Chapter 5
Relation to Complex Dynamics

In this chapter we outline how rotation sets occur in the dynamical study of complex polynomial maps. Special attention is paid to the relation with the dynamics of complex quadratic and cubic polynomials. This link provides a geometric realization of rotation sets under m_d, whose abstract theory was developed in the previous chapters.

5.1 Polynomials and Dynamic Rays

We assume the reader is familiar with the basic notions of complex dynamics, as in [21]. Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial map of degree $d \geq 2$. The ***filled Julia set*** $K(f)$ is the union of all bounded orbits of f, and the ***Julia set*** $J(f)$ is the topological boundary of $K(f)$. Both are compact non-empty subsets of the plane. The complement $\mathbb{C} \smallsetminus K(f)$ is connected and can be described as the ***basin of infinity*** for f, that is, the set of all points whose orbits under f tend to ∞. The ***Green's function*** of f is the continuous function $G : \mathbb{C} \to \mathbb{R}$ defined by

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^{\circ n}(z)|,$$

which describes the escape rate of z to ∞ under the iterations of f. It is easy to see that G satisfies the relation

$$G(f(z)) = d\, G(z)$$

with $G(z) = 0$ if and only if $z \in K(f)$. The Green's function is harmonic in the basin of ∞, with critical points at all precritical points of f. In other words, $\nabla G(z) = 0$ for some $z \in \mathbb{C} \smallsetminus K(f)$ if and only if $f^{\circ n}(z)$ is a critical point of f for some $n \geq 0$.

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There is a unique conformal isomorphism β, defined in some neighborhood of ∞, which is tangent to the identity at ∞ (in the sense that $\lim_{z\to\infty} \beta(z)/z = 1$) and conjugates the action of f to that of the power map $z \mapsto z^d$:

$$\beta(f(z)) = (\beta(z))^d \quad \text{for large} \quad |z|.$$

We call β the ***Böttcher coordinate*** of f near ∞. The modulus of β is related to the Green's function by the relation $|\beta(z)| = e^{G(z)}$ for large $|z|$. It is not hard to check that β is univalent in the domain $\{z \in \mathbb{C} : G(z) > G_0\}$, where

$$G_0 = \max\{G(c) : c \text{ is a critical point of } f\}.$$

In particular, if every critical point of f belongs to $K(f)$, then $G_0 = 0$ and β is a conformal isomorphism $\mathbb{C} \smallsetminus K(f) \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$. This happens precisely when $K(f)$ is connected.

In what follows and unless otherwise stated we assume that $K(f)$ is connected. In this case the inverse Böttcher coordinate $\psi = \beta^{-1} : \mathbb{C} \smallsetminus \overline{\mathbb{D}} \to \mathbb{C} \smallsetminus K(f)$ is a conformal isomorphism which satisfies

$$\psi(z^d) = f(\psi(z)) \quad \text{for} \quad |z| > 1. \tag{5.1}$$

By the ***(dynamic) ray*** of f at angle $t \in \mathbb{T}$ we mean the real-analytic curve

$$R(t) = \psi\big(\{re^{2\pi i t} : r > 1\}\big).$$

The functional equation (5.1) shows that

$$f(R(t)) = R(m_d(t)) \quad \text{for all} \quad t \in \mathbb{T}. \tag{5.2}$$

We say that $R(t)$ ***lands*** at $z \in J(f)$ if $\lim_{r\to 1} \psi(re^{2\pi i t}) = z$. It follows from (5.2) that if $R(t)$ lands at z, then $R(m_d(t))$ lands at $f(z)$. Similarly, if f has local degree k at $w \in f^{-1}(z)$, then there are k preimages $\{t_1, \dots t_k\}$ of t under m_d such that each $R(t_i)$ lands at w. A ray may or may not land, but the set of angles t for which $R(t)$ lands has full Lebesgue measure on the circle.

The ***impression*** $\hat{R}(t)$ of the ray $R(t)$ is the set of all $w \in \mathbb{C}$ for which there is a sequence $z_n \in \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ such that $z_n \to e^{2\pi i t}$ and $\psi(z_n) \to w$. It is not hard to check that $\hat{R}(t)$ is a non-empty compact connected subset of $J(f)$. Every point of the Julia set belongs to at least one impression. We say that the impression $\hat{R}(t)$ is ***trivial*** if it reduces to a single point $\{z\}$. In this case, $R(t)$ necessarily lands at z (a landing ray, however, may well have a non-trivial impression). Furthermore, it is easily seen that

$$\limsup_{n\to\infty} \hat{R}(t_n) \subset \hat{R}(t) \quad \text{whenever} \quad t_n \to t. \tag{5.3}$$

(As usual, the limsup on the left is the set of all $p \in \mathbb{C}$ such that every neighborhood of p meets infinitely many of the $\hat{R}(t_n)$.) We will also use the following separation property later on: Suppose the rays $R(t')$, $R(t'')$ land at z and W is one of the two connected components of $\mathbb{C} \smallsetminus (R(t') \cup R(t'') \cup \{z\})$. If a third ray $R(t)$ is contained in W, then $\hat{R}(t) \subset W \cup \{z\}$.

A point $z \in K(f)$ is the landing point of two or more rays if and only if $K(f) \smallsetminus \{z\}$ is disconnected. More precisely, z has $2 \leq n \leq \infty$ distinct rays landing on it if and only if $K(f) \smallsetminus \{z\}$ has n connected components [18]. If z has finite forward orbit under f, the number of rays landing on it can be arbitrarily large (see the case of a parabolic fixed point below). But if the forward orbit of z is infinite, there is an upper bound $C(d)$ for the number of rays that can land at z (one can take $C(d) = 2^d$, and the bound improves to $C(d) = d$ if z is not precritical [15]).

The ***multiplier*** of a fixed point $\zeta = f(\zeta)$ is the derivative $f'(\zeta)$. We call ζ ***attracting, repelling,*** or ***indifferent,*** according as the modulus $|f'(\zeta)|$ is less than, greater than, or equal to 1. An indifferent fixed point is called ***parabolic*** if its multiplier is a root of unity. The multiplier and type of a periodic point ζ of period n can be defined analogously by treating ζ as a fixed point of the iterate $f^{\circ n}$.

Suppose the angle $t \in \mathbb{T}$ is periodic of period $q \geq 1$ under m_d, so t is rational of the form $i/(d^q - 1)$. According to the Douady-Hubbard landing theorem [21], the ray $R(t)$ lands at a periodic point of f with period dividing q, and this periodic point is necessarily repelling or parabolic. Conversely, every repelling or parabolic periodic point of f is the landing point of finitely many rays whose angles are periodic under m_d of the same period.

As a special case, if $u_i = i/(d-1) \pmod{\mathbb{Z}}$, it follows that for each $0 \leq i \leq d-2$ the ***fixed ray*** $R(u_i)$ lands at a repelling or parabolic fixed point $\zeta_i = f(\zeta_i)$. When ζ_i is parabolic, the multiplier $f'(\zeta_i)$ is necessarily 1. Of course the fixed points $\zeta_0, \ldots, \zeta_{d-2}$ need not be distinct.

The study of dynamic rays when $K(f)$ is disconnected is a bit more complicated (an example of this case will be briefly discussed in Sect. 5.4). In this case at least one critical point of f escapes to ∞ and the Green's function G has infinitely many critical points outside $K(f)$. We can still define the dynamic rays $\{R(t)\}_{t \in \mathbb{T}}$ partially near ∞ by pulling back the radial lines under the Böttcher coordinate

$$\beta : \{z \in \mathbb{C} : G(z) > G_0\} \to \{z : |z| > e^{G_0}\}.$$

These partial rays are the trajectories of the gradient vector field ∇G near ∞, so they can be extended in backward time. Such an extended trajectory either avoids the critical points of G and tends to $K(f)$, or it eventually tends to such a critical point (namely an escaping precritical point of f). We call the ray ***smooth*** or ***bifurcated*** accordingly. For all but countably many $t \in \mathbb{T}$ the ray $R(t)$ is smooth. In this case $R(m_d(t))$ is also smooth and the relation (5.2) holds. On the other hand, for a countably infinite set of angles t the ray $R(t)$ is bifurcated. Under the iterations of f every bifurcated ray eventually maps to a smooth ray passing through a critical value of f.

5.2 Rotation Sets and Indifferent Fixed Points

This section will study polynomial maps of degree $d \geq 2$ with connected Julia set which have an indifferent fixed point of multiplier $e^{2\pi i\theta} \neq 1$. Every such map is affinely conjugate to a monic polynomial of the form

$$f : z \mapsto e^{2\pi i\theta} z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + z^d, \tag{5.4}$$

where the indifferent fixed point is placed at the origin. We consider two cases depending on the nature of the fixed point 0.

The parabolic case. First suppose 0 is a parabolic fixed point so θ is rational of the form p/q in lowest terms. Then there are finitely many rays landing at 0, each being periodic of period q. We can label these rays as

$$R(t_1), R(t_2), \ldots, R(t_{Nq})$$

where $N \geq 1$ and $0, t_1, \ldots, t_{Nq}$ are in positive cyclic order. Using the form of the multiplier, it is easily seen that $f(R(t_j)) = R(t_{j+Np})$, or $m_d(t_j) = t_{j+Np}$ for every j, where as usual the indices are taken modulo Nq. It follows that $\{t_1, \ldots, t_{Nq}\}$ is the union of N disjoint q-cycles under m_d, each with the combinatorial rotation number p/q.

The following lemma ties up the situation with rotation sets:

Lemma 5.1 *The set X of the angles $t \in \mathbb{T}$ for which the ray $R(t)$ lands at 0 is a rotation set under m_d with $\rho(X) = p/q$.*

Proof Label $X = \{t_1, \ldots, t_{Nq}\}$ as above. For $1 \leq i \leq N$, let C_i denote the q-cycle

$$t_i \mapsto t_{i+Np} \mapsto t_{i+2Np} \mapsto \ldots \mapsto t_{i+(q-1)Np}$$

under m_d. Evidently X is the disjoint union of $C_1, \ldots, C_N$ and these cycles are superlinked in the sense of Sect. 2.3. By Lemma 2.25, X is a rotation set with $\rho(X) = \rho(C_i) = p/q$. □

The deployment invariant of X can be described dynamically as follows. Two adjacent rays $R(t_j)$ and $R(t_{j+1})$ together with their common landing point 0 divide the plane into two open sectors. By definition, the ***(dynamic) wake*** W_j is the sector that contains the rays $R(t)$ with $t \in (t_j, t_{j+1})$ (thus, W_j is the sector defined by going counter-clockwise from $R(t_j)$ to $R(t_{j+1})$). The gap $I_j = (t_j, t_{j+1})$ of X corresponds to the part of the boundary of the wake W_j on the circle at ∞. By Lemma 2.13, the multiplicity n_j of I_j is the number of fixed rays that are contained in W_j. It is also the number of the critical points of f in W_j (see [12], where this invariant is called the "critical weight" of W_j, and compare Theorem 5.10 for a similar case). In particular, I_j is a major gap if and only if W_j contains a fixed point ζ_i, or equivalently a critical point. As there are $d-1$ fixed rays, there are at most $d-1$ indices $1 \leq j \leq Nq$ for which $n_j \neq 0$. Form the non-decreasing list of integers $0 \leq s_1 \leq s_2 \leq \cdots \leq s_{d-1} = Nq$ in which each index $1 \leq j \leq Nq$ appears n_j times. It then follows from Lemma 3.5 that $(s_1, \ldots, s_{d-1})$ is the signature $s(X)$ as defined in Sect. 3.2 and therefore $(s_1/(Nq), \ldots, s_{d-1}/(Nq))$ is the cumulative deployment vector $\sigma(X)$.

Since the multiplier of the fixed point 0 is a q-th root of unity, the q-th iterate of f has the local expansion

$$f^{\circ q}(z) = z + a\, z^m + O(z^{m+1}) \quad \text{for some} \quad a \neq 0 \quad \text{and} \quad m > 1.$$

The integer m, the algebraic multiplicity of 0 as the root of the equation $f^{\circ q}(z) - z = 0$, is necessarily of the form $kq + 1$ for some $1 \leq k \leq N$. According to Leau and Fatou [21], there are bounded Fatou components $U_1, \ldots, U_{kq}$ arranged as kq "petals" around the common boundary point 0. If we choose labeling counter-clockwise, we have $f(U_j) = U_{j+kp}$ for every j, taking indices modulo kq, so the U_j are permuted with combinatorial rotation number p/q. Every point in the union $U_1 \cup \cdots \cup U_{kq}$ has an infinite orbit that tends to 0. Conversely, every infinite orbit converging to 0 must eventually enter this union. It follows from this local picture that the ***petal number*** kq of the parabolic fixed point is bounded above by the ***ray number*** Nq. The bound $N \leq d - 1$ of Theorem 2.27 now shows that

$$q \leq \text{petal number } kq \leq \text{ray number } Nq \leq (d-1)q.$$

In the quadratic case $d = 2$ it follows that the petal number and ray number are both q, while in the cubic case $d = 3$ these numbers can be q or $2q$ (see Fig. 5.1 for the case $(k, N) = (1, 1)$ and $(1, 2)$, and Fig. 5.9 for the case $(k, N) = (2, 2)$).

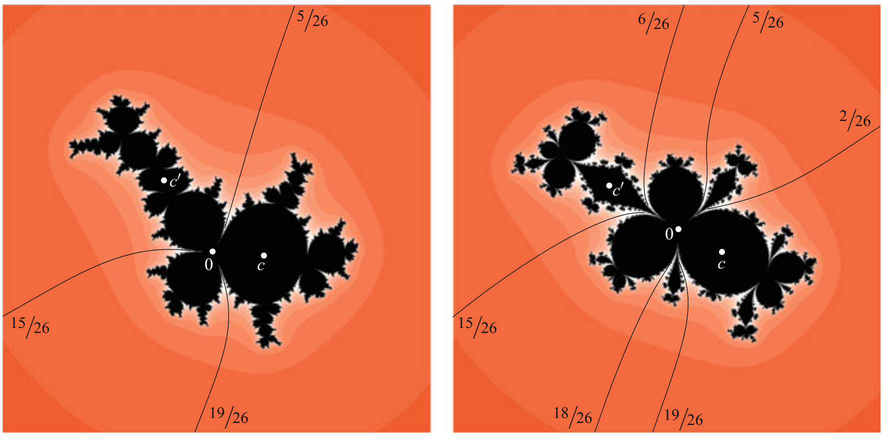

Fig. 5.1 Examples of parabolic points with multiplier $\lambda = e^{2\pi i/3}$ and petal number 3. Left: The cubic $z \mapsto \lambda z - (0.04 + 0.85i)z^2 + z^3$ with ray number 3. Right: The cubic $z \mapsto \lambda z + (0.23 - 0.20i)z^2 + z^3$ with ray number 6. The critical points c, c' are marked as white dots

The "good" Siegel case. Now suppose 0 is a linearizable fixed point, so it belongs to a bounded Fatou component Δ in which the action of f is conjugate to the irrational rotation $z \mapsto e^{2\pi i\theta} z$. The domain Δ is called the ***Siegel disk*** of f centered at 0. We will assume that the boundary $\partial\Delta$ is a Jordan curve containing at least one critical point of f. This is certainly the case if θ is an irrational number of bounded type, that is, if the partial quotients in the continued fraction expansion $\theta = [a_1, a_2, a_3, \ldots]$ form a bounded sequence (compare [8] and [31]).[1] To avoid topological complications and focus on the combinatorial aspects of the constructions, we further make the following assumption:

The Limb Decomposition Hypothesis There is a countable collection of disjoint non-trivial compact connected subsets of $K(f)$, called ***limbs***, such that

(LD1) $K(f)$ is $\overline{\Delta}$ union all the limbs,
(LD2) Each limb meets $\overline{\Delta}$ at a single point on $\partial\Delta$ called its ***root***,
(LD3) For each $\varepsilon > 0$ there are at most finitely many limbs with diameter $> \varepsilon$.[2]

We denote by $L(p)$ the limb with root $p \in \partial\Delta$.

Lemma 5.2 *A point $p \in \partial\Delta$ is a root if and only if $K(f) \smallsetminus \{p\}$ is disconnected.*

Proof For every root p the non-empty set $L(p) \smallsetminus \{p\}$, which is clearly closed in $K(f) \smallsetminus \{p\}$, is also open in there by the condition (LD3) above. It follows that $K(f) \smallsetminus \{p\}$ is disconnected. Conversely, if $K \smallsetminus \{p\}$ is disconnected for some $p \in \partial\Delta$, there are two distinct rays landing at p. These rays together with their landing point divide the plane into two open sectors, one containing Δ and the other containing a non-trivial subset of $K(f)$ which necessarily lies in a single limb. It easily follows that p is the root of this limb. □

Lemma 5.3 *The set of roots is backward-invariant and therefore everywhere dense on $\partial\Delta$.*

Proof Take a root p and let z be the unique point on $\partial\Delta$ such that $f(z) = p$. There are small neighborhoods U of z and U' of p such that $f : U \to U'$ acts as the power $w \mapsto w^k$ for some $k \geq 1$. Take two distinct rays landing at p, take their intersections with U' and pull them back under f to obtain $2k \geq 2$ arcs in U landing at z. Each such arc is necessarily contained in a ray because of the functional equation (5.1). It follows that $K \smallsetminus \{z\}$ is disconnected and therefore z is a root by Lemma 5.2. This proves backward-invariance of roots. Density of roots is now immediate since $f|_{\partial\Delta} : \partial\Delta \to \partial\Delta$ is conjugate to an irrational rotation. □

Every root p has infinite forward orbit since $f|_{\partial\Delta}$ is conjugate to an irrational rotation. It follows that there are at least 2 and at most 2^d rays landing at p. These

[1] It is conjectured that $\partial\Delta$ is a Jordan curve containing a critical point for almost every rotation number θ. This has been proved in the quadratic case in [25].

[2] The limb decomposition hypothesis is believed to hold for almost every rotation number θ (and at least for θ of bounded type), but so far this has been rigorously verified only for $d = 2$ where the whole Julia set is known to be locally connected; see [23] and [25].

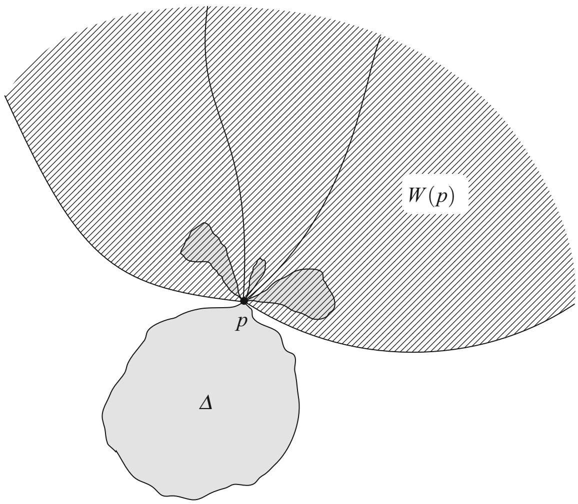

Fig. 5.2 The wake $W(p)$ with the root p on the boundary of the Siegel disk Δ

rays together with their landing point p divide the plane into finitely many open sectors. There is a unique sector that contains Δ which we call the ***co-wake*** with root p and denote by $V(p)$. The complement $W(p) = \mathbb{C} \smallsetminus \overline{V(p)}$ is called the ***(dynamic) wake*** with root p. Thus $W(p)$ is bounded by two rays landing at p and contains $L(p) \smallsetminus \{p\}$ (see Fig. 5.2). Notice that distinct wakes are disjoint. Every point in the plane is either in $\overline{\Delta}$, or in a unique wake, or else on a unique ray which is outside all wakes.

Lemma 5.4 *Every ray $R(t)$ that is outside all wakes lands at a point $z \in \partial\Delta$. Moreover,*

(i) If z is not a root, then $\hat{R}(t) = \{z\}$.
(ii) If z is a root, then $\hat{R}(t) \subset L(z)$ so $\hat{R}(t) \cap \partial\Delta = \{z\}$.

Proof Let us first make the extra assumption that the ray $R = R(t)$ is not a boundary ray of any wake. Suppose the impression $\hat{R}$ contains a point $z \notin \partial\Delta$. Then z belongs to a limb $L(p)$, and since $z \neq p$, we have $z \in W(p)$. Since by our assumption R is disjoint from $\overline{W(p)}$, it must be contained in the co-wake $V(p)$. But then $\hat{R} \subset V(p) \cup \{p\}$, which implies $z \in V(p)$, contradicting $z \in W(p)$. This proves $\hat{R} \subset \partial\Delta$. If the impression $\hat{R}$ is non-trivial, by connectivity it must contain an open subarc $T \subset \partial\Delta$. By Lemma 5.3, there are distinct roots $p, p' \in T$. The open set $\mathbb{C} \smallsetminus (\overline{W(p)} \cup \overline{W(p')} \cup \overline{\Delta})$ has two connected components and R is contained in one of them, say H. It follows that $T \subset \hat{R} \subset \overline{H} \cap \partial\Delta$. But the intersection $\overline{H} \cap \partial\Delta$ is one of the two closed subarcs of $\partial\Delta$ with endpoints p, p', neither of which contains the open arc T. The contradiction proves that $\hat{R}$ is a single point on $\partial\Delta$.

Now consider the case where R is one of the two boundary rays of a wake $W(z)$. An argument similar to the above paragraph shows that $\hat{R} \subset L(z) \cup \partial\Delta$. If $\hat{R}$ contained a point of $\partial\Delta$ other than z, it would have to contain a non-degenerate open arc in $\partial\Delta$. A similar argument as before would then yield a contradiction. This shows $\hat{R} \subset L(z)$ and completes the proof. □

Corollary 5.5 *Every non-root $z \in \partial\Delta$ belongs to the impression of a unique ray. This ray has trivial impression and therefore lands at z.*

Proof Let $R(t)$ be any ray whose impression contains z. Then $R(t)$ is outside all wakes since $R(t) \subset W(p)$ would imply $\hat{R}(t) \subset W(p) \cup \{p\}$ which in turn would imply $z = p$ is a root. It follows from the previous lemma that $\hat{R}(t) = \{z\}$. To see uniqueness, simply note that if $\hat{R}(s)$ also contained z for some $s \neq t$, then by the above observation $\hat{R}(s) = \{z\}$. As the landing point of two distinct rays, z would disconnect $K(f)$ and therefore would be a root by Lemma 5.2. □

Let $\iota : \mathbb{C} \to \overline{\Delta}$ be the map that is the identity on $\overline{\Delta}$, sends every wake to its root and sends every ray outside all wakes to its landing point (Lemma 5.4).

Lemma 5.6 *$\iota : \mathbb{C} \to \overline{\Delta}$ is a retraction.*

Proof We need only check continuity of ι at every point z that does not belong to Δ or any wake. First consider the easier case where $z \in \partial\Delta$. Take a sequence $z_n \notin \overline{\Delta}$ that tends to z. Each z_n belongs to a limb $L(p_n)$ and we may assume that these limbs are distinct. Since $\operatorname{diam}(L(p_n)) \to 0$ by (LD3), it easily follows that $\iota(z_n) = p_n \to z = \iota(z)$.

Now consider the case where z belongs to a ray $R(t)$ outside all wakes. Take any sequence $z_n \to z$. For large n, each z_n belongs to a unique ray $R(t_n)$, where $t_n \to t$. We distinguish two cases:

Case 1 After passing to a subsequence, every ray $R(t_n)$ is outside all wakes. Then, by (5.3) and Lemma 5.4,

$$\limsup_{n\to\infty}\{\iota(z_n)\} = \limsup_{n\to\infty} \hat{R}(t_n) \cap \partial\Delta \subset \hat{R}(t) \cap \partial\Delta = \{\iota(z)\}.$$

This proves $\iota(z_n) \to \iota(z)$.

Case 2 After passing to a subsequence, each $R(t_n)$ lies in some wake $W(p_n)$. Then the impression $\hat{R}(t_n)$ is contained in the limb $L(p_n)$ whose diameter tends to 0 as $n \to \infty$. Hence $\limsup_{n\to\infty} \hat{R}(t_n)$ coincides with the set of all accumulation points of the sequence of roots $\{p_n = \iota(z_n)\}$. Again, by (5.3) and Lemma 5.4, $\limsup_{n\to\infty} \hat{R}(t_n) \subset \hat{R}(t) = \{\iota(z)\}$, and we conclude that $\iota(z_n) \to \iota(z)$. □

Recall that for $0 \leq i \leq d-2$ the fixed point $\zeta_i \in J(f)$ is the landing point of the fixed ray $R(u_i)$. Let $w_i = \iota(\zeta_i) \in \partial\Delta$. Since the ζ_i do not belong to $\overline{\Delta}$, they lie in wakes, so every w_i must be a root. We call $\{w_0, \ldots, w_{d-2}\}$ the ***marked roots*** of f. Take the unique conformal isomorphism $h : \Delta \to \mathbb{D}$ which fixes 0 and sends w_0 to 1. According to Carathéodory, since $\partial\Delta$ is a Jordan curve, h extends to a homeomorphism between the closures [21]. Note that $h \circ f \circ h^{-1} : \mathbb{D} \to \mathbb{D}$ fixes 0 and has derivative $e^{2\pi i\theta}$ at the origin, so by the Schwarz lemma,

$$h(f(z)) = e^{2\pi i\theta} h(z) \quad \text{for all} \quad z \in \Delta.$$

We define the ***internal angle*** of a point $z \in \partial\Delta$ as the unique $\alpha \in \mathbb{T}$ such that $h(z) = e^{2\pi i\alpha}$. By the above conjugacy relation, the internal angle of $f(z)$ will then be $\alpha + \theta \pmod{\mathbb{Z}}$.

Let $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$ denote the internal angles of the marked roots $w_1, w_2, \dots, w_{d-1} = w_0$. The following is the analog of Lemma 5.1:

Theorem 5.7 *The set X' of all angles $t \in \mathbb{T}$ for which the ray $R(t)$ lands on $\partial\Delta$ contains a unique minimal rotation set X for m_d, with $\rho(X) = \theta$. Moreover, the cumulative deployment vector of X satisfies*

$$\sigma(X) = (\alpha_1, \dots, \alpha_{d-1}) \qquad \pmod{\mathbb{Z}^{d-1}}. \tag{5.5}$$

The proof will show that the difference $X' \smallsetminus X$ consists of at most countably many isolated points.

Proof For each root $p \in \partial\Delta$ let $I(p)$ be the open interval of angles $t \in \mathbb{T}$ for which $R(t) \subset W(p)$. Set $X = \mathbb{T} \smallsetminus \bigcup_p I(p)$. By Lemma 5.4 the compact set X is contained in X' and the difference $X' \smallsetminus X$ consists of the at most countable set of angles of rays within some wake that land at a root.

Let $\psi : \mathbb{C} \smallsetminus \overline{\mathbb{D}} \to \mathbb{C} \smallsetminus K(f)$ be the inverse Böttcher coordinate of f near ∞. Define $\varphi : \mathbb{T} \to \mathbb{T}$ by letting $\varphi(t)$ be the internal angle of the point $\iota(\psi(2e^{2\pi i t})) \in \partial\Delta$. The map φ is continuous by the previous lemma, and is surjective by Corollary 5.5. Using the fact that distinct rays cannot cross, it is not hard to see that φ is monotone of degree 1, with the collection of intervals $\{I(p) : p \text{ is a root}\}$ as its plateaus. If $R(t)$ lands at $z \in \partial\Delta$ with internal angle α, then $R(m_d(t))$ lands at $f(z)$ with internal angle $\alpha + \theta$. This proves

$$\varphi \circ m_d = r_\theta \circ \varphi \quad \text{on} \quad X.$$

Furthermore, if the fiber $\varphi^{-1}(\alpha)$ is non-trivial, then $h^{-1}(e^{2\pi i\alpha})$ is a root, so its preimage $h^{-1}(e^{2\pi i(\alpha-\theta)})$ is also a root by Lemma 5.3, which proves the fiber $\varphi^{-1}(\alpha - \theta)$ is non-trivial as well. It now follows from Theorem 2.35 that X is a minimal rotation set for m_d with $\rho(X) = \theta$, and φ is the canonical semiconjugacy associated with X.

The claim (5.5) on $\sigma(X)$ follows from Lemma 3.3 since α_i, the internal angle of $w_i = \iota(\zeta_i) = \iota(\psi(2e^{2\pi i u_i}))$, is just the image $\varphi(u_i)$. □

Remark 5.8 The set X' of all rays landing on $\partial\Delta$ is closed and m_d-invariant, and every forward orbit in it has the combinatorial structure of an orbit under r_θ. Yet X' may fail to be a rotation set. For example, the cubic polynomial

$$f(z) = e^{\pi i(\sqrt{5}-1)} z + a z^2 + z^3 \quad \text{with} \quad a \approx 0.44437107 - 0.35184284\, i$$

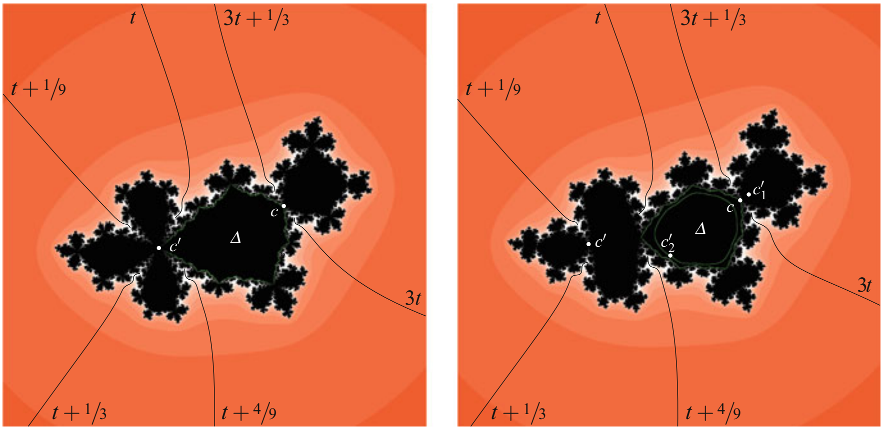

Fig. 5.3 Left: Filled Julia set of the cubic map f in Remark 5.8 with both critical points c, c' on the boundary of the Siegel disk $\varDelta$ in the center of the picture, where $f(c') = c$. Right: A small perturbation of f in Remark 5.12 for which $c' \mapsto c_1' = f(c') \mapsto c_2' = f^{\circ 2}(c') \in \varDelta$

has both critical points c, c' on $\partial\varDelta$ with $f(c') = c$ as shown in Fig. 5.3 left. The critical point c' is the landing point of four rays at angles $t, t + \frac{1}{9}, t + \frac{1}{3}, t + \frac{4}{9}$ which map under f to the two rays at angles $3t, 3t + \frac{1}{3}$ landing at c_1. Here $t \approx 0.30762195$. The set X' in this example is not a rotation set since the complement of these six rays already fails to contain two disjoint open intervals of length $\frac{1}{3}$ (Corollary 2.16). However, removing $t + \frac{1}{9}, t + \frac{1}{3}$ and all their preimages from X' will yield a minimal rotation set X.

Remark 5.9 The congruences in (5.5) determine $\sigma(X)$ uniquely from the knowledge of the internal angles $\alpha_1, \ldots, \alpha_{d-1}$ except when $\alpha_i = 0 \pmod{\mathbb{Z}}$ for all i. This corresponds to the case where there is a single marked root $w_0 = \cdots = w_{d-2}$ which is necessarily a critical point of local degree d (compare Corollary 5.11 below). This type of ambiguity has already been pointed out in Remark 3.4 and can now be understood from the dynamical standpoint. For example, when $d = 4$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0 \pmod{\mathbb{Z}}$, we have the possible candidates

$$\sigma(X) = (0, 0, 1) \quad \text{or} \quad (0, 1, 1) \quad \text{or} \quad (1, 1, 1)$$

which correspond to quartic polynomials which are conjugate by the 120° rotation around the origin. Dynamically, these cases can be distinguished by the position of the Siegel disk $\varDelta$ among the three fixed rays $R(0)$, $R(\frac{1}{3})$, $R(\frac{2}{3})$ (see Fig. 5.4).

Let us collect some corollaries of Theorem 5.7. As before, let $w_i = \iota(\zeta_i)$ $(0 \leq i \leq d-2)$ be the marked roots of f. To simplify the notation, we denote the limb

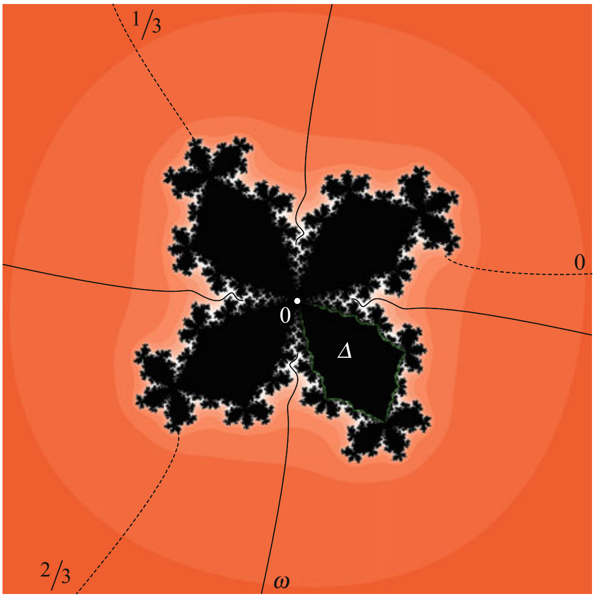

Fig. 5.4 Filled Julia set of a unicritical quartic polynomial $f(z) = z^4 + c$ with a Siegel disk $\varDelta$ of the golden mean rotation number. Here the corresponding rotation set X has $\sigma(X) = (0, 0, 1)$. Conjugating f with the 120° and 240° rotations around the origin yields quartics with $\sigma(X) = (1, 1, 1)$ and $(0, 1, 1)$. In this example, $c \approx 0.59612528 - 0.46108628\, i$ and $\omega \approx 0.68914956$

$L(w_i)$ by L_i, the wake $W(w_i)$ by W_i and the gap $I(w_i)$ by I_i. The following can be thought of as the irrational counterpart of a result of Goldberg and Milnor in [12]:

Theorem 5.10 *Let X be the minimal rotation set of Theorem 5.7.*

(i) $I_0, \dots, I_{d-2}$ *are the major gaps of X.*
(ii) The multiplicity n_i *of* I_i *is the number of fixed rays in* W_i*. It is also the number of subscripts* $0 \le j \le d-2$ *for which* $w_j = w_i$*.*
(iii) The limb $L_i = \overline{W_i} \cap K(f)$ *contains* n_i *critical points of* f *counting multiplicities.*

Proof By the proof of Theorem 5.7 every I_i is a gap of X. Since W_i contains the fixed ray $R(u_i)$, the gap I_i contains the fixed point u_i of m_d, so it must be major. By Lemma 2.13, the multiplicity n_i of I_i is the number of fixed rays in W_i or the number of times w_i appears in the list $w_0, \dots, w_{d-2}$. Since there are $d-1$ fixed rays, the sum $\sum n_i$ over distinct I_i's is $d-1$ so $I_0, \dots, I_{d-2}$ account for all major gaps of X by Theorem 2.7. This proves (i) and (ii).

The proof of (iii) is based on an idea of [12]. Let $I_i = (t, t')$, so W_i is bounded by the rays $R(t)$ and $R(t')$. Let η be a small loop around w_i which intersects each of $R(t)$ and $R(t')$ once, say at $\psi(r_1 e^{2\pi i t})$ and $\psi(r_1 e^{2\pi i t'})$. Fix a large radius r_2. Construct a positively oriented Jordan curve by going out along $R(t)$ from

$\psi(r_1 e^{2\pi i t})$ to $\psi(r_2 e^{2\pi i t})$, then following the equipotential curve $\{\psi(r_2 e^{2\pi i s}) : t \leq s \leq t'\}$, then going down along $R(t')$ from $\psi(r_2 e^{2\pi i t'})$ to $\psi(r_1 e^{2\pi i t'})$, and finally going counter-clockwise along η from $\psi(r_1 e^{2\pi i t'})$ back to $\psi(r_1 e^{2\pi i t})$. Round off the four corners of this curve to obtain a smooth positively oriented Jordan curve γ. The number of the critical points of f in W_i is the number of roots of f' inside γ. By the argument principle, this is the winding number of the closed curve $f' \circ \gamma$ around 0, which is one less than the number of full counter-clockwise turns that the tangent vector to image curve $f \circ \gamma$ makes when γ is traversed once. By the construction of γ, this number is at least $n_i - k_i + 1$, where $k_i \geq 1$ is the local degree of f at w_i. Taking into account the fact that w_i itself is a critical point of multiplicity $k_i - 1$ if $k_i > 1$, it follows that the number N_i of the critical points of f in the limb L_i is at least $(n_i - k_i + 1) + (k_i - 1) = n_i$. Since the sums $\sum N_i$ and $\sum n_i$ over distinct I_i's are $d - 1$, it follows that $N_i = n_i$ for all i, as required. □

Corollary 5.11

(i) Every critical point $c \in \partial\Delta$ is a marked root. Moreover, the algebraic multiplicity of c (as a root of f') is at most the multiplicity of the corresponding gap $I(c)$.
(ii) Every marked root w_i whose corresponding gap $I_i = I(w_i)$ is taut must be a critical point.
(iii) A point on $\partial\Delta$ is a root if and only if it is pre-critical.

Proof First suppose $c \in \partial\Delta$ is a critical point. By Corollary 5.5 the critical value $f(c)$ is the landing point of at least one ray $R(t)$. As in the proof of Lemma 5.3, take small neighborhoods U of c and U' of $f(c)$ such that $f : U \to U'$ acts as the power $w \mapsto w^k$ for some $k \geq 2$. The intersection $R(t) \cap U'$ pulls back under f to the intersection of k rays $R(t_1), \ldots, R(t_k)$ with U, all landing at c, where $t_1, \ldots, t_k$ are among the d preimages of t under m_d. This proves that $K(f) \smallsetminus \{c\}$ is disconnected, hence c is a root by Lemma 5.2. Moreover, the wake $W(c)$ contains all $R(t_i)$'s in its closure, so $|I(c)| \geq (k-1)/d$. Hence $I(c)$ is a major gap of X, and the root c is marked by Theorem 5.10(i). The multiplicity n of $I(c)$ is the integer part of $d\,|I(c)|$, so $n \geq k - 1$. (Alternatively, we could invoke Theorem 5.10(iii) to conclude that $n \geq k - 1$.) This proves (i).

To verify (ii), suppose I_i is a taut gap of the form $(t, t' = t + n_i/d)$. Then w_i is the landing point of the rays $R(t), R(t')$. Under f, these rays map to the same ray $R(m_d(t)) = R(m_d(t'))$ landing at $f(w_i)$. This shows f is not injective in any neighborhood of w_i, which proves w_i is a critical point.

For (iii), first note that by part (i) and the backward invariance in Lemma 5.3, all precritical points on $\partial\Delta$ are roots. Conversely, consider any root p so $I(p)$ is a gap of the minimal rotation set X of Theorem 5.7. Since $\rho(X)$ is irrational, Theorem 2.10 shows that there is a $k \geq 0$ such that $g_X^{\circ k}(I(p)) = I(f^{\circ k}(p))$ is a taut gap. By part (ii), $f^{\circ k}(p)$ is a critical point. □

Remark 5.12 Here are three comments related to various parts of the above corollary: (i) The algebraic multiplicity of a critical point $c \in \partial\Delta$ can be strictly less than the multiplicity of the gap $I(c)$. This happens precisely when the wake $W(c)$ contains a critical point of f. (ii) If a marked root w_i is critical, the gap I_i may be loose. For example, the cubic map f in Remark 5.8 has both critical points c, c' on $\partial\Delta$ with $f(c') = c$, where $I(c)$ is taut and $I(c')$ is loose (see Fig. 5.3 left). (iii) Marked roots can be non-critical. For example, one can perturb the above map to obtain a cubic with $c \in \partial\Delta$ and $f^{\circ 2}(c') \in \Delta$ (thus the critical point c' is "captured" by the Siegel disk Δ). Here the second marked root $f^{-1}(c) \cap \partial\Delta$ is non-critical. Figure 5.3 right shows one such perturbation where

$$f(z) = e^{\pi i(\sqrt{5}-1)} z + az^2 + z^3 \quad \text{with} \quad a \approx 0.54716981 - 0.31132075\, i.$$

The two examples before and after perturbation have identical minimal rotation sets X. We will discuss this phenomena in more detail in Sect. 5.4.

Corollary 5.13 *Suppose all critical points of f are on $\partial\Delta$. Then these critical points are precisely the marked roots $w_0, \dots, w_{d-2}$, and the algebraic multiplicity of each w_i is equal to the multiplicity of its corresponding gap.*

Proof By Corollary 5.11 all critical points of f are marked roots. Let $c_1, \dots, c_k$ be the distinct critical points of multiplicities $\alpha_1, \dots, \alpha_k$. Let $n_1, \dots, n_k$ be the multiplicities of the corresponding gaps. By Corollary 5.11(i), $\alpha_i \leq n_i$ for all i. Hence, by Theorem 2.7, $d - 1 = \sum \alpha_i \leq \sum n_i \leq d - 1$. It follows that $\alpha_i = n_i$ for all i and $\{c_1, \dots, c_k\} = \{w_0, \dots, w_{d-1}\}$. □

It would be interesting to investigate how the preceding constructions should be modified for indifferent fixed points with arbitrary irrational rotation numbers. The difficulty arises when the fixed point 0 is the center of a "wild" Siegel disk or is non-linearizable (a so-called "Ceremer point"). In this case, the natural candidate for the rotation set X would be the minimal set of angles of dynamic rays whose impressions meet $\partial\Delta$ in the Siegel case and the fixed point 0 in the Cremer case. But in the absence of some kind of control on the Julia set of such maps, proving analogous results seems out of reach even for quadratic polynomials.

5.3 The Quadratic Family

This section and the next illustrate the relation between indifferent fixed points and rotation sets in the low-degree cases $d = 2$ and $d = 3$, in both dynamical and parameter planes. The abstract analyses of these rotation sets, carried out in Sects. 4.5 and 4.6, come to life in these concrete realizations.

The case $d = 2$ is more straightforward and rather well-known. Consider the monic quadratic polynomial

$$P = P_\theta : z \mapsto e^{2\pi i\theta} z + z^2 \tag{5.6}$$

with an indifferent fixed point at the origin. When θ is rational of the form $p/q \neq 0$ in lowest terms, the parabolic fixed point 0 is the landing point of precisely q rays $R(t_1), \dots, R(t_q)$, where $X_{p/q} = \{t_1, \dots, t_q\}$ is the unique minimal rotation set under doubling with rotation number p/q. If as usual we assume $0, t_1, \dots, t_q$ are in positive cyclic order, it follows that the unique critical point $c = -e^{2\pi i\theta}/2$ lies in the wake bounded by $R(t_1)$, $R(t_q)$, corresponding to the longest gap of $X_{p/q}$. Similarly, the critical value $v = P(c) = -e^{4\pi i\theta}/4$ lies in the wake bounded by $R(t_{1+p})$, $R(t_{q+p})$, corresponding to the shortest gap of $X_{p/q}$ (compare Fig. 5.5 left).

When θ is an irrational of bounded type (or more generally belongs to the full-measure set $\mathcal{E}$ in [25]), the Julia set $J(P)$ is locally connected. In this case the boundary of the Siegel disk Δ of P centered at 0 is a Jordan curve containing c, and the limb decomposition hypothesis automatically holds. It follows from the general results of the previous section that the set of angles of the rays that land on $\partial\Delta$ is precisely the minimal rotation set X_θ under doubling. Note that X_θ is a Cantor set with a single major gap of length $\frac{1}{2}$ bounded by the angles $\omega, \omega' = \omega - \frac{1}{2}$, where $0 < \omega = \omega(\theta) < \frac{1}{2}$ is the leading angle of X_θ as defined in Sect. 4.5. By Corollary 5.11, both rays $R(\omega)$, $R(\omega')$ land at the critical point c which is the unique marked root. The precritical point $P^{-n}(c) \cap \partial\Delta$ is then the root whose corresponding wake defines the gap of X_θ of length $\frac{1}{2^n}$ (see Fig. 5.5 right).

The realization of rotation sets in the dynamical plane allows an alternative route to Lemma 4.24. The binary expansion $0.b_0b_1b_2\cdots$ of the leading angle $\omega = \omega(\theta)$ of X_θ is characterized by the condition $b_k = 1$ if and only if $2^k\omega \in (\frac{1}{2}, 1)$. If $\theta = p/q \neq 0$ and $X_{p/q} = \{t_1, \dots, t_q\}$ as above, then $0 \in (t_q, t_1)$ and

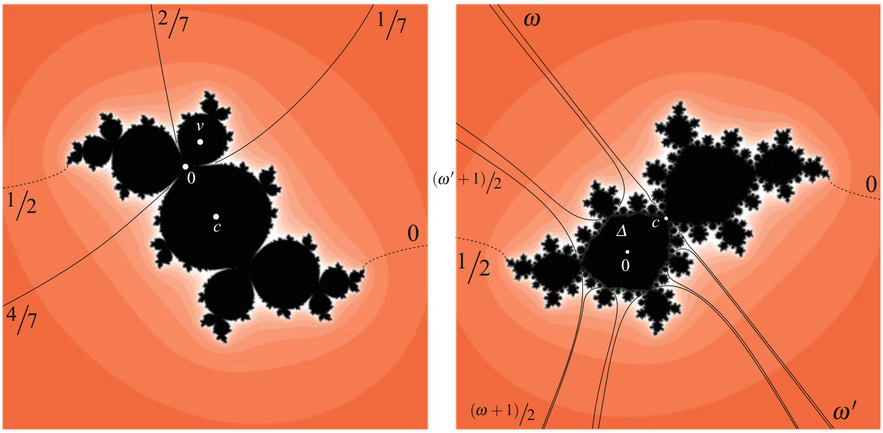

Fig. 5.5 Filled Julia set of the quadratic polynomial $z \mapsto e^{2\pi i\theta}z + z^2$ with the corresponding minimal rotation set X_θ under doubling. Left: The parabolic case $\theta = \frac{1}{3}$. Right: The Siegel case $\theta = \frac{(\sqrt{5}-1)}{2}$. Shown here are the wakes rooted at the critical point c and its first five preimages on $\partial\Delta$, which define the major gap (ω', ω) of X_θ and the five minor gaps of lengths 2^{-k} for $2 \leq k \leq 6$

$\frac{1}{2} \in (t_{q-p}, t_{q-p+1})$. Hence $t_1, \ldots, t_{q-p} \in (0, \frac{1}{2})$ while $t_{q-p+1}, \ldots, t_q \in (\frac{1}{2}, 1)$. Thus,

$$2^k \omega = t_{1+kp} \in \left(\frac{1}{2}, 1\right) \iff 1 + kp \pmod{q} \text{ is in } \{q - p + 1, \ldots, q\}.$$

This is clearly equivalent to $k\theta \in [-\theta, 0)$.

A similar argument works when θ is an irrational and P has a "good" Siegel disk. In this case, $0 \in (\omega', \omega)$ and $\frac{1}{2} \in ((\omega' + 1)/2, (\omega + 1)/2)$, so $2^k\omega \in (\frac{1}{2}, 1)$ if and only if $2^k\omega \in ((\omega + 1)/2, \omega')$. But the pair $R(\omega), R(\omega')$ land at c with the internal angle 0 and the pair $R((\omega+1)/2), R((\omega'+1)/2)$ land at the preimage $P^{-1}(c) \cap \partial\Delta$ with the internal angle $-\theta$. It follows that $2^k\omega \in ((\omega+1)/2, \omega')$ precisely when $k\theta$, the internal angle of $P^{\circ k}(c)$, is in the interval $(-\theta, 0)$.

The parameter space of quadratic polynomials provides a complete catalog of all rotation sets under doubling. To see this, it will be convenient to represent our quadratics in the normal form $f_c(z) = z^2 + c$ where $c \in \mathbb{C}$. The connectedness locus

$$\mathscr{M}_2 = \{c \in \mathbb{C} : K(f_c) \text{ is connected}\},$$

commonly known as the ***Mandelbrot set***, is non-empty, compact, and full. If β_c denotes the Böttcher coordinate of f_c near ∞, the ***Douady-Hubbard map*** $\Phi : \mathbb{C} \smallsetminus \mathscr{M}_2 \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ which assigns to each c outside $\mathscr{M}_2$ the Böttcher coordinate $\beta_c(c)$ of the critical value $f_c(0) = c$, is a conformal isomorphism. By the ***parameter ray*** of $\mathscr{M}_2$ at angle $t \in \mathbb{T}$ we mean the real-analytic curve

$$\mathscr{R}(t) = \Phi^{-1}\big(\{re^{2\pi i t} : r > 1\}\big).$$

We say $\mathscr{R}(t)$ lands at $z \in \partial\mathscr{M}_2$ if $\lim_{r\to 1} \Phi^{-1}(re^{2\pi i t}) = z$.

Each quadratic P_θ in (5.6) is affinely conjugate to f_c with $c = c(\theta) = e^{2\pi i\theta}/2 - e^{4\pi i\theta}/4$. As θ varies in $[0, 1]$, the image $c(\theta)$ traces out a cardioid on the boundary of $\mathscr{M}_2$ that is prominently visible in Fig. 5.6. When $\theta \neq 0$ is rational, $c(\theta)$ is the landing point of the two parameter rays $\mathscr{R}(2\omega), \mathscr{R}(2\omega')$. (Recall that (ω', ω) is the major gap of X_θ.) If θ is irrational, then $c(\theta)$ is the landing point of the unique parameter ray $\mathscr{R}(2\omega) = \mathscr{R}(2\omega')$. One may interpret this by saying that $c(\theta)$ is always the landing point of the parameter ray at angle $2\omega(\theta)$, which is a strictly increasing function of θ that jumps by $1/(2^q - 1)$ at every rational $\theta = p/q$ (Corollary 4.26). When θ is rational, the two parameter rays $\mathscr{R}(2\omega), \mathscr{R}(2\omega')$ together with their landing point $c(\theta)$ define the ***parameter wake*** $\mathscr{W}(\theta)$, characterized by the property that the dynamic rays with angles in X_θ land at a fixed point of f_c if and only if $c \in \mathscr{W}(\theta) \cap \mathscr{M}_2$ (for a detailed treatment see [20] and compare Fig. 5.6).

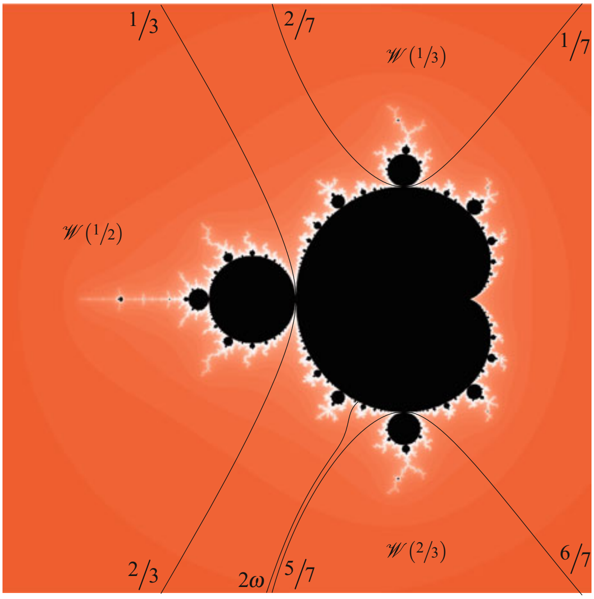

Fig. 5.6 The Mandelbrot set $\mathscr{M}_2$ and its parameter wakes $\mathscr{W}(\frac{1}{3})$, $\mathscr{W}(\frac{1}{2})$ and $\mathscr{W}(\frac{2}{3})$. Also shown is the parameter ray $\mathscr{R}(2\omega)$ landing at the quadratic that is affinely conjugate to $e^{2\pi i\theta}z + z^2$. Here $\theta = \frac{(\sqrt{5}-1)}{2}$ and $\omega = \omega(\theta) \approx 0.35490172$

Remark 5.14 The family of degree d unicritical polynomials $z \mapsto z^d + c$ exhibits very similar features in relation with rotation sets. As an example, the cubic map $f_c : z \mapsto z^3 + c$ has an indifferent fixed point of multiplier $e^{2\pi i\theta}$ if and only if

$$c = \pm c(\theta) \quad \text{where} \quad c(\theta) = -\frac{1}{3\sqrt{3}}\, e^{3\pi i\theta} + \frac{1}{\sqrt{3}}\, e^{\pi i\theta}.$$

The maps $f_{c(\theta)}$ and $f_{-c(\theta)}$ are conjugate by the 180° rotation $z \mapsto -z$. The angles of the dynamic rays of $f_{c(\theta)}$ that land on the indifferent fixed point when θ is rational, or on the boundary of the Siegel disk when θ is a suitable irrational, form the rotation set $X_{\theta,1}$ under tripling. The rotation set associated with the conjugate map $f_{-c(\theta)}$ is of course $X_{\theta,0}$. As θ varies in $[0, 1]$, the images $\pm c(\theta)$ trace out an algebraic curve (a nephroid) on the boundary of the corresponding connectedness locus $\mathscr{M}_3$ which bounds the central hyperbolic component containing $c = 0$. The analog of the Douady-Hubbard map is a conformal isomorphism $\mathbb{C} \smallsetminus \mathscr{M}_3 \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$, which can be used to define parameter rays in the c-plane. The boundary point $c(\theta)$ is the landing point of the parameter ray at angle $3\omega(\theta, 1)$, which strictly increases from 0 to $\frac{1}{2}$, jumping by $2/(3^q - 1)$ at every rational $\theta = p/q$ (Corollary 4.32). Similarly, $-c(\theta)$ is the landing point of the parameter ray at angle $3\omega(\theta, 0) = 3\omega(\theta, 1) + \frac{1}{2}$, which strictly increases from $\frac{1}{2}$ to 1 with similar jumps at every rational θ. As in the case of the Mandelbrot set, there is an analogous notion of parameter wakes for $\mathscr{M}_3$ and their dynamical characterization (see Fig. 5.7).

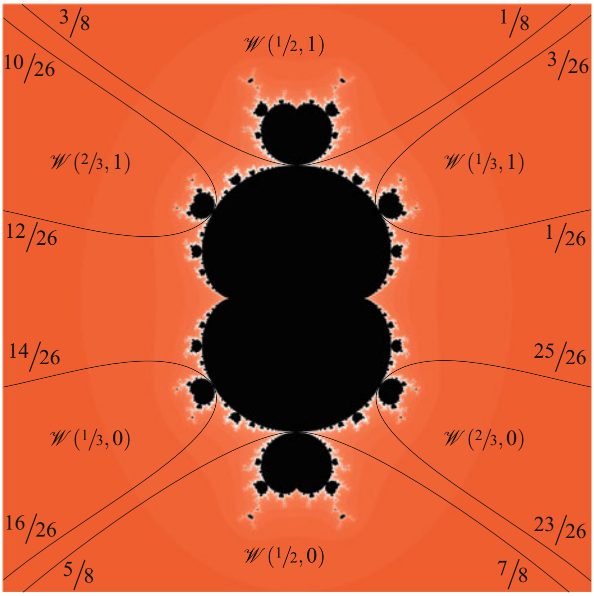

Fig. 5.7 The connectedness locus $\mathscr{M}_3$ of the unicritical cubic family $\{f_c : z \mapsto z^3 + c\}_{c \in \mathbb{C}}$, with selected parameter rays and wakes. Here $\mathscr{W}(p/q, \delta) \cap \mathscr{M}_3$ for $\delta = 0, 1$ is precisely the set of parameters c for which the dynamical rays at angles in $X_{p/q,\delta}$ land at a fixed point of f_c

5.4 The Cubic Family

This section is somewhat expository and contains outlines of the results. Consider the space of monic cubic polynomials with an indifferent fixed point of multiplier $e^{2\pi i\theta}$ at the origin. Each such cubic has the form

$$f_a : z \mapsto e^{2\pi i\theta} z + az^2 + z^3 \quad \text{for some} \quad a \in \mathbb{C}. \tag{5.7}$$

Note that f_a and f_{-a} are affinely conjugate by the involution $z \mapsto -z$. One could thus look at the quotient of the a-plane under $a \mapsto -a$ (equivalently, work with the parameter a^2). However, for our purposes in this section we prefer to treat f_a and f_{-a} as distinct cubics.

The connectedness locus of this cubic family is defined by

$$\mathscr{M}_3(\theta) = \{a \in \mathbb{C} : K(f_a) \text{ is connected}\}.$$

It is not hard to verify that $\mathscr{M}_3(\theta)$ is a compact, connected and full subset of $\mathbb{C}$ which is invariant under the involution $a \mapsto -a$ [30].

When $a \in \mathscr{M}_3(\theta)$, both critical points of f_a belong to the filled Julia set $K(f_a)$. When $a \notin \mathscr{M}_3(\theta)$, exactly one of the critical points, labeled c_a, belongs to $K(f_a)$ while the other, labeled e_a, escapes to ∞. The escaping critical value $v_a = f_a(e_a)$

has two preimages under f_a: the critical point e_a itself (with multiplicity 2) and a regular point $\hat{e}_a$ which we call the escaping ***co-critical point***. The Böttcher coordinate β_a of f_a near ∞ is defined and holomorphic in some neighborhood of $\hat{e}_a$. The analog of the Douady-Hubbard map $\Phi : \mathbb{C} \smallsetminus \mathscr{M}_3(\theta) \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ defined by

$$\Phi(a) = \beta_a(\hat{e}_a)$$

is a conformal isomorphism [6]. We define the ***parameter ray*** at angle $t \in \mathbb{T}$ by

$$\mathscr{R}(t) = \{\Phi^{-1}(re^{2\pi i t}) : r > 1\}.$$

We study the realization of rotation sets under m_3 in the dynamical plane of f_a as well as the parameter a-plane. The discussion is presented in two cases depending on whether θ is rational or an irrational of bounded type. We will outline the first case only briefly, as our main interest is the case of cubics with Siegel disks.

The parabolic case. Let us assume θ is rational of the form $p/q \neq 0$ in lowest terms. By the discussion of Sect. 5.2, the q-th iterate of f_a has the form

$$f_a^{\circ q}(z) = z + A(a)\, z^{q+1} + \cdots + z^{3^q}.$$

Here $A(a)$ is a polynomial of degree q in a with simple roots. Moreover, A is an even function if q is even, and odd function if q is odd. If $A(a) \neq 0$, the petal number of the parabolic point 0 is q and its ray number is q or $2q$. If, on the other hand, $A(a) = 0$, then the above expression reduces to

$$f_a^{\circ q}(z) = z + B(a)\, z^{2q+1} + \cdots + z^{3^q}.$$

where $B(a) \neq 0$, so the petal and ray numbers are both $2q$. In this case, we say f_a has a ***degenerate parabolic*** fixed point at 0.

By Lemma 5.1 the set X_a of angles of the dynamic rays of f_a that land at 0 is a rotation set under tripling with $\rho(X_a) = p/q$, which consists of one or two q-cycles. The deployment vector of X_a has the form $\delta(X_a) = (\delta_a, 1 - \delta_a)$, where $\delta_a \in [0, 1]$ is the ***deployment probability*** of f_a, i.e., the probability that a dynamic ray $R_a(t)$ of f_a landing on 0 has its angle t in $(0, \frac{1}{2})$. Note that by symmetry,

$$\delta_{-a} = 1 - \delta_a \qquad a \in \mathscr{M}_3(p/q).$$

First suppose the ray number is q, so X_a is a single q-cycle $\{t_1, \ldots, t_q\}$. Thus, in the notation of Sect. 4.6, $X_a = X_{p/q,i/q}$ for some $0 \leq i \leq q$. If we assume $0, t_1, \ldots, t_q$ are in positive cyclic order, it follows that one critical point of f_a lies in the wake bounded by the dynamic rays $R_a(t_q), R_a(t_1)$, the other in the wake bounded by $R_a(t_i), R_a(t_{i+1})$. Thus, the deployment probability $\delta_a = i/q$ is determined by the "combinatorial distance" i between the two critical points of f_a (that is, how many wakes they are apart). Figure 5.1 left illustrates this case with $p/q = i/q = \frac{1}{3}$.

Next consider the case where the ray number is $2q$, so $X_a = \{t_1, \ldots, t_{2q}\}$. Under tripling, each t_j maps to t_{j+2p} so X_a splits into two q-cycles. As these q-cycles are compatible, Theorem 3.16 shows that

$$X_a = X_{p/q,i/q} \cup X_{p/q,(i+1)/q}$$

for some $0 \leq i \leq q-1$. Now one critical point of f_a lies in the wake bounded by $R_a(t_{2q}), R_a(t_1)$, the other in the wake bounded by $R_a(t_{2i+1}), R_a(t_{2i+2})$. Thus, similar to the above case, the deployment probability $\delta_a = (2i+1)/(2q)$ is determined by the combinatorial distance $2i+1$ between the two critical points of f_a. Figure 5.1 right illustrates this case with $p/q = i/q = \frac{1}{3}$.

Turning the attention to the parameter space, one can identify the following types of the interior components for $\mathscr{M}_3(p/q)$:

- ***adjacent***, where the two critical points belong to the same attracting petal at 0;
- ***bi-transitive***, where the two critical points belong to different attracting petals at 0 in the same cycle;
- ***capture***, where the orbit of one critical point eventually hits the cycle of attracting petals at 0;
- ***hyperbolic-like***, where the orbit of one critical point converges to an attracting cycle.

Conjecturally, every interior component of $\mathscr{M}_3(p/q)$ is of one of the above types. In fact, the only possibility to rule out is a “queer” component in a small copy of the Mandelbrot set in $\mathscr{M}_3(p/q)$ in which the interior of $K(f_a)$ is the basin of attraction of 0 but the Julia set $J(f_a)$ has positive measure and admits an invariant line field.

Let $a_0, \ldots, a_{q-1}$ denote the degenerate parabolic parameters, i.e., simple roots of the equation $A(a) = 0$. There is a chain of interior components $C_0, C_1, \ldots, C_q$ of $\mathscr{M}_3(p/q)$ such that $\partial C_{i-1} \cap \partial C_i = \{a_i\}$ for $1 \leq i \leq q$. Here $C_i = -C_{q-i}$, with C_0 and C_q of adjacent type and $C_1, \ldots, C_{q-1}$ of bi-transitive type (see Fig. 5.8). For every parameter $a \in C_i$, we have $\delta_a = i/q$.

The deployment probability δ_a can be determined throughout the connectedness locus $\mathscr{M}_3(p/q)$. Each degenerate parabolic parameter a_i is the landing point of four parameter rays whose angles are those of the dynamic rays of f_{a_i} that bound the Fatou components containing its co-critical points. Using the general results of Sect. 4.6 it is not hard to find explicit formulas for these angles in terms of the leading angles $\omega(p/q, i/q)$ and $\omega(p/q, (i+1)/q)$. An example of this computation for $p/q = \frac{2}{3}$ and $i = 0$ is shown in Fig. 5.9.

These $4q$ parameter rays together with their landing points $\{a_0, \ldots, a_{q-1}\}$ divide the a-plane into $3q+1$ ***parameter wakes*** $\mathscr{W}_0, \ldots, \mathscr{W}_q, \Omega_0^{\pm}, \ldots, \Omega_{q-1}^{\pm}$. Here $\mathscr{W}_i$ contains C_i and the pair $\Omega_i^{\pm}$ separate $\mathscr{W}_i$ from $\mathscr{W}_{i+1}$ (see Fig. 5.8). We have $X_a = X_{p/q,i/q}$ if $a \in \mathscr{W}_i \cap \mathscr{M}_3(p/q)$, and $X_a = X_{p/q,i/q} \cup X_{p/q,(i+1)/q}$ if $a \in \Omega_i^{\pm} \cap \mathscr{M}_3(p/q)$. Thus,

$$\delta_a = \begin{cases} \frac{i}{q} & \text{if } a \in \mathscr{W}_i \cap \mathscr{M}_3(p/q) \\ \frac{2i+1}{2q} & \text{if } a \in \Omega_i^{\pm} \cap \mathscr{M}_3(p/q). \end{cases}$$

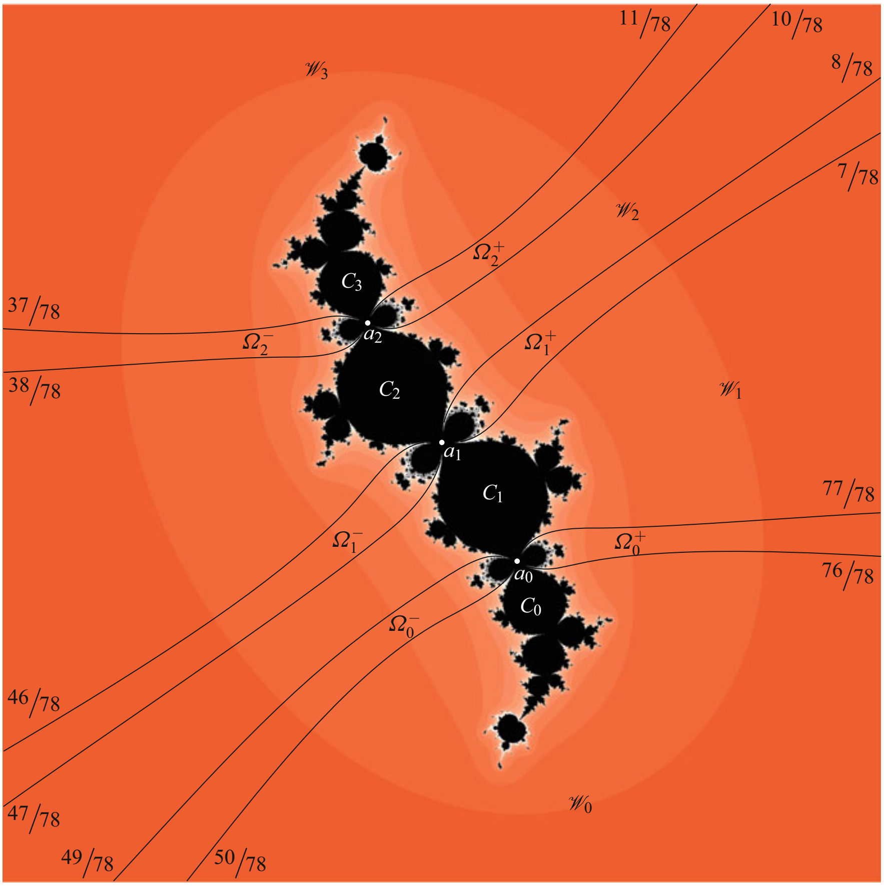

Fig. 5.8 The parabolic connectedness locus $\mathscr{M}_3(\frac{2}{3})$ and the chain of interior components C_0, C_1, C_2, C_3. The twelve parameter rays landing on the degenerate cubics a_0, a_1, a_2 define the ten wakes $\mathscr{W}_0, \mathscr{W}_1, \mathscr{W}_2 \mathscr{W}_3$ and $\Omega_0^\pm, \Omega_1^\pm, \Omega_2^\pm$. The deployment probability δ_a takes the value $i/3$ on $\mathscr{W}_i \cap \mathscr{M}_3(\frac{2}{3})$ and $(2i+1)/6$ on $\Omega_i^\pm \cap \mathscr{M}_3(\frac{2}{3})$

A detailed analysis of the landing properties of some of the parameter rays of $\mathscr{M}_3(p/q)$ can be found in [3].

The "good" Siegel case. Now suppose θ is an irrational of bounded type, so the fixed point 0 of f_a is the center of a Siegel disk Δ_a. The boundary $\partial \Delta_a$ is then a Jordan curve (in fact a quasicircle) passing through one or both critical points of f_a.

One can easily identify the following two types of interior components of the connectedness locus $\mathscr{M}_3(\theta)$:

- ***capture***, where the orbit of one critical point eventually hits the Siegel disk;
- ***hyperbolic-like***, where the orbit of one critical point converges to an attracting cycle.

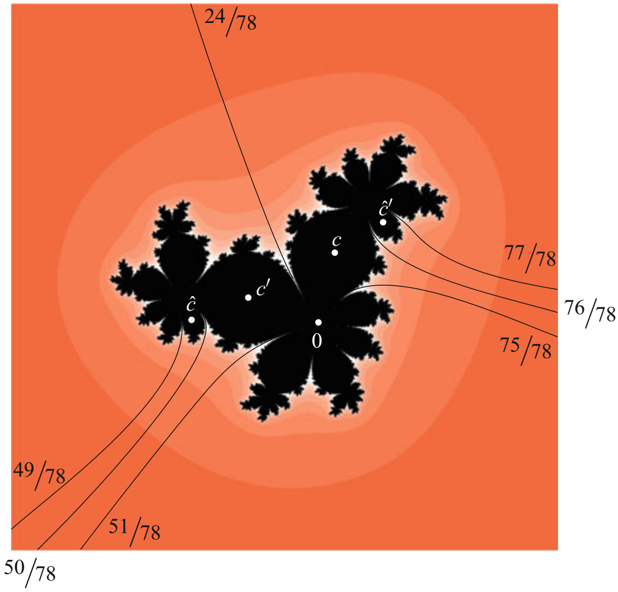

Fig. 5.9 Filled Julia set of the degenerate parabolic f_a in $\mathscr{M}_3(\frac{2}{3})$ with $X_a = X_{\frac{2}{3},\frac{0}{3}} \cup X_{\frac{2}{3},\frac{1}{3}} = \{\frac{24}{78}, \frac{51}{78}, \frac{60}{78}, \frac{69}{78}, \frac{72}{78}, \frac{75}{78}\}$ and $\delta_a = \frac{1}{6}$. Here $a \approx 0.68308975 - 1.08669099\,i$. The ray pairs at angles $(\frac{75}{78}, \frac{24}{78})$ and $(\frac{24}{78}, \frac{51}{78})$ bound the Fatou components containing the critical points c and c', respectively. It follows that the ray pairs at angles $(\frac{75}{78} - \frac{1}{3} = \frac{49}{78}, \frac{24}{78} + \frac{1}{3} = \frac{50}{78})$ and $(\frac{24}{78} + \frac{2}{3} = \frac{76}{78}, \frac{51}{78} + \frac{1}{3} = \frac{77}{78})$ bound the Fatou components containing the co-critical points $\hat{c}$ and $\hat{c}'$, respectively

As in the rational case, it is conjectured that every interior component of $\mathscr{M}_3(\theta)$ has one of these types. In Fig. 5.10 left the capture components are the blue bulbs, while the hyperbolic-like components are the grey bulbs that belong to a small copy of the Mandelbrot set.

The following is proved in [30]:

Theorem 5.15 *There is a closed embedded arc $\Gamma(\theta) \subset \mathscr{M}_3(\theta)$ with the property that $a \in \Gamma(\theta)$ if and only if $\partial\Delta_a$ contains both critical points of f_a.*

The arc $\Gamma(\theta)$ is clearly invariant under the involution $a \mapsto -a$. The endpoints of $\Gamma(\theta)$ are the parameters $\pm\sqrt{3e^{2\pi i\theta}}$ corresponding to the cubics with a double critical point. We denote by a_0 the endpoint in the lower half-plane, so $-a_0$ is the other endpoint in the upper half-plane. The midpoint of $\Gamma(\theta)$ is the parameter $a = 0$ corresponding to the cubic with centered critical points. See Fig. 5.10 right.[3]

[3]In [30] the cubics are given in the normal form

$$z \mapsto e^{2\pi i\theta} z\Big(1 - \frac{1}{2}\big(1 + \frac{1}{c}\big)z + \frac{1}{3c}z^2\Big) \qquad c \in \mathbb{C}^*,$$

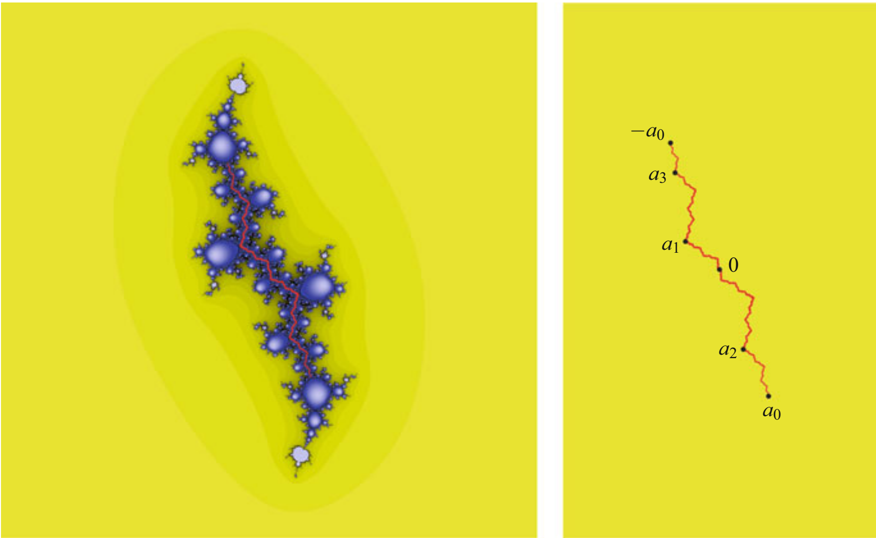

Fig. 5.10 Left: The cubic connectedness locus $\mathscr{M}_3(\theta) \subset \mathbb{C}$. Right: The arc $\Gamma(\theta) \subset \mathscr{M}_3(\theta)$. Here $\theta = \frac{(\sqrt{5}-1)}{2}$

The arc $\Gamma(\theta)$ is parametrized by the internal angle between the two critical points (as defined in Sect. 5.2). More precisely, if $a \in \Gamma(\theta)$ and if the internal angles of the critical points of f_a are 0 and $\tau_a \in [0, 1]$, where $\tau_{a_0} = 0$ and $\tau_{-a_0} = 1$, then the map $a \mapsto \tau_a$ is a homeomorphism $\Gamma(\theta) \to [0, 1]$.

Here are two alternative characterizations of $\Gamma(\theta)$:

- $\Gamma(\theta)$ is the set of parameters near which the boundary $\partial\Delta_a$ fails to move holomorphically. In fact, if U is a disk which does not intersect $\Gamma(\theta)$, then the critical point of f_a that lies on $\partial\Delta_a$ depends holomorphically on $a \in U$, so its forward orbit moves holomorphically over U. By the λ-lemma [16], this holomorphic motion extends to a holomorphic motion of the closure of this forward orbit, which is just $\partial\Delta_a$. On the other hand, if U is a disk that does intersect $\Gamma(\theta)$, the critical point on $\partial\Delta_a$ cannot be followed holomorphically in U, which shows $\partial\Delta_a$ does not move holomorphically over U (although it still moves continuously in the Hausdorff topology [30]).
- Let $\mathrm{rad}(a)$ denote the conformal radius of the Siegel disk Δ_a relative to its center 0. The function $a \mapsto \log \mathrm{rad}(a)$ is continuous and subharmonic in $\mathbb{C}$ and harmonic off $\Gamma(\theta)$ (see [5] and [32]). The arc $\Gamma(\theta)$ can be described as the support of the generalized Laplacian $4\partial\overline{\partial} \log \mathrm{rad}$. This has been proved by I. Zidane and independently by the author (unpublished).

with marked critical points at 1 and c. The punctured c-plane is a double-cover of the a^2-plane, branched at $c = \pm 1$. In this normalization, $\Gamma(\theta)$ appears as a Jordan curve passing through these branch points, and is invariant under the involution $c \mapsto 1/c$.

An adaptation of the work of Petersen in [23], using complex a priori bounds for critical circle maps, proves that for every $a \in \Gamma(\theta)$ the Julia set of f_a is locally connected and has measure zero. Thus, along $\Gamma(\theta)$ the Julia set is tame enough to allow the general constructions of Sect. 5.2 to go through. In particular, it follows from Theorem 5.7 that we can assign to each $a \in \Gamma(\theta)$ a minimal rotation set X_a under tripling with $\rho(X_a) = \theta$, consisting of angles of the dynamic rays of f_a which land on $\partial\Delta_a$. Notice the symmetry

$$X_{-a} = X_a + \frac{1}{2} \pmod{\mathbb{Z}}. \tag{5.8}$$

For each $a \in \Gamma(\theta)$ consider the deployment vector $\delta(X_a) = (\delta_a, 1 - \delta_a)$, where $\delta_a \in [0, 1]$ is the deployment probability of f_a, i.e., the probability that a dynamic ray $R_a(t)$ landing on $\partial\Delta_a$ has its angle t in $(0, \frac{1}{2})$. It follows from the symmetry relation (5.8) that

$$\delta_{-a} = 1 - \delta_a \qquad a \in \Gamma(\theta).$$

At the two endpoints $a = \pm a_0$ of $\Gamma(\theta)$ the cubic f_a has a double critical point whose wake contains both dynamic rays $R_a(0)$ and $R_a(\frac{1}{2})$. At any other $a \in \Gamma(\theta)$ the critical points of f_a are distinct and we label them as $*_a$ and $*'_a$ by requiring that the wake $W(*_a)$ contains $R_a(0)$ and the wake $W(*'_a)$ contains $R_a(\frac{1}{2})$. Under this labeling, the internal angle of $*_a$ will be 0 and that of $*'_a$ will be τ_a.

The following is an immediate corollary of Theorem 5.7:

Theorem 5.16 *For every parameter $a \in \Gamma(\theta)$, the deployment probability of X_a is the internal angle between the two critical points of f_a:*

$$\delta_a = \tau_a.$$

Thus, starting at the endpoint a_0 of $\Gamma(\theta)$ in the lower half-plane and moving to the other endpoint $-a_0$, the probability δ_a increases monotonically and takes all values between 0 and 1. In particular, the family $\{X_a\}_{a \in \Gamma(\theta)}$ spans all minimal rotation sets under tripling with $\rho(X_a) = \theta$.

For each integer $n \geq 1$, let a_n be the unique parameter on $\Gamma(\theta)$ for which $\delta_{a_n} = n\theta \pmod{\mathbb{Z}}$ (the first few a_n are shown in Fig. 5.10 right). Using Theorem 5.16, it is readily seen that $f_{a_n}^{\circ n}(*_{a_n}) = *'_{a_n}$. By Theorem 4.31, the rotation set X_{a_n} has a taut gap of length $\frac{1}{3}$ corresponding to the wake $W(*'_{a_n})$ and a loose gap of length $\frac{1}{3} + \frac{1}{3^{n+1}}$ corresponding to the wake $W(*_{a_n})$ (compare Fig. 5.12). Of course by symmetry the parameters $-a_n$ have similar dynamical description, with $*_a$ and $*'_a$ exchanged. Namely, $\delta_{-a_n} = -n\theta \pmod{\mathbb{Z}}$, $f_{-a_n}^{\circ n}(*'_{-a_n}) = *_{-a_n}$, and X_{-a_n} has a taut gap of length $\frac{1}{3}$ corresponding to $W(*_{-a_n})$ and a loose gap of length $\frac{1}{3} + \frac{1}{3^{n+1}}$ corresponding to $W(*'_{-a_n})$.[4]

[4]Each parameter $\pm a_n$ is the "root" of a capture component in which the $(n+1)$-st iterate of one critical point hits the Siegel disk. We will not be using this fact in our presentation.

We can combinatorially describe $\Gamma(\theta)$ by specifying the angles of the candidate parameter rays that presumably land on it. This description is related to rotation sets under tripling, much like what we have seen in the case of the boundary of the main cardioid of the Mandelbrot set. It will be convenient to use Theorem 5.16 to parametrize $\Gamma(\theta)$ by the deployment probability. For each $\delta \in [0, 1]$, let $a(\delta) \in \Gamma(\theta)$ be the unique parameter with $\delta_{a(\delta)} = \delta$. Thus, $a(\frac{1}{2}) = 0$ and in terms of our previous notation, $a(0) = a_0$, $a(1) = -a_0$, and $a(\pm n\theta) = \pm a_n$ for all $n \geq 1$. If $\delta \neq n\theta \pmod{\mathbb{Z}}$ for all n, there are two angles $-\frac{1}{6} < s(\delta) < \frac{1}{6}$ and $\frac{1}{3} < t(\delta) < \frac{2}{3}$ such that the parameter rays $\mathscr{R}(s(\delta))$ and $\mathscr{R}(t(\delta))$ land at $a(\delta)$ (thus, in Fig. 5.14, $\mathscr{R}(s(\delta))$ lands at $a(\delta)$ from the right side of $\Gamma(\theta)$ while $\mathscr{R}(t(\delta))$ lands there from the left side). These angles can be expressed in terms of the leading angle $\omega(\theta, \delta)$ of $X_{a(\delta)} = X_{\theta,\delta}$ studied in Sect. 4.6:

$$t(\delta) = \omega(\theta, \delta) + \frac{1}{3}$$

$$s(\delta) = \omega(\theta, 1 - \delta) - \frac{1}{6}$$

This can be seen by examining Fig. 5.11 which illustrates the angles of the dynamic rays landing at the co-critical points of $f_{a(\delta)}$. Notice that by symmetry,

$$t(\delta) = s(1 - \delta) + \frac{1}{2}.$$

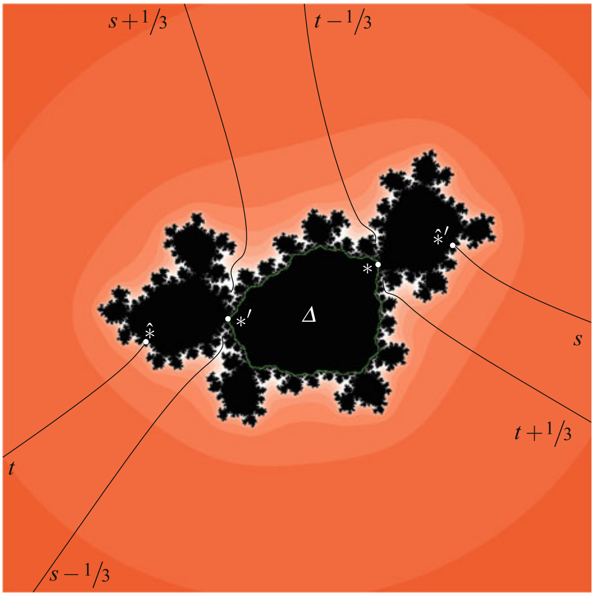

Fig. 5.11 Filled Julia set of a typical cubic map f_a with $a \in \Gamma(\theta)$, where the critical points $*, *'$ have disjoint orbits on $\partial\Delta$. Here the rays at angles $t \pm \frac{1}{3}$ land at $*$ and those at angles $s \pm \frac{1}{3}$ land at $*'$. If δ is the deployment probability of the associated rotation set X_a, we have $t - \frac{1}{3} = \omega(\theta, \delta)$ and $s - \frac{1}{3} = \omega(\theta, 1 - \delta) + \frac{1}{2}$. Thus, the rays landing at the co-critical points $\hat{*}, \hat{*}'$ have angles $t = \omega(\theta, \delta) + \frac{1}{3}$ and $s = \omega(\theta, 1 - \delta) - \frac{1}{6}$, respectively

Recall from Theorem 4.33 that the leading angle $\delta \mapsto \omega(\theta, \delta)$ is a decreasing, left-continuous function with a jump discontinuity of size $\frac{1}{3^{n+1}}$ at $\delta = n\theta \pmod{\mathbb{Z}}$ for each $n \geq 0$. Moreover,

$$\omega(\theta, 0) = \omega(\theta, 0^+) + \frac{1}{3} = \omega(\theta, 1) + \frac{1}{2}.$$

It follows from the above formulas that $s(\delta)$ is increasing and $t(\delta)$ is decreasing as a function of δ. For each $n \geq 1$ the angle $t(\delta)$ has a jump discontinuity of size $\frac{1}{3^{n+1}}$ at $\delta = n\theta \pmod{\mathbb{Z}}$, while $s(\delta)$ remains continuous there, and similarly, $s(\delta)$ has a jump discontinuity of size $\frac{1}{3^{n+1}}$ at $\delta = -n\theta \pmod{\mathbb{Z}}$, while $t(\delta)$ remains continuous there. These values of δ correspond to the parameters $\pm a_n$ along $\Gamma(\theta)$ and the aforementioned discontinuity suggests that every a_n with $n \geq 1$ is the landing point of three parameter rays at angles

$$\begin{aligned} t_n^- &= \omega(\theta, n\theta) + \frac{1}{3} - \frac{1}{3^{n+1}} \\ t_n^+ &= \omega(\theta, n\theta) + \frac{1}{3} \\ s_n &= \omega(\theta, -n\theta) - \frac{1}{6} \end{aligned}$$

while the parameter $-a_n$ is the landing point of the three parameter rays at angles

$$\begin{aligned} s_n^- &= \omega(\theta, n\theta) - \frac{1}{6} - \frac{1}{3^{n+1}} \\ s_n^+ &= \omega(\theta, n\theta) - \frac{1}{6} \\ t_n &= \omega(\theta, -n\theta) + \frac{1}{3}. \end{aligned}$$

These computations are illustrated in Fig. 5.12 which shows the angles of the dynamic rays that land at the co-critical points of f_{a_n}.

Finally, the endpoint a_0 of $\Gamma(\theta)$ is the landing point of the two parameter rays at angles

$$\begin{aligned} t_0^- &= \omega(\theta, 1) + \frac{1}{2} \\ t_0^+ &= \omega(\theta, 1) + \frac{5}{6}, \end{aligned}$$

while the other endpoint $-a_0$ is the landing point of the two parameter rays at angles

$$\begin{aligned} s_0^- &= \omega(\theta, 1) \\ s_0^+ &= \omega(\theta, 1) + \frac{1}{3}. \end{aligned}$$

Compare Fig. 5.13 which provides a justification for these formulas.

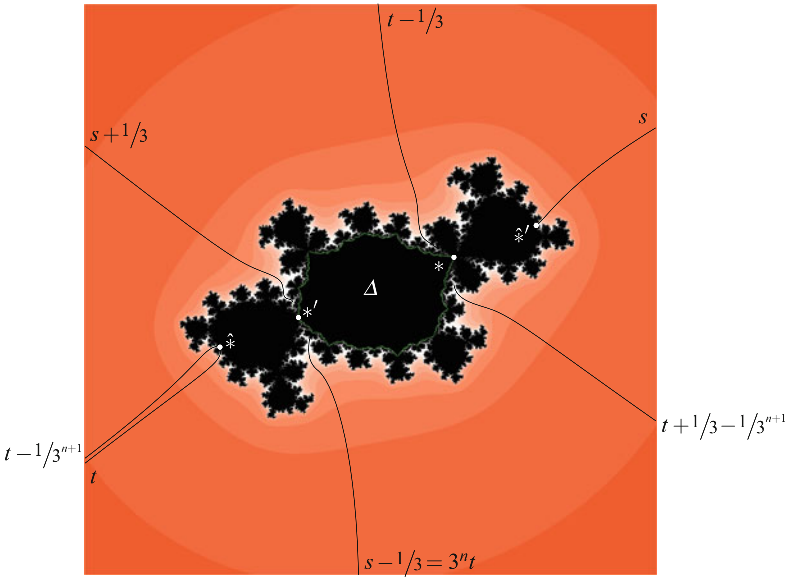

Fig. 5.12 Filled Julia set of the cubic map f_{a_n}, where the n-th iterate of the critical point $*$ hits the critical point $*'$. Here the rays at angles $s \pm \frac{1}{3}$ land at $*'$ and those at angles $t \pm \frac{1}{3}$ and $t \pm \frac{1}{3} - \frac{1}{3^{n+1}}$ land at $*$ (although only two of them, shown in the picture, are present in the rotation set X_{a_n}). We have $t - \frac{1}{3} = \omega(\theta, n\theta)$ and $s - \frac{1}{3} = \omega(\theta, -n\theta) + \frac{1}{2}$. Thus, the ray at angle $s = \omega(\theta, -n\theta) - \frac{1}{6}$ lands at the co-critical point $\hat{*}'$ and the rays at angles $t = \omega(\theta, n\theta) + \frac{1}{3}$ and $t - \frac{1}{3^{n+1}} = \omega(\theta, n\theta) + \frac{1}{3} - \frac{1}{3^{n+1}}$ land at the co-critical points $\hat{*}$

By Theorem 4.35, the above angles can be expressed rationally in terms of the (transcendental) base angle $\omega = \omega(\theta, 1)$. It follows that

$$t_n^+ = \frac{(3^n + 1)\omega + A_n}{2 \cdot 3^n} + \frac{1}{3}$$
$$s_n = \frac{(3^n + 1)\omega - B_n}{2} - \frac{1}{6},$$

where A_n, B_n are the integers defined by (4.16).

Example 5.17 For the golden mean $\theta = \frac{(\sqrt{5}-1)}{2}$, the base angle $\omega = \omega(\theta, 1)$ can be effectively computed with desired precision using the formula (4.13):

$$\omega \approx 0.128099593431 \cdots$$

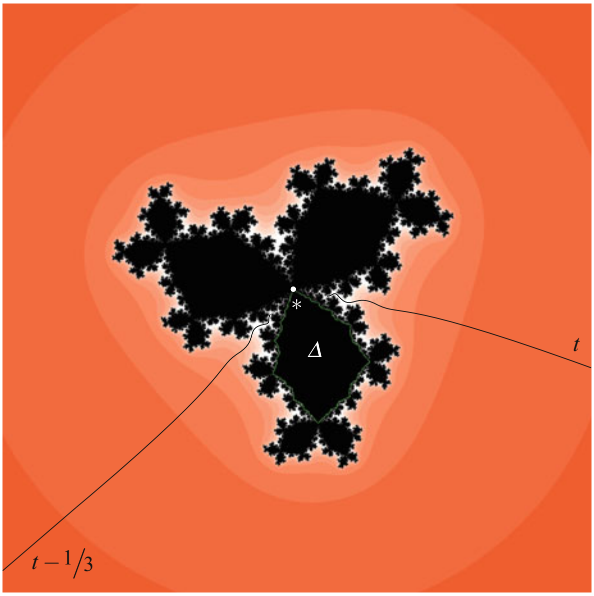

Fig. 5.13 Filled Julia set of the cubic map f_{a_0} with a double critical point $* = *'$ (which also coincides with the co-critical points $\hat{*} = \hat{*}'$). Here the rays at angles $t = \omega(\theta, 1) + \frac{5}{6}$ and $t - \frac{1}{3} = \omega(\theta, 1) + \frac{1}{2}$ land at $*$

Using the formula (4.16) it is easy to compute the integers A_n, B_n. Here are the results for $1 \leq n \leq 5$:

$$A_1 = 3^0 = 1 \qquad B_1 = 0$$
$$A_2 = 3^0 + 3^1 = 4 \qquad B_2 = 3^0 = 1$$
$$A_3 = 3^0 + 3^1 = 4 \qquad B_3 = 3^1 = 3$$
$$A_4 = 3^0 + 3^1 + 3^3 = 31 \qquad B_4 = 3^0 + 3^2 = 10$$
$$A_5 = 3^0 + 3^1 + 3^3 + 3^4 = 112 \qquad B_5 = 3^0 + 3^1 + 3^3 = 31.$$

The corresponding angles are listed in Table 5.1. Figure 5.14 shows selected parameter rays at these angles.

We can extend this picture to parameters outside the arc $\Gamma(\theta)$. One possible approach is to show that when θ is of bounded type, the filled Julia sets $K(f_a)$ for $a \in \mathscr{M}_3(\theta)$ satisfy the limb decomposition hypothesis in Sect. 5.2 so the rotation set X_a is well defined. This is already known for many parameters in $\mathscr{M}_3(\theta)$, including the hyperbolic-like ones, and is surely true for all capture parameters. An alternative route, which is outlined below, is to approach $\mathscr{M}_3(\theta)$ from outside, allowing disconnected Julia sets.

Outside the connectedness locus, the filled Julia set $K(f_a)$ consists of countably many homeomorphic copies of the filled Julia set of the quadratic polynomial

Table 5.1 Angles of some parameter rays which "land" on the arc $\Gamma(\theta)$ for $\theta = \frac{(\sqrt{5}-1)}{2}$

Angle	In terms of $\omega = \omega(\theta, 1)$	Approximate value
t_0^-	$\omega + \frac{1}{2}$	0.628099593431
t_0^+	$\omega + \frac{5}{6}$	0.961432926764
t_1^-	$\frac{2}{3}\,\omega + \frac{7}{18}$	0.474288617843
t_1^+	$\frac{2}{3}\,\omega + \frac{1}{2}$	0.585399728954
s_1	$2\omega - \frac{1}{6}$	0.089532520195
t_2^-	$\frac{5}{9}\,\omega + \frac{14}{27}$	0.589684959314
t_2^+	$\frac{5}{9}\,\omega + \frac{5}{9}$	0.626721996351
s_2	$5\omega + \frac{1}{3}$	0.973831300488
t_3^-	$\frac{14}{27}\,\omega + \frac{32}{81}$	0.461483739804
t_3^+	$\frac{14}{27}\,\omega + \frac{11}{27}$	0.473829418816
s_3	$14\omega - \frac{5}{3}$	0.126727641367
t_4^-	$\frac{41}{81}\,\omega + \frac{253}{486}$	0.585416666634
t_4^+	$\frac{41}{81}\,\omega + \frac{85}{162}$	0.589531892972
s_4	$41\omega - \frac{31}{6}$	0.085416664004
t_5^-	$\frac{122}{243}\,\omega + \frac{410}{729}$	0.626727642244
t_5^+	$\frac{122}{243}\,\omega + \frac{137}{243}$	0.628099384356
s_5	$122\omega - \frac{44}{3}$	0.961483731915

$P : z \mapsto e^{2\pi i\theta}z + z^2$ and uncountably many points. In particular, the connected component K_a of $K(f_a)$ containing the Siegel disk Δ_a, called the ***little filled Julia set***, is homeomorphic to $K(P)$. More precisely, let $G_a : \mathbb{C} \to \mathbb{R}$ be the Green's function of f_a as defined in Sect. 5.1, and U_a and V_a be the connected components of $G_a^{-1}[0, G_a(e_a))$ and $G_a^{-1}[0, G_a(e_a)/3)$ containing K_a, respectively (recall that e_a is the escaping critical point). Then U_a and V_a are Jordan domains with $K_a \subset V_a \subset \overline{V_a} \subset U_a$ and the restriction $f_a : V_a \to U_a$ is a degree 2 branched covering (see Fig. 5.15). According to Douady and Hubbard, this restriction is hybrid equivalent to the quadratic P, namely, there is a quasiconformal homeomorphism $\phi_a : U_a \to \phi_a(U_a)$ which satisfies $\phi_a \circ f_a = P \circ \phi_a$ in V_a, with $\phi_a(K_a) = K(P)$ and $\overline{\partial}\phi_a = 0$ a. e. on K_a (see for example [30] or [6]).

When a is outside $\mathscr{M}_3(\theta)$, it belongs to the parameter ray $\mathscr{R}(t)$ for a unique $t \in \mathbb{T}$ called the ***external angle*** of a. It follows that the dynamic rays $R_a(t \pm \frac{1}{3})$ are bifurcated and crash into the escaping critical point e_a. Let N_t be the countable dense set of angles whose forward m_3-orbit hit either of $t \pm \frac{1}{3}$. If $u \notin N_t$, the ray $R_a(u)$ is smooth. If $u \in N_t$, the ray $R_a(u)$ is bifurcated and crashes into an iterated preimage of e_a (only once if neither $t \pm \frac{1}{3}$ is periodic under m_3, infinitely many times

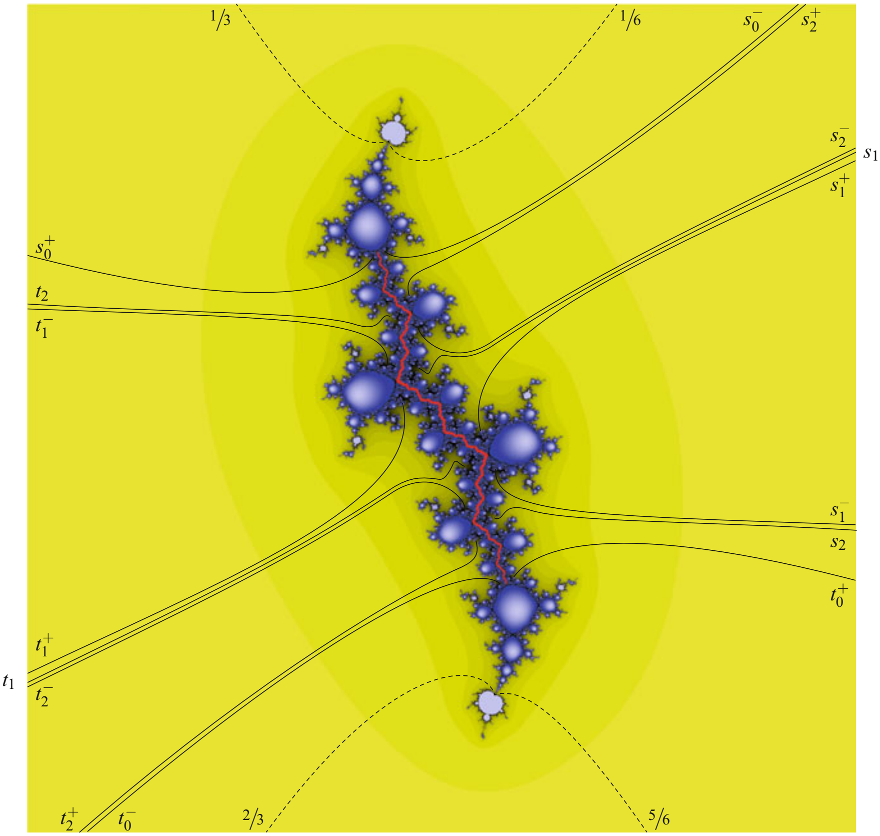

Fig. 5.14 Some parameter rays which "land" on the roots of capture components along the arc $\Gamma(\theta)$. Here $\theta = \frac{(\sqrt{5}-1)}{2}$

otherwise). For each $u \in N_t$ we can define the ***limit rays*** $R_a(u^{\pm})$ as the pointwise limits

$$R_a(u^+) = \lim_{\substack{v \to u^+ \\ v \notin N_t}} R_a(v) \quad \text{and} \quad R_a(u^-) = \lim_{\substack{v \to u^- \\ v \notin N_t}} R_a(v),$$

with one always turning to the right at a bifurcation point, the other always turning to the left. Every point of the little filled Julia set K_a is accumulated by at least one smooth or limit ray. When $u \in N_t$, only one of $R_a(u^+)$ or $R_a(u^-)$ can accumulate on K_a and we agree to denote this simply by $R_a(u)$.

Consider the compact set

$$Y_t = \left\{u \in \mathbb{T} : m_3^{\circ i}(u) \notin \left(t + \frac{1}{3}, t - \frac{1}{3}\right) \text{ for all } i \geq 0\right\}.$$

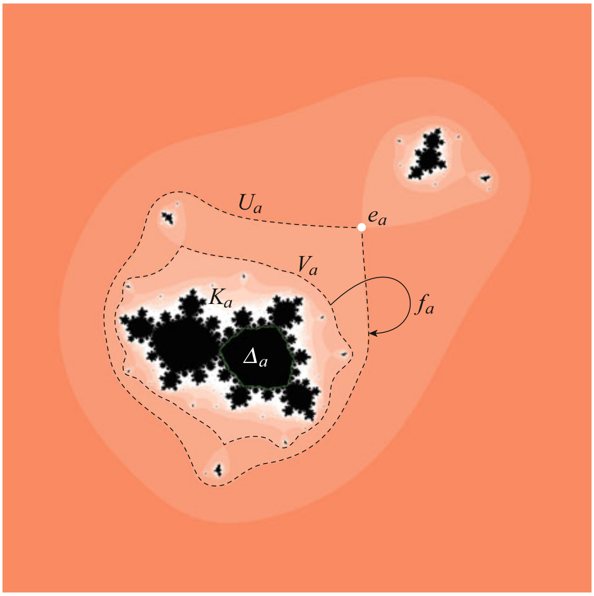

Fig. 5.15 Filled Julia set of a cubic f_a outside the connectedness locus $\mathscr{M}_3(\theta)$. The restriction $f_a : V_a \to U_a$ is a degree 2 branched covering hybrid equivalent to the quadratic $z \mapsto e^{2\pi i\theta} z + z^2$

It is not hard to show that Y_t contains a maximal m_3-invariant Cantor set A_t characterized by the property that $u \in A_t$ if and only if the (smooth or limit) ray $R_a(u)$ accumulates on K_a. Every endpoint of a gap of A_t belongs to N_t and the inclusion $A_t \supset Y_t \smallsetminus N_t$ holds. According to [2], there exists a degree 1 monotone map $h : \mathbb{T} \to \mathbb{T}$, with plateaus over the gaps of A_t, which satisfies

$$h \circ m_3 = m_2 \circ h \quad \text{on} \quad A_t. \tag{5.9}$$

The following is a special case of the main result of [26]:

Theorem 5.18 *The ray $R_a(u)$ with $u \in A_t$ lands at $z \in K_a$ if and only if the ray $R(h(u))$ of the quadratic P lands at $\phi_a(z) \in K(P)$.*

Since $K(P)$ is locally connected [23], it follows that all rays $R_a(u)$ with $u \in A_t$ land on K_a. In particular, since every point on the boundary of the Siegel disk of P is the landing point of one or two rays, and since $h|_{A_t}$ is at most 2-to-1, we see that every point of $\partial \Delta_a$ is the landing point of at most four (smooth or limit) rays. An argument similar to Sect. 5.2 for connected Julia sets then shows that the set of angles of rays landing on $\partial \Delta_a$ contains a minimal rotation set $X_a \subset A_t$ under tripling, with $\rho(X_a) = \theta$. Let us investigate the relation between the deployment probability $\delta_a \in [0, 1]$ of X_a and the external angle t of a.

We may assume without loss of generality that $s_0^+ = \omega + \frac{1}{3} < t \leq t_0^+ = \omega + \frac{5}{6}$ (the complementary case is treated by symmetry). Then the interval $(t + \frac{1}{3}, t - \frac{1}{3})$ of length $\frac{1}{3}$ is contained in the major gap I_0 of X_a that contains the fixed point 0. It will be convenient to first study the case where $X_a \cap N_t \neq \emptyset$, so at least one of the angles $t \pm \frac{1}{3}$ belongs to X_a. Since no angle in X_a is periodic under m_3, the rays $R_a(t \pm \frac{1}{3})$ crash at e_a and then join as a single smooth path to land at a point $w_a \in \partial\Delta_a$ which is characterized by the property that the internal angle from the non-escaping critical point $c_a \in \partial\Delta_a$ to w_a is δ_a. Here are the possibilities:

Case 1. $\delta_a = 0$. Then $w_a = c_a$. We either have $I_0 = (t, t-\frac{1}{3})$ where $t = \omega+\frac{5}{6} = t_0^+$, or $I_0 = (t+\frac{1}{3}, t)$ where $t = \omega + \frac{1}{2} = t_0^-$ (see Fig. 5.16a, b).

Case 2. $\delta_a = n\theta \pmod{\mathbb{Z}}$ for some $n \geq 1$. Then $c_a = f_a^{\circ n}(w_a)$. We either have

$$I_0 = \Big(t + \frac{1}{3} - \frac{1}{3^{n+1}}, t - \frac{1}{3}\Big), \quad \text{where} \quad t = \omega(\theta, n\theta) + \frac{1}{3} = t_n^+,$$

or

$$I_0 = \Big(t + \frac{1}{3}, t - \frac{1}{3} + \frac{1}{3^{n+1}}\Big), \quad \text{where} \quad t = \omega(\theta, n\theta) + \frac{1}{3} - \frac{1}{3^{n+1}} = t_n^-$$

(see Fig. 5.16c, d which show the case $n = 1$).

Case 3. $\delta_a = -n\theta \pmod{\mathbb{Z}}$ for some $n \geq 1$. Then $w_a = f_a^{\circ n}(c_a)$ and we have $I_0 = (t+\frac{1}{3}, t-\frac{1}{3})$ where $t = \omega(\theta, -n\theta) + \frac{1}{3} = t_n$ (see Fig. 5.16e which shows the case $n = 1$).

Case 4. $\delta_a \neq n\theta \pmod{\mathbb{Z}}$ for all integers n. In this case c_a and w_a have disjoint orbits on $\partial\Delta_a$, and we have $I_0 = (t+\frac{1}{3}, t-\frac{1}{3})$ where $t = t(\delta_a)$ (see Fig. 5.16f).

Using monotonicity of $\delta \mapsto \omega(\theta, \delta)$, it is easy to see that the above cases classify X_a for all external angles t except when $t \in (t_n^-, t_n^+)$ for some $n \geq 0$. As a corollary, we obtain

Corollary 5.19 *If the external angle t of $a \notin \mathscr{M}_3(\theta)$ lies in (t_n^-, t_n^+) for some $n \geq 0$, then X_a is contained in the set*

$$Y_t \smallsetminus N_t = \Big\{u \in \mathbb{T} : m_3^{\circ i}(u) \notin \Big[t + \frac{1}{3}, t - \frac{1}{3}\Big] \textit{ for all } i \geq 0\Big\}.$$

In particular, every dynamic ray $R_a(u)$ with $u \in X_a$ is smooth.

It remains to determine X_a when t belongs to such an interval. We will need a preliminary observation:

Lemma 5.20 *Corollary 5.19 holds if we replace X_a with the rotation set $X_{\theta,n\theta}$.*

Proof We know that $X_{\theta,n\theta}$ has a loose gap $I_0 = (\alpha + \frac{1}{3} - \frac{1}{3^{n+1}}, \alpha - \frac{1}{3})$ containing 0 and a taut gap $(\beta + \frac{1}{3}, \beta - \frac{1}{3})$ containing $\frac{1}{2}$. Here

$$\alpha = \omega(\theta, n\theta) + \frac{1}{3} \quad \text{and} \quad \beta = \omega(\theta, -n\theta) - \frac{1}{6}$$

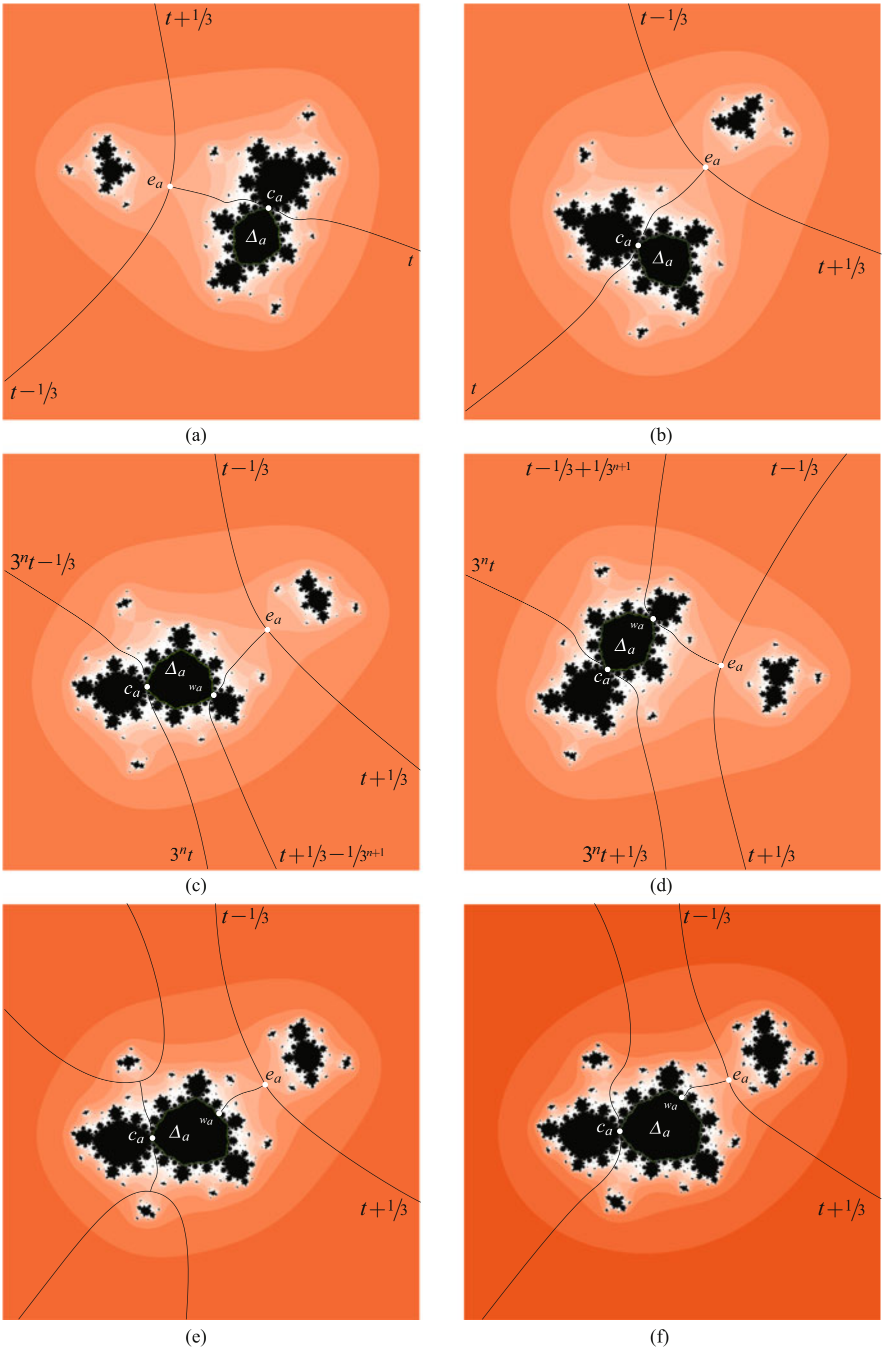

Fig. 5.16 Possible types of cubics f_a with $a \notin \mathscr{M}_3(\theta)$ which have a non-smooth ray landing on $\partial\Delta_a$. **(a)** $\delta_a = 0, t = t_0^+$. **(b)** $\delta_a = 0, t = t_0^-$. **(c)** $\delta_a = n\theta, t = t_n^+$. **(d)** $\delta_a = n\theta, t = t_n^-$. **(e)** $\delta_a = -n\theta, t = t_n$. **(f)** $\delta_a \neq n\theta, t = t(\delta_a)$

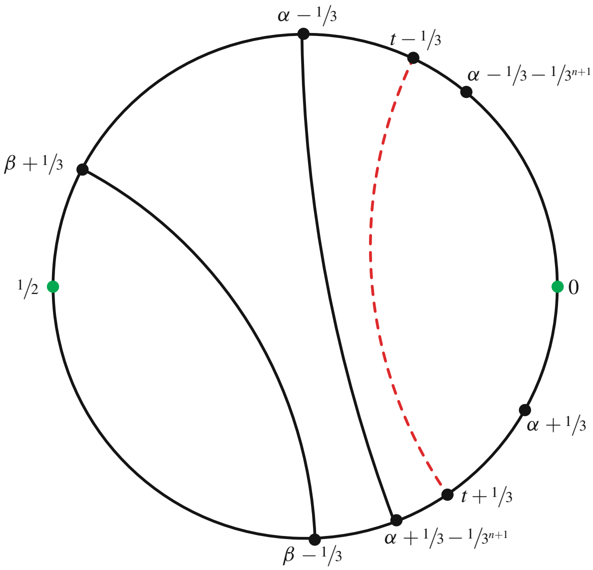

Fig. 5.17 Major gaps of $X_{\theta,n\theta}$ and the proof of Lemma 5.20

(see Fig. 5.17). We have

$$t_n^- = \omega(\theta, n\theta) + \frac{1}{3} - \frac{1}{3^{n+1}} = \alpha - \frac{1}{3^{n+1}} \quad \text{and} \quad t_n^+ = \omega(\theta, n\theta) + \frac{1}{3} = \alpha,$$

so the assumption $t_n^- < t < t_n^+$ implies $[t + \frac{1}{3}, t - \frac{1}{3}] \subset I_0$. Since the forward m_3-orbit of every $u \in X_{\theta,n\theta}$ avoids I_0, it must avoid the subinterval $[t + \frac{1}{3}, t - \frac{1}{3}]$, which implies $u \in Y_t \smallsetminus N_t$. □

Theorem 5.21 *If the external angle t of $a \notin \mathscr{M}_3(\theta)$ lies in (t_n^-, t_n^+) for some $n \geq 0$, then $X_a = X_{\theta,n\theta}$.*

Proof By Corollary 5.19, $X_a \subset Y_t \smallsetminus N_t \subset A_t$. The semiconjugacy h of (5.9) has plateaus over the gaps of A_t, so it is injective on X_a. Hence h maps X_a homeomorphically onto an m_2-invariant Cantor set $C = h(X_a)$. If φ is the canonical semiconjugacy associated with X_a, the composition $\varphi \circ h^{-1}$ is a well-defined degree 1 monotone map of the circle since each fiber of h maps to a single point under φ. Since $\varphi \circ h^{-1}$ semiconjugates $m_2|_C$ to the rotation r_θ, it follows that C is a rotation set for m_2 with $\rho(C) = \theta$. Similarly, by Lemma 5.20 $X_{\theta,n\theta} \subset Y_t \smallsetminus N_t \subset A_t$ and an identical argument shows that $C' = h(X_{\theta,n\theta})$ is also a rotation set for m_2 with $\rho(C') = \theta$. By the uniqueness of rotation sets under doubling, $C = C'$. It follows from injectivity of h that $X_a = X_{\theta,n\theta}$. □

Assuming that the rays $\mathscr{R}(t_n^\pm)$ in fact land at a_n, we can define the ***parameter wake*** $\mathscr{W}_n$ as the connected component of $\mathbb{C} \smallsetminus (\mathscr{R}(t_n^-) \cup \mathscr{R}(t_n^+) \cup \{a_n\})$ which does not meet $\Gamma(\theta)$. Using monotonicity of $\delta \mapsto \omega(\theta, \delta)$ it is easy to see that distinct parameter wakes are disjoint. Theorem 5.21 can be restated as saying that $X_a = X_{\theta,n\theta}$ whenever $a \in \mathscr{W}_n \setminus \mathscr{M}_3(\theta)$. We can show that this holds for every $a \in \mathscr{W}_n$ (this contains the claim that X_a is well defined for $a \in \mathscr{W}_n \cap \mathscr{M}_3(\theta)$). The argument uses holomorphic motions as follows.

A dynamic ray $R_a(u)$ moves holomorphically over the parameter $a \in \mathbb{C}$ as long as it remains smooth (see [6], Proposition 2). Lemma 5.20 shows that every ray $R_a(u)$ with $u \in X_{\theta,n\theta}$ is smooth for $a \in \mathscr{W}_n \setminus \mathscr{M}_3(\theta)$. Since $R_a(u)$ is trivially smooth for $a \in \mathscr{M}_3(\theta)$, it follows that this ray moves holomorphically over the entire parameter wake $\mathscr{W}_n$. By the λ-lemma, this motion extends to a holomorphic motion of the closure $\overline{R_a(u)}$ over $\mathscr{W}_n$. But for $a \in \mathscr{W}_n \setminus \mathscr{M}_3(\theta)$ this closure is $R_a(u)$ union its landing point on $\partial \Delta_a$. Since $\partial \Delta_a$ also moves holomorphically over $\mathscr{W}_n$, it follows that $R_a(u)$ lands on $\partial \Delta_a$ for every $a \in \mathscr{W}_n$, as required.

Away from the endpoints $\pm a_0$ of $\Gamma(\theta)$ the critical points of f_a can be continued analytically as a function of a (however, going around $\pm a_0$ will swap the two critical points, so the monodromy is non-trivial). In particular, the escaping and non-escaping critical points of f_a for $a \in \mathscr{W}_n \setminus \mathscr{M}_3(\theta)$ extend to holomorphic functions $a \mapsto e_a, c_a$ defined for all $a \in \mathscr{W}_n$. The preceding paragraph then shows that e_a belongs to the dynamical wake $W(f_a^{-n}(c_a))$ whenever $a \in \mathscr{W}_n$. It seems likely that this property is the dynamical characterization of the parameter wake $\mathscr{W}_n$.

To summarize, we have identified the dependence of δ_a on a in the following cases:

- If $a \in \overline{\mathscr{W}_0}$, then $\delta_a = 0$.
- If $a \in \overline{-\mathscr{W}_0}$, then $\delta_a = 1$.
- If $a \in \overline{\mathscr{W}_n \cup \mathscr{R}(s_n)}$ for some $n \geq 1$, then $\delta_a = n\theta \pmod{\mathbb{Z}}$.
- If $a \in \overline{-\mathscr{W}_n \cup \mathscr{R}(t_n)}$ for some $n \geq 1$, then $\delta_a = -n\theta \pmod{\mathbb{Z}}$.
- If $a \in \overline{\mathscr{R}(t(\delta)) \cup \mathscr{R}(s(\delta))}$ where $\delta \neq n\theta \pmod{\mathbb{Z}}$ for all n, then $\delta_a = \delta$.

It is conjectured that an analog of the limb decomposition hypothesis in Sect. 5.1 holds in this cubic parameter space, in the sense that the ***parameter limbs*** $\mathscr{L}_n = \mathscr{M}_3(\theta) \cap \overline{\mathscr{W}_n}$ have shrinking diameters as $n \to \infty$. Under this assumption, the connectedness locus $\mathscr{M}_3(\theta)$ would be the union of the arc $\Gamma(\theta)$ together with the parameter limbs $\pm\mathscr{L}_n$ for all $n \geq 0$, and the five cases above would describe δ_a (hence X_a) for every $a \in \mathbb{C}$.