Chapter 6
Mass Dimension for Infinite Networks

In this chapter we consider a sequence $\{\mathbb{G}_t\}_{t=1}^{\infty}$ of complex networks such that $\Delta_t \equiv diam(\mathbb{G}_t) \rightarrow \infty$ as $t \rightarrow \infty$. A convenient way to study such networks is to study how the "mass" of $\mathbb{G}_t$ scales with $diam(\mathbb{G}_t)$, where the "mass" of $\mathbb{G}_t$, which we denote by N_t, is the number of nodes in $\mathbb{G}_t$. The fractal dimension used in [73] to characterize $\{\mathbb{G}_t\}_{t=1}^{\infty}$ is

$$d_M \equiv \lim_{t \to \infty} \frac{\log N_t}{\log \Delta_t} , \tag{6.1}$$

and d_M is called the *mass dimension*. An advantage of d_M over the correlation dimension d_C is that it is sometimes much simpler to compute the network diameter than to compute $C(n, s)$ for each n and s, as is required to compute $C(s)$ using (5.3).

A procedure is presented in [73] that uses a probability p to construct a network that exhibits a transition from fractal to non-fractal behavior as p increases from 0 to 1. For $p = 0$, the network does not exhibit the small-world property and has $d_M = 2$, while for $p = 1$ the network does exhibit the small-world property and $d_M = \infty$. The construction, illustrated by Fig. 6.1, begins with $\mathbb{G}_0$, which is a single arc, and $p \in [0, 1]$. Let $\mathbb{G}_t$ be the network after t steps. The network $\mathbb{G}_{t+1}$ is derived from $\mathbb{G}_t$. For each arc in $\mathbb{G}_t$, with probability p we replace the arc with a path of three hops (introducing the two nodes c and d, as illustrated by the top branch of the figure), and with probability $1 - p$ we replace the arc with a path of four hops (introducing the three new nodes c, d, and e, as illustrated by the bottom branch of the figure). For $p = 1$, the first three generations of this construction yield the networks of Fig. 6.2. For $p = 0$, the first three generations of this construction yield the networks of Fig. 6.3. This construction builds upon the construction in [51] of (u, v) trees.

E. Rosenberg, *A Survey of Fractal Dimensions of Networks*, SpringerBriefs in Computer Science, https://doi.org/10.1007/978-3-319-90047-6_6

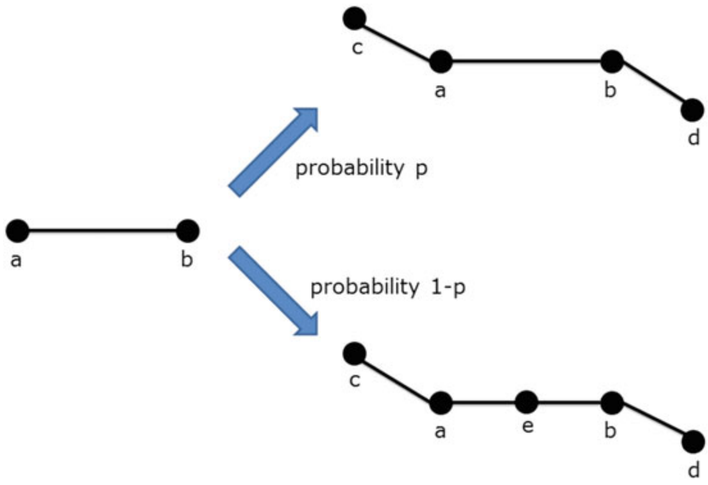
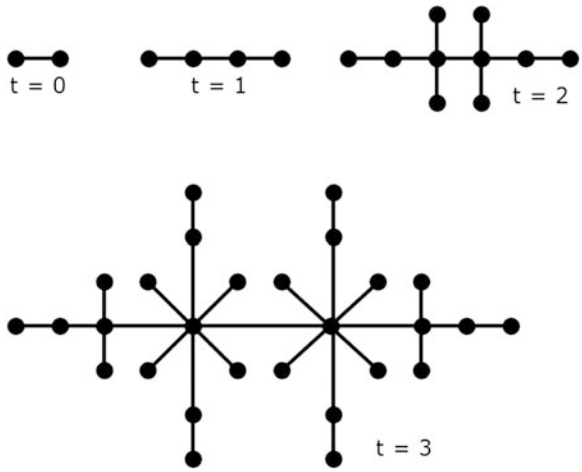


Fig. 6.1 Network that transitions from fractal to non-fractal behavior

Fig. 6.2 Three generations with $p = 1$



Let N_t be the expected number of nodes in $\mathbb{G}_t$, let A_t be the expected number of arcs in $\mathbb{G}_t$, and let Δ_t be the expected diameter of $\mathbb{G}_t$. The quantities N_t, A_t, and Δ_t depend on p, but for notational simplicity we omit that dependence. Since each arc is replaced by three arcs with probability p, and by four arcs with probability $1 - p$, for $t \geq 1$ we have

$$\begin{aligned} A_t &= 3pA_{t-1} + 4(1-p)A_{t-1} = (4-p)A_{t-1} \\ &= (4-p)^2 A_{t-2} = \ldots = (4-p)^t A_0 = (4-p)^t \,, \end{aligned} \tag{6.2}$$

where the final equality follows as $A_0 = 1$, since $\mathbb{G}_0$ consists of a single arc.

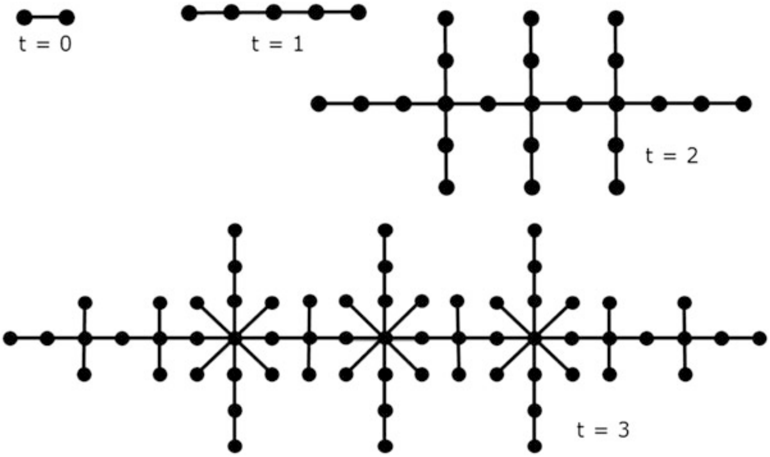


Fig. 6.3 Three generations with $p = 0$

Let x_t be the number of new nodes created in the generation of $\mathbb{G}_t$. Since each existing arc spawns two new nodes with probability p and spawns three new nodes with probability $1 - p$, from (6.2) we have

$$x_t = 2pA_{t-1} + 3(1-p)A_{t-1} = (3-p)A_{t-1} = (3-p)(4-p)^{t-1}\,. \quad (6.3)$$

Since $\mathbb{G}_0$ has two nodes, for $t \geq 1$ we have

$$N_t = 2 + \sum_{i=1}^{t} x_i \;=\; 2 + \sum_{i=1}^{t} (3-p)(4-p)^{i-1}$$
$$= 2 + (3-p)\frac{(4-p)^t - 1}{(3-p)} \;=\; (4-p)^t + 1\,. \quad (6.4)$$

Now we compute the diameter Δ_t of $\mathbb{G}_t$. We begin with the case $p = 1$. For this case, distances between existing node pairs are not altered when new nodes are added. At each time step, the network diameter increases by 2. Since $\Delta_0 = 1$ then $\Delta_t = 2t+1$. Since $N_t \sim (4-p)^t$, then the network diameter grows as the logarithm of the number of nodes, so $\mathbb{G}_t$ exhibits the small-world property for $p = 1$. From (6.1) we have $d_M = \infty$.

Now consider the case $0 \leq p < 1$. For this case, the distances between existing nodes are increased. Consider an arc in the network $\mathbb{G}_{t-1}$, and the endpoints i and j of this arc. With probability p, the distance between i and j in $\mathbb{G}_t$ is 1, and with probability $1 - p$, the distance between i and j in $\mathbb{G}_t$ is 2. The expected distance between i and j in $\mathbb{G}_t$ is therefore $p + 2(1-p) = 2 - p$. Since each $\mathbb{G}_t$ is a tree, for $t \geq 1$ we have

$$\Delta_t = p\Delta_{t-1} + 2(1-p)\Delta_{t-1} + 2 = (2-p)\Delta_{t-1} + 2$$

and $\Delta_0 = 1$. This yields [73]

$$\Delta_t = \left(1 + \frac{2}{1-p}\right)(2-p)^t - \frac{2}{1-p}\,. \tag{6.5}$$

From (6.1), (6.4), and (6.5),

$$d_M = \lim_{t\to\infty} \frac{\log N_t}{\log \Delta_t} = \lim_{t\to\infty} \frac{\log[(4-p)^t + 1]}{\log\left[\left(1+\frac{2}{1-p}\right)(2-p)^t - \frac{2}{1-p}\right]} = \frac{\log(4-p)}{\log(2-p)}\,, \tag{6.6}$$

so d_M is finite, and $\mathbb{G}_t$ does not exhibit the small-world property. For $p = 0$ we have $d_M = \log 4/\log 2 = 2$. Note that $\log(4-p)/\log(2-p) \to \infty$ as $p \to 1$.

6.1 Transfinite Fractal Dimension

A deterministic recursive construction can be used to create a self-similar network, called a (u, v)-*flower*, where u and v are positive integers [51]. By varying u and v, both fractal and non-fractal networks can be generated. The construction starts at time $t = 1$ with a cyclic graph (a ring), with $w \equiv u + v$ arcs and w nodes. At time $t + 1$, replace each arc of the time t network by two parallel paths, one with u arcs, and one with v arcs. Without loss of generality, assume $u \le v$. Figure 6.4 illustrates three generations of a $(1, 3)$-flower. The $t = 1$ network has four arcs. To generate the $t = 2$ network, arc a is replaced by the path $\{b\}$ with one arc, and also by the path $\{c, d, e\}$ with three arcs; the other three arcs in Fig. 6.4a are similarly replaced. To generate the $t = 3$ network, arc d is replaced by the path $\{p\}$ with one arc, and also by the path $\{q, r, s\}$ with three arcs; the other fifteen arcs in Fig. 6.4b are similarly replaced. The self-similarity of the (u, v)-flowers follows from an equivalent method of construction: generate the time $t + 1$ network by making w copies of the time t network, and joining the copies at the hubs.

Let $\mathbb{G}_t$ denote the (u, v)-flower at time t. The number of arcs in $\mathbb{G}_t$ is $A_t = w^t = (u + v)^t$. The number N_t of nodes in $\mathbb{G}_t$ satisfies the recursion $N_t = wN_{t-1} - w$; with the boundary condition $N_1 = w$ we obtain [51]

$$N_t = \left(\frac{w-2}{w-1}\right) w^t + \left(\frac{w}{w-1}\right)\,. \tag{6.7}$$

Consider the case $u = 1$. Let Δ_t be the diameter of $\mathbb{G}_t$. It can be shown [51] that for $(1, v)$-flowers and odd v we have $\Delta_t = (v-1)t + (3-v)/2$ while in general, for $(1, v)$-flowers and any v,

$$\Delta_t \sim (v-1)t\,. \tag{6.8}$$

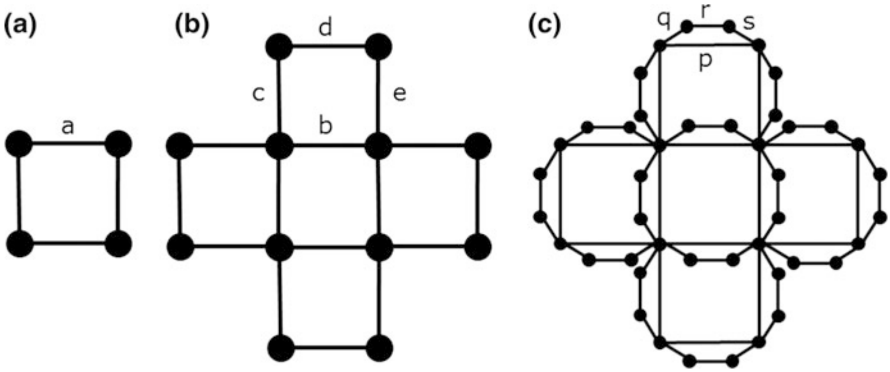


Fig. 6.4 Three generations of a $(1, 3)$-flower

Since $N_t \sim w^t$ then $\Delta_t \sim \log N_t$, so $(1, v)$-flowers enjoy the small-world property. By (6.1), (6.7), and (6.8), for $(1, v)$-flowers we have

$$d_M = \lim_{t\to\infty} \frac{\log N_t}{\log \Delta_t} = \lim_{t\to\infty} \frac{\log w^t}{\log t} = \infty\,, \tag{6.9}$$

so $(1, v)$-flowers have an infinite mass dimension.

We want to define a new type of fractal dimension that is finite for $(1, v)$-flowers and for other networks whose mass dimension is infinite. For $(1, v)$-flowers, from (6.7) we have

$$N_t \sim w^t = (1 + v)^t$$

as $t \to \infty$, so $\log N_t \sim t \log(1 + v)$. From (6.8) we have $\Delta_t \sim (v - 1)t$ as $t \to \infty$. Since both $\log N_t$ and Δ_t behave like a linear function of t as $t \to \infty$, but with different slopes, let d_E be the ratio of the slopes, so

$$d_E \equiv \frac{\log(1 + v)}{v - 1}\,. \tag{6.10}$$

From (6.10), (6.8), and (6.7), as $t \to \infty$ we have

$$d_E = \frac{t \log(1 + v)}{t(v - 1)} = \frac{\log(1 + v)^t}{t(v - 1)} = \frac{\log w^t}{t(v - 1)} \sim \frac{\log N_t}{\Delta_t}\,, \tag{6.11}$$

from which we obtain

$$N_t \sim e^{d_E \Delta_t}\,. \tag{6.12}$$

Define $\alpha_t \equiv \Delta_{t+1} - \Delta_t$. From (6.12),

$$\frac{N_{t+1}}{N_t} \sim \frac{e^{d_E \Delta_{t+1}}}{e^{d_E \Delta_t}} = e^{d_E \alpha_t} . \tag{6.13}$$

Writing $N_t = N(\Delta_t)$ for some function $N(\cdot)$, we have

$$N_{t+1} = N(\Delta_{t+1}) = N(\Delta_t + \alpha_t) .$$

From this and (6.13) we have

$$N(\Delta_t + \alpha_t) \sim N(\Delta_t) e^{d_E \alpha_t} , \tag{6.14}$$

which says that, for $t \gg 1$, when the diameter increases by α_t, the number of nodes increases by a factor which is exponential in $d_E \alpha_t$. As observed in [51], in (6.14) there is some arbitrariness in the selection of e as the base of the exponential term $e^{d_E \alpha_t}$, since from (6.10) the numerical value of d_E depends on the logarithm base. If (6.14) holds as $t \to \infty$ for a sequence of self similar graphs $\{\mathbb{G}_t\}$ then d_E is called the *transfinite fractal dimension*, since this dimension "usefully distinguishes between different graphs of infinite dimensionality" [51]. Self-similar networks such as $(1, v)$-flowers whose mass dimension d_M is infinite, but whose transfinite fractal dimension d_E is finite, are called *transfinite fractal* networks, or simply *transfractals*. Thus $(1, v)$-flowers are transfractals with transfinite fractal dimension $d_E = \log(1 + v)/(v - 1)$.

Finally, consider (u, v)-flowers with $u > 1$. It can be shown [51] that $\Delta_t \sim u^t$. Using (6.7) we have

$$\lim_{t \to \infty} \frac{\log N_t}{\log \Delta_t} = \lim_{t \to \infty} \frac{\log w^t}{\log u^t} = \frac{\log(u + v)}{\log u} ,$$

so

$$d_M = \frac{\log(u + v)}{\log u} .$$

Since d_M is finite, these networks are fractals, not transfractals, and these networks do not enjoy the small-world property.