Chapter 4
Optimal Control of FBSDE with Partially Observable Information

In this chapter, we study an optimal control problem with state process governed by a nonlinear FBSDE and with partially observable information, i.e., Problem B introduced in Section 1.2. For simplicity, we take the dimensions $n = m = k = \tilde{k} = 1$. Using a direct method and a Malliavin derivative method, we establish two versions of the stochastic maximum principle for the characterization of the optimal control. To demonstrate the applicability, we work out an illustrative example within the framework of recursive utility and then solve it via the stochastic maximum principle and the stochastic filtering.

4.1 A Direct Method

4.1.1 Some Prior Estimates

Recall that Problem B consists of the state equation

$$\begin{cases} dx^v(t) = & b(t,x^v(t),v(t))dt + \sigma(t,x^v(t),v(t))dW(t) \\ & \quad +\tilde{\sigma}(t,x^v(t),v(t))d\tilde{W}^v(t), \\ -dy^v(t) = & g(t,x^v(t),y^v(t),z^v(t),\tilde{z}^v(t),v(t))dt \\ & \quad -z^v(t)dW(t) - \tilde{z}^v(t)dY(t), \\ x^v(0) = & x_0, \quad y^v(T) = f(x^v(T)), \end{cases} \tag{4.1}$$

and the cost function

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l(t,x^v(t),y^v(t),z^v(t),\tilde{z}^v(t),v(t))dt + \phi(x^v(T)) + \gamma(y^v(0))\right]. \tag{4.2}$$

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The information is provided by the observation equation

$$\begin{cases} dY(t) = & h(t, x^v(t))dt + d\tilde{W}^v(t), \\ Y(0) = & 0, \end{cases} \tag{4.3}$$

Recall also that the process $Z^v(t)$ is given by (1.16) which helps to transfer $\tilde{W}^v$ into a Brownian motion under a new probability measure $\mathbb{P}^v$.

Let $\varepsilon \in (0,1)$ and $v(\cdot)$ such that $v(\cdot) + u(\cdot) \in \mathscr{U}_{ad}$. By the convexity of U, $u + \varepsilon v \in \mathscr{U}_{ad}$. Denoted by $(x^{u+\varepsilon v}(\cdot), y^{u+\varepsilon v}(\cdot), z^{u+\varepsilon v}(\cdot), \tilde{z}^{u+\varepsilon v}(\cdot))$ and $Z^{u+\varepsilon v}(\cdot)$ the solutions of (4.1) and (1.16) along with the control $u(\cdot) + \varepsilon v(\cdot)$. Making use of the Burkholder–Davis–Gundy (BDG) inequality and Gronwall's inequality, we get the following estimates.

Lemma 4.1. *Under (H1.6), for any $v(\cdot) \in \mathscr{U}_{ad}$ there is a constant $C > 0$ such that the solutions of (1.15) and (1.16) satisfy*

$$\begin{aligned} \sup_{0 \le t \le T} \mathbb{E}|x^v(t)|^8 &\le C\left(1 + \sup_{0 \le t \le T} \mathbb{E}|v(t)|^8\right), \\ \sup_{0 \le t \le T} \mathbb{E}|y^v(t)|^2 &\le C\left(1 + \sup_{0 \le t \le T} \mathbb{E}|v(t)|^2\right), \\ \mathbb{E}\left(\int_0^T |z^v(t)|^2 dt + \int_0^T |\tilde{z}^v(t)|^2 dt\right) &\le C\left(1 + \sup_{0 \le t \le T} \mathbb{E}|v(t)|^2\right), \\ \mathbb{E}|Z^v(t)|^\ell &< \infty, \quad \forall \ell > 0. \end{aligned}$$

Lemma 4.2. *Under (H1.6), there is a constant $C > 0$ such that*

$$\begin{aligned} &\sup_{0 \le t \le T} \mathbb{E}|x^{u+\varepsilon v}(t) - x(t)|^8 \le C\varepsilon^8, \quad \sup_{0 \le t \le T} \mathbb{E}|y^{u+\varepsilon v}(t) - y(t)|^2 \le C\varepsilon^2, \\ &\mathbb{E}\int_0^T |z^{u+\varepsilon v}(t) - z(t)|^2 dt \le C\varepsilon^2, \quad \mathbb{E}\int_0^T |\tilde{z}^{u+\varepsilon v}(t) - \tilde{z}(t)|^2 dt \le C\varepsilon^2, \\ &\sup_{0 \le t \le T} \mathbb{E}|Z^{u+\varepsilon v}(t) - Z(t)|^2 \le C\varepsilon^2. \end{aligned}$$

We introduce the variational equations

$$\begin{cases} dZ^1(t) = \left(Z^1(t)h(t, x(t)) + Z(t)h_x(t, x(t))x^1(t)\right) dY(t), \\ Z^1(0) = 0 \end{cases} \tag{4.4}$$

and

$$
\left\{
\begin{aligned}
dx^1(t) = {} & \{[b_x(t,u) - \tilde{\sigma}_x(t,u)h(t,x(t)) - \tilde{\sigma}(t,u)h_x(t,x(t))]\, x^1(t) \\
& + [b_v(t,u) - \tilde{\sigma}_v(t,u)h(t,x(t))]\, v(t)\}\, dt \\
& + \left[\sigma_x(t,u)x^1(t) + \sigma_v(t,u)v(t)\right] dW(t) \\
& + \left[\tilde{\sigma}_x(t,u)x^1(t) + \tilde{\sigma}_v(t,u)v(t)\right] dY(t), \\
-dy^1(t) = {} & \left[g_x(t,u)x^1(t) + g_y(t,u)y^1(t) + g_z(t,u)z^1(t)\right. \\
& \left. + g_{\tilde{z}}(t,u)\tilde{z}^1(t) + g_v(t,u)v(t)\right] dt \\
& - z^1(t)dW(t) - \tilde{z}^1(t)dY(t), \\
x^1(0) = 0, & \quad y^1(T) = f_x(x(T))x^1(T),
\end{aligned}
\right.
\tag{4.5}
$$

where we used the notation convention of the last chapter. For example,

$$
b_x(t,u) = b_x(t,x(t),u(t)) \text{ and } g_z(t,u) = g_z(t,x(t),y(t),z(t),\tilde{z}(t),u(t)).
$$

For any $v(\cdot) \in \mathscr{U}_{ad}$, it is easy to see that under (H1.6), (4.4) and (4.5) admit a unique solution, respectively.

Lemma 4.3. *Under (H1.6), it follows that*

$$
\mathbb{E}|x^1(t)|^8 < \infty, \quad \mathbb{E}|Z^1(t)|^4 < \infty. \tag{4.6}
$$

Let

$$
\phi^\varepsilon(t) = \frac{\phi^{u+\varepsilon v}(t) - \phi(t)}{\varepsilon} - \phi^1(t) \text{ with } \phi = x, y, z, \tilde{z}, Z. \tag{4.7}
$$

Note that ϕ^ε defined in (4.7) is for $\varepsilon \in [0,1)$, and it should not be confused with ϕ^1 defined in (4.5).

Lemma 4.4. *If (H1.6) holds, then*

$$
\begin{aligned}
& \lim_{\varepsilon\to 0} \sup_{0\le t\le T} \mathbb{E}|x^\varepsilon(t)|^4 = 0, \quad \lim_{\varepsilon\to 0} \sup_{0\le t\le T} \mathbb{E}|Z^\varepsilon(t)|^2 = 0, \\
& \lim_{\varepsilon\to 0} \mathbb{E}\int_0^T |z^\varepsilon(t)|^2 dt = 0, \quad \lim_{\varepsilon\to 0} \mathbb{E}\int_0^T |\tilde{z}^\varepsilon(t)|^2 dt = 0, \\
& \lim_{\varepsilon\to 0} \sup_{0\le t\le T} \mathbb{E}|y^\varepsilon(t)|^2 = 0.
\end{aligned}
$$

Proof. It follows from (1.15) and (4.5) that

$$
dx^\varepsilon(t) = b^\varepsilon(t)dt + \sigma^\varepsilon(t)dW(t) + \tilde{\sigma}^\varepsilon(t)dY(t),
$$

where

$$
\begin{aligned}
b^{\varepsilon}(t) = & \left(\frac{b(t,u+\varepsilon v)-b(t,u)}{\varepsilon} - b_x(t,u)x^1(t) - b_v(t,u)v(t)\right) \\
& - \left(\frac{\tilde{\sigma}(t,u+\varepsilon v)-\tilde{\sigma}(t,u)}{\varepsilon} - \tilde{\sigma}_x(t,u)x^1(t) - \tilde{\sigma}_v(t,u)v(t)\right) h(t,x) \\
& - \left(\frac{h(t,x^{u+\varepsilon v})-h(t,x)}{\varepsilon} - h_x(t,x)x^1(t)\right)\tilde{\sigma}(t,u) \\
& - \frac{\tilde{\sigma}(t,u+\varepsilon v)-\tilde{\sigma}(t,u)}{\varepsilon}\left(h(t,x^{u+\varepsilon v})-h(t,x)\right),
\end{aligned}
$$

$$
\sigma^{\varepsilon}(t) = \frac{\sigma(t,u+\varepsilon v)-\sigma(t,u)}{\varepsilon} - \sigma_x(t,u)x^1(t) - \sigma_v(t,u)v(t),
$$

and

$$
\tilde{\sigma}^{\varepsilon}(t) = \frac{\tilde{\sigma}(t,u+\varepsilon v)-\tilde{\sigma}(t,u)}{\varepsilon} - \tilde{\sigma}_x(t,u)x^1(t) - \tilde{\sigma}_v(t,u)v(t).
$$

Denote

$$
\Theta = (t, x+\varepsilon\lambda(x^{\varepsilon}+x^1), u+\varepsilon\lambda v) \text{ and } \Xi = (t, x+\varepsilon\lambda(x^{\varepsilon}+x^1)).
$$

It is easy to show that

$$
\begin{aligned}
\sigma^{\varepsilon}(t) = & \, x^{\varepsilon}(t)\int_0^1 \sigma_x(\Theta)d\lambda + x^1(t)\left(\int_0^1 \sigma_x(\Theta)d\lambda - \sigma_x(t,u)\right) \\
& + v(t)\left(\int_0^1 \sigma_v(\Theta)d\lambda - \sigma_v(t,u)\right).
\end{aligned}
$$

Denote by $\gamma^{\varepsilon}(t)$ the maximum of

$$
\left|\phi_X(t,x+\varepsilon\lambda(x^{\varepsilon}+x^1),u+\varepsilon\lambda v) - \phi_X(t,x,u)\right| \tag{4.8}
$$

for ϕ and X runs over σ, $\tilde{\sigma}$, b, h and x, v, respectively. Then, by (H1.6) and Lemmas 4.2 and 4.3, we have

$$
|\sigma^{\varepsilon}(t)| \leq K\left(|x^{\varepsilon}(t)| + \left(|x^1(t)|+|v(t)|\right)\gamma^{\varepsilon}(t)\right). \tag{4.9}
$$

Similarly, we can prove that

$$
|\tilde{\sigma}^{\varepsilon}(t)| \leq K\left(|x^{\varepsilon}(t)| + \left(|x^1(t)|+|v(t)|\right)\gamma^{\varepsilon}(t)\right),
$$

and

$$
|b^{\varepsilon}(t)| \leq K\left(|x^{\varepsilon}(t)| + \left(|x^1(t)|+|v(t)|\right)\left(\gamma^{\varepsilon}(t)\vee\varepsilon\right)\right).
$$

According to Hölder's inequality and the BDG inequality, we derive

$$\mathbb{E}|x^\varepsilon(t)|^4 \le C\mathbb{E}\int_0^t |x^\varepsilon(s)|^4 ds$$
$$+C\int_0^T \left(\mathbb{E}|x^1(t)|^8 + \mathbb{E}|v(t)|^8\right)^{1/2} \left(\mathbb{E}\left(\gamma^\varepsilon(t)\vee\varepsilon\right)^8\right)^{1/2} dt.$$

Note that $\mathbb{E}\left(\gamma^\varepsilon(t)^8\right) \to 0$. By Gronwall's inequality, we obtain the first limit. The second can be proved similarly.

To prove the other limit, we note that

$$-dy^\varepsilon(t) = g^\varepsilon(t)dt - z^\varepsilon(t)dW(t) - \bar{z}^\varepsilon(t)dY(t),$$

where

$$g^\varepsilon(t) = \varepsilon^{-1}\left(g(t, x^{u+\varepsilon v}, y^{u+\varepsilon v}, z^{u+\varepsilon v}, \bar{z}^{u+\varepsilon v}, u+\varepsilon v) - g(t,x,y,z,\bar{z},u)\right)$$
$$-\left(g_x(t)x^1(t) + g_y(t)y^1(t) + g_z(t)z^1(t) + g_{\bar{z}}(t)\bar{z}^1(t) + g_v(t)v(t)\right).$$

Applying Itô's formula, we get

$$d|y^\varepsilon(t)|^2 = \left(-2g^\varepsilon(t)y^\varepsilon(t) + |z^\varepsilon(t)|^2 + |\bar{z}^\varepsilon(t)|^2\right)dt$$
$$+2y^\varepsilon(t)z^\varepsilon(t)dW(t) + 2y^\varepsilon(t)\bar{z}^\varepsilon(t)dY(t).$$

Taking integral and then expectation, we have

$$\begin{aligned}\mathbb{E}|y^\varepsilon(t)|^2 - \mathbb{E}|y^\varepsilon(T)|^2 &= \mathbb{E}\int_t^T 2g^\varepsilon(s)y^\varepsilon(s)ds - \mathbb{E}\int_t^T \left(|z^\varepsilon(s)|^2 + |\bar{z}^\varepsilon(s)|^2\right)ds\\
&\le \delta\mathbb{E}\int_t^T |g^\varepsilon(s)|^2 ds + \delta^{-1}\mathbb{E}\int_t^T |y^\varepsilon(s)|^2 ds\\
&\quad -\mathbb{E}\int_t^T \left(|z^\varepsilon(s)|^2 + |\bar{z}^\varepsilon(s)|^2\right)ds, \qquad (4.10)\end{aligned}$$

where $\delta > 0$ is an arbitrary constant.

Similar to (4.9), we can prove that $\mathbb{E}|y^\varepsilon(T)|^2 \to 0$ and

$$\mathbb{E}|g^\varepsilon(s)|^2 \le K\mathbb{E}\left(|x^\varepsilon(s)|^2 + |y^\varepsilon(s)|^2 + |z^\varepsilon(s)|^2 + |\bar{z}^\varepsilon(s)|^2\right)$$
$$+K\mathbb{E}\left(\left(|x^1(s)|^2 + |y^1(s)|^2 + |z^1(s)|^2 + |\bar{z}^1(s)|^2 + |v(s)|^2\right)\left(\tilde{\gamma}^\varepsilon(s)\vee\varepsilon\right)\right),$$

where K is a constant which may depend on x^1 etc., and $\tilde{\gamma}^\varepsilon$ is defined as (4.8) with $\phi = g$ and X runs over $x,\ y,\ z,\ \bar{z},\ v$.

Note that $\tilde{\gamma}^\varepsilon$ is bounded and convergent to 0, by the dominated convergent theorem, we have

$$\int_0^T \mathbb{E}\left(\left(|x^1(s)|^2 + |y^1(s)|^2 + |z^1(s)|^2 + |\bar{z}^1(s)|^2 + |v(s)|^2\right)\left(\tilde{\gamma}^\varepsilon(s)\vee\varepsilon\right)\right)ds \to 0.$$

Taking δ small enough such that $\delta K < 1$ in (4.10), it then follows from Gronwall's inequality that the last three identities of the lemma hold. □

4.1.2 Stochastic Maximum Principle

The following assumption and adjoint equations will be needed in deriving the stochastic maximum principle.

(H4.1) *(i) For any t, τ such that $t+\tau \in [0,T]$, and bounded $\mathscr{F}_t^Y$-measurable random variable ν, we formulate the control process $v(s) \in U$, with*

$$v(s) = \nu I_{[t,t+\tau)}(s), \quad s \in [0,T],$$

where $I_{[t,t+\tau)}(s)$ is the indicator function on the set $[t,t+\tau]$.

(ii) For any $v(s) \in \mathscr{F}_s^Y$ with $v(s)$ bounded, $s \in [0,T]$, there is an $\varepsilon > 0$ such that $u(\cdot)+\varepsilon v(\cdot) \in \mathscr{U}_{ad}$ for $\varepsilon \in (-1,1)$.

We formulate the adjoint equations

$$\begin{cases} dp(t) = [g_y(t,u)p(t) - l_y(t,u)]\,dt \\ \qquad\qquad + [g_z(t,u)p(t) - l_z(t,u)]\,dW(t) \\ \qquad\qquad + [(g_{\tilde{z}}(t,u) - h(t,x(t)))\,p(t) - l_{\tilde{z}}(t,u)]\,d\tilde{W}(t), \\ -dq(t) = \{[b_x(t,u) - \tilde{\sigma}(t,u)h_x(t,x(t))]\,q(t) \\ \qquad\qquad + \sigma_x(t,u)k(t) + \tilde{\sigma}_x(t,u)\tilde{k}(t) + h_x(t,x(t))\tilde{Q}(t) \\ \qquad\qquad - g_x(t,u)p(t) + l_x(t,u)\}\,dt \\ \qquad\qquad - k(t)dW(t) - \tilde{k}(t)d\tilde{W}(t), \\ p(0) = -\gamma_y(y(0)), \quad q(1) = -f_x(x(T))p(T) + \phi_x(x(T)), \end{cases} \tag{4.11}$$

and

$$\begin{cases} -dP(t) = l(t,u)dt - Q(t)dW(t) - \tilde{Q}(t)d\tilde{W}(t), \\ \quad P(T) = \phi(x(T)). \end{cases} \tag{4.12}$$

Hereinafter we adopt the notation $\tilde{W}(\cdot) = \tilde{W}^u(\cdot)$. Note that the appearance of the driving Brownian motion $\tilde{W}^v(\cdot)$ in (4.1) makes adjoint equations (4.12) and (4.11) dramatically different from the classical FBSDEs. Moreover, (1.25) is used to treat the terms induced by partially observable information, which is unnecessary in the cases of Peng [66], Øksendal and Sulem [61], Wu [100], and Yong [107].

We now state the first maximum principle for optimal control of Problem B.

Theorem 4.1. *Let (H1.6), (H1.7), and (H4.1) hold. Assume that $u(\cdot)$ is a local minimum for $J(v(\cdot))$, in the sense that for all process $v(\cdot)$ such that $v(\cdot)+u(\cdot) \in \mathscr{U}_{ad}$,*

$$\mathscr{J}(\varepsilon) = J(u(\cdot)+\varepsilon v(\cdot)), \quad \varepsilon \in [0,1)$$

attains its minimum at $\varepsilon = 0$. Suppose that for any $v(\cdot) \in \mathscr{U}_{ad}$, the functions ϕ, $\phi_x \in \mathscr{L}^2_{\mathbb{F}}(\Omega;\mathbb{R})$, l, l_x, l_y, l_z, $l_{\tilde{z}}$, $l_v \in \mathscr{L}^2_{\mathbb{F}}(0,T;\mathbb{R})$. Furthermore, suppose that

(1.25) and (1.26) admit unique solutions $(P(\cdot),Q(\cdot),\tilde{Q}(\cdot)) \in \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}^3)$ *and* $(p(\cdot),q(\cdot),k(\cdot),\tilde{k}(\cdot)) \in \mathscr{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^4)$*, respectively. Then for any* $\nu \in U$ *we have*

$$\mathbb{E}^u\left[H_v(t,x(t),y(t),z(t),\tilde{z}(t),u(t);p(t),q(t),k(t),\tilde{k}(t),\tilde{Q}(t))(\nu-u(t))|\mathscr{F}_t^Y\right] \geq 0,$$

where the Hamiltonian function $H : [0,T]\times\mathbb{R}^4\times U\times\mathbb{R}^5 \to \mathbb{R}$ *is defined by*

$$\begin{aligned} H(t,x,y,z,\tilde{z},v;p,q,k,\tilde{k},\tilde{Q}) &= b(t,x,v)q+\sigma(t,x,v)k+\tilde{\sigma}(t,x,v)\tilde{k}+h(t,x)\tilde{Q} \\ &\quad -\Big(g(t,x,y,z,\tilde{z},v)-h(t,x)\tilde{z}\Big)p+l(t,x,y,z,\tilde{z},v). \end{aligned} \tag{4.13}$$

Proof. Note that

$$\begin{aligned} 0 &\leq \frac{d}{d\varepsilon}\mathscr{J}(\varepsilon)\Big|_{\varepsilon=0} \\ &= \lim_{\varepsilon\to 0}\frac{J(u(\cdot)+\varepsilon v(\cdot))-J(u(\cdot))}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\Big\{\int_0^T\left[\left(Z^{u+\varepsilon v}(t)-Z(t)\right)l(t,u)\right. \\ &\qquad \left.+Z^{u+\varepsilon v}(t)\left(l(t,u(t)+\varepsilon v(t))-l(t,u)\right)\right]dt \\ &\qquad +\left(Z^{u+\varepsilon v}(T)-Z(T)\right)\phi(x(T))+Z^{u+\varepsilon v}(T)\left(\phi(x^{u+\varepsilon v}(T))-\phi(x(T))\right) \\ &\qquad +\gamma(y^{u+\varepsilon v}(0))-\gamma(y(0))\Big\}. \end{aligned} \tag{4.14}$$

To deal with $Z(\cdot)$, let $\Gamma(\cdot)=Z^1(\cdot)Z^{-1}(\cdot)$. Making use of (1.16) and (4.4), by Itô's formula, we get

$$\begin{cases} d\Gamma(t) = h_x(t,x(t))x^1(t)\left(dY(t)-h(t,x(t))dt\right) \\ \qquad\quad = h_x(t,x(t))x^1(t)d\tilde{W}(t), \\ \Gamma(0)=0. \end{cases} \tag{4.15}$$

Applying Itô's formula to $P(\cdot)\Gamma(\cdot)$, $p(\cdot)y^1(\cdot)$, and $q(\cdot)x^1(\cdot)$, respectively, we derive

$$\begin{aligned} &\mathbb{E}^u\left[\Gamma(T)\phi(x(T))+\int_0^T\Gamma(t)l(t,u)dt\right] \\ &= \mathbb{E}^u\int_0^T\tilde{Q}(t)h_x(t,x(t))x^1(t)dt, \end{aligned} \tag{4.16}$$

$$\begin{aligned} &\mathbb{E}^u\left[p(T)f_x(x(T))x^1(T)+\gamma_y(y(0))y^1(0)\right] \\ &= -\mathbb{E}^u\int_0^T\left[l_y(t,u)y^1(t)+l_z(t,u)z^1(t)+l_{\tilde{z}}(t,u)\tilde{z}^1(t)\right]dt \\ &\quad -\mathbb{E}^u\int_0^T\left[g_v(t,u)v(t)+g_x(t,u)x^1(t)\right]p(t)dt \end{aligned} \tag{4.17}$$

and

$$
\begin{aligned}
&\mathbb{E}^u\left[\phi_x(x(T))x^1(T) - p(T)f_x(x(T))x^1(T)\right] \\
= &\mathbb{E}^u \int_0^T g_x(t,u)x^1(t)p(t)dt \\
&- \mathbb{E}^u \int_0^T \left[l_x(t,u) + \tilde{Q}(t)h_x(t,x)\right] x^1(t)dt \\
&+ \mathbb{E}^u \int_0^T \left[b_v(t,u)q(t) + \sigma_v(t,u)k(t) + \tilde{\sigma}_v(t,u)\tilde{k}(t)\right] v(t)dt.
\end{aligned}
\tag{4.18}
$$

By Lemmas 4.2 and 4.4, we may continue (4.14) with

$$
\begin{aligned}
0 \le &\mathbb{E}^u\left[\phi_x(x(T))x^1(T) + \gamma_y(y(0))y^1(0)\right] \\
&+ \mathbb{E}^u\left[\phi(x(T))\Gamma(T) + \int_0^T \Gamma(t)l(t,u)dt\right] \\
&+ \mathbb{E}^u \int_0^T \left[l_x(t,u)x^1(t) + l_y(t,u)y^1(t)\right] dt \\
&+ \mathbb{E}^u \int_0^T \left[l_z(t,u)z^1(t) + l_{\tilde{z}}(t,u)\tilde{z}^1(t)\right] dt \\
&+ \mathbb{E}^u \int_0^T l_v(t,u)v(t)dt.
\end{aligned}
\tag{4.19}
$$

Substituting (4.16), (4.17), and (4.18) into (4.19) and recalling Condition (H4.1), we have

$$
\begin{aligned}
0 \le &\mathbb{E}^u \int_0^T \left[b_v(t,u)q(t) + \sigma_v(t,u)k(t) + \tilde{\sigma}(t,u)\tilde{k}(t)\right] v(t)dt \\
&+ \mathbb{E}^u \int_0^T \left[l_v(t,u) - g_v(t,u)p(t)\right] v(t)dt \\
= &\mathbb{E}^u \int_t^{t+\tau} \nu H_v(s,x,y,z,\tilde{z},u;p,q,k,\tilde{k},\tilde{Q})ds.
\end{aligned}
\tag{4.20}
$$

Differentiating with respect to τ, we get

$$
\mathbb{E}^u\left[\nu H_v(t,x,y,z,\tilde{z},u;p,q,k,\tilde{k},\tilde{Q})|\mathscr{F}_t^Y\right] \ge 0.
$$

The proof is then completed. □

4.2 A Malliavin Derivative Method

We now state the second maximum principle for optimal control of Problem B.

Theorem 4.2. *Let (H1.6), (H1.7), and (H4.1) hold. Assume that $u(\cdot)$ is a local minimum for $J(v(\cdot))$, in the sense that for all processes $v(\cdot)$ with $u(\cdot) + v(\cdot) \in \mathscr{U}_{ad}$,*

$$\mathscr{J}(\varepsilon) = J(u(\cdot) + \varepsilon v(\cdot)), \quad \varepsilon \in [0,1]$$

attains its minimum at $\varepsilon = 0$. *Suppose that (1.27) admits the unique solution* $\bar{p}(\cdot) \in \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{D}_{1,2})$. *Assume that* ϕ, $\phi_x \in \mathbb{D}_{1,2}$, l, l_x, *and* $\Psi(t,s)$ *are in* $\mathbb{L}_{1,2}(\mathbb{R})$ *for all* $0 \le t \le s \le T$. *Then for any* $\nu \in U$ *we have*

$$\mathbb{E}^u\left[\bar{H}_v(t,x(t),y(t),z(t),\tilde{z}(t),u(t);\bar{p}(t),\bar{q}(t),\bar{k}(t),\bar{\tilde{k}}(t))(\nu - u(t))\middle|\mathscr{F}_t^Y\right] \ge 0,$$

where $\bar{H}_v$ *is defined by*

$$\begin{aligned}\bar{H}_v(t,x,y,z,\bar{z},v;\bar{p},\bar{q},\bar{k},\bar{\tilde{k}}) &= b_v(t,x,v)\bar{q} + \sigma_v(t,x,v)\bar{k} + \tilde{\sigma}_v(t,x,v)\bar{\tilde{k}} \\ &\quad - g_v(t,x,y,z,\tilde{z},v)\bar{p} + l_v(t,x,y,z,\bar{z},v).\end{aligned}$$

Proof. If $u(\cdot)$ is a local minimum for $J(v(\cdot))$, then

$$\begin{aligned}0 &\le \frac{d}{d\varepsilon}\mathscr{J}(\varepsilon)\Big|_{\varepsilon=0} \\ &= \mathbb{E}^u\left(\phi(x(T))\Gamma(T) + \int_0^T \Gamma(t)l(t,u)dt\right) \\ &\quad + \mathbb{E}^u\left[(\phi_x(x(T)) - \bar{p}(T)f_x(x(T)))x^1(T)\right] \\ &\quad + \mathbb{E}^u\left(\bar{p}(T)f_x(x(T))x^1(T) + \gamma_y(y(0))y^1(0)\right) \\ &\quad + \mathbb{E}^u\int_0^T \left(l_x(t,u)x^1(t) + l_y(t,u)y^1(t)\right)dt \\ &\quad + \mathbb{E}^u\int_0^T \left(l_z(t,u)z^1(t) + l_{\tilde{z}}(t,u)\tilde{z}^1(t)\right)dt \\ &\quad + \mathbb{E}^u\int_0^T l_v(t,u)v(t)dt.\end{aligned} \tag{4.21}$$

According to (4.15), Lemmas A.7 and A.8, we have

$$\begin{aligned}\mathbb{E}^u(\phi(x(T))\Gamma(T)) &= \mathbb{E}^u\left(\phi(x(T))\int_0^T h_x(t,x)x^1(t)d\tilde{W}(t)\right) \\ &= \mathbb{E}^u\int_0^T h_x(t,x)x^1(t)D_t^{(\tilde{W})}\phi(x(T))dt\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\mathbb{E}^u\int_0^T \Gamma(t)l(t,u)dt &= \mathbb{E}^u\int_0^T l(t,u)\int_0^t h_x(s,x)x^1(s)d\tilde{W}(s)dt \\ &= \mathbb{E}^u\int_0^T\int_0^t h_x(t,x)x^1(t)D_s^{(\tilde{W})}l(t,u)dsdt \\ &= \mathbb{E}^u\int_0^T\left(\int_t^T D_t^{(\tilde{W})}l(s,u)ds\right)h_x(t,x)x^1(t)dt.\end{aligned} \tag{4.23}$$

Note that, in deriving the last line in (4.23), we used Fubini's theorem. It then follows from (4.22) and (4.23) that

$$
\begin{aligned}
&\mathbb{E}^u \left(\phi(x(T))\Gamma(T) + \int_0^T \Gamma(t) l(t,u) dt \right) \\
&= \mathbb{E}^u \int_0^T \left(D_t^{(\tilde{W})} \phi(x(T)) + \int_t^T D_t^{(\tilde{W})} l(s,u) ds \right) h_x(t,x) x^1(t) dt \qquad (4.24) \\
&= \mathbb{E}^u \int_0^T h_x(t,x) x^1(t) D_t^{(\tilde{W})} \Pi(t) dt.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbb{E}^u \left[(\phi_x(x(T)) - \bar{p}(T) f_x(x(T))) x^1(T) \right] \qquad (4.25) \\
&= \mathbb{E}^u \Big\{ (\phi_x(x(T)) - \bar{p}(T) f_x(x(T))) \\
&\quad \times \Big[\int_0^T \big((b_x(t,u) - \tilde{\sigma}(t,u) h_x(t,x)) x^1(t) + b_v(t,u) v(t) \big) dt \\
&\quad + \int_0^T \big(\sigma_x(t,u) x^1(t) + \sigma_v(t,u) v(t) \big) dW(t) \\
&\quad + \int_0^T \big(\tilde{\sigma}_x(t,u) x^1(t) + \tilde{\sigma}_v(t,u) v(t) \big) d\tilde{W}(t) \Big] \Big\} \\
&= \mathbb{E}^u \int_0^T (\phi_x(x(T)) - \bar{p}(T) f_x(x(T))) \\
&\quad \times \Big\{ (b_x(t,u) - \tilde{\sigma}(t,u) h_x(t,x)) x^1(t) + b_v(t,u) v(t) \\
&\quad + \big(\sigma_x(t,u) x^1(t) + \sigma_v(t,u) v(t) \big) D_t^{(W)} (\phi_x(x(T)) - \bar{p}(T) f_x(x(T))) \\
&\quad + \big(\tilde{\sigma}_x(t,u) x^1(t) + \tilde{\sigma}_v(t,u) v(t) \big) D_t^{(\tilde{W})} (\phi_x(x(T)) - \bar{p}(T) f_x(x(T))) \Big\} dt.
\end{aligned}
$$

By (4.5), and Lemmas A.7 and A.8, we have

$$
\begin{aligned}
&\mathbb{E}^u \int_0^T l_x(t,u) x^1(t) dt \\
&= \mathbb{E}^u \int_0^T \int_0^t \Big\{ l_x(t,u) \left[(b_x(s,u) - \tilde{\sigma}(s,u) h_x(s,x)) x^1(s) + b_v(s,u) v(s) \right] \\
&\quad + \big(\sigma_x(s,u) x^1(s) + \sigma_v(s,u) v(s) \big) D_s^{(W)} l_x(t,u) \\
&\quad + \big(\tilde{\sigma}_x(s,u) x^1(s) + \tilde{\sigma}_v(s,u) v(s) \big) D_s^{(\tilde{W})} l_x(t,u) \Big\} ds dt. \qquad (4.26)
\end{aligned}
$$

Simple calculations from (4.26) then yield that

$$
\begin{aligned}
&\mathbb{E}^u \int_0^T l_x(t,u)x^1(t)dt \\
&= \mathbb{E}^u \int_0^T \left\{ \int_t^T l_x(s,u)ds \left[(b_x(t,u) - \tilde{\sigma}(t,u)h_x(t,x))\, x^1(t) + b_v(t,u)v(t) \right] \right. \\
&\quad + \left(\sigma_x(t,u)x^1(t) + \sigma_v(t,u)v(t) \right) \int_t^T D_t^{(W)} l_x(s,u)ds \\
&\quad \left. + \left(\tilde{\sigma}_x(t,u)x^1(t) + \tilde{\sigma}_v(t,u)v(t) \right) \int_t^T D_t^{(\tilde{W})} l_x(s,u)ds \right\} dt. \qquad (4.27)
\end{aligned}
$$

By (4.25) and (4.27) we then have

$$
\begin{aligned}
&\mathbb{E}^u \left[(\phi_x(x(T)) - \bar{p}(T)f_x(x(T)))\, x^1(T) + \int_0^T l_x(t,u)x^1(t)dt \right] \\
&= \mathbb{E}^u \int_0^T \left\{ \Sigma(t) \left[(b_x(t,u) - \tilde{\sigma}(t,u)h_x(t,x))\, x^1(t) + b_v(t,u)v(t) \right] \right. \\
&\quad + \left(\sigma_x(t,u)x^1(t) + \sigma_v(t,u)v(t) \right) D_t^{(W)} \Sigma(t) \\
&\quad \left. + \left(\tilde{\sigma}_x(t,u)x^1(t) + \tilde{\sigma}_v(t,u)v(t) \right) D_t^{(\tilde{W})} \Sigma(t) \right\} dt. \qquad (4.28)
\end{aligned}
$$

Applying Itô's formula to $\bar{p}(\cdot)y^1(\cdot)$, we derive

$$
\begin{aligned}
&\mathbb{E}^u \left[\bar{p}(T)f_x(x(T))x^1(T) + \gamma_y(y(0))y^1(0) \right] \\
&= -\mathbb{E}^u \int_0^T \left[l_y(t,u)y^1(t) + l_z(t,u)z^1(t) + l_{\tilde{z}}(t,u)\tilde{z}^1(t) \right] dt \qquad (4.29) \\
&\quad - \mathbb{E}^u \int_0^T \left[g_v(t,u)v(t) + g_x(t,u)x^1(t) \right] \bar{p}(t)dt.
\end{aligned}
$$

Inserting (4.24), (4.28), and (4.29) into (4.21), we have

$$
\begin{aligned}
0 &\le \frac{d}{d\varepsilon} \mathscr{J}(\varepsilon) \Big|_{\varepsilon=0} \\
&= \mathbb{E}^u \int_0^T \Big[\Sigma(t) \left(b_x(t,u) - \tilde{\sigma}(t,u)h_x(t,x) \right) + \sigma_x(t,u)D_t^{(W)}\Sigma(t) \\
&\quad + \tilde{\sigma}_x(t,u)D_t^{(\tilde{W})}\Sigma(t) + h_x(t,x)D_t^{(\tilde{W})}\Pi(t) - g_x(t,x,y,z,\tilde{z},u)\bar{p}(t) \Big] x^1(t)dt \qquad (4.30) \\
&\quad + \mathbb{E}^u \int_0^T \Big[\Sigma(t)b_v(t,u) + \sigma_v(t,u)D_t^{(W)}\Sigma(t) + \tilde{\sigma}_v(t,u)D_t^{(\tilde{W})}\Sigma(t) \\
&\quad + l_v(t,u) - g_v(t,u)\bar{p}(t) \Big] v(t)dt.
\end{aligned}
$$

Since (4.30) holds for any admissible control $v(\cdot)$, hereafter we take

$$
v(s) = \nu I_{(t,t+\tau]}(s),
$$

where $v = v(\omega)$ is a bounded $\mathscr{F}_t^Y$-measurable random variable, $0 \le t \le t+\tau \le T$. In this situation, it is easy to see from (4.5) that

$$x^1(s) = 0, \quad \text{for } 0 \le s \le t. \tag{4.31}$$

Then (4.30) can be written as

$$0 \le \mathscr{J}_1(\tau) + \mathscr{J}_2(\tau) \tag{4.32}$$

with

$$\begin{aligned} \mathscr{J}_1(\tau) = \mathbb{E}^u \int_t^T \Big[&\Sigma(s)\left(b_x(s,u) - \tilde{\sigma}(s,u)h_x(s,x)\right) + \sigma_x(s,u)D_s^{(W)}\Sigma(s) \\ &+ \tilde{\sigma}_x(s,u)D_s^{(\tilde{W})}\Sigma(s) + h_x(s,x)D_s^{(\tilde{W})}\Pi(s) \\ &- g_x(s,u)\bar{p}(s)\big] x^1(s)ds \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \mathscr{J}_2(\tau) = \mathbb{E}^u \int_t^{t+\tau} v \Big[&\Sigma(s)b_v(s,u) + \sigma_v(s,u)D_s^{(W)}\Sigma(s) + \tilde{\sigma}_v(s,u)D_s^{(\tilde{W})}\Sigma(s) \\ &+ l_v(s,u) - g_v(s,u)\bar{p}(s)\Big] ds. \end{aligned} \tag{4.34}$$

Note that with the special control $v(s) = vI_{(t,t+\tau]}(s)$, we arrive at

$$\begin{aligned} dx^1(s) = x^1(s)\Big\{ &[b_x(s,u) - \tilde{\sigma}(s,u)h_x(s,x)]\,ds \\ &+ \sigma_x(s,u)dW(s) + \tilde{\sigma}_x(s,u)d\tilde{W}(s)\Big\}, \quad \text{for } s \ge t+\tau. \end{aligned}$$

Solving the above equation, we get

$$x^1(s) = x^1(t+\tau)\Phi(t+\tau, s),$$

where

$$\begin{aligned} x^1(t+\tau) = v \int_t^{t+\tau} &\left(b_v(r,u)dr + \sigma_v(r,u)dW(r) + \tilde{\sigma}_v(r,u)d\tilde{W}(r)\right) \\ &+ \int_t^{t+\tau} x^1(r)\left[\left(b_x(r,u) - \tilde{\sigma}(r,u)h_x(r,x)\right)dr \right. \\ &\left. + \sigma_x(r,u)dW(r) + \tilde{\sigma}_x(r,u)d\tilde{W}(r)\right]. \end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{d\tau}\mathscr{J}_1(\tau)\Big|_{\tau=0} &= \frac{d}{d\tau}\mathbb{E}^u\left[\int_{t+\tau}^T H_x(s)x^1(t+\tau)\Phi(t+\tau,s)ds\right]_{\tau=0} \\
&= \int_t^T \frac{d}{d\tau}\mathbb{E}^u\left[H_x(s)x^1(t+\tau)\Phi(t+\tau,s)\right]_{\tau=0}ds \\
&= \int_t^T \frac{d}{d\tau}\mathbb{E}^u\left[x^1(t+\tau)\Psi(t,s)\right]_{\tau=0}ds. \\
&= \mathscr{J}_{11}+\mathscr{J}_{12},
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{J}_{11} = \int_t^T \frac{d}{d\tau}\mathbb{E}^u\Bigg\{\Psi(t,s)\int_t^{t+\tau} x^1(r)\left[(b_x(r,u)-\tilde{\sigma}(r,u)h_x(r,x))\,dr\right. \\
\left.+\sigma_x(r,u)dW(r)+\tilde{\sigma}_x(r,u)d\tilde{W}(r)\right]\Bigg\}_{\tau=0}ds
\end{aligned}\tag{4.35}$$

and

$$\begin{aligned}
\mathscr{J}_{12} = \int_t^T \frac{d}{d\tau}\mathbb{E}^u\Bigg\{v\Psi(t,s)\int_t^{t+\tau}\left[b_v(r,u)dr\right. \\
\left.+\sigma_v(r,u)dW(r)+\tilde{\sigma}_v(r,u)d\tilde{W}(r)\right]\Bigg\}_{\tau=0}ds.
\end{aligned}\tag{4.36}$$

According to (4.31), Lemmas A.7 and A.8, it is not difficult to derive that

$$\mathscr{J}_{11}=0$$

and

$$\begin{aligned}
\mathscr{J}_{12} = \mathbb{E}^u\int_t^T v\Bigg(\Psi(t,s)b_v(t,u)+\sigma_v(t,u)D_t^{(W)}\Psi(t,s) \\
+\tilde{\sigma}_v(t,u)D_t^{(\tilde{W})}\Psi(t,s)\Bigg)ds.
\end{aligned}\tag{4.37}$$

Similarly,

$$\begin{aligned}
\frac{d}{d\tau}\mathscr{J}_2(\tau)\Big|_{\tau=0} = \mathbb{E}^u\Big\{v\Big[\Sigma(t)b_v(t,u)+\sigma_v(t,u)D_t^{(W)}\Sigma(t) \\
+\tilde{\sigma}_v(t,u)D_t^{(\tilde{W})}\Sigma(t)+l_v(t,u)-g_v(t,u)\bar{p}(t)\Big]\Big\}.
\end{aligned}\tag{4.38}$$

From (4.21), (4.37), and (4.38), we get

$$\begin{aligned}
0 &\le \frac{d}{d\varepsilon}\mathscr{J}(\varepsilon)\Big|_{\varepsilon=0} \\
&= \mathbb{E}^u\Big\{v\Big[b_v(t,u)\bar{q}(t)+\sigma_v(t,u)\bar{k}(t)+\tilde{\sigma}_v(t,u)\bar{\bar{k}}(t) \\
&\quad +l_v(t,u)-g_v(t,u)\bar{p}(t)\Big]\Big\}.
\end{aligned}$$

The proof is then completed. □

4.3 A Recursive Utility Optimization Problem

This section focuses on illustrating Theorem 4.2 within the framework of recursive utility. For convenience, we let $\tilde{C}(t) = 0$ in (1.7), $0 \le t \le T$.

The aim of the policymaker is to find a control strategy $u(\cdot) \in \mathscr{U}_{ad}$ so that

$$J(u(\cdot)) = \min_{v(\cdot)\in\mathscr{U}_{ad}} \mathbb{E}^v \left[\frac{1}{2}\int_0^T (v(t) - M(t))^2\, dt - y^v(0)\right] \tag{4.39}$$

subject to (1.7), (1.8) and Definition 1.2, where $M(\cdot)$ is a pre-set target, and $y^v(\cdot)$ is a generalized recursive utility resulting from x and v. In the sense of El Karoui et al. [19], $y^v(\cdot)$ can be regarded as the solution of

$$\begin{cases} -dy^v(t) = g(t, x^v(t), y^v(t), z^v(t), \tilde{z}^v(t))dt - z^v(t)dW(t) - \tilde{z}^v(t)dY(t), \\ \quad y^v(T) = f(x^v(T)), \end{cases}$$

where f and g satisfy (H1.6). The example captures the scenario where the policymaker has two objectives: on one hand, the concern of the policymaker is to prevent the control strategy $v(\cdot)$ from large deviations so as to stabilize the related economic scheme, on the other hand, he/she would like to optimize the recursive utility. Note that utility functional (4.39) is inspired by Shi and Wu [75], where an optimization problem with complete information was studied.

With this setup, it is easy to see from (1.7) and (1.8) that

$$\begin{gathered} b(t,x,v) = A(t)x + B(t)v,\ \sigma(t,x,v) = C(t)v + D(t), \\ \tilde{\sigma}(t,x,v) = \tilde{D}(t),\ h(t,x) = \tfrac{1}{\beta}\alpha(t,x) - \tfrac{1}{2}\beta. \end{gathered}$$

The new adjoint processes are written as

$$\begin{aligned} \bar{q}(t) &= -f_x(x(T))\bar{p}(T) + \int_t^T H_x(s)\Phi(t,s)ds, \\ \bar{k}(t) &= D_t^{(W)}\bar{q}(t), \quad \bar{\tilde{k}}(t) = D_t^{(\tilde{W})}\bar{q}(t) \end{aligned} \tag{4.40}$$

with

$$\begin{cases} d\bar{p}(t) = \bar{p}(t)\Big[g_y(t,x(t),y(t),z(t),\tilde{z}(t))dt + g_z(t,x(t),y(t),z(t),\tilde{z}(t))dW(t) \\ \qquad\qquad + \left(g_{\tilde{z}}(t,x(t),y(t),z(t),\tilde{z}(t)) - \dfrac{1}{\beta}\alpha(t,x(t)) + \dfrac{1}{2}\beta\right) d\tilde{W}(t)\Big], \\ \bar{p}(0) = 1, \end{cases}$$

$$\begin{aligned} H_x(t) = &\ -f_x(x(T))\bar{p}(T)\left[A(t) - \frac{1}{\beta}\tilde{D}(t)\alpha_x(t,x(t))\right] + \frac{1}{\beta}\alpha_x(t,x(t))D_t^{(\tilde{W})}\Pi(t) \\ &- g_x(t,x(t),y(t),z(t),\tilde{z}(t))\bar{p}(t), \end{aligned}$$

$$\Pi(t) = \frac{1}{2}\int_t^T (u(s) - M(s))^2 ds$$

and

$$\Phi(t,s) = \exp\left\{\int_t^s \left[A(r) - \frac{1}{\beta}\tilde{D}(r)\alpha_x(r,x(r))\right] dr\right\}.$$

According to Theorem 4.2 and (4.40), we have

Proposition 4.1. *Let $H_x(t)\Phi(t,s) \in \mathbb{L}_{1,2}(\mathbb{R})$, $0 \le t \le s \le T$. If $u(\cdot)$ is an optimal control strategy, then it is necessary to satisfy*

$$u(t) = M(t) - B(t)\mathbb{E}^u\left[\bar{q}(t)|\mathscr{F}_t^Y\right] - C(t)\mathbb{E}^u\left[D_t^{(W)}\bar{q}(t)|\mathscr{F}_t^Y\right], \tag{4.41}$$

where $\bar{q}(\cdot)$ is the solution of (4.40).

Note that a more explicit representation of (4.41) strongly depends on the specific structure of the distributions $\mathbb{E}^u\left[\bar{q}(t)|\mathscr{F}_t^Y\right]$ and $\mathbb{E}^u\left[D_t^{(W)}\bar{q}(t)|\mathscr{F}_t^Y\right]$. To illustrate this point, let us consider a special case of Proposition 4.1 in detail.

(H4.2) *Assume that g is independent of (x,y), and*

$$g(t,z,\tilde{z}) = c(t)z + \tilde{c}(t)\tilde{z}, f(x) = x \text{ and } \alpha(t,x) = \alpha(t),\ \forall(t,z,\tilde{z}) \in [0,T]\times\mathbb{R}^2,$$

where $c(\cdot)$, $\tilde{c}(\cdot)$, and $\alpha(\cdot)$ are deterministic and bounded.

It follows from (4.40) that

$$\bar{q}(t) = \bar{p}(T)\bar{A}(t), \quad D_t^{(W)}\bar{q}(t) = c(t)\bar{A}(t)\bar{p}(T),$$

with

$$\bar{A}(t) = -\int_t^T A(s)e^{\int_t^s A(r)dr}ds - 1.$$

Next, let

$$\hat{\bar{p}}_{s,t} = \mathbb{E}^u[\bar{p}(s)|\mathscr{F}_t^Y], \quad 0 \le t \le s \le T$$

be the optimal extrapolation of $\bar{p}(\cdot)$ with respect to

$$\mathscr{F}_t^Y = \sigma\{\tilde{W}(r); 0 \le r \le t\}.$$

Then (4.41) is rewritten as

$$u(t) = M(t) - \bar{A}(t)\Big(B(t) + c(t)C(t)\Big)\hat{\bar{p}}_{1,t}, \tag{4.42}$$

where

$$\hat{\bar{p}}_{s,t} = 1 + \int_0^t \bar{c}(r)\hat{\bar{p}}(r)d\tilde{W}(r)$$

with

$$\hat{\bar{p}}(r) = e^{\int_0^r \bar{c}(\theta)d\tilde{W}(\theta) - \frac{1}{2}\int_0^r \bar{c}^2(\theta)d\theta} \text{ and } \bar{c}(r) = \tilde{c}(r) - \frac{1}{\beta}\alpha(r) + \frac{1}{2}\beta.$$

Furthermore, the optimal cost functional can be derived in terms of (4.39) and (4.42).

We now summarize the result as follows.

Corollary 4.1. *Under (H4.2), the optimal control strategy of the underlying problem is given uniquely by (4.42).*

4.4 Notes

The earliest research on partially observable optimal control can be traced back to Florentin [22]. Since this paper was published in 1962, numerous people have contributed to this field. The interested reader is referred to Davis and Varaiya [16], Fleming and Pardoux [21], Bensoussan [6], Elliott et al. [20], Zhang and Xie [113], Tang [78], Shen et al. [72], and references cited therein for the development in various subjects, especially in maximum principle as well as dynamic programming.

However, prior to the beginning of 21st century, almost all the combined problems of control and filtering were formulated under the assumption that the state processes solve (forward) SDEs. With the rapid development and broad application of FBSDE in stochastic control theory, it is nature to say whether we can establish a combined model of filtering and control of FBSDE. Starting about from 2003, Zhen Wu and his graduate students at the School of Mathematics and System Sciences (now named the School of Mathematics), Shandong University, began to focus on exploring such a model. After about 5 years, the first result on Kalman–Bucy filtering of a special class of fully coupled FBSDEs was published, while a backward separation approach was proposed and was used to solve a partially observable LQ control problem driven by SDE in Wang and Wu [84]. At almost the same time, the first partially observable optimal control model of FBSDE was established by Wang and Wu [84] and Wu [100] from the viewpoint of mathematical finance, and then was studied by them via combining the backward separation approach with the maximum principle. Along this line, there are a few interesting papers to extend the model in several aspects, especially in maximum principle and nonlinear backward stochastic differential filtering equation. See, e.g., the doctoral dissertation of Wang [82], the survey paper of Wang et al. [93] for more details on these aspects. Note that how to obtain a dynamic programming principle corresponding to the partially observable forward-backward stochastic control model is also valuable topic. As far as we know, it has, however, not been explored so far.

The results introduced in this chapter are taken mainly from Wang et al. [88]. Similar to Chapter 3, some versions of verification theorem for optimality of Problem B can be derived. We omit them for the length of the book.