Sum-Intersection Property of Sobolev Spaces

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Dedicated to Haim Brezis and Louis Nirenberg, with deep esteem and affection

1 Introduction

In connection with the factorization of unimodular Sobolev maps, Haim Brezis and the author observed the following property of Sobolev spaces [5]. Let $1 < p < \infty$ and $0 < \lambda < 1$. Then every function $f \in W^{1,p}(\mathbb{R}^N)$ can be decomposed as

$$f = g + h, \text{ with } g \in (W^{\lambda,p/\lambda} \cap W^{1,p})(\mathbb{R}^N) \text{ and } h \in (W^{p,1} \cap W^{1,p})(\mathbb{R}^N). \quad (1)$$

We will present in appendix a proof of this fact using factorization. We will also explain there how (1) is related to functional calculus (superposition operators) in Sobolev spaces.

Decomposition (1) has a flavor of interpolation, and indeed we have for example when $p = 2$ the equality [20, Section 2.4.3, Theorem, p. 66]

$$W^{1,2} = [W^{\lambda,2/\lambda}, F^2_{1,1}]_{\theta,2}, \text{ with } \theta := 1/(2-\lambda). \quad (2)$$

[We will recall in the next section the definition of the Triebel-Lizorkin spaces $F^s_{p,q}$.] Using (2) and the embedding $F^2_{1,1} \hookrightarrow W^{2,1}$ (see the next section), we find that $W^{1,2} \subset W^{\lambda,2/\lambda} + W^{2,1}$. However, this does not yield the stronger conclusion $W^{1,2} \subset (W^{\lambda,2/\lambda} \cap W^{1,2}) + (W^{2,1} \cap W^{1,2})$. Actually, one cannot derive the equality $Z = (X \cap Z) + (Y \cap Z)$ merely from the inclusion $Z \subset X + Y$ (take e.g. $X = \mathbb{R} \times \{0\}$, $Y = \{0\} \times \mathbb{R}$ and $Z = \{(x, x);\ x \in \mathbb{R})\}$.

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We address here the following question. Let $0 \leq s, s_1, s_2 < \infty$, and $1 \leq p_1, p, p_2 \leq \infty$. Assume that

$$W^{s,p}(\mathbb{R}^N) \subset W^{s_1,p_1}(\mathbb{R}^N) + W^{s_2,p_2}(\mathbb{R}^N) \text{ for any } N. \tag{3}$$

Is it true that

$$W^{s,p}(\mathbb{R}^N) = (W^{s_1,p_1} \cap W^{s,p})(\mathbb{R}^N) + (W^{s_2,p_2} \cap W^{s,p})(\mathbb{R}^N) \text{ for any } N? \tag{4}$$

We emphasize the fact that we ask for N-independent properties. For example, by the Sobolev embeddings we have $W^{1,1} \subset L^2$ when $N = 1$ or 2, but not for $N \geq 3$, and thus (3) does not hold for $s_1 = s_2 = 0$, $s = 1$, $p_1 = p_2 = 2$, $p = 1$.

Our first results characterize *most* of the triples $T = (W^{s_1,p_1}, W^{s,p}, W^{s_2,p_2})$ such that (3) and (4) hold.

Proposition 1 *Assume that* (3) *holds. Then there exists some* $\theta \in [0, 1]$ *such that*

$$s \geq \theta s_1 + (1 - \theta)s_2, \tag{5}$$

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{(1 - \theta)}{p_2}. \tag{6}$$

Proposition 2 *Assume that for some* $\theta \in [0, 1]$ *we have* (6) *and* $s > \theta s_1 + (1-\theta)s_2$. *Then both* (3) *and* (4) *hold.*

On the other hand, (3) and (4) trivially hold when (5)–(6) are satisfied with $\theta = 0$ or 1, since we then have either $W^{s,p} \hookrightarrow W^{s_2,p_2}$, or $W^{s,p} \hookrightarrow W^{s_1,p_1}$. We next investigate the case where

$$s = \theta s_1 + (1 - \theta)s_2, \ \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2} \text{ for some } \theta \in (0, 1). \tag{7}$$

In this case, (3) holds *most* of the time, but *not always*. For example, when $N = 1$ we have

$$W^{1/2,2}(\mathbb{R}) \not\subset W^{1,1}(\mathbb{R}) + L^{\infty}(\mathbb{R}), \tag{8}$$

i.e., (3) does not hold for the triple $T = (W^{1,1}, W^{1/2,2}, L^{\infty})$. Indeed, for $N = 1$ we have $W^{1,1} \hookrightarrow L^{\infty}$, and thus $W^{1,1} + L^{\infty} = L^{\infty}$. However, $W^{1/2,2} \not\subset L^{\infty}$.

Definition 1 A triple $T = (W^{s_1,p_1}, W^{s,p}, W^{s_2,p_2})$ is *admissible* if it satisfies (7).

An admissible triple T is *irregular* if $s_1 \neq s_2$, $1 < p < \infty$ and (exactly) one of the spaces W^{s_1,p_1}, W^{s_2,p_2} is of the form $W^{k,\infty}$ with $k \in \mathbb{N}$. T is *regular* otherwise.

Thus $T = (W^{1,1}, W^{1/2,2}, L^{\infty})$ (which corresponds to the example occurring in (8)) is irregular.

Our main result is the following

Theorem 1 *Let T be a regular triple. Then both* (3) *and* (4) *hold.*

Equivalently, for every regular triple T we have

$$W^{s,p}(\mathbb{R}^N) = (W^{s_1,p_1} \cap W^{s,p})(\mathbb{R}^N) + (W^{s_2,p_2} \cap W^{s,p})(\mathbb{R}^N), \ \forall N. \tag{9}$$

For most of the regular triples, (4) follows *automatically* from (3), as explained in Proposition 3 below. Thus, in particular, the conclusion of the theorem follows whenever T is as in Proposition 3 and $W^{s,p}$ can be obtained by interpolation from W^{s_1,p_1} and W^{s_2,p_2}. However, when T is admissible $W^{s,p}$ need not be an interpolation space between W^{s_1,p_1} and W^{s_2,p_2}, at least for the standard real and complex methods [20, Sections 2.4.2–2.4.7, p. 64–73]; thus one cannot derive Theorem 1 directly from Proposition 3. We will present, in Sect. 3, a proof of Theorem 1 which does not rely on interpolation and establishes simultaneously (3) and (4).

Definition 2 A Sobolev space $W^{s,p}$ is *exceptional* if $s \in \mathbb{N}$ and either $p = 1$ or $p = \infty$. It is *ordinary* otherwise.

Proposition 3 *Assume that $W^{s,p}$, W^{s_1,p_1} and W^{s_2,p_2} are all three ordinary Sobolev spaces. Assume that for some (fixed) N we have $W^{s,p}(\mathbb{R}^N) \subset W^{s_1,p_1}(\mathbb{R}^N) + W^{s_2,p_2}(\mathbb{R}^N)$. Then for such N we have*

$$W^{s,p}(\mathbb{R}^N) = (W^{s_1,p_1} \cap W^{s,p})(\mathbb{R}^N) + (W^{s_2,p_2} \cap W^{s,p})(\mathbb{R}^N).$$

We now turn to irregular T's. At least in some special cases (see (8) and, more generally, the triples $T = (W^{1,1}, W^{1/p,p}, L^\infty)$, with $1 < p < \infty$), (3) does not hold for such triples. We do not know the characterization of irregular triples T for which (3) and/or (4) do not hold. For irregular triples, we were only able to establish a weaker form of (4), in which the space $W^{k,\infty}$ is replaced by a slightly larger space, modeled on *bmo* (the *local BMO* space whose definition will be recalled in the next section).

Theorem 2 *Let T be an irregular triple, and assume e.g. that $p_2 = \infty$ (and thus s_2 is an integer). Let $1 < q_2 < \infty$. Then*

$$W^{s,p}(\mathbb{R}^N) = (W^{s_1,p_1} \cap W^{s,p})(\mathbb{R}^N) + (F^{s_2}_{\infty,q_2} \cap W^{s,p})(\mathbb{R}^N). \tag{10}$$

In particular, when $s_2 = 0$ (and thus $W^{s_2,p_2} = L^\infty$) we have

$$W^{s,p}(\mathbb{R}^N) = (W^{s_1,p_1} \cap W^{s,p})(\mathbb{R}^N) + (bmo \cap W^{s,p})(\mathbb{R}^N). \tag{11}$$

When $s_2 > 0$, we have

$$\begin{aligned} W^{s,p}(\mathbb{R}^N) =& (W^{s_1,p_1} \cap W^{s,p})(\mathbb{R}^N) \\ &+ (\{f \in W^{s_2-1,\infty};\ D^{s_2-1} f \in bmo\} \cap W^{s,p})(\mathbb{R}^N). \end{aligned} \tag{12}$$

In the special case $s_2 = 0$, $s \notin \mathbb{N}$, $p_1 = 1$, Theorem 2 was established in [5, Chapter 6].

Remark 1 The question of the validity of (3)–(4) is somewhat *dual* to the one of the validity of the Gagliardo-Nirenberg inequalities. There, one asks whether the inclusion

$$W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,p_2}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N) \tag{13}$$

leads, for some appropriate $\theta \in [0, 1]$, to the estimate

$$\|f\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\mathbb{R}^N)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{R}^N)}. \tag{14}$$

In the spirit of our Proposition 1, one may prove that the validity of (13) for every N requires

$$s \le \theta s_1 + (1-\theta)s_2, \tag{15}$$

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{(1-\theta)}{p_2} \tag{16}$$

for some $\theta \in [0, 1]$. If we have either "$<$" in (15) or $\theta \in \{0, 1\}$, then we have both (13) and (14); this follows from the main result in [4]. As in our situation, the interesting case is the one of admissible triples. In that case, (15) and (16) hold when s_1, s, s_2 are integers, as established in the seminal contributions of Gagliardo [11] and Nirenberg [15]. It turns out that (15) and (16) hold for *most* of the admissible triples, but not all of them. A characterization of the admissible triples for which (15) and (16) hold has been obtained in [4]; see also [8, 12, 16] for older partial results.

Remark 2 As one may expect, whenever it is possible to decompose $f = f_1 + f_2$ with $f_1 \in (W^{s_1,p_1} \cap W^{s,p})(\mathbb{R}^N)$ and $f_2 \in (W^{s_2,p_2} \cap W^{s,p})(\mathbb{R}^N)$, we also have a norm control for f_1 and f_2 in terms of $\|f\|_{W^{s,p}}$. A simple example of such decomposition with norm control is the following. For $f \in L^2(\mathbb{R}^N)$, set $f_1 := f\,\mathbb{1}_{\{x;\, |f(x)| > \|f\|_{L^2(\mathbb{R}^N)}\}}$ and $f_2 := f\,\mathbb{1}_{\{x;\, |f(x)| \le \|f\|_{L^2(\mathbb{R}^N)}\}}$. Then clearly $f_1 \in (L^1 \cap L^2)(\mathbb{R}^N)$ and $f_2 \in (L^\infty \cap L^2)(\mathbb{R}^N)$, and in addition we have the norm controls

$$\|f_1\|_{L^1(\mathbb{R}^N)} \le \|f\|_{L^2(\mathbb{R}^N)},\ \|f_1\|_{L^2(\mathbb{R}^N)} \le \|f\|_{L^2(\mathbb{R}^N)},$$
$$\|f_2\|_{L^\infty(\mathbb{R}^N)} \le \|f\|_{L^2(\mathbb{R}^N)},\ \|f_2\|_{L^2(\mathbb{R}^N)} \le \|f\|_{L^2(\mathbb{R}^N)}.$$

Note however that the map $f \mapsto (f_1, f_2)$ is not linear. Likewise, in general we will construct nonlinear decompositions.

Our text is organized as follows. In Sect. 2, we recall some basic facts on function spaces, instrumental for our purposes. The proofs of Propositions 1, 2 and 3 and of Theorems 1 and 2 are presented in Sect. 3. A final appendix presents the factorization theory and its connections with the sum-intersection property and with the functional calculus in Sobolev spaces.

2 Basic Properties of Triebel-Lizorkin Spaces

Definition 3 Let $\psi \in C_c^\infty(\mathbb{R}^N)$ be such that $\psi = 1$ in $B_1(0)$ and $\operatorname{supp}\psi \subset B_2(0)$. Define $\psi_0 = \psi$ and, for $j \geq 1$, $\psi_j(x) := \psi(x/2^j) - \psi(x/2^{j-1})$. Set $\varphi_j := \mathscr{F}^{-1}\psi_j \in \mathscr{S}$.[1] Then for each temperate distribution f we have

$$f = \sum_j f_j \quad \text{in } \mathscr{S}', \text{ with } f_j := f * \varphi_j. \tag{17}$$

$f = \sum f_j$ is *"the" Littlewood-Paley* decomposition of $f \in \mathscr{S}'$.

Note that $\mathscr{F} f_j = \psi_j \mathscr{F} f$ is compactly supported, and therefore $f_j \in C^\infty$ for each j.

Definition 4 Starting from for Littlewood-Paley decomposition, we define the *Triebel-Lizorkin spaces* $F^s_{p,q}$ as follows: for $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leq \infty$, we let

$$\|f\|_{F^s_{p,q}} := \left\| \left\| \left(2^{sj} f_j(x)\right)_{j\geq 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)}, \quad F^s_{p,q} := \{f \in \mathscr{S}';\ \|f\|_{F^s_{p,q}} < \infty\}.$$

Same definition when $p = q = \infty$.

This definition has to be changed when $p = \infty$ and $1 < q < \infty$ [20, Section 2.3.4, p. 50]: we let

$$\|f\|_{F^s_{\infty,q}} = \inf\left\{ \operatorname*{esssup}_{x\in\mathbb{R}^N} \left\| \left\| \left(2^{sj} f_j(x)\right)_{j\geq 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^\infty(\mathbb{R}^N)} ;\ f_j \in L^\infty(\mathbb{R}^N),\ f = \sum f_j * \varphi_j \right\},$$

the latter equality being in the sense of $\mathscr{S}'$.

Most of the Sobolev spaces can be identified with Triebel-Lizorkin spaces [20, Section 2.3.5] and [17, Section 2.1.2].

Theorem 3 *The following equalities of spaces hold, with equivalence of norms:*

1. *If $s > 0$ is not an integer and $1 \leq p \leq \infty$, then $W^{s,p}(\mathbb{R}^N) = F^s_{p,p}$.*
2. *If $s \geq 0$ is an integer and $1 < p < \infty$, then $W^{s,p}(\mathbb{R}^N) = F^s_{p,2}$.*

[1] Equivalently, we have $\varphi_0 = \mathscr{F}^{-1}\psi$ and, for $j \geq 1$, $\varphi_j(x) = 2^{Nj}\varphi_0(2^j x) - 2^{N(j-1)}\varphi_0(2^{j-1}x)$.

When $s \geq 0$ is an integer and either $p = 1$ or $p = \infty$, the Sobolev space $W^{s,p}$ cannot be identified with a Triebel-Lizorkin space.

Theorem 3 is usually used in conjunction with Lemma 1 below. The reason is that, in practice, we do not know the Littlewood-Paley decomposition of f, but only a Nikol'skij decomposition of f.

Definition 5 A *Nikol'skij decomposition* of $f \in \mathscr{S}'$ is a representation of the form $f = \sum f^j$ in $\mathscr{S}'$, with $\operatorname{supp} \mathscr{F} f^j \subset \begin{cases} B_{2^{j+1}}(0) \setminus B_{2^{j-1}}(0), & \text{if } j \geq 1 \\ B_2(0), & \text{if } j = 0 \end{cases}$.

Note that in particular the Littlewood-Paley decomposition $f = \sum f_j$ is a Nikol'skij decomposition.

Lemma 1 *1. Let $1 < p < \infty$, $1 < q \leq \infty$ and $s \in \mathbb{R}$. Consider a sequence (f^j) such that $\left\| \left\| \left(2^{sj} f^j(x)\right)_{j\geq 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)} < \infty$. Then $f = \sum f^j * \varphi_j$ converges in $\mathscr{S}'$ and*

$$\|f\|_{F^s_{p,q}} \lesssim \left\| \left\| \left(2^{sj} f^j(x)\right)_{j\geq 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)}. \tag{18}$$

2. Same conclusion if $1 \leq p = q \leq \infty$.

Proof It suffices to consider finite sums, and to establish (18) in this case. We start with item 2, which is easier. Note that $f \in L^p(\mathbb{R}^N)$, and thus $f \in \mathscr{S}'$.

Let $f = \sum_{j\geq 0} f_j$ be the Littlewood-Paley decomposition of f. Since $\varphi_j * \varphi_k = 0$ if $|j-k| \geq 2$, we find that

$$f_j = f * \varphi_j = \sum_k f^k * \varphi_k * \varphi_j = \sum_{|k-j|\leq 1} f^k * \varphi_k * \varphi_j, \tag{19}$$

and thus

$$\begin{aligned} \|f_j\|_{L^p(\mathbb{R}^N)} &\leq \sum_{|k-j|\leq 1} \|f^k * \varphi_k * \varphi_j\|_{L^p(\mathbb{R}^N)} \\ &\leq \sum_{|k-j|\leq 1} \|f^k\|_{L^p(\mathbb{R}^N)} \|\varphi_k * \varphi_j\|_{L^1(\mathbb{R}^N)} \leq C \sum_{|k-j|\leq 1} \|f^k\|_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{20}$$

We obtain (18) with $p = q$ from (20).

We now consider item *1*. From (19), we find that

$$|f_j(x)| \leq \sum_{|k-j|\leq 1} |f^k * \varphi_k * \varphi_j(x)| \leq C \sum_{|k-j|\leq 1} \mathscr{M} f^k(x). \tag{21}$$

Here, $\mathscr{M}$ is the standard maximal operator, and we used the inequality [18, Proposition, p. 24]

$$|f * \rho_\varepsilon(x)| \le C_\rho \mathscr{M} f(x),\ \forall\, \rho \in \mathscr{S},\ \forall\, \varepsilon > 0.$$

Using (21), we find that

$$\|f\|_{F^s_{p,q}} \lesssim \left\| \left\| \left(2^{sj}\mathscr{M} f^j(x)\right)_{j\ge 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)} \lesssim \left\| \left\| \left(2^{sj} f^j(x)\right)_{j\ge 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)},$$

the latter inequality being the Fefferman-Stein vectorial maximal inequality [10].

Definition 6 We define, for $f \in L^1_{loc}(\mathbb{R}^N)$,

$$\|f\|_{bmo} := \sup_{|B|\le 1} \int_B |f| + \sup_{|B|\le 1} ⨍_B ⨍_B |f(x) - f(y)|\, dxdy,$$

the sup being taken over the balls of volume ≤ 1. We set

$$bmo := \{f \in L^1_{loc}(\mathbb{R}^N);\ \|f\|_{bmo} < \infty\}.$$

With its natural norm, bmo is the *local BMO space*.

Then we have [21, Theorem, p. 47] $bmo = F^0_{\infty,2}$. Using this equality, Definition 4 and the embedding $\ell^q \hookrightarrow \ell^2$, $0 < q < 2$, we obtain the following

Corollary 1 *If* $f = \sum_{j\ge 0} f^j * \varphi_j$ *in* $\mathscr{S}'$ *and* $0 < q < 2$, *then*

$$\|f\|^2_{bmo} \le C \operatorname*{esssup}_{x\in\mathbb{R}^N} \sum_j |f^j(x)|^2 \le C \operatorname*{esssup}_{x\in\mathbb{R}^N} \left(\sum_j |f^j(x)|^q\right)^{2/q}, \tag{22}$$

for some C *independent of the* f^j*'s.*

Corollary 2 *For* $1 < q < 2$, *we have* $F^0_{\infty,q} \hookrightarrow bmo$.

As we noticed above, when $s \in \mathbb{N}$ the space $W^{s,1}$ is not a Triebel-Lizorkin space. However, we have the following

Lemma 2

1. *When* $s \ge 0$, *we have* $F^s_{1,1} \hookrightarrow W^{s,1}(\mathbb{R}^N)$.
2. *More generally, for every* $s \ge 0$ *and* $1 \le p < \infty$ *we have* $F^s_{p,1} \hookrightarrow W^{s,p}(\mathbb{R}^N)$. *The same holds when* $p = \infty$ *and* $s > 0$ *is not an integer.*
3. *When* $k > 0$ *is an integer and* $1 < q \le 2$, *we have*

$$F^k_{\infty,q} \hookrightarrow \{f \in W^{k-1,\infty}(\mathbb{R}^N);\ D^{k-1} f \in bmo\}.$$

Proof We start with $p = 1$. When s is not an integer, we actually have equality. When $s = 0$ and $f \in F^0_{1,1}$, we have $\|f\|_{L^1(\mathbb{R}^N)} \le \sum_{j\ge 0}\|f_j\|_{L^1(\mathbb{R}^N)} = \|f\|_{F^0_{1,1}} < \infty$. When $s \ge 1$ is an integer, we use the fact that [20, Section 2.3.8, Theorem (ii), pp. 58–59]

$$\|f\|_{F^s_{1,1}} \sim \sum_{j=0}^{s}\|D^j f\|_{F^{s-j}_{1,1}} \ge \sum_{j=0}^{s}\|D^j f\|_{F^0_{1,1}} \ge \sum_{j=0}^{s}\|D^j f\|_{L^1(\mathbb{R}^N)} = \|f\|_{W^{s,1}(\mathbb{R}^N)}.$$

When $1 < p < \infty$, the desired inclusion follows from

$$F^s_{p,1} \hookrightarrow F^s_{p,q} = W^{s,p}(\mathbb{R}^N) \text{ (with } q = 2 \text{ or } q = p, \text{ according to } s).$$

Similarly if $p = \infty$ and s is not an integer.

Finally, if $p = \infty$ and s is an integer, we argue as for $p = 1$, relying on Corollary 2 and [20, Section 2.3.8, Remark 2, p. 60].

We now briefly recall the characterization of Triebel-Lizorkin spaces in terms of wavelets.

Let ψ_0, ψ_1 be respectively a father and mother (sufficiently smooth) wavelets. For $G \in \{0,1\}^N$, $j \in \mathbb{N}$ and $m \in \mathbb{Z}^N$, let $\psi^j_{G,m}(x) := 2^{Nj/2}\prod_{r=1}^{N}\psi_{G_r}(2^j x_r - m_r)$, $x \in \mathbb{R}^N$. Let, for $f \in \mathscr{S}'$,

$$\lambda^j_{G,m} := \begin{cases} 0, & \text{if } j > 0 \text{ and } G = \{0\}^N \\ 2^{Nj/2}\,(f, \psi^j_{G,m}), & \text{otherwise} \end{cases}.$$

Recall [22, Section 3.1.3] that $f = \sum_{j,G,m} 2^{-Nj/2}\lambda^j_{G,m}\psi^j_{G,m}$ in the sense of $\mathscr{S}'$. Conversely, if

$$f = \sum_{G,m}\mu^0_{G,m}\psi^0_{G,m} + \sum_{j>0,\, G\ne\{0\}^N,\, m} 2^{-Nj/2}\mu^j_{G,m}\psi^j_{G,m} \text{ in the sense of } \mathscr{S}',$$

then the wavelet coefficients $\lambda^j_{G,m}$ of f are given by

$$\lambda^j_{G,m} = \begin{cases} 0, & \text{if } j > 0 \text{ and } G = \{0\}^N \\ \mu^j_{G,m}, & \text{otherwise} \end{cases}.$$

Let, for $j \in \mathbb{N}$ and $m \in \mathbb{Z}^N$, $Q_{j,m}$ be the cube $\prod_{r=1}^{N}[2^{-j}(m_r - 1), 2^{-j}(m_r + 1)]$. Set, for $0 < q < \infty$, $s \in \mathbb{R}$,

$$g(x) = g^s_{p,q}(x) := \left(\sum 2^{sqj} |\lambda^j_{G,m}|^q \mathbb{1}_{Q_{j,m}}(x)\right)^{1/q}. \tag{23}$$

When $q = \infty$, we replace the ℓ^q norm by the sup norm.

Then one may read the smoothness of f in terms of the integrability properties of g. The following statement is a rephrasing of [22, Theorem 1.64, p. 33].

Theorem 4

1. *Let* $-\infty < s < \infty$, $1 \le p < \infty$, $0 < q \le \infty$. *Then* $\|f\|_{F^s_{p,q}} \sim \|g^s_{p,q}\|_{L^p(\mathbb{R}^N)}$.
2. *Same conclusion if* $p = q = \infty$.
3. *In particular, if* $s > 0$ *is not an integer and* $1 \le p \le \infty$, *then* $\|f\|_{W^{s,p}(\mathbb{R}^N)} \sim \|g^s_{p,p}\|_{L^p}$.
4. *If* $s \ge 0$ *is an integer and* $1 < p < \infty$, *then* $\|f\|_{W^{s,p}(\mathbb{R}^N)} \sim \|g_{s,2}\|_{L^p}$.

Let us note that when $p = q$, this norm equivalence takes a particularly simple form. More specifically, we have

$$\|f\|^p_{F^s_{p,p}} \sim \sum_{j,G,m} 2^{(sp-N)j} |\lambda^j_{G,m}|^p, \ -\infty < s < \infty, \ 1 \le p < \infty \tag{24}$$

$$\|f\|_{F^s_{\infty,\infty}} \sim \sup_{j,G,m} 2^{sj} |\lambda^j_{G,m}|, \ -\infty < s < \infty. \tag{25}$$

Our next result relies on properties of the Besov spaces $B^s_{p,q}$. In order to keep this section short, we will be rather sketchy.

Lemma 3 *Let* $0 \le s < \infty$, $1 \le p \le \infty$ *and* $\varepsilon > 0$. *Then, with* $g^s_{p,q}$ *as in* (23), *we have*

$$\|f\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|g^{s+\varepsilon}_{p,p}\|_{L^p(\mathbb{R}^N)}, \tag{26}$$

$$\|g^{s-\varepsilon}_{p,p}\|_{L^p(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}. \tag{27}$$

Proof (Sketch of Proof) The above estimates are equivalent to the embeddings

$$F^{s+\varepsilon}_{p,p} \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow F^{s-\varepsilon}_{p,p}. \tag{28}$$

When s is not an in an integer, we have $W^{s,p}(\mathbb{R}^N) = F^s_{p,p}$, and the conclusion is clear.

When s is an integer and $1 \le p \le \infty$, the Littlewood-Paley decomposition $f = \sum_j f_j$ of f satisfies [7, Lemma 2.1.1]

$$\|f_0\|_{L^p(\mathbb{R}^N)} \le \|f\|_{L^p(\mathbb{R}^N)}, \ 2^{sj}\|f_j\|_{L^p(\mathbb{R}^N)} \lesssim \|D^s f\|_{L^p(\mathbb{R}^N)}, \ \forall j \ge 1. \tag{29}$$

Thus $\sup_j 2^{sj}\|f_j\|_{L^p(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}$, i.e., we have the embedding

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow B^s_{p,\infty}. \tag{30}$$

On the other hand, we have [19, Chapter 5, Lemma 3.14]

$$\|D^s f_j\|_{L^p(\mathbb{R}^N)} \lesssim 2^{sj}\|f_j\|_{L^p(\mathbb{R}^N)}, \ \forall\, j \geq 0,$$

and thus

$$\|f\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \sum_j \left(\|f_j\|_{L^p(\mathbb{R}^N)} + \|D^s f_j\|_{L^p(\mathbb{R}^N)}\right) \lesssim \sum_j 2^{sj}\|f_j\|_{L^p(\mathbb{R}^N)}.$$

Equivalently, we have the embedding

$$B^s_{p,1} \hookrightarrow W^{s,p}(\mathbb{R}^N). \tag{31}$$

We obtain (28) via (30)–(31) and the following elementary embeddings [20, Section 2.3.2, Proposition 2, p. 47]

$$F^{s+\varepsilon}_{p,p} = B^{s+\varepsilon}_{p,p} \hookrightarrow B^s_{p,1} \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow B^s_{p,\infty} \hookrightarrow B^{s-\varepsilon}_{p,p} = F^{s-\varepsilon}_{p,p}.$$

□

3 Proofs

Proof (Proof of Proposition 1) In order to prove the existence of some θ such that (6) holds, we have to establish the double inequality

$$\min\{p_1, p_2\} \leq p \leq \max\{p_1, p_2\}. \tag{32}$$

We argue by contradiction. Assume first that $p > \max\{p_1, p_2\}$. Let

$$f(x) = \frac{2}{(1+x^2)^{(1+\varepsilon)/(2p)}}, \ \forall\, x \in \mathbb{R}.$$

Clearly, $f \in L^p(\mathbb{R})$, and more generally $f \in W^{k,p}(\mathbb{R})$ for every integer k. It follows that $f \in W^{s,p}(\mathbb{R})$ for every $s \geq 0$. On the other hand, for every f_1, f_2 such that $f = f_1 + f_2$ and every x we have either $|f_1(x)| \geq f(x)/2$ or $|f_2(x)| \geq f(x)/2$. We find that

$$|f_1(x)|^{p_1} + |f_2(x)|^{p_2} \gtrsim f(x)^{p_1} + f(x)^{p_2} := g(x).$$

Since, for sufficiently small ε, we have $g \notin L^1(\mathbb{R})$, we find that $f \notin L^{p_1}(\mathbb{R}) + L^{p_2}(\mathbb{R})$. Therefore, $f \notin W^{s_1,p_1}(\mathbb{R}) + W^{s_2,p_2}(\mathbb{R})$, which is a contradiction.

Assume next that $p < \min\{p_1, p_2\}$. Let $p < r < \min\{p_1, p_2\}$. Let N be sufficiently large such that $W^{s,p}(B) \not\subset L^r(B)$; here, B is a ball in $\mathbb{R}^N$. By a standard extension argument, there exists some $f \in W^{s,p}_c(\mathbb{R}^N)$ such that $f \notin L^r(\mathbb{R}^N)$. Such an f does not belong to $L^r_{loc}(\mathbb{R}^N)$, and thus does not belong to $L^{p_1}(\mathbb{R}^N)+L^{p_2}(\mathbb{R}^N)$. We find that $f \notin W^{s_1,p_1}(\mathbb{R}^N) + W^{s_2,p_2}(\mathbb{R}^N)$, again a contradiction.

We thus know that (32) holds, or equivalently, that (6) holds for some θ.

We next proceed to the proof of (5). Assume first that $p_1 = p_2 = p$. Then θ is not determined by (6), and its existence is equivalent to $s \geq \min\{s_1, s_2\}$. Arguing by contradiction, assume that $s < \min\{s_1, s_2\}$. Let $s < \rho < \min\{s_1, s_2\}$. If $f \in W^{s,p}(\mathbb{R}) \setminus W^{\rho,p}(\mathbb{R})$, then

$$f \notin W^{s_1,p}(\mathbb{R}) + W^{s_2,p}(\mathbb{R}) = W^{\min\{s_1,s_2\},p}(\mathbb{R}) \subset W^{\rho,p}(\mathbb{R}),$$

a contradiction.

Assume next that $p_1 \neq p_2$. Then θ is determined by (6). Argue again by contradiction and assume that $s < \theta s_1 + (1-\theta)s_2$. Set $\sigma := \theta s_1 + (1-\theta)s_2 > s$. Consider some $\varepsilon > 0$ such that $s + \varepsilon < \sigma - \varepsilon$. In view of Lemma 3, in order to contradict (3) it suffices to establish, for some appropriate N, the non inclusion

$$F^{s+\varepsilon}_{p,p} \not\subset F^{s_1-\varepsilon}_{p_1,p_1} + F^{s_1-\varepsilon}_{p_2,p_2}. \tag{33}$$

With no loss of generality, we may assume that

$$1 \leq p_1 < p_2 \leq \infty. \tag{34}$$

We will treat separately the cases $p_2 < \infty$ and $p_2 = \infty$.

Set, in all cases,

$$\alpha := \frac{\dfrac{s_1-\varepsilon}{p_2} - \dfrac{s_2-\varepsilon}{p_1}}{\dfrac{1}{p_1} - \dfrac{1}{p_2}} = \frac{\dfrac{s_1}{p_2} - \dfrac{s_2}{p_1}}{\dfrac{1}{p_1} - \dfrac{1}{p_2}} + \varepsilon. \tag{35}$$

Proof of (33) *When* $p_2 < \infty$. We rely on the following

Claim. For appropriate $C_1, C_2 > 0$, we have

$$[a + b = S,\, S \geq C_1 2^{\alpha j}] \implies [2^{(s_1-\varepsilon)jp_1}|a|^{p_1} + 2^{(s_2-\varepsilon)jp_2}|b|^{p_2} \geq C_2 2^{(\sigma-\varepsilon)jp}]. \tag{36}$$

Granted the claim, we conclude as follows. Consider some $f \in \mathscr{S}'$ such that for every j, G and m we have either $\lambda^j_{G,m} = 0$ or $|\lambda^j_{G,m}| \geq C_1 2^{\alpha j}$, with C_1 as in (36). The claim combined with (24) implies that for *every possible* decomposition $f = f_1 + f_2$ we have

$$\|f_1\|^{p_1}_{F^{s_1-\varepsilon}_{p_1,p_1}} + \|f_2\|^{p_1}_{F^{s_2-\varepsilon}_{p_2,p_2}} \gtrsim \|f\|^p_{F^{\sigma-\varepsilon}_{p,p}}. \tag{37}$$

We are now in position to obtain a contradiction. Let N be sufficiently large such that $(\sigma-\varepsilon+\alpha)p < N$. Let $\delta := N-(\sigma-\varepsilon+\alpha)p > 0$. Fix some $G_0 \in \{0, 1\}^N\setminus\{0\}^N$. For every $j \in \mathbb{N}$, consider a set $M_j \subset \mathbb{Z}^N$ such that $\#M_j \sim 2^{\delta j}$. Set

$$f := \sum_{j,\, m\in M_j} 2^{-Nj/2} C_1 2^{\alpha j} \psi^j_{G_0,m}.$$

By (24), we have

$$\|f\|^p_{F^{s+\varepsilon}_{p,p}} \sim \sum_j 2^{((s+\varepsilon+\alpha)p-N+\delta)j} = \sum_j 2^{-((\sigma-\varepsilon)-(s+\varepsilon))jp} < \infty,$$

while

$$\|f\|^p_{F^{\sigma-\varepsilon}_{p,p}} \sim \sum_j 2^{((\sigma-\varepsilon+\alpha)p-N+\delta)j} = \sum_j 1 = \infty.$$

We complete the proof of (33) when $p_2 < \infty$ using the two above inequalities and (37).

Proof of (33) *When* $p_2 = \infty$ *and* $\theta \in (0, 1]$. This time we have $\alpha = -(s_2 - \varepsilon)$. We modify the definition of f by setting

$$f := \sum_{j,\, m\in M_j} 2^{-Nj/2} j 2^{\alpha j} \psi^j_{G_0,m}.$$

Assume, by contradiction, that $f = f_1 + f_2$ for some $f_1 \in F^{s_1-\varepsilon}_{p_1,p_1}$ and $f_2 \in F^{s_2-\varepsilon}_{\infty,\infty}$. Write $f_1 = \sum_{j,G,m} 2^{-Nj/2} a^j_{G,m}\psi^j_{G,m}$, $f_2 = \sum_{j,G,m} 2^{-Nj/2} b^j_{G,m}\psi^j_{G,m}$.

Since $f_2 \in F^{s_2-\varepsilon}_{\infty,\infty}$, we have

$$|b^j_{G,m}| \le C2^{-(s_2-\varepsilon)j} = C2^{\alpha j},\ \forall\, j, G, m.$$

Since $a^j_{G_0,m} + b^j_{G_0,m} = j2^{\alpha j}$, $\forall\, j$, $\forall\, m \in M_j$, for sufficiently large j_0 we have

$$|a^j_{G_0,m}| \ge \frac{1}{2} j2^{\alpha j},\ \forall\, j \ge j_0,\ \forall\, m \in M_j.$$

Inserting this into (24) and using the fact that

$$(s_1 - \varepsilon + \alpha)p_1 - N + \delta = (s_1 - s_2)p_1 - \theta(s_1 - s_2)p = (s_1 - s_2)(p_1 - \theta p) = 0$$

(since $p_1 = \theta p$), we find that

$$\|f\|^{p_1}_{F^{s_1-\varepsilon}_{p_1,p_1}} \gtrsim \sum_{j\ge j_0,\, m\in M_j} j^{p_1} 2^{((s_1-\varepsilon+\alpha)p_1-N)j}$$
$$\sim \sum_{j\ge j_0} j^{p_1} 2^{((s_1-\varepsilon+\alpha)p_1-N+\delta)j} = \sum_{j\ge j_0,\, m\in M_j} j^{p_1} = \infty.$$

On the other hand, we have

$$\|f\|^{p}_{F^{s+\varepsilon}_{p,p}} \sim \sum j^p 2^{((s+\varepsilon+\alpha)p-N+\delta)j} = \sum j^p 2^{-((\sigma-\varepsilon)-(s+\varepsilon))jp} < \infty.$$

This leads to a contradiction and completes the proof of (33) when $p_2 = \infty$ and $\theta \in (0, 1]$.

Proof of (33) *When* $p_2 = \infty$ *and* $\theta = 0$. This is similar to the case $p_2 = \infty$ and $\theta \in (0, 1]$. We have $\alpha = -(s_2 - \varepsilon) = -(\sigma - \varepsilon) < -(s + \varepsilon)$. Consider $f := \sum_{j,m} 2^{-Nj/2} j 2^{\alpha j} \psi^j_{G_0,m}$. [This time, the sum in m is over all $m \in \mathbb{Z}^N$.] We then have $f \in F^{s+\varepsilon}_{\infty,\infty}$. Arguing by contradiction, we obtain that f cannot be decomposed as $f = f_1 + f_2$ with $f_1 \in F^{s_1-\varepsilon}_{p_1,p_1}$ and $f_2 \in F^{s_2-\varepsilon}_{\infty,\infty}$. Indeed, as in the previous case, if $f_2 \in F^{s_2-\varepsilon}_{\infty,\infty}$ then for large j_0 we have

$$\|f_1\|^{p_1}_{F^{s_1-\varepsilon}_{p_1,p_1}} \gtrsim \sum_{j\ge j_0} \sum_{m\in\mathbb{Z}^N} j^{p_1} 2^{(s_1\alpha p_1-N)j} = \infty.$$

Proof of the Claim. Let $S > 0$. The function

$$[0,\infty) \ni t \mapsto g(t) := 2^{(s_1-\varepsilon)j}(1-t)S + 2^{(s_2-\varepsilon)jp_2/p_1} t^{p_2/p_1} S^{p_2/p_1}$$

is convex, and its derivative at the origin is negative. Thus g has a global minimum at the point t_0 where $g'(t_0) = 0$. Solving the equation $g'(t) = 0$, we find that $t_0 = C_1 2^{\alpha j} S^{-1}$, with $C_1 > 0$ independent of j. Provided that $S \ge C_1 2^{\alpha j}$, we have $t_0 \le 1$, and therefore the first term in $g(t)$ is non negative. For such S, we thus have

$$g(t) \ge g(t_0) \ge 2^{(s_2-\varepsilon)jp_2/p_1}(t_0)^{p_2/p_1} S^{p_2/p_1} = c\, 2^{(s_2-\varepsilon+\alpha)p_2/p_1 j} = c 2^{(\sigma-\varepsilon)p/p_1 j},$$
$$\forall\, t \ge 0,$$

with $c > 0$ independent of S.

Let now a, b be such that $a + b = S \ge C_1 2^{\alpha j}$. Then

$$2^{s_1 jp_1}|a|^{p_1} + 2^{s_2 jp_2}|b|^{p_2} \ge 2^{s_1 jp_1}\underline{a}^{p_1} + 2^{s_2 jp_2}\underline{b}^{p_2},$$

where

$$\underline{a} := \begin{cases} a, & \text{if } 0 \le a, b \le S \\ 0, & \text{if } a < 0 \text{ and } b > S \\ S, & \text{if } a > S \text{ and } b < 0 \end{cases},\quad \underline{b} := \begin{cases} b, & \text{if } 0 \le a, b \le S \\ S, & \text{if } a < 0 \text{ and } b > S \\ 0, & \text{if } a > S \text{ and } b < 0 \end{cases}. \tag{38}$$

Therefore, it suffices to prove (36) under the extra assumption that $0 \le a, b \le S$. Write $a = (1-t)S$, $b = tS$, with $t \in [0,1]$. We then have

$$2^{(s_1-\varepsilon)jp_1}a^{p_1} + 2^{(s_2-\varepsilon)jp_2}b^{p_2} \sim \left(2^{(s_1-\varepsilon)j}a + 2^{(s_2-\varepsilon)jp_2/p_1}b^{p_2/p_1}\right)^{p_1}$$
$$= [g(t)]^{p_1} \ge [g(t_0)]^{p_1} \ge c^{p_1}2^{(\sigma-\varepsilon)jp}.$$

□

Proof of Proposition 2 Assuming Theorem 1. As already noticed in the proof of Proposition 1, when $p_1 = p_2 = p$ or when $\theta \in \{0,1\}$, properties (3) and (4) are trivially true. We may thus assume that $p_1 \ne p_2$ and $\theta \in (0,1)$. Set $\lambda := s - (\theta s_1 + (1-\theta)s_2) > 0$. For $0 < \varepsilon < \dfrac{\lambda}{\theta}$, let $\delta > 0$ satisfy $\theta\varepsilon + (1-\theta)\delta = \lambda$. Then we may pick ε such that neither $s_1 + \varepsilon$ nor $s_2 + \delta$ is an integer. Thus the triple $T := (W^{s_1+\varepsilon,p_1}, W^{s,p}, W^{s_2+\delta,p_2})$ is regular. Granted Theorem 1, this implies

$$W^{s,p}(\mathbb{R}^N) = (W^{s_1+\varepsilon,p_1} \cap W^{s,p})(\mathbb{R}^N) + (W^{s_2+\delta,p_2} \cap W^{s,p})(\mathbb{R}^N)$$
$$\subset (W^{s_1,p_1} \cap W^{s,p})(\mathbb{R}^N) + (W^{s_2,p_2} \cap W^{s,p})(\mathbb{R}^N).$$

□

Proof of Proposition 3. Decompose $f \in W^{s,p}(\mathbb{R}^N)$ as $f = f_1 + f_2$, with $f_1 \in W^{s_1,p_1}(\mathbb{R}^N)$ and $f_2 \in W^{s_2,p_2}(\mathbb{R}^N)$. Write, in the sense of $\mathscr{S}'$,

$$f = \sum_{j,G,m} 2^{-Nj/2}\lambda^j_{G,m}\psi^j_{G,m},$$
$$f_1 = \sum_{j,G,m} 2^{-Nj/2}a^j_{G,m}\psi^j_{G,m},\ \ f_2 = \sum_{j,G,m} 2^{-Nj/2}b^j_{G,m}\psi^j_{G,m}.$$

In the spirit of (38), define

$$\underline{a}^j_{G,m} := \begin{cases} a^j_{G,m}, & \text{if } 0 \le a^j_{G,m}, b^j_{G,m} \le \lambda^j_{G,m} \\ 0, & \text{if } a^j_{G,m} < 0 \text{ and } b^j_{G,m} > \lambda^j_{G,m} \\ \lambda^j_{G,m}, & \text{if } a^j_{G,m} > \lambda^j_{G,m} \text{ and } b^j_{G,m} < 0 \end{cases},\ \ \underline{f}_1 := \sum_{j,G,m} 2^{-Nj/2}\underline{a}^j_{G,m}\psi^j_{G,m}$$

$$\underline{b}^j_{G,m} := \begin{cases} b^j_{G,m}, & \text{if } 0 \le a^j_{G,m}, b^j_{G,m} \le \lambda^j_{G,m} \\ \lambda^j_{G,m}, & \text{if } a^j_{G,m} < 0 \text{ and } b^j_{G,m} > \lambda^j_{G,m} \\ 0, & \text{if } a^j_{G,m} > \lambda^j_{G,m} \text{ and } b^j_{G,m} < 0 \end{cases},\ \ \underline{f}_2 := \sum_{j,G,m} 2^{-Nj/2}\underline{b}^j_{G,m}\psi^j_{G,m}.$$

Then $f = \underline{f}_1 + \underline{f}_2$, and Theorem 4 implies that

$$\left\|\underline{f}_1\right\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)} < \infty, \ \left\|\underline{f}_1\right\|_{W^{s_1,p_1}(\mathbb{R}^N)} \lesssim \|f_1\|_{W^{s_1,p_1}(\mathbb{R}^N)} < \infty,$$
$$\left\|\underline{f}_2\right\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)} < \infty, \ \left\|\underline{f}_2\right\|_{W^{s_2,p_2}(\mathbb{R}^N)} \lesssim \|f_2\|_{W^{s_2,p_2}(\mathbb{R}^N)} < \infty.$$

□

Proof of Theorem 1. The case where $p_1 = p_2$ is trivial, since we then have $W^{s,p} \subset W^{\min\{s_1,s_2\},p}$.

We may thus assume that

$$1 \le p_1 < p < p_2 \le \infty. \tag{39}$$

We further distinguish between the cases $s_1 = s_2$ and $s_1 \neq s_2$, and also between $p_2 < \infty$ and $p_2 = \infty$.

Given $f \in W^{s,p}(\mathbb{R}^N)$, we write $f = \sum_{j,G,m} 2^{-Nj/2}\lambda^j_{G,m}\psi^j_{G,m}$.

Case 1. $s_1 = s_2 = s \notin \mathbb{N}$. Set

$$f_1 := \sum_{j,G,m} 2^{-Nj/2} a^j_{G,m}\psi^j_{G,m}, \ f_2 := \sum_{j,G,m} 2^{-Nj/2} b^j_{G,m}\psi^j_{G,m},$$

with

$$a^j_{G,m} := \begin{cases} \lambda^j_{G,m}, & \text{if } |\lambda^j_{G,m}| \ge 2^{-sj} \\ 0, & \text{if } |\lambda^j_{G,m}| < 2^{-sj} \end{cases}, \ b^j_{G,m} := \begin{cases} 0, & \text{if } |\lambda^j_{G,m}| \ge 2^{-sj} \\ \lambda^j_{G,m}, & \text{if } |\lambda^j_{G,m}| < 2^{-sj} \end{cases}.$$

Since $p_1 < p$, we have

$$|a^j_{G,m}|^{p_1} \le 2^{sj(p_1-p)}|\lambda^j_{G,m}|^p. \tag{40}$$

Using (40), the fact that s is not an integer and (24), we find that

$$\|f_1\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}, \ \|f_1\|^{p_1}_{W^{s_1,p_1}(\mathbb{R}^N)} \lesssim \|f\|^p_{W^{s,p}(\mathbb{R}^N)}. \tag{41}$$

Similarly, if $p_2 < \infty$ then we have

$$\|f_2\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}, \ \|f_2\|^{p_2}_{W^{s_2,p_2}(\mathbb{R}^N)} \lesssim \|f\|^p_{W^{s,p}(\mathbb{R}^N)}. \tag{42}$$

On the other hand, if $p_2 = \infty$ then

$$\|f_2\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}, \ \|f_2\|_{W^{s_2,\infty}(\mathbb{R}^N)} \lesssim 1. \tag{43}$$

We complete this step via (41)–(43).

Remark 3 The estimates (41)–(43) are nonlinear, while one would expect linear estimates. Actually, it is possible to obtain linear estimates by cutting the coefficients $\lambda^j_{G,m}$ at height $A2^{-sj}$ instead of 2^{-sj}, with $A := \|f\|_{W^{s,p}(\mathbb{R}^N)}$. The corresponding decomposition satisfies

$$\|f_1\|_{W^{s,p}(\mathbb{R}^N)} + \|f_1\|_{W^{s_1,p_1}(\mathbb{R}^N)} + \|f_2\|_{W^{s,p}(\mathbb{R}^N)} + \|f_2\|_{W^{s_2,p_2}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}.$$

Similar observations apply to all the other cases.

Case 2. $s_1 = s_2 = s \in \mathbb{N}$. In this case, we follow the ideas of DeVore and Scherer [9] concerning the interpolation theory of classical spaces, in the form presented in Bennett and Sharpley [1, Section 5.5, pp. 347–362].

We claim that it suffices to decompose every $f \in (W^{s,p} \cap C^\infty)(\mathbb{R}^N)$ as $f = f_1 + f_2$, with

$$\|f_1\|_{W^{s,1}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}, \ \|f_1\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}, \tag{44}$$

$$\|f_2\|_{W^{s,\infty}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}, \ \|f_2\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}. \tag{45}$$

Indeed, if this holds then Hölder's inequality implies that

$$\|f_1\|_{W^{s,p_1}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}, \ \|f_2\|_{W^{s,p_2}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}, \ 1 \le p_1 < p_2 \le \infty, \tag{46}$$

and then a density argument shows that (44)–(46) hold without the extra assumption $f \in C^\infty$; this settles this case.

We next proceed to the construction of f_1 and f_2. Let $\mathscr{M}$ denote the standard maximal (uncentered) operator. Set $H_\ell(x) := \sum_{|\alpha|=\ell} |\partial^\alpha f(x)|$ and $H(x) := \sum_{\ell=0}^k H_\ell(x)$. Let $\Omega := \{x \in \mathbb{R}^N; \mathscr{M}H(x) > \tau\}$ and $M := \mathbb{R}^N \setminus \Omega$. Thus M is closed and $H(x) \le \tau, \forall\, x \in M$.

Let c be such that $\|\mathscr{M}g\|_{L^p(\mathbb{R}^N)} \le c\|g\|_{L^p(\mathbb{R}^N)}, \forall\, g \in L^p(\mathbb{R}^N)$. If $\tau := c\|H\|_{L^p(\mathbb{R}^N)} \sim \|f\|_{W^{s,p}(\mathbb{R}^N)}$, then

$$|\Omega| \le \frac{1}{\tau^p} \int_\Omega (\mathscr{M}H)^p(x)\, dx \le \frac{1}{\tau^p} \|\mathscr{M}H\|^p_{L^p(\mathbb{R}^N)} \le 1. \tag{47}$$

We then let f_2 be the Whitney extension of $f_{|M}$ and set $f_1 := f - f_2$. More specifically, let (Q_j) be a Whitney covering of Ω with cubes of size ℓ_j and centers y_j. Let $Q_{j,t}$ denote the cube of center y_j and size $t\ell_j$. Recall the following properties of the Whitney covering:

$$(Q_{j,9/8}) \text{ is a covering of } \Omega, \ Q_{j,4} \text{ intersects } M, \ \forall\, j, \ \sum_j \mathbb{1}_{Q_j}(x) \le C(N), \ \forall\, x. \tag{48}$$

Let (ϕ_j) be an adapted Whitney partition of unity in Ω, i.e.,

$$\operatorname{supp}\phi_j \subset Q_{j,9/8},\ \forall\, j,\ \text{and}\ |\partial^\alpha \phi_j| \lesssim (\ell_j)^{-\alpha},\ \forall\, \alpha \in \mathbb{N}^N. \tag{49}$$

Let $x_j \in M \cap Q_{j,4}$ and set

$$T_j(x) := \sum_{|\alpha|\le s-1} \partial^\alpha f(x_j)\frac{(x-x_j)^\alpha}{\alpha!},$$

the Taylor expansion of order $s-1$ of f around x_j. Then we set $f_2 := \begin{cases} f, & \text{in } M \\ \sum T_j\phi_j, & \text{in } \Omega \end{cases}$.

This f_2 satisfies [1, Theorem 5.10, p. 355] $f_2 \in W^{s,\infty}(\mathbb{R}^N)$ and

$$\|f_2\|_{W^{s,\infty}(\mathbb{R}^N)} \lesssim \tau \sim \|f\|_{W^{s,p}(\mathbb{R}^N)}. \tag{50}$$

On the other hand, using the fact that $|\Omega| \le 1$ (by (47)), we find that for every $1 \le r \le p$ the function f_1 satisfies

$$\begin{aligned} \|f_1\|_{W^{s,r}(\mathbb{R}^N)} &= \|f - f_2\|_{W^{s,r}(\Omega)} \le \|f\|_{W^{s,r}(\Omega)} + \|f_2\|_{W^{s,r}(\Omega)} \\ &\lesssim \|f\|_{W^{s,r}(\Omega)} + \tau \lesssim \|f\|_{W^{s,p}(\Omega)} + \tau \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}. \end{aligned} \tag{51}$$

Combining (50)–(51), we also have

$$\|f_2\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^N)}. \tag{52}$$

We obtain (44) (and complete this case) from (50)–(52).

Case 3. $s_1 \ne s_2$ and $p_2 < \infty$. This is somewhat the *general* case. We will prove below that

$$F^s_{p,q} = (F^{s_1}_{p_1,q_1} \cap F^s_{p,q}) + (F^{s_2}_{p_2,q_2} \cap F^s_{p,q}), \tag{53}$$

under the assumptions

$$\begin{aligned} &-\infty < s_1, s, s_2 < \infty,\ s_1 \ne s_2,\ 0 < p_1 < p < p_2 < \infty \\ &\text{such that (7) holds},\ 0 < q_1, q, q_2 < \infty. \end{aligned} \tag{54}$$

In view of Theorem 3 and Lemma 2, this is stronger than the conclusion of Theorem 1.

We now proceed to the proof of (53). Throughout the calculations we perform in this case, we assume (54).

Define, in the spirit of (35),

$$\alpha := \frac{\dfrac{s_1}{p_2} - \dfrac{s_2}{p_1}}{\dfrac{1}{p_1} - \dfrac{1}{p_2}}. \tag{55}$$

Let us first note that the proportionality condition (7) leads to the following identities

$$\alpha = \frac{\dfrac{s}{p_2} - \dfrac{s_2}{p}}{\dfrac{1}{p} - \dfrac{1}{p_2}} = \frac{\dfrac{s_1}{p} - \dfrac{s}{p_1}}{\dfrac{1}{p_1} - \dfrac{1}{p}} \tag{56}$$

and

$$(s_1 + \alpha)p_1 = (s + \alpha)p = (s_2 + \alpha)p_2. \tag{57}$$

In addition, we have

$$\text{either } s_1 + \alpha,\ s + \alpha,\ s_2 + \alpha > 0, \text{ or } s_1 + \alpha,\ s + \alpha,\ s_2 + \alpha < 0. \tag{58}$$

Given a sequence (x_j) of nonnegative numbers, set, for $i = 1, 2$,

$$\begin{aligned} S_i(x) &:= \sum 2^{s_i j q_i}(x_j)^{q_i},\ g_i(x) := [S_i(x)]^{p_i/q_i}, \\ T(x) &:= \sum 2^{sjq}(x_j)^q,\ h(x) := [T(x)]^{p/q}. \end{aligned} \tag{59}$$

Lemma 4 *There exists some finite constant C such that*

$$[x_j \le 2^{\alpha j},\ \forall j] \implies g_2(x) \le Ch(x).$$

Lemma 5 *There exists some finite constant C such that*

$$[\forall j,\ x_j \ge 2^{\alpha j} \text{ or } x_j = 0] \implies g_1(x) \le Ch(x).$$

Granted the two lemmas, we proceed to the proof of (53).

Let $f \in F^s_{p,q}$ and write, in the sense of $\mathscr{S}'$, $f = \sum_{j,G,m} 2^{-Nj/2}\lambda^j_{G,m}\psi^j_{G,m}$. Set

$$f_1 := \sum_{|\lambda^j_{G,m}| > 2^{\alpha j}} 2^{-Nj/2}\lambda^j_{G,m}\psi^j_{G,m},\ f_2 := \sum_{|\lambda^j_{G,m}| \le 2^{\alpha j}} 2^{-Nj/2}\lambda^j_{G,m}\psi^j_{G,m}. \tag{60}$$

Clearly, $f_1, f_2 \in F^s_{p,q}$.

We next note that, for each x and j, there exists some subset $M(j,x)$ of $\mathbb{Z}^N$, say $M(j,x) = \{m^\ell_{j,x}\}_{\ell=1}^k$ (with $k := 3^N$ independent of j and x), such that $m \notin M(j,k) \implies x \notin Q_{j,m}$. This implies that for all $x \in \mathbb{R}^N$ we have

$$\sum_{j,G,m} 2^{\sigma j \rho} \left| a^j_{G,m} \right|^\rho \mathbb{1}_{Q_{j,m}}(x) \sim \sum_{j,G,\ell} 2^{\sigma j \rho} \left| a^j_{G,m^\ell_{j,x}} \right|^\rho \mathbb{1}_{Q_{j,m^\ell_{j,x}}}(x), \ \forall \sigma, \ \forall \rho, \ \forall a^j_{G,m}. \tag{61}$$

Applying Lemmas 4 and 5 with $x_j := \left| \lambda^j_{G,m^\ell_{j,x}} \right| \mathbb{1}_{Q_{j,m^\ell_{j,x}}}(x)$ and using (60)–(61), we find that

$$\| f_1 \|^{p_1}_{F^{s_1}_{p_1,q_1}} \lesssim \| f \|^p_{F^s_{p,q}}, \ \| f_2 \|^{p_2}_{F^{s_2}_{p_2,q_2}} \lesssim \| f \|^p_{F^s_{p,q}}. \tag{62}$$

It thus remains to prove Lemmas 4 and 5.

Proof of Lemma 4. Define $A := (s_2 + \alpha) q_2$, $B := (s + \alpha) q$. By (57), we have either $A, B > 0$, or $A, B < 0$.

Set $a_j := 2^{-\alpha j} x_j \in [0, 1]$. Then

$$S_2(x) = \widetilde{S}_2(a) := \sum 2^{Aj} (a_j)^{q_2}, \ g_2(x) = \widetilde{g}_2(a) := \left[\widetilde{S}_2(a) \right]^{p_2/q_2},$$
$$T(x) = \widetilde{T}(a) := \sum 2^{Bj} (a_j)^q, \ h(x) = \widetilde{h}(a) := \left[\widetilde{T}(a) \right]^{p/q}.$$

Let J be an arbitrary nonnegative integer, and set

$$A^2_J := \{ a = (a_j)_{j \ge 0}; \ a_j \in [0, 1], \ \forall j, \ \text{and } a_j = 0, \ \forall j > J \}. \tag{63}$$

In order to establish the lemma, it suffices to prove that

$$\widetilde{g}_2(a) \le C \widetilde{h}(a), \ \forall a \in A^2_J, \tag{64}$$

provided C does not depend on J.

Fix J. For $a \in A^2_J$, $a \not\equiv 0$, set $\widetilde{f}_2(a) := \dfrac{\widetilde{g}_2(a)}{\widetilde{h}(a)}$. Since $\widetilde{f}_2$ is homogeneous of degree $p_2 - p > 0$, it attains its maximum at some a such that at least one of the a_j's equals 1. For this a, set

$$\Lambda_1 := \{ j \le J; \ a_j = 0 \}, \ \Lambda_2 := \{ j \le J; \ a_j = 1 \}, \ \Lambda_2 := \{ j \le J; \ 0 < a_j < 1 \}.$$

By the above, we have $\Lambda_2 \ne \emptyset$. Set $m := \min \Lambda_2$ and $M := \max \Lambda_2$.

Step 1. Proof of the lemma when $\Lambda_3 = \emptyset$. Assume first that $A, B > 0$. Then

$$\widetilde{S}_2(a) = \sum_{j \in \Lambda_2} 2^{Aj} \le \sum_{j \le M} 2^{Aj} \lesssim 2^{AM}, \ \widetilde{T}(a) = \sum_{j \in \Lambda_2} 2^{Bj} \ge 2^{BM}.$$

We find that

$$\widetilde{f}_2(a) \lesssim \frac{(2^{AM})^{p_2/q_2}}{(2^{BM})^{p/q}} = 1,$$

since $A\dfrac{p_2}{q_2} = B\dfrac{p}{q}$ (by (57)).

If $A, B < 0$, we have similarly $\widetilde{S}_2(a) \lesssim 2^{Am}$ and $\widetilde{T}(a) \geq 2^{Bm}$, and therefore $\widetilde{f}_2(a) \lesssim 1$.

Step 2. Proof of the lemma when $\Lambda_3 \neq \emptyset$. Set $\ell := \min \Lambda_3$, $L := \max \Lambda_3$.

If $j \in \Lambda_3$, then $\dfrac{\partial}{\partial a_j}\widetilde{f}_2(a) = 0$, and thus

$$p_2 2^{Aj}[\widetilde{S}_2(a)]^{p_2/q_2-1}(a_j)^{q_2-1}[\widetilde{T}(a)]^{p/q} = p 2^{Bj}[\widetilde{T}(a)]^{p/q-1}(a_j)^{q-1}[\widetilde{S}_2(a)]^{p_2/q_2},$$

which implies that

$$(a_j)^{q_2-q} = C_1 2^{(B-A)j},\ \forall\, j \in \Lambda_2,\ \text{with } C_1 = C_1(a) \text{ constant.} \tag{65}$$

Step 2.1. Proof of the lemma when $\Lambda_3 \neq \emptyset$ and $q_2 = q$. By (65), the quantity $2^{(B-A)j}$ does not depend on $j \in \Lambda_3$. On the other hand, since $q_2 = q$ we have $B - A = (s - s_2)q \neq 0$. Thus Λ_3 contains only one element, $\Lambda_3 = \{\ell\} = \{L\}$. We find that

$$\widetilde{g}_2(a) = \sum_{j\in\Lambda_2} 2^{Aj} + 2^{A\ell}(a_\ell)^q,\ \widetilde{h}(a) = \sum_{j\in\Lambda_2} 2^{Bj} + 2^{B\ell}(a_\ell)^q.$$

As in Step 1, when $A, B > 0$ we find that

$$\widetilde{f}_2(a) \lesssim \frac{\left(2^{AM} + 2^{A\ell}(a_\ell)^q\right)^{p_2/q}}{\left(2^{BM} + 2^{B\ell}(a_\ell)^q\right)^{p/q}} \lesssim \frac{2^{Ap_2/qM} + 2^{Ap_2/q}(a_\ell)^{p_2}}{2^{Bp/qM} + 2^{Bp/q}(a_\ell)^p} \leq 1,$$

the latter inequality following from $A\dfrac{p_2}{q_2} = B\dfrac{p}{q}$, $p_2 > p$ and $0 < a_\ell < 1$.

The case where $A, B < 0$ is handled similarly.

Step 2.2. Proof of the lemma when $\Lambda_3 \neq \emptyset$ and $q_2 \neq q$. Define $\gamma := \dfrac{B-A}{q_2-q}$. It follows from (65) that

$$a_j = C_2 2^{\gamma j},\ \forall\, j \in \Lambda_3. \tag{66}$$

Let us note that

$$A + \gamma q_2 = A + \frac{B-A}{q_2-q}q_2 = \frac{Bq_2 - Aq}{q_2-q} = qq_2\frac{s-s_2}{q_2-q} \neq 0.$$

We therefore have the following four possibilities:

1. $A, B > 0$, $A + \gamma q_2 > 0$.
2. $A, B > 0$, $A + \gamma q_2 < 0$.
3. $A, B < 0$, $A + \gamma q_2 > 0$.
4. $A, B < 0$, $A + \gamma q_2 < 0$.

We complete Step 2.2 in one of these cases, and let to the reader the three other ones, which are similar. Assume e.g. that $A, B > 0$ and $A + \gamma q_2 < 0$. In this case we obtain an information on C_2 by letting, in (66), $j = \ell$. [If $A + \gamma q_2 > 0$, we take $j = L$.] Since $0 < a_\ell < 1$, we have $0 < C_2 2^{\gamma\ell} < 1$, and thus $C_2 = C_3 2^{-\gamma\ell}$, with $0 < C_3 < 1$. We find that

$$a_j = C_3 2^{\gamma(j-\ell)},\ \forall\, j \in \Lambda_3,\ \text{ for some } C_3 \in (0, 1). \tag{67}$$

Since $A > 0$ and $A + \gamma q_2 < 0$, we find that

$$\begin{aligned}\widetilde{S}_2(a) &\le \sum_{j\le M} 2^{Aj} + \sum_{j\ge\ell} 2^{(A+\gamma q_2)j}(C_3)^{q_2} 2^{-\gamma q_2 \ell}\\ &\lesssim 2^{AM} + 2^{(A+\gamma q_2)\ell}(C_3)^{q_2} 2^{-\gamma q_2\ell} = 2^{AM} + 2^{A\ell}(C_3)^{q_2},\end{aligned}$$

while

$$\widetilde{T}(a) \ge 2^{BM} + 2^{B\ell}(C_3)^q.$$

We find that

$$\widetilde{f}_2(a) \lesssim \frac{\left(2^{AM} + 2^{A\ell}(C_3)^{q_2}\right)^{p_2/q_2}}{\left(2^{BM} + 2^{B\ell}(C_3)^q\right)^{p/q}} \lesssim \frac{2^{Ap_2/q_2 M} + 2^{Ap_2/q_2\ell}(C_3)^{p_2}}{2^{Bp/qM} + 2^{Bp/q\ell}(C_3)^p} \le 1,$$

since $A\dfrac{p_2}{q_2} = B\dfrac{p}{q}$, $0 < C_3 < 1$ and $p_2 > p$.

The proof of Lemma 4 is complete. □

Sketch of Proof of Lemma 5. This is very much similar to the proof of Lemma 4. This time, we have $a_j \in \{0\} \cup [1, \infty)$. With $C := (s_1 + \alpha)q_1$, we set $\widetilde{S}_1(a) := \sum 2^{Cj}(a_j)^{q_1}$ and

$$A_J^1 := \{a = (a_j)_{j\ge 0};\ a_j = 0 \text{ or } a_j \ge 2^{\alpha j},\ \forall\, j,\ \text{ and } a_j = 0,\ \forall\, j > J\}.$$

If $a \in A_J^1$, $a \not\equiv 0$, we set $\widetilde{f}_1(a) := \dfrac{[\widetilde{S}_1(a)]^{p_1/q_1}}{[T(a)]^{p/q}}$. We have to prove that $\widetilde{f}_1(a) \lesssim 1$, $\forall\, J$, $\forall\, a \in A_J^1$, $a \not\equiv 0$. This is obtained following the same strategy as in the proof of Lemma 4, considering, for a maximum point a of $\widetilde{f}_1$, the sets

$$\Lambda_1 := \{j \le J;\ a_j = 0\},\ \ \Lambda_2 := \{j \le J;\ a_j = 1\},\ \ \Lambda_3 := \{j \le J;\ 1 < a_j < \infty\}.$$

The key ingredients are that C and B are either both positive or both negative, respectively the fact that, when $q \ne q_1$, the quantity $C + \dfrac{B-C}{q_1 - q} q_1$ does not vanish.

Details are left to the reader. □

Case 4. $s_1 \ne s_2$ and $p_2 = \infty$. This is very much similar to Case 3. We prove the equality

$$F^s_{p,q} = (F^{s_1}_{p_1,q_1} \cap F^s_{p,q}) + (F^{s_2}_{\infty,\infty} \cap F^s_{p,q}) \tag{68}$$

under the assumptions

$$\begin{aligned} &-\infty < s_1, s, s_2 < \infty,\ \ s_1 \ne s_2,\ \ 0 < p_1 < p < p_2 = \infty \\ &\text{such that (7) holds},\ \ 0 < q_1, q < \infty. \end{aligned} \tag{69}$$

[For an improvement of (68) under more restrictive conditions of p_1, see the proof of Theorem 2.]

In view of Theorem 3 and Lemma 2, this implies Case 4. In order to prove (68), we decompose $f \in F^s_{p,q}$ as in (60). By Theorem 4 and Lemma 5, we have $f_1 \in F^{s_1}_{p_1,q_1} \cap F^s_{p,q}$. On the other hand, since $p_2 = \infty$ we have $\alpha = -s_2$, and then clearly (60) implies that $f_2 \in F^{s_2}_{\infty,\infty} \cap F^{s,p}$.

The proof of Theorem 1 is complete. □

Proof of Theorem 2. We will prove the following version of (68): we have

$$F^s_{p,q} = (F^{s_1}_{p_1,q_1} \cap F^s_{p,q}) + (F^{s_2}_{\infty,q_2} \cap F^s_{p,q}) \tag{70}$$

under one of the following assumptions

$$\begin{aligned} &-\infty < s_1, s, s_2 < \infty,\ \ s_1 \ne s_2,\ \ 1 < p_1 < p < p_2 = \infty \\ &\text{such that (7) holds},\ \ 0 < q < \infty,\ \ 1 < q_1, q_2 < \infty \end{aligned} \tag{71}$$

or

$$\begin{aligned} &-\infty < s_1, s, s_2 < \infty,\ \ s_1 \ne s_2,\ \ 1 = p_1 < p < p_2 = \infty \\ &\text{such that (7) holds},\ \ 0 < q < \infty,\ \ q_1 = 1,\ \ 1 < q_2 < \infty. \end{aligned} \tag{72}$$

Granted (70), we obtain the conclusion of Theorem 2 via Theorem 3, Corollary 2 and Lemma 2.

We now proceed to the proof of (70).

Let $f \in F^s_{p,q}$, and let $f = \sum f_j$ be the Littlewood-Paley decomposition of f. Set $f^j := \sum_{|k-j|\le 1} f_k = \sum_{|k-j|\le 1} f * \varphi_k * \varphi_j$. Taking into account the fact that

$\varphi_j * \varphi_k = 0$ if $|j-k| \geq 2$ and that $\sum_k \varphi_k = \delta$ in the sense of $\mathscr{S}'$, we find that

$$\sum_j f^j * \varphi_j = \sum_{j,k} f * \varphi_k * \varphi_j = \sum_j f * \varphi_j = f. \tag{73}$$

On the other hand, we clearly have

$$\left\| \left\| \left(2^{sj} f^j(x)\right)_{j\geq 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)} \lesssim \left\| \left\| \left(2^{sj} f_j(x)\right)_{j\geq 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)} = \|f\|_{F^s_{p,q}}. \tag{74}$$

Define

$$\delta := \frac{1}{p(s-s_2)} \neq 0. \tag{75}$$

Let us note that (7) and (75) imply the identity

$$\frac{p_1}{p} + (s_1 - s)\delta p_1 = 1. \tag{76}$$

Given $x \in \mathbb{R}^N$, let $h(x) := \left(\sum 2^{sjq}|f^j(x)|^q\right)^{p/q}$, so that $h < \infty$ a.e. Whenever $h(x) < \infty$, define $J = J(x)$ as follows: J is the least non negative integer such that $2^J \geq [h(x)]^\delta$.

Lemma 6 *Let δ and J be as above.*

1. If $\delta > 0$, then

$$\left(\sum_{j<J} 2^{s_1 j q_1}|f^j(x)|^{q_1}\right)^{p_1/q_1} \lesssim h(x) \text{ and } \left(\sum_{j\geq J} 2^{s_2 j q_2}|f^j(x)|^{q_2}\right)^{1/q_2} \lesssim 1. \tag{77}$$

2. If $\delta < 0$, then

$$\left(\sum_{j>J} 2^{s_1 j q_1}|f^j(x)|^{q_1}\right)^{p_1/q_1} \lesssim h(x) \text{ and } \left(\sum_{j\leq J} 2^{s_2 j q_2}|f^j(x)|^{q_2}\right)^{1/q_2} \lesssim 1. \tag{78}$$

Granted Lemma 6, we complete the proof of Theorem 2 as follows. Assume e.g. that $\delta > 0$, the case $\delta < 0$ being similar. Define, for a.e. $x \in \mathbb{R}^N$,

$$g^j(x) := \begin{cases} f^j(x), & \text{if } j < J(x) \\ 0, & \text{if } j \geq J(x) \end{cases}, \quad h^j(x) := \begin{cases} 0, & \text{if } j < J(x) \\ f^j(x), & \text{if } j \geq J(x) \end{cases}. \tag{79}$$

Combining (73), (74), Lemma 6 and Lemma 1, we find that the series $f_1 := \sum g^j * \varphi_j$ and $f_2 := h^j * \varphi_j$ converge in $\mathscr{S}'$, that $f = f_1 + f_2$, and that $f_1 \in F^{s_1}_{p_1,q_1} \cap F^s_{p,q}$, $f_2 \in F^{s_2}_{\infty,q_2} \cap F^s_{p,q}$. □

Proof of Lemma 6. We consider only the case $\delta > 0$, the case $\delta < 0$ being similar. Set $M := [h(x)]^\delta$. We let to the reader the case where $M < 1$ and thus $J = 0$ and the first sum in (77) vanishes. Assuming that $M \geq 1$, we have $2^J \sim M$ and

$$|f^j(x)| \leq 2^{-sj}[h(x)]^{1/p} = 2^{-sj}M^{1/(\delta p)},\ \forall\, j \geq 0. \tag{80}$$

Since $\delta > 0$, we have $s > s_2$, and thus $s_1 > s > s_2$. Using (80), we find that

$$\begin{aligned}\sum_{j<J} 2^{s_1 j q_1}|f^j(x)|^{q_1} &\lesssim M^{q_1/(\delta p)}\sum_{j<J} 2^{(s_1-s)jq_1} \lesssim M^{q_1/(\delta p)}2^{(s_1-s)Jq_1}\\ &\sim M^{q_1[1/(\delta p)+(s_1-s)]}.\end{aligned} \tag{81}$$

Combining (76) and (81), we find that

$$\left(\sum_{j<J} 2^{s_1 j q_1}|f^j(x)|^{q_1}\right)^{p_1/q_1} \lesssim \left[M^{1/\delta}\right]^{p_1/p+(s-1-s)\delta} = M^{1/\delta} = h(x),$$

i.e., the first inequality in (77) holds.

For the second inequality, we note that (80) leads to

$$\begin{aligned}\sum_{j\geq J} 2^{s_2 j q_2}|f^j(x)|^q_2 &\lesssim M^{q_2/(\delta p)}\sum_{j\geq J} 2^{(s_2-s)jq_2}\\ &\lesssim M^{q_2/(\delta p)}2^{(s_2-s)Jq_2} \sim M^{q_2[1/(\delta p)+(s_2-s)]} = 1,\end{aligned}$$

the latter equality following from the definition of δ. □

Appendix: Factorization, Functional Calculus, Sum-Intersection

The lifting problem for $\mathbb{S}^1$-valued Sobolev maps is the following. Let B be a ball in $\mathbb{R}^N$. Let $s > 0$ and $1 \leq p \leq \infty$. Is it possible to lift *every* map $u \in W^{s,p}(B;\mathbb{S}^1)$ as $u = e^{\imath\varphi}$ with $\varphi \in W^{s,p}(\mathbb{B};\mathbb{R})$? This question has been completely answered in [3]. The answer depends on s, p and N. For example, in $W^{1,p}(B;\mathbb{S}^1)$ the answer is positive if $N = 1$ or [$N \geq 2$ and $p \geq 2$], but negative if [$N \geq 2$ and $1 \leq p < 2$]. Factorization is a substitute to lifting, but is also valid and relevant if the answer to the lifting problem is positive. Special cases of factorization were announced in [13]. The general case is presented in [5] and asserts the following. Let $s > 0$ and

$1 \le p < \infty$. Then *every* map $u \in W^{s,p}(B;\mathbb{S}^1)$ can be *factorized* as $u = e^{\imath\varphi} v$, with $\varphi \in W^{s,p}(B;\mathbb{R})$ and $v \in F^{sp}_{1,1}(B;\mathbb{S}^1)$.

Factorization has the following application announced in the introduction. Let $p > 1$ and consider some $f \in W^{1,p}(B;\mathbb{S}^1)$. Set $u := e^{\imath f} \in W^{1,p}(B;\mathbb{S}^1)$. Let $0 < \lambda < 1$. Since $u \in W^{1,p} \cap L^\infty$, we also have $u \in W^{\lambda,p/\lambda}$ (by Gagliardo-Nirenberg). Factorization implies that $u = e^{\imath\varphi} v$, with $\varphi \in W^{\lambda,p/\lambda}$ and $v \in F^p_{1,1} \hookrightarrow W^{p,1}$.

We note that

$$W^{p,1}(B;\mathbb{R}) \ni v = e^{\imath(f-\varphi)}, \text{ with } f \in W^{1,p}(B;\mathbb{R}) \text{ and } \varphi \in W^{\lambda,p/\lambda}(B;\mathbb{R}). \tag{82}$$

We next invoke the following delicate result [5]. If $f_1 \in W^{s_1,p_1}(B;\mathbb{R})$, $f_2 \in W^{s_2,p_2}(B;\mathbb{R})$ are such that

$$s_1 p_1 \ge 1,\ s_2 p_2 \ge 1,\ e^{\imath(f_1+f_2)} \in W^{s_3,p_3},\ \text{with } s_3 \ge 1,$$

then $f_1 + f_2 \in W^{s_3,p_3} \cap W^{s_3 p_3,1}$. In our case, this implies that $\psi := f - \varphi \in W^{p,1} \cap W^{1,p}$, and thus $\varphi = f - \psi \in W^{1,p}$. Finally, $f = \varphi + \psi$, with $\varphi \in W^{\lambda,p/\lambda} \cap W^{1,p}$ and $\psi \in W^{p,1} \cap W^{1,p}$.

Our Theorem 1 yields the same conclusion without factorization.

Let us also note that not only factorization leads to a sum-intersection property, but sum-intersection is *necessary* for factorization to hold. Indeed, let $p \ge 2$ and $u \in W^{1,p}(B;\mathbb{S}^1)$. Then we may write $u = e^{\imath f}$ with $f \in W^{1,p}(B;\mathbb{R})$ [2]. Assume that we want to factorize $u = e^{\imath\varphi} v$ with $\varphi \in W^{\lambda,p/\lambda}(B;\mathbb{R})$ and $v \in W^{p,1}(B;\mathbb{S}^1)$. The first step consists of splitting (assuming this is possible) $f = \varphi + \psi$, with $\varphi \in W^{\lambda,p/\lambda}$ and $\psi \in W^{p,1}$. However, this decomposition does not imply that $v := e^{\imath\psi} \in W^{p,1}(B;\mathbb{S}^1)$. Indeed, if $s > 1$ and $\rho > 1$, then a map $g \in W^{\sigma,\rho}$ satisfies $e^{\imath g} \in W^{\sigma,\rho}$ if and only if g satisfies the extra-assumption $g \in W^{1,\sigma\rho}$ [6]. In our case, this implies that factorization in $W^{\lambda,p/\lambda}(B;\mathbb{S}^1)$ *requires* the sum-intersection property of the triple $T = (W^{1,p}, W^{\lambda,p/\lambda}, W^{p,1})$. However, one cannot reduce factorization to sum-intersection, since in general $W^{s,p}(B;\mathbb{S}^1)$ does not have the lifting property.

Sum-intersection property has the following implication related to lifting, presented in [5]. If $sp < 1$, then maps in $W^{s,p}(B;\mathbb{S}^1)$ can be lifted within $W^{s,p}$ [3]. Factorization leads to a better result. Indeed, let $u \in W^{s,p}(B;\mathbb{S}^1)$ and let $\varphi \in W^{s,p}(B;\mathbb{R})$ be a lifting of u. Write, as in Theorem 2, $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1 \in BMO \cap W^{s,p}$ and $\varphi_2 \in W^{sp,1} \cap W^{s,p}$. Set $v := e^{\imath\varphi_2} \in W^{sp,1}$. Then v has a lifting $\varphi_3 \in W^{sp,1} \cap L^\infty$ [14]. By Gagliardo-Nirenberg, we also have $\varphi_3 \in W^{s,p}$, and clearly $\varphi_3 \in BMO$ (since $\varphi_3 \in L^\infty$). Finally, $u = e^{\imath\psi}$, where $\psi := \varphi_1 + \varphi_3$ satisfies the improved regularity $\psi \in W^{s,p} \cap BMO$.

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