Chapter 11
Zeta Dimension

In this final chapter we consider the use of the zeta function

$$\zeta(\alpha) = \sum_{i=1}^{\infty} i^{-\alpha} \tag{11.1}$$

to define the dimension of a network. The zeta function has a rich history [8, 17]. It was studied by Euler in 1737 for non-negative real α and extended in 1859 by Riemann to complex α. We will consider the zeta function only for non-negative real α. The zeta function converges for $\alpha > 1$ and diverges otherwise. It is a decreasing function of α and $\zeta(\alpha) \to 1$ as $\alpha \to \infty$ [54].

The zeta function has been used to define the fractal dimension of a finite complex network [54]. Although the zeta dimension of a network has not enjoyed widespread popularity, it has interesting connections to the Hausdorff dimension. Recall from (7.5) in Chap. 7 that $\partial\mathbb{N}(n, r)$ is the set of nodes whose distance from node n is exactly r, and $|\partial\mathbb{N}(n, r)|$ is the number of such nodes. Define the *graph surface function* by

$$S_r \equiv \frac{1}{N} \sum_{n \in \mathbb{N}} |\partial\mathbb{N}(n, r)| \, , \tag{11.2}$$

so S_r is the average number of nodes at a distance r from a random node in the network. Define the *graph zeta function* $\zeta_{\mathbb{G}}(\alpha)$ [54] by

$$\zeta_{\mathbb{G}}(\alpha) \equiv \frac{1}{N} \sum_{x \in \mathbb{N}} \sum_{\substack{y \in \mathbb{N} \\ y \neq x}} dist(x, y)^{-\alpha} \, . \tag{11.3}$$

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Since $\mathbb{G}$ is a finite network, then $\zeta_{\mathbb{G}}(\alpha)$ is finite, and the graph zeta function and the graph surface function are related by

$$\begin{aligned}\zeta_{\mathbb{G}}(\alpha) &= \frac{1}{N}\sum_{x\in\mathbb{N}}\sum_{r\geq 1}|\partial\mathbb{N}(x,r)|r^{-\alpha}\\ &= \sum_{r\geq 1}\Big(\frac{1}{N}\sum_{x\in\mathbb{N}}|\partial\mathbb{N}(x,r)|\Big)r^{-\alpha}\\ &= \sum_{r\geq 1}S_r r^{-\alpha}\,. \end{aligned} \tag{11.4}$$

The function $\zeta_{\mathbb{G}}(\alpha)$ is decreasing in α. For $\alpha = 0$ we have

$$\zeta_{\mathbb{G}}(0) = (1/N)\sum_{x\in\mathbb{N}}(N-1) = N-1\,.$$

For a given $x \in \mathbb{N}$, if $dist(x, y) > 1$ then $dist(x, y)^{-\alpha} \to 0$ as $\alpha \to \infty$, so

$$\lim_{\alpha\to\infty}\sum_{\substack{y\in\mathbb{N}\\ y\neq x}} dist(x,y)^{-\alpha} = \lim_{\alpha\to\infty}\sum_{\substack{y\in\mathbb{N}\\ dist(x,y)=1}} dist(x,y)^{-\alpha} = \delta_x\,, \tag{11.5}$$

where δ_x is the node degree of x. Thus $\zeta_{\mathbb{G}}(\alpha)$ approaches the average node degree as $\alpha \to \infty$.

Since (11.3) defines $\zeta_{\mathbb{G}}(\alpha)$ only for a finite network, we would like to define $\zeta_{\mathbb{G}}(\alpha)$ for an infinite network $\mathbb{G}$. If $\mathbb{G} = \lim_{N\to\infty}\mathbb{G}_N$, where $\mathbb{G}_N$ has N^E nodes, then we can define $\zeta_{\mathbb{G}}(\alpha) \equiv \lim_{N\to\infty}\zeta_{\mathbb{G}_N}(\alpha)$, as is implicitly done in [54]. For example, $\mathbb{G} = \lim_{N\to\infty}\mathbb{G}_N$ holds when $\mathbb{G}$ is an infinite E-dimensional rectilinear lattice and $\mathbb{G}_N$ is a finite E-dimensional rectilinear lattice for which each edge has N nodes. Table 11.1, from [54], provides $\zeta_{\mathbb{G}}(\alpha)$ for an infinite rectilinear lattice in $\mathbb{R}^E$ and the L_1 norm. Here Γ denotes the gamma function, so $\Gamma(E) = (E-1)!$

We are interested in infinite networks $\mathbb{G}$ for which $\zeta_{\mathbb{G}}(\alpha)$ can be infinite. Since $\zeta_{\mathbb{G}}(\alpha)$ is a decreasing function of α, if $\zeta_{\mathbb{G}}(\alpha)$ is finite for some value α, it is finite for $\alpha' > \alpha$. If $\zeta_{\mathbb{G}}(\alpha)$ is infinite for some value α, it is infinite for $\alpha' < \alpha$. Thus there is at

Table 11.1 S_r and $\zeta_{\mathbb{G}}(\alpha)$ for an infinite rectilinear lattice in $\mathbb{R}^E$

E	S_r	$\zeta_{\mathbb{G}}(\alpha)$
1	2	$2\zeta(\alpha)$
2	$4r$	$4\zeta(\alpha-1)$
3	$4r^2+2$	$4\zeta(\alpha-2)+2\zeta(\alpha)$
4	$(8/3)r^3+(16/3)r$	$(8/3)\zeta(\alpha-3)+(16/3)\zeta(\alpha-1)$
$r\to\infty$	$O\big(2^E r^{E-1}/\Gamma(E)\big)$	$O\big(2^E\zeta(\alpha-E+1)/\Gamma(E)\big)$

most one value of α for which $\zeta_{\mathbb{G}}(\alpha)$ transitions from being infinite to finite, and the *zeta dimension* d_Z of $\mathbb{G}$ is the value at which this transition occurs. This definition parallels the definition in Sect. 1.2 of the Hausdorff dimension as that value of d for which $v^{\star}(d)$ transitions from infinite to finite. If for all α we have $\zeta_{\mathbb{G}}(\alpha) = \infty$ then d_Z is defined to be ∞.

Example 11.1 Let $\mathbb{G}$ be an infinite rectilinear lattice in $\mathbb{R}^3$. From Table 11.1 we have

$$\zeta_{\mathbb{G}}(\alpha) = 4\zeta(\alpha - 2) + 2\zeta(\alpha) .$$

Since $4\zeta(\alpha - 2) + 2\zeta(\alpha) < \infty$ only for $\alpha > 3$, then

$$d_Z = \inf\{\alpha \mid \zeta_{\mathbb{G}}(\alpha) < \infty\} = 3 . \quad \square$$

Example 11.2 As in [55], consider a random graph in which each pair of nodes is connected with probability p. For any three distinct nodes x, y, and z, the probability that z is not connected to both x and y is $1 - p^2$. The probability that x and y are not both connected to some other node is $(1 - p^2)^{N-2}$, which approaches 0 as $N \to \infty$. Thus for large N each pair of nodes is almost surely connected by a path of length at most 2. For large N, each node has $p(N - 1)$ neighbors, so from (11.2) we have $S_1 \approx p(N - 1)$. For large N, the number S_2 of nodes at distance 2 from a random node is given by $S_2 \approx (N - 1) - S_1 = (N - 1)(1 - p)$. Hence

$$\zeta_{\mathbb{G}_N}(\alpha) \approx p(N - 1) + (N - 1)(1 - p)2^{-\alpha} .$$

Since $\lim_{N\to\infty} \zeta_{\mathbb{G}_N}(\alpha) = \infty$ for all α then $d_Z = \infty$. $\square$

An alternative definition of the dimension of an infinite graph, using the zeta function, but not requiring averages over all the nodes of the graph, is given in [55]. For $n \in \mathbb{N}$, define

$$\zeta_{\mathbb{G}}(n, \alpha) = \sum_{\substack{x\in\mathbb{N} \\ x\neq n}} dist(n, x)^{-\alpha} .$$

There is exactly one value of α at which $\zeta_{\mathbb{G}}(n, \alpha)$ transitions from being infinite to finite; denote this value by $d_Z(n)$. The alternative definition of the zeta dimension is

$$d_Z \equiv \limsup_{n\in\mathbb{N}} d_Z(n) .$$

This definition is not always identical to the above definition of d_Z as the value at which $\zeta_{\mathbb{G}}(\alpha)$ transitions infinite to finite [55].