as the reasoning used on the SD problem. The a priori codes used were *same*, *different*, *unable to code for sameness*, or *missing due to a skipped problem.*

Conceptual Metaphor Coding Although students may be most likely to express the same conceptual metaphor they experienced during instruction, this should not be assumed. Each model encourages students to enact a particular metaphor; however, students may conceive of numbers using a different metaphor because the assigned model differed from their prior conceptions or other reasons. Consequently, whether students expressed a particular conceptual metaphor was documented blind to student condition and then compared to the assigned model only after this analysis.

Students could use one or more conceptual metaphors or no metaphor. We categorized a student's response as a COLLECTING OBJECTS metaphor if the words or the drawings involved particular objects or things (e.g., "take away negatives" or "take out more"; drawings of circles or actual objects). Documentation of a MOVING ON A PATH metaphor occurred if students indicated with words or drawings that something or someone moved along a drawn or imagined line or path (e.g., "walk six," "go lower into the negatives"; drawn arrows or hops along the line) and if the locations or points on a number line were considered numbers (e.g., "here is zero"). MEASURING was documented if static distances were shown (e.g., the distance between two numbers on a number line). If no reasoning was provided, this was coded as no metaphor expressed, regardless of whether an answer was provided or the entire problem was skipped. The no metaphor code was also used if these conceptual metaphors were not found, such as if the reasoning stated a generalization or rule. In the rare instances in which the reasoning did not fit into any of these categories or where an explanation could not be distinguished between types, coders documented this as unable to code metaphor.

Findings

To provide an overview, I first compare frequency of student success by integer model on the delayed posttest SD problem that was similar to instruction ($N = 160$). Then I analyze the subset of students who successfully answered the SD problem (*n* = 56) to determine whether they extended their understanding of single-digit subtraction to the DD problem. This analysis of extension beyond instruction was explored in several ways. First I explore whether students changed their reasoning approach from SD to DD subtraction as well as the relationship of the change to accuracy on the DD problem. Then I detail how students' expression of conceptual metaphor related to their solution accuracy and the conceptual metaphor they enacted during instruction. In the rest of the findings for efficiency, SD will stand for the "single-digit subtraction problem" and DD as "double-digit subtraction problem."

Subtraction Problems Similar to Instruction

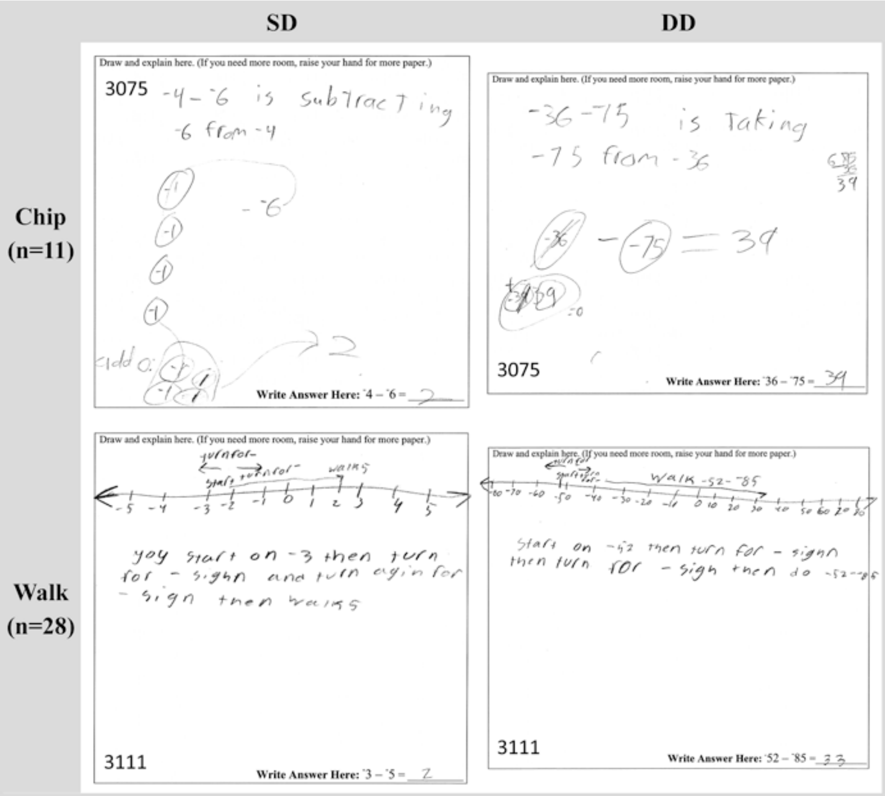
This section reports on student responses to subtraction items that have positive solutions. Students who learned with the walk-it-off model provided positive solutions 1.8 times more than those who had learned with the chip model (36 walk and 20 chip). Close to half of the students who learned with the walk-it-off model provided positive solutions to these subtraction problems (45%), whereas a quarter of the students who learned with chips did so (25%). The students' pretests were then checked to consider if any of the students correctly answered this type of subtraction problem prior to instruction. Only two students (3.6%) accurately answered SD at pretest with a reasonable supporting explanation, and they also correctly answered DD. Both of these students experienced the walk-it-off model instruction and continued to accurately answer both SD and DD at delayed posttest.

Extension Beyond Instruction

When considering the entire sample of 160 EDT delayed posttests, 35% (28/80) of students who learned with the walk-it-off model successfully extended their SD understanding beyond the instructional tasks to DD, whereas 14% (11/80) of students who learned with chips did so.

Frequencies of Extension Success Categories About 70% of the 56 students who were successful on SD extended this knowledge to DD. Table 5.1 displays whether students extended their knowledge from SD to DD by integer model and the strength of that evidence. Recall from the Methods section that coders were blind to student condition at the time of analysis. I retrospectively added model condition to Table 5.1 during the next phase of analysis and reporting.

Successful Extension Frequency Five weeks after instruction, about 2.5 times as many students who learned with the walk-it-off model (*n* = 28) than students who learned with chips (*n* = 11) successfully extended their understanding from SD to DD. Figure 5.3 displays SD and DD explanations of a student from each model who successfully extended. Looking only at the strongest evidence of successful exten-



Draw and explain here. (If you need more room, raise your hand for more paper.)

-4 - ⁻6 is subtracting -6 from -4

Write Answer Here: ⁻4 – ⁻6 = 2

Draw and explain here. (If you need more room, raise your hand for more paper.)

-36 - ⁻75 is Taking -75 from -36

Write Answer Here: ⁻36 – ⁻75 = 39

Draw and explain here. (If you need more room, raise your hand for more paper.)

you start on -3 then turn for - sighn and turn agin for - sign then walks

Write Answer Here: ⁻3 – ⁻5 = 2

Draw and explain here. (If you need more room, raise your hand for more paper.)

start on -52 then turn for - sighn then turn for - sign then do -52-⁻85

Write Answer Here: ⁻52 – ⁻85 = 33

Fig. 5.3 Examples of students from each integer model who successfully extended their reasoning from SD to DD

sion in the first row of Table 5.1, four times as many students who learned with the walk-it-off model as chips provided clear evidence that they extended their understanding from SD to DD (if the two students extended at pretest are removed, the rate is 3.6 times).

Unsuccessful Extension Frequency Notice in Table 5.1 that equal numbers of students who learned with chip or walk-it-off intentionally skipped the DD task, providing no evidence of extension. I considered these skipped items unsuccessful extension, because in typical classroom instruction, a skipped problem is counted incorrect. Similar numbers of students using either model gave responses demonstrating they did not extend their understanding (see Table 5.1).

Analysis of Extension Reasoning Overall, the students who were successful used the same reasoning for SD and DD. Only students who changed their reasoning approach were unsuccessful at extending to double-digit magnitudes greater than

Table 5.1 DD extension status for students who successfully solved the SD by integer model

Extension	Chip	Walk	Total
Successful			
Evidence of extension	5	20	25
Potential extension	6	8	14
Unsuccessful			
No evidence of extension	5	5	10
Evidence did not extend	4	3	7
Total	20	36	56

20. Maintaining or switching reasoning approaches did not seem to depend on the integer model the students experienced during instruction, which the figures will help illustrate.

During instruction, I did not explicitly teach students how to represent their physical motions in written form. This would have unnecessarily privileged written representations over the physical enactments. While solving tasks during instruction, students were required only (a) to physically enact the models as a trio and (b) write solutions as equations (e.g., -3 – -7 = 4) on each individual's papers. During whole-class discussions of their group work, we typically reenacted their processes with physical chips on a document camera or by walking on a number line. Near the end of the unit, students were encouraged to only use paper and pencil during group work (although physically enacting the models was still an option), so it was only at the end of the unit, when students themselves drew to represent the models, that we put these drawn representations on the board as a shared representation.

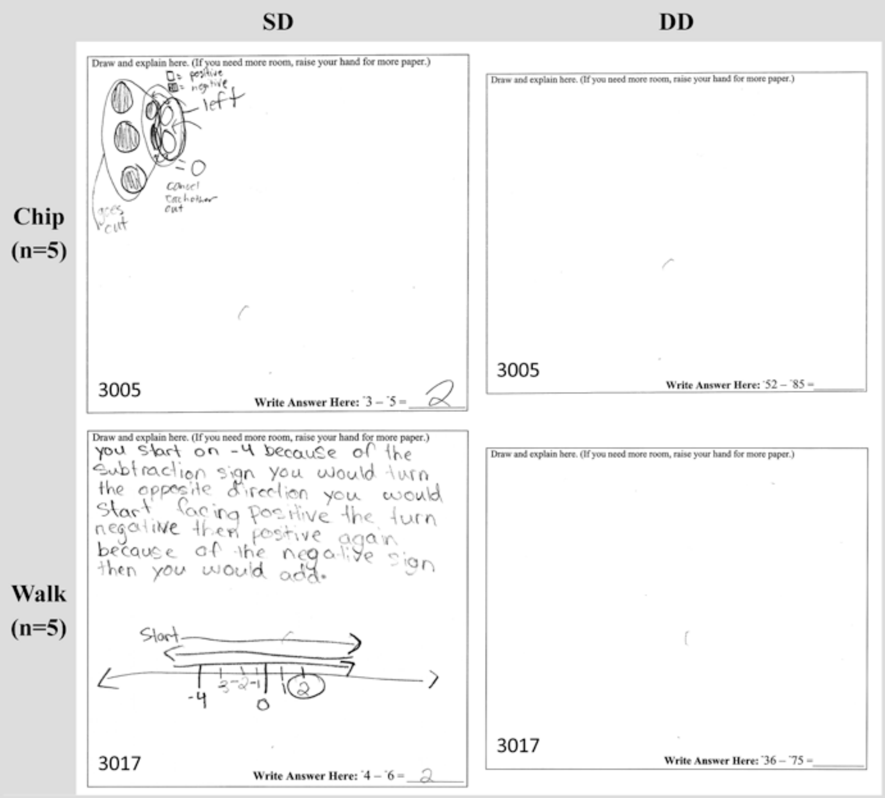
Successful Student Reasoning Single-Digit to Double-Digit Overall students who were successful on DD applied the same reasoning they used on SD. Of the 39 students who successfully extended from SD to DD, 35 explicitly expressed the same reasoning on the DD question as they did on the SD. Revisit Fig. 5.3 to see examples of students from each model whose delayed posttest responses showed successful extension. The remaining four students who were successful on SD and DD provided accurate solutions but chose not to explain or draw their reasoning on one of the problems. Not only did students continue with the same reasoning, but students who successfully extended from SD to DD typically used the conceptual metaphor they enacted during their instruction, which the student responses in Fig. 5.3 portray. All chip model students coded as successful extension (n = 11) expressed a COLLECTING OBJECTS metaphor on SD and also DD. Most of the successful walk-it-off model students (n = 26) expressed a MOVING ON A PATH metaphor on both SD and DD.

The 11 students who used a COLLECTING OBJECTS metaphor successfully on SD and DD each drew positive and negative chips. To solve -4 – -6, some students drew four negatives and six negatives to represent the minuend and subtrahend, respectively, and then determined how to continue. Student 3075 (see Fig. 5.3) provides an example of one of the students who used the more advanced strategy of only drawing the necessary chips to have enough chips to remove for the subtrahend. Student 3075 drew four negatives to represent the minuend and then either knew two more negatives were sufficient, or through the process of drawing the objects near the bottom of the work space, the student was able to recognize two negatives were sufficient and drew a line to continue connecting those two extra negatives with the original four. The student added two more positives to represent "add 0."

Student 3111 on SD and DD was able to identify the minuend as a starting location or point on the number line path and then treat the subtraction operation and the subtrahend as directed movements consistent with the MOVING ON A PATH metaphor enacted with the walk-it-off model. Some walk-it-off students, such as student 3145, explicitly described in words how to move relative to their current position by going in "the opposite direction." Although student 3111 did not use the word "opposite," he explained on both SD and DD why he turned the opposite direction (see Fig. 5.3). The drawing of -52 as a point on the number line supplemented by the written explanation shows the student recognized the number -52 as a location on a path "Start on -52," whereas each of the other "-" symbols signaled him to turn. If we look only at the written words, this idea of "turn" may seem ambiguous. However, note at the top of the workspace the student first drew the number line with labels to describe how to move on the path. Only after this did the student supplement this drawing with written explanations to describe the reason for turning, such as "turn for the – sign then turn for the – sign." The arrows in the drawings clearly show the direction the student meant when he used the word "turn."

In Fig. 5.3 on SD, notice that both the student who learned with the chip model and the student who learned with the walk-it-off model represented each integer using the respective conceptual metaphor. The chip student showed each -1 with a circle around it as though -1 was an object or chip, and the student who used the walk-it-off model showed a tick mark to represent the positions of each integer on the path from -5 to 5. Both of these students exemplify that they continued to use the same respective conceptual metaphor of COLLECTING OBJECTS or MOVING ON A PATH in a more abstract way for DD. Rather than showing 36 objects to represent the number 36, student 3075, who used the COLLECTING OBJECTS metaphor, expressed -36 as a single unit object that required an additional unit object representing 39 negatives in order to make it possible to remove 75 negatives. Similarly, student 3111, who used the walk-it-off model, marked multiples of ten on the number lines instead of each tick mark and then positioned the starting and ending locations appropriately relative to these multiples.

Unsuccessful Student Reasoning Single-Digit to Double-Digit Of the 17 students who unsuccessfully extended knowledge from SD to DD, 11 skipped DD. Figure 5.4 shows the detailed explanations of a student from each model who was successful on SD but skipped DD.



Draw and explain here. (If you need more room, raise your hand for more paper.)

3005

Write Answer Here: $^{-}3 - {}^{-}5 =$ 2

Draw and explain here. (If you need more room, raise your hand for more paper.)

3005

Write Answer Here: $^{-}52 - {}^{-}85 =$

Draw and explain here. (If you need more room, raise your hand for more paper.)

you start on -4 because of the subtraction sign you would turn the opposite direction you would start facing positive the turn negative then positive again because of the negative sign then you would add.

3017

Write Answer Here: $^{-}4 - {}^{-}6 =$ 2

Draw and explain here. (If you need more room, raise your hand for more paper.)

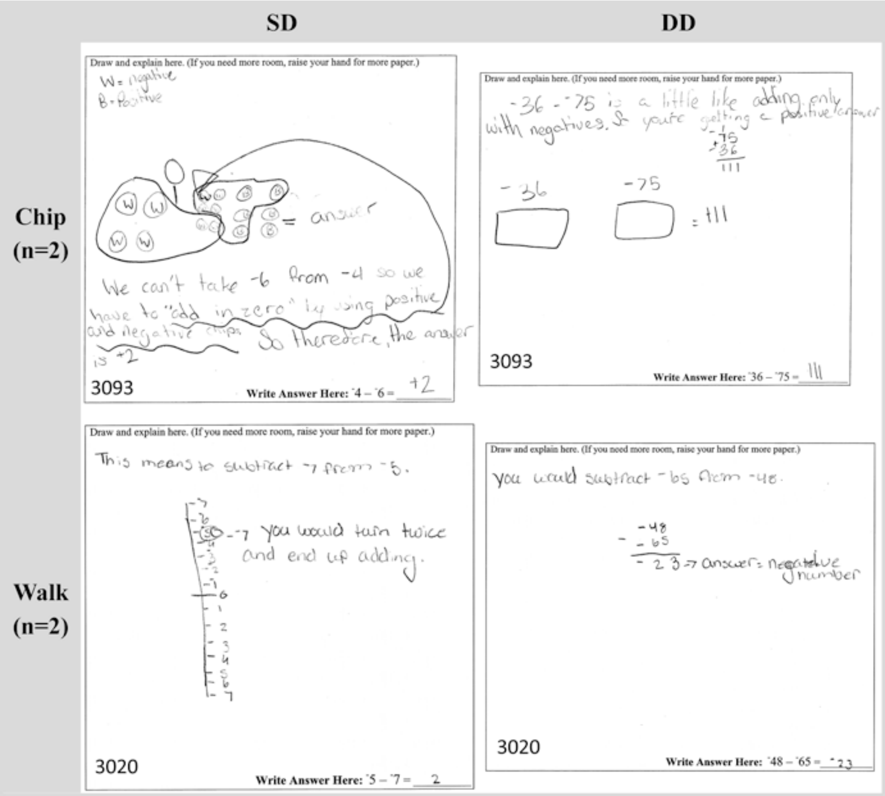
3017

Write Answer Here: $^{-}36 - {}^{-}75 =$

Fig. 5.4 Examples of students from each integer model who successfully provided positive solutions to SD but intentionally skipped DD (no evidence of extension)

Student 3005 represented each numeral in the subtraction problem using a COLLECTING OBJECTS metaphor and treated the colors used in chip model instruction as appropriately arbitrary representations (he used black for negatives, whereas during the study instruction, black represented positives). Using few words, student 3005 explained more than most other chip model students did about the ideas used to solve the problem. Although student 3005 did not explain why two negatives and two positives were added to the three negatives, he circled them and labeled these "0" and wrote under it "cancel each other out." The student also used the phrase "goes out" to represent subtraction of the five objects circled to take out. The student then indicated that what is left are the two objects the student keyed as positives.

In spite of the diagrams that seem to indicate the student not only understands procedures for subtraction using a COLLECTING OBJECTS metaphor but may have some conceptual understanding, the student skipped the DD problem providing no evidence that the student could extend understanding to more difficult problems. For student 3005 to take a similar approach on DD would have meant drawing 52 negatives and then draw many more to discover how many extra negatives and



Draw and explain here. (If you need more room, raise your hand for more paper.)

W = negative
B = Positive

We can't take -6 from -4 so we have to "add in zero" by using positive and negative chips So therefore, the answer is +2

3093

Write Answer Here: ⁻4 – ⁻6 = +2

Draw and explain here. (If you need more room, raise your hand for more paper.)

-36 - ⁻75 is a little like adding only with negatives. & youre getting a positive answer

$$\begin{array}{r} -75 \\ +36 \\ \hline 111 \end{array}$$

= 111

3093

Write Answer Here: ⁻36 – ⁻75 = 111

Draw and explain here. (If you need more room, raise your hand for more paper.)

This means to subtract ⁻7 from ⁻5.

-7 you would turn twice and end up adding.

3020

Write Answer Here: ⁻5 – ⁻7 = 2

Draw and explain here. (If you need more room, raise your hand for more paper.)

you would subtract ⁻65 from -48.

$$\begin{array}{r} -48 \\ - \quad -65 \\ \hline -23 \end{array}$$

→ answer = negative number

3020

Write Answer Here: ⁻48 – ⁻65 = ⁻23

Fig. 5.5 Examples of students from each integer model who successfully answered SD using a conceptual metaphor consistent with instruction and then unsuccessfully switched to using a traditional subtraction algorithm for DD

positives were needed. The effort demands of using a COLLECTING OBJECTS metaphor in which he treated each -1 as its own object may have been a reason the student did not persevere to attempt this problem.

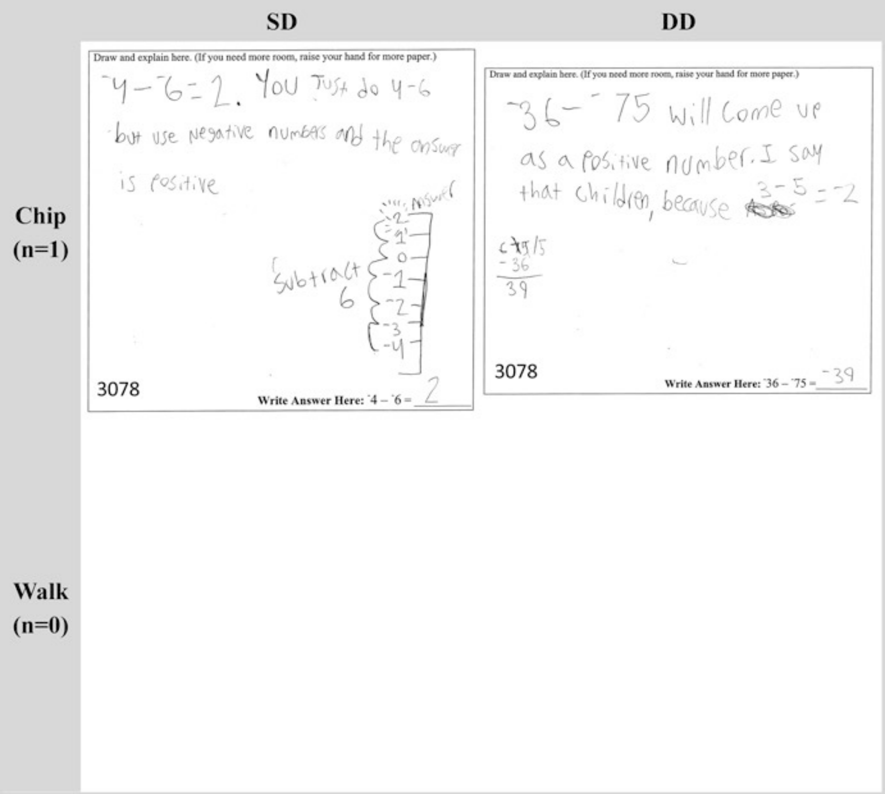
On SD student 3017 treated the number -4 as a point or location on a path and started thinking about and described in writing that the subtraction operation makes "you…turn the opposite direction." The student seemed to realize she had not told the fictitious student audience which way to face to begin, so she added that you start positive, so then turning the opposite direction means you turned negative, and then "because of the negative sign" that means to return to the positive direction. The student associated each symbol with the reasons for direction changes. This student, like student 3020 in Fig. 5.5, then communicated that this process resulted in the concept of what the fictitious audience student would recognize as "add" on a number line. Perhaps similar to student 3005, the effort of making a mark to represent each numeral (and insufficient room) may have been the reason why student 3017 did not use MOVING ON A PATH metaphor for DD.

Students who unsuccessfully attempted DD, in spite of being successful on SD, changed the way they reasoned to explain DD. The most common change was that they abandoned use of a conceptual metaphor (four students). The two students who had learned with chips and the two students who had learned with the walk-it-off model each used the model from their respective instruction, and thus also the respective conceptual metaphor, to answer SD. When they explained DD, however, each of these students used vertical algorithms. Figure 5.5 shows an example student from each model who accurately answered SD but then inaccurately answered DD in this way. Mathematically we can "take -6 from -4." It is only when a person uses a COLLECTING OBJECTS metaphor to think of the number -6 as six objects that it is in some sense true that "we can't take -6 from -4" as student 3093 stated. In other words, we cannot take 6 negative chips away when there are too few negative chips available in the representational space. This shows evidence that when solving problems with single-digits, the students' ideas of negative numbers were firmly rooted in the COLLECTING OBJECTS metaphor promoted by the model the student experienced during instruction. On the DD item, due to the rectangles drawn, one might wonder if the student from the chip model may have been trying to think of this problem in terms of COLLECTING OBJECTS; however, the student did not appear to use these drawings as she did on SD. As several students shared when turning in their tests, the student may have drawn something to satisfy the directions to draw rather than to reason. Although instruction included a few problems in the range from -20 to 20 and the student was able to replicate prior use of objects for SD, when asked to perform a DD, student 3093 abandoned a COLLECTING OBJECTS metaphor to unsuccessfully use an abstract vertical algorithm.

Note that although student 3020's work was not completely correct because she drew a vertical number line reverse of the culturally accepted and disciplinary version of a number line, the directions and distances the student moved on the path were consistent with the path she drew. In other words, how the student used a MOVING ON A PATH metaphor to reason with the number line tool was correct in relation to the drawn representation. Student 3020 did not use words to explain in as many details as some other students why she turned twice. Notice, however, the ways the student did represent the meaning. She circled -5 as the starting location and then positioned the rest of the problem next to that. So with the remaining symbols of the problem she explained "– -7" by writing, "You would turn twice," which indicates she understood that each "-" symbol meant to turn the other direction. More significantly, the student's words expressed a generalization that this idea of turning twice when subtracting a negative number on the number line in this problem means a person "end[s] up adding."

On SD, students 3093 and 3020 each from different models represented numbers using an individual symbol for every number they considered, which was effective in terms of accuracy but was an inefficient and effortful strategy. The strategy would have been unwieldy to apply to DD because the students would have needed to represent each integer as a point on a path or with individual objects.

Notice that although the student who had used chips and the student who used walk-it-off abandoned their conceptual metaphor in favor of a vertical algorithm on



Draw and explain here. (If you need more room, raise your hand for more paper.)

-4 – -6 = 2. You Just do 4-6 but use negative numbers and the answer is positive

Write Answer Here: -4 – -6 = 2

Draw and explain here. (If you need more room, raise your hand for more paper.)

-36 – -75 will come up as a positive number. I say that children, because 3 – 5 = -2

75
-36
39

Write Answer Here: -36 – -75 = -39

Fig. 5.6 The students from each integer model who successfully used a conceptual metaphor on SD inconsistent with the instructional model and then unsuccessfully extended to DD

DD, the students did this differently. When rewriting the expression, student 3093 changed the problem to an inequivalent addition problem with the larger magnitude number on top, as is typical with whole-number problems. Although the student attempted to explain and may have recognized from prior experience and the SD that the solution would be positive, the student work did not support the positive solution. In contrast, the walk-it-off model student accurately converted the horizontal subtraction problem to a vertical problem representation, but as her solution for DD evinces, student 3020 had not yet generalized that subtracting a negative number would mean addition as a universal abstraction.

There are two other interesting cases displayed in Figs. 5.6 and 5.7, respectively. Only a single individual used the same conceptual metaphor on the pretest and delayed posttest SD that was inconsistent with the conceptual metaphor the student enacted during model-based instruction. Student 3078 used a chip model during instruction. However, on the delayed posttest SD, this student used the MOVING ON A PATH metaphor consistent with the conceptual metaphor that he had

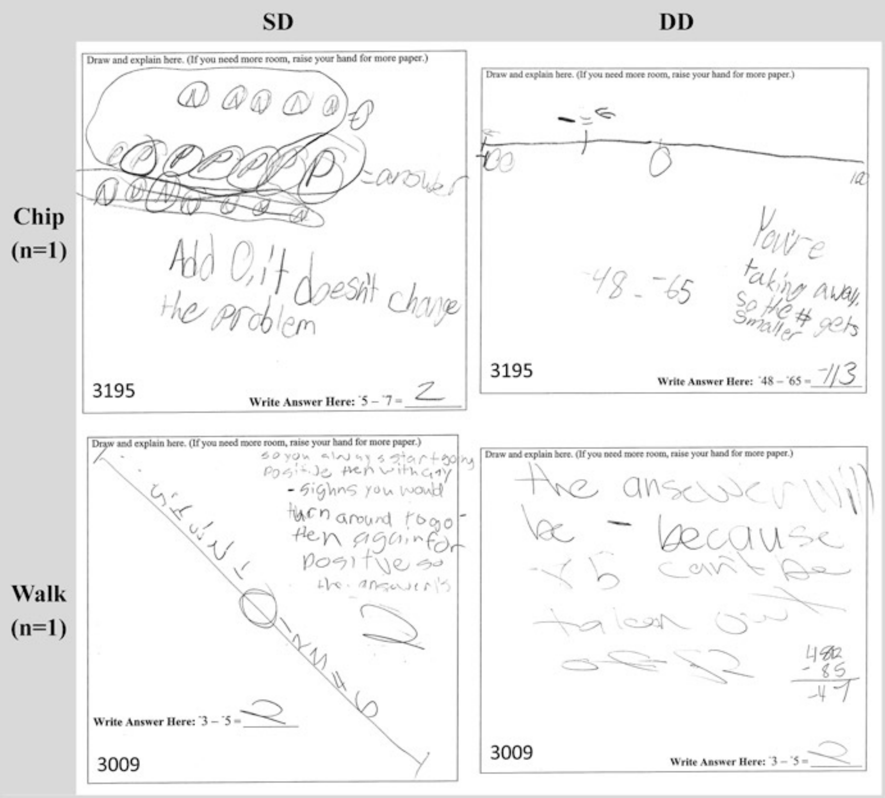


Fig. 5.7 Examples of students from each integer model who switched conceptual metaphor expressed from SD to DD, from a conceptual metaphor consistent with the integer model of instruction to using the other conceptual metaphor not experienced in the lesson

expressed on the pretest but inconsistent with the chip model instruction (see Fig. 5.6). When solving DD, he switched to no metaphor by calculating with a vertical subtraction algorithm. Although he said the solution would "come up as a positive number," the solution the student gave was negative ("-39"), similar to others who used a traditional subtraction algorithm. Note in Fig. 5.6 that none of the students who learned with the MOVING ON A PATH metaphor-based model expressed a COLLECTING OBJECTS metaphor on SD.

Figure 5.7 shows a unique case of the one student from each model who, when faced with DD, used a different conceptual metaphor than that which helped them be successful on SD. Student 3195 used a COLLECTING OBJECTS metaphor consistent with the chip model he used during class but unsuccessfully switched to a MOVING ON A PATH metaphor to explain DD. Student 3009 explained SD using a MOVING ON A PATH metaphor with the walk-it-off model but then abandoned this metaphor and instead seemed to draw on a COLLECTING OBJECTS metaphor, also unsuccessfully. Note the language that student 3009 used in DD. Rather than referring to numbers as points or distance on a number line as in SD, she discussed numbers as things that cannot be removed with the language "taken out of." In this case, the COLLECTING OBJECTS metaphor interfered with her ability to determine a positive solution.

Discussion

Skill-based assessments, particularly forced-choice items common with standardized tests, lead to false-positives in which students obtain accurate answers for inaccurate reasons. In contrast, the EDT subtraction items afforded insights into student reasoning. Since prior to instruction only 2 of the 56 students were correct on single-digit subtraction, the study instruction of both chip and walk-it-off models can be inferred as the reason that these students who learned with either model had longer-term success on the delayed posttest. After just eight lessons of all four operations (addition, subtraction, multiplication, and division), the walk-it-off model in this study facilitated more students than the chip model to overcome the difficult overgeneralization that subtraction leads to a smaller solution.

As previously noted, large-scale data showed that only 40% of 14-year-olds accurately gave positive solutions to subtraction problems (Periasamy & Zaman, 2009, pp. 376–377). Students, however, could have used inaccurate reasoning to correctly answer these forced-choice items. Indeed, if the current study of 11–12-year-olds had included the 19 students with positive solutions but inaccurate reasoning, the accuracy rate would have been 47%. Given the documented difficulties older students have, it was unsurprising that many students in this study still found these counterintuitive problems difficult after using either model. Although this chapter exclusively reported the assessment of these subtraction problem-structures, these constituted a small portion of their integer instruction, because during the eight lessons students experienced all possible problem-structures of all four operations. Yet, within this limited timeframe, the walk-it-off model better supported students to solve these counterintuitive subtraction items compared to the reported large-scale data of older students as well as compared to a chip model in this study. Recall that almost half of the students who used the walk-it-off model compared to about one-quarter of students who learned with chips provided answers greater than zero with reasoning that supported those positive solutions.

Moreover, in spite of the 5-week delay, the walk-it-off model more so than the chip model also facilitated more students to extend this knowledge to larger double-digit problems with the same problem-structure. Recall that four times more students who learned with the walk-it-off model provided clear evidence of extending this knowledge with supporting explanations. This research empirically supported others' theoretical critiques that the chip model breaks for subtraction (Vig, Murray, & Star, 2014). The results of the current analysis are also consistent with the larger study in which Nurnberger-Haag (2015) assessed calculation accuracy without attending to reasoning and found that this chip model could interfere with accuracy on some problem types for all operations except addition.

Interpreting Results in Terms of Theory

Empirical and theoretical work from embodied cognition affords perspectives as to why (a) some conceptual metaphor-based model may support learning and (b) reasons the walk-it-off model may have better supported this sample of students to make sense of the counterintuitive effect that subtraction could make a solution, not only larger but also positive.

Conceptual Metaphors Support Initial Learning of Integer Arithmetic Chiu (2001) found that initial learners were more successful when their reasoning about integer arithmetic was grounded in some conceptual metaphor. That is, compared to adults, initial learners were more accurate when they conceived of integer operations grounded in more real-world experience than abstract rules, regardless of which metaphor the initial learners used (Chiu, 2001). Results from the current study provide additional evidence to support Chiu's assertion, because 26 out of the 28 successful extenders expressed the same conceptual metaphor (either COLLECTING OBJECTS or MOVING ON A PATH) on single-digit and double-digit subtraction items. Conversely, those students who used a conceptual metaphor to find a positive solution for single-digit subtraction, but did not use that conceptual metaphor for double-digit subtraction, were unable to extend what they knew. This is also consistent with Chiu's (2001) finding that initial learners who used abstract approaches were less successful than those who used a conceptual metaphor to problem solve. This finding that using a conceptual metaphor aids success is important, because it contributes additional empirical evidence to the sometimes philosophical or anecdotal justification for recommendations about whether arithmetic should be taught only at the abstract symbolic level using algebraic rules or with more "concrete" methods (e.g., French, 2001; Freudenthal, 1973; Uttal, Scudder, & DeLoache, 1997).

Instructional Activities Need to Support Conceptual Metaphors Due to the difficulty of integer arithmetic compared to whole-number arithmetic, for such counterintuitive subtraction problems as explored here, instructional support may be needed to effectively reason with any conceptual metaphor. Both evidence of student success and failure to extend support this claim. For example, the 39 students shown in Fig. 5.3 successfully solved DD by reenacting the instructed conceptual metaphor at delayed posttest. In contrast, students may struggle to extend their reasoning if their instructional support promotes a model different from what they already used effectively (as shown by the student in Fig. 5.6). Another potential barrier for educators and researchers to consider is how to support students who may successfully use a conceptual metaphor to continue using the same conceptual metaphor in more efficient ways. The contrast of strategies and success between students in Fig. 5.5 who drew every object or tick mark compared to students in Fig. 5.3 suggests that some students may need support to use efficient ways of thinking with either metaphor. Finally, the students shown in Fig. 5.7 provided evidence that after they had experienced an instructional model that allowed them to enact a particular conceptual metaphor, they were successful using this metaphor to solve problems similar to instruction; however, for DD if they switched to a different metaphor that they had not experienced during instruction, they were no longer successful. In other words, regardless of which metaphor students used (whether students successfully used a COLLECTING OBJECTS or MOVING ON A PATH metaphor) on the easier single-digit subtraction problem, when these students switched to the other conceptual metaphor for which they had not had instructional support, they were no longer successful. Future research should compare conceptual metaphor-based integer models to each other as this study did, as well as compare conceptual metaphor-based integer models to approaches that exclusively scaffold students' conceptual metaphor-based thinking without introducing a model.

MOVING ON A PATH Metaphor Better Supports Initial Learning The results presented here suggest that when educators choose how to begin integer instruction, a MOVING ON A PATH metaphor may be more helpful than a COLLECTING OBJECTS metaphor to help students develop concepts of subtraction with negative numbers. This is consistent with another prior report, which did not draw on CMT, which described that using objects to think about integer arithmetic was more difficult than using a number line (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011). From an embodied perspective, Chiu (2001) found that children used a metaphor of motion more frequently than an objects metaphor both as a strategy to solve problems and in explanations to show their understanding. Thus, consistent with embodied as well as constructivist theories of learning, this suggests a MOVING ON A PATH metaphor may be the most likely candidate from which to begin integer instruction.

Potential factors not accounted for in this analysis such as working memory and spatial visualization ability leave open questions for future research (Moreno & Mayer, 1999; Raghubar, Barnes, & Hecht, 2010). It behooves us to investigate if students who have stronger working memory and visualization skills are better able to use any COLLECTING OBJECTS metaphor-based models, because of the number of objects that needs to be manipulated, whether physically or through mental visualization or drawing. Compared to the number of objects needed to represent single-digit quantities, COLLECTING OBJECTS may require these skills in order to keep track and move around the larger number of objects required to conceive of a number like -85 and the additional objects needed to cancel some of these quantities. Future research could explore if these demands to track increasing quantities of objects are more difficult than simply moving farther down a path. Moreover, since robust integer understanding requires integration of multiple conceptual metaphor-based ways of thinking as well as abstracted ways of thinking (Chiu, 2001), much research is needed to consider sequencing and connecting instruction that draw on these multiple ways of conceiving of integer arithmetic.

Model Movements in Relation to Conceptual Metaphor As we know, however, there are multiple factors involved with each model that was used. A more explanatory view is likely complex. Certain models may better support student learning and thinking due to interactions of factors such as the underlying conceptual metaphor a model promotes and model movements students enact. The prior publications that claimed a number line model was more problematic than a cancellation model used the number line in typical ways (see, e.g., Küchemann, 1981; Liebeck, 1990) rather than the model movements students enacted in this study. An analysis of student learning with the walk-it-off model or a chip model on all four integer arithmetic operations grouped by whether particular problem-structures required students to move in ways that contradicted the problem operations demonstrated that these contradictory chip model movements interfered with student success (Nurnberger-Haag, 2015). The current findings about student learning of integer subtraction fit with other research in embodied cognition that the congruence or consistency of humans' physical movements with the target ideas influences thinking and learning (Anelli, Lugli, Baroni, Borghi, & Nicoletti, 2014; Goldin-Meadow, Cook, & Mitchell, 2009; Kontra, Fischer, Lyons, & Beilock, 2015). If students were to enact the chip model to do any of the subtraction problems analyzed in this study, they had to move in inconsistent ways: they had to add (i.e., put in additional chips) in order to subtract (i.e., have enough negatives to remove). The fact that so many fewer students who learned with the chip model were initially successful on the single-digit problems similar to their instruction suggests future research is needed to investigate sensemaking with either model. Perspectives that consider the model movements in relation to the conceptual metaphor are likely needed. It may be useful to consider dynamic systems approaches (e.g., Smith, 2005) for future investigations of integer learning with models, because integer arithmetic may be grounded in multiple conceptual metaphors (Chiu, 2001; Lakoff & Núñez, 2000), typical integer models draw on different conceptual metaphors (Kilhamn, 2011; Nurnberger-Haag, 2013) or integrate multiple conceptual metaphors (e.g., Tsang, Blair, Bofferding, & Schwartz, 2015), and different model movements may differentially impact learning (Nurnberger-Haag, 2015).

Conclusions

The subtraction problems analyzed in this chapter involve a difficult subtraction problem-structure because they have solutions that contradict students' generalizations from whole-number arithmetic that subtraction always makes a number smaller (Karp, Bush, & Dougherty, 2014). This overgeneralization is based on prior experiences with taking away concrete objects and other ways of subtracting natural numbers. Yet, those students who were most successful overcoming this generalization did so not by abstract rules but by expressing a conceptual metaphor grounded in embodied experiences of enacting an integer model during instruction.

When students found themselves in situations of subtracting negative numbers that warranted a solution greater than zero, walking the other way compared to taking it away better helped students find their way to those positive solutions. More walk-it-off students provided positive solutions to single-digit subtraction problems (e.g., -3 – -5) similar to instruction, and more of the students who used this model were able to apply their understanding to new subtraction problems with larger magnitude (e.g., -36 – -75). This was the case even though the entire unit of instruction on all four operations was shorter than the length of prior research studies that exclusively taught only addition or addition and subtraction operations (e.g., Linchevski & Williams, 1999; Stephan & Akyuz, 2012; Thompson & Dreyfus, 1988). Consequently, future research should investigate instruction of all operations (four primary operations and opposite operators) without models compared to models, as well as compare models that enact the same conceptual metaphors to uncover relative merits of different approaches during typical length units.

Procedural fluency with negative numbers is currently far from reality for most students across the world (Ryan & Williams, 2007; Periasamy & Zaman, 2009). We need students to develop this procedural fluency with the conceptual understanding that allows them to recognize whether solutions would be positive or negative even if students use a calculator for the actual calculations (National Council of Teachers of Mathematics, 2014). This analysis began to address such issues, focusing primarily on procedural fluency of subtraction and the underlying conceptual metaphor used to reason about those procedures. For example, between those using a COLLECTING OBJECTS metaphor or a MOVING ON A PATH metaphor, it was only students using a MOVING ON A PATH metaphor consistent with the walk-it-off model who commented that subtracting the negative number resulted in "adding" (see Figs. 5.4 and 5.5). These realizations were based on the symbol meanings of the "-" signs indicating to turn the opposite direction twice on a number line. Given that students who used chip model or walk-it-off had the same instructional tasks, such data suggests the need for future research about how to support these generalizations. Such comments may reveal ways to scaffold students to make more formal generalizations grounded in a conceptual understanding through recognizing patterns of solutions for particular problem types rather than simply memorizing that subtraction means adding the opposite.

More nuanced analyses with new data are needed related to varied conceptual understandings. Research methods that interview students about their written explanations would be needed to ascertain reasons students in either model chose not to articulate in their explanations why (a) when using a COLLECTING OBJECTS metaphor that zero can be represented with both positives and negatives or (b) why it made sense when using a MOVING ON A PATH metaphor that subtraction signs and the negative sign of the subtrahend could mean turn or go in the "opposite direction." One potential reason is that students simply learned procedures of their respective model to answer these skill-based problems with little conceptual understanding. Alternatively, although chip models and walk-it-off models involve procedures, the students may have connected these procedures with meaning so well as their concepts consolidated during the five-week delay that they did not believe these ideas required explanation. Methodological designs that plan for such data analysis using theoretical frameworks such as cognitive demand (Stein, Grover, & Henningsen, 1996) in relation to embodied cognition might offer such insights as to what, if any, aspects are simply procedural or what aspects of model use facilitate intuitive understandings of integer arithmetic.

The findings of this study offer insights about conceptual metaphors as a factor of student thinking that could be used as part of a learner profile to select cases of students to participate in teaching experiments. In order to truly begin instruction from an individual student's current thinking, we need to consider student thinking in terms of broader theories of cognition such as CMT. Such approaches would likely support educators and researchers to come closer to designing instruction that capitalizes on an individual's current thinking when instruction begins. Furthermore, this study in light of Wood's (2010) finding that students who drew on different conceptual metaphors to explain their ideas about fractions may fail to communicate with each other suggests that research that draws on social theories of learning would benefit from considering how students who think in terms of different conceptual metaphors communicate with each other about integer operations. Thus, to optimize instructional approaches that begin from each student's thinking, future investigations are warranted in which educators group students within heterogeneous achievement levels but homogenously in terms of conceptual metaphor profiles. Including the conceptual metaphors students express prior to instruction as part of learner profiles might also be important to account for in statistical analyses in larger-scale studies that otherwise do not draw on embodied cognitive perspectives. If we consider the bases of students' conceptions of integers in these ways in terms of the conceptual metaphors on which they draw, this might help us truly offer instructional practices that build on students' thinking and differentiate integer instruction in real classrooms in ways not previously considered.

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Chapter 6

Different Differences: Metaphorical Interpretations of "Difference" in Integer Addition and Subtraction

Cecilia Kilhamn

Abstract Mathematically speaking, a *difference* is the result of a subtraction. However, when the number domain is extended from natural numbers to integers, the separation of the magnitude of a number from its value creates "different differences," where the connection to subtraction is no longer straightforward. Based on video-recorded lessons and individual student interviews with 21 students in a Swedish year 8 class, a conceptual metaphor analysis of the discourse shows how ambiguous the term *difference* can be and how an implicit use of metaphors can create confusion in relation to addition and subtraction with integers.

Seeking the Difference: A Vignette

We enter a classroom where a subtraction problem is being discussed.[1] On the board the teacher has written: 283 – 275 = __. As the students solve the problem, different strategies emerge. Some students explain that they "jump back" or "take away" by starting on 283 and jumping back 275 steps, ending up on 8. Another student's suggestion is "to start on 275 and jump up 8 steps to 283, so the answer is 8," saying that he "found the difference between 275 and 283 because the numbers are close." The teacher represents the two strategies on the board, using an open number line as shown in Fig. 6.1, describing the second strategy as "adding-on."

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[1] The vignette is a transcript of a classroom video that was recorded and distributed as part of a professional development program for teaching basic arithmetic (Dolk & Fosnot, 2006). The purpose of the video is to illustrate how teachers can introduce the number line model for subtraction to enhance mathematical reflection. This video is not part of the empirical data reported in the result section.

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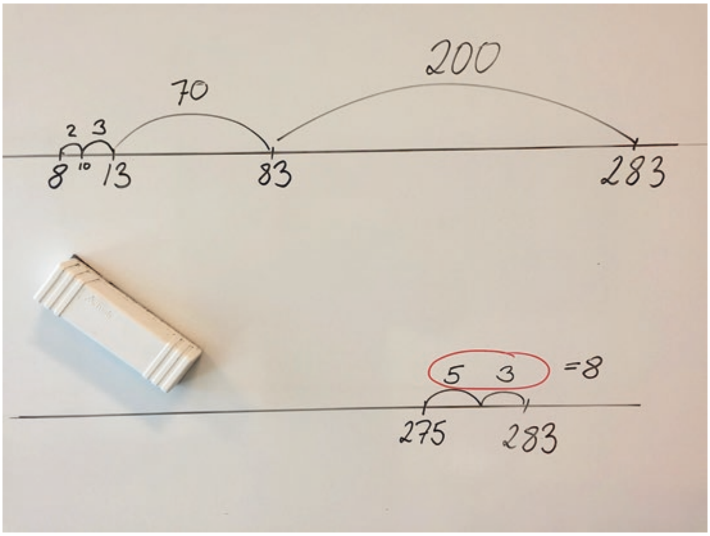

Fig. 6.1 Two strategies for solving 283 – 275 using the open number line

Then the teacher asks the students to compare the two strategies and discuss why it seems to work to do addition with a subtraction problem. In the "adding-on" strategy represented on the bottom line in Fig. 6.1, students describe 8 as "the difference." In the "take-away" strategy represented on the top line in Fig. 6.1, they do not; when moving backward, 8 is referred to as "the answer," not the "difference." One student says, "It is different differences, in the first, 275 is the difference between 283 and 8, and in the second, 8 is the difference between 283 and 275." So, the teacher asks if the first strategy, counting down steps 275 from 283, is, in fact, an illustration of another problem where 275 is the difference. He writes on the board:

283 – 8 = 275
275 + 8 = 283

This vignette highlights the complexity of subtraction and the elusive meaning of the word *difference*. Moving backward or counting down does not always generate what is perceived of as a difference. For a student who solves subtraction thinking about a backward motion or using a counting-down strategy, the difference is the subtrahend rather than the answer.

Introduction

This chapter reports on difficulties concerning integer addition and subtraction that are closely related to the meaning of the word *difference* and metaphorical interpretations of numbers and operations in the switch from natural numbers to integers. Teaching and learning about integers, sometimes referred to as signed or directed numbers, is an ongoing discussion in mathematics education research (e.g., Freudenthal, 1983; Glaeser, 1981; Heeffer, 2008; Mumford, 2010; Schubring, 2005) and is considered of vital importance for the learning of algebra (Vlassis, 2002). When students encounter the new number domain, they often inappropriately use natural number reasoning, interpreting operations with integers through their experiences with positive numbers (e.g., Bruno & Martinón, 1999; Petit, Laird, & Marsden, 2010). The result of a subtraction is called "difference." In the domain of natural numbers, the difference is the result of a subtraction of a smaller number from a greater number. It is the difference in magnitude when two numbers are compared. When a greater positive number is subtracted from a smaller positive number, the result becomes negative, and an extension of the number domain is needed. In the new number domain, the meaning of difference is not always the same, nor is it always clear which of the two numbers is the greater number. For example, -3 is greater than -5, -3 > -5, by order, whereas -5 has a larger magnitude (absolute value) than -3, |-5| > |-3|, an issue central to students' synthetic mental models for integer order and value (Bofferding, 2014). Hence, for integers, the separation of magnitude from the ordered value of a number creates new differences. Consequently, the connection between the idea of a difference and the result of a subtraction is ambiguous.

Subtraction situations with natural numbers are characterized in terms of change, or compare/equalize (Fuson, 1992). In change situations, a state or set is transformed into a different state, described as a state-translation-state problem (Marthe, 1979; Wessman-Enzinger & Tobias, 2015). Such situations are commonly described as take-away situations where the difference is the answer to the question: "What is left?" In comparison situations, state-state-translation or state-state-state situations (Marthe, 1979; Wessman-Enzinger & Tobias, 2015), there are two states or sets, and the difference is what can be added to or removed from one to make it equal to the other. In comparison situations, the *difference* is the answer to questions of the type: How many more? How much less? Take-away situations dominate in natural number arithmetic, but they create difficulties when a larger number is taken away from a smaller number. On the other hand, compare situations may also be confusing since the direction of the comparison is not intuitive. Regarding the transition from natural numbers to integers, Fuson (1992) writes:

> … consideration of the full range of addition and subtraction situations requires an extension to the integers (including negative as well as positive whole numbers), which necessitates an avoidance of terminology or educational practices in the lower grades that interfere with later comprehension of these integers. (pp. 246–247)

A *difference*, according to Fuson, already involves an integer quality since it is relative, containing information about both who has more or less and the quantity of the difference. Switching from take-away situations to comparison situations is critical for subtraction of negative numbers (Kullberg, 2010; Marton & Pang, 2006). Also Wessman-Enzinger and Tobias (2015) advocate for the importance of posing problems of different types, since state-translation-state problems do not facilitate reasoning about all types of integer subtractions. The language used to describe subtraction situations and solution strategies may differ in different situations. Since numbers and mathematical operations are abstract entities, we speak of them metaphorically (English, 1997; Pimm, 1981; Sfard, 1994). Expressions such as "take away," "jumping backward," "smaller number," and "difference" all relate back to embodied experiences that may be different for different people. Although metaphors and analogies are essential for us in order to talk about abstract entities, they may also attach features to these entities that are misleading in new situations. When investigating young students' reasoning about integer addition and subtraction, Bofferding (2010) found that the language used to describe solution processes revealed conflicting conceptions of addition and subtraction. What happens, for instance, when we switch from take-away situations to comparison situations or from natural numbers to integers? Does terminology used in the classroom also change or are students expected to be able to transfer the abstract meaning of words they have connected to subtraction? Are teachers aware of the metaphorical meanings of the words they use, and do they make these meanings explicit in the classroom discourse? These are some of the questions that initiated a study of metaphorical interpretations of the idea of *difference* reported in this chapter.

Conceptual Metaphors for Arithmetic

In recent years, researchers have highlighted the importance of metaphors in mathematics education (Danesi, 2003; English, 1997; Frant, Acevedo, & Font, 2005; Parzysz, Pesci, & Bergsten, 2005). Lakoff and Núñez (2000), for example, claim that mathematics would not exist without its metaphors. Building on a theory of conceptual metaphors (Lakoff & Johnson, 1980), Lakoff and Núñez assert that basic arithmetic is understood through four "grounding metaphors." Since mathematical objects are created through these metaphors, the objects inherit the structure of the experiences that shape the source domains of the metaphors. While metaphors help us make sense of concepts by providing coherent structure, they highlight some features but hide others. Since different metaphors are used to structure different aspects of a concept, several metaphors are needed to fully understand a rich concept. Each metaphor contributes unique and sometimes contradictory features. The number 5 can, for example, be described as the fifth in an ordered row (an ordinal number) or as five objects in a set (a cardinal number). These two features of five are not the same: they draw meaning from different experiences. When speaking of five, we use our experiences as source domains for different metaphors, which exist simultaneously and come to the forefront in different situations. All abstract mathematical objects are, according to the theory, conceptually grounded in firsthand embodied experiences of four grounding metaphors for arithmetic (Lakoff & Núñez, 2000). Three of these that are clearly in play in the classroom discourse analyzed in the study are:

- Measurement, where numbers are seen as length of segments
- Motion along a path, where numbers are seen as point locations or movements
- Object collection, where numbers are seen as collections or sets of objects

Describing *difference* as a distance between two numbers is using the *measurement metaphor*, whereas moving up or down on the number line is using experiences of *motion along a path* to describe numbers and operations. In the vignette above, these two metaphors are blended. Additionally, the words "take away," originating in the *object collection* metaphor, are used when moving backward on the number line. Removing objects, rather than movement, is the embodied experience for "taking away."[2]

In previous research (Küchemann, 1981), the most difficult subtractions involving integers were found to be items of two types:

(1) -2 – -5 = __ (44% correct solutions)
(2) -2 – +3 = __ (36% correct solutions)

In both items, interpreting subtraction as taking away objects is difficult. It does not make sense to take away more negative objects than you have to start with (1) or to take away positive objects from negative objects (2). The first of the two test items can be seen as a distance on a number line (measurement), whereas the second cannot, since the distance is 5 but the difference is -5. If seen as movements on a number line where subtraction is usually mapped as a backward movement, the first item involves subtracting a negative, which would imply a backward movement (sometimes illustrated by a car that turns around and then reverses, ending up advancing its position). Since the direction relates to both the bearing and the movement, it becomes ambiguous. Similar items were problematic in the study reported here, where analysis of the classroom discourse reveals an implicit and incoherent use of metaphors. In order to better understand students' difficulties, the following research question was posed: What metaphors involving *a difference* appear in the classroom discourse when adding and subtracting integers, and how do these metaphors influence students' sensemaking?

Methods

Data Collection

The empirical data described in this chapter was collected as part of a larger project investigating students' development of number sense, where 1 class of 21 students was studied over a period of 3 years (school years 6–8, ages 12–15) using participant observation and recurrent individual interviews (Kilhamn, 2011). The focus of this chapter is the data collected during year 8 when negative numbers were introduced. In total, seven lessons in year 8 contained work on the topic of negative numbers. These were all video recorded, transcribed verbatim, and translated into English by the author. At the end of year 8, each student was asked to comment on

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[2] For a more thorough analysis of these metaphors, see Kilhamn (2011).

episodes from these videos during a stimulated recall interview. The textbook unit on negative numbers was seen as part of the classroom discourse since a substantial part of the video-recorded lessons included individual work in the textbook.[3] All students and their parents provided informed consent. The teacher was a qualified mathematics and science teacher with many years of experience teaching mathematics in years 7–9. Since the aim of the research project was to investigate authentic teaching with an interest in student sensemaking, no experimental or design-based elements were included.

Analytical Framework

Difference as a mathematical idea appeared in the classroom discourse in various forms. Here, discourse is defined as the use of specific words, narratives, procedures, and visual mediators (Sfard, 2008). The analytical focus is the metaphorical meanings of the word *difference* and other related words such as subtraction, addition, negative number, large number, and small number. The words and narratives that were produced, along with related visual mediators in the form of illustrations and mathematical symbols, were analyzed in terms of their metaphorical underpinnings. For example, a procedure that is part of this discourse is the use of sign rules when rewriting a symbolic expression including several minus signs. The classroom discourse includes the voices of the teacher, the textbook, and the students. Hence, teacher instruction, student work, and textbook activities were included in the analysis whenever the idea of *difference* appeared. I paid special attention to situations where conflicting narratives emerged. A conflict may appear when what was previously considered true is questioned, so that a change of discourse is necessary in order to resolve the conflict (Sfard, 2008). For example, in the domain of natural numbers, the smallest number is 1, but when an extension is made to include integers, there is no longer a smallest number. Discursive changes to resolve this conflict could either be changes in the narrative about what is actually true about numbers or a change in what defines the word smallest, giving it different meanings in different number domains. To understand a metaphor, the analogy from which it is created needs to be "unpacked." The analytical tool chosen here is adapted from Lakoff and Núñez (2000), in which the metaphor is described in terms of features of the source domain (e.g., objects, temperatures, distances) that are used to communicate about the target domain (i.e., the mathematical content), along with the attributes of the target domain to which they correspond. In particular, mappings of the word *difference* are analyzed in various metaphors.

Before looking at the classroom discourse, I describe and analyze two metaphors mapping the difference between numbers. These two metaphors were frequently

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[3] Carlsson, S., Hake, K. B., & Öberg, B. (2002). *Matte direkt, år 8.* Stockholm: Bonniers.

referred to in the classroom discourse analyzed in this chapter but were seldom made explicit or contrasted. The description of the two metaphors is followed by a number of excerpts illustrating these metaphors within the classroom discourse, along with a discussion of problems and misunderstandings that arose as a result.

Analysis of Two Metaphors

In the following analysis of the difference between numbers in two metaphors, it is necessary to distinguish between the two different aspects of size of a number. The term *magnitude* refers to what is often written as the absolute value: $|a|$ for the magnitude of any number *a*. The signed numbers +8 and -8 are said to have the same magnitude, |8|. Each number also has a *value* that relates to the order of numbers in the number system, such that …-2 < -1 < 0 < +1 < +2…. Thus, -8 is less than 8, -8 < +8, but the magnitudes are equal, |-8| = |+8|. Although -3 is said to be a smaller number than -2, it is also commonly referred to as a "larger negative number" (Hertel & Wessman-Enzinger, 2017). Bofferding (2010) observed these conflicting interpretations among young students dealing with integer addition. She writes:

> … adding a smaller positive to a larger negative number (e.g., -8 + 4) results in a larger number further to the right on the number line but which has a smaller absolute value. Likewise adding a negative to a negative number (e.g., -2 + -5) results in a number with a larger absolute value but which is further to the left on the number line. (p. 708)

In the classroom discourse explored here, the distinction between these two aspects of size was never made explicit. The two metaphors that frequently appeared in the discourse involving negative numbers were the object collection metaphor and the measurement metaphor.

Object Collection Metaphor: Comparing Sets

The teacher used the word *difference* for situations she identified as a comparison between sets. In the object collection metaphor, *difference* is mapped onto subtraction when the number of objects in two collections is compared. In the domain of natural numbers, this could be visualized as in Fig. 6.2. In the figure, both collections are illustrated as separate sets, but the metaphor is similar if the smaller set is included in the larger set, as in the problem: There are 8 apples, 5 are green, how many are not green? The difference is the result of subtracting the smaller number from the larger number. In the domain of natural numbers, the magnitude and the value coincide, and many students learn as a rule to always subtract the smaller from the larger. The meaning of the word *difference* is clear in a comparison situation. In the domain of integers, this mapping can be done in a similar fashion if *both* numbers are negative.

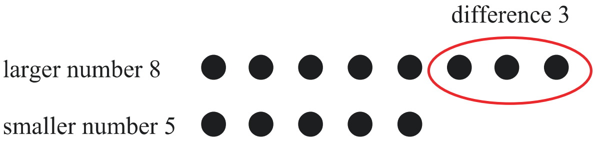

Fig. 6.2 Object collection metaphor. The difference is a comparison of two sets of objects and shows the result of the **subtraction** |8| – |5|= |3|. In the domain of natural numbers, this illustrates 8 – 5 = 3, and in the domain of integers, it illustrates -8 – -5 = -3

This mapping is consistent with the meaning of subtraction of positive numbers (Bruno & Martinón, 1999). However, in this mapping it is the magnitudes that are considered rather than the values. The smaller magnitude is subtracted from the larger magnitude. If the numbers are negative, the magnitude is opposite in relation to the value (i.e., the number with the smaller value has the greater magnitude). Finding the difference between the two numbers -8 and -5, for example, is similar to finding the difference between a debt of 8 and a debt of 5. The difference is a debt of 3 and is mapped onto the subtraction -8 – -5 = -3. The difference, considered as a magnitude, can be said to be smaller than the original collection (i.e., a debt of 3 is a smaller debt than a debt of 8, even if the value of -3 is larger than that of -8). The mapping, however, only works for subtractions $(-a) - (-b)$ where $|a| > |b|$; it does not work as a mapping for the subtraction -5 – -8. In this metaphor, the difference is a collection of objects, and the smallest collection of objects is the empty collection, which means that the difference is always a magnitude.

A special case of the object collection metaphor is when objects of different types (positive and negative) are compared. Figure 6.3 illustrates the situation of finding the difference between a negative and a positive number in an object collection metaphor. An example of this mapping is when the teacher refers to -8 and +5 as “a debt of 8” and “a gain of 5.” The inconsistent feature is that *difference* is mapped onto the addition of -8 and +5, not the subtraction (i.e., -8 + +5 = -3 or +5 + -8 = -3). The idea is that opposites “pair off,” and the difference consists of the objects that are left unpaired. This notion of “pairing off” relates to the aspect of total zero described by Gallardo and Hernández (2006). In mathematical terms, this is represented as a *sum* rather than a *difference*, but in the teacher’s discourse, it is talked of as a *difference*, drawing on the same idea of comparing sets that was the origin of the term *difference* in the object collection metaphor. With objects of different types, it is, however, no longer relevant to see one set as included in the other.

The metaphor just described is embedded in the sign rules formulated in the *Brahmasphuta-siddhanta* (628, as cited in Mumford, 2010), where it is written concerning addition of integers, “[*The sum*] of two positives is positive, of two negatives, negative; of a negative and a positive [*the sum*] is their *difference*…” and concerning subtraction, “[if] a larger [number is to be subtracted] from a smaller, their *difference* is reversed—negative becomes positive and positive negative”

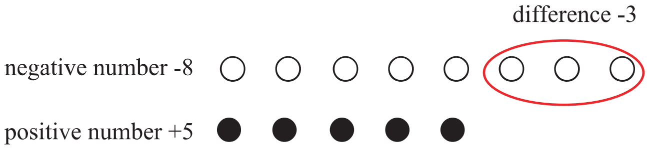

Fig. 6.3 Object collection metaphor in the domain of integers. The difference here is the result of the **addition** -8 + +5 = -3, where five of each type pair off and three negatives remain

[emphasis inserted] (p. 123). The word *difference* can in this way come to be associated with both addition and subtraction. However, as we shall see in the excerpts, the teacher tries to bridge this inconsistency by saying that the students should "think subtraction" 8 – 5 = 3 when they "write addition" (-8) + 5 = (-3). They will know what sign to put in the answer by considering which type of objects there were more of at the start.

Measurement Metaphor: Distance

In the video-recorded lessons analyzed in this study, the textbook was frequently used as a teaching tool, and students spent a large portion of the lesson time working individually in their textbooks. In addition to the teacher's frequent use of an object collection metaphor, the textbook emphasized a measurement metaphor to describe *difference*. In this metaphor, in the domain of natural numbers, difference is mapped onto the distance between two locations or points. It is mapped onto the subtraction of the smaller number from the larger number, but the *difference* itself is a distance and as such can only be a magnitude. As the source domain does not normally include experiences of negative distances, it does not allow for subtraction of a larger number from a smaller number.

Figure 6.4 shows how the number domain is extended to integers in the textbook. The *difference* is still referred to as the distance between two points and is mapped onto the subtraction of the smaller value from the larger value as for natural numbers. This can, for instance, be written as 3 – -4 = 7. Contrary to the metaphor in the natural number domain, the distance here could also be mapped onto addition of the magnitudes (i.e., 3 + 4 = 7).

The textbook illustrates the equivalence between the two expressions as shown in Fig. 6.4. In the expression 3 – -4 = 7, the numbers 3 and -4 are interpreted as locations and the difference is the distance between them. In the expression 3 + 4 = 7, in fact, all three numbers are interpreted as distances: two distances added together becomes a longer distance. In the classroom, the teacher used this picture to help students visualize the sign rule of "two negatives make a positive" (i.e., changing 3 – -4 into 3 + 4). An assumption in this metaphor is that a distance

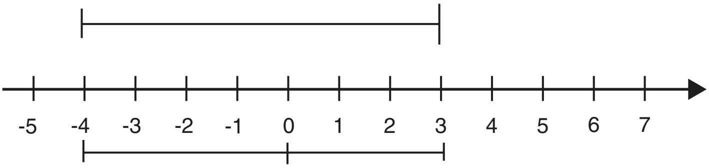

Fig. 6.4 Measurement metaphor involving integers. The difference is a result of the **subtraction** 3 – -4 = 7 or the **addition** 3 + 4 = 7

cannot be negative, so it only works when subtracting a smaller value from a larger value. There is nothing in the source domain of our experiences of distances that maps onto a subtraction of a number with a larger value from a smaller value, such as -10 – 6 = -16. When referring to the difference as a distance between two points, the calculations -10 – 6 and 6 – -10 would be the same since the distance between the numbers is 16 in both cases, just as the distance between two cities is the same whichever direction you travel. That conclusion is in conflict with other narratives about subtraction as noncommutative. Consequently, in order to know whether the answer is -16 or +16, it is necessary to extend the metaphor to include direction. In the textbook, the introduction of this metaphor was followed by non-contextualized tasks with the structure $a – b$ where $a > b$. No problem included a subtraction with a negative difference, and nothing was said about the restrictions connected to the metaphor, so the inherent conflict did not surface.

Finding the Difference Between Numbers with Different Signs

We have seen that in a situation where one number is positive and one is negative, the phrase "find the difference between the numbers" can have different meanings and induce different procedures depending on the metaphorical meaning attached to it, summarized in Fig. 6.5.

Participants in a discourse may not think in terms of the same metaphor when using the phrase "finding the difference between the numbers." This presents a dilemma, where confusion as to whether to use addition or subtraction and magnitudes or values could be the result. In both of the metaphors, an introduction of *directed differences* is possible but would entail an extension of the metaphor that is not found in the empirical data of this study. A "negative number of negative objects" and a "negative distance between locations" are word constructions far removed from intuition, creating narratives that would need to be subject to very explicit negotiation of meaning in order to be useful extensions of the metaphors.

Object collection metaphor: the difference between two sets of objects with opposite signs (e.g., -2 and +3)	A: Written as [value] + [value] but calculated as: [larger magnitude] – [smaller magnitude] The answer is a magnitude, and the sign needs to be considered separately. "the sum of a negative and a positive is their difference, if a larger [number is to be subtracted] from a smaller their difference is reversed*" Brahmasphuta-siddhanta year 628*	
Measurement metaphor: the difference between two points on either side of zero (e.g., -2 and +3)	B1, subtraction: [larger value] – [smaller value] The answer is the distance between the two points (i.e., a magnitude)	B2, addition: [magnitude] + [magnitude] The addition of two distances is a longer distance (i.e., a magnitude)

Fig. 6.5 Overview of how the two metaphors map a difference between two numbers with different signs (one positive and one negative). The different mappings are named A for the object collection metaphor and B1 and B2 for the measurement metaphor

Examples from the Classroom

This section contains episodes where the idea of "finding the difference between two numbers" surfaced as a main issue in the classroom interaction. The difference between two numbers was sometimes associated with the difference between two magnitudes in an addition of numbers with opposite signs and sometimes associated with the difference in value in a subtraction of two numbers.

Whole-Class Instruction: 4 + (-6)

In an episode from the first negative number lesson, the teacher discusses with the class how to work out the sum of 4 positive (green dots) and 6 negative (red dots). One student has suggested writing this as "four plus minus-six" and the teacher writes 4 + (-6). Another student, Ove, suggests 4 – 6. The teacher makes a distinction between "how to think" and "how to write." Although the original operation is an addition, both Ove and the teacher, in two different ways, interpret the situation as a subtraction in the following excerpt.[4]

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[4] Transcript orthography: … means a short pause; […] indicates removed words or utterances; [word] indicates what the conversation is about; (word) clarifies an action/gesture; emphasis in bold writing is added by the author as part of the analysis.

Excerpt 6.1: Whole-Class Instruction

1	Teacher	[4 + (-6)] How were you supposed to work this out?
2		It looks rather strange when there are two, Ove,
3		when there is both a plus and a minus next to each other, it looks a bit strange.
4		So how do you explain your thinking?
5		How are you supposed to **think** when you work it out?
6	Ove	You **think 4 minus 6**
7	Teacher	Well, 4 minus 6,
8		and how then, do you **think** when you work out 4 minus 6?
9	Ove	You think, eh, just **4 take away 6**, you have a **debt** then
10	Teacher	You get a **debt** then, ok, of 2.
11		The question is, we know **they are different so to speak**
12		**something which is positive and something which is negative,** is what we have here,
13		and then we must **work out the difference between them**
14		The **difference between 6 and 4 is 2**, yes
15		and **there were more debts**, there were more on the red dice that are debts.

Ove sees this as a take-away situation by interpreting the minus sign as a subtraction [Excerpt 6.1, line 9]. In his discourse, both 4 and 6 are positive numbers; money is taken away, but since we do not have enough, we end up with a debt. Only the answer is a negative number. In his metaphor, the mappings onto the operations are the same as in the domain of natural numbers. Ove can take away more than he has and end up having a debt (but he would have difficulty taking away a debt unless he had one to start with). The teacher tries to make a distinction between how to write and how to think [lines 1–5]. She sees the situation as a comparison situation between two opposite types of objects [lines 11–14]. The difference is a difference in magnitude between 6 and 4, which is a natural number subtraction, and she considers the sign separately [line 15]. The situation described by the teacher is an example of (A) in Fig. 6.5 above: the *sum* of two values of opposite kinds results in a *difference* expressed as a magnitude where the sign of the number is determined separately. (See also Fig. 6.3 for an illustration of the metaphor.) The teacher says 6 – 4 = 2 where 2 is negative, whereas Ove says 4 – 6 = a debt of 2.

George's Problem: Subtracting a Negative from a Negative, -2 – -7

When working on subtraction tasks in his textbook, George becomes uncertain and calls for the teacher, seeking support for a conjecture. He says, "A negative number minus another negative number will always be a positive number." He makes this conjecture when trying to solve a problem where he is asked to write two negative numbers in the brackets to make the equivalence true: ( ) – ( ) = 5. A long discussion follows where the teacher gives counterexamples to his conjecture and talks about situations of taking away a "smaller debt from a larger debt" and still ending up with a debt (i.e., a negative number). In all these examples, subtraction is seen as taking away one type of object from a greater amount of the same type of objects, as illustrated in Fig. 6.2. Then George asks about the task (-2) – (-7) = 5, which is a subtraction of a type that does not correspond to anything in the source domain of the metaphor the teacher had been using. Excerpt 6.2 shows how the teacher converts this into the addition problem -2 + 7 using a sign rule and then interprets the addition of two numbers with different signs metaphorically as "finding the difference between the two numbers," in line with the discourse in the whole-class introduction in Excerpt 6.1.

Excerpt 6.2: Subtracting a Negative from a Negative

1	George	[George has suggested (-2) – (-7) = 5] How about this, is it right?
2		Minus 2 minus minus-seven[5] is 5?
3	Teacher	Minus 2 minus minus-seven becomes plus 7, doesn't it?
4	George	yes
5	Teacher	And **minus 2 plus 7,**
6		the **difference between** 2 and 7 is 5,
7		**which were there more of**, positive or negative?
8	George	Positive
9	Teacher	Precisely, so therefore it's plus-five.

It is implicit in the teacher's reasoning that she needs to simplify the expression -2 – -7 into -2 + 7 by applying a sign rule. The teacher then interprets -2 + 7 within an object metaphor, talking of the numbers as objects of different kinds where the difference is mapped onto addition [Excerpt 6.2, lines 5–6]. However, it is the difference in the magnitude of the numbers that is compared, and whichever type there are more of determines the sign [Excerpt 6.2, line 7]. This is another example of (A) in Fig. 6.5.

Lina's Problem: Adding a Negative and a Positive, -12.8 + 7.88

Excerpt 6.3 is from the third lesson in the sequence about negative numbers. Tina and Lina are working with the task: (-12.8) – (-7.88). They simplify the expression using a sign rule and come up with -12.8 + 7.88 but do not know how to go on.

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[5] In a Swedish school context, negative numbers are most commonly spoken of as minus-numbers. The distinction between "minus seven" meaning subtract seven and "minus-seven" meaning negative seven is very difficult to discern. To make this clearer in the excerpts, negative numbers such as (−7) are expressed in words: minus-seven.

Excerpt 6.3: Adding a Negative and a Positive

1	Teacher	You need to take the **difference**, between, you have minus 12 point 8 and plus 8 point, eh 7 point 88. So you need to work out the difference between them.
2		You get the **difference** by pressing 12 point 8 **minus** 7 point 88.
3		And you get 4 point 92, and then you know that the answer is minus. Because you saw, 12 is bigger than 7.
4	Tina	But did you take minus now?
5	Teacher	Now I just took the **normal**, 12 point, I took the **difference** between them, 12 point 8 and 7 point 88, equals, and then I got 4 point 92.
6		And then, I know the answer is **minus** 492, 4 point 92. Because **there were more minuses**, weren't there? […]
11	Teacher	No **they are different**, so you just take the **difference**.
12	Lina	Yes… (Lina tries doing it on the calculator pressing 12.8 + 7.88)
13	Teacher	Oops, you plussed[6] them
14	Lina	But wasn't I supposed to plus, there were two minus signs?
15	Teacher	That one is minus. And that one is plus. And they should be plussed together.
16		But now it's minus 12 point, and then they are **different signs. Then it's minus**.
17	Lina	I don't understand anything.
18	Teacher	You owe me 12 kronor
19	Lina	Yes
20	Teacher	And you are given 7 kronor
21	Lina	Yes
22	Teacher	Well, then what you get, is a **difference**
23	Lina	How do you calculate it?
24	Teacher	Minus
25	Lina	You do minus?

In Excerpt 6.3, lines 1–6, the teacher is talking metaphorically about the task -12.8 + 7.88 in line with what she did in the whole-class instruction. The addition of two numbers with opposite signs is interpreted as a comparison of two collections of different types of objects (see Fig. 6.3). This comparison is then mapped onto a subtraction of magnitudes: 12.8 – 7.88. The teacher uses the term "normal difference" [line 5] indicating that she now considers only the magnitudes and treats the numbers as quantities. Lina wants to do an addition since there was a plus sign indicating addition [line 14], and despite the teacher's persistence, Lina is still surprised in the end that the teacher uses minus [line 25]. In lines 15–16, the teacher is inconsistent about whether to map onto addition or subtraction, trying to distinguish between *writing* it as an addition and *thinking* of it as a subtraction. Compared to George's problem, this task is complicated by the fact that it is the negative number that has the larger magnitude, which means that the answer is negative.

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[6] In a Swedish school mathematics context, the verbs adding and subtracting are often spoken of as "plussing" and "minusing."

Throughout the episode, the subtraction is transformed and reinterpreted several times:

(a) -12.8 – -7.88 is changed into -12.8 + 7.88 [using a sign rule].
(b) -12.8 + 7.88 is interpreted as a difference, although it is an addition [line 1, line 15], and therefore transformed into 12.8 – 7.88 [line 2, line 16].
(c) 12.8 – 7.88 is worked out on the calculator, producing the answer 4.92 [line 3], which is interpreted as -4.92 [line 3].

Although the whole episode has a procedural focus, there are clear metaphorical meanings related to the object collection metaphor in the words used by the teacher, such as *difference* and "more minuses" [line 6] and the example of owing 12 kronor and giving 7 kronor [lines 18–20]. Although we could argue that the use of a calculator afforded the possibility of simply pressing −12.8 + 7.88 and getting the answer -4.92, that strategy would not be useful without a calculator. In another episode when Olle is working on a similar task, -12.8 + 3.02, he does not have a calculator at hand and wants to work it out using a pen and pencil algorithm. The procedure is the same: he needs to work out 12.8 – 3.02 first and then interpret the answer as negative. The difference is mapped onto addition when it is written but onto subtraction of magnitudes when it is calculated

We have now seen several examples where words and narratives from an object collection metaphor introduced by the teacher influenced the classroom discourse about negative numbers. The textbook relies on a measurement metaphor using contexts such as a number line, time line, and thermometer. In the following example, we see one student, Tomas, using the textbook discourse when discussing a temperature problem.

Interpreting a Difference as More Than Zero

In the stimulated recall interview with Tomas, an episode from one of the classroom videos was viewed as a probe for discussing *difference*. The episode was taken from the last lesson on the topic of negative numbers, where students came to the whiteboard in pairs to present and solve a negative number story problem they had constructed. Tomas and Hans presented their problem:

It is 22° inside and -13° outside. What is the temperature difference?
22 – (-13) = 35. The difference is 35°.

When solving the task they used the words "difference" and "degrees" from the context of the task. They said: "You need to take the difference between twenty-two degrees and minus thirteen. And so that is thirty-five degrees. So it makes plus, two minus (pointing to the two minus signs), and so you get the difference." This episode is discussed in the interview with Tomas as shown in Excerpt 6.4 (where Int. is the interviewer).

Excerpt 6.4a: Interpreting a Difference as More Than Zero

499	Int.	How do you know that you should take 22 minus minus-thirteen?
500	Tomas	Because, that's what you need to do to work out the **difference**
501	Int.	yes?
502	Tomas	Then you need to take minus, I mean, the **big** number minus, the, **smaller**
503	Int.	yes?
504	Tomas	and so you've got minus 13 as the **smallest** number
505	Int.	yes ok, is it always the bigger number first?
506	Tomas	eh… or yes, **otherwise you get a negative number**, that can… that's **not** a … well… yes that's how it is, you need to take the big…

Tomas interprets "finding the difference" as a question of subtracting the smaller value from the larger value but works out the answer by adding the two magnitudes, as in (B2) in Fig. 6.5. To further explore Tomas's way of thinking, the interviewer introduces a new problem where the magnitude and the value diverge by asking Tomas about the difference in temperature if it is -35° outside and +20° inside; i.e., -35 < +20 but |35| > |20|. The interviewer then continues with a non-contextualized addition of a negative number and a positive number: -35 + 20.

Excerpt 6.4b: Interpreting a Difference as More Than Zero (continued)

508	Tomas	Then you take eh, that, plus 20 minus minus-thirty-five
509	Int.	Mm? and so that's biggest? [pointing to 20]
510	Tomas	Yes
		[…]
517	Int.	If I calculate this 20 minus minus-thirty-five. What does that make?
518	Tomas	That's ehm, 55
519	Int.	Mm… and how about if I work this out [writes −35 + 20] minus 35 plus 20, what does that make?
520	Tomas	Well that makes ff… minus 15
521	Int.	So then, which is the difference between these numbers, is it 55 or is it 15?
522	Tomas	Eh it's 55 […]
525	Int.	How do you know that you need to put the biggest number first?
526	Tomas	…you know…well, but you always have to kind of something to… if you've got something, some difference
527	Int.	Mm?
528	Tomas	Then you need to have one higher and one lower
529	Int.	Mm?
530	Tomas	And eh, if you want to work out, a eh…. positive number which is what you need to get
531	Int.	Mm?
532	Tomas	If you want to work out the difference that is, the only way is, take the biggest…

533	Int.	Mm, good explanation. Why does the difference have to be a positive number?
534	Tomas	Because, ehm, that's smaller. It always has to be… a… you know it always has to be… something, smaller than… bigger. So there kind of has to be a… difference between them
535	I	Mm?
536	Tomas	And so it's positive

Tomas clearly conceives of a difference as a nonnegative number or rather as a magnitude. The difference between two numbers is associated with a procedure coherent with a measurement metaphor (see Chap. 9 in this book for more about this use of measurement metaphor and temperature). He can work out -35 + 20 correctly but does not consider it a difference [Excerpt 6.4b, line 521–522], so his mapping of a difference is more in line with that of the textbook than the teacher's discourse involving an *object collection* metaphor. His metaphorical meaning of the word difference is that a difference is a number larger than zero, a positive number [line 530].

The discourse in this classroom has provided the students with two quite different metaphors involving negative numbers: an object collection metaphor and a measurement metaphor. The above analysis showed that the meaning of the word difference was elusive and at times contradictory when related to integers. As shown in Fig. 6.5, a difference could be connected to writing addition but thinking subtraction (A) or to writing subtraction but thinking addition (B1 and B2). In all cases, the difference is a magnitude, and if the answer is negative (as in some cases of A), the sign has to be dealt with separately. With regard to the measurement metaphor (B), there was no narrative in the discourse involving a negative difference

Metaphorical Meanings Affecting Student Achievement

For a teacher of mathematics, an important question is whether this metaphorical reasoning has any impact on students' achievement. We shall look at the students' solutions to a problem about temperature differences that appeared on the test at the end of the chapter on negative numbers:

Problem[7] The temperature in the freezer was -14°C
The temperature in the room was +20 °C
How many degrees difference were there between
the freezer and the room?

With a real thermometer at hand, the problem would have been trivial for a student in Sweden, where temperatures below zero are discussed every winter. However, the task was not simply to determine the difference between two

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[7] Two test versions were used: one had the numbers −12 and + 20 and the other −14 and + 20. For comparison reasons they are all referred to here as though they were − 14 and + 20.

Table 6.1 Students' solutions to a question about temperature differences ($n = 21$)

Mapping of difference	Solution	n	%
Addition	*14 + 20 = 34* With a clear reference to a visual representation of a number line	4	
	14 + 20 = 34 Without further comment	5	
Total number of correct answers		14	67
Addition	*-14 + 20 = 34* Correct result but incorrect representation. The first temperature is written as a signed number but in the calculation the magnitude is used	1	
	(-14) + 20 = 6 Adding the two values	2	
Subtraction	*20 – 14 = 6* Finding the difference between the magnitudes instead of the values	2	
	14 – 20 = 34 Correct result but incorrect representation. Writing a subtraction but calculating an addition	1	
Others	(Incorrect)	1	
Total number of incorrect answers		7	33

temperatures but also to represent it appropriately. The variety of solutions presented by the students indicates confusion concerning whether to add or subtract and whether to use magnitudes or values. Table 6.1 shows that, out of 21 students, correct answers were given by 9 students who modeled the difference as an addition of magnitudes and by 5 students who modeled it as a subtraction of values, all in line with the measurement metaphor (as described in Fig. 6.5b). Seven out of 21 students (33%) suggested mathematically incorrect calculations or representations, choosing a wrong combination of operation and magnitude/value.

Data suggests that although teacher, textbook, and students all used the same words, these had different metaphorical underpinnings and therefore created different narratives. Problems arose when the participants in the lesson were unaware of this and meanings were taken as shared although they differed. When conflicting narratives emerged, they were not discussed and never resolved, resulting in uncertainty for the students.

Discussion

Two main points about the discourse can be noted in these results. First, the teacher conveyed that there are two different discourses: what you think and what you write (e.g., thinking subtraction but writing addition). Maybe this distinction could be described more constructively as a distinction between representing the problem with signed numbers and solving it using natural numbers. Second, in these lessons most of the metaphorical reasoning came from the teacher. The students more often used mathematical terminology, which suggests that they were attempting an intra-mathematical discourse or that they were only interested in establishing a procedure. According to Sfard (2007), it is necessary to change the mathematical meta-rules in order to make sense of negative numbers; rather than justifying narratives about negative number operations through real-world examples or concrete models, students need to accept an intra-mathematical justification. In this study, the teacher guided the students into metaphorical reasoning by only using models and metaphorical justifications, thereby staying completely within the old meta-rule. However, the models used were only concrete in an imagined sort of way: there was no use of money and thermometers to justify the reasoning in a hands-on manner. Furthermore, the teacher left the concrete models when she formulated a sign rule as a procedure of exchanging two minus signs with a plus sign; but, even when students applied the rule correctly, they had problems making sense of the calculation and getting a correct answer. Many times metaphors appeared implicitly, and the teacher might not have been aware of them. The examples above illustrate how underlying and implicit metaphors are sometimes taken for granted but are actually not a shared reference. There was no explicit or shared understanding of a "difference between two signed numbers," and consequently the students were uncertain about how to solve such problems. There was no shared understanding of what "taking away a debt" meant. Font, Bolite, and Acevedo (2010) observed in a study of teachers' use of metaphors that "There was no control over metaphors while teachers were unaware of using them" (p. 148). Becoming aware of the underpinning metaphors would seem to be a necessary requirement for better understanding here.

Another important finding concerns terminology relating to the size of numbers and whether to consider magnitudes or values of signed numbers. There were no words in the classroom discourse to distinguish between the two different aspects of size that diverge for negative numbers: a smaller negative number has a larger absolute value. *Difference* was, for most of the students, associated with magnitudes and must therefore be more than zero, as Tomas expressed in Excerpt 6.4. Various researchers have emphasized the importance and difficulty of distinguishing between the magnitude and direction of negative numbers (Altiparmak & Özdoan, 2010; Ball, 1993). These results support previous results and suggest the necessity of using appropriate words to distinguish between the two meanings of size, for example, by introducing the idea of absolute value along with signed numbers. The rather imprecise discourse of the mathematics classroom could have influenced these students' achievement. Great uncertainty prevailed among the students concerning the meaning of the phrase "take the difference between the numbers," which was influenced by the different contexts where this phrase has been used and the different metaphors these contexts become source domains for. Consequently, the students did not develop a discourse rich enough to deal with "different differences."

Implications

In the empirical data for this study, integers were talked about mainly as object collections in take-away or add-on situations often involving money or as distances in measuring instances such as the distance on a number line, time line, or thermometer. When there were tasks involving adding one positive with one negative number, the numbers were interpreted as object collections and treated as magnitudes to find the difference. Very rarely did any motion along a path appear. However, in the vignette at the start of this chapter, addition and subtraction were presented as motions in terms of jumps up or down the number line. In all the metaphors discussed so far, the common problem is the connection between subtraction and the meaning of the word *difference*. When the word difference appears in a new situation with a new meaning, a conflict may emerge. One way to resolve that conflict is to spend time making the meaning of the words we use explicit, detaching them from their metaphorical meaning to allow them to represent generalized features of abstract mathematical concepts. Freudenthal (1983) writes, "however one proceeds in extending the number concept, it is necessary that the fact and the mental process of extending are made conscious" (p. 460). The extension of the number concept needs to become part of the discourse, and to this end an appropriate use of words with clear meanings in metaphorical reasoning is essential. Precisely the kind of reflective mathematizing illustrated in the introductory vignette is needed.

In the case of subtracting integers, most of the ambiguity of the word difference comes from the fact that we call the answer to a subtraction with positive numbers a difference although we represent it as something else: as the remainder in a take-away situation or the location we end up on in a backward movement. Neither of these two meanings of difference is related to the everyday meaning of difference as dissimilarity or inequity of size when comparing two things, nor are they easy to extend to subtracting integers. In the measurement metaphor, the difference is a distance between two locations, akin to an everyday experience of differences, but not generalizable to integer subtractions due to its lack of direction.

Returning to the introductory vignette, we can imagine how the discussion continues. Although it is a lesson about positive whole-number subtraction and not about subtracting negative numbers, it goes to the heart of how we talk about subtraction. We left the class with the question, "Why does it work to do addition in a subtraction problem?" The answer to that question lies in how we define the term difference. In the subtraction sentence 283 – 275 = 8, the answer 8 is called the difference between the two numbers. Not because it is the number we end up on when we move backward 275 steps. Not because it is what remains after we have removed 275 objects. The answer 8 is called the difference because it is what we need to add to the subtrahend 275 to make it the same as the minuend 283. This explanation connects addition with subtraction: $a - b = x \leftrightarrow b + x = a$. When a subtraction is rewritten as an addition, the motion in the metaphor (x) can have direction and be represented by a positive or a negative number: how much more ($+x$) or how much less ($-x$) one needs in order to be equal to the other. Taking away, moving backward, and measuring distances are simply procedures used to work out that difference, sometimes useful and sometimes not. To realize this, we need to move away from take-away situations in favor of compare/equalize situations (Fuson, 1992), whatever the metaphor for numbers may be. I will venture to describe how this could be done using two hypothetical extensions of the ideas brought forward in the introductory vignette, taking them further to illustrate the subtractions -2 – -5 = 3 and -5 – -2 = -3. In both items, the absolute difference between the numbers is the same, but the directed difference (i.e., the *difference* as the answer in a subtraction) is not. The metaphorical meaning of numbers and operations in the following illustrations can be generalized to all subtractions involving integers. The point of explicitly (re)defining subtraction and difference in this way is to connect subtraction with addition. In both examples, a [state_1 + translation = state_2] addition is connected to a [state_2 – state_1 = translation] subtraction.

Using an Object Collection Metaphor

If collection *a* contains 2 negatives and collection *b* contains 5 negatives, the subtraction $a - b$ is -2 – -5. It illustrates the difference between the two numbers interpreted as: How many objects of what kind *do we need to add* to collection *b* to make it the same as collection *a*? In other words -5 + x = -2. Adding 3 positive objects to 5 negative objects will result in 2 negative objects, given the conceptual understanding of zero as a total zero, made up of opposites (Gallardo & Hernández, 2006). The difference is +3 because -5 + +3 = -2. Within the context of debts and gains from the classroom described above, the subtraction -2 – -5 would be discussed as follows: The difference between these two economic states is what I need to add to the second one to make it equal to the first one. A debt of 5 needs to have a gain of 3 added to be equal to a debt of 2. The important issue when talking about differences of states is the order of comparison (Kullberg, 2010). Reversely, the subtraction -5 – -2 illustrates the difference between the two numbers interpreted as: How many objects of what kind do we need to add to -2 to make it the same as -5? In this case the difference is -3 because -2 + -3 = -5.

Using a Movement Along a Path Metaphor

If *a* is the location -2 (i.e., 2 below zero) and *b* is the location -5, then the subtraction $a - b$ is written as -2 – -5 and interpreted as: How far and in what direction *do I need to move* to get from *b* to *a*? In other words -5 + x = -2. In this case, the difference is +3 because -5 + +3 = -2 (see the top arrow in Fig. 6.6). Similarly, -5 – -2 illustrates the difference between the two numbers interpreted as: How far and in what direction do I need to move to get from -2 to arrive at -5? The difference is -3 because -2 + -3 = -5 (see the bottom arrow in Fig. 6.6).

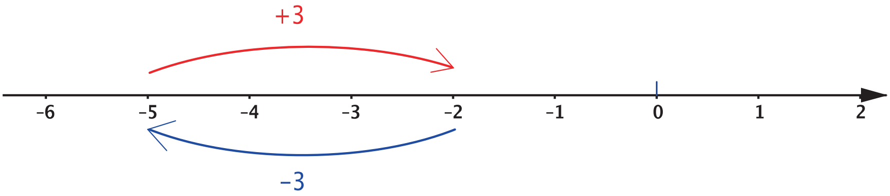

Fig. 6.6 Illustrations of directed difference. Top arrow: -2 – -5. Bottom arrow: -5 – -2

In order to understand subtraction as difference in this way, a thorough understanding of addition of integers is essential, including an operational and conceptual understanding of the number zero. Engaging students in the kind of reflective investigations about subtraction and its relation to addition described in the introductory vignette and the final examples will pave the way for the change of meta-rules that Sfard (2007) calls for when integers enter the scene.

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Chapter 7
Challenges of Promoting Conceptual Change with Instructional Contexts

Laura Bofferding

Abstract This chapter focuses on the interaction of two first graders as they attempt to make sense of a particular instructional context for learning negative numbers. The context is one where they move an elevator to a building's floors above and below ground in order to model integer addition and subtraction problems. In particular, the focus of the activity was to discover that solving problems such as 4 – 1 and 1 – 4 will result in different answers. The two students misinterpret each other, model the problems in multiple ways with the elevator (and to varying extents), but also work cooperatively at times as they complete the activity. Their efforts highlight the difficulties they encounter when working with ready-made contexts and in obtaining solutions that do not fit their prior experiences. The results present a brief view of the conceptual change process and support a stronger focus on connecting to students' prior thinking when introducing new instructional contexts.

Conceptual change is often a lengthy and difficult process, especially in cases where new concepts are counterintuitive and do not align with observable phenomenon (Vosniadou, 2013). In the case of negative integers, students have particular difficulty negotiating the values of negatives (Ball, 1993; Bofferding, 2014). Negative quantities are not tangible in the world in the way that positive quantities are (Martínez, 2006), and this is reflected in children's counting sequences which start "One, two, three…" and may sometimes include zero when counting backward (Bofferding, 2014; Clements & Sarama, 2014) – neither sequence typically includes negative numbers. Further, students' early experiences with whole numbers can create barriers for their later learning of integers. Murray (1985) captured this well when identifying eighth- and ninth-grade students who could solve a variety of negative integer problems but solved 3 – 8 as 8 – 3. For this problem type, young children will also reverse the numbers or answer 0 (Peled, Mukhopadhyay, & Resnick, 1989), arguing that there are not enough to subtract (Bofferding, 2010).

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Although it may at times seem elusive, there are mechanisms for promoting conceptual change through instruction. One such way is to present students with situations in which they have the opportunity to confront their current way of thinking with a different yet plausible one. For example, for children who think the Earth is flat, Vosniadou and Skopeliti (2014) suggest that teachers show them how the Earth can appear flat to a person standing on the ground but appears round to an astronaut in space, rather than just telling them that it is round. In this way, they can understand why they have the perception they do but that it is limited. Similarly, for children who ignore negative signs, providing contexts where they have to compare problems with and without negative signs can help them begin to notice them (Aqazade, Bofferding, & Farmer, 2016; Bofferding, 2014).

A complementary way to promote conceptual change is to have students work with others in order to encourage them to reconsider their own thinking (Vosniadou, 2007). Children naturally challenge each other as they work, providing opportunities to discuss different reasoning (Inagaki & Hatano, 2013) or explore a concept in new ways (Parks, 2015). In the latter case, Parks provides an example of two children covering a Lego plate with Legos. When Ivan sees the tricky gaps left, he starts to build upward; however, Cliff demonstrates how they can continue to fill in some of the spaces by using narrow pieces and rotating them. They then work to completely fill the bottom, sometimes moving pieces in order to meet their goal. In this case, Cliff helps Ivan consider the gaps from a new perspective, which ultimately leads him to see covering the whole plate as possible. In a similar vein, it is important to help children see problems, such as 3 – 8 from a new perspective, an integer perspective.

Framework

Conceptual Change Involving Integers

Children, through interacting with the world and other people, initially learn about whole number concepts (Clements & Sarama, 2014; Vosniadou, Vamvakoussi, & Skopeliti, 2008). They count the number of rocks they find, they learn about concepts of more as they get another cracker, they use one-to-one correspondence when giving each person a fork, and they experience order with time and lining up objects by size (Baroody, Lai, & Mix, 2006). Elsewhere, I illuminated a series of mental models that children exhibit concerning integer order and values, arising from such initial experiences with number and building to formal characterizations of integers (as determined by Western culture) (Bofferding, 2014). Next, I provide descriptions of these integer order and value mental models.

Children who exhibit an *initial mental model* for integer order and values rely on whole number cues and ignore extraneous signs (i.e., negative signs); other children might order negative numbers apart from positive ones (because they look different) but still work with them as equivalent to positive numbers (e.g., -5 = 5). Some children demonstrate an inclination toward considering negative numbers as different in value than positive numbers, primarily because they interpret the negative sign as a subtraction sign. For instance, some children will exclaim that negatives are worth zero (Schwarz, Kohn, & Resnick, 1993–1994), in some cases clarifying that the number is taken away (Bofferding, 2014). These children exhibit a *transition I mental model* since they are transitioning from treating negatives as equivalent to positives, but they do not distinguish among negative values as they are all considered worth zero (Bofferding, 2014).

When children accept that negative numbers have values in their own right (and are not just amounts being taken away), many will consider negatives to be less than positives but larger negatives to be more than smaller ones (e.g., -5 > -3). These students are classified as exhibiting *synthetic mental models* because their treatment of negatives is a synthesis of their initial mental models (i.e., numbers with larger magnitude are greater) and a modification based on new information (i.e., negatives are less than zero). Depending on the context, such thinking could be correct (e.g., in terms of coldness, -10 °F is more than -2 °F) (Bofferding & Farmer, 2018); however, in terms of formal mathematics, children need to learn that -3 > -5 unless otherwise dictated by the context of the question. Children who are grappling with this idea will sometimes switch between choosing the larger or smaller negative as greater. These children exhibit *transition II mental models*, as opposed to children who exhibit *formal mental models* and reason formally about negative integer order and values on a consistent basis.

There are potential implications of children's integer mental models in regard to their solutions to integer addition and subtraction problems. If a child exhibits an initial mental model, it would make sense for them to solve integer problems as if they are positive number problems (e.g., solve -3 + 5 as 3 + 5 = 8). Further, without knowledge of negatives, they may reverse numbers in problems, such as 2 – 5 (solving it as 5 – 2), or indicate that the answer is zero because they cannot take any more away. A child exhibiting a transition I mental model would likely treat negatives as either positive or zero (e.g., potentially solving -3 + 5 as 0 + 5 = 5). Therefore, understanding how children interpret integer order and values can provide important insight into their choices when solving integer addition and subtraction problems and vice versa. Further, knowing how children think about integers can help teachers tailor lessons to help students reconsider their integer mental models (e.g., whether the negative sign matters) and provoke the conceptual change process.

Conceptual Models and Problem Types Involving Integers

As discussed, children's experiences with numbers can expose them to new ways of thinking about integer order, values, and operations, challenging their integer mental models. Likewise, contexts that teachers or students impose on integer problems reflect underlying properties of integers in relation to operations (i.e., conceptual models) (Wessman-Enzinger & Mooney, 2014) that may interact with students'

Table 7.1 Marthe's (1979, 1982) problem types and correspondence to conceptual models

Problem types		Description	Conceptual models
State-State-State	(SSS)	Combine two static amounts; remove a static amount from a static amount	Counterbalance
State-Transformation-State	(STS)	Make a change to a static amount	Bookkeeping; translation
Transformation-Transformation-Transformation	(TTT)	Combine or undo changes	Bookkeeping; translation

mental models of integers. Therefore, it is important to consider the different integer conceptual models children might use to reason about instructional contexts. There are several integer conceptual models that children will use when thinking about integer problems (Wessman-Enzinger & Mooney, 2014) and that instructional contexts try to support.

The *counterbalance conceptual model* is rooted in discrete quantities but requires some imagination on the part of the child; adding equal amounts of positive and negative quantities results in zero, even though the actual quantities added still remain (Wessman-Enzinger & Mooney, 2014). In terms of instruction, the "chip" instructional context is often used to support counterbalance thinking. In one version of the chip instructional context, one positive chip cancels out one negative chip (e.g., Liebeck, 1990). Another context is net worth, where $1 in assets cancels out $1 in debt (Stephan & Akyuz, 2012). Based on the problem types identified by Marthe (1979, 1982; see Table 7.1 for a brief summary), problems using a chip context lend themselves to State-State-State (SSS) problems, which involve the composition of directed states (e.g., two initial and a final state). For example, 5 positive chips and 7 negative chips result in an overall "charge" of -2, much like an asset of $5 and debt of -$7 results in a net worth of -$2.

The *bookkeeping conceptual model* is also rooted in discrete quantity but further capitalizes on students' use of the counting sequence to think about solving addition and subtraction. Bookkeeping is perhaps a natural conceptual model for students as it involves thinking about gains and losses, where zero is neither a gain nor a loss (Wessman-Enzinger & Mooney, 2014). Mukhopadhyay, Resnick, and Schauble (1990) drew on children's bookkeeping conceptual models by presenting stories about a boy who gains and loses money and asking the children to comment on the boy's situation. These types of problems typically follow a State-Transformation-State (STS) problem structure, where there is an initial state, followed by a transformation, which leads to a final state (Marthe, 1979, 1982) (e.g., Henri started with a bank balance of $2 [state], lost $5 [transformation], and ended with -$2 [state]). They might also involve a combination of transformations, the Transformation-Transformation-Transformation (TTT) problem type, when gains and losses are treated relative to some unknown starting point (Marthe, 1979, 1982) (e.g., Henri earned $2 [transformation] and then lost $5 [transformation], resulting in a change of -$3). However, the challenge with the bookkeeping conceptual model is that children may not reason about negative numbers, e.g., interpreting debts as money owed (Mukhopadhyay et al., 1990; Whitacre, Bishop, Philipp, Lamb, & Schappelle, 2015).

The *translation conceptual model* incorporates students' use of the counting sequence with an added focus on movement and directed magnitude that could more explicitly support reasoning about negative integers; further, it involves work with continuous instead of discrete quantities. In this model, often supported by a number line (Caldwell, Karp, & Bay-Williams, 2011), positive numbers represent movement in one direction, negative numbers represent movement in the opposite direction, and zero represents no movement (Wessman-Enzinger & Mooney, 2014). Multiple contexts support a translation conceptual model: temperature, elevation, and movements up and down in an elevator, which typically draw on the STS and TTT problem types described by Marthe. A benefit of these contexts is that their use can build on students' use of counting to solve positive number problems, and children can leverage the counting process to solve problems with negatives (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011). Further, these contexts relate to real-life situations.

Situating the Data

Although particular instructional contexts align more with some conceptual models than others, there is no guarantee that students will use the optimal conceptual model when reasoning about them or that one particular conceptual model will spark conceptual change. In this chapter, I present a case of two students as they negotiate being asked to use a specific instructional context as they solve positive integer subtraction problems with negative answers. In particular, after illustrating their use of the elevator context for solving addition problems, I focus on how their use of the context interacts with their own integer mental models, subtraction strategies, and interpretations of the number sentences.

The data presented in this chapter comes from a larger study involving a pretest, 8-lesson intervention, posttest design. The pretest and posttest contained questions that targeted students' understanding of integer order and values, as well as of integer addition and subtraction (see Bofferding, 2014 for more details). Using methods described in Bofferding (2014), I classified students' integer mental models on both the pretest and posttest. The main data of focus involves the interaction of two students during the pair work time of one of the lessons. To analyze their interaction, I identified how they interpreted the problems they were solving (as SSS, STS, or TTT problems), what conceptual models they drew on, and the integer mental models characterizing their responses.

There is much written on children's addition and subtraction strategies that I will not repeat here (see Clements & Sarama, 2014 for a nice overview); however, before introducing the two case study students, a relevant distinction to note are two ways of thinking about the operations: either as a change in magnitude (i.e., counting up or down) or in terms of movement (i.e., going to the right or left on a number line). With positive number addition and subtraction, these two align: addition means counting to the right on the number line, which corresponds to an increase in number magnitude (and vice versa with subtraction). With negative numbers, this is not always the case. For example, "-3 + 5 = 2" involves a decrease in overall magnitude but, when starting from -3, involves a movement to the right on the number line (Bofferding, 2014).

As the focus of this chapter is on the interactions between a pair of students, social interaction also comes into play. When negotiating mathematics problems, pairs of children often do best if they are similar in terms of mathematical development, except in cases where weaker children encourage active participation out of their partners (Yackel, Cobb, & Wood, 1991). There are several potential benefits of having children work together as described by Yackel et al.:

> As the children work together and strive to communicate, opportunities arise naturally for them to verbalize their thinking, explain or justify their solutions, and ask for clarifications. Further, attempts to resolve conflicts lead to both the opportunity to reconceptualize a problem and thus construct a framework for another solution method, and the opportunity to analyze an erroneous solution method and provide a clarifying explanation. (p. 401)

Children can interact with each other's ideas with different levels of engagement. According to Webb et al. (2014), high-level engagement involves explicit acknowledgment of another child's strategy and either adding detail to it, referencing it in relation to another strategy, or challenging it based on an alternative strategy. Medium-level engagement, they suggest, involves acknowledging or challenging another person's strategy without adding to it. Finally, with low-level engagement, students either talk about or agree or disagree with another strategy in vague ways (e.g., not being specific about what aspects they agree with). The extent to which students present new ideas and challenge each other could play a role in their conceptual change. This chapter illuminates the messiness involved for children as they sort through integer ideas with a partner and highlights the challenges of promoting conceptual change while also offering some suggestions for mitigating the difficulties.

Focus Students' Before-Unit Pretest Integer Reasoning

The two girls, Dinah (ID: 207) and Indira (ID: 402), who are the focus of this chapter were first graders (6.5- and 7-year-olds) from two different classrooms at the same public elementary school in northern California. They chose to work together for this lesson. Dinah exhibited an initial, whole number mental model for integer order and values on the pretest. When filling in missing numbers on a number line, she only used positive numbers and left out zero (5, 4, 3, 2, 1, 2, 3, 4, 5), and she always chose the number with highest absolute value when determining which integer was greater. Indira exhibited a transition I mental model on the pretest, suggesting she saw a difference between numbers with and without a negative sign but was

Table 7.2 Summary of Dinah and Indira's pretest performance

Pretest section	Dinah	Indira
Empty number line	5, 4, 3, 2, 1, 0, 1, 2, 3, 4, 5	0, 1, 2, 3, 4, 5
Ordering numbers	Mostly absolute value	Negatives reversed before zero
	-3, 0, 2, 3, -5, -9 (0 = least)	-3, -5, -9, 0, 2, 3 (0 = least)
Comparing numbers	Based on absolute value	Negatives equal zero
	-5 > 3	3 > -5 (because -5 = 0)
Are these equal: 1 – 3 vs. 3 – 1?	Yes	Yes
Solving 1 – 4, 3 – 9, 6 – 8	Reversed numbers	Reversed numbers or answered 0
Arithmetic strategies	Ignore negatives	Ignore negatives; use negatives as subtraction signs
Order and value mental model	*Initial: whole number*	*Transition I*

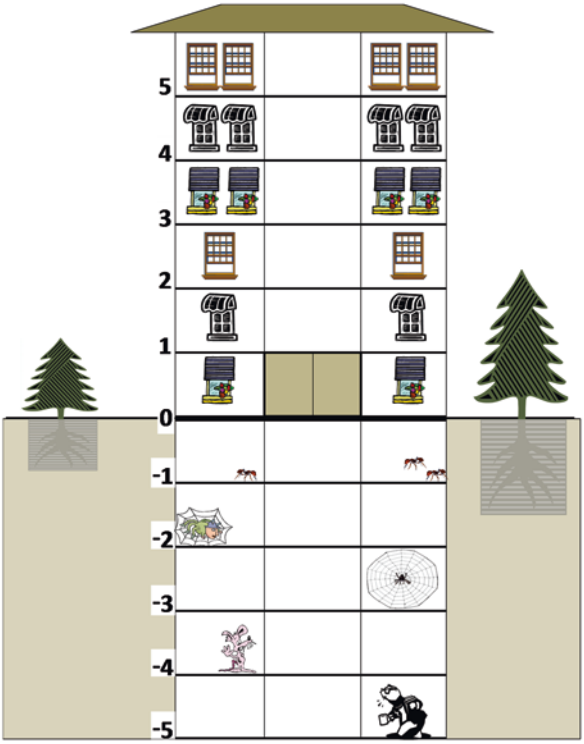
still trying to determine what that meant in terms of number values. Although she did not fill in any negative numbers on her number line (0, 1, 2, 3, 4, 5), when ordering a set of integer cards, she ordered the negative numbers before the positive but in reverse order (-3, -5, -9, 0, 2, 3). Further, when comparing integers, she interpreted negatives as worth zero. When comparing -5 and 3, she claimed, "Five minus five is zero, and three is greater," and when comparing -2 and -7, she further explained, "Seven is greater than two, but they're both zero."

In pretest interviews prior to integer instruction, when asked if the problems 3 – 1 vs. 1 – 3, 5 – 8 vs. 8 – 5, and 9 – 6 vs. 6 – 9 would have the same answer, both girls thought they would because the problems had the same numbers. Likewise, when asked to solve 3 – 9, 6 – 8, and 1 – 4, Dinah reversed the numbers and got positive answers. Indira did this for 1 – 4, but answered zero for 3 – 9 and 6 – 8. On the rest of the integer problems, the girls ignored the negative signs, although Dinah occasionally interpreted the negative sign as an indication to subtract (e.g., -9 + 2 = 7, 7+ -3 = 4). A summary of these results appears in Table 7.2.

Integer Instruction

Before the activity of interest here, the students had four other lessons related to negative numbers as part of an integer intervention. Two involved focusing on the differences between positive and negative symbols, and one involved putting integers in order. The fourth lesson involved sharing two cakes with four people and four cakes with two people to help students be receptive to the idea that if the numbers switch in a problem, it can sometimes lead to different answers. At the beginning of their fifth lesson, the students were shown a visual of an elevator in a building that had floors above and below ground (see Fig. 7.1). The students helped label the floors and showed how to move the elevator on the visual to solve 0 + 5 and 5 + 0. Students were then given a worksheet (see Fig. 7.2) and told that they

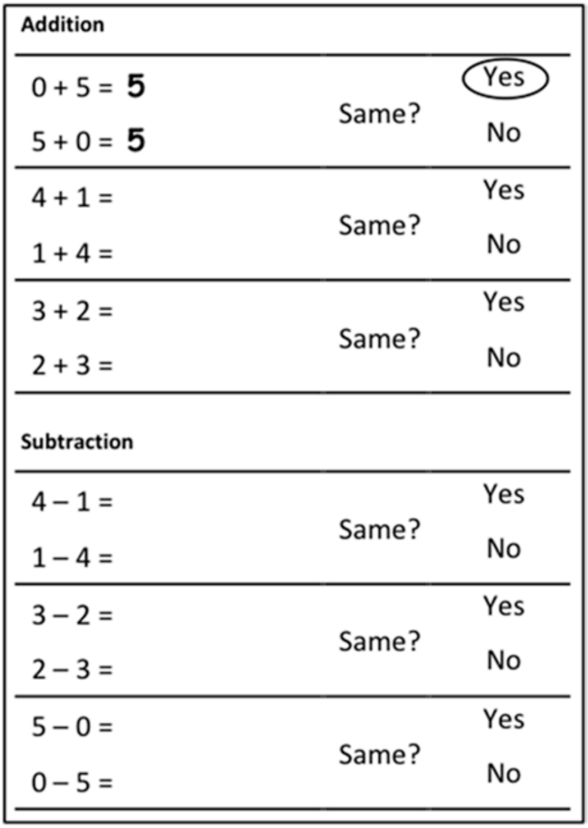
Fig. 7.1 Elevator context



should move the elevator on their boards to solve the problems and then indicate if each pair of reversed problems had the same answer. The elevator context was chosen to give students a realistic situation where subtracting a larger number from a smaller one would make sense in order to help them see that such problems are possible and to realize that subtraction problems with the numbers reversed would have different answers (e.g., 5 – 3 = 2 but 3 – 5 = -2). The following description and analysis focuses on Dinah and Indira as they work to complete the worksheet together.

A Closer Look

In the next sections, I present the results of the interaction between Indira and Dinah as they solve the problems on the worksheet illustrated in Fig. 7.2. First, I include their negotiations around the addition problems; next, I provide their debates around the subtraction problems. Their interactions, presented in the present tense (exchanges), appear on the left side of the page with my corresponding analysis (what we can learn from the exchange) on the right side.



Addition

0 + 5 = **5** 5 + 0 = **5**	Same?	[✓ Yes] No
4 + 1 = 1 + 4 =	Same?	Yes No
3 + 2 = 2 + 3 =	Same?	Yes No

Subtraction

4 − 1 = 1 − 4 =	Same?	Yes No
3 − 2 = 2 − 3 =	Same?	Yes No
5 − 0 = 0 − 5 =	Same?	Yes No

Fig. 7.2 Worksheet that students completed while moving the elevator to solve problems

Solving Addition Problems: Finding Positive Answers

Let's take a look at how the two girls begin to navigate the first set of addition problems at the top of the worksheet.

Exchange #1	*What can we learn from this exchange?*
Indira has the recording sheet in front of her, while Dinah controls the elevator. Indira reads the first problem, "Four plus one." Dinah repeats this twice while grabbing the elevator, putting it at floor four and moving it up to the fifth floor, "Equals five." Dinah hands a pencil to Indira who writes "5" on the sheet.	*The girls quickly assumed roles and effectively modeled and completed the first problem using the elevator. Dinah treated the problem as an STS problem, starting the elevator at the fourth floor and moving up one floor.*

Exchange #1	*What can we learn from this exchange?*
Dinah suggests they move onto the next problem. Both of them state, “One plus four,” and then Dinah moves the elevator piece to floor one and moves the piece up as she counts up four. However, she answers, “One.”	*Dinah again treated the problem as an STS problem but lost track of what she was supposed to report. She reported the initial, as opposed to the final, state.*
Indira moves the elevator to the ground floor and responds, “No. It’s supposed to be one plus four equals five.” She moves the elevator up to the fourth floor by counting, “One, two, three, four,” and then, as she moves the elevator up to the fifth floor, states, “Four plus one equals five.” Next to “1 + 4 =” she then writes “5.”	*Indira, exhibiting high-level engagement, took over, modeling the problem as a TTT problem. However, although she was solving “1 + 4,” she modeled “4 + 1,” starting at the ground floor, moving up four floors, and then one more floor.*
	Indira already stated the problem and answer, so her switching of the numbers may have not been a conscious choice. The carefree nature with which she and other children change problems similar to this may explain their willingness to do the same with subtraction problems. In this case, she continued to model the problem, even though she had already stated the answer. Indira’s procedure also led to the correct answer; however, the goal was for them to model the problem as written in preparation for working with subtraction.

Now we return to the girls as they begin to work on the next set of problems, comparing “3 + 2” and “2 + 3.”

Exchange #2	*What can we learn from this exchange?*
Indira reads the next problem, “Three plus two.” Dinah starts the elevator at the third floor and moves it to the fifth floor while counting, “One, two.” Indira seems to have accidentally skipped ahead as she retorts, “No, two plus three” and grabs the elevator from Dinah.	*Similar to the previous pair of problems, Dinah controlled the elevator initially, using it to show the translation from the third floor to the fifth floor. At this point, the girls no longer worked in unison.*
Dinah continues on with the original problem, stating, “Equals five,” and points to the sheet where Indira is supposed to write the answer. Ignoring Dinah, Indira places the elevator at the ground floor and moves it to the first floor and then the second floor. Dinah interrupts her, “No, see if we move there (points to the ground floor), then it’s plus two, three, and then one.”	*Indira refocused on making translations from the ground floor, but Dinah took issue with Indira’s process, insinuating that she might be moving an extra floor; although they were not working together, they still exhibited a high-level of engagement.*

Exchange #2	*What can we learn from this exchange?*
Dinah takes the elevator and places it at the first floor, then the second, and third before stopping. Then she moves it to the fourth. Indira corrects her by moving it up one more floor, saying, "Two" to indicate that Dinah had only counted on one. She writes "2" next to "3 + 2=" and then erases the answer to the problem above it (1 + 4=) and asserts, "Let's do this again."	*Perhaps because Indira then corrected Dinah (to count on "two" more instead of one), Indira overly focused on the word "two" and wrote this down as the answer. Although both girls tried to use the context, they continued to do so in different ways; Dinah focused on the initial value as a state with each floor acting as a discrete unit (reflective of a bookkeeping conceptual model), whereas Indira focused on it as a transformation where distance from zero is important (reflective of a translation conceptual model).*

We join the girls again as Indira, having gotten sidetracked, revisits a previous problem (1 + 4 =) while Dinah continues on to the next problem on the sheet (2 + 3 =).

Exchange #3	*What can we learn from this exchange?*
Indira, solving "1 + 4," places the elevator at the ground floor (zero) and moves it to the first floor, then the second floor, and then the third. Dinah, meanwhile, continues to the next problem, "2 + 3=," stating, "Two and then three." Trying to continue with her solution to "1 + 4=," Indira moves the elevator to the fourth floor. Dinah completes her thought without using the elevator, "Equals five" while Indira writes "4" for "1 + 4" and circles, "No" on the right side of the paper, indicating that "4 + 1" and "1 + 4" do not have the same answers.	*It is unclear how Indira got an answer of "4" for "1 + 4," but she continued to start her counting at the ground level. On the other hand, Dinah continued to start at the first number but began to abandon using the elevator, perhaps making the movement in her head.*
Dinah now looks back at what Indira wrote for "3 + 2 =," "Equals two? That's, that's supposed to be…" Then she looks up to "1 + 4 = 4" and points to it exclaiming, "Oh, that's an ugly four!" Dinah takes the recording sheet and gives Indira the elevator sheet. "It is not four. It is five," declares Dinah as she erases the "4" and writes "5" instead. Indira tries to defend her answer of four, but Dinah quickly moves on to review the next problem "3 + 2 = ." Indira places the elevator at the ground floor and then moves it upward while counting, "One, two, three, four, five."	*Dinah, irritated by Indira's incorrect answer and "ugly" writing, took control of writing. However, they continued to go through the problems together to check their previous answers with a medium-level of engagement.*
Dinah responds, "Oh five" and erases the "2." Confirming Indira's movements, she demonstrates the addition with her fingers. First she says "See, like three" and holds three fingers. "Three and then two more." Dinah then puts up two more fingers. "Five." She writes "5" next to both "3 + 2=" and "2 + 3=." Indira points to the right side of the paper, and Dinah circles "yes." Indira then points to the area above, and Dinah erases the "no" and circles "yes" for that section as well.	*Although Indira continued to use the elevator, Dinah switched to demonstrating addition using her fingers, reflecting a bookkeeping conceptual model (*i.e.*, gains and losses). Even though Dinah was not using the context as intended, both students' methods led to agreement, and their discussion indicated a high level of engagement.*

Solving Subtraction Problems: The Potential for Negative Answers

Let's take a look at how the two girls begin to tackle the subtraction problems, having successfully completed the addition problems. The need for negative numbers has the potential to challenge their current integer mental models.

Exchange #4	*What can we learn from this exchange?*
The girls then move on to the subtraction problems. Abandoning her prior strategy of using the elevator, Indira points to "4 – 1 =," puts up four fingers, puts one of them down, and says, "Four minus one. Three." Then, she moves the elevator from the ground floor down to floor negative three and responds, "Three."	*Interestingly, although Indira used her fingers to solve "4 – 1" (suggestive of a bookkeeping conceptual model), she switched back to using the elevator for "1 – 4."At this point, though, it is unclear whether she switched back to using a translation conceptual model or continued to use the bookkeeping conceptual model. She started at 0, which is what she did on the addition problems, reflecting a translation conceptual model. However, we did not see her move up to one before going down four (although this would make sense as she got an answer of -3). Her switch back to using the elevator suggests she knew that the problem, although it looked similar to "4 – 1," was unfamiliar.*
She points to the recording sheet, and Dinah writes down "3" for the answers to "4 – 1 =" and "1 – 4 =." Dinah confirms, "Three and three." Indira checks what Dinah wrote and exclaims, "What?!" as she motions to the elevator located next to -3.	*Indira correctly moved the elevator to floor -3 but called it "three." However, she appeared to notice that the 3 that Dinah wrote was missing a negative sign, as she indicated the location of -3, which provides further evidence that she exhibits a transition I mental model for integers.*
Dinah erases the "3" saying, "One minus four is," but Indira interrupts by saying, "One minus four," and moving the elevator back up to "0." She moves it down four, hovering around -4.	*Yet, Indira continued to struggle with the use of the elevator model; when she tried to show Dinah how she got her answer, she started at the ground floor instead of the first floor. Perhaps this reflects her desire to start at the ground floor for addition, but instead of moving up to floor 1 first, she moved down four floors. This provides further evidence that she might have been switching from relying on a translation conceptual model to a bookkeeping conceptual model.*
However, Dinah continues to ignore Indira's use of the elevator, explaining, "One minus four – there's no way you can do that, so it's zero." At this point, the girls begin an inaudible argument, and Indira makes some reference to "the teacher." Ultimately, they agree on writing "3," and Indira circles "yes" to indicate that the answers to "4 – 1" and "1 – 4" are the same.	*Relying on her knowledge of whole numbers, Dinah discounted Indira's actions and asserted that they cannot take away four, leaving them with an answer of zero. Her reasoning indicates that she still has an initial mental model for integers. Ultimately, the girls answered both problems with 3, ignoring the negative sign. They continued to show medium to high levels of engagement throughout.*

Now we return to the girls as they move on to complete the rest of the worksheet.

Exchange #5	*What can we learn from this exchange?*
Moving on to "3 – 2=," Indira puts the elevator at -3, and then as Dinah reads, "Three minus two," Indira moves the elevator to -1 and says, "One." Dinah writes down "1."	*Indira solved "3 – 2 =" by starting at -3, perhaps because the elevator was in the negatives from the previous problem. Interestingly, to subtract, she moved the elevator up (which usually indicates addition), toward numbers with smaller magnitude.*
Dinah then continues, "Two minus three – oh, it switches!" She says this excitedly, waving her hands back and forth across each other to illustrate the switching. Indira places the elevator at -2 and goes down three floors to -5 but says, "Five."	*For "2 – 3 =," Indira continued to start in the negatives, at -2, but then moved the elevator down (consistent with subtracting a positive number). Unlike with the addition problems, she modeled the subtraction ones as STS problems. Further, her tendency to interchange the positive and negative numbers suggests she is still developing an understanding of the difference between them, and her uncertainty as to whether she should move up or down suggests she was trying to make sense of whether she should pay attention to magnitude or direction primarily.*
Dinah, however, is not listening to Indira and writes down "1" instead. Indira urges, "Put a negative" and points to the "1" after "3 – 2 = ." Dinah instead puts a negative before the "1" for "2 – 3 = ." Indira tries again, saying ,"Negative" and pointing to the "1" after "3 – 2 = ." Dinah adds a negative and then circles "yes" to indicate those two problems have the same answer.	*Indira continued to see the negative sign as important because she insisted that Dinah include it in the answer. However, as the two girls were not always focused on the same problem, Dinah ended up putting negative signs on both answers. Although Dinah finally noticed the difference in the order of the numbers in the problems — "It switches" — this did not influence how she solved or answered the problems. At this point, they correct each other using a lower level of engagement, not really interacting with the changes they urge each other to make.*
Indira points to the bottom problems and as she taps them says, "Five. Five, five, five." Dinah writes "5" for the answer to both "5 – 0=" and "0 – 5 = ." Indira argues, "No, they're not the same," and points to the "no" but then changes her mind and points to "yes", saying, "Yes." Dinah circles the "yes," and Indira raises her hand to have me check their work.	*Until the last set of problems, Indira used the elevator to solve the majority of the problems, especially when subtracting a larger number from a smaller. She abandoned this with the problems involving zero, possibly because in her experience zero involves no change, and ignored the change in number order in the number sentence.*

Checking the Work

Once I (as the teacher for the activity) join the girls, I quickly notice several incorrect answers. I ask them to show and tell me how they solved "4 – 1 =" and "1 – 4 =." Indira places the elevator at 0, but I suggest she "start at one." Dinah takes over and places the elevator at floor 1. I then guide, "Go minus four." Dinah once again moves the piece down four spaces and provides the answer, "Three." I probe "What kind of three?" and point to the negative in front of the three. "Negative three," states Dinah. I encourage them to check their other answers and walk away. Let's look at what happens when I am no longer helping them.

Exchange #6	*What can we learn from this exchange?*
As I walk away, Indira adds a negative to the "3" for "4 – 1 =" (instead of "1 – 4 ="). Dinah, again annoyed with Indira's handwriting, claims, "That's not how you do a negative," reaches across, and writes a "better" negative. Indira switches the papers so she has the elevator again (and Dinah has the recording sheet).	*Indira put the negative on the answer to the wrong problem, suggesting she was not paying careful attention to the differences in the two problems. Dinah continued to take control of the writing, and Indira switched back to controlling the elevator without complaint.*
Indira puts the elevator at 2 but says, "Start at five…" Dinah interrupts stating, "Minus zero!" and puts the elevator at zero.	*Both students at this point were confused as to which problem they were addressing and not communicating effectively.*
Pointing at the recording sheet where they circled "yes" for "4 – 1" and "1 – 4" having the same answer, Indira commands, "Erase this if it's no." Dinah starts to erase the "yes" for the next set of questions but completes Indira's request with some redirection from her.	*Indira took a more commanding role at this point, continuing to re-explain to Dinah until Dinah did what she requested.*
Moving on to check 3 – 2, Indira suggests, "You start at 3." She puts the elevator on floor 3 and moves it down two. "One," she says and points to "3 – 2 = -1." "Change it." "All the time I have to erase," complains Dinah. "Okay." "Because you're the eraser, because you're the writer," retorts Indira. Next, Indira puts the elevator at 2 and moves it down to 0. "Zero. Zero," she repeats as she points to "2 – 3 =," "right up here." Dinah rechecks by putting the elevator at 2, "Two minus three. One, two, three" and moves the elevator down to -1. Confused, Indira comments, "No, it starts from two, minus three." Dinah repeats the same process and ends at -1. After a bit of argument, Indira concedes, "Okay, the negative one." "Which one, which one is it?" asks Dinah, who has lost her place on the recording sheet. Indira points to "2 – 3 =" and repeats, "Negative one." Dinah notices that they had circled "yes" for those problems being the same and corrects it, "No, that's not the same."	*Indira modeled 3 – 2 easily, interpreting it again as an STS problem. This might have been influenced by the way I prompted them to solve the problems. However, when solving 2 – 3, she prematurely stopped at zero. At this point, Dinah took a more active role in double-checking her partner with the use of the elevator, also interpreting it as an STS problem. Perhaps because she used the same interpretation as Indira, she was able to convince Indira that the correct answer was -1. This was another instance of high-level engagement between the pair. They then worked together to determine where to change the answer on the worksheet. This exchange is striking as it represents a moment when the two work together in a similar way to solve the problem.*

Exchange #6	*What can we learn from this exchange?*
Indira asserts that the final pair of problems “is the same” and raises her hand to have me check their work again. However, Dinah tells her, “We need to correct it first.” She points to floor five, saying, “Five.” Indira moves the elevator to floor five. “Minus zero,” reads Dinah. “Zero – Five.” At this point I arrive and ask about the last problem “0 – 5.” “If you start at zero and you go minus five…” Indira puts the elevator at zero, and I prompt, “Zero minus five.” Indira moves the elevator down five spaces to -5. “Negative five,” answers Dinah. Indira adds a negative sign to their answer. I ask, “What do you notice about subtraction?” “Wait,” says Indira and changes their answer to “no” for whether “5 – 0” and “0 – 5” have the same answers. After I repeat the question, Dinah responds, “Um, they’re gonna be not the same.”	*Dinah, perhaps because of her personality, insisted that they check the final set of problems. Dinah adeptly used the STS interpretation and answered with negatives. She was also able to articulate that the answers were not the same with the subtraction problems. Indira was confident in her answers to “5 – 0 = 5” and “0 – 5 = 5” before Dinah made her check the answers but eventually came to agree with Dinah.*

Recap

Initially, when solving the addition problems, Dinah interpreted the problems as STS problems and started the elevator at the initial number while Indira modeled them as TTT problems and started the elevator at zero. Due to the differences in their methods, Dinah questioned what Indira was doing and often seemed suspicious of her answers. Her attention to what Indira was doing led her to correct one of Indira’s answers and then check her other ones. Dinah also abandoned the use of the elevator, opting to use fingers instead – which may have seemed more natural to her given her interpretation of the problems as STS and bookkeeping problems.

When they moved to solving the subtraction problems, Indira appeared to switch back and forth between interpreting problems from a translation to a bookkeeping point of view, sometimes starting the elevator at zero as before but not moving it the initial value. Dinah continued to use fingers as she had for addition, and this led her to answering 0 on problems where the elevator would have moved to a negative floor. This was consistent with her whole number mental model. Indira switched to interpreting the problems as STS problems and incorrectly started the elevator in the negatives on multiple occasions. Yet, her use of negatives started moving her toward a synthetic mental model; she was inconsistent in whether she moved the elevator up or down from the negative floors. Although Dinah noticed that the order of the numbers in the problems was switched, this did not affect her solutions.

After I helped them check their work and rethink about how to use the elevator, Dinah went back to modeling the problems as STS problems with the elevator and moving into the negatives, helping her begin to accept negatives as a different type of number. She used this process to correct Indira.

Focus Students' End of Unit Posttest Reasoning

Following this lesson, the girls participated in three more lessons that focused on helping them think about adding a positive number as moving in a more positive direction, adding a negative as moving in a more negative direction, subtracting a positive as moving in a less positive direction, and subtracting a negative as moving in a less negative direction. By the posttest, Dinah had progressed slightly and exhibited an initial, absolute value mental model for integer order and values. Unlike on the pretest, she could now fill in negative values on the number line (although she left out zero on one). However, she continued to order the integers as if they were all positive and chose the larger integer based on magnitude. Indira, on the other hand, exhibited a transition II mental model (as opposed to transition I on the pretest). She correctly filled in negatives on the number lines. When ordering the integers, she always put negatives to the left of (or below) positives, although she was not consistent in whether she ordered them by reversed magnitude versus magnitude (-2, -4, -8, 1, 5, 7 — similar to her response on the pretest — versus -9, -7, -5, 0, 3, 8). However, she correctly and consistently determined which of two integers was greater.

When asked if the problems 3 – 1 vs. 1 – 3, 5 – 8 vs. 8 – 5, and 9 – 6 vs. 6 – 9 would have the same answer, both girls thought they would because the problems had the same numbers, just as they had on the pretest. Likewise, when asked to solve 3 – 9, 6 – 8, and 1 – 4, they both reversed the numbers and got positive answers. Dinah gave positive answers for the rest of the integer problems (e.g., -7 + -1 = 8), whereas Indira gave several negative answers but often switched between counting up or counting down for similar problems. For example, Indira solved -7 + -1 = -8 but then -6 + -4 = -2. See Table 7.3 for a summary of the posttest results.

Overall, both Dinah and Indira made some conceptual progress. Dinah went from not using any negatives to labeling them as points to the left of zero on the number line. Meanwhile, Indira transitioned from interpreting negatives as worth zero (and not part of calculations) to using them in her arithmetic solutions.

Discussion and Implications

As exemplified through the case of Indira and Dinah, conceptual change is difficult because multiple concepts are often at play within any one activity. Further, these concepts are frequently intertwined. More specifically, in this activity, students worked with the elevator model, which supported both bookkeeping and translation

Table 7.3 Summary of Dinah and Indira's posttest performance

Pretest section	Dinah	Indira
Empty number line	-4, -3, -2, -1, 1, 2, 3, 4, 5	-3, -2, -1, 0, 1, 2, 3, 4, 5
Ordering numbers	Absolute value	Negatives reversed before zero
	1, -2, -4, 5, 7, -8	-2, -4, -8, 1, 5, 7
	Absolute value	Negatives ordered correctly
	0, 3, -5, -6, 8, -9	-9, -7, -5, 0, 3, 8
Comparing numbers	Based on absolute value	Correct
Are these equal: 1 – 3 vs. 3 – 1?	Yes	Yes
Solving 1 – 4, 3 – 9, 6 – 8	Reversed numbers	Reversed numbers
Arithmetic strategies	Ignore negatives	Some negative answers; confused about direction to count
Order and value mental model	*Initial: absolute value*	*Transition II*

conceptual models, providing the girls with the opportunity to consider the numbers involved as discrete or continuous quantities. The model also allowed them to interpret the problems as STS or TTT problem types. On top of these interpretations, their strategies were also influenced by how willing they were to accept negative numbers based on their integer mental models. These elements both supported and constrained their thinking, in particular as they shifted among them.

Indira began the elevator activity already distinguishing between positive and negative numbers. She had a strong focus on the movement of the elevator for the addition problems, always starting at zero. In this way, she interpreted the problems as a series of translations, reflecting continuous quantities and a translation conceptual model. With the shift to subtraction, she also changed her use of the problems. She switched to starting at the initial number (as she interpreted it) and then moving and modeling them as STS problems and reflecting discrete quantities and a bookkeeping conceptual model. Although she started on the negative-numbered floors for the subtraction problems instead of their positive counterparts, her willingness to use the negative-numbered floors illuminates her acceptance of negatives. Yet, she did not notice that she was starting on the negative floors as opposed to the positive ones. On the other hand, she did insist that Dinah add a negative to one of the answers, indicating that negative signs hold relevance to her. Her switch in methods from those reflective of a translation conceptual model on the addition problems to those reflective of a bookkeeping conceptual model on the subtraction problems might reflect a decreasing level of comfort with the subtraction problems. If interpreting problems in terms of bookkeeping is generally easier, it may explain why Wessman-Enzinger and Mooney (2014) found that more students used bookkeeping contexts than translation contexts.

Dinah began the elevator activity at a point when she did not distinguish between positive and negative numbers. She also relied on interpreting the problems as involving discrete quantities. Instead of using the elevator, she often began by using recall or her fingers to show the answers to the addition problems. When she did use the elevator, she modeled the problems as STS problems. Her focus on discrete quantities, together with ignoring the negative signs, was reflected in her reluctance to consider the possibility that the answer to 1 – 4 could be less than (or below) zero. Unintentionally drawing on a bookkeeping conceptual model, I walked the girls though solving the subtraction problems as STS problems. This aligned with Dinah's way of interpreting the problems, and once she accepted that I wanted her to act out the problem, she quickly took it up and used negative answers.

Challenges with Imposed Instructional Contexts

Dinah's interactions, in particular, illustrate that ready-made contexts based on the conceptual models (the elevator context) may not be accepted by students, especially without repeated encouragement or multiple opportunities to grapple with these contexts over time. Even though the point of the activity was to use the elevator to solve the problems, Dinah continually referred to her prior knowledge or use of fingers to get the answers, even when Indira got a conflicting answer using the elevator. Although she used the elevator at the end after I intervened, the process did not lead to lasting change; she continued to provide only positive answers days later on the posttest. This provides evidence that instruction around a particular idea might be more productive if it builds on conceptual models and contexts students are already using. In Dinah's case, this could involve building off her use of fingers to align with counting and using a bookkeeping conceptual model. In Indira's case, this could involve helping her interpret the subtraction problems as TTT problems. This could have supported her thinking about the numbers as continuous quantities and drawn more attention to the initial numbers, helping her see that they were positive. Further, teachers could be more explicit about the differences between the models and methods they are asking students to use and the models and methods students typically use.

Indira's reluctance to revisit "5 – 0 = 5" and "0 – 5 = 5" provides insight into why students so readily answer with positive answers or do not use the model they are being asked to use. They feel so confident in their answers! Her prior work with zero up to this point would have led her to believe that the presence of zero does not impact the other number; they do not see a conflict with their current understanding (Vosniadou & Skopeliti, 2014). Therefore, the problem seemed easy – no elevator needed. Likewise, she used her fingers for "4 – 1" (using a familiar method for a familiar problem) but then used the elevator for "1 – 4," a problem that likely did not look familiar to her. Dinah had a hard time accepting the negative numbers (arguing that you can't subtract a larger number from a smaller one) even when Indira showed her how to use the elevator to get a negative answer. Although not feasible in this research lesson, highlighting such tensions would be important to do in classroom discussions to draw students' attention to solving the problems in more than one way in order to check their answers.

Incremental Conceptual Change

The pairing of these two girls resulted in interesting interactions, especially compared to other groupings in the class. Both displayed rather strong personalities – perhaps they felt more comfortable with each other because they chose to work together – and were willing to challenge each other on different aspects of finding the answers; this led to a high level of engagement (Webb et al., 2014), which helped them resolve conflicts (Yackel et al., 1991). Dinah and Indira exhibited behaviors tending toward initial mental models at the beginning, although Indira had some knowledge of negatives initially, putting her at a higher mental model level. Further, the two used different methods to solve the worksheet problems. Indira was more willing to use the elevator (and where it ended up) to support her assertions, while Dinah appealed to reasoning, often based on her use of fingers and knowledge of whole number problems. Indira's assertions to use the negative signs may have helped Dinah become more aware of them, while Dinah's objections may have helped Indira solidify her arguments (although they may also have led her to using more recall later rather than thinking about how to solve problems using the elevator). Perhaps due to their high level of engagement with each other's solutions, they were able to push each other to participate more (Yackel et al., 1991), which often corresponds to a higher level of achievement (Webb et al., 2014). On the other hand, if Dinah had been paired with someone who used their fingers to count into the negatives, it is possible she would have gained even more because she initially relied on justifying her answers with her fingers but insisted that they could not take larger numbers away from smaller ones. Therefore, it is important to continue thinking about ways to create student pairings that could best support conceptual change for all. It might be that pairing students at different developmental levels can still be effective if the higher-level student helps the weaker student build on his or her methods in productive ways.

The elevator model supported both bookkeeping and translation conceptual models. Further, it helped Indira make sense of unfamiliar problems and begin to use negative numbers. Dinah even used it effectively by the end of the lesson and noticed that the answers were not the same if the numbers switched places in subtraction problems. Although they both seemed successful at the end, neither of them provided negative answers for similar subtraction problems (e.g., 1 – 4) on the posttest. The incremental progress in conceptual change they seemed to make did not translate to long-term change, or it remained bound to that particular context. Yet, they both did exhibit some conceptual change from pretest to posttest and had mini breakthroughs during the activity that reflected changes in conceptual model use. The anecdote presented here reinforces the difficulty of shifting away from a whole number mental model of integers (initially exhibited by Dinah who eventually began to see the importance of the negative sign). It also shows the importance of the transition mental models. Students in transition from one mental model to the next may exhibit behaviors of either the less or more advanced mental model. Indira benefitted from using negatives in ways that contradicted her prior notion of negatives as indicating a number being taken away. However, both students would likely have benefitted from making more connections back to the use of the elevator in later lessons to help them make sense of problems such as “1 – 4.” Promoting conceptual change takes time, and repeated opportunities to grapple with the ideas in ways that both build on and challenge their current way of thinking can help support this process.

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Part III
Guiding the Journey: (Prospective) Teachers' Thinking About Integer Addition and Subtraction

Chapter 8
Nuances of Prospective Teachers' Interpretations of Integer Word Problems

Laura Bofferding and Nicole M. Wessman-Enzinger

Abstract This chapter identifies the ways in which 15 prospective teachers engage the strands of mathematical proficiency as they solve word problems involving integer addition and subtraction. The prospective teachers, through think-aloud interviews, demonstrated a strong focus on solving problems using procedures, which some did not explain and others explained in detail. Number line representations were popular ways to illustrate solution methods, especially to highlight distances to and from zero. Further, some problems elicited a variety of strategies, while others mainly elicited procedures. The collective think-aloud data reveal strong, interconnected strands that could help individuals reflect on procedural versus conceptual knowledge and how best to explain and make connections among the ideas involved in the problems.

> Negative eight minus negative five…it's like a double negative, kind of, um, I'll have to make it a plus positive five, because you can't subtract negative five. I don't really know how to explain that, but I was taught that somewhere. (Interview with Jackie, October 16, 2012)

As illustrated in the quote from Jackie in the opening vignette, prospective teachers (PTs), both elementary and secondary, face a daunting task as they prepare to teach mathematics. Not only must they make sense of students' thinking and foster their learning, but they also have to confront their own mathematical learning and consider how to effectively explain and troubleshoot concepts that they may not have had the opportunity to learn sufficiently. Yet, even though PTs may benefit from some relearning of mathematics concepts, they have a rich knowledge and experiential base that they draw on, such as the procedural knowledge that subtracting a negative number can be solved by instead adding a positive number. Further, each PT can bring important insight to mathematics pedagogy courses, one place

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where building on PTs' collective knowledge should be capitalized on. In this chapter, we use the context of integer addition and subtraction to explore PTs' individual and collective mathematical proficiency.

Framework for Mathematical Proficiency

The strands of mathematical proficiency provide a helpful framework for analyzing PTs' individual and collective knowledge because they capture the type of knowledge and characteristics we want all students and teachers to exhibit (National Research Council [NRC], 2001). The five strands—procedural fluency, conceptual understanding, strategic competence, adaptive reasoning, and productive disposition—support each other, and all are needed for successful mathematical development.

Procedural Fluency

"*Procedural fluency* refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (NRC, 2001, p. 121). When working with integer addition and subtraction, rules such as "adding a negative is equivalent to subtracting a positive" or "subtracting a negative is the same as adding a positive" help people solve problems efficiently, but these rules are not always used with mathematical integrity (Hertel & Wessman-Enzinger, 2017). Yet, PTs are fairly adept at using procedural rules to solve integer addition and subtraction number problems (Bofferding & Richardson, 2013; Steiner, 2009). Based on dissertation data, Steiner (2009) found that 79 PTs were able to solve symbolic integer addition problems correctly 94–100% of the time. For subtraction problems with negative numbers, their accuracy fell to 76–92% correct. Kajander and Holm (2013) explored PTs' responses to 5 – (-3) and found that just over 50% of the 128 participants articulated a rule that two negatives make a positive in order to solve the problem. Others did not explain how they got their answer or gave an incorrect explanation.

Through utilizing think-alouds (e.g., Ericsson & Simon, 1993), Bofferding and Richardson (2013) found that PTs manipulated equations in multiple ways in order to make the problems easier to solve, depending on their preferred strategy. Some PTs tended to change the equations using procedural rules so that they contained either all positive or all negative numbers. Others changed the problems so that they involved adding or subtracting a positive number only. Finally, the one PT who relied on a canceling strategy preferred to change problems so that she was adding positives or adding a positive and negative so that she could cancel easily.

Being able to translate word problems into appropriate equations is also important. In a study of 137 PTs in their second year of study in Spain, Almeida and

Bruno (2014) found that PTs' accuracy in solving word problems differed depending on the type of word problem and whether the second or third quantity was missing. For example, PTs had 92% accuracy on solving change problems where the answer was missing, 88% accuracy on change problems where the change was missing, and 68% accuracy on compare problems where the answer was missing.

Conceptual Understanding

"*Conceptual understanding* refers to an integrated and functional grasp of mathematical ideas" (NRC, 2001, p. 118) and involves "the comprehension and connection of concepts, operations, and relations" (National Council of Teachers of Mathematics [NCTM], 2014, p. 7). In fact, NCTM (2014) describes conceptual understanding as the foundation on which procedural fluency develops and encourages teachers to "build procedural fluency from conceptual understanding" (p. 10). Recommendations include utilizing discourse and purposeful tasks to elicit student thinking as a way to use conceptual understanding to leverage the development of procedures (NCTM, 2014). Although recommendations state procedural understanding should develop from conceptual understanding (NCTM, 2014), Sfard (2001) posited that procedural understanding and conceptual understanding often work synergistically together. Further, Star (2005) pointed to the difficult nuances of distinguishing between procedural and conceptual understanding.

In terms of what conceptual understanding could mean for integers specifically, Kilhamn (2009) delineated mathematical ideas that are important for understanding integer operations conceptually (which she refers to as number sense). The first component of conceptual understanding for integer operations she described is intuitions associated with numbers. For example, negative integers can be interpreted as *taking away*, which is developed in the whole number domain. This type of reasoning may work with negatives but can break down for number sentences like 0 – -5 (Bofferding & Wessman-Enzinger, 2017).

The second component for understanding integers conceptually is the ability to make magnitude comparisons. For instance, she describes a child who, when comparing -2 and -4, may struggle to determine which is "larger." Kilhamn (2009) differentiates between magnitude (absolute value) and relative size (order). This is consistent with the magnitude-based and order-based reasoning described in this book (see, e.g., Chap. 2 in this book) and also the results illustrated in Bofferding (2010, 2014). Even PTs are drawn to the magnitude of negative numbers and decimals. In a study of 94 PTs' understanding of negative decimal placement on a number line, Widjaja, Stacey, and Steinle (2011) highlighted 1 PT who reversed the order of the negative decimals, placing -1.2 closer to 0 than -0.5. Additionally, they presented cases of some PTs who ordered decimals in the form of -0.X correctly, while others placed them immediately to the right of zero or placed them as if they were positive. Similar strategies emerged when students placed negative integers on a number line (Bofferding, 2014). These responses may reflect PTs' mental number lines (Bofferding, 2014; Kilhamn, 2009).

The third component Kilhamn (2009) described for conceptual understanding of integers is the ability to benchmark or recognize patterns between the numbers. Kilhamn pointed to the example of how integers often neutralize each other, with zero representing a point of symmetry. A few PTs in Bofferding and Richardson's (2013) study explained that they solved -4 + 6 by first adding 4, the opposite of -4, and then adding two more. Such strategies demonstrate an understanding of additive inverses as well as decomposition of numbers.

Finally, the last conception that Kilhamn highlighted is possessing knowledge of the effects of operations on numbers. Bofferding and Richardson (2013) found that most PTs relied on procedural explanations for operations; however, one PT solved -8 – -8 conceptually by explaining, "When you subtract a number from itself you get zero" (p. 116). Such generalizations demonstrate a solid understanding of quantity within takeaway situations. However, Kilhamn also pointed out that interpreting subtraction as takeaway can be limiting and advocates for more work around the distance metaphor (see, e.g., Chap. 6 in this book).

Strategic Competence

"*Strategic competence* refers to the ability to formulate mathematical problems, represent them, and solve them… They should know a variety of solution strategies as well as which strategies might be useful for solving a specific problem" (NRC, 2001, p. 124). For example, it can be efficient to draw on counterbalance conceptual models (Wessman-Enzinger, 2015; Wessman-Enzinger & Mooney, 2014) when adding positive and negative numbers, although counterbalance situations are more difficult when they involve subtraction problems where the subtrahend has a larger absolute value than the minuend (e.g., -2 – 5).

PTs often used horizontal or vertical number lines to help illustrate relations in the problems (Almeida & Bruno, 2014; Bofferding & Richardson, 2013). When determining the distance between a negative and positive point, some PTs calculated to and from zero (Almeida & Bruno, 2014; Bofferding & Richardson, 2013; Peled, Mukhopadhyay, & Renick, 1989), utilizing a divided number line model (Peled et al., 1989), while others used a continuous calculation, often counting from one point to the next to determine the distance (Almeida & Bruno, 2014; Bofferding & Richardson, 2013; Peled et al., 1989). Bofferding and Richardson (2013) also described one PT who used a hills and holes context (i.e., where one hill cancels out one hole) to reason correctly about the operations.

Solving contextual problems can be more difficult because the context first needs to be translated into a numerical problem. When solving integer word problems involving temperature, a group of PTs in Spain used six main strategies: using an operation with positive numbers (and discussing how it fit the negative situation), using an operation with negative numbers, using a number line, counting, just giving a verbal explanation without demonstrating a more specific strategy, and making a drawing (Almeida & Bruno, 2014). Although PTs often used one method, they sometimes combined multiple methods. In one case, a PT made a drawing of a sun next to the morning temperature and a drawing of a moon next to the change in temperature, used an operation with negative numbers, and drew a number line to show the result in the drop in temperature. Further, they solved the problems from a positive perspective (e.g., calculating 4 – -5 = 9 and indicating the temperature dropped 9 degrees instead of -5 – 4 = -9) almost as often as they solved the problems from a negative perspective (Almeida & Bruno, 2014). These results are similar to those found with secondary students, who may take the view of the lender or borrower in money contexts (Whitacre et al., 2015).

Almeida and Bruno (2014) also illustrated that some of the PTs incorrectly formulated the problems, either because they misinterpreted the problems or because they were "number grabbing" (National Research Council, 2001, p. 124) or just choosing an operation with the numbers given in the problem. Instead of starting at 2 m (location of a bird) and finding the altitude 6 m below that (altitude of a fish), they assumed the fish was *at* -6 m and found the distance between the two (Almeida & Bruno, 2014).

Adaptive Reasoning

"*Adaptive reasoning* refers to the capacity to think logically about the relationships among concepts and situations" (NRC, 2001, p. 129). PTs justify their algorithms in different ways (Bofferding & Richardson, 2013), and the use of contexts can help PTs reason about the meaning of integer operations. For example, Bofferding and Richardson (2013) described one PT who counted further into the negatives for problems such as -3 – 5 and -5 – 9 because she thought of a hole (negative number) that was getting deeper; therefore, she mapped taking away a positive with digging a hole. This same PT was able to explain why subtracting a negative is equivalent to adding by reasoning that taking away a hole is similar to adding a hill.

The use of analogies is a common way both PTs and children will justify solutions to integer problems (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016; Bofferding & Richardson, 2013; Bofferding & Wessman-Enzinger, 2017). For example, several PTs solved -7 + -1 = -8 by adding 7 + 1 = 8 and then making the answer negative, indicating that the two problems are solved in similar ways (Bofferding & Richardson, 2013). This reasoning works well for those who understand that -7 + -1 is equivalent to -(7 + 1); however, children who do not understand this relation will also use the same reasoning but misapply it to other situations. For example, they may argue that -3 + 1 = -4 because 3 + 1 = 4, and the 3 was negative (Bofferding, 2010; Fagnant, Vlassis & Crahay, 2005; Schwarz, Kohn, & Resnick, 1993–1994). Therefore, PTs need to be careful about how they use these justifications with students.

Productive Disposition

"*Productive disposition* refers to the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (NRC, 2001, p. 131). Prospective teachers and teachers need a productive disposition of inquiry in order to provide better opportunities and authentic mathematical experiences for their students (NRC/NSF, 1996; NRC, 1999). Part of being a doer of mathematics includes a disposition of inquiry in mathematics and authentic mathematical experiences and includes a willingness to try new problems or unsolved problems. Furthermore, dispositions of learners and doers of mathematics include curiosity and wonder (The Math Forum, 2016). In general, many PTs are eager to develop their mathematical understanding (Holm & Kajander, 2011). However, in one study, PTs with high mathematics anxiety were less likely to agree that they could teach mathematics effectively (Bursal & Paznokas, 2006). Therefore, even if they have a productive disposition toward their own mathematics learning, they may still have anxiety about their mathematics teaching.

Capturing PTs' Integer Reasoning

The 15 prospective teachers who participated in this research came from a large public university in the Midwest. Eight of the PTs were training to be secondary teachers, and seven were training to become elementary teachers. Two of the PTs were male, but to maintain anonymity, all are referred to as "she" in this chapter. Although the PTs were at various points of completion in their programs, they had all taken some mathematics content coursework in the mathematics department.

Each of these PTs participated in an individual, structured think-aloud interview as they solved integer problems. The focus of this chapter is on their solutions to eight word problems (see Table 8.1). The word problems were chosen to represent equalize, join, and separate problem types (Fuson, 1992). PTs saw each problem on a separate piece of paper and were instructed to say aloud everything they were thinking in their heads, which helped make their thinking available to us (Ericsson, 2006; Ericsson & Simon, 1993). Aside from transcribing the interviews, we also collected any written work they generated.

Analyzing PTs' Integer Reasoning

To aid in our coding, we created a spreadsheet and listed PTs' responses to each word problem on a different sheet. For each word problem, we created columns for the five strands of competency and added three additional columns: one for

Table 8.1 Eight integer word problems that PTs solved while talking aloud

Word problem	Equivalent numerical problem and notes
1) Kyle has -2 points. Jill has 9 points. Who is winning? How many more points does Kyle need to get to catch up to Jill?	9 – -2 (equalize; also equivalent to a join, change unknown problem: -2 + __ = 9)
2) Andy has 6 points. Joan has -7 points. How many more points does Andy have than Joan?	6 – -7 (equalize; also equivalent to a join, change unknown problem)
3) Sam started with 6 points. Then he lost 8 points. What is his score?	6 – 8 (separate)
4) Devin had -3 points and then lost 5 more. What is her score?	-3 – 5 (separate)
5) Ina started with -1 point and gets 8 points. What is her score?	-1 + 8 (join)
6) Eric has -6 points. Aki has -2 points. Who is winning? How many more points does Eric need to get to catch up to Aki?	-2 – -6 (equalize; also equivalent to a join, change unknown problem: -6 + __ = -2)
7) Brianna started with a -4-point card.[a] Her opponent took -3 points from her. What is her score?	-4 – -3 (separate)
8) Paola started with 7-point card. Then she drew a -3-point card. What is her score?	7 + -3 (join)

[a] For some PTs, we had to clarify that she had a card hand worth -4 points

language use, one for illustrations they made on their paper (written) or talked about using (either explicitly drawn or implicitly used), and one for the equations they wrote on their paper (written) or talked about verbally (explicit or implicit).

Utilizing Tesch's (1990) steps for developing codes and the coding process, we coded the think-aloud data across participants, going through the PTs' responses one word problem at a time (Tesch referred to this as clustering topics), identifying the use of various strands of mathematical proficiency (i.e., procedural knowledge, conceptual knowledge, strategic competence, adaptive reasoning, productive dispositions). Both authors coded each separately. We then compared our codes and discussed any differences to decide on the final codes after discussion. When coding and negotiating the final codes we decided under which strand, if any, each sentence within a PT's think-aloud would fall. In some cases one statement fell under multiple strands, which is not surprising given the interdependent nature of the strands.

Strategies

As part of a second round of coding, we identified a set of strategies that PTs used when solving the problems.

Translation Strategies coded under the larger heading of translation involved finding the distance from one number to the other number, such as on a number line. There were several subcategories to this code.

Counting This strategy refers to instances when a PT started at one number and counted to the other number, such as when determining how many more points one person had than another (e.g., Schwarz et al., 1993–1994). It also refers to instances when they started at one number and counted a change, such as when determining a person's new score.

Referent: 0 This strategy was often seen when PTs were determining how many more points one person had than another. Instead of counting each number individually, some PTs identified the distance between each score and 0 and added those distances together (e.g., the nullification strategy in Schwarz et al., 1993–1994).

Subitize Distance Rather than count or chunk the distance based on a reference point, some PTs knew the distance based on the numbers. This strategy is similar to recalling the answer except that PTs indicated that they were interpreting the distance between two points.

Counterbalance The counterbalance strategy focused on comparing the magnitudes of the quantities involved, usually when positive and negative quantities canceled each other out (e.g., Wessman-Enzinger, 2015).

Analogy Analogy strategies were ones where PTs considered the magnitude of the quantities and either compared the problem to a whole number one they knew or solved the problem as a whole number problem. Then, often referring to the context, they made their answers negative (e.g., the symmetrization strategy in Schwarz et al., 1993–1994).

Procedure Solving a problem using a procedure involved the PTs manipulating the equivalent equations for a word problem (e.g., Gallardo, 1994; Guerrero & Martinez, 1982; Ryan, Williams, & Doig, 1998). This could involve an algebraic equation manipulation, where PTs set up an equation with the missing value listed as an unknown (i.e., "x"). Then, they solved for the unknown by adding or subtracting values from each side of the equation. It could also involve setting up the initial problem and changing the operation and sign (e.g., changing subtracting a negative into adding a positive, see the computational strategy in Bishop, Lamb, Philipp, Whitacre, and Schappelle's chapter of this book).

Recall Instances where the PT stated the answer to the problem without any other explanation were coded as recall. If the PT indicated the answer first and then indicated that you *could* figure it out another way, we gave them credit for both recall and their other stated strategies.

PTs' Collective Integer Reasoning

Utilizing the First Four Strands of Proficiency

Across the range of problems, PTs' use of procedural fluency, conceptual understanding, strategic competence, and adaptive reasoning varied. Next, we illuminate some common ways they used these strands of proficiency on three representative problems: one where PTs relied primarily on procedural fluency, one where they used a mix of procedural fluency and conceptual understanding, and one where they use zero as a referent.

Finding the New Score: 7 + -3 The problem for 7 + -3 (#8 in Tables 8.1 and 8.2) is one for which PTs primarily relied on procedural fluency with varying levels of detail. Two of the PTs immediately indicated that the answer would be 4, while seven others clarified it would be 4 because the problem is 7 – 3. Those seven PTs more explicitly translated the situation into an equation, a hallmark of strategic reasoning. A strict reading of the problem, however, aligns with the problem 7 + -3, if one is maintaining consistency of number sentence to problem type (Wessman-Enzinger, in press). Six PTs, nearly half of the participants, articulated this and further indicated that 7 + -3 is 7 – 3, which is why the answer would be 4. Adrianna's response was typical of these PTs: "So we're taking seven and then adding a negative three which is seven minus three, which is four."

Although representing the problem in either way displays strategic competence, such an explanation does not make explicit the reason *why* the two expressions are equivalent, a key part of a justification that would indicate deep analytical reasoning. Jackie mentioned that the problem "kinda looks like seven minus negative three," misspeaking by saying "minus" instead of "plus" and focusing on the visual aspects of the signs. Two others provided a more explicit connection by clarifying that they could change the problems because adding a negative is "the same thing as" subtracting a positive. One of the PTs, Anna, even circled the plus and negative signs in her written equation to draw attention to them. As mentioned before, most of the PTs' explanations for this problem tended toward procedural rules; however, Kelsey talked about more conceptual ideas related to the problem. She determined that there were more positives than negatives by focusing on the two magnitudes (seven positives versus three negatives) and used this knowledge to argue for why the answer would be positive.

Losing Points: -3 – 5 Compared to the previous problem (7 + -3), PTs' solutions for -3 – 5 (#4 in Tables 8.1 and 8.2) involved a greater mix of procedural fluency and conceptual understanding while also drawing on analogies and number line representations. Six PTs made procedural statements only, providing either just the answer (-8) or talking about the equation or equations they would use. For example, Ramona described the problem as -3 – 5, "He started with negative three then lost five more, so you're going to be subtracting five, which is negative eight." The remaining PTs used additional strategies alongside the procedures. Adrianna also

Table 8.2 Strategies participants used for each word problem

	Translation						
Word problem	Count	Referent: 0	Subitize distance	Counterbalance	Analogy	Procedure	Recall
1) Kyle has -2 points. Jill has 9 points. Who is winning? How many more points does Kyle need to get to catch up to Jill? (9 – -2 or 2 + __ = 9)	Adrianna	Addison			Anna*	Ashley	Kirsten
	Jackie	Alexa				Audrey	Ramona
		Anna*				Rochelle	Joanna
		Karey					
		Kelsey					
		Octavia					
		Ophelia					
2) Andy has 6 points. Joan has -7 points. How many more points does Andy have than Joan? (6 – -7)	Alexa*	Addison				Adrianna	Ramona
	Jackie	Alexa*				Ashley	
		Anna				Rochelle	
		Audrey					
		Karey					
		Kirsten					
		Kelsey					
		Octavia					
		Ophelia					
		Joanna					
3) Sam started with 6 points. Then he lost 8 points. What is his score? (6 – 8)		Anna	Octavia*	Addison	Alexa	Adrianna	Kirsten
		Ashley		Kelsey*	Audrey	Kelsey*	Ramona
		Ophelia		Rochelle	Karey		
		Joanna			Jackie		
					Octavia*		

4) Devin had -3 points and then lost 5 more. What is her score? (-3 – 5)	Karey			Anna*	Adrianna	Alexa	Addison
					Anna*	Kelsey	
					Audrey	Rochelle*	Ashley
					Rochelle*	Octavia	Kirsten
					Jackie*	Jackie*	Ramona
							Ophelia
							Joanna
5) Ina started with -1 points and gets 8 points. What is her score? (-1 + 8)		Karey		Anna	Joanna	Adrianna	Addison
		Jackie		Kelsey		Alexa	Kirsten
						Ashley	Ramona
						Audrey	
						Rochelle	
						Octavia	
						Ophelia	
6) Eric has -6 points. Aki has -2 points. Who is winning? How many more points does Eric need to get to catch up to Aki? (-2 – -6; -6 + __ = -2)	Adrianna		Addison		Kelsey*	Anna	Alexa
	Karey		Kelsey*		Ophelia	Ashley*	Ashley*
	Jackie*		Jackie*			Audrey	Kirsten
			Octavia			Rochelle	Ramona
							Joanna
7) Brianna started with a -4-point card.* Her opponent took -3 points from her. What is her score? (-4 – -3)				Addison*	Alexa*	Addison*	Kirsten
				Anna	Audrey*	Adrianna	Ramona
				Rochelle*	Karey	Alexa*	Ophelia
					Kelsey*	Ashley	Joanna
					Rochelle*	Audrey*	
					Jackie	Kelsey*	
						Rochelle*	
						Octavia	

(continued)

Table 8.2 (continued)

Word problem	Translation			Counterbalance	Analogy	Procedure	Recall
	Count	Referent: 0	Subitize distance				
8) Paola started with 7-point card. Then she drew a -3-point card. What is her score? (7 + -3)				Kelsey		Adrianna	Addison
						Alexa	Kirsten
						Anna	
						Ashley	
						Audrey	
						Karey	
						Ramona	
						Rochelle	
						Octavia	
						Ophelia	
						Jackie	
						Joanna	

*Designates PTs who used more than one strategy identified the problem as -3 – 5 but then used procedural rules to create an equivalent problem (-3 + -5), saying, "We add the two numbers together—three and five—and then add a negative so that's negative eight." She used an analogy to help make sense of the problem, thinking strategically about the relations between problems. However, she did not explain why she could use the procedure she did. On the other hand, Kelsey relied on a more procedural connection between -3 – 5 and -3 + -5, explaining that this happens "when you do the slash and dash" (referring to changing the minus to a plus and adding a negative sign to 5). Yet, drawing on analytical reasoning, she also provided a justification based on conceptual knowledge, stating, "You can kind of think of it [5] as a negative already since he's losing it…it's just going to take him further down into the negatives."

A couple of the PTs did not provide explicit equations in their explanations but explained their solutions using more conceptual descriptions. Karey drew a number line and counted to show the result of losing five more points. She connected the idea of "losing" with moving to the left on the number line, a translation, and counted, "Negative four, negative five, negative six, negative seven, negative eight." Karey demonstrated strategic competence in terms of the representation, but her counting method was less efficient than reasoning about the relations among the numbers. Anna indicated that she would "minus 5" but drew on counterbalance as a strategy by using a hills and holes context: "So then we have a hole of negative three and we are taking five more away, which is a total of eight, because five plus three is eight." In a traditional use of the counterbalance conceptual model, the PT would need to add five zero pairs of hills and holes so that she could remove five hills, leaving her with eight holes. Instead, she appears to have rethought the problem as -3 + -5 or getting five more holes. Unfortunately, her discussion around the answer was decontextualized and did not include reference to the "8" representing holes (i.e., -8), and she wrote the answer as positive. A stronger focus on analytical reasoning may have helped her attune to her incomplete reasoning.

How Many More Points: 6 – - 7 PTs' solutions to 6 – -7 (#2 in Table 8.1) were particularly interesting in terms of their mention of positive versus negative quantities as well as their use of zero as a reference point. 11 out of the 15 PTs explicitly drew on conceptual understanding and referred to positive quantities being greater or better than negative quantities when explaining which person was winning the game. Such responses involved "A positive number is larger than a negative number" (Addison); "The negative number is of lesser value than a positive number" (Joanna); "He's above zero, and Joan is less than zero" (Ashley); and "Positive points are good points" (Jackie). Drawing on her analytical reasoning to make sense of the context of Joan's negative score, Kelsey reflected, "I would assume that Joan has lost points." Audrey also indicated that "negative typically is bad unless you're playing golf." Not all PTs had an easy time with the context surrounding this problem. Two of the PTs (Karey and Kirsten) argued that it is not possible to have negative points, yet they continued to solve the problem anyway.

Adrianna clearly illustrated strategic competence as she explained the meaning of the problem, “When I see how many more points or how much more is something than another thing um I take the first thing minus the second thing.” Similarly, Addison clarified that she needed “to see the difference” between the two numbers. In order to find this difference, Jackie counted the tick marks from -7 to 6, while ten of the PTs added the distances to and from the endpoints (-7 and 6) to 0. In two of these cases, the PTs did not explicitly mention zero; for example, Kirsten explained, “I knew you had to go back 6, then 7 more, so 13.” Both of these PTs started at six and chunked backward; yet, they knew that the answer would be positive. These PTs thought about how many points ahead the winner was. In other cases, as with Kelsey, PTs justified that they were determining how many points the loser would need to get to the winner. Therefore, they started at -7 and chunked forward. In terms of analytical reasoning, Octavia’s justification of her procedure was less convincing. She said, “From zero, it’s six points, and seven points from zero.” In relation to the context, this explanation may not be as clear to students because the PT added two distances from zero as opposed to finding the distance between the two scores.

Aside from PTs who found the difference using a number line, the other four PTs focused less on visualizing the context and solved the problem procedurally by manipulating the equations. Ramona explained, “You’re gonna take the winner’s points, six, subtract them from the loser’s points, which is negative seven. You’re gonna get positive thirteen.” However, the equation -7 – 6 would get -13. She got confused about which number needed to be subtracted from which number. The other three all set up the problem as 6 – -7. Both Adrianna and Ashley indicated that this is the same as solving six plus seven, while Rochelle more explicitly explained, “When you’re subtracting a negative, you’re actually adding the number.” Although such explanations can be helpful, it may be difficult for students to understand *why* that is the case. This is where utilizing the number line can support PTs’ analytical reasoning. Karey justified this relation saying, “To get to zero, Joan has to like gain seven more points, and then to get to where Andy is… she has to gain six more, so it’s seven plus six, thirteen.” Such explanations could add meaning to the procedural equation manipulations.

Strategy Use Across Problems

Although an element of strategic competence is that one should make use of “a variety of solution strategies,” one also needs to identify “which strategies might be useful for solving a specific problem” (National Research Council, 2001, p. 124). Table 8.2 illustrates various strategies that PTs used for each problem type.

It is interesting to note that problems like #8 (7 + -3) were not as productive for eliciting a variety of strategies from PTs. In fact, other than recalling the answer or using procedural rules, only one PT used a more conceptually driven strategy on this problem. In stark contrast, problem #3 (6 – 8) elicited all of the strategies, and problem #2 (6 – -7) had the largest number of PTs reasoning about distances in relation to 0. Finally, problem #7 (-4 – -3) involved the greatest use of analogy as a strategy with six PTs using this strategy. Of the analogies, three of the PTs related -4 – -3 to 4 – 3 and reasoned similarly to Karey who said, "Cause if I think about having four things and someone takes away three, then I'm gonna have one left. If it's negatives it works the same way." The other three PTs had a variation of the reasoning expressed by Alexa, who changed the problem to -4 + 3 and then changed it again to 3 – 4. From here, she reasoned that 3 – 4 is like 4 – 3 but in the other direction; so, since 4 – 3 = 1, 3 – 4 = -1. Table 8.3 presents a summary of the strategies each participant used across all of the word problems.

Three of the PTs overwhelmingly relied on using procedures or recall to solve the problems. In particular, Ramona used recall for all of the problems except for problem #8 (7 + -3), on which she first used a procedural rule to change the problem to 7 – 3. Ashley also relied heavily on procedural rules and recall, only using reasoning about distances relative to zero for the problem 6 – 8. It is interesting that this problem prompted reasoning related to zero as this strategy was most prevalent on the first two problems, especially 6 – -7. Traditionally, problems similar to 6 – 8 are hard problems for upper elementary students to solve because they are used to thinking that they cannot subtract a larger number from a smaller one (Murray, 1985). It may be that tying the problem to number order helped Ashley overcome this inclination. Finally, Kirsten primarily recalled answers but used zero as a referent on 6 – -7. Yet, she explained this strategy after stating the answer, which suggests she may have been using similar processes on other problems but doing it so quickly that she did not talk about them. She acknowledged on number problems given before those discussed here that she was forgetting to talk aloud.

The rest of the PTs used reasoning that made conceptual elements of the problems more explicit on at least two problems, with Kelsey leading the way in terms of number of strategies used. She used all of the above strategies, except counting and recall. In fact, she also used consistent strategies for similar problem types, suggesting strong strategic competence. When solving the first two problems exploring a difference in points between two people, 9 – -2 and 6 – -7, she calculated the distance using a referent to 0; likewise, when solving a similar problem with two negative point values, problem #6 (-2 – -6), she subitized the distance. She solved problems #3, 5, and 8 using magnitude reasoning. Although problem #3 (6 – 8) did not initially involve addition of a positive and negative number like the other two problems, she interpreted the problem as 6 + -8. On -3 – 5, she changed the problem to adding a negative and recalled the answer, and for -4 – -3, she constructed an analogy with 4 – 3. She clearly attended to the different meanings of the problems, and her strategies reflected that.

Productive Dispositions

The PTs provided evidence of productive dispositions in multiple ways as they considered what children or students thought, connected their reasoning to the real world, persevered in problem-solving, changed their minds about solutions flexibly, and critiqued the wording of some of the problems. Table 8.4 illustrates

Table 8.3 Participants' use of strategies

	Translation						
Participants	Count	Referent: 0	Subitize distance	Counterbalance	Analogy	Procedure	Recall
Addison		X	X	X		X	X
Adrianna	X				X	X	
Alexa	X	X			X	X	X
Anna		X		X	X	X	
Ashley		X				X	X
Audrey		X			X	X	
Jackie	X	X	X		X	X	
Joanna		X			X	X	X
Karey	X	X			X	X	
Kirsten		X					X
Kelsey		X	X	X	X	X	
Octavia		X	X		X	X	
Ophelia		X			X	X	X
Ramona						X	X
Rochelle				X	X	X	

Table 8.4 Evidence that PTs considered what children or students might think

Number sentence	Prospective teacher	PTs' statements
6 – -7	Anna	So students might originally think that, oh, seven is greater than six, so clearly Joan has to be winning, but the negative is important no matter how negative it is. A negative number is still less than any positive number.
-1 + 8	Karey	A kid would probably go like one, two, three, four, five, six, seven.
-2 – -6	Anna	A more negative number is less than a less negative number. So in this case, which would really confuse students, a negative two points is better than the negative six points.... So that means that Eric needs four points to catch up to Aki, which can be done by counting as well on fingers, which I'm sure students would do.
-3 – 5	Jackie	So this one um, cause of the wording is like you could easily get confused with that, I mean I shouldn't but, kids could get confused with that, um, you could just change this two, like take out the word lost, and then um do like negative three then negative five more points, and so that would be like three plus five equals eight, but they're both negatives, so it would be three.

some of the ways across the different problem types that the PTs considered the potential responses and thoughts of students who might encounter similar problems.

These PTs demonstrated awareness that although mathematics should make sense, there are ways in which it may not make sense to children. They considered that children might find comparing the magnitude of 6 and -7 challenging and anticipated that children might get confused with the wording of a problem. They also suggested that children might incorporate counting strategies as a logical way to make sense of the problems.

In addition to considering children's thinking, the PTs also connected their number sentences and word problems to the real world, further demonstrating that mathematics should make sense. Table 8.5 illustrates some of the statements that PTs made that connect to the real world.

The questions PTs had pointed to important issues about the contexts of the problems, because in the cases of the "Who is winning?" questions, the answers would differ depending on if the game was golf (as some wondered) or another game where the highest (positive) number is best. Their questions illuminate ways the questions could possibly be strengthened by indicating the specific game being played or indicating more clearly what a winning score is. The question about having a -4-point card and losing negative three points was also worded poorly; PTs had to think about the -4 as being broken up into separate cards. Yet, even in these cases, the PTs persisted. Table 8.6 illustrates how PTs persevered through solving problems.

Overall, when PTs identified that they did not know something, they kept moving forward without much pause, suggesting they had confidence in their abilities to solve the problems. They either rectified their initial confusion or found a different way of solving the problem.

Table 8.5 Evidence that PTs made connections to the real world

Number sentence	Prospective teacher	PTs' statements
6 – -7	Audrey	What game are they playing? Golf? They could be playing golf. Who's winning? How many more points does the winner have than the loser? Depends on what game we're playing.
2 – -6	Audrey	Are we still assuming that we're not playing golf?
7 + -3	Karey	Oh, so it's like a game, for some reason I was thinking of like, of like a soccer game or something for points.
7 + -3	Kirsten	What are these people playing?
-4 – -3	Kirsten	Wait a minute, how can this 4, this is a...Oh, ok. So maybe these are like four cards with -1s on each.
-4 – -3	Ophelia	So what's the card?

Table 8.6 Evidence that PTs persevered

Number sentence	Prospective teacher	PTs' statements
6 – -7	Kirsten	So, in my head I'm thinking, well you can't have negative six points um, he has six points and Joan has negative seven. Andy is winning.
-2 – -6	Ashley	I don't know if it's -2 – -6? Which is -2 + 6, which is 62, which is four.
-2 – -6	Kelsey	I don't know what the problem would be. But on the number line, you would just have to move four spots to get to negative two for them to be tied.
6 – 8	Jackie	It'd probably come out looking like six plus minus eight, I mean negative eight, something like that, but I'm just going to do um, I'm going to flip it, and do eight minus six.
-4 – -3	Ashley	I don't understand the question. I guess it's negative one.

Discussion

Building on Collective Knowledge

Although the PTs were not asked to explain how to solve the problem as they would to a student, their think-aloud data provides an indication of how they think about the problems. Therefore, if they are inclined to rely on procedural knowledge, they need to be aware of this so that they can better prepare additional conceptual explanations for students. On an individual level, many of the PTs' responses to the questions could be concerning, such as relying on how an expression looks to determine if they are equivalent, misreading an addition problem as a subtraction problem (as with reading 7 + -3 as 7 – -3), or using an analogy without explanation (as with -3 – 5 compared to -3 + -5). However, if we look at the collective responses, a richer picture emerges.

Although most of the PTs relied on procedural rules to solve 7 + -3, one PT shared a strategy that involved comparing the magnitudes of 7 and -3 and drawing on a counterbalance strategy or neutralizing with a magnitude perspective. Collectively, the counterbalance strategy from Kelsey paired with the procedures provided by other PTs provides an opportunity to think about how procedural knowledge may build from conceptual knowledge for problems such as 7 + -3. Taken together, their responses provide a nice link from how 7 + -3 could translate into 7 – 3.

Similarly, for -3 – 5, Kelsey provided a clear explanation for why -3 – 5 is equivalent to -3 + -5, drawing on reasoning that both operations include a movement further into the negatives. This adds a level of explicitness to Adrianna's analogy and also helps illustrate why subtracting hills is equivalent to adding holes. Paired with the number lines they drew, the strategies used by the PTs are rich and could help students approach the problem in several ways. However, in order for PTs to capitalize on their peers' strategies, they would need to share and discuss the connections in their own content or pedagogy classes.

PTs' responses to 6 – -7 highlighted the potential importance of utilizing a number line representation with students. Although counting strategies for determining how many more points 6 is from -7 are productive, a referent to 0 was an efficient strategy for the PTs. As teachers, these PTs will have to make sense of all of the counting strategies: counting from 6 to -7, counting from -7 to 6, finding the distance from 6 from -7, moving backward from 6 to -7 on a number line, and utilizing a referent to 0. In terms of directed distances, moving backward from 6 to -7 should result in -13, unless the count is contextualized. PTs need to be aware of these nuances in order to know when to prompt students for more information. In addition to making sense of all of these strategies, PTs as teachers will also need to connect these strategies and decide what to focus classroom discussion on. And, when they pose a problem, like problem #2 with 6 – -7, they will need to consider what strategy or strategies they are hoping to elicit through that problem. Through the collective responses of the PTs, they illustrated a preference for a referent to 0 but also supplied a variety of other ways that children may think about this problem as well.

Having discussions in mathematics content or pedagogy courses around problems, such as the ones PTs solved here, provides an important opportunity for PTs to learn from each other by hearing explanations that enrich their own understanding and by having the opportunity to challenge each other's explanations and encourage more detailed explanations. The collective knowledge of the PTs is comprehensive and powerful for leveraging these types of conversations.

Choosing Productive Problems

Ultimately, PTs will need to choose which problems to pose in their own classrooms. The results around PTs' strategy use presented here provide an interesting perspective on the utility of different integer word problem types. For instance, the strategies used for #8 (7 + -3) to other problems such as #3 (6 – 8) or #6 (-2 – -6) in

Table 8.2 are quite different. With the exception of all but one PT, the strategies used with 7 + -3 consisted of recall or procedures. However, with problems like 6 – 8 and -2 – -6, PTs employed all (or nearly all) of the strategies. Therefore, some problem types may elicit the use of a variety of strategies more than others. If PTs are hoping to support discourse that includes such a variety of strategies, then they will need to make conscious decisions about what problems to include in their discussions. On the other hand, if they want to have discussions around the equivalence of one problem type to another, then focusing on a problem such as 7 + -3 might be helpful.

Productive Dispositions

Contexts, and particularly contexts for integer operations, can be interpreted in different ways. PTs sought to understand the context and make sense of the meaning. When asked, for example, who was winning a game, as a collective group, they contributed that winning depends on the game. Due to the nature of integers, sometimes having the most negative points wins (e.g., golf), and sometimes having the most positive points wins. Questioning contexts and making sense of the world around them are an important part of mathematical reasoning that mathematics teacher educators and researchers can draw out—capitalizing on what they are doing well and leveraging discussions about that to help others think in new ways.

Too often literature and discourse around PTs assume a deficit perspective, especially when it comes to mathematics. We challenge deficit perspectives here by arguing that mathematics teacher educators can leverage the collective groups' responses to build a stronger understanding for all PTs. Mathematics teacher educators can do this whether the PTs' strength is in applying procedures, using illustrations to support their reasoning, or making their understanding explicit. As a collective, PTs are naturally good at many things: they persevere, make connections to the world and children, think about what makes sense, and connect written problems to symbolic representations. Mathematics teacher educators can bring these into focus through targeted discussions, helping the PTs work toward a richer conceptualization of mathematics.

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Chapter 9
Prospective Teachers' Attention to Children's Thinking About Integers, Temperature, and Distance

Jennifer M. Tobias, Nicole M. Wessman-Enzinger, and Dana Olanoff

Abstract The study reported on in this chapter describes the justifications that elementary and middle school prospective teachers (PTs) made as they examined the temperature story that a Grade 5 student posed for an integer subtraction number sentence. The ways that the PTs made sense of the student's story that used integer subtraction as distance are described, providing further insight into the ways that PTs may reason about temperature stories in relation to an integer subtraction number sentence. PTs' justifications focused on attributes like order, rather than a magnitude discrepancy in the story. PTs need more experience examining stories for integer addition and subtraction in order to promote discussion and reflection on the various complexities of posing stories for integer addition and subtraction number sentences: consistency, realism, and subtraction as distance.

Nineteenth-century German mathematician, Leopold Kronecker, is credited for saying, "God made the integers, all else is the work of man" (Leopold Kronecker, n.d.). Part of that *work of man* includes the development of the Celsius and Fahrenheit scales. Both of these linear scales include negative integers that are often used as a context in the teaching and learning of integers (e.g., Almeida & Bruno, 2014; Wessman-Enzinger & Salem, 2018). This chapter focuses on the nuanced knowledge that prospective teachers (PTs) may engage in as they work with integer subtraction and make sense of children's thinking in the context of temperature.

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Theoretical Perspective: Specialized Content Knowledge for Teaching Integer Subtraction

Understanding the different models of subtraction, like take-away and distance (Selter, Prediger, Nührenbörger, & Hußmann, 2012), is a component of mathematical knowledge for teaching (MKT). MKT highlights the types of mathematical content and pedagogical knowledge specific to the teaching of mathematics (Ball & Bass, 2003; Ball, Thames, & Phelps, 2008). The MKT construct illustrates knowledge that goes beyond the notion of "more content" and points to very specific kinds of content knowledge, like specialized content knowledge (SCK) and knowledge of content and students (KCS). SCK and KCS include understanding the mathematical errors that students may make (Ball et al., 2008) or attending to the mathematics embedded in student thinking (Jacobs, Lamb, & Philipp, 2010). Important SCK and KCS knowledge for integer addition and subtraction includes using the integers with appropriate contexts and recognizing when and how children use various contexts appropriately or inappropriately in relation to their model of subtraction. For example, a realistic monetary context for -5 – -2 = -3 could include removing debts and thinking about subtraction as take-away (e.g., removing 2 dollars of debt from 5 dollars of debt).

PTs often have experiences with take-away subtraction; however, they often do not have sufficient experience with subtraction as distance (see, e.g., Chap. 6). Subtraction as distance is one model that supports interpreting integer number sentences, especially in the context of finding the difference between two temperatures. Because it is important for PTs to both use subtraction as distance (Tillema, 2012) and the context of temperature (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA and CCSSO], 2010) with integers, we highlight results from a study where PTs made sense of a temperature story posed by a Grade 5 student about an integer subtraction number sentence. Specifically, we describe the ways that PTs attended to a singular child's temperature story involving distance and temperature for an integer subtraction number sentence.

Research tells us that PTs struggle to think conceptually about integer addition and subtraction; often they focus only on procedures (Bofferding & Richardson, 2013). When PTs are asked to reason conceptually and pose stories for integer addition and subtraction number sentences, they often do not use temperature contexts (Wessman-Enzinger & Tobias, 2015). When they do use temperature, PTs often struggle to realistically pose stories involving subtracting a negative number (Wessman-Enzinger & Salem, 2018; Wessman-Enzinger & Tobias, 2015). Yet, in their work as teachers, PTs will need to be able to pose stories for integer number sentences and work within the context of temperature (NGA & CCSSO, 2010). Additionally, PTs will need to make sense of stories that children pose, judge whether or not the stories are realistic, and evaluate children's use of mathematical operations.

Research has shown that both children and PTs have difficulties when posing stories for integer number sentences (e.g., Mukhopadhyay, 1997), and they find temperature stories particularly challenging (Wessman-Enzinger & Tobias, 2015). Some of these difficulties include posing unrealistic stories and changing the number sentence to a different number sentence (Kilhamn, 2009; Mukhopadhyay, 1997; Wessman-Enzinger & Mooney, 2014). Mukhopadhyay (1997) asked children to solve problems involving negative integers and tell a story that matched the equations. She hypothesized that the difficulties they had could be attributed to the various mental models the students were possibly employing. If the mental model a child draws on, for example, does not support a continuous number line with negatives, they may pose an unrealistic story for a number sentence like 6 – 8 = □ (e.g., "I had six pencils and lost eight pencils.")

Similarly, Kilhamn (2009) asked PTs to solve and describe their thinking for number sentences (e.g., -8 – -3= □). Kilhamn found that only a small subset of PTs incorporated a model or context in their mathematical explanations; those who did use a model or context used either number lines or temperature to explain their reasoning.

In addition to posing temperature stories as teachers, PTs will also need to make sense of the stories that children pose about temperatures if they use problem posing. Yet, we know that attending to children's thinking is difficult for teachers (Jacobs et al., 2010)—so making sense of a child's posed temperature story is likely a productive, yet cognitively demanding, task for PTs.

The studies above included both children and PTs posing stories, but very few of them explicitly focused on a particular context. Focusing on a singular context provides robust insight into PTs' conceptual understanding of the integers (Wessman-Enzinger & Tobias, 2015). We chose temperature as a context for a number of reasons: (a) teachers will need to use temperature (NGA & CCSSO, 2010), (b) temperature is one of the more complex contexts (Schwarz, Kohn, & Resnick, 1993–1994), and (c) we lack insight into how PTs attend to children's thinking about integers and temperature. Although we are gaining insight into the ways PTs reason about integers (Almeida & Bruno, 2014; Bofferding & Richardson, 2013), this chapter adds to the existing research by focusing on how PTs made sense of a child's thinking in a temperature context for an integer subtraction problem that highlights subtraction as distance (Selter et al., 2012). Next, we share a more in-depth literature review. This guided the development of our study and provided insight into what we know about the SCK and KCS of PTs and about thinking about integer subtraction.

Challenges of Thinking About Integer Subtraction

Research has only recently begun to focus on PTs' efforts to make sense of integer subtraction (e.g., Almeida & Bruno, 2014; Bofferding & Richardson, 2013). Although research has shown that both children and PTs have demonstrated sophisticated reasoning about integers, thinking and learning about integers are not without challenges. Perhaps the negative integers are challenging, even for university students and PTs (Piaget, 1948), because when children learn about the negative integers in later grades, they often have to overcome more than a decade of experiences of operating with positive numbers to accommodate the negative integers. Additionally, the negative integers are already naturally challenging due to their lack of physical embodiment (Martínez, 2006). Although the literature spans different ages and grade levels, the commonalities of the challenges about integer addition and subtraction are consistent, and we can expect PTs to have similar difficulties with integer subtraction. These challenges include the following:

- Analogies to whole numbers may break down, for example, 2–7 may seem impossible to students (e.g., Bazzini, 1990; Booth, 1989).
- Subtraction is sometimes interpreted as commutative like addition (e.g., Bell, O'Brien, Shiu, 1980; Bofferding, 2010; Murray, 1985).
- More and less have different meanings with integers (e.g., Bell, 1984; Bofferding, 2010, 2014; Bofferding & Farmer, 2018; Guerrero & Martinez, 1982).
- Studies have shown that students sometimes perform operations with the negatives by omitting signs and then adding them in later (e.g., Ayres, 2000; Bell et al., 1980).
- In some contexts, the negative sign may only denote locations and not subtraction (e.g., Bell, 1984; Gallardo, 1995).
- Changing direction and passing through zero change the difficulty of a problem (e.g., Bell, 1993; Bell et al., 1980; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Bofferding, 2010, 2014; Bofferding & Hoffman, 2014; Gallardo, 1995).
- Students have difficulty distinguishing between states and transformations (e.g., Gallardo, 2003; Marthe, 1982).

We discuss these challenges in more detail below.

Analogies: Subtraction and Commutativity Students often make analogies from problems with only positive integers to problems with negative integers, and one challenge is that their analogies may break down (Bofferding & Wessman-Enzinger, 2017). For example, students may incorrectly make an analogy, comparing 2 – 7 to 7 – 2 (Bofferding, 2010). That is, some students extend the commutative property of addition to subtraction, incorrectly equating 2 – 7 to 7 – 2. Or, some students may think that problems like 2 – 7 are impossible (e.g., Bazzini, 1990; Booth, 1989). Students have to overcome their whole number experiences and recognize that subtraction is not commutative like addition. If students only use positive integers in instruction, subtraction might seem as if it is commutative (e.g., Bell et al., 1980; Bofferding, 2011; Murray, 1985). In relation to the study reported on in this chapter, analogies do not work well with temperature problems. For instance, -6 – -5 is a number sentence that students often compare to 6–5 and solve with the take-away model of subtraction (Bofferding & Wessman-Enzinger, 2017). However, this analogy is not appropriate in the context of temperature, as you cannot interpret -6 – -5 using a take-away model of subtraction.

Language: More or Less in the Context of Integers and Temperature Another common challenge is that *more* and *less* can have different meanings in the context of negative integers (e.g., Bell, 1984, Bofferding, 2010, 2014; Bofferding & Farmer, 2018; Guerrero & Martinez, 1982). That is, as you move right along the number line, numbers become *more* positive and *less* negative. Similarly, as you move left along the number line, numbers become *more* negative and *less* positive (Bofferding, 2014). Bofferding and Farmer (2018) related this to temperature when they had students consider temperatures like -2 degrees and -5 degrees and consider which of these temperatures is *most hot* or *most cold*. Although this is about comparison of order and magnitude and our chapter is about subtraction, this research highlights the importance of considering the nuances of language PTs engage in when making sense of children's reasoning about integer subtraction in the context of temperature.

Signs: Confounding Signs, Relative Numbers, and Translations Another challenge that students have to navigate is the use of the minus sign (i.e., "-"). The minus sign has multiple meanings (i.e., unary, binary, opposite) that students may confound (Bofferding, 2014; Gallardo & Rojano, 1994; Vlassis, 2004, 2008). Students often operate with the negative integers by simply omitting or "ignoring" the minus sign and then adding it later (e.g., Ayres, 2000; Bell et al., 1980). For example, to solve -5 – 4, a student may solve 5 – 4 = 1 and then apply a minus sign back in the number sentence (-5 – 4 = -1), resulting in an incorrect answer. Additionally, the minus sign is used to denote location, as well as subtraction, and this can be confusing to students (e.g., Bell, 1984; Gallardo, 1995). For example, for problems like 10 – -2, some students may interpret this as subtracting twice (Bofferding, 2010). Students may think that 10 – -2 represents the problem 10 – 2 – 2, rather than representing the distance between two locations, 10 and -2, on a number line.

Integer Subtraction Number Sentences

Although subtraction with integers is challenging, not all integer subtraction number sentences are equally challenging (e.g., Hativa & Cohen, 1995; Marthe, 1982). Hativa and Cohen (1995) illustrated that some number sentence types are more intuitive than others. For example, -a – -b, where a > b > 0, is an easier number sentence type than -a – b because students can use analogies (e.g., -5 – -2 is like 5 – 2, but -5 – 2 is not like 5 – 2). Similarly, Marthe (1982) found that number sentences like $x + b = c$ were more challenging for students when b and c had different signs. Overall, researchers have identified that different number sentence types for addition and subtraction of integers have varying difficulties (see, e.g., Chap. 3). However, they have not all agreed on what constitutes a different problem type. For example, Mukhopadhyay, Resnick, and Schauble (1990) consider -2 + 5 and -7 + 5 to be the same number sentence type. However, others (e.g., Bofferding, 2010; Peled, 1991) recognize that magnitude affects the problem type (see Table 9.1),

Table 9.1 Number sentence types for integer subtraction in literature

Number sentence types without magnitude	Number sentence types with magnitude
a, b > 0	a, b, A, B, c > 0 where the capital letter indicates the greatest magnitude in the number sentence
a – b	-A – b, -a – B, c – c
a – -b	A – -b, a – -B, c – -c
-a – b	-A – b, -a – B, -c – c
-a – -b	-A – -b, -a – -B, -c – -c

differentiating between -2 + 5 and -7 + 5. With magnitude considerations, there are 12 different number sentence types for integer subtraction. If zero is included in these problem types (i.e., a – 0, 0 – a, -a – 0, 0 – -a), then there are 16 different problem types.

When considering the number sentence types in the second column of Table 9.1, Murray (1985) found that number sentence types like -A – -b, -a – -B, -A – b, and A – -b were the most challenging for students. Other researchers found that the problem type -A – -b was not as challenging for students (e.g., Wheeler, Pearla, Bell, & Gattengo, 1981), and Peled (1991) found that problem types A – -b and a – -B were the most challenging for students. In our study reported here, we used the number sentence -14 – -20 = □, of the form -a – -B. We used a problem where the subtrahend (-B) was negative to encourage the PTs to look at integer subtraction in the context of distance, rather than as temperature increasing or decreasing (i.e., where the subtrahend would be positive).

Conceptions About Integer Subtraction

Recent research on students' conceptions and strategies for subtraction of integers (Bofferding, 2010, 2011; see, also, Chaps. 3 and 8) and investigations into students' ways of reasoning when solving open number sentences with integers (Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014) have focused on the productive ways that students can operate with the subtraction of integers but also point out ways that whole number reasoning can interfere with extending that knowledge to the integers. For example, curriculum materials for mathematics in elementary school, including reform curricula, advocate for the use of "fact family" instruction. This type of instruction focuses on highlighting the relations between operations that create a "family" of facts (e.g., 2 + 3 = 5, 5 – 3 = 2). Bofferding (2011) challenged the present role of fact family instruction by advocating for the use of negative integers with young children. Addition is commutative with addition (e.g., 2 + 3 = 3 + 2 = 5); however, subtraction is not commutative with subtraction (e.g., 3–2 ≠ 2–3). Since negative numbers are not typically taught in first grade, many students develop the misconception that subtraction is commutative because addition is commutative. Likewise, students may pose similar temperature stores for 3–2 and 2–3, and PTs need to understand that 3–2 and 2–3, although not commutative, can also represent a similar context. Consider the following two stories and how they are similar, even though they represent different number sentences:

> *Temperature Story 1*: It is 2 degrees in Fairbanks, Alaska and 3 degrees in Juno, Alaska. What is the difference in the temperatures of the two cities? (3 – 2 = 1, where 1 represents that there is 1 degree difference in the temperatures.)
>
> *Temperature Story 2*: It is 2 degrees in Fairbanks, Alaska and 3 degrees in Juno, Alaska. What is the difference in temperature from Fairbanks to Juno? (2 – 3 = -1, where -1 represents that it is 1 degree colder in Fairbanks.)

These different stories point to the necessity for PTs to also be able to make sense of the magnitude of integers. Bishop, Lamb, Philipp, Whitacre, Schappelle, and Lewis (2014) and Bofferding and Richardson (2013) found that both children and PTs tended to use magnitude-based reasoning about the integers. That is, the children typically thought about the magnitude of -3 in comparison with the magnitude of 6 in the problem -3 + 6. In fact, Kilhamn (2009) argued that the ability to make numerical magnitude comparisons is an important component to understanding the integers (see also Bofferding, 2014 and Chap. 2).

Although tied to whole number reasoning and reasoning about magnitudes, conceptions about integers are also grounded in definitions of subtraction; two ways that subtraction can be conceptualized are as take-away and as distance (Selter et al. 2012). Subtraction as take-away has limitations with integer subtraction because for problem types like 1 – -3, when taking away, discrete objects may not physically exist (Bofferding & Wessman-Enzinger, 2017). For this reason, reasoning about subtraction as distance and directed distance supports integer subtraction. But, reasoning about integer subtraction with directed distance points to another challenge: students have difficulty distinguishing between states and translations with the integers (e.g., Marthe, 1982; Gallardo, 2003; see, also, Chaps. 6 and 7). That is, -2 can represent a state, like a temperature of -2 degrees Fahrenheit or a location on a number line, or -2 can represent a translation of dropping 2 degrees Fahrenheit or a movement two units left on a number line (Schwarz et al., 1993–1994; Wessman-Enzinger & Tobias, 2015). (Schwarz et al., 1993–1994) highlighted this as an important component to utilizing temperature contexts. When using temperature and integer subtraction, there are many dimensions to coordinate. Prospective teachers need to coordinate the realism of the temperature story (Mukhopadhyay, 1997) with the structure of the number sentence (Kilhman, 2009; Roswell & Norwood, 1999) and interpret the language of integers contextually with temperature (Bofferding & Farmer, 2018).

Although many researchers have highlighted the challenges that students have with changing directions and passing through zero on number line (e.g., Bell, 1993; Bell et al., 1980; Bofferding, 2010, 2014; Bofferding & Hoffman, 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Gallardo, 1995), students may conceptualize integer subtraction with distance or movement (Bofferding & Wessman-Enzinger, 2017). When considering number sentences like -1 – -3, students tend to think about crossing zero on the number line if the problem is interpreted as a movement of three units from -1 to 2 (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014). However, if students interpret -1 – -3 as the distance between -1 and -3 on the number line, the problem may not be as challenging (Bofferding & Wessman-Enzinger, 2017). In terms of temperature, conceptualizing -1 – -3 as a distance between two temperatures is important for prospective teachers to experience (Wessman-Enzinger & Salem, 2018) because it exposes them to more challenging number sentence types and subtraction as distance, a model with which they have less experience.

Research Question

As a field we need to learn more about how PTs make appropriate sense of integers in temperature contexts (CCSSO & NGA, 2010) and, specifically, how they make sense of children's posed stories for integer subtraction in the context of temperature. The research question that guided our work for this study was:

> What do PTs attend to as they evaluate a child's temperature story for integer subtraction that uses distance?

Methods

While enrolled in an introductory mathematics content course designed specifically for prospective teachers, elementary and middle school PTs (100 total) participated in a study focused on integer addition and subtraction. The design of the mathematics content course promoted conceptually oriented discourse around number and operations. Preparing PTs to become mathematics educators, we[1] encouraged PTs to solve problems in multiple ways, present their own solution strategies, and *understand the reasoning of others* as a part of this course (Cobb & Yackel, 1996).

Data Collection

We collected data across four sections of the course over two academic semesters, Fall 2013 and Spring 2014, during a unit focused on integer addition and subtraction. Instruction on integer addition and subtraction immediately followed instruction on whole number operations. The PTs participated in activities that included creating contexts for number sentences, discussing the validity of contexts in

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[1] The first two authors were the instructors for this course.

Table 9.2 The Parker Task

Student	Number sentence	Story
Parker	-14 – -20 = □	The freezer is -20°. The refrigerator is -14°.
		The freezer is 6° colder than the refrigerator.

Note. The table shows the number sentence given to a fifth-grade student and the story written. Decide if the story makes sense with the number sentence provided. Write a "Y" for yes, if you think that the story matches the number sentence. Write "No" for no if you think the story does not match the number sentence. Explain your reasoning.

relation to temperature, as well as examining other PTs' and children's written work with integer addition and subtraction (for further discussion of some of these activities, see Wessman-Enzinger & Tobias, 2015). We also collected PTs' work on in-class activities, homework, and an end-of-unit exam.

Although we collected data for several activities, the focus of this chapter is on the findings from an integer addition and subtraction problem given on the end-of-unit exam, which included integers. The exam question required PTs to analyze stories written by Grade 5 students for different integer number sentences. This chapter focuses on the story written by Parker, for the number sentence -14 – -20 = □ (see Table 9.2).

Parker's story and the corresponding number sentence used a state-state-translation problem type instead of the commonly utilized state-translation-state problem type (Wessman-Enzinger & Tobias, 2015). Therefore, the story supported thinking about subtraction as distance rather than subtraction as take-away (Selter et al., 2012)—a challenging concept for students.

Research has shown that students may avoid the use of negative numbers in context; for example, a student may discuss $60 of debt rather than having -$60 (Whitacre et al., 2015). This story posed by Parker contains a similar challenge—stating *6° colder*, which is equivalent to a -6° difference. Using *6° colder* potentially could be the fifth grader avoiding the use of negative. Furthermore, "6° colder" aligns to the number sentence -20 – -14 = -6, rather than -14 – -20 = 6. Thus, one version of a consistent story for -14 – -20 = 6 includes framing the refrigerator as *6° warmer* than the freezer.

Data Analysis

We examined and analyzed the PTs' responses for the Parker Task, looking at both their answers to whether the story made sense with the corresponding number sentence and their justifications. We used constant comparative methods (Merriam, 1998) to analyze PTs' responses. During our analysis of Parker, we examined the

Table 9.3 Codes for Parker Task

The Parker Task		
Consistency	Does the PT mention a consistency issue, e.g., that the story shows -20 – -14 = -6 instead of -14 – -20 = 6? (yes = 1/no = 0)	PT includes: (0) no consistency issue; (1) order of the numbers -20 and -14; (2) 6 versus -6; (3) confounds magnitude (e.g., states, “-14 is bigger than -20”); (4) others.
Distance	Does the PT mention difference or distance in their justification? (yes =1/no =0)	

PTs’ written language for how they attended to consistency and subtraction as distance. We also determined which reason, of those they listed, was the main justification for why the PT indicated the student’s answer was correct or incorrect.

We individually coded each PT’s response. Within our analysis of PTs’ responses, we looked for two things:

- Did the PT attend to the consistency issues in relating the story to the number sentence?
- Did the PT mention the relation between the idea of distance in the story and the operation of subtraction?

After coding for each theme, the authors then met weekly to discuss their codes. All disagreements were negotiated and resolved. Table 9.3 illustrates our coding scheme centered on consistency and subtraction as distance.

Results and Discussion

Seventy-three out of 100 PT responses included a statement that Parker’s story matched the number sentence -14 – -20 = 6, 26 PTs said his story did not match, and 4 PTs said both yes and no. Our results indicated that when determining if Parker’s story was consistent with the number sentence, many PTs attended to the order or magnitude of -20 and -14 or the answer of 6 versus -6 (see Table 9.4). PTs also discussed procedures when making consistency explanations. Additionally, approximately half (51%) of the PTs mentioned *difference* or *distance* in their explanations (see Table 9.4). What follows is a discussion highlighting the varying explanations the PTs wrote.

Consistency

The Story Matched the Number Sentence PTs who reasoned that the story matched indicated this was because both numbers (-14 and -20) and/or the solution of 6 was accounted for and represented in Parker’s story. The following two examples illustrate where PTs reason that the -14, -20, and/or 6 are accounted for in Parker's story:

> Yes, because the number sentence shows what the freezer and refrigerator are.
>
> This does match up because the answer to the number sentence is 6. And the answer in the story is 6. It also makes sense because the -20 and the -14 are accounted for in the solution.

Both of the PTs' responses above indicate that they compared the number sentence to Parker's story only in terms of the numbers given in the problem, not if the story was consistent (i.e., referencing an operation) with the number sentence itself. For many PTs, this was enough evidence for them to conclude that Parker was correct.

Other PTs confounded the magnitude of -14 and -20, using that as a justification for whether the story matched the number sentence. One PT, for instance, discussed the "largeness" of the numbers, that -14 is greater than -20:

> Parker's number sentence and story did match up. Parker fully understood the concept of subtracting & negative and took the temperature of the freezer minus the temperature of the fridge to figure out how much colder the freezer was than the fridge. Parker also understood that to get the correct answer the larger number -14 must come first. This is why Parker's story and number sentence correctly match up.

This PT, focusing on the order of the numbers, concluded that -14 should be first in the number sentence because "the larger number -14 must come first"; thus Parker's story matched.

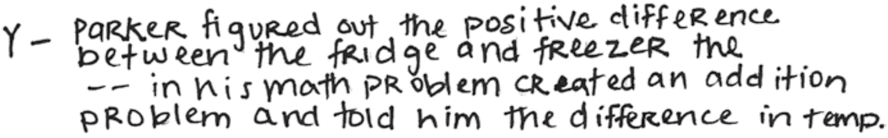
Other PTs stated that the story matched but provided reasoning that was different than the order or magnitude of the numbers. In particular, eight PTs' explanations that supported the story matching included the argument that Parker's story matched because subtracting a negative was the same as adding, thus giving you the correct answer of 6. It is noteworthy to point out that neither the number sentence nor the story posed by Parker included addition, yet eight PTs referenced addition, an example of which is shown in Fig. 9.1.

Rather than connecting Parker's story in the temperature context to subtracting a negative number, these types of explanations related the problem to an equivalent problem expressed as addition to conclude that Parker was correct.

The Story Did Not Match the Number Sentence The PTs that responded that Parker's story did not match the number sentence (26 out of 100) also had varied reasoning for why Parker was incorrect. The majority of responses that said that the story did not match indicated that the story showed -20 – -14 = -6 instead of -14 – -20 = 6, as in the explanation below:

> The number sentence doesn't match the story because the numbers were talked about in the story the opposite way than in the number sentence.

Though many PTs mentioned the order of the -14 and -20, a few of the PTs stated that Parker was solving for a different number sentence. The following explanations, for example, support different number sentences:



Y – Parker figured out the positive difference between the fridge and freezer the -- in his math problem created an addition problem and told him the difference in temp.

Fig. 9.1 PT referenced that subtracting a negative is the same as adding

Table 9.4 Classifications of written explanations from the PTs

Consistency explanations	Difference or distance explanations
Does not mention a consistency issue	*Mentions difference or distance*
Number of PTs = 59	Number of PTs = 49
Mentions the order of the numbers -20 and -14	*Does not mention difference or distance*
Number of PTs = 25	Number of PTs = 51
References 6 versus *-6* (i.e., *6 degrees colder* versus *6 degrees warmer*)	
Number of PTs = 1	
Confounds magnitude (e.g., *-14 is bigger than -20*)	
Number of PTs = 2	
Other reasoning	
Number of PTs = 19	

Note. The counts in the first column sum over 100 because we assigned some PTs' explanations more than one code

> No, I would say this number sentence does not match up because as I read this problem I would do -20 minus 6, because since it is getting colder we subtract, equals -14 (the temperature of the refrigerator).
>
> No, although Parker would get the right answer his equation is also wrong. You can't subtract a negative in temperature because you can't go down a negative degree. If the freezer is 6° colder than the refrigerator than the correct equation would be -14 (refrigerator) – 6 (how much colder it is) = -20. Therefore, although his answer would be right, the equation is wrong.
>
> Parker should have made his # [number] sentence to match his story, such as putting -20 + 14 = -6.° But, if you look at Parker's problem differently, it can seem right because even though he put the refrigerator before the freezer, he is making a comparison of the 2 to get how much colder the freezer is and it is 6° colder.

Though the first PT created a new number sentence that was mathematically incorrect (stating -20 – 6 = -14), she related the subtraction in the problem to *getting colder* or dropping a temperature, thus interpreting Parker's story as take-away instead of comparison. This was the same reasoning the second PT used when writing, "…you can't go down a negative degree." Both PTs' responses attended to the idea that Parker was incorrect because the order of the entire number sentence did not match his story. Similarly, the third explanation described the problem as comparison and stated that though the order of the numbers was switched in the story that making a comparison between the refrigerator and freezer to determine that the freezer is 6 degrees colder can make his story seem correct.

These explanations highlight different number sentences for Parker's story: -20 – 6 = -14, -14 – 6 = -20, and -20 + 14 = -6, respectively. In the topic of subtraction as comparison versus take-away, there is more than one number sentence to represent a subtraction as comparison situation. Both the second and third PTs' responses alluded to comparison, whereas the first discussed take-away as getting colder.

Notably, only two PTs focused on the solution of 6. They explained that when Parker stated that the freezer is 6 degrees colder, this would actually represent -6 not positive 6. The following PT compared 6 versus -6 to decide that the story does not match the number sentence:

> I said no for Parker. I like how he was thinking but his last part would make the answer -6 not 6. I would change that part to say the fridge is 6 warmer than the freezer. This would give you a positive 6.

By changing the wording of the solution to talk about the refrigerator being warmer than the freezer, the PT noticed that Parker actually discussed a solution of -6 not positive 6 as represented in the number sentence.

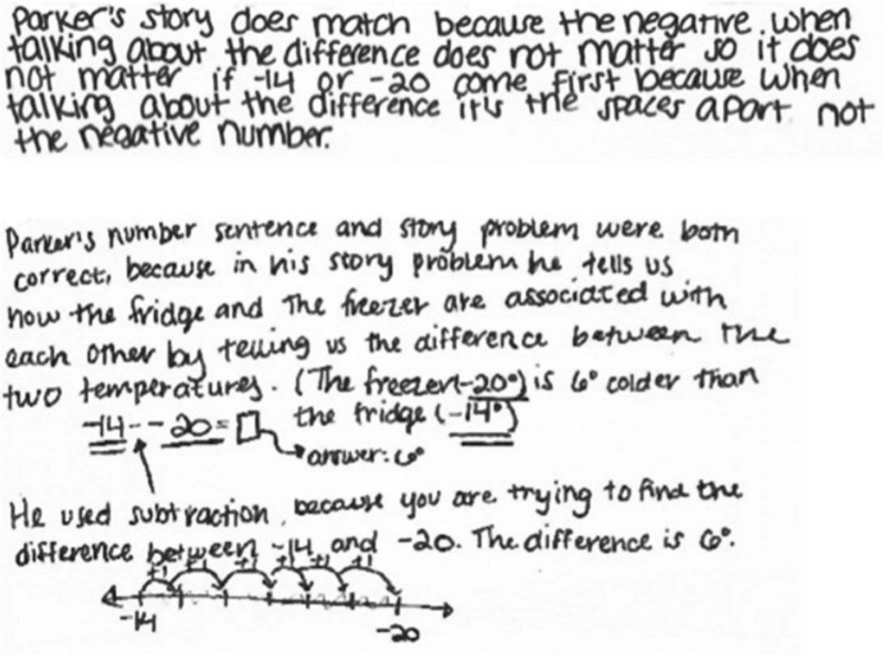
In summary, only about a quarter of the PTs responded that Parker's story did not match, and their justifications mostly centered on the order of -14 and -20. But, the order of the numbers is only part of the story. Although -20 – -14 and -14 – -20 may represent the same distance, interpreting the -6 and 6 is not the same. Yet, only two prospective teachers mentioned the discrepancy between 6 and -6. Overall, most PTs responded that the story did match the number sentence. The story represents difference of temperatures, which is a strong use of integer subtraction. However, the nuance of *6 degrees colder* is an important observation for PTs to make if they are to interpret student work and lead discussions around mathematical work.

Distance

Fifty-one of the PTs related Parker's story to either the difference or distance between the refrigerator and freezer temperature. PTs that used the distance or difference either briefly mentioned that this is what Parker was finding in his story or used this as a main justification for why he was correct (see Fig. 9.2).

By relating Parker's story to finding a distance or difference between -14 and -20, PTs either stated that he was correct because the order of the numbers does not matter or that the difference between the two temperatures is 6°; thus Parker was correct. If the PTs did not mention distance, they generally mentioned consistency issues instead.

The first response from a PT in Fig. 9.2 mentions the order of -14 and -20 in relation to the difference "because when talking about the difference it's the spaces apart not the negative number." This type of reasoning is correct if we are



Parker's story does match because the negative. when talking about the difference does not matter so it does not matter if -14 or -20 come first because when talking about the difference it's the spaces apart not the negative number.

Parker's number sentence and story problem were both correct, because in his story problem he tells us how the fridge and the freezer are associated with each other by telling us the difference between the two temperatures. (The freezer-20°) is 6° colder than the fridge (-14°)

-14 - -20 = □ answer: 6°

He used subtraction, because you are trying to find the difference between -14 and -20. The difference is 6°.

Fig. 9.2 PTs used difference/distance to justify why Parker is correct

referencing the absolute value or magnitude. But, with integers, we can use *directed distance*. With directed distance, -20 – -14 and -14 – -20 both represent distance; but -20 – -14 = -6 is moving -6 units from -14 to -20, and -14 – -20 = 6 is moving +6 units from -20 to -14. In relation to Parker's temperature context and the number sentence, this means the distinction between *6° colder* and *6° warmer* matters. The second response in Fig. 9.2 highlights this distance between the two numbers, -14 and -20, and emphasizes the directed distance of moving from -14 to -20 with arrows above the number line. It is interesting to note that this PT ordered these numbers in an unconventional way, with -14 to the left and -20 to the right. In this way, the PT shows the temperature getting *colder* despite not mentioning the 6 versus -6.

Overall, although potentially due to the written nature of this task, it is noteworthy that none of the PTs mentioned directed distance. Directed distance is an important part of incorporating negative integers into distance models of subtraction. The majority of PTs responded that the numbers in the story matched the numbers in the number sentence as their main justification for whether the number sentence matched Parker's temperature story. Likewise, the majority of PT responses that claimed Parker's story did not match provided justification that the numbers in Parker's story were presented in the opposite order as they were in the number sentence. We found few PTs that presented other reasoning such as the solution of 6 versus -6 or confounded the magnitudes of -14 and -20. Other PTs used the presence of difference or distance of the two numbers in the story to determine if Parker was correct or not but did not mention the directed distance.

Implications

Our study is an inaugural description of the ways PTs attend to children's thinking about integers in a temperature context involving distance. We purposefully selected a difficult problem type (-a – -B) for determining what understandings of integers PTs use. This helps us understand how PTs analyze children's thinking about integers and temperature; it also faciliates our thinking around what is needed for integer instruction with PTs. PTs will teach integers in the future, both within and without a temperature context. Specifically, PTs will need to (a) support children in developing a distance conception of subtraction and (b) make sense of children's thinking about integers, temperature, and subtraction as distance.

Results indicated that very few PTs discussed the directed distance of 6 or -6. Yet, directed distance, not just order and operations, is a crucial component to understanding integers (Bofferding, 2010, 2014). Although many PTs noted the relation between distance and subtraction, the PTs did not attend to the direction. The assessment in this chapter occurred at the end of the integer instruction in our courses, where thinking about integers conceptually was promoted and PTs even posed stories themselves for integers. Consequently, we think that our future instruction with PTs should include opportunities for explicit discussions about directed distance while analyzing children's thinking in contexts.

Our results also indicate that although context plays an important role in helping PTs understand different types of subtraction problems, not every PT attended to the context of the problem. In fact, only one PT recognized that Parker's story indicated -6 with "the freezer is 6 degrees colder" as opposed to +6. Also, 23 PTs disregarded the context and instead focused on the order of the numbers in the problem. This is a surprising result to us because this exam question came after weeks of instruction focused on problem solving and reasoning with number concepts and operations. Consequently, more research is needed to determine the ways in which PTs develop an understanding of integers by analyzing children's thinking from both a conceptual and procedural approach.

Instruction with integers is often overlooked in courses for PTs but can aid in PTs' understanding of what it means to add and subtract numbers in multiple contexts. Much can be learned about subtraction as distance with integers. Since the same contexts for whole numbers do not necessarily work for integers, we cannot assume that PTs will readily connect their experiences in whole number contexts to integer contexts. Likewise, our results indicate that even when presented with a familiar context, like temperature, the context may not be taken into account and PTs may not readily extend it to making sense of children's thinking. Thus, it is important that PTs engage with integers for developing the skills necessary for teaching mathematics deeply and conceptually.

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Chapter 10
Using Models and Representations: Exploring the Chip Model for Integer Subtraction

Eileen Murray

Abstract Mathematics educators advocate for the use of models as an instructional practice that can potentially aid in building students' understanding of difficult topics. Integers and integer operations are historically problematic for students and are critically important in both arithmetic and the future study of algebra. In this chapter, I explore one particular model for working with integers, a discrete model using colored chips. By doing so, I illustrate a tension that arises when deciding how to present a model during instruction and examine key questions about the use of models in mathematics teaching. How should mathematics teachers build connections between students' informal understandings of mathematical content and formal mathematics? In what ways can teachers encourage students to use a model in ways that make sense while challenging the utility of the model for related problems?

One instructional practice mathematics educators have advocated for as a potentially effective way to help build students' understanding of mathematics is the use of models (Moyer-Packenham & Westenskow, 2013; Ross & Willson, 2012; Sowell, 1989). *Models* are symbols, pictures, diagrams, graphs, or concrete materials that help students manage, document, communicate, or interpret mathematical ideas and phenomena (Beswick, 2011; Jones, 2010). Models provide opportunities for students to use their everyday, intuitive knowledge to explore and make sense of mathematics (McNeil & Jarvin, 2007). Some research has shown that the use of models can help children construct meaningful ideas, enhance performance in class activities and assessments, and improve attitudes toward mathematics (Bolyard & Moyer-Packenham, 2012; Clements, 1999; Moyer, 2001).

Nevertheless, the use of models does not guarantee successful mathematical learning. In order for effective learning to take place, teachers need to be knowledgeable about the use of models and understand that mathematics does not exist in the models themselves; rather, the models allow for the exploration of mathematics

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(Clements, 1999; Kamii, Lewis, & Kirkland, 2001; Moyer, 2001). Furthermore, if models are not recognized as valuable tools for mathematical learning, teachers may not create conditions conducive to allowing students to construct their own knowledge through interactions with the materials (McNeil & Jarvin, 2007; Moyer & Jones, 2004). Therefore, it is important that educators are aware of how models best represent mathematical concepts and students' learning of these concepts (Hall, 1998).

Understanding integers and integer operations is one important topic for which educators have promoted the use of models, but questions about the effectiveness and value of certain models for helping students understand this mathematical topic persist (Murray, 2011; Stephan & Akyuz, 2012; Vig, Murray, & Star, 2014). Nonetheless, models may be helpful in this domain because students have difficulty understanding negative numbers (Vlassis, 2008), accepting negatives' usefulness and place in the number system, and interpreting negative numbers in particular real-world situations (Whitacre et al., 2011; Wessman-Enzinger & Mooney, 2014).

Operations with integers are a critically important concept for school mathematics, in both arithmetic and the future study of algebra (Peled & Carraher, 2008). However, students frequently struggle when trying to understand negative integers and operations with them—sometimes overgeneralizing their experiences with positive numbers (Bofferding, 2014; Whitacre et al., 2011). Moreover, the difficulties students continue to face when attempting to create meaning for particular operations and conventions for integers call into question the basic premise that students can transfer their learning of integers from concrete contexts to abstract domains (Stephan & Akyuz, 2012). These hurdles and other difficulties with negative integers have been a consistent theme in mathematics education literature for decades (Bishop et al., 2014; Gallardo, 2002; Hefendehl-Hebeker, 1991; Thomaidis, 1993; Vlassis, 2008).

The goal here is not to enter into the conversation about the effectiveness of models, either generally or for teaching integer arithmetic. Rather, by exploring one particular discrete model for working with integers that uses colored chips, I aim to illustrate an inescapable tension that arises in the instructional use of models: deciding whether to present models the way curricula dictate or the way teachers understand them or feel students understand them. In the section below, I provide information about models, including the theories underlying their use and research on a chip model for teaching integer operations. I then describe a teaching scenario that I experienced and use this instructional context to consider key questions about the use of models in mathematics teaching. For example, how should teachers build connections between students' informal understandings of mathematical content and formal mathematics? In what ways can teachers encourage students to use a model in ways that make sense while challenging the utility of the model for various (related) problems?

Background

The use of models has been documented in mathematics classrooms for decades (e.g., National Council of Teachers of Mathematics, 2000). However, much still needs to be understood with respect to how models are used and how teachers and students understand them. The current work sought to understand one possible model for integer operations, methods of implementation, and the mathematics behind each method. In the sections that follow, I provide a more detailed definition of what I consider a model in this context and give examples of research on models while highlighting the conflicts that arise within this literature.

Definition of Models

I employ the term *model* for use in the mathematics classroom as any structure of symbols, images, or concrete objects that represent mathematical concepts. That is, models can be "materials, visual sketches, paradigmatic situations, schemes, diagrams, and even symbols" (Van den Heuvel-Panhuizen, 2003, p. 13), which become useful when they are rooted in realistic, imaginable contexts and remain flexible so they can be applied on a more advanced or general level. In this study, I focus on a subset of models, *manipulatives*, rather than on all forms discussed in the literature.

Mathematicians and mathematics educators talk about manipulatives as manipulative materials (Hall, 1998; Moyer, 2001), manipulative models (Clements, 1999), concrete manipulatives (Clements, 1999), or concrete materials or representations (Hall, 1998). Manipulatives are said to be "physical objects used to represent mathematical ideas" (National Research Council, 2001, p. 9) or "concrete objects use to help students understand abstract concepts" (McNeil & Jarvin, 2007, p. 310). Moyer (2001) describes manipulative materials as visual and tactile objects that students maneuver through hands-on experiences. In her definition of concrete representations, Sowell (1989) includes "materials such as beansticks, Cuisenaire rods, geoboards, paper folding, or other manipulative material under the supervision of a treatment administrator" (p. 499).

For the purposes of this work, I use the term manipulative as a way to capture physical tools[1] that are used to help support students' learning and that are manifestations of mathematical concepts. Such tools are pre-constructed and presented to students, rather than constructed by the students themselves, and can be used to represent abstract mathematical ideas explicitly and concretely. Whether one prefers the term representation, manipulative, or model, mathematics educators have long been researching theories underlying their use and debating their benefits in the teaching and learning of mathematics.

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[1] Recent work by Moyer et al. (2013) discusses virtual manipulatives, but this is outside the scope of this work and therefore is not discussed.

Manipulative Use in Mathematics Instruction

Manipulatives have numerous benefits for the teaching and learning of mathematics. One such benefit includes enhancing students' memory and understanding through the physical actions they employ when using manipulatives (McNeil & Jarvin, 2007). Many studies endorse the use of manipulatives to help students communicate using mathematics, develop abstract reasoning, access real-world knowledge and experience, and discover concepts through student-centered activities (Carbonneau, Marley, & Selig, 2013; Uribe-Flórez & Wilkins, 2016; Van den Heuvel-Panhuizen, 2003). The use of concrete representations can improve students' mathematical abilities, problem solving, and reasoning skills (Pape & Tchoshanov, 2001). Similarly, concrete representations help transform everyday knowledge to mathematical understandings (Linchevski & Williams, 1999).

Effective instructional use of manipulatives provides students with the control and flexibility needed to make connections among different pieces of knowledge and should "have characteristics that mirror, or are consistent with, cognitive and mathematical structures" (Clements, 1999, p. 50). However, as with any curriculum or instructional approach, implementation matters—the use of manipulatives themselves does not guarantee success (Carbonneau et al., 2013). Students could fail to link the use of manipulatives with the appropriate mathematical concepts if they learn to use manipulatives in rote ways (Clements, 1999; Moyer, 2001). This failure to engage students in substantive mathematics as they use manipulatives is one reason why manipulatives may not be employed successfully (McNeil & Jarvin, 2007), as the connection between the manipulative and the mathematical concepts may not be transparent enough for students to grasp (Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009; Uttal, Scudder, & DeLoache, 1997). Because of these difficulties, manipulatives are not a panacea for instruction and teachers should take steps to explicitly build connections between their students' informal understandings and formal mathematical symbols or concepts.

Another problem with using manipulatives resides in their dual representation—a manipulative is a representation of a mathematical concept or procedure as well as an object in its own right. Some studies have shown that perceptually rich manipulatives (e.g., actual objects) or representations of an actual object (e.g., fake money) might hinder learning because of the irrelevant information contained in the manipulative or because students may not recognize and interpret the dual nature of the manipulative (Uttal et al., 2009). This dual representation is difficult for many students to understand and may overwhelm students' limited cognitive resources.

Additionally, the nature of instruction can influence the effect of manipulatives on mathematics learning. Some researchers have highlighted the importance of instructional guidance because of the difficulties students have with understanding how the physical manipulation of objects is related to the mathematical concepts being addressed (Martin, 2009; Sarama & Clements, 2009). Related to the dual nature of manipulatives, some research has shown that more direct instruction and "the scripted manipulation of objects helps students establish connections between concrete representations and their abstract referents (i.e., words), which in turn enhances comprehension" (Carbonneau et al., 2013, p. 395). Others say this guidance might impede learning because it can confine students to particular interpretations (Martin, 2009). Further research is needed to untangle this debate. One idea may be to look more carefully at how learning objectives and teachers' understanding of manipulatives might play a role in the level of instructional guidance.

Beyond the issues with manipulatives themselves, the concept of integers is a difficult one for students and may compound instructional difficulties. In the next section, I discuss research on integers and integer operations to help situate this content in student learning and connect it to the work with manipulatives.

Research on Teaching Integer Operations

For decades mathematicians and mathematics educators have considered the difficulties most students have with learning how to interpret and manipulate integers (e.g., Hefendehl-Hebeker, 1991; Vlassis, 2008). Students may have difficulties with the "meaning of the numerical system and the direction and multitude of the number," the "meaning of arithmetic operations," and/or the "meaning of the minus sign" (Altiparmak & Özdoğan, 2010, p. 31). Others have shown how problems with algebraic symbols can be the root of students' difficulties with negative numbers (Lamb et al., 2012; Vlassis, 2008). In particular, negative numbers are fundamentally linked with algebraic language and symbols and thus with how the minus sign is used. The different uses of the minus sign (e.g., unary operator, binary operator, and symmetric operator for negative numbers) make it necessary for students to be able to be flexible in their thinking about the sign in order to make sense of variables, algebraic expressions, calculations, and algebraic equations.

Prior to first encountering negative numbers, students work with only positive numbers that can be directly applied to numerous real-world situations, including counting objects and measuring quantities. Thus, as students learn about negative numbers, they may have difficulties if they continue to think of the negative numbers as measuring physical quantities. The shift of moving beyond the concrete requires time and considerable conceptual change, which could make the use of manipulatives a natural tool for instruction. By using discrete manipulatives, students may work with integers in a familiar setting within a context that helps them understand the conceptual jump from counting numbers to integers.

Flores (2008) used colored chips and zero pairs (i.e., additive inverses) to help students learn about integer subtraction. The use of discrete objects could allow students to take a concrete approach to negative numbers and help them internalize their understandings. Hayes and Stacey (1999) found that students who learned about negative numbers using integer tiles performed better on a posttest when compared to students who used the number line. They concluded that instruction focused on "neutralization and neutral pair" or a zero pair method[2] made operations "easier

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[2] The zero pair method is based on the principle that opposites sum to zero. By combining an equal number of red and black chips, students learn that such collections of chips will always add to zero. Therefore, students can add zero pairs, when necessary, to problems in order to help complete particular operations.

to demonstrate, explain and understand, countering to some extent a criticism of the model due to the difficulty that some students find with such, according to some critics, 'unnatural' processes" (p. 8). Liebeck (1990) reported on students using counters to keep score in a card game in which cards instructed students to add either *scores* (black counters representing positive numbers) or *forfeits* (red counters representing negative numbers). Her goal was to see how students would understand different representations of the same number, e.g., 3 = 4 + -1, and different representations of manipulations with the counters, e.g., 3 – -1 = 3 + 1. Liebeck found the students who were taught using counters were able to construct their own strategies for operations on negative numbers and to reproduce, apply, and extend their knowledge on assessments.

As the above section illustrates, there have been many attempts to use manipulatives for instruction of integers and integer operations in an effort to alleviate difficulties students encounter as they learn about negative numbers and to help build deeper understandings of underlying concepts. But while there has been literature discussing manipulatives and how to use them with students, less attention has been paid to how the *teachers themselves* view the manipulatives and the instructional challenges that are inherent in their use. In particular, with so many manipulatives to choose from, teachers are required to know a great deal about the affordances and constraints of different types (Vig et al., 2014). As the landscape of manipulative use becomes more complex, mathematics educators must take a step back to look more closely at *teachers'* use and understanding of manipulatives. The study described here seeks to make a contribution to this area by considering carefully teachers' knowledge of and use of manipulatives for teaching integer operations.

Method

Through a series of after-school meetings with two seventh-grade mathematics teachers, the goal of this work was to help teachers plan for and understand the implementation of the chip model for integer operations. The research question guiding this work is as follows: How does a seventh-grade teaching team understand and implement the chip model for integer subtraction? Below I describe the larger study within which this work is situated, the research site along with the nature of the mathematics curriculum, two of the teachers who participated in this study, and the data sources and analysis for this study.

Context

This work is situated within the context of a larger study, which was designed to add to the research base on effective teaching strategies that incorporate higher-order thinking skills for all learners. I used a professional development strategy incorporating the cyclical process of teaching (Loucks-Horsley, Hewson, Love, & Stiles, 1998; Roth McDuffie & Mather, 2006; Silver, Clark, Ghousseini, Charalambous, & Sealy, 2007; Smith, 2001), which is used to provide teachers with practice-based experiences from which they can learn about instructional strategies and student thinking. The particular cyclical process in this work is called a *reflective teaching cycle* ([RTC], Smith, 2001) and consists of three stages: planning, teaching or acting, and reflecting. In planning, teachers decide what to teach and spend time understanding the mathematics in the task, their students' prior knowledge, their mathematical goals, and how they can achieve them. After planning, teachers enact their idea by implementing their plans in the classroom. During the *teaching* or *acting* phase, teachers make decisions about how to engage students in learning and what changes to make in their pedagogy, if any. After teaching, teachers *reflect* on the type of thinking students engaged in during the lesson and how deeply they grappled with the mathematics. Teachers consider what students did and said to help them gain access to students' understanding of the central mathematical ideas of the lesson to aid in this reflection.

The RTC is effective when used in a one-on-one setting with a teacher and teacher educator (Roth, Mather, & Reynolds, 2004) or in large professional development workshops (Silver et al., 2007). For the larger study, I extended this literature base by engaging a team of two seventh-grade mathematics teachers in a series of RTCs and examining how this series of cycles influenced teachers' selection and implementation of tasks that had the potential to facilitate higher-order thinking. During the course of the larger study, the teachers worked through two units of instruction. The work presented here focuses on cycles from the unit on rational numbers, specifically those that revolved around integer addition and subtraction.

Research Site

The study was conducted in an urban school district in the southeastern area of the United States. At the time of the larger study, the district served over 12,000 students in 14 elementary, 4 middle, and 2 high schools. Of these institutions, nearly 32% did not meet adequate yearly progress (AYP)[3] as described in the No Child Left Behind (NCLB) legislation for the 2008–2009 school year, including three of the four middle schools in this study. The student population was approximately 53% African American, 21% Hispanic, 20% White, 2% Asian/Pacific Islander, and 4% multiracial, with close to 70% of its students listed as economically disadvantaged.

The study took place in Helix Middle School (pseudonym), which enrolled close to 500 students. The school's demographics were slightly different from the district

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[3] Adequate yearly progress is a measure by which schools are held accountable for student performance under the 2001 No Child Left Behind (NCLB) federal law. NCLB was the version of the Elementary and Secondary Education Act in the United States from 2002 to 2015. The current law is the Every Student Succeeds Act.

with a population of approximately 58% African American, 32% Hispanic, 7% White, and over 90% economically disadvantaged. The school was on its fourth principal in 4 years and had not met AYP for the previous 6 years.

At Helix, each teacher instructed five classes a day and was allotted one period for planning. Two of the five planning periods each week were allotted for content planning—one day for content teachers to plan and the other day for them to share and modify plans with special education teachers. All mathematics teachers in the district used the *Connected Mathematics 2* (CMP2) textbook as their main resource. In the present study, I focus on the seventh-grade instructional unit, *Accentuate the Negative*, focused on positive and negative rational numbers. The goals for this unit include using appropriate notation to indicate positive and negative numbers, understanding the relations between a number and its additive inverse, and developing algorithms for adding, subtracting, multiplying, and dividing positive and negative numbers. In the early part of the unit, students are encouraged to explore and use two models to represent addition and subtraction of integers: the number line and the chip model. The chip model uses both physical and drawn representations to provide students with experiences manipulating integers as they learn how to perform operations on integers.

According to the instructor materials for this unit, the objective in using the chip model for integer subtraction within that curricular material is to learn how to flexibly rename integers using different combinations of positive and negative chips. For example, when posed with the problem -4 – 2, the CMP2 suggests students rename -4 as −6 + 2 in order to have positive chips from which to subtract 2 (i.e., -4 – 2 = (-6 + 2) – 2 = −6 + (2 – 2) = -6 + 0 = -6). See Fig. 10.1, where each black chip represents positive one and each red chip represents negative one.

This general approach of renaming integers in order to anticipate upcoming operations is quite common in CMP2 and in the use of chip models generally. In the present problem of -4 – 2, note that -4 can be renamed in many ways, such as -5 + 1, -6 + 2, -7 + 3, etc. However, cognizant of the impending need to subtract 2, the renaming of -4 as -6 + 2 could be viewed as optimal, as one that has the two black chips available to subtract. This use of the chip model requires a *particular* renaming that indicates awareness of the specific nature of the upcoming computations. As a result, how one renames -4 is different for the problem -4 – 2 versus the problem -4 – 3; in the former, -4 is renamed as -6 + 2, while in the latter, -4 is renamed as -7 + 3.

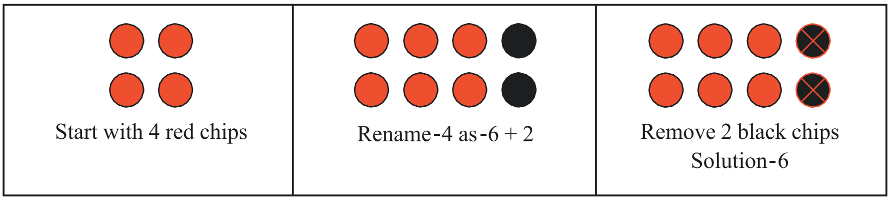

Fig. 10.1 Chip model for -4 – 2.

Participants

The two seventh-grade mathematics teachers, Clark and Tess, both began teaching at Helix the same year. In their first year, both were assigned to the eighth grade and moved together to the seventh grade in their second year. During this study, Clark and Tess were in their second year teaching seventh grade.

Clark is an African American man in his 50s who had been teaching for 19 years at both elementary and middle school levels. Tess is a White woman in her 40s coming from an educational psychology background and who was in her third year of teaching. Since Clark and Tess had been working together in the same grade level for 3 years, they had developed a good working relationship and enjoyed talking with each other about teaching. They were eager to engage in professional conversations because they were still learning about the seventh-grade curriculum and were thus excited to participate in the study.

Data Sources

I collected data during each RTC. As stated above, each cycle included planning and reflection meetings with the teachers. During these meetings, we collectively planned for instruction and reflected on the teachers' practice. Clark and Tess had the opportunity to critically examine their own and each other's practice, challenge each other's thinking, give support, and provide feedback on pedagogical strategies and classroom management. I acted as a facilitator and participant observer by prompting the teachers with questions to better understand their reasoning for selecting or implementing tasks in particular ways, suggesting different pedagogical strategies, and considering how the teachers' actions might affect student learning. Transcripts and notes from the planning and reflection meetings helped me address the research questions by allowing me to investigate the influence of our conversations on the teachers' selection and implementation of mathematical tasks. My notes also helped me record any mathematical ideas that were discussed through drawn representations or symbolic calculations. During the teaching phase of each cycle, I observed both teachers independently implementing the planned lesson. These observations provided me with information about how the teachers introduced problematic aspects of the tasks and used questioning during instruction. This information allowed me to cater future cycles to the needs of the teachers and gave me the opportunity to observe the influence the cycles were having on the teachers' decisions and pedagogical moves.

In all, we engaged in seven cycles from September to November 2009, which spanned two instructional units (see Table 10.1).

For this chapter, I report on three of the four cycles (RTCs 4, 5, 6) that occurred during the implementation of the *Accentuate the Negative* unit. During these meetings, the teachers discussed lessons concerning the addition and subtraction of

Table 10.1 Reflective teaching cycles

RTC	Unit	Planning	Observation	Reflection
1	Variables and patterns	Wednesday, 9/16	Thursday, 9/17	Thursday, 9/17
2	Variables and patterns	Tuesday, 9/22	Wednesday, 9/23	Thursday, 9/24
3	Variables and patterns	Tuesday, 9/29	Thursday, 10/1	Thursday, 10/1
4	Variables and patterns Accentuate the negative	Tuesday, 10/13	Thursday, 10/15	Tuesday, 10/20
5	Accentuate the negative	Tuesday-Wednesday, 10/20–21	Friday, 10/23	Tuesday, 10/27
6	Accentuate the negative	Wednesday, 10/28	Thursday, 10/29	Thursday, 10/29
7	Accentuate the negative	Wednesday, 11/4	Thursday, 11/5	Friday, 11/6

integers, including the use of a discrete chip model to support students' meaning making of these operations. Meetings generally occurred after school and lasted 60–90 min. In this chapter, I focus specifically on how the seventh-grade teaching team understood and implemented a discrete model for integer subtraction.

Data Analysis

I analyzed the data using thematic analysis (Schwandt, 2007) in order to code sections of the transcripts according to emerging themes. I began my analysis by reading through all transcripts from the planning and reflection meetings and divided the text into *episodes*. An episode is a section of text in which the conversation was about a particular topic, and each episode would end when the focus of the talk changed. For example, if the teachers were discussing what tasks they would be using and then began to consider the mathematical goals of the task, I would create two episodes: one for the task they would be using and one for the discussion about the mathematical goals. During this process, I took notes about each episode that reflected general observations of the conversation as well as my initial attempt to categorize the episodes based on the larger study's research questions. This open coding allowed me to develop main themes that became organizational categories for my data (Emerson, Fretz, & Shaw, 1995; Maxwell, 2005). Because the larger study focused on the teachers' facilitation or hindrance of higher-order thinking, my initial analysis concentrated on how the episodes were related to higher-order thinking. I developed substantive categories (Maxwell, 2005) to describe the nature of the episodes to help me develop a general theory of what was happening during the reflective teaching cycles.

Following the organizational and substantive categorization of the data, I reexamined the data by creating documents that included one planning meeting transcription, observation notes from each teacher's class, and one reflection meeting transcription for each cycle. I read the documents together to get a sense of the events that transpired in each cycle and wrote narratives to tell the story of each individual cycle. During this analysis, I found the series of conversations the teachers and I had during the *Accentuate the Negative* instructional unit about using a chip model to subtract integers particularly interesting, in that the teachers' selection and implementation of tasks was different for this unit than previous ones. I thus decided to further investigate the ways in which the teachers and I discussed this chip model to explore the ways the teachers approached this content and how their understanding and use of the chip model impacted the nature of the opportunities their students had to learn about operations with integers. The goal of this additional analysis was to understand each of the possible methods of implementation and the mathematics behind each method.

Results

As noted above, Clark and Tess were teaching a seventh-grade unit of CMP2 entitled *Accentuate the Negative*. Because of the timing of the cycles, the teachers and I did not plan for every lesson in the unit but did discuss many of the issues that arose in their classrooms around the content of the unit. Table 10.2 outlines the nature of the conversations during the meetings and the ways in which the conversations focused on integers and integer subtraction.

In the sections to follow, I discuss the ways Clark and Tess understood and implemented the chip model with subtraction. These descriptions are based on a holistic analysis of the three cycles summarized above. That is to say, while the original units of analysis for the larger study were the cycles themselves, for the purpose of this analysis, I use these three cycles together and the discussions in the meetings to describe how the teachers understood and implemented the chip model. My own observation notes as well as the teachers' self-reported actions in the classroom also support this implementation analysis.

Tess

Tess liked the instructor material's description of how to use the chip model; particularly how the investigations helped her students formulate understandings of negative numbers and negative number operations. She felt that this approach would be successful for her students, and as a result, her instruction closely aligned with the curriculum's teaching guide. Specifically, Tess highlighted how to add the optimal number of zero pairs in integer subtraction problems in her classes, which matched the curriculum's suggested use of chips. She felt this particular strategy would help her students develop a deeper understanding of integers and become comfortable with manipulating them before formalizing algorithms for addition and subtraction. For example, as Tess reflected on the benefit of the chip model, she talked about the feeling of quantity. "So seeing all those reds means the feeling of

Table 10.2 Summary of planning and reflection meetings in RTCs 4, 5, and 6

RTC	Planning	Reflection
4	Plan for initial lesson of unit. Objectives: understand opposites on the number line; the existence of negative numbers	Attend to how the task of finding the difference between scores was confusing for the students because of the word "difference"
		Start to think about planning for next investigations—specifically a discrete model with X's and O's for positive and negative numbers. Start to discuss zero pairs
		Clark reports on situation in an unobserved class where he explained to the students how to subtract using the chip model (discussed below as Clark's method)
5	Revisit discussion from last reflection meeting—using chip model for integer subtraction	Press teachers about their use of subtraction in the lesson that was focused on addition. Tess, in particular, explains her motivation is for her students to "move flexibly" between 8 plus negative 12 and 8 minus 12
	Plan for next lesson in unit. Objectives: formulate algorithms for addition of positive and negative numbers, have students think about the meaning of the operations, and use different representations (discrete model and number line)	
6	Reflect on classes not observed and situations in which integer subtraction arose. Teachers recount pedagogical strategies used, including using the chip model (Clark) and contexts (Tess) to understand subtraction	Talk about the "turn turn" and "change change" ideas that came up in Tess's class as well as the mention of "two negatives makes a positive"
	Plan different lessons for teachers. Tess's lesson will include a review of integer addition and subtraction along with an assessment	Teachers attribute students' difficulties with chip model to their problems with understanding the use of zero pairs

negative is a stronger negative. The feeling of quantity. You don't necessarily get that with the number line" (Tess, reflection meeting, RTC 5).

Tess wanted to use the investigations focused on addition and subtraction with the chip model to help her students build an intuitive understanding of concepts. She argued that the students were not supposed to be "learning how to subtract" but rather "learning how to do that [subtraction] on a chipboard." However, as Tess used the model to help students perform integer subtraction, she provided steps for the students to follow, which allowed them to determine the optimal number of zero pairs needed for renaming integers. Thus, in Tess's instruction, she did not allow students to rename integers in any way that would work but rather led them toward a particular renaming, which seemed to work for her students. Tess described her classes:

> My kids understand it the way the book has it and the way I've been teaching it. I've got some pretty low kids. Because it seems like it's more directly what it's saying. Like, an operation is an operation. You're adding or subtracting. So you start with your 5 and then

> you do what you have to do to it. So, you don't do another step first. You either see, "Can I take away 3 [i.e., 5 – 3]? Yeah. I could just take away 3 or, if I can't take away 7 [i.e., 5 – 7] and in that case I'm gonna have to add 2 zero pairs or, yeah. Because I can take away 5 and then 2 positives and then I'd end up with negative 2. (Tess, planning meeting RTC 5)

In her arguments for this method, she said that it would help students develop number sense and be better equipped to develop algorithms for integer subtraction. Tess's instruction provided a scripted way to use the chip model, but she did not seem to explicitly draw connections between the use of the model and the mathematical concepts. As discussed above, such scripted manipulation may help students develop understandings of mathematical concepts (Carbonneau et al., 2013), but explicit connections are necessary for students to develop a sense of the interconnectedness of mathematical ideas (Clements, 1999). Therefore, it is unclear whether Tess's students built links between their formal and informal understandings of integer subtraction (McNeil & Jarvin, 2007) and whether Tess was able to strike the appropriate balance between too much and too little direction when students use manipulatives (Glenberg, Brown, & Levin, 2007; Marley, Levin, & Glenberg, 2007; Marley, Szabo, Levin, & Glenberg, 2011).

Tess's understanding of the function of minus sign as a unary, binary, or symmetric may have also impacted her decisions to script the use of the chip model and not draw explicit connections between the model and the mathematics. In our planning meeting during RTC 5, in which we discussed the investigation where students would think about the meaning of the operations and use different representations (discrete model and number line), the teachers discussed the meaning of *opposite*. I suggested prompting students to create various number sentences equaling the same value using "opposite signs." Clark asked if the negative sign could be called the opposite sign, to which I replied, "It's just opposite numbers because if I start with a negative, then opposite's positive. So, it's not the opposite sign" (Murray, planning meeting, RTC 5). Tess then recalled talking to students about addition and subtraction being opposites and described that students typically had trouble with the ideas of opposite and inverse. This language is indicative of the confusion of the function of the minus sign in different contexts. Clark and I were discussing the symmetric function of the minus sign, representing the action of taking the opposite of a number, while Tess was talking about the binary function of the minus sign as an operational sign (Vlassis, 2008). This confusion continued to surface in the reflection meeting when Tess mentioned the importance of her students' ability "to see the flexibility between eight plus negative 12 of eight minus 12" (Tess, reflection meeting, RTC 5).

During the sixth cycle, the multidimensionality of the minus sign continued to impact Tess's thinking and instruction. For this cycle, Tess decided to spend part of the time reviewing integer addition and subtraction before giving a short assessment. During the observation, students began to talk about rules they were using to decide on solution strategies for basic problems. Ideas such as "turn turn" or "change change" surfaced as students struggled with understanding how to subtract a negative number. At one point, one student mentioned, "Taking away negative makes positive," while another said, "Two negatives make a positive." To combat misuse of these ideas, Tess told her students to instead consider the number sentences 5–3 and 5 – -3 on the number line because the chips were “too hard.” She then provided the students with a context (not recorded in notes) for why -7 – 8 was different from -7 + -8. Even so, 10 min after this discussion, Tess told her students that they could change 10 – 2 to 10 + -2 without providing a rational.

The problems Tess’s students encountered with the use of negative numbers and the chip model seemed to be linked to the use of the minus symbol and the inconsistent language used to explain what was happening in different problems. In order for students to make sense of integers and operations with negative numbers, students should be flexible in their understanding of the different functions of the minus sign (Vlassis, 2008), something not apparent in these lessons or Tess’s instruction.

Clark

As Clark came to understand integer subtraction using the chip model, he had concerns about the model’s ability to help his students with the operation. As a result of Clark’s experiences with the unit in his first year of teaching the seventh grade, during this second year, Clark expressed doubts that his students would be able to select the *particular* renaming for a given problem that is inherent to the suggested use of chips. Clark recalled a situation from the previous year in which another mathematics educator, Dr. White, had helped him understand the chips for the first time.

> Yeah, because see remember last year when we taught this we kept talking about doing the opposite? Remember that? And I remember, and I guess I’m so stuck on the, the pairs, the zero pairs because I learned that from Dr. White and it’s, it’s like, “This is easy. We can do it this way!” You know? And that was the biggest difference. Because the kids were confused last year because we kept saying, “Adding the opposite. Adding the opposite.” It was always adding the opposite. … Add the opposite. … Inverse, that kind of a thing. And that got the kids confused. [Clark, planning meeting, RTC 5]

Through this experience, Clark developed his own way of understanding the use of the chips to teach his class in the current year. Dr. White had helped Clark see how one could think about a simple subtraction problem, such as 2 – 1, as 2 subtracts a *positive* 1. Therefore, in using the chips and zero pairs, “if I got 2 minus 1 and I got the zero pairs there, I’ll take one positive away, all right, I got a negative and I still got 2 positives over here. I take them off and I got one left. And that’s, that stuck with me because, man, this is an easier way to teach it” (Clark, planning meeting, RTC 5).

We can see from this comment that Clark was trying to get in front of the problem students ran into in figuring out the renaming of integers in order to use the chips. That is, from Clark’s perspective, his students were, and would, be confused about why the curriculum materials suggested renaming for 2 – 5 (optimally renaming the 2 as -3 + 5) versus 5 – 2, which did not require renaming. In other words, Clark felt that his students would not see why some problems required renaming when using the chip model while others did not. Therefore, as an alternative to the curriculum's prescribed way of using the chip model, Clark implemented the chip model in a way that he said made sense and worked every time and that he could explain. Clark wanted a method where students would perform the same steps for each problem—instead of sometimes renaming and sometimes not. In Clark's words, while his method might involve an extra step, "it's a step that they'll [the students] do consistently and they're more apt to get the problem right because they're doing the same thing over and over again. Then they'll start remembering" (Clark, reflection meeting RTC 4). Below I describe the method Clark taught his students to use whenever they came in contact with an integer subtraction problem.

Clark described what he told his students in class when he taught them his method:

> [W]henever I'm doing subtraction, I'm thinking any number after the subtraction sign is a pair, what we call a zero pair, a set of zero pairs… So, let's illustrate that, and let's see. Start out with five pluses. …[and] because we have two over here, we're going to set up two sets of pairs—one positive, one negative. …But what does the subtraction tell me? Subtract the two what? The two positives. Now, I've gotten rid of my two positives. …So what do I do next? I create my pairs. I'm getting rid of my pairs. And what's my answer? I got three positives left. (Clark, reflection meeting, RTC 4)

Clark's method deviates from the recommended way of using the chip model in two ways. First, it is less dependent on problem conditions, given that the same steps can be applied for all subtraction problems, with no concern for the optimal renaming. Second, Clark's method is less efficient for some problems. I elaborate on these differences below by considering how Clark's method and the recommended method for using the chip model differ for the problems 5 – 2 and 5 – -2.

For 5 – 2, the method advocated by the curriculum would suggest that students take the 5 positive chips, rewrite 5 as the sum of 3 and 2 (e.g., separate the 5 chips into groups of 3 and 2), and then remove the 2 chips to represent subtraction. Mathematically, this way of using the chips can be represented as: 5 – 2 = (3 + 2) -2 = 3 + (2 – 2) = 3 + 0 = 3. For the problem 5 – -2, instead of renaming 5 as 3 + 2, the optimal renaming is 7 + -2. By doing this, two chips can be removed to represent subtraction: 5 – -2 = (7 + -2) – -2 = 7 + (-2 – -2) = 7 + 0 = 7. See Fig. 10.2 for an illustration of Clark's method versus the CMP2 method for this problem.

As noted above, to use chips to model integer subtraction as the curriculum advocates, it is helpful to determine how to optimally rename the minuend for each individual problem. As a result, note that the task of renaming 5 as 7 + -2 is nontrivial for students; thus, it may seem necessary for students to already have some sense of the relative magnitudes of the numbers in order to successfully model integer subtraction using chips. Clark's method eliminates the students' need to consider the optimal number of zero pairs students should add in order to subtract two numbers. This results in a series of steps that are identical for 5 – 2 and for 5 – -2 (see Fig. 10.3). For all subtraction problems, one only needs to create zero pairs based on the magnitude of the subtrahend (e.g., 2, in both 5 – 2 and 5 – -2) and then proceed with the subtraction.

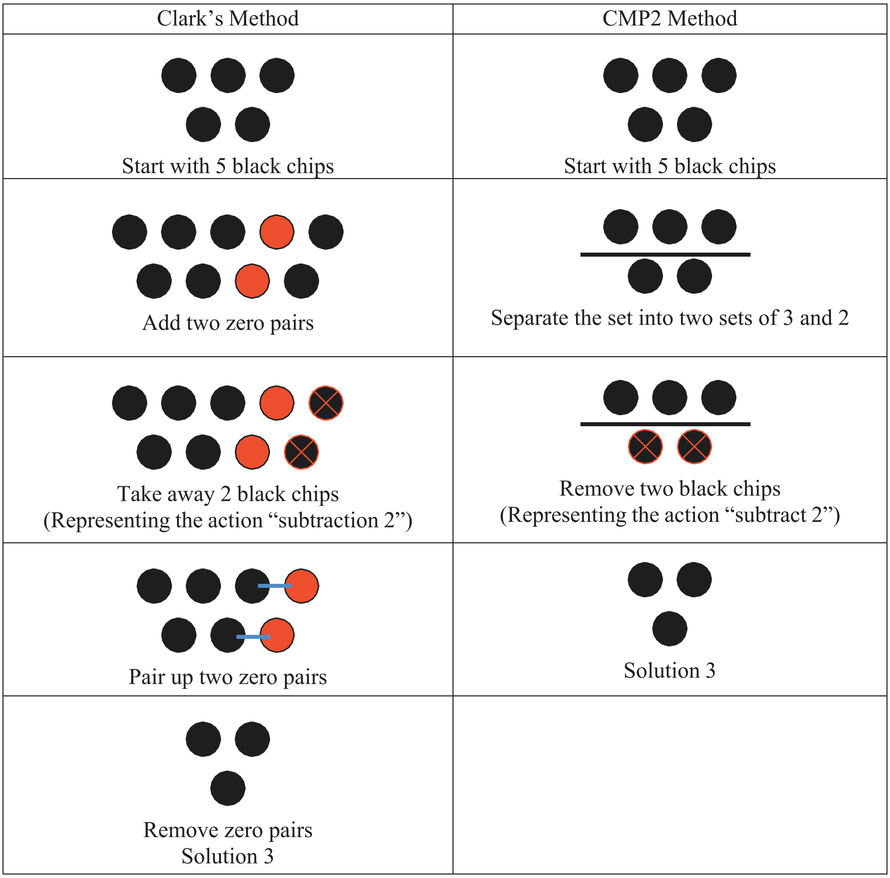

Fig. 10.2 Clark's method of integer subtraction versus CMP2 method for 5 – 2

It is clear that Clark's method is generalizable but arguably not maximally efficient. For some problems, students perform unnecessary actions by adding zero pairs to the problem where they are not needed. For 5 – -2, Clark's method is as efficient as the curriculum recommended method. But for 5 – 2, adding two zero pairs is unnecessary and introduces extra steps in the modeling process. For other problems, such as 2 – 5, Clark's method has students adding more zero pairs than the CMP2 curriculum would advocate (e.g., five instead of three), which may be considered inefficient or unnecessary. Thus, while in Clark's method, zero pairs are used in all problems, CMP2 would say that for some problems, zero pairs are unnecessary, and in others, students should add the minimum number necessary to complete a given operation.

Clark felt that the benefits of having a generalizable method outweighed these inefficiencies. Clark liked that his students were able to consistently compute correct answers without having to worry about "changing signs and all this other stuff."

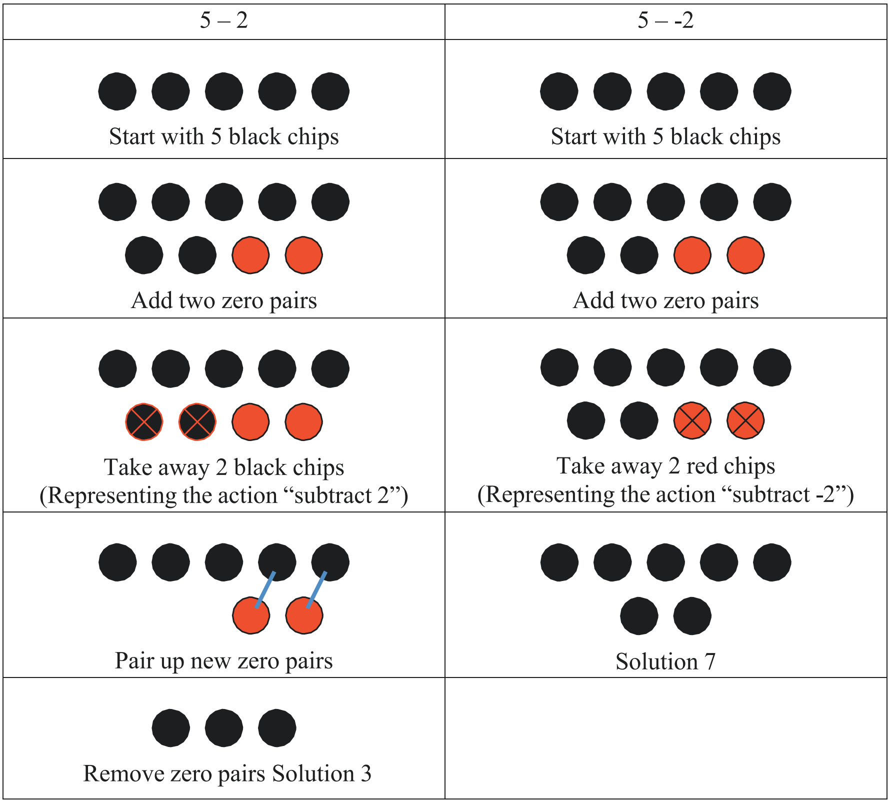

Fig. 10.3 Clark's method for 5 – 2 and 5 – -2

He felt that his method made it easier for his students to use the model for subtraction because it was "consistent" and was "just as good" as how the curriculum described the use of the model. In Clark's words:

> There are two of them because each number's gonna represent, I'm gonna represent these pairs for each one of these numbers so I can show you how to do this. …and it makes it easier for the kids to see it, and it's more consistent…because the kids see, "Okay, it says subtract a negative 2 from here so I'm gonna take these two negatives and now I got nothing but positives left. If it say subtract a positive 2 I take the two positives out and then I got nothing left…" (Clark, reflection meeting, RTC 4)

According to CMP2, the intended goal of the model is for students to understand how integers can be represented using different combinations of chips. In understanding these representations, students would determine which renaming was most useful in particular subtraction problems. In essence, CMP2 intends for students to consider *when* and *why* zero pairs are needed in order to develop a sense of integer subtraction. Clark's method removed this decision-making process. This modification could be seen as reducing the level of thinking needed to engage in the task or (on the other hand) as an asset because it made the model more generalizable since it could be used for all problems algorithmically and successfully. Therefore, just as with Tess's scripted instruction, similar questions arise with respect to how Clark's instruction may not have provided students with a way to make meaningful connections between the manipulative and the mathematical structure (Clements, 1999).

In addition to the use of zero pairs and renaming integers, the issue of the minus sign of a binary or symmetric operator is also a factor in Clark's use of the chips for integer subtraction. This issue surfaced when Clark first introduced his method to us in the reflection meeting in cycle 4. Both Tess and I saw how Clark's method was actually changing the nature of the problem. In particular, Tess claimed that Clark's method changed the problem from subtraction (5 – 2) to addition (5 + -2). Clark was not considering the minuend and subtrahend. Instead, he started with the minuend and added zero to get the expression 5 + (-2 + +2). The next step was to remove +2 (which formed a zero pair with the original minuend of 2) to end up with 5 + -2. In this way, Clark was changing the function of the minus sign from binary to symmetric to help make the use of the chips easier for his students. In addition, because Clark was subtracting the +2 from the zero pair rather than the original minuend, Tess believed that he was not highlighting the correct subtraction. Tess felt that these two issues fundamentally changed the mathematical goals of the investigation, which were to help students understand opposites on the number line, understand the existence of negative numbers, and have students think about the meaning of addition and subtraction using the number line and the chip model.

Discussion

As apparent from the preceding sections, Clark and Tess's understanding and implementation of the chip model for subtraction were different in some ways and similar in others. The differences lay in the way in which the teachers wanted their students to use zero pairs (optimal number of zero pairs versus adding the magnitude of the subtrahend) and how they wanted their students to consider the purpose of and use of the chips (exploring the nature of integers and subtraction versus performing computation to get the correct answer). The similarities lay in the way in which the teachers actually had their students engage in activities and their own understanding of the different functions of the minus sign. In particular, both teachers provided steps on how to use the chip model, and both teachers moved between the binary and symmetric functions of the minus sign without explicit attention to this difference.

Clark's method was fundamentally different than the intended use of the model. Tess understood this difference as relating to how students were instructed to use the chips in the curriculum—that adding the optimal number of zero pairs only when necessary was integral to the curriculum's suggested use of chips. Tess wanted to implement the intended curriculum and believed that by doing so she would enable her students to gain a deeper understanding of the operation of subtraction. According to her, deviating from this would only confuse the students and detract from the mathematical goals of the lesson. Tess felt that Clark's method was essentially asking students to memorize a process with the model that might lead to students misusing or misapplying the method. Tess wanted to make sure she followed the pacing of the curriculum and instead wanted to use the current investigation to help her students build an intuitive understanding of operations and the chip model. In Tess's words:

> Clark's putting a rule in there instead of teaching with what the numbers are asking you to do, which is what I understood from reading the book, is it's intuitively asking you to start here and then what you're gonna do. But now I have to think of a rule. Well, is this addition or subtraction? And what am I gonna do in either case? And that's not what, with my kids, I'm not presenting it as we're learning subtraction and addition yet because that's the next chapter. And you don't have to do that.

As the two teachers discussed their difference of opinion about how to use the chips to model subtraction, at no point did they try to connect the steps their students were using to the mathematical structure of the number sentences, and it did not seem that this structure was transparent to the teachers. The apparent lack of seeing the structure of the number sentences could be heard in the teachers' discussion of opposite and inverse, as well as their repeated references equating subtraction to adding the opposite. We also never discussed what the zero pairs might mean to the students or how their use might help the students deepen their understanding of integer subtraction. Therefore, it is unclear if the teachers succeeded in building connections between the manipulative and the operation, an important component of successful teaching with manipulatives (Clements, 1999; Moyer, 2001). Moreover, it is unclear whether the teachers understood if the chip model was being used in conjunction with the mathematical structure of the numerical expressions, another necessary factor (Carbonneau et al., 2013).

As the discussion of Clark's method continued throughout cycles 4, 5, and 6, we collectively examined his method, considered what it meant mathematically, and debated how his pedagogical strategy might be affecting student learning. Throughout the conversations, the arguments from both sides never really changed. Clark continued to defend his choice with evidence of his students' success and ability to use the model for computations and generalizations about the operation. For example, during the sixth reflection meeting, Clark shared a story from his classroom in which he responded to a student computing the problem, -5 – -7. At first, the student computed the answer as -12 using the number line. Clark encouraged the student to try the problem again using the chip model. As the student did so, Clark had him describe his steps.

Clark: So he took away seven negatives. I said, "Okay, so now what?" He said, "Now I got to take these pairs and pair them together." And he said, "Oh, I got a positive two." I said, "So what happened with the number line and the chips? They don't coincide, so something's wrong. What's going on?" I'm trying to make him look at it. And he said, "Um … I'm adding." "Oh," he said, "I'm adding here." And I said, "Huh?"

Eileen: How did he know it was adding?

Clark: Because he said, "The only way this could be a positive two, I had to add. Something had to be positive somewhere." And so he looked at it and he said, "Well, this had to be positive now, so that's adding."

The student calculated the answer using the chip model and saw that it was not the same as his previous answer. He decided the chip method was correct without considering why his answers were inconsistent. Clark asked the student to think about what had happened by evaluating why the statements were contradictory. The student instead made a generalization about subtracting a negative number, which is what Clark wanted his students to do; that is, he wanted them to use the method to draw conclusions about computations. The question remains, however, what did the student actually understand? Did the student understand the meaning of the zero pairs he was using? Did the student understand the difference in the function of the minus sign? Why did the student accept the chip model answer above the number line calculation? Based on this reported classroom event, it is unclear whether this concrete approach to negative numbers actually helped Clark's student internalize his understanding about integer subtraction, and unlike Flores (2008) or Hayes and Stacey (1999), I cannot be sure if Clark's instruction focused on the zero pair method made subtraction easier to explain and understand for his student.

Despite this evidence, Tess continued to argue that Clark's method was not successful because it did not follow the intended curriculum, changed the problem, and altered the mathematical goals of the lesson. The conversation below is representative of both teachers' thinking.

Tess: My kids understand it the way the book has it and the way I've been teaching it. I've got some pretty low kids. Because it seems like it's more directly what it's saying. Like, an operation is an operation. You're adding or subtracting. So you start with your 5 and then you do what you have to do to it. So, you don't do another step first. You either see "Can I take away 3? Yeah. I could just take away 3 or" If I can't take away 7 and in that case I'm gonna have to add 2 zero pairs or, yeah. Because I can take away 5 and then 2 positives and then I'd end up with negative 2. But I don't see why I would add 7 zero pairs and then…Does that make sense? Are you following me?

Clark: I'm hearing you all and I'm understand what you're saying. I understand exactly what you're saying.

Tess: Okay. So I don't need to write it.

Clark: But I'm just looking at it…some consistency because there are gonna be some cases where you gotta use the zero pairs and what I was trying to do was eliminate the guess work and saying, "Okay, if I did this, this way every time, and it does work every time, then I can see my patterns and then I can generalize what is really going on here."

Eileen: I guess I felt like at this point in the unit that we did want the guesswork. That we did want them troubling with 'what do these numbers mean and how can I deal with these quantities?' I felt like we did want them to puzzle with these ideas and then later, you know, [Clark: Come back to it.] in 2 and 3, then it becomes "All right, we've struggled with this. We've puzzled this. Now let's go and look for patterns we can use to actually perform the operations."

Tess: That's what I was gonna say, is that the chip thing falls away, right? Once you learn how to do it you don't need the chips ever again, really ever again.

When connecting these arguments to the teachers' understanding of the manipulative and its use in this context, it seems that Clark's understanding focused on allowing students to learn and implement a routine, which did not vary based on the particular problems and, when done correctly, resulted in the right answer. His goal was to provide students with a way to experience success without considering how to rename integers, use zero pairs, or understand the function of the minus sign. Alternatively, Tess understood the model to be a stepping stone toward understanding the operation and eventually establishing rules or algorithms for integer operations. The goal, to her, was to develop knowledge beyond simply being able to calculate using the chips. However, as Tess introduced the model and helped students use it to perform integer subtraction, she also provided steps for the students to follow and also did not make clear the function of the minus sign and how it may change as students considered problems in different ways (e.g., 10 – 2 versus 10 + -2). In essence, Tess appeared to proceduralize her method in a way that was not dissimilar to how Clark proceduralized his method. The steps that Tess instructed her students to follow involved determining the optimal number of zero pairs needed for renaming integers. In her arguments for this method, she said that it would help students develop number sense and be better equipped to develop algorithms for integer subtraction. But as I observed her classes, it was unclear if her students were engaging in thinking about what the renaming was, why they were doing it, or how they were to come up with the answers—or instead whether they were merely following the steps in the method that Tess demonstrated and formalized for them. Therefore, the assumed benefit of students' thinking about the optimal renaming of the integers may have been lost. Furthermore, Tess's procedure for her method was more complicated and difficult to implement than Clark's, given that Tess's procedure necessarily included a conditional step of when and how to rename an integer in a problem. Finally, neither teacher addressed the function of the minus sign and how they were (or were not) changing it in different number sentences.

Conclusion

As the above arguments suggest, the mathematical goals for the use of chip models need to be better understood. Tess's use of her chip model appears to be quite common; my experience in this study, as well as in other professional development work, has made me aware of how many teachers see the optimal renaming feature as integral to proper use of the chip model. Teachers think that students can gain deeper understanding of the nature of integers and integer operations by determining the optimal number of zero pairs needed to solve a given problem. This perception is also explicitly supported and legitimized by CMP2.

However, Clark's modification to this method sheds light on how the optimal renaming feature of the model may not be its most important feature. Optimal renaming became, in Tess's class, merely a way that the use of the chip model turned into a procedure for computing the answers to subtraction problems. For Tess, there was one right way to rename the subtrahend, and the task was to determine this one correct way to use the model. As such, I began to wonder if Tess's use of the model was actually any better than Clark's—it may, in fact, be worse (because Tess's procedure was more difficult to implement than Clark's).

Through observations and conversations with Tess and Clark, I came to believe that the focus on the optimal renaming of the subtrahend is misguided, as doing so may not even encourage students to develop flexibility with integers (which in turn helps in developing understanding of subtraction). In order for this to happen, the model itself needs to be flexible in order for students to be able to reinvent the model and use it independently, adapt it to new situations, and relate the model to their own informal strategies (Van den Heuvel-Panhuizen, 2003).

The goal of the current analysis is not to evaluate the chip model but to raise questions about how teachers implement the chip model and understand its use in integer arithmetic. I use the case of Clark and Tess to question whether certain assumptions about the model's use, especially the optimal renaming aspect, are valid and interpreted by teachers. Moreover, my interactions with Clark and Tess suggest that there may be some taken-for-granted assumptions about models in general. Certain models seem to have found a home in the elementary curriculum, including chip models for integer subtraction, the number line positional model for integer multiplication and division, and the area model for binomial multiplication. Although the number of commonly used and accepted models has grown, it is also the case that the use of such models is based on limited research on teachers' implementation and understanding of models as well as the numerous ways that models can be interpreted and implemented in the classroom.

Among these many models teachers can use in curricula and standards, should teachers accept them as is or should teachers devote time to carefully considering the optimal use of each model? If teachers do not think about the models they are expected to use from a variety of viewpoints—mathematically appropriate and concrete versus abstract—teachers may not reflect in a way that challenges their taken-for-granted assumptions about the model or mathematics. In addition, there are still unanswered questions about models, their effectiveness, and the best ways to use them. For example, while there is agreement on the importance of models for building connections between students' informal understandings and formal mathematics (McNeil & Jarvin, 2007), it is not at all clear exactly how these connections are best made. Moreover, teachers must possess strong content knowledge in order to use models appropriately and well. Clark and Tess's content knowledge could have been one factor in their decisions to use the chips in particular ways. Teachers' content knowledge in general could be a factor in the tension teachers feel in using models and understanding them. In the end, teachers should be aware of the arguments for and against any model and its use to think critically about how and why they should use them in their own instruction. Clearly, we need more research into how teachers are engaging in model use and how they understand models. We also need to pay more attention to how teachers view models and the instructional challenges that are inherent in their use. As this research unfolds, the mathematics education community will be better situated to understand the instructional challenges that are inherent in teachers' need to be familiar with and competent with a growing number of new models.

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Part IV
Commentaries

Chapter 11
Commentary on Chapters 1 to 3: Using Meaningful Analogies to Reflect on and Make Sense of Integers

Arthur J. Baroody

Abstract Ginsburg (1977) observed that children typically develop surprisingly powerful informal (everyday) knowledge of mathematics and that mathematical learning difficulties often arise when formal instruction does not build on this existing knowledge. By using meaningful analogies teachers can help connect new formal instruction to students' existing informal knowledge and thus comprehend and learn it. The chapters by Bofferding et al., Wessman and Enzinger, and Bishop et al. vividly underscore these key points in the domain of integer addition and subtraction. These chapters richly illustrate how children can make sense of this domain even before formal instruction. How—in the absence of meaningful instruction (e.g., meaningful analogies)—their informal knowledge has key limitations. How playing games that embody a meaningful analogy can be help children construct a deeper understanding of integer addition and subtraction. In brief, relating integer addition and subtraction to a meaningful analogy has important implications for both assessing and promoting this critically important knowledge.

Given my training as a teacher and as an educational and developmental psychologist (particularly with the tuition of Herbert P. Ginsburg), my focus is on how we can make formal, largely written, mathematics meaningful to students. Ginsburg (1977) observed that an important source of mathematical learning difficulties is the gap between formal instruction and children's existing informal (everyday) knowledge of mathematics. More generally, if school instruction fails to relate new content to what already makes sense to children, they may well not comprehend the new instruction and may learn the new content by rote, in a partial fashion, or not all—none of which are good long-term pedagogical options.

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Consider, for example, the topic of integer addition and subtraction—a topic that is baffling for many students (and teachers). Although children's existing (whole-number) knowledge of addition and subtraction may support some similar aspects of integer addition and subtraction, it does not take into account how integer addition and subtraction are different from whole-number addition and subtraction (Glancy & Pettis, 2017). Thus, children's informal knowledge may not be helpful, or actually interfere, with their formal learning. To make matters worse, formal instruction on addition and subtraction of integers too often makes little or no attempt to relate the topic to children's informal knowledge but instead focuses on memorizing rules by rote. As a result, with only the rarest of exceptions, my former preservice teachers were either not prepared to help children understand the rules of integer addition or subtraction or had forgotten the less intuitive rules, namely, how to subtract negative integers (e.g., -5 – [-3] or -3 – [-5]) or integers with different signs (e.g., +5 – [-3] or [-3] – [+5]). As Bofferding, Aqazade, and Farmer (this volume) found with elementary-level students, many of my preservice teachers did not clearly distinguish between operation signs (e.g., the minus sign) and directional signs (the negative sign).

In effort to make integer addition and subtraction meaningful to my preservice teachers, I relied heavily on analogies such as everyday experiences of credit and debt or net worth (Baroody & Coslick, 1998). One particularly valuable aspect of such analogies is that they required a student to distinguish explicitly between operation signs (e.g., the minus sign) and directional signs (the negative sign). For instance, for +5 – (-3), the positive integer +5 represents a "credit," the negative integer -3 represents a "debit," and the minus sign (still) represents "take away" (the operation of subtraction). Another particularly valuable aspect of such analogies is that students can accurately operate on integers without rules and can, indeed, rediscover for themselves the rules for adding and subtracting integers. The value of rediscovering such rules for themselves involves students in (inductive and deductive) reasoning, can help students better appreciate the true nature of mathematics, and can be empowering for students. Importantly, the meaningful learning and memorization of the rules makes it more likely students will retain the rules and apply them flexibly and appropriately to new problems.

Bofferding, Aqazade, and Farmer: Chap. 1

As a moderate constructivist, I am particularly appreciative that Bofferding et al. quoted Davis (1996) on the value of play as a means of exploring new ideas (finding order or devising new meaning) in a supportive (at least somewhat bounded or structured) manner. As Bofferding et al. suggest, however, not all play situations are of equal pedagogical value (Baroody, Clements, & Sarama, in press). For example, Aze's (1989) location movement activity (a child moves on a number line as classmates indicate how much to add or subtract) can model the addition or subtraction of positive integers but not negative integers. Using blocks as an aid may not help with cases where more blocks need to be removed than are present (e.g., 3 take away 4). In contrast, Wessman-Enzinger and Mooney's (2014) counterbalance conceptual model, which may be supported by chips of two colors, can represent addition and subtraction of both positive and negative integers.

Relating a concrete model to a meaningful analogy can transform it into a powerful learning tool. For instance, a "car analogy" can transform the previously mentioned number line-based location movement model into one that can represent the addition and subtraction of both positive and negative integers. With this analogy, the first integer represents where a car (facing right) is on a number line. A plus sign indicates that the car continues to face right, whereas a minus sign indicates that the car turns around and faces left. If the second term is a positive integer, the car drives forward the magnitude of the integer; if a negative integer, the car drives in reverse the distance indicated.

Liebeck's (1990) model with two-color counters is related to an analogy of sorts—a black counter equals a score for a team, and a red counter indicates a (score) forfeit. For instance, +3 + (-1) = +2 translates into three scores plus a forfeit, and +3 – (-1) = +4 translates into three scores and the reversal of (taking away) a forfeit. The equation -5 – (-3) = -2 translates into five forfeits and three reversals, which would seem to require rather poor officiating, for a final score of two forfeits. The equation -1 – (-3) = +2 translates into the potentially difficult to imagine situation of one forfeit and reversing (taking away) three forfeits for a net score of two. From an informal change take-away view of subtraction, how can one start with 1 forfeit and then reverse more forfeits than are available? Nevertheless, Bofferding et al., referring to results from Williams, Linchevski, and Kutscher's (2008) study, noted children "were able to reason that removing a bead from one team corresponded to gaining a bead on the other team." However, this insight does not translate easily into formal symbolism. Removing a bead from one team, which would seem to suggest – (+1), corresponds to or equals (=) gaining a bead for the other team, which would seem to suggest + (+1) but – (+1) $\neq$ + (+1).

An analogy for black and red counters that is consistent with an informal change take-away view of subtraction and readily translates into formal equations is the "charged-particle analogy." For -3 – (-5), this analogy requires imagining starting with a box with three red counters or a net charge of -3. For the analogy to work with such a problem, a student must assume the box also contains an infinite but equal number of black and red counters that effectively neutralize each other. The minus sign indicates that counters need to be removed (taken away) from the box. (A plus sign indicates that counters need to be added to the box.) The -5 indicates that five red counters (negative charges) need to be removed from the box—three counters representing the net negative charge of -3 and two more red counters from the infinite pair of neutral charges. This now leaves two black chips (positive charges) unpaired with a negative charge for a new net charge of +2. For a similar analogy, see Glancy and Pettis' (2017) buoys and anchors model.

Bofferding et al.'s number path and integer arithmetic tasks have ecological validity because integers and integer addition or subtraction are too often introduced to children in school without real-world contexts (i.e., in a relatively sterile manner).

Consider, though, whether using analogies in the task instructions may have changed the results. What if the instructions of the number path task had been modified to include, say, "The 5 here indicates *5* degrees (or *5* feet above sea level)? Fill in the missing numbers on the thermometer (measuring stick) below." It seems possible that putting the task into a meaningful context (e.g., using the analogy of temperature or elevation) might have connected better with children's existing (informal) knowledge or prompted more playful and effective logical reasoning than a task with no context. Might it have significantly reduced the number of students who stopped after entering positive numbers or whole numbers, used repeating patterns, filled in the first 12 spaces with 0, entered a -0 as well as 0, or started the negative numbers with a number other than -1? Future research will tell.

Bofferding et al. indicated that meaningful analogies, such as temperature, elevation, or net worth, would be helpful with children who already use a negative number ray to help think about integer order and explore the continuum from most negative to most positive. Indeed, such analogies might be helpful with all the children in their sample, including the less developmentally advanced ones. For example, for the half that used only positive or whole numbers to fill in the number path, Bofferding et al. recommended: "Teachers could encourage students to consider what numbers go on the other side of zero and to think about how to notate them and to provide them with opportunities to think about numbers continuing indefinitely in both directions." Such efforts might benefit greatly by using familiar analogies such as temperature, elevation, or net worth. Such analogies might also benefit students who leveraged patterns or symmetry with comparing "patterns in increasing magnitudes from zero (e.g., 1, 2, 3 versus -1, -2, -3) and exploring how they are similar yet different in terms of how they are ordered (i.e., -3, -2, -1 versus 1, 2, 3)."

Meaningful analogies, such as credit-debit (net worth), may be especially helpful for students who otherwise assume subtraction is commutative (e.g., treat 1 – 4 as 4 – 1), ignore confusing symbolism, or treat negative numbers as zero. Yes, "acknowledgement of negatives as their own type of number (located separately from positives on a number path) is a key step," and a meaningful analogy such as credit-debit (net worth) might be invaluable in taking this step. Integers after all were invented because there are many circumstances where the whole numbers, which indicate magnitude only, are inadequate. Some situations, such as the business world, require knowing the direction of a number as well as its magnitude. For example, a credit of 1 million dollars is very different than a debit of a million dollars. Not coincidentally, integers may have been first invented to deal with such differences in the business world.

Wessman-Enzinger: Chap. 2

As a long-time proponent of using games to teach mathematics throughout the elementary curriculum (Baroody, 1987; Baroody & Coslick, 1998), I was delighted to read Wessman-Enzinger's analysis of how games can prompt reflection about integers. The *Integers: Draw or Discard* card game presents a player with the option of drawing an integer card or discarding an integer card. A particularly good feature of the game included a clear distinction between operations (drawing a card = adding a positive or negative integer and discarding a card = subtracting an integer) and direction of magnitude. Another good feature is that the game does not rely merely on luck to win; a thoughtful strategy plays an important role also. Yet other particularly good features are that the game embodies the counterbalance conceptual model (or the charged-particles analogy) relatively directly but as structured play. In brief, the *Integers: Draw or Discard* card game provides a rich and motivating opportunity to explore informally the effects of addition or subtraction on integers. For instance, children quickly learned that discarding (subtracting) a large negative card had a larger impact on the net value than drawing (adding) a small negative and that discarding a negative was the equivalent of adding its magnitude to their net score (e.g., discarding -3 resulted in a net of 3).

Wessman-Enzinger's analysis underscores the important role teachers can and should play in using math games as an instructional tool. Games are not a substitute for good teaching but a tool in the service of good teaching. Perhaps unsurprisingly, the children initially did not connect the teacher-taught counterbalance conceptual model to their game playing. For example, in justifying a draw of cards involving -7 and 8 having a net effect of one, Jace argued, "because eight *minus* seven equals one." He treated the -7 as *minus seven* instead of, as the model implies, the *addition of negative seven*. Moreover, the children did not equate discarding a card with subtraction. To help students make these connections, which they may not have made otherwise, the teacher encouraged them to write number sentences. Without a clear model or analogy to guide their actions, the students initially struggled to do so. When they finally started writing number sentences, they still did not equate discarding with subtraction but instead treated discarding, say, -3 as adding 3.

In a paragon of constructivist teaching, the teacher did not simply tell (impose) the connection with counterbalance conceptual model but *prompted* the students to think how they could preserve the negative value of a negative card. The teacher did this by encouraging the students to consider a fictitious hand, such as -3, -5, and 8 with the option of drawing a -2 card. The students concluded discarding -5 was better than drawing -2. Although some students represented this move as 0 (the net of the held cards) + 5 (the effect of discarding -5), one student did hit upon 0 – -5 = 5.[1]

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[1] Teachers of older students with less time to make the connection could ask students to write an equation representing holding a -3 card, drawing -5 and 8 cards, and the net: -3 + -5 + 8 = 0. They could then be encouraged to write an equation for discarding (subtracting) the -5 card: -3 + -5 + 8 – -5 = 0 – -5 = 5. A teacher could also encourage students to consider the parallels with a charged-particles analogy: holding a -3 card is analogous to starting with a net charge of -3 in the box,

The debate over which is "bigger" provided an opportunity for a teacher to note that both Kim, who advocated for -8, and Alice or Jace, who advocated for -4, were—mathematically speaking—correct. On one hand, -8 is the more negative and, in terms of *magnitude alone* or *absolute value*, greater than -4. On the other hand, -4 is the least negative and, in terms of ordering integers by *magnitude and direction*, is greater than -8. The use of an analogy could serve to drive this distinction home (e.g., $8 of debt is more than $4 of debt, but $4 of debt is better than $8 of debt in terms of net worth). Such a discussion would not only help students understand concepts such as absolute value and integers more deeply, it would underscore the importance of being mathematically precise and that "the correct answer" depends on the parameters specified.

Consider also the opportunity provided by Alice's comment, "I think it's kind of the middle." A teacher could prompt:, "Can you think of an example in the real world where zero is in the middle?" Three example responses are:

- A zero balance indicates one has neither credits nor debits.
- Zero degrees Celsius separates above freezing from below freezing temperatures.
- An altitude of zero feet (sea level) separates above sea level and below sea level.

Note that, whereas a net worth of $0 is consistent with children's informal view that that 0 means nothing, 0° Celsius is not the absence of a subjective perception of hot or cold and an altitude of 0 feet does not indicate the absence of a height. Such a discussion might have dissuaded Alice from concluding that, "zero's like not a number because it's nothing."

Burghardt's (2011) elements of play raise issues and do not address a key criterion for using play as a pedagogical tool. The criterion spontaneous or pleasurable seems to equate two, not necessarily compatible, aspects of play. For example, a teacher-introduced game such as *Integers: Draw or Discard* may not be spontaneous (performed as a result of a sudden *inner* impulse and without *external* stimulus), but it was pleasurable and, thus, welcomed and engaging. Put differently, it does not seem that play must be both spontaneous *and* pleasurable to be of pedagogical value. The criterion "initiated in the absence of stress" overlooks that some forms of entertainment, such as playing a competitive game, a cooperative game against a time or other limit, or riding a roller coaster, are undertaken to experience tension (albeit without real consequences).

More importantly, for pedagogical purposes, it is critical to distinguish between play for the sake of play (spontaneous free play), which has a useful role of its own, and play for the purpose of learning or setting the table for learning academic knowledge (e.g., teacher-imposed math game). In his critiques of progressive education, Dewey (1963) distinguished between *educative experiences*, which lead to worthwhile learning or a basis for later learning, and *mis-educative experiences*,

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drawing a -5 and 8 card is analogous to *adding* five negative (-5) and eight positive charges to the box, which results in a neutral net charge. Discarding a -5 is equivalent to *taking away* five negative charges, which leaves five positive charges uncoupled for a net charge of +5. Note that the informal language of the charged-particles analogy translates relatively directly into formal equations.

which—even if fun—do not promote learning. Clearly, much of the children's effort when playing the *Integers: Draw or Discard* can be characterized as educative experiences. However, the children's spontaneous, imaginative, and creative play involving "pretend" point totals did not appear to advance thinking about integers—though one could argue that it advanced their number sense in some way.

Bishop, Lamb, Philipp, Whitacre, and Schappelle: Chap. 3

The chapter by Bishop et al. is based on a semi-structured clinical interview method (a standard set of initial questions with flexible follow-up questions to explore a child's thinking). The method is difficult to implement, but it can—as Bishop et al.'s results show—provide a rich source of data on children's thinking. On one hand, one general positive impression of their report is that children can use a wide variety of strategies—including impressive informal or self-invented strategies—to solve integer addition and subtraction problems. On the other hand, their report reveals important gaps in children's knowledge of integer addition and subtraction. This is not surprising for younger children, who have not had formal instruction on the topic. The knowledge gaps of older students who had received formal instruction, however, are a testament to the poor quality of integer instruction in this country (e.g., the gap between such instruction and children's informal knowledge).

Consistent with Piaget's (1964) principle of assimilation and moderate novelty principle, many young children—without formal instruction—determined the outcome of integer addition or subtraction in cases for which their knowledge of whole-number arithmetic was applicable but not in cases for which was not (see also Glancy & Pettis, 2017). The chapter by Bishop et al. illustrates vividly how children might apply existing knowledge. Consider the case of Fran-Olga, who confronted with the novel equation of -3 + 6 = □, concluded she could count up to six for an answer of 3 or count down to six for answer of -9. She correctly settled on the former because she knew addition and subtraction were inverse operations, recalled that for a prior "minussing problem (3 – 5) she had counted down, and reasoned that for the present 'plussing'" problem (-3 + 6) she should count in the opposite direction. Reasoning with logical necessity—comparing a novel problem to known problem and making the appropriate adjustments so that the underlying logic of the system and strategy remain consistent—is impressive for any elementary student. Moreover, Fran-Olga correctly answered -5 – -3 using analogy-based reasoning: negative numbers behave like positive numbers. If 5 – 3 is 2, then -5 – -3 could be 2 also (J. Bishop, personal communication, November 6, 2017).

However, while applying emergent reasoning, Fran-Olga incorrectly answered the highly novel equation 6 – -2 = □ (J. Bishop, personal communication, November 6, 2017). What accounts for Fran-Olga's strategy choices and why she was able to solve -5 – -3 = □ but not the similar problem 6 – -2 = □?[2] It appears that the second

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[2] Glancy and Pettis (2017) similarly found a discrepancy in success between (-5) – (-2), which was relatively easy, and (-3) – (-5), (-5) – 3, and 5 – (-2), which were relatively difficult.

grader did *not understand* integers or operations on them and, as a result, resorted to using local strategies on an ad hoc basis, rather than apply a general and coherent strategy. In such situations, children often attempt to relate a novel problem to what they do know. If their initial search does not help, they sometimes search other aspects of known knowledge. For $-3 + 6 = \square$, Fran-Olga was able to assimilate the problem to her extant knowledge of inverse operations and counting and used logical necessity. Perhaps unable to apply this knowledge and way of reasoning (WoR) to $-5 - -3 = \square$, she searched for other extant knowledge and another solution method. This led her to hit upon analogy-based reasoning (perhaps ignoring the directional symbols and treating the problem as $5 - 3$). In both cases where she could find a connection to whole-number operations (moderately novel problems), assimilation efforts *fortuitously* afforded her correct solutions to problems she probably did not understand. Unable to connect the highly novel equation $6 - -2 = \square$ to her knowledge of operations on whole numbers, she was unable to apply successfully logical necessity or analogy-based reasoning and (unsuccessfully) drew on what she knew to answer as best she could (emergent reasoning). Still, given that the rules for subtracting negative integers do not conform to the rules governing the subtraction of whole numbers, it is commendable she found way to solve $-5 - -3 = \square$, if not $6 - -2 = \square$.

Nevertheless, Fran-Olga's reasoning is based on conceptually shaky grounds and, as a result, is logically incoherent. The strategy for (correctly) solving $-3 + 6 = \square$ was based on the *overly broad* rule that "plussing" means count to the right and "minusing" means counting in the opposite direction. Unfortunately, this overly broad rule results in the wrong answer for $-5 - -3 = \square$ and is *logically inconsistent* with the relatively meaning-free analogy-based reasoning used to solve the problem correctly. Similarly, if the overly broad rule were applied to $6 - -2 = \square$, it would require disregarding the directional symbol for -2 and moving in the wrong direction.

The case of Fran-Olga illustrates why mathematics educators emphasize the integration of conceptual understanding, procedural knowledge, and (strategic) reasoning (Battista, 2016; National Research Council, 2001). Conceptual understanding guides the invention and application of a procedure or WoR, and understanding a child's present understanding or developmental level provides guidance on how to design instruction to help her advance to the next developmental level. Research indicates that children can reason logically about familiar ("concrete") or—as the case of Fran-Olga illustrates—moderately familiar situations but not unfamiliar ("abstract") ones (Ennis, 1975; Evans, 1982). In this respect, young children are very much like adults who do or do not understand (are or are not familiar with) how a television works trying to troubleshoot a TV malfunction.

Consider what Fran-Olga might have done if negative integers were related to the analogy of a debt—perhaps making it possible to more fully assimilate integer addition and subtraction problems and reason in a logically consistent manner. She may have reasoned that adding a credit of six to a debt of three cancels the debt and leaves a surplus of three. The beauty of credit-debit analogy is that it is easily applicable and logically consistent across the range of integer addition and subtraction problems, including -5 – -3 = □ (if you start with a debt of $5 and take away $3 of the debt, it leaves you only $2 in debt) and 6 – -2 = □ (a credit of $6 and take away a previous debt of $2 results in an improved net worth of $8). Indeed, it is possible that the girl might have recognized that taking away a debt was equivalent to adding a credit.

A limitation of the WoR taxonomy as presented—at least for pedagogical purposes—is that it does not differentiate between a WoR or a particular strategy that is based on conceptual understanding and those that are not. For example, the problem with knowing only that Oscar uses an order-based WoR or that Cole uses a computational WoR is that we do not know if their procedures have been memorized by rote or meaningfully memorized (i.e., can be connected to conceptual understandings). This distinction is critical because the former typically results in routine expertise—knowledge that can be applied to familiar but not unfamiliar problems or contexts—and the latter permits adaptive expertise—knowledge that can be applied flexibly and creatively even to novel problems or in new contexts (Hatano, 2003). Put differently, as a network of conceptual and procedural knowledge, adaptive expertise is more likely than routine expertise to permit a child to reason coherently and successfully across a range of related new problems. Another advantage of adaptive expertise over routine expertise is that the former is significantly more likely to be retained or, if forgotten, reconstructed.

Bishop et al.'s data paint a particularly gloomy picture of formal integer instruction. The participants used logical necessity, which requires applying conceptual understanding, on less than 1% of all problems posed. They relied on meaningful analogies only about 3% of the time. In contrast, participants relied most heavily on the computational WoR—40% of the time. This is not bad in itself, but what is troubling is that given the opportunity to explain or justify their rule, participants in the examples reportedly did not logically relate it to meaningful everyday experiences (meaningful analogies) or meaningful mathematical concepts. For example, note that Cole's justification for -3 + 6 = 3 is entirely in terms of rules. Asked to justify his boom-boom ("magical?") strategy (to take away a negative integer, add the integer), Gabriel verges on a meaningful explanation ("Because you're taking away something that's negative?") but—apparently because he did not understand the rule—ultimately deployed the rationale provided by uncomprehending students, parents, and teachers everywhere ("It just works"). Such cases underscore the observations of Ginsburg (1977) 31 years ago that the main problem with formal mathematics instruction is the gap between formal instruction and children's existing informal knowledge and those of Brownell (1935) 83 years ago that school instruction relies too heavily on memorization of arithmetic by rote instead of meaningful memorization. Although one student had a cute mnemonic for recalling the rules of adding and subtracting integers (sung to the tune of *Row*, *Row*, *Row Your Boat*), would not it be better for students in the long run for them to understand such operations in terms of a meaningful analogy?

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Chapter 12
Commentary on Chapters 4 to 7: Students' Learning of Integer Addition and Subtraction Using Models

Kaye Stacey

Abstract These four chapters describe studies of using models for integer addition and subtraction. The models draw principally on the two grounding metaphors of object collection and motion along a path. A strength of all chapters is detailed analysis of how the models are and can be implemented and how they influence student's learning. Together the chapters demonstrate that teaching with any model requires care and attention to achieve conceptual clarity. Teachers need to act entirely within in the world of the model, only later adding other models to round out the mathematical picture. The teaching of integers is a key point for maintaining students' confidence that mathematics is a subject that makes sense. These chapters demonstrate how use of models can assist in this goal.

These four chapters all describe studies of students' learning about integer addition and subtraction by using models. Chapter 4 by Michelle Stephan and Didem Akyuz provides a case study of three students with disabilities learning to use a financial model of assets, debts, and net worth. Chapter 5 by Julie Nurnberger-Haag compares the outcomes of using two models—positive and negative chips and walking on a number line. Chapter 6 by Cecilia Kilhamn looks specifically at the surprisingly complex idea of 'difference' in the context of money and measurement models. Chapter 7 by Laura Bofferding provides a developmental sequence of mental models to frame a study of very young students learning about integers with an elevator model.

The variety of models discussed above immediately draws attention to the fact that there are many models for teaching about integers and that it is almost universally agreed that teaching with a model is essential for this topic. Some of the models are sets of physical objects to be manipulated, some are visual representations, and others are abstract. The four chapters look at which models can be best used, how they should be used and for whom, and how they impact learning. Together the

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chapters reinforce an important message: that models are helpful but that teaching with any model requires care and attention to achieve conceptual clarity. The chapters also point to areas where further research would be fruitful.

My initial reading of these four chapters highlighted the diversity in these studies. Although all were concerned with teaching similar content (addition and subtraction of integers), they worked with students with a large age range from year 1 to year 8. Teaching time for the integer topic varied from 1 week to 5 weeks. Two chapters are mainly case studies within teaching interventions: one reports an experiment involving eight classes, and the other reports observations of 'normal' teaching. The theoretical perspectives were also varied and included deep content-related cognitive analysis, sociocultural analysis, theories of embodied cognition, conceptual change, and conceptual metaphors. This variety is an example of how important problems in the teaching of mathematics benefit from research from many different perspectives.

In the following sections, I discuss some of the major themes that emerge from a reading of these four chapters. Before doing this, I want to raise two concerns. Firstly, the topic name 'integers' is not a good one, even though it sounds like important mathematics. All four chapters focused almost exclusively on calculations with positive and negative whole numbers—integers. Yet research shows that working with negative numbers that are not integers needs special teaching. Many students need to be taught explicitly even about the ordering of negative decimals and fractions (see, e.g. Widjaja, Stacey, & Steinle, 2011). Naming this topic 'negative numbers' or 'directed numbers' might encourage proper attention to this. Of course, it is sensible to start with negative integers only, to minimize cognitive load by avoiding calculation difficulties, but gradually all number types should be included. There are only a few instances of nonintegers in these chapters. A related point arises in Nurnberger-Haag's Chap. 5, where students are tested to see how well they can generalize ideas taught with small numbers to double digit numbers. Many students had difficulty, and some work samples showed false generalizations as students tried to extend addition and subtraction algorithms to negative integers. Teachers need to address the full range of numbers explicitly.

The second criticism is a criticism of our field of mathematics education in general, which is illustrated by these four chapters. Mathematics education will advance in a more scholarly way when the research literature is more cohesive and studies build more directly on each other. One of the ways to do this is to develop careful and consistent terminology with agreed meanings and, as far as possible, to use similar frameworks unless there is a strong imperative for variation. Many examples of the lack of cohesion in the research literature are demonstrated within these four chapters. Terms such as value and magnitude are used inconsistently, and absolute value, the mathematical standard, could be used more often. The cancellation models of Chap. 5 are the counterbalance models of Chap. 7. Models and metaphors, mental models, conceptual models, and instructional contexts are not always clearly distinguished or defined. The classification of additive situations by Marthe (1979) (e.g. state-transformation-state (STS)) was used in two chapters, but this (or an alternative) could have been used to advantage throughout. Drawing together the ideas in the chapters of this book could be a stimulus towards more cohesion and agreement on frameworks. The chapters highlight the need, but they also offer possibilities such as Bofferding's developmental list of mental models and the classifications used in each chapter. Across mathematics education, we need to identify good terminology and frameworks to make standard and then use them consistently to build a stronger mathematics education edifice.

The Main Challenges the Chapters Address

Underlying all four chapters is the goal of helping students build a strong understanding of the meaning of negative numbers as objects and then their addition and subtraction. Also shared is the conviction that building this understanding requires the use of one or more models. The initial idea of a negative number—having less than zero things—is very puzzling but quite wonderful, and there are instances reported in the chapters of students who do not yet accept it. In general, however, using a negative number just as a label for an 'underground' number (Groves & Stacey, 1998) in a context is not difficult even for very young children. This is well demonstrated by Bofferding in Chap. 7 using the context of an elevator going above and below ground. In many simple situations, there is an obvious purpose in using these 'directed numbers' as labels with the number component indicating distance from an often arbitrary zero point and the sign + or – indicating in which direction. Moreover, even very young children can work within such a context to answer questions such as 'Where is the elevator if it starts at floor -2, then goes up 6 floors?' and 'Which floor is highest, floor 3 or floor -5?' and 'Which floor is furthest from the ground level 0: floor 3 or floor -5?' and 'How many floors are between floor -3 and floor 7?' In context, there is little difficulty in finding locations after various movements or in determining order (both which is highest and which is furthest from the origin).

However, the difficulty ramps up very quickly when the labels transform to become numbers, interacting with others; this is first done through the operations of addition and subtraction. The easy questions become much more challenging when they are mapped onto the formal system of addition and subtraction. As pointed out in Kilhamn's Chap. 6, the real reasons for extending the labels to numbers and defining the operations in the standard way are inherently intra-mathematical: to create a set of directed numbers all seamlessly obeying the same rules. For example, a seamless number system enables us to calculate 29 + 36 – 46 as 29 + (-10) = 19 instead of only as (29 + 36) – 46 = 65 – 46 = 19. Before negative numbers were accepted, there had to be many special cases of formulas (e.g. for solving a quadratic equation) so that subtraction of a greater from a lesser could be avoided at every stage. Students at the age for learning integers are insufficiently mathematically sophisticated to appreciate what is really being achieved by this seamless number system. The outcome is that dealing with abstract 'naked number' expressions (and soon algebraic expressions) becomes the main skill to be mastered in the integer topic, and the main teaching challenge is to give number expressions ‘concrete’ meaning. Decades of research into student performance have pinpointed where the most errors occur, especially with expressions involving counter-intuitive actions such as (-2) – (-7) and (-2) – (+7). All the chapters build on this research.

The Models

Taken together, the four chapters give a broad overview of the models available for teaching about integers. A strength of all chapters is the detailed analysis of how the models are implemented and how they assist student’s learning. Drawing on the observations that abstract mathematics can only be understood in terms of metaphors, two chapters classify available models through the ‘grounding metaphors’ (Lakoff & Núñez, 2000) for arithmetic that they employ: especially motion along a path and object collection.

Object Collection Models

Object collection is the most fundamental metaphor for early learning about natural number and in the most basic version, number corresponds to the cardinality of the set of objects, addition corresponds to combining sets, and subtraction to taking objects from a set (the simplest idea of subtraction). Several chapters point to the difficulties that arise when students’ conceptions are tied too narrowly to the basic object collection metaphor. Extending this model leads to various instantiations of the counterbalance/cancellation model, often implemented with positive and negative chips that annihilate each other. The advantage of this model is that number is again represented by cardinality of a set of objects (after cancellation). Addition can again be represented by combining sets, again with cancellation as required. Very importantly, subtraction can again be interpreted as taking away one set from another. This interpretation is straightforward when the numbers have the same sign (e.g. taking away two positive chips from seven positive chips or two negative chips from seven negative chips), but it is more complex when the two numbers have different signs (e.g. taking two positive chips from three negative chips). In those cases, additional chips with a total value of zero must be added to enable the ‘take-away’ step. This is an example of the very important mathematical manoeuvre of replacing a number by an equivalent form of the same number. Other examples are to replace a fraction by an equivalent fraction in order to add, or to replace 63 by (50 + 13) when subtracting 29 with the standard decomposition subtraction algorithm. It is also important in algebra, for example, to add and subtract a term (with net addition 0) in order to 'complete the square'. However, it is well known as a difficult step for students in all of these contexts, and the discussions of this model in these chapters highlight this difficulty for integers. In recommending an approach, a curriculum developer must weigh up the benefit of encountering an important mathematical principle against the cost of losing some students.

Nurnberger-Haag in Chap. 5 reports a very careful comparison of the subtraction success of students using the chip model and a number line model, importantly measured at a delayed post-test. She gives many insights into the working of both models. She draws on the theory of embodied cognition to propose that the incongruity in the chip model of putting in (adding) extra chips to subtract is one of the reasons why the chip model students were less successful. More studies such as this are called for, since there is inconsistency in the research. Hayes (1998), for example, found chip model students had an advantage over number line students for a year after initial instruction.

Number Line Models

The various number line models draw on Lakoff and Núñez's (2000) 'motion along a path' metaphor with some aspects of measurement metaphor. These models can be fully abstract (moving along a 'number line' without any other referents) or draw on real-world contexts (elevator, height above sea level, temperature scales, walking north and south, walking east and west, etc.). In each model, a complexity is that numbers are represented by, and represent, both locations and movements. Operations are modelled by, and model, movements along the line with the associated number specifying magnitude and direction. Vertical number lines also tap into the fundamental up-down conceptual metaphor that originates in bodily experience (Lakoff & Johnson, 1980).

Complications that arise through the dual representation of number by location and movement arise in the analysis within each of the chapters. A good example is provided by Bofferding in Chap. 7, in the careful analysis of how two girls work together. One girl interprets 4 + 1 as a state of an elevator being transformed: start at the fourth floor and move up one. The other interprets it as two movements being carried out: start at zero, move the elevator up four floors and then up another, reaching floor 5. Of course, both of these interpretations are valid, and they are interchangeable for experts, but they can be quite different for learners. The detailed analysis of how the two girls work together shows miscommunication within the pair. There are several other instances of contradictory uses of the model along with resolution of some conflicts, and it leads to a call for more research on how to create student pairings that minimize miscommunication and best support conceptual change for all. Bofferding's observation applies as much to teachers' interactions with students as it does to interactions between students.

Nurnberger-Haag in Chap. 5 also presents a careful discussion of the different ways in which symbolic expressions such as -3 – (-7) can be embodied/enacted on a number line. She believes the differences to be sufficiently significant to talk about number line models (plural), not 'the number line model'. She carefully justifies her choice of the 'walk-it-off' model, which simplifies the rules that students must learn to model an integer expression. Her empirical study demonstrates its success. Some number line models have very complicated rules about interpretations of the '–' sign for which direction to face along the number line and whether to walk backwards or forwards. Some textbooks I have seen present rules that are very hard to remember, appear quite arbitrary, and fail to distinguish between the unary and the binary minus sign. A number line model for interpreting multi-term arithmetic expressions is not simple! One of the main messages from this and other chapters (especially Chap. 6) is that teachers need to be thoroughly prepared and clear about how they will present and talk about a model.

Models to Explain or to Illustrate

In Chap. 4, Stephan and Akyuz provide a thorough discussion of the learning, class participation, and problem solving methods of three students who struggle to learn mathematics. The students had previously learned about integers, including rules for calculating expressions such as -4 – (-8), and used a horizontal number line. They found that these students with disabilities improved to mastery in 5 weeks, mainly using a financial model (labelled the bookkeeping model in Chap. 7). Stephan and Akyuz use a hybrid model where finance is a real-world context (model) and a vertical number line is a visual but abstract model. They need three real-world concepts to represent numbers: assets and debts and net worth (the combination of assets and debts). Mostly, assets and debts are like movements on a line in a number line model, and net worth is like position, but in practice students must think more flexibly than this (as they must with number line models). Initially debts were seen as positive amounts of money; later they were symbolized with a negative sign. The vertical number line began as a model of performing transactions on net worths; it later became a model for reasoning with integer quantities.

In this chapter, Stephan and Akyuz emphasize that they want mathematical activity to be grounded in experientially real contexts as in the other chapters but that their main goal is to develop intellectual autonomy, so that students can use the model to learn concepts and skills and also use it to work out how to solve problems for themselves. To do this, they need to build the relevant real-world concepts, which may at first sight appear quite simple (e.g. net worth is a combination of assets and debts) but provide challenge for their students. Also, they need to make a strong link to mathematical notation (e.g. taking away a debt increases net worth, \$5000 – (-\$3000) = \$8000) because the goal is to work with 'naked' number expressions.

This chapter highlights that instruction that might use the same model can in practice be very different in intent. It is possible to teach with models using a direct instruction approach. For example, students might be told unmotivated rules for marching up and down a number line to illustrate an expression such as 5 – (-3) and then physically practise following these rules to get answers. Students may 'understand' that 5 – (-3) = 8 because that is where you arrive after following the given marching rules: a concrete embodiment of rules which are as apparently arbitrary as symbolic rules. Alternatively, with a different quality of understanding, they may 'understand' that 5 – (-3) = 8 because this makes sense in the real-world situation that the numbers are modelling: removing a debt is the same as gaining an asset, the difference between a temperature of 5 °C and of -3 °C. Just as being told that a particular teaching episode uses a model does not determine the quality of understanding that students might have gained from the instruction. A model can be used to explain and construct the symbolic rules for integer expressions, or it can be used to illustrate given rules. Mathematics can model the real world, but in school it is often the real world that models mathematics.

In Search of Conceptual Clarity

Each of the chapters, through detailed analysis of student learning, highlights the need for conceptual clarity in teaching episodes. Kilhamn's Chap. 6 is the strongest example, providing a careful analysis of the use of the word 'difference' in classroom instruction and its links to subtraction. Analysis of videos of classroom discourse (verbal and written) reveals implicit and incoherent use of metaphors, even from a respected teacher. Over seven lessons, the word difference was unsurprisingly mapped onto subtraction when the sizes of two collections were compared, but the teacher also used the word difference for addition as in (-8) + 5. The opposites paired off and the difference was represented by the objects left unpaired (-3). (Written symbolically, the teacher is mentally calculating (-8) + 5 = -(8 – 5).) The teacher suggested students 'write addition' but 'think subtraction'. Another complexity is that when the word 'difference' is used in a measurement sense, as the distance between two points on a number line, the metaphor of distance is inadequate to deal with differences (answers to subtraction) being either positive or negative: it needs to be extended. The transcripts revealed that there was no shared understanding of a difference between (signed) numbers. It was also the case that many times metaphors appeared implicitly in the discourse with participants probably unaware of them. Kilhamn makes the important point that metaphors might be taken for granted, but they are not always a shared reference. Using metaphors in a consistent way that builds students understanding requires teachers to live fully in the world of the model, without drawing on 'outside' ideas that do not make sense in the world in which the students are expected to work.

Conceptual clarity in instruction with models seems important. However, over the years of school, students need to be able to draw on multiple useful models and metaphors to discuss and think about mathematics. As Kilhamn notes in Chap. 6, 'While metaphors help us make sense of concepts by providing coherent structure, they highlight some features, but hide others. Since different metaphors are used to structure different aspects of a concept, several metaphors are needed to fully understand a rich concept'.

Conclusion

Negative numbers are a mind-boggling topic. How can there be less than zero things? How can it be that a minus times minus equals a plus? How can we calculate with these amazing new numbers, and what does it mean? How can taking away make something bigger? Addition and subtraction of integers (or directed numbers more generally) is certainly a conceptually fascinating topic. The difficulty for learners is the need, for a variety of reasons, to come to operate with integer expressions and, soon after, with algebraic expressions that strongly draw on this new knowledge. This can be an entirely symbolic exercise, driven by rules. There are well-documented dangers in this. Students who just learn rules come to accept that it is not possible to reason autonomously to solve mathematical problems. And this is fatal to mathematical development. The teaching of integers is a key point for sense-making, and these chapters have demonstrated the care with which this needs to be done with the many models that are available.

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Chapter 13

Commentary on Chapters 8 to 10: Teachers' Knowledge and Flexibility—Understanding the Roles of Didactical Models and Word Problems in Teaching Integer Operations

Irit Peled and Anat Klemer

Abstract The crucial role of teachers in introducing integers to children is highlighted in chapters 8–10, comprising this section. The three chapters discuss (prospective) teachers' conceptions of integer equations, of children's thinking about integer expressions, and of the role of some didactical models used in teaching integer addition and subtraction. These different aspects of teacher knowledge and conceptions draw an important picture of characteristics and issues that should be taken into account by teacher educators in preparing teachers for teaching integers. In the first part of our commentary we highlight the main contributions of each of the chapters, focusing on the central findings and on important issues brought up by each chapter. The second part offers a meta-perspective of some of the issues by discussing more general educational implications. In this part we also take the opportunity to express our own insights emerging and associated with the ideas presented in the three chapters.

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The crucial role of teachers in introducing integers to children is highlighted in Chaps. 8, 9, and 10, comprising this section. The three chapters discuss (prospective) teachers' conceptions of integer equations, of children's thinking about integer expressions, and of the role of some didactical models used in teaching integer addition and subtraction. These different aspects of teacher knowledge and conceptions draw an important picture of characteristics and issues that should be taken into account by teacher educators in preparing teachers for teaching integers.

If we were asked to solve the problem -4 – 2 = □, we might think about -4 as a point on the number line and view the second minus sign as a subtraction sign asking us to perform a subtraction transformation on -4. We would then subtract 2 by moving "down" from -4 to -6. Some of us could possibly think about -4 – 2 = □ as involving the distance between two points on the number line, 2 and -4, and figure out that the transformation -6 takes us from 2 to -4. However, if we were teachers, we might be expected to think about the problem quite differently. Chapter 10 presents a textbook solution of this problem using a didactical model (i.e., a two-colored chip model). Without getting into all the details, the model offers a sense of quantity, where positive or negative quantities can be taken away from positive or negative quantities correspondingly. It suggests reorganizing the given number to enable quantity manipulation by using a renaming method. In the given case, -4 can be renamed in many ways (e.g., -8 + 4, -7 + 3, -6 + 2), and the student is expected to choose an optimal "new name" of a number for subtraction. In this case it would be renaming -4 as -6 + 2, getting -4 – 2 = (-6 + 2) – 2 = -6 + (2 – 2) = -6 + 0 = -6.

This example demonstrates only some of the complexities emerging in teaching integers: the minus sign has several meanings, the operation of subtraction has different definitions and constructs, and a didactical model offered by the textbook might seem to be complicating the solution. As can be seen, some of these difficulties are inherent in the integer topic itself making it difficult for textbook writers. In addition, and unlike earlier encountered numbers and operations, textbook writers face the difficulty of finding meaningful contexts for introducing integers. These problematic issues demonstrate the immense difficulty involved in teaching integers and the importance of the research presented in this section.

In Part A of our commentary, we summarize the main contributions of each of the chapters, focusing on the central findings and on important issues brought up by each chapter.

Part B offers connections between the chapters together with a meta-perspective of some of the issues by discussing more general educational implications. In this part we also take the opportunity to express our own insights emerging and associated with the ideas presented in the three chapters.

Part A: Findings, Issues, and Conclusions in Chaps. 8, 9, and 10

The three chapters have different foci and different research tools. Chapters 8 and 9 focus on teacher knowledge and conceptions, revealing them through their own solution of word problems (Chap. 8) or through their evaluation of children's answers (Chap. 9). Chapter 10 investigates teacher instructional approaches by interviewing and observing two teachers as they introduce a certain didactical model in class.

Bofferding and Wessman-Enzinger: Chap. 8

This chapter aims to identify PT's mathematical knowledge related to integer addition and subtraction as it is exhibited through their solution of word problems. The **research questions** ask about the personal and the collective mathematical knowledge of PTs related to this topic.

The solutions of 15 PTs (7 elementary and 8 secondary teachers) to 8-word problems were analyzed using a framework of 5 strands of mathematical proficiency. The strands include procedural and conceptual knowledge, strategic knowledge (e.g., use of tools such as representations), adaptive knowledge (e.g., making connections), and productive disposition (e.g., seeing the role of mathematics).

The first 6-word problems included problems such as *Andy has 6 points. Joan has -7 points. How many more points does Andy have than Joan?* Although the problem did not describe clearly the meaning of the points, they were regarded by the PTs as numbers on the number line, and the question was perceived to be about the difference between the numbers. The PTs were expected to write an equation without much deliberation. This was especially evident in the last two problems, where the situation was even less clear than in the first problems. Problem 7 asked: *Briana started with a -4-point card. Her opponent took -3 points from her. What is her score?* The authors noted that for some PTs, they had to clarify that *card* meant *a card hand worth -4 points*, but no other explanations were reported. This point will be discussed in the second part, where we discuss the process and the goals of mathematizing a word problem.

The **results** present both individual and collective knowledge. On the one hand, the PTs exhibited a rich repertoire of solution strategies including counting up, using 0 as a point of reference, and more. They expressed ideas for neutralizing amounts, for example, as they counterbalanced positive with negative values in performing -4 + 6 by splitting the 6 into +4 and +2 so that the +4 neutralizes the -4. Another way to cancel amounts was demonstrated by one of the PTs who explained that -8 – -8 = 0 *because when you subtract a number from itself, you get zero.* Yet, on the other hand, most of the PTs' explanations were based on their procedural knowledge, and there was a tendency to directly write a simple expression such as 7 – 3 rather than 7 + -3 without explaining why this was "legitimate." The authors noted that some of the responses could be concerning and that only a few PTs could offer conceptual explanations to actions such as the abovementioned simplification.

The authors offer two main **conclusions**: (a) the difference between individual PT knowledge and collective knowledge has led to suggesting that teacher educators *leverage the collective groups' responses* by creating opportunities such as targeted discussions for sharing knowledge and building a stronger understanding for all PTs and (b) the observed differences between word problems in eliciting strategies has led to recommending making conscious decisions about what problems to include in class discussions.

An additional recommendation involves the authors' belief that in preparing for working with students, teachers should be able to offer conceptual explanations related to the procedures or concepts they are about to teach. Therefore, they recommend making PTs aware of their own explanations, especially if they tend to rely on procedural knowledge.

Tobias, Wessman-Enzinger, and Olanoff: Chap. 9

In this chapter the authors use prospective teacher evaluations of a student's answer as a window into PTs' conceptions and their views of children's work in integer subtraction in the context of temperature. The task is aimed at answering the specific **research question**: What do PTs attend to when they evaluate a child's temperature story for integer subtraction that uses distance?

The reactions of 100 elementary and middle school PTs to the story written by a fifth grade student named Parker were analyzed. Given the expression -14 – -20 = Parker wrote: *The freezer is -20°. The refrigerator is -14°. The freezer is 6° colder than the refrigerator.* The PTs were supposed to decide whether the story matched the symbolic expression and to explain their answer.

As can be seen, the numbers in the symbolic expression are represented in the story as points on a number line. The authors use the term "states" referring to earlier work— thus the meaning of subtraction is the move or the distance between the states. Parker concludes that "The freezer is 6° colder than the refrigerator" rather than "The refrigerator is 6° warmer than the freezer." Both descriptions fit the given states' relationship but seem to be evaluated differently on the "fit the given symbolic expression" criteria.

The authors analyzed the PTs responses while looking for evidence of noting that Parker uses a distance (or difference) meaning. They also looked for evidence of PTs' realization of Parker's "inconsistency" in writing a story that is supposedly represented by -20 – -14 = -6 rather than -14 – -20 = +6. This attitude toward the mapping between a symbolic expression and a word problem will be discussed further on together with a similar mapping issue mentioned in Chap. 8.

The **results** indicated that about half of the PTs noted the distance meaning, but only about a quarter of the PTs thought that the story did not match the symbolic expression. The latter claim was supported by different types of explanations. Most of the PTs mentioned the order of numbers in the story (that does not correspond to their order in the equation), and only 2 PTs mentioned that the story represented a solution of -6 rather than 6.

The authors seem to be disappointed with the results. They **conclude** both PTs' ability to make sense of children's thinking and their awareness of central factors in a context that is used for teaching integers (such as temperature) are deficient. Therefore, the authors suggest that teacher education should include opportunities for analyzing children's thinking in such a context and conduct explicit discussions of relevant features.

Murray: Chap. 10

While Chaps. 8 and 9 investigate teacher conceptions about integer equations, this chapter deals with instruction and specifically with the use of a certain didactical model, a chip model, for teaching integer addition and subtraction. The research **goal** is to call attention to the tension between curricula and instruction when a teacher has to decide whether to use a didactical model exactly in the way presented in the textbook or make adjustments on her own initiatives, using her personal understanding of the model and her experience and knowledge about children's thinking.

Two 7th grade mathematics teachers from the same school were observed as they used the same curriculum (Connected Mathematics Project 2 [CMP2]) teaching addition and subtraction of integers using a chip model. The **research questions** were focused on these two teachers asking how they used the chip model and how, in turn, their instruction impacted the nature of their students' opportunities to learn these integer operations.

As it **turned out** (or perhaps this is the reason for choosing them), the two teachers, Tess and Clark, used the chip model in different ways. While the curriculum intended that students use a renaming strategy and choose an optimal number renaming in a given integer subtraction problem, both teachers offered their students different procedures, circumventing the need to make such a choice. **Tess** taught her students a specific procedure for determining exactly when and how to add zero pairs. **Clark** used his own understanding of the model in creating and teaching a procedure that was more general and often less efficient than the textbook procedure.

The author **concludes** that teacher understanding and implementation of a model, such as the chip model, are an important issue that should be considered beyond the question whether a certain model has merits. In addition, the author highlights the need for teachers to understand the assumptions and roles of a given didactical model and of the roles of models in general. She suggests that this understanding would be supported by strong content knowledge and by familiarity with the features of a variety of models.

Part B: Beyond Research Findings—Emerging Central Issues

The summaries of the three chapters, while being very brief, present sufficient evidence of the chapters' significant contribution to teacher education. Still, in this section, we aim to show that their contribution goes beyond the research findings and the answers to the research questions. In a similar manner, to Chap. 8's observations of individual knowledge versus collective teacher knowledge, when the three chapters are viewed together, their collective observations highlight crucial issues.

Each of the chapters offers a window to teacher knowledge related to integers and at the same time gives an example of a task that enables teacher educators to diagnose this knowledge. In Chap. 8, PTs are asked to mathematize and solve word problems that involve integers; Chap. 9 asks them to evaluate student work of writing stories that fit given symbolic integer equations; and Chap. 10 describes an observation and reflection process that includes teacher interviews.

Using these different windows to teacher knowledge and conceptions, the combined research findings converge to several central topics that should be of concern to teacher educators: (a) the relation between word problems and integer symbolic expressions and (b) the role of didactical models for integer instruction. The range of conceptions demonstrated in the chapters with regard to these issues and the fact that in some cases we do not see eye to eye with the authors' explicit or implicit beliefs support the necessity of discussing these topics.

1. Context and symbolic expressions with integers

In their first encounters with numbers and with the operations of addition and subtraction, children are given many everyday examples that either define or apply the new mathematical concepts in contexts that are familiar to them. This practice is difficult to employ with integers. In a way, this supposed downside might have an advantage: it forces us to better differentiate between the mathematical world and the real world and clarify the role of mathematical concepts in mathematizing situations.

To say it more explicitly, the many everyday situations that involve putting things together, for example, become almost automatically associated with the mathematical additive structure. In addition, the natural language of these situations is very similar to the mathematical language of the mathematical operations. These strong relations inhibit us from perceiving the process of problem-solving as involving a decision about mathematizing the given story, i.e., fitting a mathematical structure to the situation. We were warned about it by Nesher (1980) who describes the problematic nature of word problems. Nesher gives an example of children solving a problem involving mixing jugs of water at 40° and 80°, thinking that the combined mixture should be (40 + 80)°. Their answer reflects the didactical contract or rules of the problem-solving game. We use this example to make us aware of not offering situations that require some deliberation on whether addition is relevant. With the less instantly available contexts for integers, hopefully the offered word problems would be of a different nature.

Although much less prevailing than with natural numbers, some helpful relevant contexts are still available for integers. While a few centuries ago even mathematicians found it difficult to accept the concept of negative numbers, nowadays the existence of some realistic contexts that make use of numbers with a negative sign facilitates the recognition of their existence. Such contexts can potentially facilitate the conception of numbers that do not necessarily stand for a quantity. Why "potentially"? Because while the immediate contexts that come to mind (Ball, 1993) are temperature (with the use of below zero temperature) and money (because one might owe money), they might not be similarly accepted. For example, as described by Ball (1993) and Peled and Carraher (2008), much confusion arises in symbolizing debt situations.

Besides the fact that temperature is a context that actually uses negative numbers, this context has another critical merit. As discussed in Chap. 9, it is essential to introduce the difference or distance meaning of subtraction in addition to the more familiar take-away meaning, and the temperature context offers an opportunity to do so. Thus, it is of no surprise that two of the three chapters discuss this context.

Early encounters with the difference/distance concept begin with natural number comparison and with "compare" word problems. The latter are considered the most difficult problems among additive one step word problems. There are several sources for their difficulty, and one of the central reasons is the conceptual difficulty to accept the use of subtraction in determining by how much one set is bigger than another set. Hatano and Inagaki (1998) describe an argument between a first grader and his classmates who want to subtract the number of girls in class from the number of boys in order to answer the question *how many boys more than girls*. The first grader cannot accept this operation claiming that one cannot "take away" girls from boys.

On top of the conceptual difficulty involved with accepting subtraction as a relevant mathematical model for finding the difference or distance between two values, the temperature context introduces another difficulty (i.e., the direction of the distance). Chapter 8 refers to Almeida and Bruno (2014) who asked PTs to solve word problems involving temperature and detailed different strategies used by the PTs. One of the problems asked was about the change in temperature given that the morning temperature was 4° and the night temperature was −5°. Although Almeida and Bruno accept as correct different ways with different equations as long as the explanation and conclusion were right, their findings are described in Chap. 8 as follows: *They [the PTs] solve the problems from a positive perspective (*e.g., *calculating 4 – -5 = 9 and indicating the temperature dropped 9 degrees* ***instead*** *[our highlight] of -5 – 4 = -9) almost as often as they solved the problems from a negative perspective.*

Chapter 9 deals with a similar correspondence between a temperature story and an equation. In section A, we presented Parker's problem and the PTs' task to check whether the story that Parker wrote matches the number sentence he was given. In their analysis, the authors check PTs' responses for consistency, namely, *Does the PT mention that the story shows -20 – -14 = −6* ***instead*** *[our highlight] of -14 – -20 = 6*, and in the discussion, they declare that the true inconsistency in Parker's story is that +6 was used contextually with "colder."

Indeed, there is some conflict between the mathematical order defined on degree of temperature (a bigger number goes with a higher temperature) and the dimension of "coldness." When we turn Parker's story into a problem: *The freezer is -20° and the refrigerator is -14°. How much colder is the freezer than the refrigerator?*; this conflict makes it natural to make a transformation within the situation saying that the question is equivalent to asking how much warmer the refrigerator is than the freezer. Thus, we get the expression *-14 – -20 = 6*, the expression Parker was given originally, from which we can deduce that the freezer is 6° colder than the refrigerator. Note that a similar transformation, studied by Verschaffel (1994), is used in compare word problems, where the relation is given in an inconvenient direction.

With regard to the authors' concern that the given number sentence involved the number +6 but was verbally described in the story by "6 degrees colder," we turn to research discussing the beliefs developed by children following traditional problem-solving. These experiences led children to undesired beliefs about word problems as exhibited in Noga's story problem. They also resulted in about a third of the children in a US national assessment test (Carpenter, Lindquist, Matthews, & Silver, 1983) answering that "31 remainder 12" busses are needed to move soldiers in busses. According to Schoenfeld (1987), their performance showed that they developed wrong beliefs about "what mathematics is all about."

Our point is that the symbolic expression is only a means to get to the realistic conclusion and solve the given word problem. The number sentence used in the solution of a word problem does not have to represent the realistic action or the exact numbers described in the story. For example, if the story involves an action of reducing an amount, the operation in the symbolic expression (e.g., in some "change" additive problems) does not have to be subtraction.

Actually this discussion shows that we are missing the point. This conclusion is strengthened when we turn to the 8-word problems given to the PTs in Chap. 8. The authors call the problems "word problems," but they are very sterile; no explanation is given about the situation, and the added value of giving the "word problems" rather than directly presenting the mathematical expressions is unclear. Problems of this type made Noga, a first grader, compose the problem: *In the morning I read 20 books, and in the afternoon, I read 10 more books. How many books did I read?* She explained: *All the teacher really wants is that we write a mathematical expression.*

Not only was the situation not explained, but also the PTs' answers were evaluated in a way that misses the idea of mathematizing a word problem. Problem #8 asks: *Paola started with 7-point card. Then she drew a -3-point card. What is her score?* The authors' expected symbolic expression is 7 + -3 = 4; this expression might be expected (and even then its use is not a must) in special situations where four different actions are described (as in the didactical postman model where bills can be given or taken away and checks can be given or taken away), and there is a rational and realistic reason to represent each action differently.

The story in problem #8 seems to involve a game where one picks up a card (or roles a dice) and the card might tell her to move forward (+3) or backward (-3). The natural action is simple addition in the first case and simple subtraction in the second.

Thus, the natural expression for the problem is 7 – 3. The authors seem to think differently, and when such an answer is given, they expect the PT to give an explanation for supposedly moving from 7 + -3 to 7 – 3.

The expectation of a very specific equation in both chapters indicates the use of word problems as a tool to introduce meanings to symbolic expressions. As summarized in part A, the authors have a good cause exhibited in relevant research questions. They aim to expose teacher conceptions and teacher knowledge about integer operations and succeed in bringing valuable research findings. However, such work with teachers should be done with caution to avoid creating undesired beliefs about the roles and goals of word problems and about the evaluation of children's solutions.

Being reminded of Nesher's (1980) temperature example mentioned earlier, we would like teacher educators to introduce PTs to integer word problems that encourage analyzing the situation, making assumptions and considerations, and deliberating about the mathematical structure that might be relevant. These problems are termed *modeling problems* since they involve a modeling process, and much research is done nowadays on their effect.

When we introduce such a problem in teaching integer operations, a mathematical model using integers might be "competing" against another model (e.g., a problem might involve a game situation similar to problem #8 in Chap. 8) such as a simple additive structure. We believe that the arguments brought up in such a "competition" would deepen the understanding of each of the competing mathematical structures. Thus, at the same time, we would acquire an important goal of problem-solving, namely, the ability to decide when an integer model is relevant, and also promote the understanding of the mathematical concepts of integers and integer operations.

2. The role of didactical models (e.g., the chip model)

Speaking of goals leads us naturally to the second central issue, the use of didactical models, a term used by Thompson (2002), in teaching integer operations. This issue is strongly connected to word problems via the fact that a concept has different senses or meanings. In general, when a concept has several meanings (termed *constructs* in Kieren's (1976) fraction analysis) this means that there exist different types of applications, i.e., word problems involving situations of different categories where the concept can be applied.

What does this have to do with didactical models? Going back to the child in Hatano and Inagaki's (1998) example, this child was given a word problem involving the situation of a difference (or distance). He was reluctant to use subtraction because he had been taught [only] the take-away meaning of subtraction and was most likely exposed to word problems of the take-away category. As exhibited in this example, when children are taught a new concept using a didactical model that has a certain meaning, they can be expected to encounter difficulty in applying the concept in word problems that are associated with another meaning.

This means that the choice of a didactical model, such as the chip model discussed in Chap. 10, involves a decision made by the program and textbook writers about meaning and applications. Since the teacher, like the two teachers described in Chap. 10, is supposed to introduce the subject using the textbook's model, she should know (a) that the role of the model is not just to define and help children perform integer operations and (b) that the model has some specific meaning she should be aware of.

It is difficult to tell how much of this knowledge the two teachers possess. Their reasons for offering alternative procedures for using the chip model seem to attribute much importance to the correct performance of integer operations. Although they do it in different ways, both teachers are sensitive to their students' difficulties and create safe procedures for them. This focus on procedures is understandable in light of the fact that the described chip model is an artifact that uses sterile noncontextual manipulatives.

In contrast, Linchevski and Williams (1999) combined a version of a chip model with a disco context in creating a didactical model that uses RME (Realistic Mathematics Education) principles (Gravemeijer, 1999). Linchevski and Williams are mentioned in Chap. 10 as an example of using concrete manipulatives demonstrating that *concrete representations help transform everyday knowledge to mathematical understanding*. However, it should be clarified that the source of power of the disco model is not in being concrete but in being constructed using RME principles. That is, it is designed in a way that facilitates students' reinvention of mathematical rules or mathematical structures based on their analysis and organization of the situation. For example, according to Linchevski and Williams, students who participated in the disco experiment came up with ideas equivalent to the use of a zero pair on their own as a strategy invented to cope with an encountered chip shortage.

Chapter 10 discusses and stresses the importance of teacher extensive general knowledge about models and their roles. Our current discussion of didactical models supports this opinion and points to several important teacher knowledge components such as the meaning of the model and the nature of the model, for example, whether it is artificial or realistic. The chapter also suggests that strong mathematical knowledge helps teachers in acquiring the desired knowledge. Indeed, such knowledge is crucial since we would like teachers to understand the mathematical principles behind models and their associated procedures.

Both Clark and Tess circumvent procedural steps that are based on the renaming principle used in the textbook's procedure. It is difficult to tell whether they appreciate it and how aware they are of the mathematical principles behind different procedures. We experienced discussions with PTs and teachers in which they expressed reluctance to use an alternative multi-digit subtraction procedure because of being change averse and without being able to offer principle-based arguments for their decisions. Specifically, they were not able to tell that the traditional procedure is based on renaming the minuend while the alternative procedure (called the French or the Italian subtraction) is based on adding the same amount to the minuend and subtrahend.

Good teachers should have the knowledge and ability to understand the principles and the goals behind procedures, the meanings of models, and the relations between models and word problems. Their meta-knowledge of principles and structures would enable them to make helpful connections between different procedures that are based on similar principles (Peled & Segalis, 2005; Peled & Zaslavsky, 2008) and thus support children's meaningful learning.

Concluding Remarks

All three chapters in this section make significant contributions to preservice and in-service teacher education for promoting the learning of integer operations. In addition to the different specific findings on the nature of teacher knowledge and practice in this subject, all chapters agree on the need to improve the quality of this knowledge. Special attention is given to the need to promote conceptual knowledge and sensitivity to children's work, related to integer operations. One of the chapters suggests doing that by taking advantage of collective knowledge and encouraging teacher group discussions.

A more global view of the chapters and of the description of PTs and teacher work exposes some rigidity in their beliefs and practice. This is exhibited in the focus on procedural knowledge, in evaluating and constructing the mapping between stories and symbolic equations, and in the effort made by the two teachers to facilitate their students' procedural fluency. These actions imply focusing on integer operations as a central goal, while the didactical model and the word problems serve as means to construct and strengthen them.

This is a legitimate goal as long as teachers also see the larger plan and are aware of the goals and principles behind their actions. As is also true in life, there is a tight connection between goal clarification and flexibility. Without seeing the bigger picture, we might fanatically stick to some action rules that do not serve our main goals. Our role as teacher educators, as we and the chapters' authors probably agree, is to develop teachers' deep knowledge about the roles and goals of all the players and contributors (including models and word problems) in constructing integer operations' knowledge.

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Chapter 14
Conclusion: Reflecting on the Landscape: Concluding Remarks

Nicole M. Wessman-Enzinger and Laura Bofferding

Abstract The conclusion contains a response to the chapters and commentary in this book describing the thinking, models, and metaphors for integer addition and subtraction. This response includes three main sections: establishing landmarks, valuing emergent thinking, and critiquing integer instructional models. First, we further discuss the need to establish landmarks, or use clearly defined language (e.g., order, magnitude, strategies), in our work with integers. Second, we suggest that valuing emergent thinking within the research on thinking and learning of integer operations is important and entails less focus on correct strategies and places more value on the development of integer understanding. And, last, we critique the consistent rhetoric of "meaningful" for both contexts and instructional models by highlighting that what is meaningful to children may not be meaningful to teachers and researchers (and vice versa). We end the conclusion by posing questions for future research in the realm of thinking and learning within integer addition and subtraction.

The ideas presented throughout this book offer a view of the integer landscape—a landscape of multiple theoretical perspectives (e.g., realistic mathematics education, metaphors, play), various mathematical topics (e.g., order, absolute value, addition, subtraction), and different domains (e.g., children's reasoning about integers versus prospective teacher's reasoning about integers). Throughout this book, one can traverse the integer landscape and experience a conversation about the thinking and learning of integer addition and subtraction.

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Revisiting the Commentaries

Defining the Landmarks

> The map? I will first make it. (Patrick White, Voss)

A map of the integer landscape needs established landmarks. Within her commentary, Stacey challenges us to connect our work by using common language—an essential element to map building. What are the important "landmarks" needed on our map? What language can we agree on as a community in order to navigate the integer landscape and move forward? Throughout the chapters and across the integer literature, researchers use a multitude of words to describe how children think about and solve problems as well as to describe the conditions provided for their instruction and for generating solutions. Such words include strategies, ways of reasoning, mental models, conceptual models, conceptual metaphors, integer models, instructional contexts, and others.

Models and Contexts Conceptual models involve broad ways of thinking about integer problems in terms of the various roles of zero, which numbers are manipulated, and how they are manipulated (see, e.g., Wessman-Enzinger, 2015; Wessman-Enzinger & Mooney, 2014). Generally, conceptual models represent larger categories under which integer models (e.g., chip models, number line models) fall. In turn, these integer models often embody one or more conceptual metaphors (e.g., COLLECTING OBJECTS, MEASURING). For example, the counterbalance conceptual model is a general way to describe the underlying features of the two-colored chip model with zero pairs (Flores, 2008), the floats and anchors model (Glancy & Pettis, 2017), the balloon model (Janvier, 1985), and the charged particle model (Battista, 1983). Bofferding, in Chap. 7 of this book, describes one such instantiation of a conceptual model as an instructional context instead of an integer model, to emphasize that children did not create these integer models on their own; however, *instructional model* might be more appropriate. Moving forward, this may be an important distinction to make: between children's use of models (which we could call integer models) and teachers' use of models (which we could call instructional models). As described by Nurnberger-Haag in Chap. 5 of this book, these models, and the conceptual model of counterbalance more globally, involve a COLLECTING OBJECTS conceptual metaphor (treating both positives and negatives as things with rules for how they interact). Solving integer problems using an integer model or instructional model, which draw on a conceptual model, involves the use of one or more strategies, two of which were described for the chip model in Chap. 10 of this book by Murray.

On a similar but slightly different path, in Chap. 3 of this book, Bishop, Lamb, Philipp, Whitacre, and Schappelle (2016) describe ways of reasoning, which are other broad ways of thinking that draw on different mathematical properties (e.g., order) than the conceptual models. Therefore, the strategies that fall under a particular way of reasoning may or may not involve integer models or instructional models. For example, using movements along a number line or counting by ones are strategies that fall under the order-based way of reasoning (see also Schwarz, Kohn, & Resnick, 1993–1994). However, the former strategy involves an integer model; whereas, the latter one does not. Diverging from both conceptual models and ways of reasoning, mental models as used by Bofferding (2014; see also Chap. 7) focus on conceptualizations of concepts inherent to numbers (i.e., symbols, linear order, and absolute value) and operations. Use of conceptual models, ways of reasoning, and strategies can help clarify students' mental models. Do we need all of these ways of describing integer problems and integer thinking? Are there ways of merging them? If we do not merge them, then at the very least, we need to be more explicit about the meaning of the terms we do use.

Strategies Students use a variety of strategies to solve integer addition and subtraction problems; however, these strategies are not consistently named or defined. For example, in this book, *logical necessity*, defined in Chap. 3, includes comparing related problems and using a fundamental mathematical property (e.g., inverse operations) to solve the related problem. Within Chap. 8, some of these types of strategies are still considered *analogies*, as they involve comparisons between related problems. Based on these definitions, comparing to -5 – 4 to -5 + -4 could be an analogy or logical necessity.

Within this book and the literature, the use of *analogy*, a strategy highlighted by many researchers, has been used haphazardly at times. One definition of analogy used in the commentary from Baroody and other integer literature (e.g., Bishop et al., 2016) involves relating integers to contexts. For example, they consider making sense of integers with temperature, debits and credits, or other models as an analogy. In Chap. 8 and other integer literature (e.g., Bofferding, 2011; Murray, 1985; Wessman-Enzinger, 2017), the use of analogy involves the act of constructing comparisons with other numerical problems (e.g., making an analogy by comparing 2 + 3 and -2 + -3). Knowing that children draw upon analogy-based reasoning (Vosniadou, 1989) and that children may use different types of comparisons (Bishop et al., 2016; Bofferding, 2011; Bofferding & Wessman-Enzinger, 2017), perhaps we should consider drawing on prior definitions of analogy outside of the domain of integer addition and subtraction. Vosniadou (1989), for example, characterized the mechanism behind analogical reasoning as the following: (a) "retrieving a source system (Y), which is similar to X in some way"; (b) "mapping a relational structure from Y to X"; and (c) "evaluating the applicability of this relationship structure for X" (p. 422). She also highlighted two different types of analogies: between-domain and within-domain analogies. It is possible that theoretical descriptions of analogy like this could help guide us as a community of integer researchers.

For integers, within-domain and between-domain analogies depend upon what we call different "domains." If we consider whole numbers or positive integers to be a different domain than integers (positive and negative), then comparing problems like -2 + -3 to 2 + 3 are between-domain analogies and comparing problems like -5 – 4 to -5 + -4 are within-domain analogies. When we consider contexts, it could be that comparing integers to a context is a between-domain analogy, and the numerical analogies are within-domain analogies. Making these decisions about and using theory related to analogical reasoning could help researchers frame their work more cohesively, build on each other's work, and learn more deeply about children's analogical reasoning.

Order and Magnitude How much is a negative worth and how do we talk about it? Cardinality refers to the size of a set. Quantity also suggests a positive amount of things that are countable, although some researchers talk about negative quantities (which might be counterintuitive to students). Both magnitude and absolute value refer to the size of numbers in terms of distance from zero (where direction is not considered), also in terms of positive numbers or zero. Value, while similar to absolute value, is a broader term, although some specify positive values versus negative values. There are two cases for which vocabulary around the worth of integers would be helpful: the instance where we need to talk about numbers' relative distances from zero and the instance where we need to talk about negatives' worth in terms of order, where smaller negatives are considered larger, mathematically, than larger negatives. We propose, as Kaye suggested in her commentary, that we adopt *absolute value* for the first instance related to relative distances, and for the second instance related to order, we suggest that we adopt *linear value*.

Valuing Emergent Thinking

> A path is a prior interpretation of the best way to traverse a landscape. (Rebecca Solnit, Wanderlust: A History of Walking)

Our prior interpretations of integer paths include visions of determining "best models" or goals of "correct" answers (e.g., -7 < -5, recognizing that 0 is a number). What if getting correct answers mattered less? And, valuing *all* thinking mattered more? When thinking about integers matters more than achieving correctness or participating in the cultural norms of integer instruction, then valuing emergent thinking is a higher priority. In his commentary, Baroody pointed out the concerns of overlooking emergent thinking and focusing on the "gloomy" state of integer instruction centered on achievement. Similarly, Peled and Klemer in their commentary pointed out the dangers of expecting students to write certain number sentences, when many expressions may represent certain situations (e.g., -20 – -14 versus -14 – -20). They, instead, suggest an open perspective of allowing for discourse and finding the various perspectives of a problem.

Even when students struggle, they are capable of accomplishing robust mathematics. We see this theme throughout the book. For instance, the young children in Chap. 1 constructed notation or ordered integers unconventionally but did so in sophisticated and playful ways that may contribute to deeper reasoning about integers later. Consider how the students in Chap. 2 struggled to write subtraction number sentences yet constructed unprompted conversations about linear values versus absolute values. Consider the vast array of strategies produced by children in Chap. 3. Specifically, in Chap. 3 and other literature (e.g., Bishop et al., 2014), we read about emergent thinking. Yet, we know little about the nuances of that emergent thinking or what distinguishes it from other ways of thinking. We need to value emergent thinking enough to not only explore it more but also find ways to incorporate or capitalize on emergent thinking in school instruction. It may be that a focus on mental models and conceptual change could help in this process (e.g., Bofferding, 2014).

Meaningful Integer Instruction

> There are always two people in every picture: the photographer and the viewer. (Ansel Adams)

To those who critiqued Ansel Adam's landscape photography as having "no people," he notoriously responded that there are people: the photographer and the viewer. In some ways, this is like the use of *meaningfulness* with integer instructional models. It is easy to critique the lack of meaning within a context or advocate for a particular instructional model. But, similar to the photographs of landscapes, there are two viewers: the teacher-researcher and the student. Where a teacher or researcher sees meaning within an integer instructional model, a student may not; where a student sees meaning within an integer instructional model, a teacher or researcher may not. As we consider the meaningfulness of integer instructional models, we need to be conscientious of the different viewers and viewpoints.

In his commentary, Baroody pointed to the necessity of meaningful integer instruction and encouraged the use of temperature or debits and credits. Yet, language complicates the use of models. Temperature is an example of a context that is meaningful, yet complicated—consider the difficulties of identifying the "least warm temperature" with negative integers (Bofferding & Farmer, 2018). In addition to complications with language, there are complications with the instructional models themselves. In fact, in their commentary, Peled and Klemer point out that children may not use negatives with certain instructional models, such as debits and credits, which is why some students have an easy time with them (e.g., Mukhopadhyay, Resnick, & Schauble, 1990; Whitacre et al., 2015). This complication is why Stephan and Akyuz (see Chap. 4) use net worth, as an alternative to debits and credits alone; it forces students to operate with both positive and negative integers.

When students do connect integers to contexts, they may make unconventional connections (Mukhopadhyay et al., 1990; Wessman-Enzinger & Mooney, 2014). Students may talk about "wanting chocolate bars" or "losing pencils" instead of gains or losses of money. Although students' unconventional connections might seem contrived, researchers have also critiqued typical integer instructional models and contexts for being contrived (Ball, 1993). Thus, we are left wondering: What constitutes meaningful instructional models for integers? The results in Chap. 7 point to the idea that top-down instructional models may not be as productive as bottom-up student-generated models without some explicit connection to students' thinking.

For the difficulties with two-colored chip models, Peled and Klemer point us to the pairing of this model with a disco experiment (Linchevski & Williams, 1999) using Realistic Mathematics Education (RME) tenants (see, e.g., Chap. 4 for RME and model use). Although there are difficulties with the two-colored chip model use, as illustrated in Chap. 10, Peled and Klemer suggest that students can reinvent strategies for two-colored chip models if supported with meaningful contexts. Thus, in order to support discourse and reinventions of mathematics, we need to first learn how others think to find the meaningful contexts. And, it is possible that the use of a chip model from students might not be reinventions but *actual* inventions that may differ from what we presently use or advocate for in curricula. What students may find meaningful may not already be meaningful to the teacher-researcher. Students may play with language as they explore models: "That's the warmest cold temperature." Or, they may play with ideas: "What is biggest, -5 or -7?" (linear value based versus absolute value based). Through students' playful engagement, we can learn what is a truly *meaningful* instructional model. Using this information, together with an understanding of how teachers (and prospective teachers) reason and use models within their curricula and classrooms, researchers and teacher leaders will be better prepared to structure effective learning opportunities for educators.

On the Horizon

As we look back and survey the landscape, we see the beauty of the land and all that has been accomplished with integer addition and subtraction. But, we also need to look forward and think about ways that we can both preserve and enhance the landscape. Preserving the landscape includes continuing research agendas that have established strong roots in the domain of integer addition and subtraction; enhancing the landscape requires us to think innovatively in ways that extend and build on the current work. Any of the following questions could help preserve or move the field toward the horizon:

- How can reasoning about integers be done in playful ways so integers may be integrated into school mathematics sooner?
- How is prerequisite knowledge (i.e., order, linear value, absolute value) related to reasoning about addition and subtraction?
- How are different (student-invented) strategies for integer additions and subtraction supported in classrooms?
- How do students' strategies (and the various frameworks) for integer addition and subtraction relate to their strategies for integer multiplication and division?
- How do we support prospective teachers in learning this rich and dynamic topic (i.e., integers and children's reasoning about integers), especially when integers are overlooked in current teacher preparation programs?

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