Exploring the Numerics of Branch-and-Cut for Mixed Integer Linear Optimization

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1 Introduction

The branch-and-cut algorithm for mixed integer linear optimization problems (MILPs) combines aspects of the branch-and-bound algorithm with the cutting plane algorithm to strengthen the initial LP relaxation (see [4] for a complete description of these operations and the definitions of these terms). While branching increases the number of subproblems to be solved and should thus be avoided in principle, the addition of too many cutting planes often results in an LP relaxation with undesirable numerical properties. Recent research into the viability of solving MILPs using a pure cutting plane approach has provided some insight into how and why this happens and has explored techniques to generate a sequence of valid inequalities whose addition to the LP relaxation is less likely to cause difficulties [5, 9].

In general, branching and cutting must be used carefully in concert with each other to maintain numerical stability. The effect of these operations on numerics is not well understood, however, and is difficult to control directly. There exists a number of approaches to effectively combine the branching and cutting operations. In some solvers, cutting is only done at the root node, while in others, cuts are added throughout the tree. As with any numerical process, implementations of these solution algorithms use floating-point arithmetic and are subject to accumulation of roundoff errors within the computations. Without appropriate handling of these

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errors, the algorithms may return unreliable results, failing to behave or terminate as expected.

Modern MILP solvers use a wide range of techniques to mitigate the difficulties associated with numerical errors. For example, it is standard practice to discard or modify cuts whose coefficients differ significantly in magnitude, since these inequalities are likely to degrade the conditioning of the LP relaxation. This and other techniques help to ensure that the LP relaxation will have better numerical properties and increases the computational stability of the algorithm.

It is well understood that the addition of cutting planes has the potential to negatively impact the numerical properties of the LP relaxation, even after steps have been taken to improve their reliability. On the other hand, branching may counteract this effect to some extent, leading to a more stable algorithm overall. In this paper we seek to carefully investigate the impact of both branching and cutting on the numerical properties of the LP subproblems solved in the branch-and-cut algorithm. The purpose of this work is both to confirm existing folklore, namely that branching improves condition and cutting degrades it, as well as to explore the potential for directly controlling numerical properties through judicious algorithmic choices.

In Sect. 2 we discuss the choice and computation of the basis matrix condition numbers as a measure of numerical stability. In Sect. 3 we describe computational results regarding how branching and cutting affect the condition numbers. Section 4 discusses some implications of our findings and ongoing work.

2 Condition Numbers

The *condition number* of a numerical problem is a bound on the relative change (in terms of a given norm) in the solution to a problem that can occur as a result of a change in the input (see [3] for formal definitions). For example, the condition number of a matrix A is $\kappa(A) = \|A\|_2 \|A^{-1}\|_2$ and yields a bound on how much the solution to the linear system of equations $Ax = b$ might change, relative to a change in the right hand side vector b. For LPs, a handful of different condition numbers have also been defined; a comprehensive treatment of condition numbers for LPs, along with much more general discussion regarding the concept of problem condition, is given in [3].

When LPs are solved by the simplex method, a sequence of basis matrices are encountered (see [4]), each corresponding to a square system of linear equations. Although condition numbers can be defined for LPs themselves, it is the condition number of the basis matrices encountered during a simplex solve (particularly the optimal basis) that is the most relevant measure of numerical stability of the branch-and-cut algorithm. A primary reason for this is that the solution to the LP relaxation is obtained by solving a system of equations involving the basis matrix so that the condition number of this matrix determines the multiplicative effect of numerical errors in the computed cuts.

After applying cutting planes or branching at a node, the resulting modified LP is re-solved. In general, we expect that the newly added cuts or branching inequalities will be binding at the new basic solution, which means that these additional constraints are a factor in determining the conditioning of the basis. Thus, measuring the condition number of these linear systems and how they change as a result of the added cuts or branching inequalities should give some insight into the numerical behavior of the simplex algorithm and ultimately the branch-and-cut algorithm. In this paper, we are looking for overall trends (how much does the addition of cuts *generally* degrade the conditioning), so we consider these numbers in the aggregate and provide some suggestions for visualizing this data.

Since we are interested in an accurate picture, we use the 2-norm power iteration method to determine condition numbers. This method provides an accurate answer, though it is unlikely to be efficient enough for practical use. An excellent discussion on algorithms for condition number estimation is given in [6, Chap. 15].

3 Experiments

To study the effect of cuts on conditioning, we solved a subset of instances from MIPLIB 3 [2], MIPLIB 2003 [1], and MIPLIB 2010 [7] test sets, collecting detailed statistics. The solver used was SCIP 4.0 with the LP solver SoPlex 3.0 [8] (with slight modifications to allow access to the condition number information). We used a time limit of one hour and a node limit of 10,000.

To get a clearer picture, we deactivated many advanced features, such as primal heuristics, domain propagation, and conflict analysis. Furthermore, we only generated Gomory cuts and disabled all other cutting plane generators. While SCIP only applies cutting planes at the root by default, we enabled cut generation at all nodes in order to study how this affects conditioning. Note that although cuts are generated throughout the tree, SCIP still uses a scoring strategy to determine which inequalities should actually be added.

In what follows, we first study how the condition number of the basis matrix evolves at the root node, where the initial LP relaxation is solved and initial rounds of cuts are added, and then study how the condition number of the basis matrices are affected by branching and cutting as the algorithm progresses.

3.1 Root Node Analysis

In general, we expect the condition number of the basis matrix to degrade as a result of operations performed in the root node and our initial computations are aimed at confirming this. Figure 1 shows the condition number of each basis matrix encountered during each iteration of the solution of the initial LP relaxation and during each iteration of the re-solve occurring after adding each round of cuts for

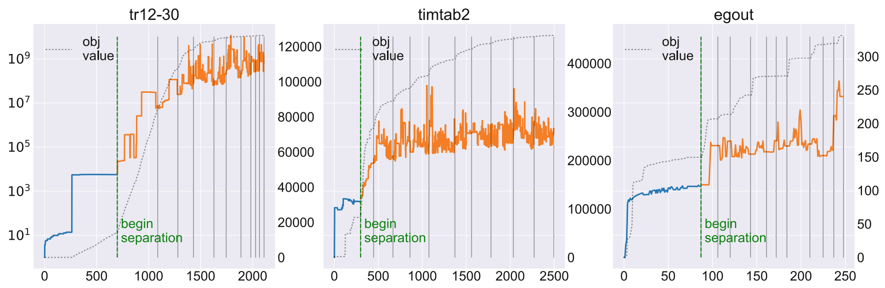
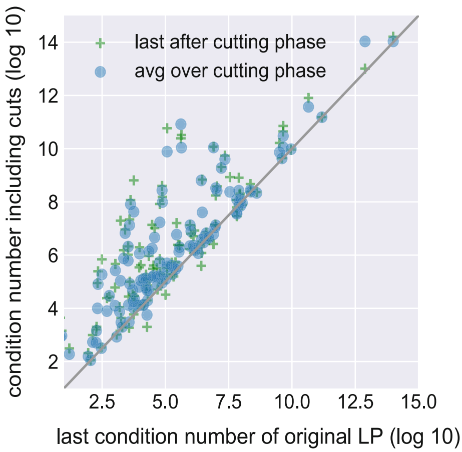


Fig. 1 Condition number development (vertical axis, in log scale) for every simplex iteration in the root node (horizontal axis) including re-optimizations after adding cutting planes in multiple rounds (vertical lines). A plot of objective values at each iteration is overlaid as a dashed gray line with the scales given to the right of each plot

Fig. 2 Root node: Comparison of condition numbers of the original LP and including cutting planes



selected instances from our test set. One can observe that during the early iterations—especially of the initial relaxation in the root node—the condition numbers of the basis matrices grow quickly. This is expected, as more structural variables are pivoted into the basis, while slack variables are pivoted out. Since the initial basis is always the identity matrix, which has condition number 1, the conditioning can only degrade at first. After the initial optimization, the MILP solver tries to generate Gomory cuts. This computation involves the basis matrix itself, so an ill-conditioned basis matrix can prevent precise calculation of the coefficients of the new constraint. Moreover, adding these new rows to the LP often deteriorates its condition number even further as can be seen in Fig. 1. This sample of instances clearly shows the expected behavior.

Figure 2 is a visualization of the difference between the condition number of the optimal basis of the original LP and two other numbers: (1) the average over all bases encountered during the cutting procedure and (2) the condition number of the final optimal basis. While for some instances there is a slight improvement after

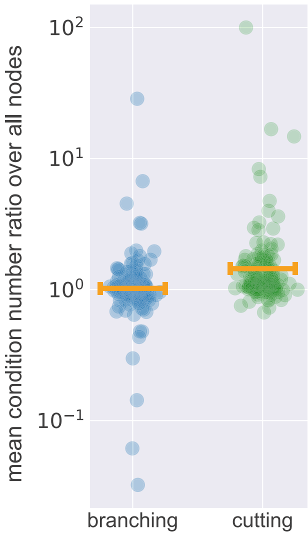


Fig. 3 Effects of branching and cutting

adding cuts, in most cases addition of cuts leads to an increased condition number, as expected.

3.2 *Tree Analysis*

One way in which the addition of cuts can cause basis matrices to become poorly conditioned is if the associated hyperplanes are nearly parallel; addition of many such cuts may lead to a *tailing off* of the cutting plane algorithm as many similar cuts are generated and the process stalls. Although branching also involves imposing a special kind of "cut" to the resulting subproblems, these branching constraints have a simple form (the coefficient vector is a unit vector), which makes them quite attractive from a numerical point of view. In particular, they are mutually orthogonal and unlikely to degrade the conditioning much in general. As such, we may be tempted to hope that the addition of this special kind of inequality may even improve conditioning.

Despite the apparent plausibility of this hypothesis, our experiments do not fully support it, though they do show a significant difference between the effect of branching versus cutting, as expected. In Fig. 3, we show how branching and cutting impact the numerical stability. The left plot shows the average relative change in the condition number as a result of the addition of the branching constraints. Similarly, the right plot shows the average relative change in conditioning resulting from the addition of cuts. In each case, we took the difference between the condition numbers of the optimal basis matrices before and after either branching or cutting. Each dot then represents the average across all nodes for a given instance. The bar represents the mean over all instances. While branching does not seem to have a significant

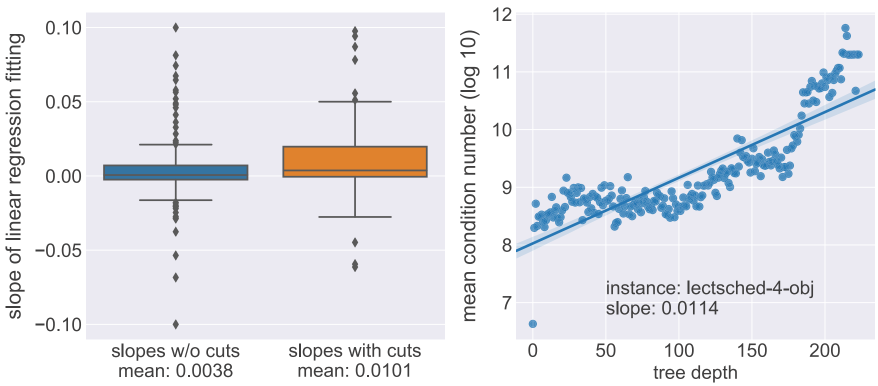


Fig. 4 Condition number development in the tree. Left: Distribution of linear regression slopes of all instances in the test set. Right: Single instance example

effect on average, adding cutting planes clearly leads to an increase in the condition number. Thus, despite the observation that branching does not appear to degrade the condition number in the same way as cut generation, it does not appear to help it either.

In Fig. 4 we visualize how condition numbers degrade generally as a function of the depth of a given node. The idea is to determine whether conditioning generally degrades consistently as the tree gets deeper. The right figure plots the average condition number across all nodes at a given depth, along with a regression line showing the average degradation in the log of the condition number per level in the tree for a single instance. The left figure shows the distribution of slopes of this same linear regression across all instances both with cuts and with a pure branch-and-bound.

It appears that in general, the condition number often has a strong positive correlation with the tree depth if cuts are added throughout the solving process. When cutting is disabled this effect is much less strong. One has to be aware that the behavior of a single instance might be much different from what the trend predicts.

4 Outlook

In this paper, we presented a preliminary exploration of the numerical behavior of SCIP, a state-of-the-art MILP solver. In the future, we hope to do similar explorations with other solvers to determine what the overall behavior is and where additional control of the numerical stability might have an impact. The eventual goal is to determine whether it is possible to more directly estimate the impact of certain algorithmic choices on numerical behavior and whether this could lead to improved control mechanisms.

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