Chapter 3
The Deployment Theorem

The main result of this chapter is that a minimal rotation set for m_d is uniquely determined by its rotation number together with an invariant called the "deployment vector" which, roughly speaking, describes how the points of the rotation set are deployed relative to the $d-1$ fixed points of m_d. This was first proved in the rational case by Goldberg [11] and was later extended to the irrational case by Goldberg and Tresser [13] using a Farey tree machinery. By contrast, our presentation here builds upon the ideas developed in the previous chapter and treats both rational and irrational cases in a unified fashion. Various applications of this result will be discussed in the next chapter.

3.1 Preliminaries

To begin the discussion, consider a minimal rotation set X for m_d with $\rho(X) = \theta \neq 0$ and the standard monotone map g_X. Let $\varphi : \mathbb{T} \to \mathbb{T}$ be the combinatorial semiconjugacy between g_X and r_θ if θ is rational, or the Poincaré semiconjugacy between g_X and r_θ if θ is irrational. In either case, we have the semiconjugacy relation

$$\varphi \circ m_d = r_\theta \circ \varphi \quad \text{on} \quad X.$$

Recall that φ is normalized by $\varphi(0) = 0$ and its plateaus are precisely the gaps of X. We refer to φ as the ***canonical semiconjugacy*** associated with X.

It follows from the discussion in Sect. 1.5 that there is a unique m_d-invariant Borel probability measure μ supported on X. This measure, which henceforth will be called the ***natural measure*** of X, is related to the canonical semiconjugacy by

$$\varphi(t) = \mu[0, t] \pmod{\mathbb{Z}}.$$

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If $\theta = p/q$ in lowest terms so X is a q-cycle, then μ is just the uniform Dirac measure on X which assigns a mass of $1/q$ to each point of X. On the other hand, if θ is irrational so X is a Cantor set, then μ is the (well-defined) pull-back of Lebesgue measure under φ.

Recall that the $d-1$ fixed points of m_d are denoted by

$$u_i = \frac{i}{d-1} \pmod{\mathbb{Z}}.$$

Set

$$\delta_i = \mu[u_{i-1}, u_i) \qquad 1 \leq i \leq d-1.$$

Then $(\delta_1, \ldots, \delta_{d-1})$ is a probability vector, that is, it belongs to the $(d-2)$-dimensional simplex

$$\Delta^{d-2} = \left\{(x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} : x_i \geq 0 \text{ and } \sum_{i=1}^{d-1} x_i = 1\right\}.$$

Definition 3.1 The vector $\delta(X) = (\delta_1, \ldots, \delta_{d-1}) \in \Delta^{d-2}$ is called the ***deployment vector*** of the minimal rotation set X.

Here is a more explicit description for the components of $\delta(X)$. If $\rho(X) = p/q$ in lowest terms, the component δ_i is the fraction of points of X that fall between the fixed points u_{i-1} and u_i:

$$\delta_i = \frac{1}{q} \#\big\{t \in X : t \in [u_{i-1}, u_i)\big\}.$$

If $\rho(X)$ is irrational, it follows from unique ergodicity that δ_i is the fraction of time that the orbit of every $t \in X$ spends in $[u_{i-1}, u_i)$:

$$\delta_i = \lim_{n \to \infty} \frac{1}{n} \#\big\{0 \leq k \leq n-1 : m_d^{\circ k}(t) \in [u_{i-1}, u_i)\big\}$$

(compare (1.11)).

Observe that the deployment vectors of the rotation sets

$$X + \frac{1}{d-1},\; X + \frac{2}{d-1},\; \ldots,\; X + \frac{d-2}{d-1} \pmod{\mathbb{Z}}$$

are obtained by cyclically permuting the components of $\delta(X)$. For example, if X is a rotation set under m_4 with $\delta(X) = (\delta_1, \delta_2, \delta_3)$, then $\delta(X + \frac{1}{3}) = (\delta_3, \delta_1, \delta_2)$ and $\delta(X + \frac{2}{3}) = (\delta_2, \delta_3, \delta_1)$.

Closely related is the ***cumulative deployment vector*** $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1}) \in [0, 1]^{d-1}$ whose components are defined by

$$\sigma_i = \delta_1 + \cdots + \delta_i \qquad 1 \leq i \leq d-1 \tag{3.1}$$

and therefore satisfy $0 \leq \sigma_1 \leq \cdots \leq \sigma_{d-1} = 1$. In terms of the natural measure μ, the number σ_i is just $\mu[u_0, u_i)$. Whether we use $\delta(X)$ or $\sigma(X)$ is solely a matter of preference, as each of these vectors determines the other uniquely.

Let

$$\begin{aligned} N_0 &= \#\{1 \leq i \leq d-1 : \sigma_i = 0\} \\ N_1 &= \#\{1 \leq i \leq d-1 : \sigma_i = 1\}, \end{aligned}$$

so the components of $\sigma(X)$ begin with $N_0 \geq 0$ zeros and end in $N_1 \geq 1$ ones. It is easy to check that the major gap I_0 of X containing the fixed point $u_0 = 0$ contains precisely the fixed points $u_{-N_1+1}, \ldots, u_{N_0}$. It follows from Lemma 2.13 that $N_0 + N_1$ is the multiplicity of I_0.

Remark 3.2 We can assign a deployment vector to every rotation set X, even if it is not minimal: If X is rational, consider the finitely many cycles $C_1, \ldots, C_N$ that are contained in X (Corollary 2.27) and define $\delta(X)$ to be the average $(1/N) \sum_{i=1}^{N} \delta(C_i)$. If X is irrational, define $\delta(X) = \delta(K)$, where K is the unique minimal rotation set contained in X (Theorem 2.33).

Lemma 3.3 *Let X be a minimal rotation set for m_d with $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$, and let $\varphi : \mathbb{T} \to \mathbb{T}$ be the canonical semiconjugacy associated with X. Then,*

$$\sigma_i = \varphi(u_i) \pmod{\mathbb{Z}} \quad \textit{for all} \quad 1 \leq i \leq d-1. \tag{3.2}$$

Proof Let μ be the natural measure of X, so $\varphi(t) = \mu[u_0, t] \pmod{\mathbb{Z}}$ for all $t \in \mathbb{T}$. Since $\rho(X) \neq 0$ by the assumption, X contains none of the fixed points u_i, so $\mu\{u_i\} = 0$ for every i. Hence $\varphi(u_i) = \mu[u_0, u_i] = \mu[u_0, u_i) = \sigma_i \pmod{\mathbb{Z}}$, as required. □

Remark 3.4 The congruences (3.2) allow us to determine $\sigma(X)$ from the knowledge of the $d-1$ points $\varphi(u_1), \ldots, \varphi(u_{d-1})$ on $\mathbb{T}$ except when $\varphi(u_i) = 0 \pmod{\mathbb{Z}}$ for all i because in this case we cannot decide whether each σ_i is 0 or 1. For example, when $d = 4$, each of the vectors

$$\sigma(X) = (0, 0, 1) \quad \text{or} \quad (0, 1, 1) \quad \text{or} \quad (1, 1, 1)$$

would correspond to a minimal rotation set whose canonical semiconjugacy satisfies

$$\varphi(u_1) = \varphi(u_2) = \varphi(u_3) = 0 \pmod{\mathbb{Z}}.$$

This ambiguity can be dealt with, for example, by looking at a lift of φ. Alternatively, we can work with rotation sets for which $\sigma_1 \neq 0$ so every σ_i lies in (0, 1]. This condition can always be achieved by simply rotating the set: If the components of $\sigma(X)$ begin with a string of 0's of length N_0, replace X by its rotated copy $X - N_0/(d-1)$.

3.2 Deployment Theorem: The Rational Case

Throughout this section we assume that X is a minimal rational rotation set, that is, a q-cycle $\{t_1, \ldots, t_q\}$ under m_d with $\rho(X) = p/q$ in lowest terms. As usual, we label the points of X so that $0, t_1, \ldots, t_q$ are in positive cyclic order (in particular, $0 \in (t_q, t_1)$) and the subscripts are taken modulo q.

Lemma 3.5 *The interval $I_j = (t_j, t_{j+1})$ is a major gap of X of multiplicity n if and only if $j/q \pmod{\mathbb{Z}}$ appears exactly n times as a component of $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$.*

Note that since $0/q = q/q \pmod{\mathbb{Z}}$, this generalizes our previous observation that the multiplicity of $I_0 = I_q$ is $N_0 + N_1$.

Proof According to Lemma 2.13, I_j is a major gap of multiplicity n if and only if it contains exactly n fixed points. Under the canonical semiconjugacy associated with X, each such fixed point maps to j/q. The result now follows from Lemma 3.3. □

The main result of this section asserts that a minimal rational rotation set is uniquely determined by its rotation number and deployment vector. To motivate the main idea of the proof, we begin with an example.

Example 3.6 Suppose we want to find a 5-cycle $X = \{t_1, \cdots, t_5\}$ under m_4 with $\rho(X) = \frac{1}{5}$ and $\delta(X) = (\frac{3}{5}, 0, \frac{2}{5})$. Let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$. By Lemma 3.5, the knowledge of the cumulative deployment vector $\sigma(X) = (\frac{3}{5}, \frac{3}{5}, \frac{5}{5})$ tells us that I_3 is a major gap of multiplicity 2, $I_5 = I_0$ is a major gap of multiplicity 1, and the remaining I_j are minor (see Fig. 3.1). Since $\rho(X) = \frac{1}{5}$, we know that I_j maps to I_{j+1}. It follows from Lemma 2.8 that

$$
\begin{aligned}
\ell_2 &= 4\ell_1 \\
\ell_3 &= 4\ell_2 = 4^2\ell_1 \\
\ell_4 &= 4\ell_3 - 2 = 4^3\ell_1 - 2 \\
\ell_5 &= 4\ell_4 = 4^4\ell_1 - 8 \\
\ell_1 &= 4\ell_5 - 1 = 4^5\ell_1 - 33.
\end{aligned}
$$

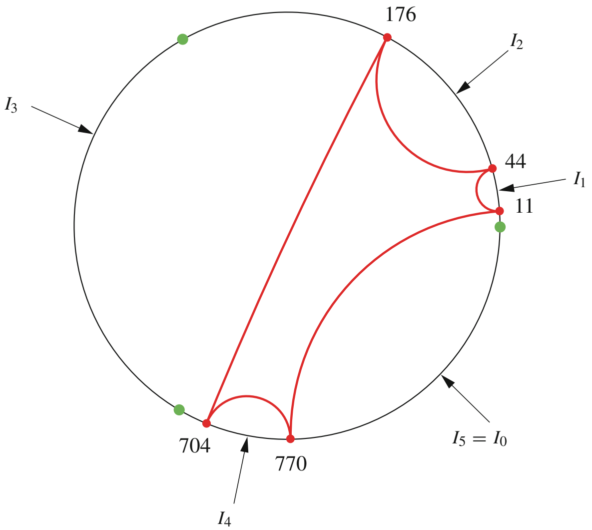

Fig. 3.1 The unique minimal rotation set X under m_4 with $\rho(X) = \frac{1}{5}$ and $\delta(X) = (\frac{3}{5}, \frac{0}{5}, \frac{2}{5})$ (angles in X are given in multiples of $\frac{1}{1023}$). Here X has cumulative deployment vector $\sigma(X) = (\frac{3}{5}, \frac{3}{5}, \frac{5}{5})$, and major gaps I_3 and $I_5 = I_0$ of multiplicities 2 and 1, respectively, which are also the number of fixed points of m_4 (shown as green dots) they contain

The last equation can be solved uniquely for ℓ_1, which in turn determines every ℓ_j:

$$\ell_1 = \frac{33}{1023}, \quad \ell_2 = \frac{132}{1023}, \quad \ell_3 = \frac{528}{1023}, \quad \ell_4 = \frac{66}{1023}, \quad \ell_5 = \frac{264}{1023}.$$

Since $\ell_1 = t_2 - t_1 = 4t_1 - t_1 = 3t_1$, we find t_1 and therefore every t_j:

$$t_1 = \frac{11}{1023}, \quad t_2 = \frac{44}{1023}, \quad t_3 = \frac{176}{1023}, \quad t_4 = \frac{704}{1023}, \quad t_5 = \frac{770}{1023}.$$

It is easily checked that this 5-cycle has the required rotation number and deployment vector. The uniqueness automatically follows from the above computation.

In general, the method of Example 3.6 can be described more formally as follows. Suppose we are looking for a minimal rotation set $X = \{t_1, \dots, t_q\}$ for m_d with $\rho(X) = p/q \neq 0$ and $\delta(X) = (\delta_1, \dots, \delta_{d-1})$. Let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$. Set n_j to be the multiplicity of I_j if I_j is major, and $n_j = 0$ otherwise. Then the relations $\ell_{j+p} = d\ell_j - n_j$ hold for every j (recall that all subscripts are taken modulo q). Introduce the vectors

$$\boldsymbol{\ell} = (\ell_1, \dots, \ell_q) \quad \text{and} \quad \boldsymbol{n} = (n_1, \dots, n_q)$$

in $\mathbb{R}^q$ and denote by $T : \mathbb{R}^q \to \mathbb{R}^q$ the isometry

$$T(x_1, x_2, \dots, x_q) = (x_{1+p}, x_{2+p}, \dots, x_{q+p}).$$

Notice that T is determined by the rotation number while $\boldsymbol{n}$ is determined by the deployment vector (Lemma 3.5). The q relations above can then be written as the non-homogeneous linear equation

$$T(\boldsymbol{\ell}) = d\boldsymbol{\ell} - \boldsymbol{n} \tag{3.3}$$

which can be easily solved for $\boldsymbol{\ell}$ by applying T repeatedly on each side and using the fact that $T^{\circ q} = \text{id}$. The result is

$$\boldsymbol{\ell} = \frac{1}{d^q - 1} \sum_{i=0}^{q-1} d^{q-i-1} \, T^{\circ i}(\boldsymbol{n}). \tag{3.4}$$

Since $\boldsymbol{n} \neq \boldsymbol{0}$ and since the addition $j \mapsto j + p \pmod q$ acts transitively on $\mathbb{Z}_q$, the right hand sum has strictly positive components, so the above formula gives a unique solution $\boldsymbol{\ell}$ of (3.3) with $\ell_j > 0$ for all j. Once the gap lengths ℓ_j are known, we can find the t_j by noting that the counterclockwise distance from t_j to $t_{j+p} = dt_j \pmod{\mathbb{Z}}$ is the sum $\ell_j + \cdots + \ell_{j+p-1}$. The method produces a unique candidate q-cycle X, but one still needs to verify that this X has indeed the required rotation number and deployment vector.

There is an alternative way to solve (3.3) which, despite its appearance, will turn out more advantageous. Write (3.3) as

$$\boldsymbol{\ell} = \frac{1}{d} T(\boldsymbol{\ell}) + \frac{1}{d}\boldsymbol{n}$$

which can then be turned into

$$\boldsymbol{\ell} = \frac{1}{d}\Big(\frac{1}{d} T^{\circ 2}(\boldsymbol{\ell}) + \frac{1}{d} T(\boldsymbol{n})\Big) + \frac{1}{d}\boldsymbol{n} = \frac{1}{d^2} T^{\circ 2}(\boldsymbol{\ell}) + \frac{1}{d^2} T(\boldsymbol{n}) + \frac{1}{d}\boldsymbol{n}.$$

Continuing this way and using the fact that $T^{\circ k}(\boldsymbol{\ell})/d^k \to 0$ as $k \to \infty$, we obtain the series solution

$$\boldsymbol{\ell} = \sum_{k=0}^{\infty} d^{-(k+1)} \, T^{\circ k}(\boldsymbol{n}). \tag{3.5}$$

The vectors $\boldsymbol{\ell}$ and $\boldsymbol{n}$ can be thought of as positive measures supported on the subset

$$S = \left\{ \frac{j}{q} \pmod{\mathbb{Z}} : 0 \leq j \leq q-1 \right\} \cong \mathbb{Z}_q$$

of the circle by identifying ℓ_j with $\boldsymbol{\ell}\{j/q\}$ and n_j with $\boldsymbol{n}\{j/q\}$. Under this identification, $\boldsymbol{\ell}$ is just the push-forward of Lebesgue measure under the canonical semiconjugacy associated with X. Lemma 3.5 can then be translated into the statement that

$$\boldsymbol{n} = \sum_{i=1}^{d-1} \mathbb{1}_{\sigma_i},$$

where $\mathbb{1}_x$ is the unit mass at x. Thus, for each $k \geq 0$,

$$T^{\circ k}(\boldsymbol{n}) = \sum_{i=1}^{d-1} \mathbb{1}_{\sigma_i - kp/q}$$

and (3.5) can be written as

$$\boldsymbol{\ell} = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - kp/q}. \tag{3.6}$$

This means that to find $\boldsymbol{\ell}$ we start with a point mass $1/d$ at each σ_i and spread it around S by taking pull-backs under the rigid rotation $r_{p/q}$, each time dividing the mass by d. The measure $\ell_j = \boldsymbol{\ell}\{j/q\}$ is the sum of $d-1$ infinite series, each representing the contribution from the initial mass concentrated at one of the σ_i. This slightly disguised form of the solution (3.5) will be used in the proof of Theorem 3.7 below. Why do we use (3.6) instead of the simpler formula (3.4)? Because this formulation allows us to construct the cycle explicitly and to verify that it has the given rotation number and deployment vector. More importantly, it generalizes without any modification to the irrational case discussed in the next section, thus allowing a unified treatment of both rational and irrational cases of the deployment theorem.

Theorem 3.7 (Goldberg) *For every fraction $0 < p/q < 1$ in lowest terms and every vector $(\delta_1, \ldots, \delta_{d-1}) \in \Delta^{d-2}$ with $q\delta_i \in \mathbb{Z}$ there is a unique minimal rotation set X for m_d such that $\rho(X) = p/q$ and $\delta(X) = (\delta_1, \ldots, \delta_{d-1})$.*

Proof It will be convenient to use the notation $\equiv$ for congruence modulo $\mathbb{Z}$, so we write $m_d(t) \equiv dt$, $r_\theta(t) \equiv t + \theta$ and so on. We may also assume $\delta_1 \neq 0$; the general case will follow by cyclically permuting the components of $(\delta_1, \ldots, \delta_{d-1})$ and rotating the corresponding rotation set. Define $\sigma_i = \delta_1 + \ldots + \delta_i$ for $1 \leq i \leq d-1$. Then $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{d-1} = 1$ and each σ_i is congruent to some element of the set $S = \{j/q \pmod{\mathbb{Z}} : 0 \leq j \leq q-1\}$. Motivated by (3.6), we consider the atomic probability measure ν supported on S defined by

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - kp/q}. \tag{3.7}$$

Notice that $\nu\{\sigma_i\} > 1/d$. More precisely, if some $j/q \in S$ appears exactly n times as a σ_i, then $n/d < \nu\{j/q\} < (n+1)/d$. The lower bound is immediate from the definition. The upper bound holds since the contribution of the remaining terms of (3.7) to $\nu\{j/q\}$ is at most

$$(d-1)\sum_{k=0}^{\infty} d^{-(kq+2)} = \frac{d-1}{d^2} \cdot \frac{d^q}{(d^q-1)} < \frac{1}{d}.$$

The same argument also proves that $0 < \nu\{j/q\} < 1/d$ whenever j/q is not congruent to any of the σ_i.

Let, as before, $N_1 \geq 1$ be the number of indices $1 \leq i \leq d-1$ for which $\sigma_i = 1$. Define

$$\psi_j = \nu\Big[0, \frac{j}{q}\Big) = \nu\Big\{\frac{0}{q}\Big\} + \cdots + \nu\Big\{\frac{j-1}{q}\Big\} \qquad 1 \leq j \leq q, \tag{3.8}$$

so $N_1/d < \psi_1 < \cdots < \psi_{q-1} < \psi_q = 1$. Set

$$a = \frac{N_1 - \nu[0, p/q)}{d-1} = \frac{N_1 - \psi_p}{d-1} \tag{3.9}$$

and

$$t_j \equiv \psi_j - a \qquad 1 \leq j \leq q.$$

We show that $X = \{t_1, \ldots, t_q\}$ is the desired rotation set.

The relation

$$\nu\Big(B + \frac{p}{q}\Big) \equiv d\nu(B) \tag{3.10}$$

for every set $B \subset \mathbb{T}$ is easily verified from the definition of ν. It implies

$$\nu\Big[0, \frac{j+p}{q}\Big) \equiv \nu\Big[0, \frac{p}{q}\Big) + \nu\Big[\frac{p}{q}, \frac{j+p}{q}\Big) \equiv \nu\Big[0, \frac{p}{q}\Big) + d\nu\Big[0, \frac{j}{q}\Big),$$

which yields the relation

$$\psi_{j+p} \equiv d\psi_j + \psi_p$$

for all j. Thus,

$$\begin{aligned} t_{j+p} &\equiv \psi_{j+p} - a \equiv d\psi_j + \psi_p - a \\ &\equiv dt_j + (d-1)a + \psi_p \equiv dt_j + N_1 \equiv dt_j. \end{aligned} \tag{3.11}$$

Since $t_1, \dots, t_q$ are in positive cyclic order, this proves that X is a q-cycle under m_d with combinatorial rotation number p/q. It follows from Corollary 1.16 that X is a rotation set with $\rho(X) = p/q$.

Next, we verify that $\delta(X) = (\delta_1, \dots, \delta_{d-1})$ or equivalently $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1})$. First note that $\psi_p > N_1/d$, so $N_1 - \psi_p < N_1(d-1)/d$, so $0 < a < N_1/d < \psi_1$. This shows that $0 \in (t_q, t_1)$. Suppose there is an n-fold incidence of the form

$$\frac{j}{q} = \sigma_i = \sigma_{i+1} = \cdots = \sigma_{i+n-1}.$$

Then, by our earlier remark,

$$\frac{n}{d} < t_{j+1} - t_j \equiv \psi_{j+1} - \psi_j = \nu\left\{\frac{j}{q}\right\} < \frac{n+1}{d},$$

which implies (t_j, t_{j+1}) is a major gap of multiplicity n, and therefore contains n fixed points of m_d by Lemma 2.13. Under the canonical semiconjugacy φ associated with X, these n fixed points all map to j/q. Thus, φ maps the fixed point set $\{u_1, \dots, u_{d-1}\}$ to the set $\{\sigma_1, \dots, \sigma_{d-1}\}$, sending n of the u_i to the same point j/q if and only if n of the σ_i collide at j/q. Since $\varphi(0) \equiv 0$, it follows from monotonicity of φ that $\varphi(u_i) \equiv \sigma_i$ for every i. Since every σ_i lies in $(0, 1]$ by our assumption $\delta_1 \neq 0$, Lemma 3.3 proves that $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1})$.

It remains to prove uniqueness. Suppose $\hat{X} = \{\hat{t}_1, \dots, \hat{t}_q\}$ is another rotation set for m_d with rotation number p/q and deployment vector $(\delta_1, \dots, \delta_{d-1})$. As we have seen in the discussion leading to (3.4) or (3.5), for each j the gap $\hat{I}_j = (\hat{t}_j, \hat{t}_{j+1})$ of $\hat{X}$ has the same length as the gap $I_j = (t_j, t_{j+1})$ of X. Hence there is a rigid rotation r_α which maps t_j to $\hat{t}_j$ for all j. We must show that $\alpha \equiv 0$. The major gaps I_0 and $\hat{I}_0 = r_\alpha(I_0)$ contain the same set of fixed points of m_d since X and $\hat{X}$ have the same deployment vector. Since the fixed points of m_d are $1/(d-1)$ apart, it follows that the distance between α and 0 is less than $1/(d-1)$. On the other hand, $r_\alpha : X \to \hat{X}$ commutes with m_d, so $d(t_j + \alpha) \equiv dt_j + \alpha$ for every j, which implies $(d-1)\alpha \equiv 0$. The only solution of this equation whose distance to 0 is $< 1/(d-1)$ is $\alpha \equiv 0$, and the proof is complete. □

Remark 3.8 The $d-1$ solutions for a of the equation $(d-1)a + \psi_p \equiv 0$, which was key in (3.11), correspond to minimal rotation sets with rotation number p/q whose deployment vectors are cyclic permutations of $(\delta_1, \dots, \delta_{d-1})$. The particular choice of a in (3.9) guarantees that this permutation is the identity.

Example 3.9 Let us revisit Example 3.6, this time using the idea of the measure ν in the proof of Theorem 3.7. Recall that we were looking for the unique 5-cycle $X = \{t_1, \cdots, t_5\}$ under m_4 with $\rho(X) = \frac{1}{5}$ and $\delta(X) = (\frac{3}{5}, 0, \frac{2}{5})$ or $\sigma(X) = (\frac{3}{5}, \frac{3}{5}, \frac{5}{5})$. We compute the atomic measure ν on the set $S = \{\frac{0}{5}, \dots, \frac{4}{5}\}$, starting with a mass $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ at $\sigma_1 = \sigma_2 = \frac{3}{5}$ and a mass $\frac{1}{4}$ at $\sigma_3 = \frac{5}{5} \equiv \frac{0}{5}$. We then spread the measure around S by pulling back under the rotation $r_{1/5}$, each time dividing the mass by 4. Since $\nu\{t + \frac{1}{5}\} \equiv 4\nu\{t\}$ for every $t \in S$ by the transformation rule (3.10), it suffices to compute ν only at $\frac{0}{5}$:

$$\nu\left\{\frac{0}{5}\right\} = \left(\frac{1}{4} + \frac{1}{4^6} + \frac{1}{4^{11}} + \cdots\right) + \left(\frac{1}{2\cdot 4^3} + \frac{1}{2\cdot 4^8} + \cdots\right) = \frac{264}{1023}$$

It follows that

$$\begin{aligned}
4\nu\left\{\frac{0}{5}\right\} &= \frac{1056}{1023} \Longrightarrow \nu\left\{\frac{1}{5}\right\} = \frac{33}{1023}\\
4\nu\left\{\frac{1}{5}\right\} &= \frac{132}{1023} \Longrightarrow \nu\left\{\frac{2}{5}\right\} = \frac{132}{1023}\\
4\nu\left\{\frac{2}{5}\right\} &= \frac{528}{1023} \Longrightarrow \nu\left\{\frac{3}{5}\right\} = \frac{528}{1023}\\
4\nu\left\{\frac{3}{5}\right\} &= \frac{2112}{1023} \Longrightarrow \nu\left\{\frac{4}{5}\right\} = \frac{66}{1023}
\end{aligned}$$

(these are just the gap lengths ℓ_j computed in Example 3.6). Thus,

$$\begin{aligned}
\psi_1 &= \nu\left\{\frac{0}{5}\right\} = \frac{264}{1023}\\
\psi_2 &= \nu\left\{\frac{0}{5}\right\} + \nu\left\{\frac{1}{5}\right\} = \frac{297}{1023}\\
\psi_3 &= \nu\left\{\frac{0}{5}\right\} + \nu\left\{\frac{1}{5}\right\} + \nu\left\{\frac{2}{5}\right\} = \frac{429}{1023}\\
\psi_4 &= \nu\left\{\frac{0}{5}\right\} + \nu\left\{\frac{1}{5}\right\} + \nu\left\{\frac{2}{5}\right\} + \nu\left\{\frac{3}{5}\right\} = \frac{957}{1023}\\
\psi_5 &= \nu\left\{\frac{0}{5}\right\} + \nu\left\{\frac{1}{5}\right\} + \nu\left\{\frac{2}{5}\right\} + \nu\left\{\frac{3}{5}\right\} + \nu\left\{\frac{4}{5}\right\} = 1.
\end{aligned}$$

Now $t_j = \psi_j - a$, where $a = (1 - \psi_1)/3 = \frac{253}{1023}$. We obtain

$$\begin{aligned}
t_1 &= \frac{264}{1023} - \frac{253}{1023} = \frac{11}{1023} & t_2 &= \frac{297}{1023} - \frac{253}{1023} = \frac{44}{1023}\\
t_3 &= \frac{429}{1023} - \frac{253}{1023} = \frac{176}{1023} & t_4 &= \frac{957}{1023} - \frac{253}{1023} = \frac{704}{1023}\\
t_5 &= \frac{1023}{1023} - \frac{253}{1023} = \frac{770}{1023},
\end{aligned}$$

which is of course the same cycle obtained by the method of Example 3.6.

Remark 3.10 A different approach to the rational case of the deployment theorem can be found in the recent work [27] which solves the general problem of realizing cyclic permutations of q objects as period q orbits of m_d. The idea is to reduce the problem to finding the stationary state of an associated Markov chain, which can then be tackled by classical Perron-Frobenius theory.

For each $q > 0$ the number of distinct vectors $(\delta_1, \ldots, \delta_{d-1}) \in \Delta^{d-2}$ with $q\delta_i \in \mathbb{Z}$ can be computed as the number of ways to deploy q identical balls in $d-1$ labeled boxes. This, in view of Theorem 3.7, gives the following

Corollary 3.11 (Goldberg) *For every fraction $0 < p/q < 1$ in lowest terms, there are*

$$\binom{q+d-2}{q} = \frac{(q+d-2)!}{q!(d-2)!}$$

distinct minimal rotation sets X under m_d with $\rho(X) = p/q$.

For $d = 2$ this number reduces to 1, proving that *there is a unique minimal rotation set under doubling with a given rational rotation number*.

The deployment theorem can be generalized to unions of cycles as follows. Suppose X is a rotation set for m_d, with $\rho(X) = p/q \neq 0$ in lowest terms, consisting of distinct q-cycles $C_1, \ldots, C_N$ (here $N \leq d-1$ by Corollary 2.27). As in Remark 3.2, we define the deployment vector and the cumulative deployment vector of X as the averages

$$\delta(X) = \frac{1}{N}\sum_{i=1}^{N} \delta(C_i) \quad \text{and} \quad \sigma(X) = \frac{1}{N}\sum_{i=1}^{N} \sigma(C_i).$$

Of course the ith components of $\delta(X)$ and $\sigma(X)$ are simply the fraction of points of X that fall within the intervals $[u_{i-1}, u_i)$ and $[u_0, u_i)$, respectively. Note that these components are now rational numbers with denominator dividing Nq.

Suppose we are looking for such a rotation set X with $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$. Let $X = \{t_1, \ldots, t_{Nq}\}$, where the points are labeled so that $0, t_1, \ldots, t_{Nq}$ are in positive cyclic order and the subscripts are taken modulo Nq. Since each cycle in X has combinatorial rotation number $p/q = Np/(Nq)$, the map m_d acts as $t_j \mapsto t_{j+Np}$ on X. As in the case $N = 1$, let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$ and n_j be the multiplicity of I_j if I_j is major, and $n_j = 0$ otherwise. Then the equations $\ell_{j+Np} = d\ell_j - n_j$ for $1 \leq j \leq Nq$ can be written in vector form as $T(\boldsymbol{\ell}) = d\boldsymbol{\ell} - \boldsymbol{n}$. Here $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_{Nq})$ is unknown, $\boldsymbol{n} = (n_1, \ldots, n_{Nq})$ is determined by the cumulative deployment vector $\sigma(X)$, and $T : \mathbb{R}^{Nq} \to \mathbb{R}^{Nq}$ is the isometry

$$T(x_1, x_2, \ldots, x_{Nq}) = (x_{1+Np}, x_{2+Np}, \ldots, x_{Nq+Np})$$

determined by the rotation number. Since $T^{\circ q} = \mathrm{id}$, the same argument as in the minimal case gives a unique solution $\boldsymbol{\ell}$ of this equation which can be expressed in either of the forms (3.4) or (3.5) or (3.6). If every component of $\boldsymbol{\ell}$ obtained this way is strictly positive, then the gap lengths are uniquely determined and an argument similar to the minimal case shows that the desired rotation set X exists and is unique. On the other hand, if the solution $\boldsymbol{\ell}$ has a zero component, then no X with the given rotation number and deployment vector can exist. Using the form (3.6) of the solution, it follows that a necessary and sufficient condition for the existence of X is that the support of the atomic measure

$$\sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - kp/q}$$

is the set $S = \{j/(Nq) \pmod{\mathbb{Z}} : 0 \le j \le Nq - 1\}$. In other words, each point of S must belong to the orbit of some σ_i under $r_{p/q}$. Using the fact that p, q are relatively prime, it is easy to see that $j/(Nq), j'/(Nq) \in S$ belong to the same orbit under $r_{p/q}$ if and only if $j = j' \pmod{N}$. Thus, S is the union of N disjoint q-cycles under $r_{p/q}$, indexed by the distinct residue classes modulo N. Consider the ***signature*** $s(X) = Nq\,\sigma(X)$, that is the integer vector $s(X) = (s_1, \dots, s_{d-1})$, where s_i is the number of points of X in $[u_0, u_i)$.[1] Then the above condition is equivalent to every residue class modulo N being represented by some s_i. This proves

Theorem 3.12 (Goldberg) *Suppose $0 < p/q < 1$ is a fraction in lowest terms, $N \ge 1$ is an integer, and $\{s_i\}_{1 \le i \le d-1}$ is an integer sequence such that $0 \le s_1 \le \dots \le s_{d-1} = Nq$. Then there is a rotation set X for m_d with rotation number $\rho(X) = p/q$ and signature $s(X) = (s_1, \dots, s_{d-1})$ if and only if every residue class modulo N is represented by some s_i. Moreover, X subject to these conditions is unique.*

Notice that this result gives an alternative proof for the inequality $N \le d - 1$ in Corollary 2.27.

Example 3.13 Consider finite rotation sets with rotation number $\frac{1}{4}$ under tripling. According to Theorem 3.12, such a rotation set is either a 4-cycle (where $N = 1$) or a union of two 4-cycles (where $N = 2$), and is uniquely determined by its signature. For $N = 1$, all five signatures $(s, 4)$ for $0 \le s \le 4$ can occur; they are realized by the following rotation sets that we already encountered in Example 2.29:

X	$s(X)$	$\delta(X)$
$C_1 : \frac{1}{80} \mapsto \frac{3}{80} \mapsto \frac{9}{80} \mapsto \frac{27}{80}$	$(4, 4)$	$(1, 0)$
$C_2 : \frac{2}{80} \mapsto \frac{6}{80} \mapsto \frac{18}{80} \mapsto \frac{54}{80}$	$(3, 4)$	$(\frac{3}{4}, \frac{1}{4})$
$C_3 : \frac{5}{80} \mapsto \frac{15}{80} \mapsto \frac{45}{80} \mapsto \frac{55}{80}$	$(2, 4)$	$(\frac{1}{2}, \frac{1}{2})$
$C_4 : \frac{14}{80} \mapsto \frac{42}{80} \mapsto \frac{46}{80} \mapsto \frac{58}{80}$	$(1, 4)$	$(\frac{1}{4}, \frac{3}{4})$
$C_5 : \frac{41}{80} \mapsto \frac{43}{80} \mapsto \frac{49}{80} \mapsto \frac{67}{80}$	$(0, 4)$	$(0, 1)$

[1] In the terminology of [11], the integers s_i define the *deployment sequence* of X.

However, for $N = 2$ only the signatures $(s, 8)$ with odd $0 \leq s \leq 8$ occur. These are realized by the following four rotation sets, also encountered in Example 2.29 as unions of compatible pairs:

X	$s(X)$	$\delta(X)$
$C_1 \cup C_2$	$(7, 8)$	$(\frac{7}{8}, \frac{1}{8})$
$C_2 \cup C_3$	$(5, 8)$	$(\frac{5}{8}, \frac{3}{8})$
$C_3 \cup C_4$	$(3, 8)$	$(\frac{3}{8}, \frac{5}{8})$
$C_4 \cup C_5$	$(1, 8)$	$(\frac{1}{8}, \frac{7}{8})$

Notice that the signatures $(0, 8), (2, 8), (4, 8), (6, 8)$ cannot occur for the rotation number $\frac{1}{4}$, although they can be realized by 8-cycles with any of the rotation numbers $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}$, or $\frac{7}{8}$.

The above example shows that the cycles C_i and the unions $C_i \cup C_{i+1}$ have distinct deployment sequences. This is a special case of the following stronger form of the uniqueness part of Theorem 3.12:

Corollary 3.14 *Suppose X, X' are finite rotation sets with the same rotation number and deployment sequence. Then $X = X'$.*

Proof Let $\rho(X) = \rho(X') = p/q$ and suppose X and X' are unions of N and N' distinct q-cycles respectively. Consider the signatures $s(X) = (s_1, \dots, s_{d-1})$ and $s(X') = (s'_1, \dots, s'_{d-1})$. The assumption $\delta(X) = \delta(X')$ shows that $s_i/N = s'_i/N'$ or

$$N' s_i = N s'_i \quad \text{for all} \quad 1 \leq i \leq d - 1.$$

By Theorem 3.12, $s_j = 1 \pmod{N}$ for some j. It follows from the above equation that N divides N'. A similar reasoning shows that N' divides N, so $N = N'$. It now follows from the uniqueness statement of Theorem 3.12 that $X = X'$. □

Corollary 3.15 *For every fraction $0 < p/q < 1$ in lowest terms, there are q^{d-2} rotation sets X for m_d with $\rho(X) = p/q$, each consisting of the maximum number $d - 1$ of distinct q-cycles.*

In particular, the upper bound in Corollary 2.27 is optimal.

Proof By Theorem 3.12 for $N = d - 1$, such X are in one-to-one correspondence with signatures $s = (s_1, \dots, s_{d-2}, (d-1)q)$ for which the unordered set $A = \{s_1, \dots, s_{d-2}\}$ reduces to $\{1, \dots, d-2\}$ modulo $d - 1$. For each $1 \leq k \leq d - 2$ such A contains exactly one element of the form $j(d-1) + k$ with $0 \leq j \leq q - 1$. Evidently there are q^{d-2} choices for A, hence for the signature s. □

Another application of Theorem 3.12 is the following characterization of compatible cycles in terms of their signature (compare §2 of [19]). It will be convenient to use the notation $\mathscr{C}_d(p/q)$ for the collection of all q-cycles under m_d with rotation number p/q.

Theorem 3.16 *Two distinct cycles $C, C' \in \mathscr{C}_d(p/q)$ are compatible if and only if the non-zero components of $s(C) - s(C')$ are all* 1 *or all* -1.

Proof First suppose C, C' are compatible. By Lemma 2.25 C, C' are superlinked, so their points alternate as we go around the circle. If μ, μ' denote the natural measures of C, C', it follows that the function

$$\chi : t \mapsto q\Big(\mu[0,t) - \mu'[0,t)\Big)$$

takes values in $\{0, 1\}$ or in $\{0, -1\}$. Thus, the non-zero components $(\chi(u_1), \ldots, \chi(u_{d-1}))$ of $s(C) - s(C')$ are all 1 or all -1.

Conversely, and without loss of generality, assume that all non-zero components of $\boldsymbol{\varepsilon} = s(C) - s(C')$ are 1. The sum $s(C) + s(C')$ has both even and odd components, so by Theorem 3.12 there is a rotation set X of size $2q$ with $\rho(X) = \rho(C) = \rho(C')$ and $s(X) = s(C) + s(C')$. Decompose X into the union of two compatible q-cycles Y, Y', where $s(Y) + s(Y') = s(C) + s(C')$. By the previous paragraph and after relabeling these cycles if necessary, we may assume that all non-zero components of $\boldsymbol{\varepsilon}' = s(Y) - s(Y')$ are 1. The relation $2s(C) + \boldsymbol{\varepsilon}' = 2s(Y) + \boldsymbol{\varepsilon}$ shows that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ have the same support (that is, their non-zero components occur at the same places), so $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}'$. It follows that $s(C) = s(Y)$ and $s(C') = s(Y')$. The uniqueness part of Theorem 3.12 then shows $C = Y$ and $C' = Y'$, which proves C, C' are compatible. □

The arithmetical criterion for realizability of signatures in Theorem 3.12 has a geometric interpretation due to McMullen. He comments in [19] that $\mathscr{C}_d(p/q)$ can be identified with the vertices of a simplicial subdivision of a $(d-2)$-dimensional simplex, with compatible cycles corresponding to adjacent vertices (compare Fig. 3.2). Below we provide a justification for this statement; Lemma 3.18 below will also play a role in the proof of Theorem 4.12 in the next chapter.

In view of Theorem 3.16 we can define a relation $\prec$ between any two compatible cycles $C, C' \in \mathscr{C}_d(p/q)$ by declaring $C \prec C'$ if the non-zero components of $s(C') - s(C)$ are all 1. Evidently a collection $C_1, \ldots, C_n$ in $\mathscr{C}_d(p/q)$ are compatible if and only if they are linearly ordered by $\prec$.

Lemma 3.17 *Suppose $C_1, \ldots, C_n$ are distinct compatible cycles in $\mathscr{C}_d(p/q)$. Then the deployment vectors $\delta(C_1), \ldots, \delta(C_n) \in \mathbb{R}^{d-1}$ are affinely independent.*

Proof After relabeling the cycles we may assume $C_1 \prec C_2 \prec \cdots \prec C_n$. Let $\boldsymbol{\varepsilon}_{i,j} = s(C_j) - s(C_i)$. The cocycle relation

$$\boldsymbol{\varepsilon}_{i,j} + \boldsymbol{\varepsilon}_{j,k} = \boldsymbol{\varepsilon}_{i,k}$$

shows that the vectors $\boldsymbol{\varepsilon}_{1,2}, \boldsymbol{\varepsilon}_{2,3}, \ldots, \boldsymbol{\varepsilon}_{n-1,n}$ have disjoint supports and therefore are linearly independent in $\mathbb{R}^{d-1}$. It follows that the vectors

$$\boldsymbol{\varepsilon}_{1,2}, \quad \boldsymbol{\varepsilon}_{1,3} = \boldsymbol{\varepsilon}_{1,2} + \boldsymbol{\varepsilon}_{2,3}, \quad \ldots, \quad \boldsymbol{\varepsilon}_{1,n} = \boldsymbol{\varepsilon}_{1,2} + \boldsymbol{\varepsilon}_{2,3} + \cdots + \boldsymbol{\varepsilon}_{n-1,n}$$

are also linearly independent.

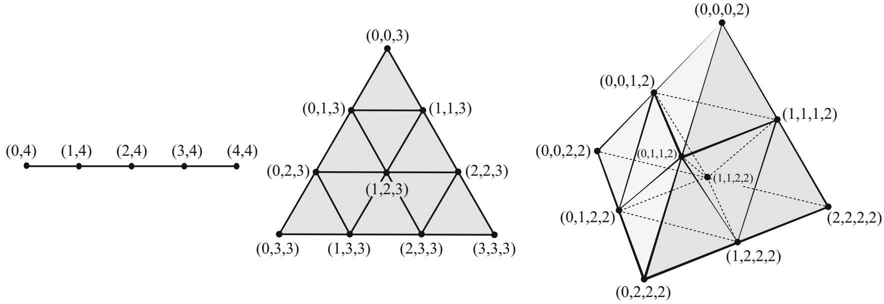

Fig. 3.2 Geometric representation of q-cycles as vertices of a subdivision Δ_q^{d-2} of the standard simplex Δ^{d-2}, following McMullen. Here each cycle is labeled by its signature and two cycles are compatible if and only if they are connected by an edge in Δ_q^{d-2}. Left: The five vertices of Δ_4^1 representing 4-cycles under m_3 with rotation number $\frac{1}{4}$ or $\frac{3}{4}$. Middle: The ten vertices of Δ_3^2 representing 3-cycles under m_4 with rotation number $\frac{1}{3}$ or $\frac{2}{3}$. Right: The ten vertices of Δ_2^3 representing 2-cycles under m_5 with rotation number $\frac{1}{2}$

To prove $\delta(C_1), \dots, \delta(C_n)$ are affinely independent, it suffices to verify the linear independence of the vectors $\{\delta(C_i) - \delta(C_1)\}_{2 \le i \le n}$. If $\sum_{i=2}^n \alpha_i (\delta(C_i) - \delta(C_1)) = 0$ for some scalars $\alpha_i \in \mathbb{R}$, then $\sum_{i=2}^n \alpha_i (\sigma(C_i) - \sigma(C_1)) = 0$, so

$$\sum_{i=2}^n \alpha_i \, \boldsymbol{\varepsilon}_{1,i} = \sum_{i=2}^n \alpha_i (s(C_i) - s(C_1)) = q \sum_{i=2}^n \alpha_i (\sigma(C_i) - \sigma(C_1)) = 0.$$

It follows from the previous paragraph that $\alpha_i = 0$ for all i. □

Recall that Δ^{d-2} is the standard simplex $\{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1} : x_i \ge 0$ and $\sum_{i=1}^{d-1} x_i = 1\}$. Fix a rotation number p/q and consider the finite set V consisting of vectors $(x_1, \dots, x_{d-1}) \in \Delta^{d-2}$ such that $q x_i \in \mathbb{Z}$ for all i. By Theorem 3.7, the assignment $C \mapsto \delta(C)$ is a bijection between $\mathscr{C}_d(p/q)$ and V. Let Δ_q^{d-2} be the collection of all convex hulls

$$[\delta(C_1), \dots, \delta(C_n)] = \Big\{ \sum_{i=1}^n \alpha_i \, \delta(C_i) : 0 \le \alpha_i \le 1 \text{ and } \sum_{i=1}^n \alpha_i = 1 \Big\},$$

where $C_1, \dots, C_n$ are distinct compatible cycles in $\mathscr{C}_d(p/q)$. Lemma 3.17 shows that $[\delta(C_1), \dots, \delta(C_n)]$ is an $(n-1)$-simplex in Δ^{d-2}.

Lemma 3.18 *Suppose $C_1, \dots, C_n$ are distinct compatible cycles in $\mathscr{C}_d(p/q)$, with $n > 1$. Then the interior of the $(n-1)$-simplex $[\delta(C_1), \dots, \delta(C_n)]$ does not meet V.*

Proof We may assume again that $C_1 \prec C_2 \prec \cdots \prec C_n$. Suppose there is a cycle $C \in \mathscr{C}_d(p/q)$ and scalars $0 < \alpha_1, \dots, \alpha_n < 1$ with $\sum_{i=1}^n \alpha_i = 1$ such that

$\sum_{i=1}^{n} \alpha_i\, \delta(C_i) = \delta(C)$. Then $\sum_{i=1}^{n} \alpha_i\, s(C_i) = s(C)$, so

$$\sum_{i=2}^{n} \alpha_i\, \boldsymbol{\varepsilon}_{1,i} = \sum_{i=2}^{n} \alpha_i (s(C_i) - s(C_1)) = s(C) - s(C_1),$$

where $\boldsymbol{\varepsilon}_{i,j} = s(C_j) - s(C_i)$ as before. Using the relation

$$\boldsymbol{\varepsilon}_{1,i} = \boldsymbol{\varepsilon}_{1,2} + \boldsymbol{\varepsilon}_{2,3} + \cdots + \boldsymbol{\varepsilon}_{i-1,i}, \tag{3.12}$$

we can rewrite this as

$$\sum_{i=2}^{n} \beta_i\, \boldsymbol{\varepsilon}_{i-1,i} = s(C) - s(C_1),$$

where $0 < \beta_i = \alpha_i + \cdots + \alpha_n < 1$. Since the vectors $\{\boldsymbol{\varepsilon}_{i-1,i}\}_{2 \le i \le n}$ have disjoint supports, the components of $\sum_{i=2}^{n} \beta_i\, \boldsymbol{\varepsilon}_{i-1,i}$ consist of the β_i and possibly some 0's. This contradicts the fact that $s(C) - s(C_1)$ is a non-zero integer vector. □

Theorem 3.19 *Δ_q^{d-2} is a simplicial subdivision of Δ^{d-2}.*

By Corollaries 3.11 and 3.15, Δ_q^{d-2} has $\binom{q+d-2}{q}$ vertices and q^{d-2} top-dimensional cells. The cases $d = 3, 4$ produce regular linear and triangular subdivisions, but the situation for $d > 4$ is not as symmetric (see Fig. 3.2).

Proof To show Δ_q^{d-2} is a simplicial complex, it suffices to check that two simplices $[\delta(C_1), \ldots, \delta(C_n)]$ and $[\delta(C'_1), \ldots, \delta(C'_m)]$ in Δ_q^{d-2} whose interiors intersect must coincide. The case $n = m = 1$ is trivial and the cases $n = 1, m > 1$ or $n > 1, m = 1$ are already covered by Lemma 3.18, so we may assume $n, m > 1$. Label the cycles so that $C_1 \prec \cdots \prec C_n$ and $C'_1 \prec \cdots \prec C'_m$. By our hypothesis, there are scalars $0 < \alpha_1, \ldots, \alpha_n < 1$ and $0 < \alpha'_1, \ldots, \alpha'_m < 1$, with $\sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{m} \alpha'_j = 1$, such that $\sum_{i=1}^{n} \alpha_i\, \delta(C_i) = \sum_{j=1}^{m} \alpha'_j\, \delta(C'_j)$. Then $\sum_{i=1}^{n} \alpha_i\, s(C_i) = \sum_{j=1}^{m} \alpha'_j\, s(C'_j)$. Letting $\boldsymbol{\varepsilon}_{i,j} = s(C_j) - s(C_i)$ and $\boldsymbol{\varepsilon}'_{i,j} = s(C'_j) - s(C'_i)$, it follows that

$$s(C_1) + \sum_{i=2}^{n} \alpha_i\, \boldsymbol{\varepsilon}_{1,i} = \sum_{i=1}^{n} \alpha_i\, s(C_i) = \sum_{j=1}^{m} \alpha'_j\, s(C'_j) = s(C'_1) + \sum_{j=2}^{m} \alpha'_j\, \boldsymbol{\varepsilon}'_{1,j},$$

or

$$\sum_{i=2}^{n} \alpha_i\, \boldsymbol{\varepsilon}_{1,i} - \sum_{j=2}^{m} \alpha'_j\, \boldsymbol{\varepsilon}'_{1,j} = s(C'_1) - s(C_1)$$

Using (3.12) and the similar relation for the $\boldsymbol{\varepsilon}'_{1,j}$, we can rewrite the above equation as

$$\sum_{i=2}^{n} \beta_i \, \boldsymbol{\varepsilon}_{i-1,i} - \sum_{j=2}^{m} \beta'_j \, \boldsymbol{\varepsilon}'_{j-1,j} = s(C'_1) - s(C_1), \tag{3.13}$$

where $0 < \beta_i = \alpha_i + \cdots + \alpha_n < 1$ and $0 < \beta'_j = \alpha'_j + \cdots + \alpha'_m < 1$. Since the vectors $\{\boldsymbol{\varepsilon}_{i-1,i}\}_{2 \le i \le n}$ have disjoint supports, the non-zero components of $\sum_{i=2}^{n} \beta_i \, \boldsymbol{\varepsilon}_{i-1,i}$ are precisely the β_i. Similarly, the non-zero components of $\sum_{j=2}^{m} \beta'_j \, \boldsymbol{\varepsilon}'_{j-1,j}$ are the β'_j. It follows that the components of the left hand side of (3.13) lie strictly between -1 and 1. Since the right hand side of (3.13) is an integer vector, the two sides must vanish. Thus, $s(C_1) = s(C'_1)$ and the finite sequences

$$1 > \beta_2 > \cdots > \beta_n = \alpha_n > 0 \quad \text{and} \quad 1 > \beta'_2 > \cdots > \beta'_m = \alpha'_m > 0$$

coincide. This implies $n = m$, $\alpha_i = \alpha'_i$ and $s(C_i) = s(C'_i)$ for all $1 \le i \le n$.

To finish the proof of the theorem, it remains to show that every $x = (x_1, \ldots, x_{d-1}) \in \Delta^{d-2}$ belongs to a simplex in Δ_q^{d-2}. Let $y = (y_1, \ldots, y_{d-1})$, where $y_i = q(x_1 + \cdots + x_i)$. Then $0 \le y_1 \le \cdots \le y_{d-1} = q$. Let $t_i \in [0, 1)$ be the fractional part of y_i. If all the t_i are zero, then $x \in V$ and we are done. Otherwise, list the non-zero elements of $\{t_1, \ldots, t_{d-1}\}$ in decreasing order as

$$t_{i_1} \ge \ldots \ge t_{i_n}, \quad \text{where} \quad 1 \le n \le d-2.$$

Here we adopt the convention that if several t_i's are equal, we list them in the order of decreasing subscripts, that is, if $t_{i_k} = t_{i_{k+1}}$, then $i_k > i_{k+1}$. Let $e_1, \ldots, e_{d-1}$ denote the unit coordinate vectors in $\mathbb{R}^{d-1}$ and define

$$\begin{aligned} v_1 &= y - (t_1, \ldots, t_{d-1}) \\ v_{k+1} &= v_k + e_{i_k} \qquad 1 \le k \le n. \end{aligned} \tag{3.14}$$

It is not hard to check that the components of each v_k form a monotonic sequence of non-negative integers ending in q, and therefore there is a unique cycle $C_k \in \mathscr{C}_d(p/q)$ with $s(C_k) = v_k$. By Theorem 3.16, $C_1, \ldots, C_{n+1}$ are compatible. Define the scalars $\{\alpha_k\}_{1 \le k \le n+1}$ by

$$\alpha_k = t_{i_{k-1}} - t_{i_k},$$

where $t_{i_0} = 1$ and $t_{i_{n+1}} = 0$. Note that the α_k are non-negative and add up to 1. It follows from (3.14) that

$$y = v_1 + \sum_{k=1}^{n} t_{i_k} \, e_{i_k} = v_1 + \sum_{k=1}^{n} t_{i_k} (v_{k+1} - v_k)$$

$$= \sum_{k=1}^{n+1} (t_{i_{k-1}} - t_{i_k})\, v_k = \sum_{k=1}^{n+1} \alpha_k\, v_k = \sum_{k=1}^{n+1} \alpha_k\, s(C_k),$$

so $x = \sum_{k=1}^{n+1} \alpha_k\, \delta(C_k)$, as required. □

3.3 Deployment Theorem: The Irrational Case

We now proceed to the irrational case of the deployment theorem. Our approach closely parallels the one presented for the rational case in the proof of Theorem 3.7.

Theorem 3.20 (Goldberg-Tresser) *For every irrational number* $0 < \theta < 1$ *and every vector* $(\delta_1, \dots, \delta_{d-1}) \in \Delta^{d-2}$ *there is a unique minimal rotation set X for* m_d *such that* $\rho(X) = \theta$ *and* $\delta(X) = (\delta_1, \dots, \delta_{d-1})$.

Thus, the space of all minimal rotation sets for m_d of a given irrational rotation number can be identified with the simplex $\Delta^{d-2} \subset \mathbb{R}^{d-1}$. When $d = 2$, it follows from this and Corollary 2.38 that *there is a unique rotation set under doubling with a given irrational rotation number*.

Proof We continue using the notation $\equiv$ for congruence modulo $\mathbb{Z}$. As in the rational case, we may assume without loss of generality that $\delta_1 \neq 0$. Set $\sigma_i = \delta_1 + \dots + \delta_i$ for $1 \leq i \leq d-1$, so $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{d-1} = 1$. We construct a degree 1 monotone map φ of the circle with the following properties:

(i) $I_s \neq \emptyset$ implies $I_{s-\theta} \neq \emptyset$, where I_s is the interior of the fiber $\varphi^{-1}(s)$;
(ii) $\varphi(dt) \equiv \varphi(t) + \theta$ whenever t is not in the closure of a plateau of φ; and
(iii) $\varphi(u_i) \equiv \sigma_i$ for $1 \leq i \leq d-1$.

Properties (i) and (ii) prove that the complement of the union of all plateaus of φ is a minimal rotation set X with $\rho(X) = \theta$ (Theorem 2.35), while property (iii) proves that $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1})$ (Lemma 3.3).

Let S be the union of the backward orbits of the σ_i under r_θ:

$$S = \{\sigma_i - k\theta \ (\mathrm{mod}\ \mathbb{Z}) : 1 \leq i \leq d-1 \text{ and } k \geq 0\}. \tag{3.15}$$

Consider the atomic probability measure ν supported on S defined by

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - k\theta}. \tag{3.16}$$

Observe that $\nu\{\sigma_i\} \geq 1/d$ for every i. More precisely, if some $s \in S$ appears exactly n times in the list $\{\sigma_1, \dots, \sigma_{d-1}\}$, then $n/d \leq \nu\{s\} < (n+1)/d$. The lower bound follows from the definition, whereas the upper bound holds since the contribution

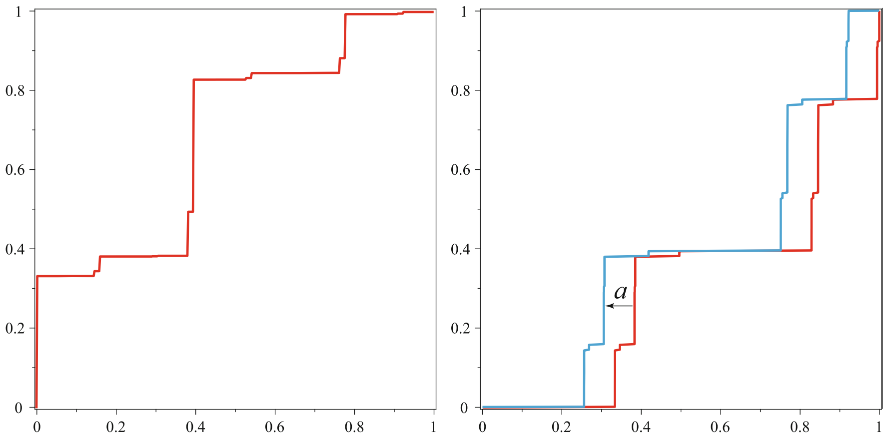

Fig. 3.3 Left: The graph of the map ψ obtained by integrating the atomic measure ν. Right: The graph of the left-inverse map ψ^{-1} along with its translation φ (in blue). In this example, $d = 3$, $\rho(X) = \frac{(\sqrt{5}-1)}{2}$, and $\delta(X) = (0.39475, 0.60525)$. Computation gives $a \approx 0.07713$

of the remaining terms of (3.16) to $\nu\{s\}$ is at most

$$(d-2)\sum_{k=1}^{\infty} d^{-(k+1)} = \frac{d-2}{d(d-1)} < \frac{1}{d}.$$

The same argument shows that $0 < \nu\{s\} < 1/d$ whenever $s \in S$ is not congruent to any of the σ_i.

The map $\psi : \mathbb{T} \to \mathbb{T}$ defined by $\psi(t) \equiv \nu[0, t)$ has degree 1, is strictly monotone, is continuous on $\mathbb{T} \smallsetminus S$ and is discontinuous at every $s \in S$ where it jumps by $\nu\{s\}$. The left-inverse ψ^{-1} extends to a *continuous* degree 1 monotone map of the circle, with a plateau I_s precisely when $s \in S$. Let $N_1 \geq 1$ be the number of indices $1 \leq j \leq d-2$ for which $\sigma_j = 1$. Set

$$a = \frac{N_1 - \nu[0, \theta)}{d-1}. \tag{3.17}$$

We show that the map $\varphi : \mathbb{T} \to \mathbb{T}$ defined by $\varphi(t) \equiv \psi^{-1}(t + a)$ has properties (i)–(iii) (see Fig. 3.3 for a typical graph of ψ and φ for the case $d = 3$).

Property (i) is immediate since $s \in S$ implies $s - \theta \in S$. The relation

$$\nu(B + \theta) \equiv d\nu(B) \tag{3.18}$$

for every Borel set B is easily verified from the definition of ν. It implies

$$\nu[0, t+\theta) \equiv \nu[0, \theta) + \nu[\theta, t+\theta) \equiv \psi(\theta) + d\nu[0, t),$$

which gives the functional equation

$$\psi(t+\theta) \equiv d\psi(t) + \psi(\theta).$$

Applying the left-inverse ψ^{-1} to both sides, we obtain

$$t+\theta \equiv \psi^{-1}(d\psi(t)+\psi(\theta)) \equiv \varphi(d\psi(t)+\psi(\theta)-a).$$

If t is not in the closure of a plateau of φ, then $\psi(\varphi(t)) = t+a$ and it follows that

$$\begin{aligned}\varphi(t)+\theta &\equiv \varphi(d(t+a)+\psi(\theta)-a) \\ &\equiv \varphi(dt+(d-1)a+\psi(\theta)) \\ &\equiv \varphi(dt+N_1) \equiv \varphi(dt).\end{aligned} \tag{3.19}$$

This proves property (ii).

To verify (iii), first note that $\psi^{-1}(t) \equiv 0$ for $t \in [0, \nu\{0\}]$. Since $\nu[0,\theta) > \nu\{0\} \geq N_1/d$, we have $N_1/d < \nu[0,\theta) < 1$, or $0 < a < N_1/d \leq \nu\{0\}$. In particular, $\varphi(0) \equiv \psi^{-1}(a) \equiv 0$. By what we have seen above, if there is an n-fold incidence $s = \sigma_i = \sigma_{i+1} = \cdots = \sigma_{i+n-1}$, the jump $\ell = \nu\{s\}$ of ψ at s satisfies the inequalities $n/d \leq \ell < (n+1)/d$. It follows that φ has a plateau of length ℓ on which it takes the constant value s. This plateau is a major gap of X, so it contains precisely n fixed points of m_d by Lemma 2.13. Thus, φ maps the fixed point set $\{u_1, \ldots, u_{d-1}\}$ to the set $\{\sigma_1, \ldots, \sigma_{d-1}\}$, sending n of the u_i to the same point s if and only if n of the σ_i collide at s. Since $\varphi(0) \equiv 0$, it follows from monotonicity of φ that $\varphi(u_i) \equiv \sigma_i$ for every i.

Finally, we prove uniqueness of X. Suppose $\hat{X}$ is any minimal rotation set with $\rho(\hat{X}) = \theta$ and $\delta(\hat{X}) = (\delta_1, \ldots, \delta_{d-1})$. Let $\hat{\varphi}$ be the canonical semiconjugacy associated with $\hat{X}$. By Lemma 3.3, $\hat{\varphi}(u_i) \equiv \sigma_i$, so $\hat{\varphi}$ takes the value σ_i on the major gap of $\hat{X}$ containing u_i. Moreover, if $\hat{X}$ has a major gap of multiplicity n, there will be an n-fold incidence between the σ_i. Since the gaps of $\hat{X}$ are precisely the plateaus of $\hat{\varphi}$, and since every gap eventually maps to a major gap, it follows that the values taken by $\hat{\varphi}$ on its plateaus form the set S in (3.15). It is now easy to see that the push-forward $\hat{\varphi}_*\lambda$ of Lebesgue measure is just the measure ν in (3.16). Since $\varphi_*\lambda = \nu$ also by the construction, the relation $\hat{\varphi}_*\lambda = \varphi_*\lambda$ must hold. Let $D \subset X$ be the countable set of the endpoints of gaps, and similarly define $\hat{D} \subset \hat{X}$. As the maps $\varphi : X \smallsetminus D \to \mathbb{T} \smallsetminus S$ and $\hat{\varphi} : \hat{X} \smallsetminus \hat{D} \to \mathbb{T} \smallsetminus S$ are bijective, the composition $\hat{\varphi}^{-1} \circ \varphi : X \smallsetminus D \to \hat{X} \smallsetminus \hat{D}$ defines a bijection $t \mapsto \hat{t}$ that preserves the cyclic order of all triples and commutes with m_d. Since for every $t_1, t_2 \in X \smallsetminus D$,

$$\lambda((t_1, t_2)) = \nu((\varphi(t_1), \varphi(t_2))) = \lambda((\hat{t}_1, \hat{t}_2)),$$

it follows that $t \mapsto \hat{t}$ is the restriction of some rigid rotation r_α to $X \smallsetminus D$. In other words, r_α maps $X \smallsetminus D$ onto $\hat{X} \smallsetminus \hat{D}$ and therefore X onto $\hat{X}$, and it commutes with m_d. To finish the proof, we must show that $\alpha \equiv 0$. The proof is identical to the rational case: Let I_0 be the major gap of X containing 0, so $r_\alpha(I_0)$ is the major gap of $\hat{X}$ containing 0 (this follows from the normalization $\varphi(0) \equiv \hat{\varphi}(0) \equiv 0$). By our construction, I_0 and $r_\alpha(I_0)$ contain the same set of fixed points of m_d, namely those which map under φ or $\hat{\varphi}$ to $\sigma_{d-1} = 1 \equiv 0$. Since the fixed points of m_d are $1/(d-1)$ apart, it follows that the distance between α and 0 must be $< 1/(d-1)$. On the other hand, r_α commutes with m_d, so $d(t+\alpha) \equiv dt + \alpha$ for every $t \in X$, which implies $(d-1)\alpha \equiv 0$. The only solution of this equation whose distance to 0 is $< 1/(d-1)$ is $\alpha \equiv 0$, and the proof is complete. □

Epilogue To conclude this chapter, let us briefly recap the main constructions related to a minimal rotation set and how they lead to the proofs of the deployment Theorems 3.7 and 3.20. Suppose X is a minimal rotation set for m_d with $\rho(X) = \theta \neq 0$, so X is a q-cycle if $\theta = p/q$ in lowest terms, and a Cantor set if θ is irrational.

- The ***canonical semiconjugacy*** associated with X is a degree 1 monotone map $\varphi : \mathbb{T} \to \mathbb{T}$, normalized by $\varphi(0) = 0$, which satisfies

$$\varphi \circ m_d = r_\theta \circ \varphi \quad \text{on} \quad X.$$

  The plateaus of φ are precisely the gaps of X.
- The ***natural measure*** of X is the unique m_d-invariant probability measure μ supported on X. It is related to the canonical semiconjugacy by

$$\varphi(t) = \int_0^t d\mu = \mu[0,t] \pmod{\mathbb{Z}}.$$

  If $\theta = p/q$ in lowest terms, then μ is the uniform Dirac measure on X:

$$\mu = \frac{1}{q} \sum_{x \in X} \mathbb{1}_x.$$

  If θ is irrational, then μ is the (well-defined) pull-back of Lebesgue measure λ under φ:

$$\lambda = \varphi_* \mu.$$

- The ***deployment vector*** of X is the probability vector $\delta(X) = (\delta_1, \ldots, \delta_{d-1}) \in \mathbb{R}^{d-1}$ defined by

$$\delta_i = \mu[u_{i-1}, u_i) \qquad 1 \leq i \leq d-1,$$

  where the $u_i = i/(d-1)$ are the fixed points of m_d.
- The ***cumulative deployment vector*** $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$ is defined by

$$\sigma_i = \mu[u_0, u_i) = \delta_1 + \cdots + \delta_i \qquad 1 \leq i \leq d-1.$$

- The ***gap measure*** of X is the push-forward ν of Lebesgue measure λ under φ:

$$\nu = \varphi_* \lambda.$$

The terminology comes from the observation that each gap I of X maps under φ to a single point s with $\nu\{s\} = |I|$. The gap measure can be expressed in terms of $\rho(X) = \theta$ and $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$ by the explicit formula

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - k\theta}. \tag{3.20}$$

In particular, ν is an atomic measure supported on the set

$$S = \{\sigma_i - k\theta \ (\mathrm{mod}\ \mathbb{Z}) : 1 \leq i \leq d-1 \text{ and } k \geq 0\},$$

which is a union of at most $d-1$ backward orbits of the rotation r_θ. Thus, S consists of the qth roots of unity if $\theta = p/q$ in lowest terms, and is dense if θ is irrational.

- The minimal rotation set X can be recovered from its rotation number (whether rational or irrational) and deployment data as follows: Form the gap measure ν as above, and let $\psi(t) = \nu[0, t)$ for $t \in \mathbb{T}$ which has a well-defined left inverse ψ^{-1}. Define $\varphi : \mathbb{T} \to \mathbb{T}$ by

$$\varphi(t) = \psi^{-1}(t + a), \quad \text{where} \quad a = \frac{N_1 - \nu[0, \theta)}{d-1}.$$

Here $N_1 \geq 1$ is the number of indices $1 \leq j \leq d-1$ for which $\sigma_j = 1$. Then φ is the canonical semiconjugacy associated with X, so X is the complement of the union of plateaus of φ.