Chapter 4
Applications and Computations

In this chapter we establish further properties of (minimal) rotation sets for m_d by exploiting the ideas and tools developed in the previous chapters, most notably the deployment theorem. We also study minimal rotation sets under doubling and tripling in some detail and carry out explicit computations. These computations will tie in with the dynamical study of quadratic and cubic polynomials in the next chapter.

4.1 Symmetries

It was already observed in Sect. 3.1 that if X is a minimal rotation set for m_d, the deployment vectors of the $d-2$ rotation sets

$$X+\frac{1}{d-1},\ X+\frac{2}{d-1},\ \ldots,\ X+\frac{d-2}{d-1} \pmod{\mathbb{Z}} \tag{4.1}$$

are obtained by cyclically permuting the components of $\delta(X)$. The uniqueness parts of the deployment Theorems 3.7 and 3.20 show at once that the converse statement is also true. In particular, if $\delta(X)$ is invariant under some cyclic permutation of its components, then X itself has a corresponding symmetry. Explicitly, suppose $\Pi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ is defined by

$$\Pi(x_1, x_2, \ldots, x_{d-1}) = (x_{d-1}, x_1, \ldots, x_{d-2}).$$

Theorem 4.1 *A minimal rotation set X for m_d has the symmetry $X = X + i/(d-1) \pmod{\mathbb{Z}}$ if and only if its deployment vector $\delta(X)$ is fixed by the iterate $\Pi^{\circ i}$.*

For example, a minimal rotation set X under tripling is ***self-antipodal*** in the sense $X = X + \frac{1}{2} \pmod{\mathbb{Z}}$ if and only if $\delta(X) = (\frac{1}{2}, \frac{1}{2})$. Moreover, there is a unique such

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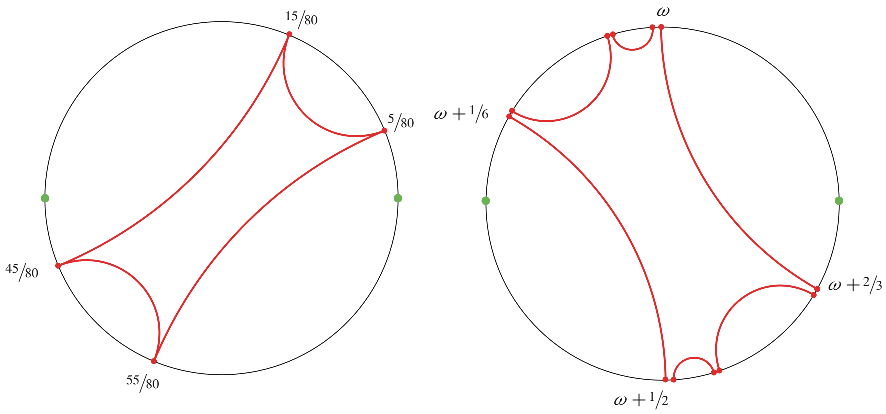

Fig. 4.1 If θ is irrational or rational with even denominator, there is a unique self-antipodal minimal rotation set X under tripling with $\rho(X) = \theta$. Left: The self-antipodal 4-cycle of rotation number $\frac{1}{4}$. Right: The self-antipodal Cantor set of the golden mean rotation number $\frac{(\sqrt{5}-1)}{2}$. Here $\omega \approx 0.25208333$ (see Sect. 4.6 for the method of such computations)

X with a given rotation number, which can only be irrational or rational with even denominator (compare Fig. 4.1).

It turns out that the sets (4.1) are the only copies of X that are rotation sets of the same rotation number:

Theorem 4.2 *Suppose both X and $X + \alpha \pmod{\mathbb{Z}}$ are rotation sets for m_d with $\rho(X) = \rho(X + \alpha)$. Then $\alpha = i/(d-1) \pmod{\mathbb{Z}}$ for some $0 \leq i \leq d-2$.*

Here the assumption $\rho(X) = \rho(X + \alpha)$ is necessary, as is illustrated by the rotation sets

$$X = \left\{\frac{5}{80}, \frac{15}{80}, \frac{45}{80}, \frac{55}{80}\right\} \quad \text{and} \quad X + \frac{1}{4} = \left\{\frac{25}{80}, \frac{35}{80}, \frac{65}{80}, \frac{75}{80}\right\}$$

under tripling for which $\rho(X) = \frac{1}{4}$ and $\rho(X + \frac{1}{4}) = \frac{3}{4}$.

Proof Denote the distinct major gaps of X by $I_1, \ldots, I_n$, so $I_1 + \alpha, \ldots, I_n + \alpha$ are the distinct major gaps of $X + \alpha$. For each $1 \leq i \leq n$, let J_i be the gap of X which maps to I_i and $\hat{J}_i$ be the gap of $X+\alpha$ which maps to $I_i+\alpha$. Evidently a gap of length ℓ for X or $X+\alpha$ belongs to $\{J_1, \ldots, J_n\}$ or $\{\hat{J}_1, \ldots, \hat{J}_n\}$ if and only if the fractional part of $d\ell$ is at least $1/d$. It follows that $\{\hat{J}_1, \ldots, \hat{J}_n\} = \{J_1 + \alpha, \ldots, J_n + \alpha\}$. We prove that in fact $\hat{J}_i = J_i + \alpha$ for every i.

Consider the standard monotone maps $g, \hat{g}$ associated with $X, X + \alpha$ and let $\varphi, \hat{\varphi}$ be the semiconjugacies between $g, \hat{g}$ and the rigid rotation r_θ, where $\theta = \rho(X) = \rho(X + \alpha)$. Recall that $\varphi, \hat{\varphi}$ map each gap of their respective rotation set to a single point. Let $\varphi(I_i) = \{t_i\}$ and $\hat{\varphi}(I_i + \alpha) = \{\hat{t}_i\}$. Then $\varphi(J_i) = \{t_i - \theta\}$ and $\hat{\varphi}(\hat{J}_i) = \{\hat{t}_i - \theta\}$. Since $X+\alpha$ is a rotation of X and since $\varphi, \hat{\varphi}$ are order-preserving, there is an orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ which maps t_i to $\hat{t}_i$ for every i and maps the set $\{t_1 - \theta, \dots, t_n - \theta\}$ onto the set $\{\hat{t}_1 - \theta, \dots, \hat{t}_n - \theta\}$. The claim $\hat{J}_i = J_i + \alpha$ is then equivalent to $h(t_i - \theta) = \hat{t}_i - \theta$. This is proved in the following

Lemma 4.3 *Suppose $t_1, \dots, t_n \in \mathbb{T}$ are distinct and $h : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism which maps the set $\{t_1 - \theta, \dots, t_n - \theta\}$ onto the set $\{h(t_1) - \theta, \dots, h(t_n) - \theta\}$ for some θ. Then $h(t_i - \theta) = h(t_i) - \theta$ for every $1 \le i \le n$.*

Proof The assumption means that the commutator $[r_\theta, h^{-1}] = r_\theta \circ h^{-1} \circ r_\theta^{-1} \circ h$ preserves the finite set $\{t_1, \dots, t_n\}$ and therefore has a well-defined combinatorial rotation number on it, which coincides with the Poincaré rotation number $\rho([r_\theta, h^{-1}])$. By Corollary 1.10, $\rho([r_\theta, h^{-1}]) = -\rho([h^{-1}, r_\theta]) = 0$. It follows that $[r_\theta, h^{-1}]$ acts as the identity on $\{t_1, \dots, t_n\}$. □

Back to the proof of the theorem, we now know that $\hat{J}_i = J_i + \alpha$ for every i. Let $J_1 = (t, s)$. Then, on the one hand, $I_1 = (dt, ds)$ so $I_1 + \alpha = (dt + \alpha, ds + \alpha)$. On the other hand, $\hat{J}_1 = J_1 + \alpha = (t + \alpha, s + \alpha)$ so $I_1 + \alpha = (dt + d\alpha, ds + d\alpha)$. It follows that $d\alpha = \alpha \pmod{\mathbb{Z}}$, or $\alpha = i/(d-1)$ for some $0 \le i \le d-2$, as required. □

Remark 4.4 The crucial point in the above proof was to use the assumption $\rho(X) = \rho(X+\alpha)$ to show that $r_\alpha \circ m_d = m_d \circ r_\alpha$ holds at some point of X, hence everywhere on the circle.

The following is an immediate corollary of Theorem 4.2:

Corollary 4.5 *For every rotation set X for m_d, the symmetry group $\{\alpha \in \mathbb{T} : X = X + \alpha \pmod{\mathbb{Z}}\}$ is a subgroup of $\mathbb{Z}/(d-1)\mathbb{Z}$.*

4.2 Realizing Gap Graphs and Gap Lengths

As an application of Theorem 3.20, we give a partial answer to the question of realizing admissible graphs as gap graphs that was raised at the end of Sect. 2.1.

Theorem 4.6 *Given an irrational number θ and an admissible graph Γ of degree d without closed paths, there exists a (minimal) rotation set X for m_d with $\rho(X) = \theta$ whose gap graph Γ_X is isomorphic to Γ.*

Proof Suppose Γ consists of α degree 0 vertices of weights $n_1, \dots, n_\alpha$ and β maximal paths $P_1, \dots, P_\beta$ of total weights $n_{\alpha+1}, \dots, n_{\alpha+\beta}$ (thus, for every $\alpha + 1 \le i \le \alpha + \beta$, the number n_i is the sum of the weights of the vertices in the path $P_{i-\alpha}$). Then $\sum_{i=1}^{\alpha+\beta} n_i = d - 1$.

Choose $\alpha + \beta$ distinct points $s_1 = 0, s_2, \dots, s_{\alpha+\beta}$ on $\mathbb{T}$ subject only to the condition that their full orbits under the rotation r_θ are disjoint. We use the s_i to produce a list L of $d - 1$ not necessarily distinct points in $\mathbb{T}$ as follows: For each

$1 \le i \le \alpha$, let L include n_i copies of the point s_i. For each $\alpha + 1 \le i \le \alpha + \beta$, consider the maximal path $P_{i-\alpha}$ which has the form

$$I_k \to I_{k-1} \to \cdots \to I_1 \quad \text{with} \quad \sum_{j=1}^{k} w(I_j) = n_i, \tag{4.2}$$

and let L include $w(I_j)$ copies of the point $s_i - (j-1)\theta$ for every $1 \le j \le k$. Represent points of L by numbers $0 < \sigma_1 \le \cdots \le \sigma_{d-2} \le \sigma_{d-1} = 1$ and let X be the minimal rotation set with $\rho(X) = \theta$ and $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$ given by Theorem 3.20. Recall that under the canonical semiconjugacy of X, each major gap of multiplicity n corresponds to an n-fold incidence $\sigma_i = \cdots = \sigma_{i+n-1}$. Using this and the selection of the list L, it is easy to see that Γ_X is isomorphic to Γ. □

Remark 4.7 The above proof shows that we have the freedom of arbitrarily prescribing the number of iterates it takes to go from each loose vertex of Γ_X to its adjacent vertex. To see this, suppose for each maximal path of Γ of the form (4.2) and each $2 \le j \le k$ we are given an integer N_j, which is to be the number of iterates it takes to map I_j to I_{j-1}. Set $N_1 = 0$, modify the list L by including $w(I_j)$ copies of the point $s_i - (N_1 + \cdots + N_j)\,\theta$ for every $1 \le j \le k$, and construct the rotation set X as before.

Remark 4.8 We would naturally want to know if every admissible graph can be realized as a gap graph even when it contains closed paths. It may seem at first glance that all closed paths of a realizable graph must have the same length, but this is not the case: Consider the rotation set

$$X = \left\{ \frac{17}{124}, \frac{18}{124}, \frac{23}{124}, \frac{53}{124}, \frac{78}{124}, \frac{79}{124}, \frac{85}{124}, \frac{90}{124}, \frac{115}{124} \right\}$$

under m_5 with $\rho(X) = \frac{2}{3}$, which is a union of three compatible 3-cycles. The cycle of gaps

$$\left(\frac{23}{124}, \frac{53}{124}\right) \mapsto \left(\frac{115}{124}, \frac{17}{124}\right) \mapsto \left(\frac{79}{124}, \frac{85}{124}\right)$$

has two major gaps of multiplicity 1, so it is represented by a closed path of length 2 in Γ_X. However, the cycle of gaps

$$\left(\frac{18}{124}, \frac{23}{124}\right) \mapsto \left(\frac{90}{124}, \frac{115}{124}\right) \mapsto \left(\frac{78}{124}, \frac{79}{124}\right)$$

has only one major gap of multiplicity 1, so it is represented by a closed path of length 1. This example also shows that the total weights around closed paths of Γ_X can be different.

We have already described possible gap lengths for rational rotation sets as the solution (3.4) of some linear equation. Using the above theorem and remark, we can provide a characterization of gap lengths in the irrational case (compare [2] where a similar result is sketched via an inductive argument):

Theorem 4.9 *A number $\ell > 0$ appears as the length of a major gap of an irrational rotation set for m_d if and only if it has the form*

$$\ell = \sum_{j=1}^{k} \frac{\alpha_j}{d^{\beta_j}}, \tag{4.3}$$

where $1 \le k \le d-1$ and $\{\alpha_j\}$, $\{\beta_j\}$ are sequences of positive integers which satisfy

$$\sum_{j=1}^{k} \alpha_j \le d-1 \quad \textit{and} \quad 1 = \beta_1 < \beta_2 < \cdots < \beta_k.$$

Proof First suppose X is an irrational rotation set for m_d with a major gap I of length ℓ and multiplicity n. If I is taut, then $\ell = n/d$, which clearly has the form (4.3). If I is loose, it is represented by a vertex in the gap graph Γ_X that belongs to a path $I = I_k \to I_{k-1} \to \cdots \to I_1$ where I_j has length ℓ_j and multiplicity n_j. For each $2 \le j \le k$, there is an integer $N_j \ge 1$ such that $I_{j-1} = g_X^{\circ N_j}(I_j)$. Hence $d^{N_j-1}(d\,\ell_j - n_j) = \ell_{j-1}$. Since I_1 is taut, $\ell_1 = n_1/d$. Using these relations, we can solve for ℓ_k to obtain

$$\ell = \ell_k = \frac{n_k}{d} + \frac{n_{k-1}}{d^{N_k+1}} + \cdots + \frac{n_1}{d^{N_2+\cdots+N_k+1}},$$

which has the form (4.3).

Conversely, suppose ℓ is a positive number of the form (4.3) for some choice of k, $\{\alpha_j\}$, and $\{\beta_j\}$. Consider the admissible graph Γ of degree d consisting of a single degree 0 vertex of weight $d - 1 - \sum_{j=1}^{k} \alpha_j$, together with a single maximal path of the form

$$I_k \to I_{k-1} \to \cdots \to I_1 \quad \text{with} \quad w(I_j) = \alpha_{k-j+1}.$$

Consider also the positive integers $N_j = \beta_{k-j+2} - \beta_{k-j+1}$ for $2 \le j \le k$. By Remark 4.7, there is a minimal irrational rotation set X, with Γ_X isomorphic to Γ, with N_j equal to the number of iterates it takes to map I_j to I_{j-1}. Then, the computation in the first part of the proof shows that the major gap I_k of X has length ℓ. □

Remark 4.10 When $d = 2$, the only possible values for the above integers are $k = \alpha_1 = 1$, confirming what we already know: An irrational rotation set under doubling has a single major gap of length $\frac{1}{2}$. For $d = 3$, there are more possibilities: If $k = 1$, then either $\alpha_1 = 1$ so $\ell = \frac{1}{3}$, or $\alpha_1 = 2$ so $\ell = \frac{2}{3}$. On the other hand, if $k = 2$, then necessarily $\alpha_1 = \alpha_2 = 1$ so $\ell = \frac{1}{3} + \frac{1}{3^{\beta_2}}$ for some $\beta_2 > 1$. Compare Theorem 4.31 for a more precise statement.

4.3 Dependence on Parameters

We begin with a preliminary observation on convergence of rotation sets:

Lemma 4.11 *Suppose $\{X_n\}$ is a sequence of rotation sets for m_d which converges in the Hausdorff metric to a compact set X. Then X is a rotation set with $\rho(X) = \lim_{n\to\infty} \rho(X_n)$. If every X_n is maximal, so is X.*

Proof Since each X_n is m_d-invariant and its complement $\mathbb{T} \smallsetminus X_n$ contains $d-1$ disjoint intervals of length $1/d$, the Hausdorff limit X must have the same properties. By Corollary 2.16, X is a rotation set. The family $\{g_n\}$ of the standard monotone maps of $\{X_n\}$ is equicontinuous since each g_n is piecewise affine with derivative bounded by d. After passing to a subsequence, we may assume that g_n converges uniformly to a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ which necessarily extends $m_d|_X$ (in fact, this shows that the entire sequence $\{g_n\}$ converges and its limit g is the standard monotone map of X). It follows from Theorem 1.11 that $\rho(X) = \rho(g) = \lim_{n\to\infty} \rho(g_n) = \lim_{n\to\infty} \rho(X_n)$.

The last assertion follows from Corollary 2.19: If the X_n are maximal, they all have $d-1$ major gaps of length $1/d$. This property persists under Hausdorff convergence, so X is maximal as well. □

Now, let $A \subset \mathbb{T} \times \Delta^{d-2}$ be the set of all pairs $\boldsymbol{a} = (\theta, \delta)$ subject to the restriction that if θ is rational of the form p/q in lowest terms, then $q\delta \in \mathbb{Z}^{d-1}$. For each $\boldsymbol{a} = (\theta, \delta) \in A$, let $X_{\boldsymbol{a}}$ denote the unique minimal rotation set for m_d with $\rho(X_{\boldsymbol{a}}) = \theta$ and $\delta(X_{\boldsymbol{a}}) = \delta$, given by the deployment theorem.

Theorem 4.12 *The assignment $\boldsymbol{a} \mapsto X_{\boldsymbol{a}}$ from A to the space of compact subsets of the circle (equipped with the Hausdorff metric) is lower semicontinuous.*

Proof Let $\boldsymbol{a}_n = (\theta_n, \delta_n) \in A$ tend to $\boldsymbol{a}_0 = (\theta_0, \delta_0) \in A$ as $n \to \infty$. Suppose $X_{\boldsymbol{a}_n}$ converges in the Hausdorff metric to a compact set $Y \subset \mathbb{T}$. We need to show that $X_{\boldsymbol{a}_0} \subset Y$. By Lemma 4.11, Y is a rotation set for m_d with $\rho(Y) = \theta_0$. Moreover, the proof of that lemma shows that the sequence $\{g_n\}$ of the standard monotone maps of $\{X_{\boldsymbol{a}_n}\}$ converges uniformly to the standard monotone map g of Y.

Let μ_n be the natural measure of $X_{\boldsymbol{a}_n}$, that is, the unique m_d-invariant probability measure supported on $X_{\boldsymbol{a}_n}$. After passing to a subsequence, we may assume that μ_n is weak* convergent to a probability measure μ. For every continuous test function $f : \mathbb{T} \to \mathbb{R}$,

$$\int_{\mathbb{T}} f\, d\mu = \lim_{n\to\infty} \int_{\mathbb{T}} f\, d\mu_n = \lim_{n\to\infty} \int_{\mathbb{T}} (f \circ g_n)\, d\mu_n = \lim_{n\to\infty} \int_{\mathbb{T}} (f \circ g)\, d\mu_n$$
$$= \int_{\mathbb{T}} (f \circ g)\, d\mu.$$

Here the first and forth equalities hold by the weak* convergence $\mu_n \to \mu$, the second equality follows from the g_n-invariance of μ_n, and the third equality holds since the uniform convergence $g_n \to g$ implies $\int(f \circ g_n)\, d\mu_n - \int(f \circ g)\, d\mu_n \to 0$ as $n \to \infty$. This proves that μ is g-invariant. For the rest of the argument, we distinguish two cases:

If $\rho(g) = \theta_0$ is irrational, it follows from the discussion in Sect. 1.5 that μ is the unique invariant probability measure supported on the Cantor attractor K of g. By Theorem 2.33, K is the unique minimal rotation set contained in Y. Let $\delta_n = (\delta_{n,1}, \dots, \delta_{n,d-1})$ and $\delta_0 = (\delta_{0,1}, \dots, \delta_{0,d-1})$. Since $\mu_n \to \mu$ and $\mu\{u_i\} = 0$ (recall that $u_i = i/(d-1)$ are the fixed points of m_d), it follows that

$$\mu[u_{i-1}, u_i) = \lim_{n\to\infty} \mu_n[u_{i-1}, u_i) = \lim_{n\to\infty} \delta_{n,i} = \delta_{0,i} \tag{4.4}$$

for every i, so $\delta(K) = \delta_0$. Since $\rho(K) = \rho(Y) = \theta_0$, the uniqueness part of Theorem 3.20 shows that $K = X_{\boldsymbol{a}_0}$. This proves $X_{\boldsymbol{a}_0} \subset Y$, as required.

In the case $\rho(g) = \theta_0$ is rational of the form p/q in lowest terms, we must modify the above argument. Let K be the support of μ. We know from Sect. 1.5 that K is a union of q-cycles of g. The Hausdorff convergence $\mathrm{supp}(\mu_n) = X_{\boldsymbol{a}_n} \to Y$ together with the weak* convergence $\mu_n \to \mu$ show that $K \subset Y$. It follows that K is a union $C_1 \cup \cdots \cup C_n$ of q-cycles in Y and therefore is a finite rotation set with $\rho(K) = p/q$. The measure μ is a convex combination $\sum_{i=1}^n \alpha_i\, \mu_{C_i}$ of the Dirac measures along the C_i, where every α_i is positive and $\sum_{i=1}^n \alpha_i = 1$. Since the limit (4.4) still holds for $1 \leq i \leq d-1$, we have

$$\sum_{i=1}^{n} \alpha_i\, \delta(C_i) = \delta_0 = \delta(X_{\boldsymbol{a}_0}).$$

By Lemma 3.18, this can happen only if $n = 1$ and $X_{\boldsymbol{a}_0} = C_1 = K$. Again, this implies $X_{\boldsymbol{a}_0} \subset Y$. □

Recall from Sect. 2.3 that a rotation set is exact if it is both maximal and minimal. Such rotation sets are necessarily irrational. Topologically, they are Cantor sets with $d-1$ major gaps of length $1/d$ (Theorem 2.37). The following lemma characterizes exactness in terms of the cumulative deployment vector:

Lemma 4.13 *Suppose X is a minimal rotation set for m_d with $\rho(X) = \theta$ and $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1})$. Then X is exact if and only if $\sigma_1, \dots, \sigma_{d-1}$ have disjoint full orbits under r_θ.*

Proof Recall from the proof of Theorem 3.20 that the lengths of the major gaps of X are the values $\nu\{\sigma_i\}$, where ν is the gap measure of X defined in (3.20). If the σ_i have disjoint orbits under r_θ, then θ is irrational and the definition of ν shows that $\nu\{\sigma_i\} = 1/d$ for each i. Conversely, a relation of the form $\sigma_i = \sigma_j - k\theta$ for $i \neq j$ and $k \geq 0$ would contribute a mass of $1/d^{k+1}$ to $\nu\{\sigma_i\}$, so the corresponding major gap would have length $\nu\{\sigma_i\} \geq 1/d + 1/d^{k+1} > 1/d$. □

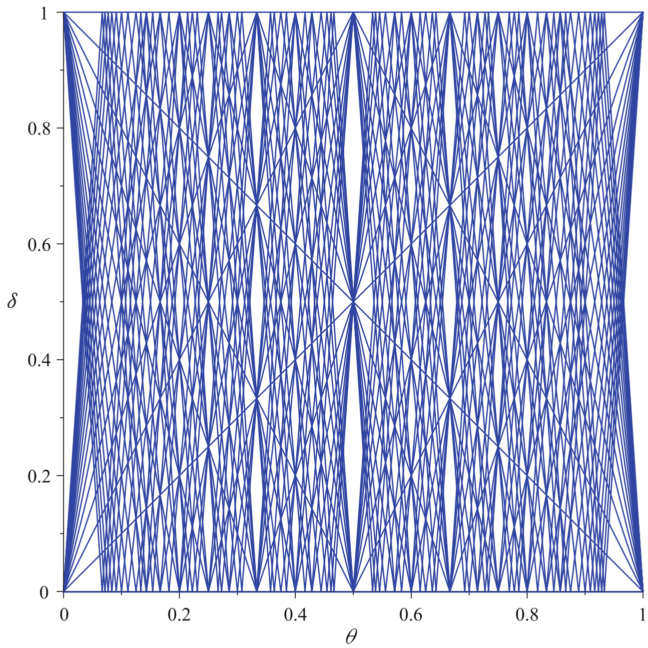

Fig. 4.2 An attempt to visualize the set of parameters $\boldsymbol{a} = (\theta, \delta)$ for which the minimal cubic rotation set $X_{\boldsymbol{a}}$ is exact. Here the deployment vector $(\delta, 1 - \delta)$ is identified with its first component $\delta \in [0, 1]$. The set of exact parameters is the complement of the union of the lines $\delta + n\theta = 0 \pmod{\mathbb{Z}}$ over all $n \in \mathbb{Z}$

Lemma 4.14 *There is a full-measure set of parameters* $\boldsymbol{a} \in A$ *for which* $X_{\boldsymbol{a}}$ *is exact.*

Figure 4.2 is an attempt to visualize this set when $d = 3$.

Proof Take any $\delta = (\delta_1, \ldots, \delta_{d-1})$ in the interior of Δ^{d-2}. Then the δ_i are positive, so the numbers $\sigma_i = \delta_1 + \cdots + \delta_i$ for $1 \le i \le d-1$ are distinct. There are countably many θ for which the orbits of the σ_i under r_θ collide. Let H_δ be the complement of this countable set in $\mathbb{T}$. Then, the union

$$H = \bigcup_{\delta} \big(H_\delta \times \{\delta\}\big)$$

has full-measure and for every $\boldsymbol{a} \in H$ the rotation set $X_{\boldsymbol{a}}$ is exact by Lemma 4.13. □

The following theorem determines when a minimal rotation set depends continuously on its rotation number and deployment vector. The possibility of such characterization was suggested to me by J. Milnor:

Theorem 4.15 *The assignment $\boldsymbol{a} \mapsto X_{\boldsymbol{a}}$ is continuous at $\boldsymbol{a}_0 \in A$ if and only if $X_{\boldsymbol{a}_0}$ is exact.*

In particular, this assignment is discontinuous at every $\boldsymbol{a}_0$ for which $X_{\boldsymbol{a}_0}$ is rational.

Proof First assume $\boldsymbol{a} \mapsto X_{\boldsymbol{a}}$ is continuous at $\boldsymbol{a}_0 \in A$. By Lemma 4.14 we can choose a sequence $\boldsymbol{a}_n \in A$ converging to $\boldsymbol{a}_0$ such that $X_{\boldsymbol{a}_n}$ is exact for every n. Since $X_{\boldsymbol{a}_n} \to X_{\boldsymbol{a}_0}$ and each $X_{\boldsymbol{a}_n}$ is maximal, Lemma 4.11 shows that $X_{\boldsymbol{a}_0}$ is maximal. As $X_{\boldsymbol{a}_0}$ is minimal by definition, we conclude that $X_{\boldsymbol{a}_0}$ is exact.

Conversely, suppose $X_{\boldsymbol{a}_0}$ is exact and take any sequence $\boldsymbol{a}_n \in A$ converging to $\boldsymbol{a}_0$. After passing to a subsequence, we may assume that $X_{\boldsymbol{a}_n}$ converges to a compact set Y in the Hausdorff metric. Theorem 4.12 shows that $Y \supset X_{\boldsymbol{a}_0}$. Since Y is a rotation set by Lemma 4.11, it follows from exactness that $Y = X_{\boldsymbol{a}_0}$. □

Example 4.16 Minimal rotation sets under the doubling map m_2 are parametrized by their rotation number. The assignment $\theta \mapsto X_\theta$ is continuous at every irrational θ since such rotation sets are exact (Corollary 2.38). To get a feel for the nature of discontinuity at rational θ, consider the n-cycle

$$X_{1/n} : \frac{1}{2^n - 1} \mapsto \frac{2}{2^n - 1} \mapsto \cdots \mapsto \frac{2^{n-1}}{2^n - 1}.$$

As $n \to \infty$, $X_{1/n}$ does not converge to $X_0 = \{0\}$, but to the maximal rotation set $\{0\} \cup \{1/2^n\}_{n \geq 1}$.

Remark 4.17 Milnor has pointed out to me that one may also study the map from the union $\mathcal{R}_d$ of all rotation sets for m_d to the set A defined as follows: The forward m_d-orbit of every $t \in \mathcal{R}_d$ eventually lands in a well-defined minimal rotation set X_t (Theorems 2.3 and 2.33), so we can assign to t the parameter $(\rho(X_t), \delta(X_t)) \in A$. This map is surjective and clearly discontinuous since $\mathcal{R}_d$ is compact (see below) but A is not.

Let $\mathcal{C}_d \subset \mathcal{R}_d$ be the union of all cycles, and $\mathcal{E}_d \subset \mathcal{R}_d$ be the union of all exact rotation sets.

Theorem 4.18

(i) *$\mathcal{R}_d$ is compact.*
(ii) *$\mathcal{C}_d$ and $\mathcal{E}_d$ are disjoint and non-compact, with $\overline{\mathcal{E}_d} \subset \overline{\mathcal{C}_d}$.*
(iii) *$\overline{\mathcal{E}_d}$ is a Cantor set.*

Proof Let $t_n \in \mathcal{R}_d$ and $t_n \to t$. Take a rotation set X_n containing t_n. After passing to a subsequence, we may assume that X_n converges to a compact set X, which is a rotation set by Lemma 4.11. Hence $t \in X \subset \mathcal{R}_d$. This proves (i).

For (ii), first note that $\mathcal{C}_d$ and $\mathcal{E}_d$ are disjoint since rational rotation sets are never exact. To see $\mathcal{E}_d$ is non-compact, take any sequence $\{X_n\}$ of exact rotation sets with

$\rho(X_n)$ tending to some rational number p/q (for example, let $\boldsymbol{a}_n = (\theta_n, \delta_n)$ where θ_n are irrational tending to p/q and δ_n have rational components, and consider the rotation sets $X_{\boldsymbol{a}_n}$ which are exact by Lemma 4.13). Some subsequence of $\{X_n\}$ converges to a compact set X which, by Lemma 4.11, is a (maximal) rotation set with $\rho(X) = p/q$. Evidently $X \subset \overline{\mathcal{E}_d}$. However, $X \cap \mathcal{E}_d = \emptyset$ since the forward orbit of any $t \in X$ eventually hits a cycle, so t cannot belong to an exact rotation set.

Now suppose X is exact and choose cycles C_n such that $\rho(C_n) \to \rho(X)$ and $\delta(C_n) \to \delta(X)$. Theorem 4.15 then shows that $C_n \to X$, so $X \subset \overline{\mathcal{C}_d}$. This proves the inclusion $\overline{\mathcal{E}_d} \subset \overline{\mathcal{C}_d}$ and also shows that $\mathcal{C}_d$ is non-compact.

For (iii), simply note that $\overline{\mathcal{E}_d}$ has no isolated point since it is the closure of a union of Cantor sets, and it is totally disconnected since it is contained in the measure zero set $\mathcal{R}_d$ (Theorem 2.5). □

Question 4.19 Does the equality $\overline{\mathcal{E}_d} = \overline{\mathcal{C}_d} = \mathcal{R}_d$ hold?

The answer is affirmative when $d = 2$ (see Theorem 4.28) and is likely to be so for all d. Indeed, the following sharper statement seems plausible: Given any maximal rotation set X for m_d there is a sequence $\{X_n\}$ of exact rotation sets for m_d such that $X_n \to X$ in the Hausdorff metric.

4.4 The Leading Angle

A minimal rotation set X is uniquely determined by any of its elements: Simply iterate any angle in X under m_d and take the closure of the resulting orbit. This section will give a recipe for computing a canonical angle in every minimal rotation set from the knowledge of its rotation number and deployment vector. The particular choice of this angle is motivated by polynomial dynamics and plays a role in the representation of rotation sets in both dynamical and parameter planes, as outlined in the next chapter.

Definition 4.20 Let X be a minimal rotation set for m_d and $I_0 = (\omega', \omega)$ be its major gap containing the fixed point 0. We call the endpoint ω of I_0 the ***leading angle*** of X.

Thus, ω is the first point of X that is met when we start at 0 and go counter-clockwise around the circle. The closed intervals $[\omega', \omega]$ and $[m_d(\omega'), m_d(\omega)]$ can be described as the fibers $\varphi^{-1}(0)$ and $\varphi^{-1}(\theta)$ of the canonical semiconjugacy φ of X, where $\theta = \rho(X)$. For convenience we identify ω', ω and their images with the representatives which satisfy the order relations $-1 < \omega' < 0 < \omega < m_d(\omega') \leq m_d(\omega) < 1$ (see Fig. 4.3).

Suppose $\rho(X) = \theta \neq 0$ and $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$. Let ν be the gap measure of X as defined in (3.20) and $N_0 \geq 0$ be the number of indices $1 \leq j < d-1$ for which $\sigma_j = 0$.

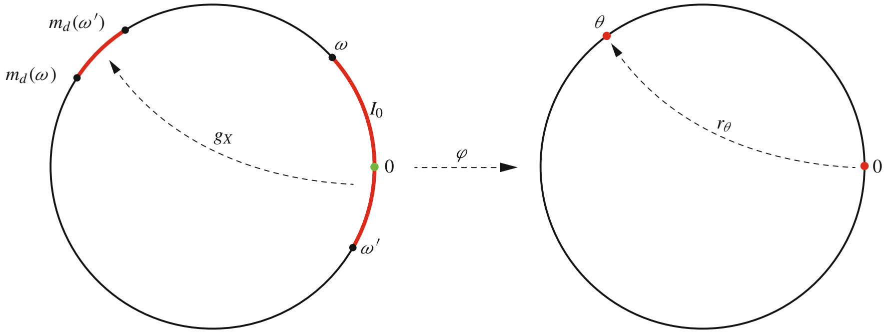

Fig. 4.3 The major gap $I_0 = (\omega', \omega)$ containing the fixed point $u_0 = 0$ and the leading angle ω of a minimal rotation set X. The closed intervals $[\omega', \omega]$ and $[m_d(\omega'), m_d(\omega)]$ are the fibers $\varphi^{-1}(0)$ and $\varphi^{-1}(\theta)$, respectively. Here φ is the canonical semiconjugacy of X and $\theta = \rho(X)$

Theorem 4.21 *The leading angle of X is given by*

$$\omega = \frac{1}{d-1}\, \nu(0, \theta] + \frac{N_0}{d-1} \tag{4.5}$$

$$= \frac{1}{d-1} \sum_{i=1}^{d-1} \sum_{0<\sigma_i - k\theta \le \theta} \frac{1}{d^{k+1}} + \frac{N_0}{d-1}$$

This formula gives an explicit algorithm for computing the base d expansion of the angle $(d-1)\omega$ (compare Lemma 4.24 below).

Proof As pointed out in the beginning of Sect. 3.2, if $N_1 \geq 1$ is the number of indices $1 \leq j \leq d-1$ for which $\sigma_j = 1$, then $n_0 = N_0 + N_1$ is the multiplicity of $I_0 = (\omega', \omega)$ as a major gap of X. Since

$$\omega' < \frac{-N_1+1}{d-1} \leq \frac{-N_1+1}{d} \leq 0 \leq \frac{N_0}{d} \leq \frac{N_0}{d-1} < \omega,$$

the gap I_0 already contains the n_0 points j/d for $j = -N_1+1, \ldots, N_0$. By Lemma 2.8, there could be no more preimages of 0 in I_0. In particular, $\omega < (N_0+1)/d$, which proves N_0 is the integer part of $d\,\omega$. Since $m_d(\omega) = d\omega \pmod{\mathbb{Z}}$, it follows that

$$m_d(\omega) = d\omega - N_0. \tag{4.6}$$

Now let $\varphi : \mathbb{T} \to \mathbb{T}$ be the canonical semiconjugacy of X and λ be Lebesgue measure on the circle. Since $\varphi_* \lambda = \nu$, we have

$$m_d(\omega) - \omega = \lambda(\omega, m_d(\omega)] = \lambda(\varphi^{-1}(0, \theta]) = \nu(0, \theta]. \tag{4.7}$$

The result follows by eliminating $m_d(\omega)$ from (4.6) and (4.7). □

Remark 4.22 A similar argument gives the following formulas for the other angles involved in Fig. 4.3:

$$m_d(\omega) = \frac{d}{d-1}\,\nu(0,\theta] + \frac{N_0}{d-1}$$

$$\omega' = \frac{1}{d-1}\,\nu[0,\theta) - \frac{N_1}{d-1}$$

$$m_d(\omega') = \frac{d}{d-1}\,\nu[0,\theta) - \frac{N_1}{d-1}.$$

We point out that the above formulas for ω, ω' can be used to compute the endpoint angles of any major gap of X. For example, if I_i is the major gap of X containing the fixed point $u_i = i/(d-1) \pmod{\mathbb{Z}}$, consider the rotation set $X - u_i$ whose deployment vector is obtained by a cyclic permutation of the components of $\delta(X)$ (see Sect. 3.1), apply the above formulas to compute the endpoints of the major gap of $X - u_i$ containing 0, and rotate them back by r_{u_i} to find the endpoints of I_i.

4.5 Rotation Sets Under Doubling

In this section we focus on the basic case $d = 2$. Theorems 3.7 and 3.20 show that for every $0 < \theta < 1$ there is a unique minimal rotation set X_θ under doubling with rotation number θ, which is a periodic orbit if θ is rational and a Cantor set if θ is irrational. The structure of X_θ in either case can be explicitly described as follows.

Let us first consider the rational case. For every fraction p/q in lowest terms, $X_{p/q}$ is a q-cycle of the form $\{t_1, \ldots, t_q\}$, where as usual the points are labeled in positive cyclic order and $0 \in (t_q, t_1)$, and the subscripts are taken modulo q. Let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$. We can compute the ℓ_j explicitly using the general formulas we developed in Sect. 3.2. Recall that $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$ and $\boldsymbol{n} = (n_1, \ldots, n_q)$ are the gap length and gap multiplicity vectors of $X_{p/q}$, respectively. Since $I_q = I_0$ is the unique major gap of $X_{p/q}$ of multiplicity 1, we have $n_q = 1$ and $n_j = 0$ for $1 \leq j < q$. According to (3.4),

$$\boldsymbol{\ell} = \frac{1}{2^q - 1} \sum_{i=0}^{q-1} 2^{q-i-1}\, T^{\circ i}(\boldsymbol{n}),$$

where $T(x_1, x_2, \ldots, x_q) = (x_{1+p}, x_{2+p}, \ldots, x_{q+p})$. Since $T^{\circ i}(\boldsymbol{n}) = (n_{1+ip}, n_{2+ip}, \ldots, n_{q+ip})$, it follows that $\ell_j = 2^{q-i-1}/(2^q - 1)$, where $0 \leq i \leq q-1$ is the unique solution of $j + ip = 0 \pmod{q}$. If $1 \le p^* \le q-1$ is the multiplicative inverse of p modulo q, it follows that $q - i = jp^* \pmod{q}$. Thus,

$$\ell_j = \frac{2^{\langle jp^* \rangle - 1}}{2^q - 1}, \quad \text{where } 1 \le \langle jp^* \rangle \le q \text{ is the unique representative of } jp^* \pmod{q}.$$

In particular, I_p and $I_q = I_0$ are the shortest and longest gaps of lengths

$$\ell_p = \frac{1}{2^q - 1} \quad \text{and} \quad \ell_q = \frac{2^{q-1}}{2^q - 1}.$$

By (4.5), the leading angle $\omega = t_1$ is given by

$$\omega = \nu\Big(0, \frac{p}{q}\Big] = \ell_1 + \cdots + \ell_p = \sum_{j=1}^{p} \frac{2^{\langle jp^* \rangle - 1}}{2^q - 1}. \tag{4.8}$$

Example 4.23 Consider the 7-cycle $X_{\frac{3}{7}} = \{t_1, t_2, \ldots, t_7\}$ under doubling. Here $q = 7$, $p = 3$ and $p^* = 5$. By the above computation, the gap lengths are

$$\ell_1 = \frac{2^{\langle 5 \rangle - 1}}{127} = \frac{16}{127} \qquad \ell_2 = \frac{2^{\langle 10 \rangle - 1}}{127} = \frac{4}{127}$$

$$\ell_3 = \frac{2^{\langle 15 \rangle - 1}}{127} = \frac{1}{127} \qquad \ell_4 = \frac{2^{\langle 20 \rangle - 1}}{127} = \frac{32}{127}$$

$$\ell_5 = \frac{2^{\langle 25 \rangle - 1}}{127} = \frac{8}{127} \qquad \ell_6 = \frac{2^{\langle 30 \rangle - 1}}{127} = \frac{2}{127}$$

$$\ell_7 = \frac{2^{\langle 35 \rangle - 1}}{127} = \frac{64}{127}.$$

(Alternatively, we could start with the minimal gap length $\ell_3 = \frac{1}{127}$ and keep doubling it until all ℓ_j are found.) The leading angle t_1 is $\ell_1 + \ell_2 + \ell_3 = \frac{21}{127}$, which, in view of the relation $t_{j+3} = 2t_j \pmod{\mathbb{Z}}$, leads to the other angles t_j:

$$t_4 = \frac{42}{127}, \quad t_7 = \frac{84}{127}, \quad t_3 = \frac{41}{127}, \quad t_6 = \frac{82}{127}, \quad t_2 = \frac{37}{127}, \quad t_5 = \frac{74}{127}.$$

Thus,

$$X_{\frac{3}{7}} = \left\{ \frac{21}{127}, \frac{37}{127}, \frac{41}{127}, \frac{42}{127}, \frac{74}{127}, \frac{82}{127}, \frac{84}{127} \right\}.$$

When θ is irrational, the unique major gap I_0 of X_θ is taut, so it has length $\frac{1}{2}$. For every $n \ge 1$ there is a unique gap of length $\frac{1}{2^{n+1}}$ which maps to I_0 after n iterates. The rotation number θ determines the cyclic order of these gaps around the circle.

Now consider the leading angle $\omega(\theta)$ of X_θ as defined in the previous section. The cumulative deployment vector of X_θ is the trivial vector $(\sigma_1) = (1)$. Hence the formula (4.5) takes the form

$$\omega(\theta) = \nu(0, \theta] = \sum_{0 < -k\theta \leq \theta} \frac{1}{2^{k+1}}. \tag{4.9}$$

If θ is rational of the form p/q in lowest terms, this sum splits into p geometric series, each taken over all $k \geq 0$ for which $-kp/q = j/q \pmod{\mathbb{Z}}$ for a given $1 \leq j \leq p$. These p series in effect correspond to the p terms of the sum (4.8). Table 4.1 illustrates the computation of $\omega(p/q)$ using both formulas for all reduces fractions with denominators up to 8.

Equation (4.9) can be interpreted as a formula for the binary expansion of the leading angle $\omega(\theta)$. Consider the intervals

$$T_0 = [0, 1-\theta) \qquad T_1 = [1-\theta, 1)$$

on the circle. The binary expansion of $\omega(\theta)$ is obtained using the itinerary of the orbit of 0 under the rotation r_θ relative to the partition $T_0 \cup T_1$:

Lemma 4.24 *The binary expansion*

$$\omega(\theta) = 0.b_0 b_1 b_2 \cdots \qquad (\text{base } 2)$$

is determined by the condition $k\theta \in T_{b_k}$ for all $k \geq 0$.

Note in particular that always $b_0 = 0$.

Proof By (4.5), $b_k = 1$ if and only if $-k\theta \in (0, \theta]$, which is equivalent to $k\theta \in [1-\theta, 1)$. □

We will see a dynamical interpretation of this lemma in the next chapter (see Sect. 5.3).

The following lemma provides yet another formula for the leading angle ω which already appears in Douady-Hubbard's work on the dynamics of the quadratic family and the Mandelbrot set. Although this formula is not computationally as efficient as (4.9), it greatly facilitates the study of the dependence of $\omega(\theta)$ on the rotation number θ:

Lemma 4.25 *The leading angle of X_θ satisfies*

$$\omega(\theta) = \frac{1}{2} \sum_{0 < p/q \leq \theta} \frac{1}{2^q - 1}, \tag{4.10}$$

where the fractions p/q in the sum are all reduced.

Table 4.1 The leading angle $\omega(p/q)$ of the cycle $X_{p/q}$ under the doubling map, for denominators $2 \leq q \leq 8$

p/q	Formula (4.8)	Formula (4.9)	$\omega(p/q)$
$\frac{1}{2}$	$\frac{2^0}{2^2-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{2j}}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{2^0}{2^3-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{3j}}$	$\frac{1}{7}$
$\frac{2}{3}$	$\frac{2^1+2^0}{2^3-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{3j-1}} + \frac{1}{2^{3j}}\right)$	$\frac{3}{7}$
$\frac{1}{4}$	$\frac{2^0}{2^4-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{4j}}$	$\frac{1}{15}$
$\frac{3}{4}$	$\frac{2^2+2^1+2^0}{2^4-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{4j-2}} + \frac{1}{2^{4j-1}} + \frac{1}{2^{4j}}\right)$	$\frac{7}{15}$
$\frac{1}{5}$	$\frac{2^0}{2^5-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{5j}}$	$\frac{1}{31}$
$\frac{2}{5}$	$\frac{2^2+2^0}{2^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{5j-2}} + \frac{1}{2^{5j}}\right)$	$\frac{5}{31}$
$\frac{3}{5}$	$\frac{2^1+2^3+2^0}{2^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{5j-1}} + \frac{1}{2^{5j-3}} + \frac{1}{2^{5j}}\right)$	$\frac{11}{31}$
$\frac{4}{5}$	$\frac{2^3+2^2+2^1+2^0}{2^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{5j-3}} + \frac{1}{2^{5j-2}} + \frac{1}{2^{5j-1}} + \frac{1}{2^{5j}}\right)$	$\frac{15}{31}$
$\frac{1}{6}$	$\frac{2^0}{2^6-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{6j}}$	$\frac{1}{63}$
$\frac{5}{6}$	$\frac{2^4+2^3+2^2+2^1+2^0}{2^6-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{6j-4}} + \frac{1}{2^{6j-3}} + \frac{1}{2^{6j-2}} + \frac{1}{2^{6j-1}} + \frac{1}{2^{6j}}\right)$	$\frac{31}{63}$
$\frac{1}{7}$	$\frac{2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{7j}}$	$\frac{1}{127}$
$\frac{2}{7}$	$\frac{2^3+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-3}} + \frac{1}{2^{7j}}\right)$	$\frac{9}{127}$
$\frac{3}{7}$	$\frac{2^4+2^2+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-4}} + \frac{1}{2^{7j-2}} + \frac{1}{2^{7j}}\right)$	$\frac{21}{127}$
$\frac{4}{7}$	$\frac{2^1+2^3+2^5+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-1}} + \frac{1}{2^{7j-3}} + \frac{1}{2^{7j-5}} + \frac{1}{2^{7j}}\right)$	$\frac{43}{127}$
$\frac{5}{7}$	$\frac{2^2+2^5+2^1+2^4+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-2}} + \frac{1}{2^{7j-5}} + \frac{1}{2^{7j-1}} + \frac{1}{2^{7j-4}} + \frac{1}{2^{7j}}\right)$	$\frac{55}{127}$
$\frac{6}{7}$	$\frac{2^5+2^4+2^3+2^2+2^1+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-5}} + \frac{1}{2^{7j-4}} + \frac{1}{2^{7j-3}} + \frac{1}{2^{7j-2}} + \frac{1}{2^{7j-1}} + \frac{1}{2^{7j}}\right)$	$\frac{63}{127}$
$\frac{1}{8}$	$\frac{2^0}{2^8-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{8j}}$	$\frac{1}{255}$
$\frac{3}{8}$	$\frac{2^2+2^5+2^0}{2^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{8j-2}} + \frac{1}{2^{8j-5}} + \frac{1}{2^{8j}}\right)$	$\frac{37}{255}$
$\frac{5}{8}$	$\frac{2^4+2^1+2^6+2^3+2^0}{2^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{8j-4}} + \frac{1}{2^{8j-1}} + \frac{1}{2^{8j-6}} + \frac{1}{2^{8j-3}} + \frac{1}{2^{8j}}\right)$	$\frac{91}{255}$
$\frac{7}{8}$	$\frac{2^6+2^5+2^4+2^3+2^2+2^1+2^0}{2^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{8j-6}} + \frac{1}{2^{8j-5}} + \frac{1}{2^{8j-4}} + \frac{1}{2^{8j-3}} + \frac{1}{2^{8j-2}} + \frac{1}{2^{8j-1}} + \frac{1}{2^{8j}}\right)$	$\frac{127}{255}$

Proof For each integer $m \geq 1$, let k_m be the largest positive integer for which $m > k_m\theta$. Then $0 < m - k_m\theta \leq \theta$, so $-k_m\theta \pmod{\mathbb{Z}}$ is in the interval $(0, \theta]$. Conversely, if $-k\theta \pmod{\mathbb{Z}}$ belongs to $(0, \theta]$, there exists an integer $m \geq 1$ such that $0 < m - k\theta \leq \theta$, so $k\theta < m \leq (k+1)\theta$, which shows $k = k_m$. Thus, by (4.9),

$$\omega(\theta) = \sum_{m=1}^{\infty} \frac{1}{2^{k_m+1}}.$$

To relate this sum to (4.10), we use an idea of Douady (compare [12] and [7]). Assign to each pair (n, m) of positive integers the weight $W(n, m) = 1/2^n$. Let W be the total weight of all (n, m) for which $m/n \leq \theta$. On the one hand,

$$W = \sum_{m=1}^{\infty} \sum_{n=k_m+1}^{\infty} W(n, m) = \sum_{m=1}^{\infty} \sum_{n=k_m+1}^{\infty} \frac{1}{2^n} = \sum_{m=1}^{\infty} \frac{1}{2^{k_m}} = 2\omega(\theta).$$

On the other hand, computing the total weight along lines with rational slope gives

$$W = \sum_{0<p/q\leq\theta} \sum_{j=1}^{\infty} W(jq, jp) = \sum_{0<p/q\leq\theta} \sum_{j=1}^{\infty} \frac{1}{2^{jq}} = \sum_{0<p/q\leq\theta} \frac{1}{2^q - 1},$$

and the result follows. □

Corollary 4.26 *The leading angle $\omega(\theta)$ of X_θ is a strictly increasing function of $0 < \theta < 1$, with $\omega(0^+) = 0$ and $\omega(1^-) = \frac{1}{2}$. Moreover,*

(i) ω has a jump discontinuity at every rational value of θ. In fact, if $\theta = p/q$ in lowest terms, then

$$\omega(p/q) = \omega(p/q^+) = \omega(p/q^-) + \frac{1}{2(2^q - 1)}.$$

(ii) ω is continuous at every irrational value of θ.

(iii) For every $0 < \theta < 1$,

$$\omega(\theta^+) + \omega((1-\theta)^-) = \frac{1}{2}.$$

Compare Fig. 4.4. There is a well-known connection between the function $\theta \mapsto \omega(\theta)$ and the quadratic family $\{z \mapsto z^2 + c\}_{c\in\mathbb{C}}$ (see Sect. 5.3).

Proof Only (iii) needs a comment, as other properties follow at once from (4.10). For $0 < \theta < 1$,

$$\begin{aligned}\omega(\theta^+) = \omega(\theta) &= \sum_{0<p/q\leq\theta} \frac{1}{2^q - 1} \\ &= \frac{1}{2} - \sum_{\theta<p/q<1} \frac{1}{2^q - 1}\end{aligned}$$

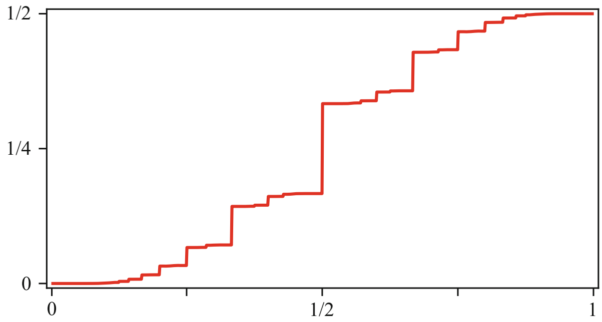

Fig. 4.4 The graph of the leading angle $\omega(\theta)$ of the minimal rotation set X_θ under doubling, as a function of the rotation number θ. Notice the jump discontinuities at every rational value of θ and the symmetry of the graph around the center point $(\frac{1}{2}, \frac{1}{4})$

$$= \frac{1}{2} - \sum_{0<(q-p)/q<1-\theta} \frac{1}{2^q - 1}$$

$$= \frac{1}{2} - \omega((1-\theta)^-),$$

as required. □

Remark 4.27 It follows from Corollary 4.26 that the map $\theta \mapsto \omega(\theta)$ has a left-inverse $\omega \mapsto \theta(\omega)$ which maps $(0, \frac{1}{2})$ monotonically onto $(0, 1)$ and has non-degenerate fibers over every rational value of θ. It is not hard to check that $\theta(\omega)$ is the rotation number of the rotation set consisting of all points in $\mathbb{T}$ whose forward orbit under doubling is contained in the closed half-circle $[\omega, \omega + \frac{1}{2}]$ (see Theorem 2.15).

The behavior of $\theta \mapsto \omega(\theta)$ makes it possible to answer Question 4.19 when $d = 2$:

Theorem 4.28 *For every maximal rotation set X under doubling there is a sequence $\{X_{\theta_n}\}$ of exact rotation sets such that $X_{\theta_n} \to X$ in the Hausdorff metric. In particular, $\overline{\mathcal{E}_2} = \overline{\mathcal{C}_2} = \mathcal{R}_2$.*

Proof If $\rho(X)$ is irrational, then X itself is exact (Corollary 2.38) and there is nothing to prove. If $\rho(X)$ is rational of the form p/q in lowest terms, then X contains the cycle $X_{p/q}$ with the major gap

$$\left(\omega(p/q) - \frac{2^{q-1}}{2^q - 1}, \omega(p/q)\right) = \left(\omega(p/q^-) - \frac{1}{2}, \omega(p/q^+)\right).$$

Corollary 2.31 then shows that the major gap of X is one of the intervals

$$I = \Big(\omega(p/q^+) - \frac{1}{2}, \omega(p/q^+)\Big) \quad \text{or} \quad J = \Big(\omega(p/q^-) - \frac{1}{2}, \omega(p/q^-)\Big).$$

Suppose the major gap of X is I. Take a decreasing sequence $\{\theta_n\}$ of irrational numbers with $\theta_n \to p/q$. The rotation sets X_{θ_n} are exact and their leading angles $\omega(\theta_n)$ tend to $\omega(p/q^+)$. By Lemma 4.11, any Hausdorff limit of $\{X_{\theta_n}\}$ is a maximal rotation set with rotation number p/q and major gap I, so it must be X. It follows that $X_{\theta_n} \to X$. If the major gap of X is J, take an increasing sequence $\{\theta_n\}$ of irrationals with $\theta_n \to p/q$, which now has the property $\omega(\theta_n) \to \omega(p/q^-)$, and conclude similarly that $X_{\theta_n} \to X$. □

4.6 Rotation Sets Under Tripling

We now consider the case $d = 3$. Theorems 3.7 and 3.20 show that for every $0 < \theta < 1$ and every $0 \leq \delta \leq 1$ there is a unique minimal rotation set $X_{\theta,\delta}$ under tripling with rotation number θ and deployment vector $(\delta, 1 - \delta)$, which is a periodic orbit if θ is rational and a Cantor set if θ is irrational. Notice that changing δ to $1 - \delta$ amounts to rotating $X_{\theta,\delta}$ by 180°:

$$X_{\theta,1-\delta} = X_{\theta,\delta} + \frac{1}{2}.$$

This means that to study the structure of $X_{\theta,\delta}$ we may restrict δ to either of the intervals $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$.

First suppose $\theta = p/q$ in lowest terms, so δ is of the form s/q for some $0 \leq s \leq q$. Then $X_{p/q,s/q}$ is a q-cycle of the form $\{t_1, \dots, t_q\}$, where the points are labeled in positive cyclic order and $0 \in (t_q, t_1)$. As before, let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$, and let $\boldsymbol{\ell} = (\ell_1, \dots, \ell_q)$ and $\boldsymbol{n} = (n_1, \dots, n_q)$ be the gap length and gap multiplicity vectors of $X_{p/q,s/q}$, respectively. The two major gaps of $X_{p/q,s/q}$ are $I_q = I_0$ and I_s containing the fixed points 0 and $\frac{1}{2}$ of m_3, respectively. We distinguish three cases:

- *Case 1.* $s = q$. The cumulative deployment vector in this case is $(\sigma_1, \sigma_2) = (1, 1)$. Evidently $I_q = I_s$ is the unique major gap of multiplicity 2, so $n_q = 2$ and $n_j = 0$ for $1 \leq j < q$. This case turns out to be completely similar to the doubling case treated in the previous section. A similar computation gives

$$\ell_j = \frac{2 \cdot 3^{\langle jp^* \rangle - 1}}{3^q - 1}, \qquad \text{where } 1 \leq \langle jp^* \rangle \leq q \text{ is the unique representative of } jp^* \pmod q.$$

In particular, I_p and I_q are the shortest and longest gaps of lengths

$$\ell_p = \frac{2}{3^q - 1} \quad \text{and} \quad \ell_q = \frac{2 \cdot 3^{q-1}}{3^q - 1}.$$

By (4.5) the leading angle $\omega = t_1$ is given by

$$\omega = \frac{1}{2}\,\nu\Big(0, \frac{p}{q}\Big] = \frac{1}{2}(\ell_1 + \cdots + \ell_p) = \sum_{j=1}^{p} \frac{3^{\langle jp^* \rangle - 1}}{3^q - 1} \tag{4.11}$$

which is analogous to the formula (4.8) for the doubling case.

- *Case 2.* $s = 0$. The cumulative deployment vector in this case is $(\sigma_1, \sigma_2) = (0, 1)$. This is similar to *Case 1* and can be reduced to it by a 180° rotation. It easily follows that the gap lengths ℓ_j are given by the same formulas as above. However, the leading angle $\omega = t_1$ is

$$\omega = \frac{1}{2}\,\nu\Big(0, \frac{p}{q}\Big] + \frac{1}{2} = \frac{1}{2}(\ell_1 + \cdots + \ell_p) + \frac{1}{2} = \sum_{j=1}^{p} \frac{3^{\langle jp^* \rangle - 1}}{3^q - 1} + \frac{1}{2}.$$

- *Case 3.* $0 < s < q$. This time I_q and I_s are distinct major gaps of multiplicity 1, so $n_q = n_s = 1$ and $n_j = 0$ for $j \neq q, s$. In this case,

$$\ell_j = \frac{1}{3^q - 1}\Big[3^{\langle jp^* \rangle - 1} + 3^{\langle (j-s)p^* \rangle - 1}\Big].$$

Note that there are now two competing candidates I_p, I_{s+p} for the shortest gap and similarly two candidates I_q, I_s for the longest gap. The choice depends on the relative size of $\langle sp^* \rangle$ and $\langle -sp^* \rangle$. In fact, the above formula shows that if $\langle sp^* \rangle < \langle -sp^* \rangle$, then the minimum and maximum gap lengths are

$$\ell_{s+p} = \frac{3^{\langle sp^* \rangle} + 1}{3^q - 1} \quad \text{and} \quad \ell_q = \frac{3^{q-1} + 3^{\langle -sp^* \rangle - 1}}{3^q - 1},$$

while if $\langle sp^* \rangle > \langle -sp^* \rangle$, the minimum and maximum gap lengths are

$$\ell_p = \frac{3^{\langle -sp^* \rangle} + 1}{3^q - 1} \quad \text{and} \quad \ell_s = \frac{3^{q-1} + 3^{\langle sp^* \rangle - 1}}{3^q - 1}.$$

If $\langle sp^* \rangle = \langle -sp^* \rangle = q/2$ (so q is even), the minimum and maximum gap lengths are

$$\ell_p = \ell_{s+p} = \frac{1}{3^{q/2} - 1} \quad \text{and} \quad \ell_q = \ell_s = \frac{3^{q/2-1}}{3^{q/2} - 1}.$$

Whatever the case, the leading angle $\omega = t_1$ can still be computed as the sum $\omega = (\frac{1}{2})(\ell_1 + \cdots + \ell_p)$ which, in view of the relation $t_{j+p} = 3t_j \pmod{\mathbb{Z}}$, would determine every angle t_j.

Example 4.29 Consider the 5-cycle $X_{\frac{3}{5},\frac{5}{5}} = \{t_1, \ldots, t_5\}$ under tripling. Here $q = 5$, $p = 3$, $p^* = 2$ and $s = 5$. By the computation in *Case 1*, the gap lengths are

$$\ell_1 = \frac{2 \cdot 3^{\langle 2 \rangle - 1}}{242} = \frac{6}{242} \quad \ell_2 = \frac{2 \cdot 3^{\langle 4 \rangle - 1}}{242} = \frac{54}{242} \quad \ell_3 = \frac{2 \cdot 3^{\langle 6 \rangle - 1}}{242} = \frac{2}{242}$$
$$\ell_4 = \frac{2 \cdot 3^{\langle 8 \rangle - 1}}{242} = \frac{18}{242} \quad \ell_5 = \frac{2 \cdot 3^{\langle 10 \rangle - 1}}{242} = \frac{162}{242}.$$

The leading angle t_1 is $(\frac{1}{2})(\ell_1 + \ell_2 + \ell_3) = \frac{31}{242}$. In view of $t_{j+3} = 3t_j \pmod{\mathbb{Z}}$, we obtain

$$t_4 = \frac{93}{242}, \quad t_2 = \frac{37}{242}, \quad t_5 = \frac{111}{242}, \quad t_3 = \frac{91}{242}.$$

Thus,

$$X_{\frac{3}{5},\frac{5}{5}} = \left\{ \frac{31}{242}, \frac{37}{242}, \frac{91}{242}, \frac{93}{242}, \frac{111}{242} \right\}.$$

Example 4.30 Now let us determine the 5-cycle $X_{\frac{3}{5},\frac{2}{5}} = \{t_1, \ldots, t_5\}$ under tripling. Here $q = 5$, $p = 3$, $p^* = 2$ and $s = 2$. By the computation in *Case 3*, the gap lengths are

$$\ell_1 = \frac{3^{\langle 2 \rangle - 1} + 3^{\langle -2 \rangle - 1}}{242} = \frac{12}{242} \qquad \ell_2 = \frac{3^{\langle 4 \rangle - 1} + 3^{\langle 0 \rangle - 1}}{242} = \frac{108}{242}$$
$$\ell_3 = \frac{3^{\langle 6 \rangle - 1} + 3^{\langle 2 \rangle - 1}}{242} = \frac{4}{242} \qquad \ell_4 = \frac{3^{\langle 8 \rangle - 1} + 3^{\langle 4 \rangle - 1}}{242} = \frac{36}{242}$$
$$\ell_5 = \frac{3^{\langle 10 \rangle - 1} + 3^{\langle 6 \rangle - 1}}{242} = \frac{82}{242}.$$

The leading angle t_1 is $(\frac{1}{2})(\ell_1 + \ell_2 + \ell_3) = \frac{62}{242}$, which gives

$$t_4 = \frac{186}{242}, \quad t_2 = \frac{74}{242}, \quad t_5 = \frac{222}{242}, \quad t_3 = \frac{182}{242}.$$

Thus,

$$X_{\frac{3}{5},\frac{2}{5}} = \left\{ \frac{62}{242}, \frac{74}{242}, \frac{182}{242}, \frac{186}{242}, \frac{222}{242} \right\}.$$

Unlike the case of the doubling map, irrational rotation numbers under tripling can have a wider variety of gap lengths depending on their deployment vector:

Theorem 4.31 *Suppose θ is irrational.*

(i) If $\delta = 0$ or 1, then $X_{\theta,\delta}$ has a single major gap of length $\frac{2}{3}$.
(ii) If $\delta = \pm n\theta \pmod{\mathbb{Z}}$ for some positive integer n, then $X_{\theta,\delta}$ has a pair of major gaps of lengths $\frac{1}{3}$ and $\frac{1}{3} + \frac{1}{3^{n+1}}$.
(iii) For all other choices of δ, $X_{\theta,\delta}$ has a pair of major gaps of length $\frac{1}{3}$.

Proof The major gaps of $X_{\theta,\delta}$ have lengths $\nu\{\delta\}$ and $\nu\{1\} = \nu\{0\}$, where

$$\nu = \sum_{k=0}^{\infty} 3^{-(k+1)} \mathbb{1}_{-k\theta} + \sum_{k=0}^{\infty} 3^{-(k+1)} \mathbb{1}_{\delta-k\theta}$$

is the gap measure of $X_{\theta,\delta}$ defined by (3.20). Since θ is irrational, the backward orbit $O_1 = \{-k\theta \pmod{\mathbb{Z}} : k \geq 0\}$ in the first sum and the backward orbit $O_2 = \{\delta - k\theta \pmod{\mathbb{Z}} : k \geq 0\}$ in the second sum consist of distinct points. However, for some values of δ the two orbits could collide. If $\delta = 0$ or 1, then $O_1 = O_2$ and there is a single major gap of length $\nu\{0\} = \frac{2}{3}$. If $\delta = n\theta \pmod{\mathbb{Z}}$ for some positive integer n, then $O_1 \subsetneq O_2$ and $\nu\{0\} = \frac{1}{3} + \frac{1}{3^{n+1}}$ and $\nu\{\delta\} = \frac{1}{3}$. Similarly, if $\delta = -n\theta \pmod{\mathbb{Z}}$ for some positive integer n, then $O_2 \subsetneq O_1$ and $\nu\{0\} = \frac{1}{3}$ and $\nu\{\theta\} = \frac{1}{3} + \frac{1}{3^{n+1}}$. For all other values of δ, $O_1 \cap O_2 = \emptyset$, so $\nu\{0\} = \nu\{\delta\} = \frac{1}{3}$. □

Let $\omega(\theta, \delta)$ denote the leading angle of $X_{\theta,\delta}$ as defined in Sect. 4.4. By the formula (4.5),

$$\omega(\theta, \delta) = \frac{1}{2} \left[\sum_{0<-k\theta\leq\theta} \frac{1}{3^{k+1}} + \sum_{0<\delta-k\theta\leq\theta} \frac{1}{3^{k+1}} \right] + \frac{N_0}{2}, \tag{4.12}$$

where $N_0 = 1$ if $\delta = 0$ and $N_0 = 0$ otherwise. One can study the function $(\theta, \delta) \mapsto \omega(\theta, \delta)$ by looking at the one-dimensional slices where θ or δ is kept fixed. The only values of δ for which $\theta \mapsto \omega(\theta, \delta)$ is defined for all $0 < \theta < 1$ are $\delta = 0$ and 1. As we have noticed before, these are similar to the doubling case. For example, when $\delta = 1$, the leading angle is given by

$$\omega(\theta, 1) = \sum_{0<-k\theta\leq\theta} \frac{1}{3^{k+1}} \tag{4.13}$$

which is similar to the formula (4.9) for the doubling map. Table 4.2 illustrates the computation of $\omega(p/q, 1)$ using formulas (4.11) and (4.13) for all reduces fractions with denominators up to 8.

Table 4.2 The leading angle $\omega(p/q, 1)$ of the cycle $X_{p/q,1}$ under the tripling map, for denominators $2 \le q \le 8$

p/q	Formula (4.11)	Formula (4.13)	$\omega(p/q,1)$
$\frac{1}{2}$	$\frac{3^0}{3^2-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{2j}}$	$\frac{1}{8}$
$\frac{1}{3}$	$\frac{3^0}{3^3-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{3j}}$	$\frac{1}{26}$
$\frac{2}{3}$	$\frac{3^1+3^0}{3^3-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{3j-1}} + \frac{1}{3^{3j}}\right)$	$\frac{4}{26}$
$\frac{1}{4}$	$\frac{3^0}{3^4-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{4j}}$	$\frac{1}{80}$
$\frac{3}{4}$	$\frac{3^2+3^1+3^0}{3^4-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{4j-2}} + \frac{1}{3^{4j-1}} + \frac{1}{3^{4j}}\right)$	$\frac{13}{80}$
$\frac{1}{5}$	$\frac{3^0}{3^5-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{5j}}$	$\frac{1}{242}$
$\frac{2}{5}$	$\frac{3^2+3^0}{3^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{5j-2}} + \frac{1}{3^{5j}}\right)$	$\frac{10}{242}$
$\frac{3}{5}$	$\frac{3^1+3^3+3^0}{3^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{5j-1}} + \frac{1}{3^{5j-3}} + \frac{1}{3^{5j}}\right)$	$\frac{31}{242}$
$\frac{4}{5}$	$\frac{3^3+3^2+3^1+3^0}{3^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{5j-3}} + \frac{1}{3^{5j-2}} + \frac{1}{3^{5j-1}} + \frac{1}{3^{5j}}\right)$	$\frac{40}{242}$
$\frac{1}{6}$	$\frac{3^0}{3^6-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{6j}}$	$\frac{1}{728}$
$\frac{5}{6}$	$\frac{3^4+3^3+3^2+3^1+3^0}{3^6-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{6j-4}} + \frac{1}{3^{6j-3}} + \frac{1}{3^{6j-2}} + \frac{1}{3^{6j-1}} + \frac{1}{3^{6j}}\right)$	$\frac{121}{728}$
$\frac{1}{7}$	$\frac{3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{7j}}$	$\frac{1}{2186}$
$\frac{2}{7}$	$\frac{3^3+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-3}} + \frac{1}{3^{7j}}\right)$	$\frac{28}{2186}$
$\frac{3}{7}$	$\frac{3^4+3^2+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-4}} + \frac{1}{3^{7j-2}} + \frac{1}{3^{7j}}\right)$	$\frac{91}{2186}$
$\frac{4}{7}$	$\frac{3^1+3^3+3^5+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-1}} + \frac{1}{3^{7j-3}} + \frac{1}{3^{7j-5}} + \frac{1}{3^{7j}}\right)$	$\frac{274}{2186}$
$\frac{5}{7}$	$\frac{3^2+3^5+3^1+3^4+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-2}} + \frac{1}{3^{7j-5}} + \frac{1}{3^{7j-1}} + \frac{1}{3^{7j-4}} + \frac{1}{3^{7j}}\right)$	$\frac{337}{2186}$
$\frac{6}{7}$	$\frac{3^5+3^4+3^3+3^2+3^1+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-5}} + \frac{1}{3^{7j-4}} + \frac{1}{3^{7j-3}} + \frac{1}{3^{7j-2}} + \frac{1}{3^{7j-1}} + \frac{1}{3^{7j}}\right)$	$\frac{364}{2186}$
$\frac{1}{8}$	$\frac{3^0}{3^8-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{8j}}$	$\frac{1}{6560}$
$\frac{3}{8}$	$\frac{3^2+3^5+3^0}{3^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{8j-2}} + \frac{1}{3^{8j-5}} + \frac{1}{3^{8j}}\right)$	$\frac{253}{6560}$
$\frac{5}{8}$	$\frac{3^4+3^1+3^6+3^3+3^0}{3^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{8j-4}} + \frac{1}{3^{8j-1}} + \frac{1}{3^{8j-6}} + \frac{1}{3^{8j-3}} + \frac{1}{3^{8j}}\right)$	$\frac{841}{6560}$
$\frac{7}{8}$	$\frac{3^6+3^5+3^4+3^3+3^2+3^1+3^0}{3^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{8j-6}} + \frac{1}{3^{8j-5}} + \frac{1}{3^{8j-4}} + \frac{1}{3^{8j-3}} + \frac{1}{3^{8j-2}} + \frac{1}{3^{8j-1}} + \frac{1}{3^{8j}}\right)$	$\frac{1093}{6560}$

The computations are identical to the doubling case in Table 4.1 once each power of 2 is replaced by the similar power of 3. In other words, the ternary expansion of $\omega(p/q, 1)$ is the same as the binary expansion of $\omega(p/q)$

An argument similar to the proof of Lemma 4.25 establishes the alternative formula

$$\omega(\theta, 1) = \frac{2}{3} \sum_{0<p/q\leq\theta} \frac{1}{3^q - 1},$$

which leads to the following analog of Corollary 4.26:

Corollary 4.32 *The leading angle $\omega(\theta, 1)$ of $X_{\theta,1}$ is a strictly increasing function of $0 < \theta < 1$, with $\omega(0^+, 1) = 0$ and $\omega(1^-, 1) = \frac{1}{6}$. Moreover,*

(i) $\omega(\theta, 1)$ has a jump discontinuity at every rational value of θ. In fact, if $\theta = p/q$ in lowest terms, then

$$\omega(p/q, 1) = \omega(p/q^+, 1) = \omega(p/q^-, 1) + \frac{2}{3(3^q - 1)}.$$

(ii) $\omega(\theta, 1)$ is continuous at every irrational value of θ.
(iii) For every $0 < \theta < 1$,

$$\omega(\theta^+, 1) + \omega((1-\theta)^-, 1) = \frac{1}{6}.$$

Compare Fig. 4.5. The function $\theta \mapsto \omega(\theta, 1)$ is related to the unicritical cubic family $\{z \mapsto z^3 + c\}_{c\in\mathbb{C}}$ (see Remark 5.14 at the end of Sect. 5.4).

Now let us fix some irrational $0 < \theta < 1$. For simplicity, let $\omega = \omega(\theta, 1)$.

Theorem 4.33 *The leading angle $\omega(\theta, \delta)$ of $X_{\theta,\delta}$ is a strictly decreasing function of $0 \leq \delta \leq 1$, with $\omega(\theta, 0) = \omega + \frac{1}{2}$ and $\omega(\theta, 1) = \omega$. Moreover,*

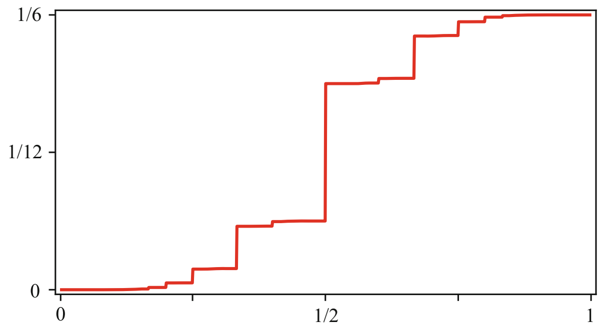

Fig. 4.5 The graph of the leading angle $\omega(\theta, 1)$ of the minimal rotation set $X_{\theta,1}$ under tripling, as a function of the rotation number θ. Notice the similarity with the graph of the leading angle $\omega(\theta)$ under doubling in Fig. 4.4

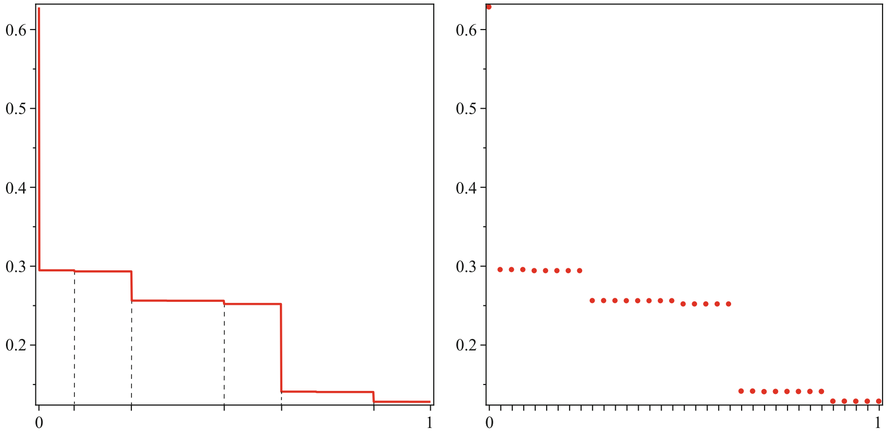

Fig. 4.6 Left: The graph of the leading angle $\omega(\theta, \delta)$ of the minimal rotation set $X_{\theta,\delta}$ under tripling, as a function of $0 \leq \delta \leq 1$. Here $\theta = \frac{(\sqrt{5}-1)}{2}$ is the golden mean. There is a jump of size $1/3^{n+1}$ at the parameter $\delta_n = n\theta \pmod{\mathbb{Z}}$ for every $n \geq 0$ (only six such jumps are visible in the figure). Right: The graph of the leading angle for the rational approximation $\frac{21}{34}$ of θ (see Remark 4.34)

(i) $\delta \mapsto \omega(\theta, \delta)$ *has a jump discontinuity at the points* $\delta_n = n\theta \pmod{\mathbb{Z}}$ *for integers* $n \geq 0$*. In fact,*

$$\omega(\theta, \delta_n) = \omega(\theta, \delta_n^-) = \omega(\theta, \delta_n^+) + \frac{1}{3^{n+1}}.$$

(ii) $\delta \mapsto \omega(\theta, \delta)$ *is continuous at every* $\delta \neq \delta_n$*.*

Compare Fig. 4.6.

Proof For each $0 < \delta \leq 1$ we have

$$\omega(\theta, \delta) = \frac{1}{2}\omega + \frac{1}{2}\sum_{k=0}^{\infty} \frac{\varepsilon_k(\theta, \delta)}{3^{k+1}},$$

where

$$\varepsilon_k(\theta, \delta) = \begin{cases} 1 & \text{if } \delta - k\theta \in (0, \theta] \\ 0 & \text{otherwise.} \end{cases}$$

Since $\varepsilon_k(\theta, \delta') \to \varepsilon_k(\theta, \delta)$ as $\delta' \to \delta^-$, it follows (say, from the dominated convergence theorem) that $\omega(\theta, \delta^-) = \omega(\theta, \delta)$, proving left-continuity at every δ. If $\delta \neq \delta_n$ for every $n \geq 0$, then $\delta - k\theta \in (0, \theta)$ or $(\theta, 1)$ for each $k \geq 0$. In either case, we have $\varepsilon_k(\theta, \delta') \to \varepsilon_k(\theta, \delta)$ as $\delta' \to \delta^+$ and right-continuity $\omega(\theta, \delta^+) = \omega(\theta, \delta)$

follows. However, suppose $\delta = \delta_n$ for some $n \geq 1$. Then the two orbit relations $\delta - n\theta = 0$ and $\delta - (n-1)\theta = \theta \pmod{\mathbb{Z}}$ show that

$$\begin{aligned} \varepsilon_n(\theta, \delta^+) &= 1 > 0 = \varepsilon_n(\theta, \delta), \\ \varepsilon_{n-1}(\theta, \delta^+) &= 0 < 1 = \varepsilon_{n-1}(\theta, \delta), \\ \varepsilon_k(\theta, \delta^+) &= \varepsilon_k(\theta, \delta) \quad \text{if} \quad k \neq n, n-1, \end{aligned}$$

where the third relation follows from the assumption that θ is irrational. It follows that

$$\omega(\theta, \delta) - \omega(\theta, \delta^+) = \frac{1}{2}\Big(- \frac{1}{3^{n+1}} + \frac{1}{3^n} \Big) = \frac{1}{3^{n+1}}.$$

Similarly, if $\delta = \delta_0 = 0$, then

$$\begin{aligned} \varepsilon_0(\theta, \delta^+) &= 1 > 0 = \varepsilon_0(\theta, \delta), \\ \varepsilon_k(\theta, \delta^+) &= \varepsilon_k(\theta, \delta) \quad \text{if} \quad k \neq 0, \end{aligned}$$

from which it follows that

$$\omega(\theta, 0) - \omega(\theta, 0^+) = \frac{1}{2} + \frac{1}{2}\Big(- \frac{1}{3} \Big) = \frac{1}{3}.$$

Finally, observe that for each n the sum $\delta \mapsto \sum_{k=0}^{n} \varepsilon_k(\theta, \delta)/3^{k+1}$ is a step function with discontinuities along $\{\delta_0, \ldots, \delta_n\}$ where it jumps to a lower value, hence is decreasing in δ. Letting $n \to \infty$, it follows that the function $\delta \mapsto \omega(\theta, \delta)$ is decreasing as well. Since the set $\{\delta_n\}_{n \geq 0}$ is dense in $[0, 1]$, we conclude that this function must be strictly decreasing. □

Remark 4.34 The parameters δ_n are precisely the values of $\delta \in [0, 1)$ for which the major gap I_0 of $X_{\theta,\delta}$ containing 0 has length $> \frac{1}{3}$. A generic perturbation $\delta = \delta_n + \varepsilon$ will replace I_0 with a major gap of length $\frac{1}{3}$ together with a nearby minor gap of length $\frac{1}{3^{n+1}}$. This gives an intuitive explanation for the nature of discontinuity of the leading angle at every δ_n.

It is not hard to check that if θ is irrational and $\delta \neq \delta_n$ for all $n \geq 0$, and if $(p_i/q_i, s_i/q_i)$ is a sequence of rational parameters that converges to (θ, δ), then $\omega(p_i/q_i, s_i/q_i) \to \omega(\theta, \delta)$. In view of this, it is natural to expect the discrete graph of $\delta \mapsto \omega(p_i/q_i, \delta)$ (consisting of $q_i + 1$ points) to resemble the graph of $\delta \mapsto \omega(\theta, \delta)$ for large i; see Fig. 4.6.

The next result shows that the values of $\omega(\theta, \delta)$ at the discontinuity points $\delta_n = n\theta \pmod{\mathbb{Z}}$ depend rationally on the "base angle" $\omega = \omega(\theta, 1)$:

Theorem 4.35 *Let $\omega = \omega(\theta, 1)$. Then, for every $n \geq 1$,*

$$\omega(\theta, n\theta) = \frac{(3^n + 1)\,\omega + A_n}{2 \cdot 3^n} \tag{4.14}$$

$$\omega(\theta, -n\theta) = \frac{(3^n + 1)\,\omega - B_n}{2}, \tag{4.15}$$

where A_n, B_n are non-negative integers (in fact, sums of distinct non-negative powers of 3):

$$A_n = \sum_{\substack{1 \leq k \leq n \\ 0 < k\theta \leq \theta}} 3^{k-1} \quad \text{and} \quad B_n = \sum_{\substack{1 \leq k \leq n \\ 0 < (k-n)\theta \leq \theta}} 3^{k-1}. \tag{4.16}$$

Proof For simplicity let Z denote the set of integers k such that $-k\theta \pmod{\mathbb{Z}}$ belongs to $(0, \theta]$. By the definition of $\omega(\theta, \delta)$ and (4.13),

$$\begin{aligned}
2\omega(\theta, n\theta) &= \sum_{k \in Z \cap [0,\infty)} 3^{-(k+1)} + \sum_{k-n \in Z \cap [-n,\infty)} 3^{-(k+1)} \\
&= \omega + \sum_{k \in Z \cap [-n,\infty)} 3^{-(k+n+1)} \\
&= \omega + \Bigg(\sum_{k \in Z \cap [0,\infty)} + \sum_{k \in Z \cap [-n,0)} \Bigg) 3^{-(k+n+1)} \\
&= (1 + 3^{-n})\,\omega + 3^{-n} \sum_{k \in Z \cap [-n,0)} 3^{-(k+1)},
\end{aligned}$$

which proves (4.14) with

$$A_n = \sum_{k \in Z \cap [-n,0)} 3^{-(k+1)} = \sum_{\substack{1 \leq k \leq n \\ 0 < k\theta \leq \theta}} 3^{k-1},$$

as in (4.16). Similarly,

$$\begin{aligned}
2\omega(\theta, -n\theta) &= \sum_{k \in Z \cap [0,\infty)} 3^{-(k+1)} + \sum_{k+n \in Z \cap [n,\infty)} 3^{-(k+1)} \\
&= \omega + \sum_{k \in Z \cap [n,\infty)} 3^{-(k-n+1)}
\end{aligned}$$

$$= \omega + \Big(\sum_{k \in Z \cap [0,\infty)} - \sum_{k \in Z \cap [0,n)} \Big) 3^{-(k-n+1)}$$

$$= (1+3^n)\omega - \sum_{k \in Z \cap [0,n)} 3^{-(k-n+1)},$$

which proves (4.15) with

$$B_n = \sum_{k \in Z \cap [0,n)} 3^{-(k-n+1)} = \sum_{\substack{1 \leq k \leq n \\ 0 < (k-n)\theta \leq \theta}} 3^{k-1},$$

as in (4.16). □

Remark 4.36 It can be shown that for every irrational θ the angle $\omega = \omega(\theta, 1)$ is transcendental (see [7] for the quadratic case and [1] for a more general result). It follows from the above theorem that all the leading angles $\omega(\theta, \pm n\theta)$ are also transcendental. These angles appear in the bifurcation loci of certain one-dimensional families of cubic polynomials (see Sect. 5.4).