Chapter 1
Monotone Maps of the Circle

Throughout this monograph the following conventions are adopted:

- The circle is represented as the quotient $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- $\pi : \mathbb{R} \to \mathbb{T}$ is the canonical projection.
- Three or more distinct points $t_1, t_2, \ldots, t_k \in \mathbb{T}$ are in ***positive cyclic order*** if there are representatives $x_i \in \pi^{-1}(t_i)$ such that $x_1 < x_2 < \cdots < x_k < x_1 + 1$.
- For a distinct pair $t_1, t_2 \in \mathbb{T}$, the interval $(t_1, t_2) \subset \mathbb{T}$ is defined as the set of all $t \in \mathbb{T}$ such that t_1, t, t_2 are in positive cyclic order. We define the intervals $(t_1, t_2]$, $[t_1, t_2)$, $[t_1, t_2]$ by adding the suitable endpoints to (t_1, t_2).
- The length of an interval $(t_1, t_2) \subset \mathbb{T}$ is always understood as its normalized Lebesgue measure, that is, the unique representative of $t_2 - t_1$ in $[0, 1)$.

Every continuous map $g : \mathbb{T} \to \mathbb{T}$ lifts under the canonical projection π to a continuous map $G : \mathbb{R} \to \mathbb{R}$, so $\pi \circ G = g \circ \pi$, and G is unique up to an additive integer. The lift G satisfies $G(x + 1) = G(x) + d$ for some integer d called the ***degree*** of g. We say that g is a ***monotone map*** if G is monotone in the usual sense (non-increasing or non-decreasing).

This chapter studies the dynamics of degree 1 monotone maps of the circle, which can be thought of as slight generalizations of orientation preserving homeomorphisms. It will be convenient to first work with lifts of such maps, i.e., continuous non-decreasing self-maps of the real line that commute with the unit translation.

1.1 The Translation Number

Suppose $G : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing map which satisfies

$$G(x + 1) = G(x) + 1 \quad \text{for all} \quad x \in \mathbb{R}. \tag{1.1}$$

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If $0 \leq x < y < 1$, then

$$(G(x) - x) - (G(y) - y) \leq (G(y) - x) - (G(y) - y) = y - x < 1$$

and

$$(G(x) - x) - (G(y) - y) \geq (G(0) - x) - (G(1) - y) = y - x - 1 > -1.$$

Since by (1.1) the function $G - \mathrm{id}_{\mathbb{R}}$ is 1-periodic, the inequality

$$|(G(x) - x) - (G(y) - y)| < 1$$

follows for all $x, y \in \mathbb{R}$. The same reasoning applied to the n-th iterate $G^{\circ n}$ shows that

$$|(G^{\circ n}(x) - x) - (G^{\circ n}(y) - y)| < 1 \quad \text{for all} \quad x, y \in \mathbb{R} \quad \text{and} \quad n \geq 1. \tag{1.2}$$

Lemma 1.1 *There exists at most one rational number p/q with $q > 0$ for which the equation $G^{\circ q}(x) = x + p$ has a solution in $x \in \mathbb{R}$.*

Proof Suppose $G^{\circ q}(x) = x + p$ and $G^{\circ n}(y) = y + m$. Then $G^{\circ nq}(x) = x + np$ and $G^{\circ nq}(y) = y + mq$. By (1.2),

$$|(G^{\circ nq}(x) - x) - (G^{\circ nq}(y) - y)| = |np - mq| < 1,$$

which implies $np = mq$. □

Consider the sets

$$\mathbb{Q}_G^- = \left\{ \frac{p}{q} : G^{\circ q}(x) > x + p \text{ for all } x \in \mathbb{R} \right\},$$

$$\mathbb{Q}_G^+ = \left\{ \frac{p}{q} : G^{\circ q}(x) < x + p \text{ for all } x \in \mathbb{R} \right\},$$

where p, q are integers with $q > 0$. Evidently $\mathbb{Q}_G^-$ and $\mathbb{Q}_G^+$ are non-empty disjoint subsets of the set $\mathbb{Q}$ of rational numbers. Furthermore,

1. If $p/q \notin \mathbb{Q}_G^- \cup \mathbb{Q}_G^+$, both equations $G^{\circ q}(x) > x + p$ and $G^{\circ q}(x) < x + p$ have solutions and so does $G^{\circ q}(x) = x + p$ by continuity. Applying Lemma 1.1, we see that the union $\mathbb{Q}_G^- \cup \mathbb{Q}_G^+$ can omit at most one rational number.
2. If $p/q \in \mathbb{Q}_G^-$ and $m/n \in \mathbb{Q}_G^+$, then $x + np < G^{\circ nq}(x) < x + mq$ for all x, so $p/q < m/n$.
3. If $p/q \in \mathbb{Q}_G^-$, since the function $G^{\circ q} - \mathrm{id}_{\mathbb{R}}$ is 1-periodic and $> p$, there is an $\varepsilon > 0$ such that $G^{\circ q}(x) > x + p + \varepsilon$ for all x. It follows by induction that $G^{\circ nq}(x) > x + np + n\varepsilon$ for all x and $n \geq 1$, which proves $(np + 1)/(nq) \in \mathbb{Q}_G^-$ as soon as $n > 1/\varepsilon$. This shows $\mathbb{Q}_G^-$ has no largest element. Similarly, $\mathbb{Q}_G^+$ has no smallest element.

Properties (1) and (2) show that the pair $(\mathbb{Q}_G^-, \mathbb{Q}_G^+)$ is a "Dedekind cut" of $\mathbb{Q}$ and

$$\sup \mathbb{Q}_G^- = \inf \mathbb{Q}_G^+.$$

We call this common value the ***translation number*** of G and denote it by $\tau(G)$. It follows from property (3) that

$$\mathbb{Q}_G^- \cup \mathbb{Q}_G^+ = \begin{cases} \mathbb{Q} \setminus \{\tau(G)\} & \text{if } \tau(G) \in \mathbb{Q} \\ \mathbb{Q} & \text{if } \tau(G) \notin \mathbb{Q}. \end{cases} \tag{1.3}$$

The terminology for $\tau(G)$ is justified by the following

Theorem 1.2 (Poincaré) *For every $x \in \mathbb{R}$,*

$$\tau(G) = \lim_{n\to\infty} \frac{G^{\circ n}(x) - x}{n}. \tag{1.4}$$

Thus, $\tau(G)$ measures the average translation per iterate that each point experiences under repeated applications of G.

Proof For any integer $n \geq 1$ we can find an integer m such that $(m-1)/n < \tau(G) < (m+1)/n$. Then $(m-1)/n \in \mathbb{Q}_G^-$ and $(m+1)/n \in \mathbb{Q}_G^+$, so

$$\frac{m-1}{n} < \frac{G^{\circ n}(x) - x}{n} < \frac{m+1}{n}$$

for all x. This gives the inequality

$$\left| \frac{G^{\circ n}(x) - x}{n} - \tau(G) \right| < \frac{2}{n} \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and} \quad n \geq 1. \tag{1.5}$$

The result follows by letting $n \to \infty$. □

Corollary 1.3 *The equation $G^{\circ q}(x) = x + p$ has a solution in $x \in \mathbb{R}$ if and only if $\tau(G) = p/q$.*

Proof Evidently $G^{\circ q}(x) = x + p$ for some x if and only if $p/q \notin \mathbb{Q}_G^- \cup \mathbb{Q}_G^+$. By (1.3), this is equivalent to $\tau(G) = p/q$. □

Corollary 1.4 *Suppose n_1, n_2, m_1, m_2 are integers with $n_1 \geq 0$ and $n_2 \geq 0$. Then*

$$n_1\tau(G) + m_1 < n_2\tau(G) + m_2 \tag{1.6}$$

if and only if

$$G^{\circ n_1}(x) + m_1 < G^{\circ n_2}(x) + m_2 \quad \textit{for all} \quad x \in \mathbb{R}. \tag{1.7}$$

Proof The case $n_1 = n_2$ is trivial, so let us assume $0 \leq n_1 < n_2$. In this case, the inequality (1.6) is equivalent to $(m_1 - m_2)/(n_2 - n_1) < \tau(G)$ which by (1.3) is equivalent to $(m_1 - m_2)/(n_2 - n_1) \in \mathbb{Q}_G^-$. The latter means $x + m_1 - m_2 < G^{\circ n_2 - n_1}(x)$ for all x, which is clearly equivalent to (1.7). The case $n_1 > n_2 \geq 0$ is treated similarly. □

1.2 The Rotation Number

Now consider a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$. By definition, this means g lifts to a continuous non-decreasing map $G : \mathbb{R} \to \mathbb{R}$ which commutes with the unit translation. All other lifts of g are of the form $G + k$ for some integer k, with the translation number $\tau(G + k) = \tau(G) + k$ by (1.4).

Definition 1.5 The ***rotation number*** $\rho(g)$ of a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ is the residue class modulo $\mathbb{Z}$ of the translation number $\tau(G)$, where $G : \mathbb{R} \to \mathbb{R}$ is any lift of g.

For convenience, we often identify $\rho(g)$ with its unique representative in $[0, 1)$. As a main example, for any $\theta \in [0, 1)$ the ***rigid rotation*** $r_\theta : \mathbb{T} \to \mathbb{T}$ defined by

$$r_\theta(t) = t + \theta \pmod{\mathbb{Z}}$$

has rotation number $\rho(r_\theta) = \theta$.

Theorem 1.6 *Let $g : \mathbb{T} \to \mathbb{T}$ be a degree 1 monotone map with $\rho(g) = \theta$. If the orbit points $r_\theta^{\circ i}(0), r_\theta^{\circ j}(0), r_\theta^{\circ k}(0)$ under the rigid rotation r_θ are in positive cyclic order, the same must be true of the orbit points $g^{\circ i}(t), g^{\circ j}(t), g^{\circ k}(t)$ for every $t \in \mathbb{T}$.*

(If θ is a fraction of the form p/q in lowest terms, we need to assume $q > 2$ in order for the theorem to have any content.)

Proof The assumption means that there are integers m_1, m_2 such that

$$i\theta < j\theta + m_1 < k\theta + m_2 < i\theta + 1.$$

If we choose a lift G of g so that $\tau(G) = \theta$, Corollary 1.4 shows that for all $x \in \mathbb{R}$,

$$G^{\circ i}(x) < G^{\circ j}(x) + m_1 < G^{\circ k}(x) + m_2 < G^{\circ i}(x) + 1.$$

Projecting down to the circle, it follows that $g^{\circ i}(t), g^{\circ j}(t), g^{\circ k}(t)$ are in positive cyclic order for every $t \in \mathbb{T}$. □

Theorem 1.7 *For every degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ and every integer $k \geq 0$,*

$$\rho(g^{\circ k}) = k\,\rho(g) \pmod{\mathbb{Z}}. \tag{1.8}$$

If g is a homeomorphism, the above formula holds for negative k as well.

Here $g^{\circ 0} = \mathrm{id}$ and $g^{\circ k}$ means the $(-k)$-th iterate of g^{-1} if $k < 0$.

Proof (1.8) is trivial for $k = 0$, so let us assume $k \geq 1$. For any lift G of g, the iterate $G^{\circ k}$ is a lift of $g^{\circ k}$ and

$$\tau(G^{\circ k}) = \lim_{n\to\infty} \frac{G^{\circ kn}(0)}{n} = k \lim_{n\to\infty} \frac{G^{\circ kn}(0)}{kn} = k\,\tau(G).$$

Taking residue classes modulo $\mathbb{Z}$ then proves (1.8). If g is a homeomorphism, so is G and the inverse G^{-1} is a lift of g^{-1}. The uniform estimate (1.5) applied to $x = (G^{-1})^{\circ n}(0)$ gives

$$\left| \frac{(G^{-1})^{\circ n}(0)}{n} + \tau(G) \right| < \frac{2}{n}$$

for all $n \geq 1$. Letting $n \to \infty$, we obtain $\tau(G^{-1}) = -\tau(G)$, which proves (1.8) for $k = -1$. The general case $k < 0$ follows from this by iteration. □

Theorem 1.8 *Either of the following assumptions on degree* 1 *monotone maps* $g, h : \mathbb{T} \to \mathbb{T}$ *implies* $\rho(g) = \rho(h)$*:*

(i) *g and h agree along some orbit, that is, there is a $t \in \mathbb{T}$ such that $g^{\circ n}(t) = h^{\circ n}(t)$ for all $n \geq 1$.*
(ii) *g and h are semiconjugate, that is, there is a degree* 1 *monotone map $\varphi : \mathbb{T} \to \mathbb{T}$ which satisfies the relation $\varphi \circ g = h \circ \varphi$.*

Proof

(i) Let $t_n = g^{\circ n}(t) = h^{\circ n}(t)$. We may assume $t_n \neq t_{n-1}$ for all n since otherwise both g, h have a fixed point and $\rho(g) = \rho(h) = 0$. Pick any $x_0 \in \pi^{-1}(t_0)$ and define x_n inductively as the smallest element of $\pi^{-1}(t_n)$ that is $> x_{n-1}$. Thus, $x_{n-1} < x_n < x_{n-1} + 1$ for all n. Take the unique lift G of g that sends x_0 to x_1 and let $y_n = G^{\circ n}(x_0)$. Applying G repeatedly on the inequalities $y_0 < y_1 < y_0 + 1$ then shows $y_{n-1} < y_n < y_{n-1} + 1$ for all n, where the inequalities remains strict by the assumption $t_n \neq t_{n-1}$. Since y_n is an integer translation of x_n and $y_0 = x_0$, it follows that $y_n = x_n$ for all n. Similarly, the unique lift H of h that sends x_0 to x_1 satisfies $x_n = H^{\circ n}(x_0)$ for all n. It follows from Theorem 1.2 that $\tau(G) = \lim_{n\to\infty}(x_n - x_0)/n = \tau(H)$, which proves $\rho(g) = \rho(h)$.

(ii) Choose lifts $G, H, \Phi : \mathbb{R} \to \mathbb{R}$ of g, h, φ such that $\Phi \circ G = H \circ \Phi$. Then $\Phi \circ G^{\circ n} = H^{\circ n} \circ \Phi$ for all n. Since Φ commutes with the unit translation, the function $\Phi - \mathrm{id}_{\mathbb{R}}$ is 1-periodic and therefore bounded on $\mathbb{R}$. It follows that

$$\lim_{n\to\infty} \frac{H^{\circ n}(\Phi(0)) - G^{\circ n}(0)}{n} = \lim_{n\to\infty} \frac{\Phi(G^{\circ n}(0)) - G^{\circ n}(0)}{n} = 0,$$

which shows $\tau(G) = \tau(H)$. □

The following can be thought of as an analog of Corollary 1.3 for arbitrary rotation numbers:

Lemma 1.9 *Let $g : \mathbb{T} \to \mathbb{T}$ be a degree 1 monotone map with $\rho(g) = \theta$. Then $g(t) = r_\theta(t)$ for some $t \in \mathbb{T}$.*

Proof Let $0 \leq \theta < 1$ and choose a lift G of g with $\tau(G) = \theta$. Suppose $G(x) > x+\theta$ for all $x \in \mathbb{R}$. Then, since the function $G - \mathrm{id}_{\mathbb{R}}$ is 1-periodic, there is an $\varepsilon > 0$ such that $G(x) > x+\theta+\varepsilon$ for all x. It follows by induction that $G^{\circ n}(x) > x+n(\theta+\varepsilon)$ for all x and all $n \geq 1$. By (1.4) this would imply $\tau(G) \geq \theta+\varepsilon$, which is a contradiction. Similarly, the assumption $G(x) < x + \theta$ for all x leads to a contradiction. Thus, $G(x) = x + \theta$ for some $x \in \mathbb{R}$. □

Here is a consequence of the above lemma that will be used in Sect. 4.1:

Corollary 1.10 *For every orientation-preserving homeomorphism $g : \mathbb{T} \to \mathbb{T}$ and every rigid rotation r_θ, the commutator $[g, r_\theta] = g \circ r_\theta \circ g^{-1} \circ r_\theta^{-1}$ has rotation number zero.*

Proof By Theorem 1.8(ii), $\rho(g \circ r_\theta \circ g^{-1}) = \rho(r_\theta) = \theta$. By Lemma 1.9, there is a $t \in \mathbb{T}$ such that $(g \circ r_\theta \circ g^{-1})(t) = r_\theta(t)$. This means $r_\theta(t)$ is a fixed point of $[g, r_\theta]$, which proves $\rho([g, r_\theta]) = 0$. □

We end this section by showing that the rotation number $\rho(g)$ depends continuously and monotonically on g. Observe that the space of continuous non-decreasing functions $\mathbb{R} \to \mathbb{R}$ which commute with the unit translation is closed in the topology of uniform convergence on the real line. Hence the space of degree 1 monotone maps $\mathbb{T} \to \mathbb{T}$ is closed in the topology of uniform convergence on the circle.

Theorem 1.11 *The mapping $g \mapsto \rho(g)$ is continuous in the topology of uniform convergence on the circle.*

Proof It suffices to check that $G \mapsto \tau(G)$ is continuous in the topology of uniform convergence on the real line. This is easy because by (1.5) this mapping is the uniform limit of the sequence of continuous mappings $G \mapsto G^{\circ n}(0)/n$. □

Now suppose we have a family $\{g_\alpha\}$ of degree 1 monotone maps of the circle depending continuously on a parameter α which varies in some interval on the real line. We say that $\{g_\alpha\}$ is a ***monotone family*** if it lifts to a continuous family $\{G_\alpha\}$ of maps of the real line such that $G_\alpha \leq G_\beta$ whenever $\alpha < \beta$. An easy induction then shows that $G_\alpha^{\circ n} \leq G_\beta^{\circ n}$ for all n, so $\tau(G_\alpha) \leq \tau(G_\beta)$. This proves

Theorem 1.12 *For every monotone family $\{g_\alpha\}$, the map $\alpha \mapsto \rho(g_\alpha)$ is monotone.*

Of course the rotation number of a monotone family can be constant. Suppose however that in the above situation $G_\beta = G_\alpha + 1$ for some $\alpha < \beta$, so $\tau(G_\beta) = \tau(G_\alpha) + 1$. Since the function $\alpha \mapsto \tau(G_\alpha)$ is continuous by (the proof of) Theorem 1.11, it assume all values in the interval $[\tau(G_\alpha), \tau(G_\alpha)+1]$ and it follows that the translation number is not constant.

Corollary 1.13 *Suppose* $g_0 : \mathbb{T} \to \mathbb{T}$ *is a degree* 1 *monotone map and*

$$g_\alpha(t) = g_0(t) + \alpha \pmod{\mathbb{Z}}$$

for $\alpha \in \mathbb{T}$. *Then the assignment* $\alpha \mapsto \rho(g_\alpha)$ *itself is a degree* 1 *monotone map.*

1.3 Dynamics in the Presence of Periodic Points

We continue assuming that $g : \mathbb{T} \to \mathbb{T}$ is a degree 1 monotone map. It is easy to see using Corollary 1.3 that $\rho(g) = p/q$ if and only if $g^{\circ q}(t) = t$ for some $t \in \mathbb{T}$. Here is a sharper statement:

Theorem 1.14 *Suppose* $g : \mathbb{T} \to \mathbb{T}$ *is a degree* 1 *monotone map with* $\rho(g) = p/q$ *in lowest terms. Then,*

(i) *g has a periodic orbit of length q.*
(ii) *All periodic orbits of g have length q.*
(iii) *If the points of a periodic orbit are labeled in positive cyclic order as $t_1, t_2, \ldots, t_q$, then $g(t_j) = t_{j+p}$, where the subscripts are taken modulo q.*

Proof By what we have seen, g has a periodic point whose period n divides q. This, in turn, implies that $\rho(g)$ is a fraction of the form $m/n \pmod{\mathbb{Z}}$. Since p and q are assumed to be relatively prime, it easily follows that $n = q$. This proves (i).

To see (ii), let t be a periodic point of g of period n. Take any $x \in \pi^{-1}(t)$ and a lift G of g with $\tau(G) = p/q$. Then $G^{\circ n}(x) = x + m$ for some integer m, where $m/n = p/q$ by Corollary 1.3. Since p and q are assumed relatively prime, we have $n = kq$ and $m = kp$ for some integer $k \geq 1$. If the minimal period n were greater than q, then either $G^{\circ q}(x) > x + p$ or $G^{\circ q}(x) < x + p$. Since G is monotone and commutes with the unit translation, it would follow inductively that $G^{\circ iq}(x) > x + ip$ or $G^{\circ iq}(x) < x + ip$ for all $i \geq 1$. This would contradict $G^{\circ kq}(x) = x + kp$. Thus $n = q$.

Finally, (iii) follows at once from Theorem 1.6 since if $a_j = j/q \pmod{\mathbb{Z}}$, the points $a_1, a_2, \cdots, a_q$ are in positive cyclic order and form the orbit of 0 under the rigid rotation $r_{p/q}$, which sends each a_j to a_{j+p}. □

For convenience we often use the term ***q*-cycle** for a periodic orbit of length q. Part (iii) of the above theorem can be expressed as a semiconjugacy relation as follows. Suppose we label the points of a q-cycle C of g as $t_1, \ldots, t_q$ in positive cyclic order. Define the piecewise constant map $\varphi : \mathbb{T} \to \mathbb{T}$ by sending each half-open interval $[t_j, t_{j+1})$ to the point $a_j = j/q \pmod{\mathbb{Z}}$. Then one has the relation

$$\varphi \circ g = r_{p/q} \circ \varphi \quad \text{on} \quad C. \tag{1.9}$$

Note that there are q different ways of labeling the points of C in positive cyclic order, giving rise to q such semiconjugacies which only differ by a rotation. In

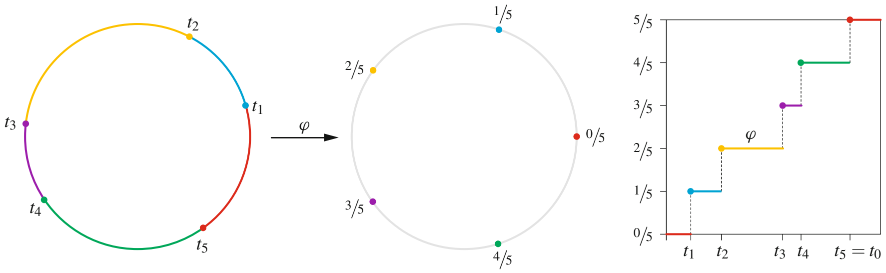

Fig. 1.1 Left: The combinatorial semiconjugacy φ associated with a 5-cycle $C = \{t_1, \ldots, t_5\}$. Right: The graph of φ. Observe that C is the complement of the union of plateaus of φ

particular, if we choose the labeling so that $0 \in [t_q, t_1)$, then $\varphi(0) = 0$. We call φ normalized this way the ***combinatorial semiconjugacy*** associated with the cycle C. To establish the analogy with the more interesting case of irrational rotation numbers to be discussed in the next section, let us comment that the cycle C can be described as the complement of the union of the "plateaus" of φ (by definition, a plateau is a maximal open interval on which the map is constant; see Sect. 1.4 and compare Fig. 1.1).

Remark 1.15 The relation (1.9) may not hold globally since g may well map a point in (t_j, t_{j+1}) to t_{j+p+1}. However, if g maps each $[t_j, t_{j+1})$ onto $[t_{j+p}, t_{j+p+1})$, then (1.9) holds on the whole circle.

The preceding discussion provides a simple characterization for the cycles that occur as periodic orbits of degree 1 monotone maps of the circle. Let C consist of q points $t_1, \ldots, t_q$ labeled in positive cyclic order and $g : C \to C$ be any transitive action. We say that C has ***combinatorial rotation number*** p/q under g if $g(t_j) = t_{j+p}$ for all j. In this case, we can extend g to an orientation-preserving homeomorphism of the circle by mapping each half-open interval $[t_j, t_{j+1})$ homeomorphically onto $[t_{j+p}, t_{j+p+1})$. Theorem 1.14(iii) then shows that $\rho(g) = p/q$.

Corollary 1.16 *A cycle can be realized as a periodic orbit of a degree* 1 *monotone map if and only if it has a well-defined combinatorial rotation number.*

See Fig. 1.2.

Recall that the ***omega limit set*** of a point $t \in \mathbb{T}$ under the action of g is the set of all accumulation points of the forward orbit of t:

$$\omega_g(t) = \bigcap_{n \geq 1} \overline{\{g^{\circ n}(t), g^{\circ n+1}(t), g^{\circ n+2}(t), \ldots\}}.$$

It is easy to see that $\omega_g(t)$ is non-empty and compact, and $g(\omega_g(t)) = \omega_g(t)$.

Theorem 1.17 *Suppose $g : \mathbb{T} \to \mathbb{T}$ is a degree* 1 *monotone map with $\rho(g) = p/q$ in lowest terms. Then $\omega_g(t)$ is a q-cycle for every $t \in \mathbb{T}$.*

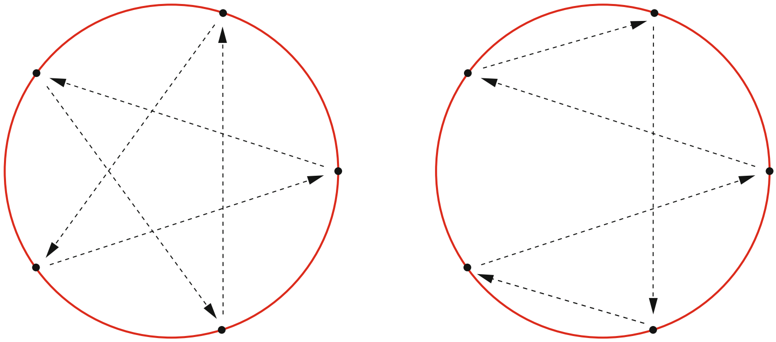

Fig. 1.2 Every q-cycle under a degree 1 monotone map of the circle has a well-defined combinatorial rotation number of the form p/q, where p and q are relatively prime. The 5-cycle on the left has combinatorial rotation number $\frac{2}{5}$, while the one on the right, having no combinatorial rotation number, cannot be realized as a periodic orbit of any degree 1 monotone map

Proof Let $E = \{t \in \mathbb{T} : g^{\circ q}(t) = t\}$. By Theorem 1.14, E is non-empty and every $t \in E$ has period q, so $\omega_g(t) = \{t, g(t), \ldots, g^{\circ q-1}(t)\}$. If $t \notin E$, then t belongs to a connected component J of the open set $\mathbb{T} \smallsetminus E$. The iterate $g^{\circ q}$ maps the interval J onto itself, keeping the endpoints fixed but moving all the interior points (note however that a point in J may map to an endpoint). An easy calculus exercise shows that one endpoint t' of J is attracting under $g^{\circ q}$ and the other is repelling. It follows that $g^{\circ nq}(t) \to t'$ as $n \to \infty$. But then $g^{\circ i+nq}(t) \to g^{\circ i}(t')$, which proves $\omega_g(t) = \{t', g(t'), \ldots, g^{\circ q-1}(t')\}$. □

1.4 Dynamics in the Absence of Periodic Points

We now turn to the case of irrational rotation numbers.

Theorem 1.18 (Poincaré) *Suppose $g : \mathbb{T} \to \mathbb{T}$ is a degree 1 monotone map with $\rho(g) = \theta$ irrational. Then there exists a degree 1 monotone map $\varphi : \mathbb{T} \to \mathbb{T}$ which satisfies $\varphi \circ g = r_\theta \circ \varphi$. Moreover, φ is unique up to postcomposition with a rigid rotation.*

We call the unique such φ normalized by $\varphi(0) = 0$ the ***Poincaré semiconjugacy*** between g and r_θ.

Proof Lift g to a map $G : \mathbb{R} \to \mathbb{R}$ with $\tau(G) = \theta$. We will construct a map $\Phi : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) Φ is continuous and non-decreasing;
(ii) $\Phi(x + 1) = \Phi(x) + 1$ for all x;
(iii) $\Phi(G(x)) = \Phi(x) + \theta$ for all x.

The quotient map $\varphi : \mathbb{T} \to \mathbb{T}$ will then have the desired property.

Consider the set

$$\Lambda = \{G^{\circ n}(0) + m : n, m \text{ are integers with } n \geq 0\}. \tag{1.10}$$

Since $\tau(G) = \theta$ is irrational, Corollary 1.4 shows that each element of Λ has a unique representation of this form. Define $\Phi : \Lambda \to \mathbb{R}$ by

$$\Phi(G^{\circ n}(0) + m) = n\theta + m.$$

The image $\Phi(\Lambda)$ is dense in $\mathbb{R}$ since θ is irrational, and Φ is strictly increasing on Λ by Corollary 1.4. Extend Φ to the real line by

$$\Phi(x) = \sup_{y \in \Lambda \cap (-\infty, x]} \Phi(y).$$

Clearly Φ is non-decreasing, so it has one-sided limits $\Phi(x^-) \leq \Phi(x^+)$ at every x. If the inequality were strict at some x, the image $\Phi(\Lambda)$ would omit all points in the interval $(\Phi(x^-), \Phi(x^+))$, with the possible exception of $\Phi(x)$ if $x \in \Lambda$, which contradicts density of $\Phi(\Lambda)$. Thus, Φ is continuous everywhere.

Properties (ii) and (iii) clearly hold when $x \in \Lambda$, and by continuity they hold when $x \in \overline{\Lambda}$. If (a, b) is a connected component of $\mathbb{R} \smallsetminus \overline{\Lambda}$, the definition of Φ shows that Φ is constant in (a, b). If $x \in (a, b)$, invariance of $\overline{\Lambda}$ under the unit translation gives

$$\Phi(x + 1) = \Phi(a + 1) = \Phi(a) + 1 = \Phi(x) + 1,$$

while monotonicity gives

$$\Phi(G(a)) \leq \Phi(G(x)) \leq \Phi(G(b)) \Longrightarrow \Phi(a) + \theta \leq \Phi(G(x)) \leq \Phi(b) + \theta.$$

Since $\Phi(a) = \Phi(b) = \Phi(x)$, we obtain $\Phi(G(x)) = \Phi(x) + \theta$. This proves that (ii) and (iii) hold for all $x \in \mathbb{R}$.

Uniqueness follows since Φ is uniquely determined by its values on Λ, which in turn are uniquely determined by $\Phi(0)$. □

Since the Poincaré semiconjugacy φ constructed above is a monotone map, each fiber $E_s = \varphi^{-1}(s)$ is either a point or a closed non-degenerate interval. It follows that the interior I_s of E_s is either empty or an open interval. In the latter case we call I_s a ***plateau*** of φ.[1] We can visualize a plateau as a maximal open interval on which the graph of φ is a horizontal line.

[1]Let us emphasize that our plateaus are *open* intervals, a convention that is not commonly adopted in the literature.

Lemma 1.19 *Let φ be the Poincaré semiconjugacy between g and r_θ, given by Theorem 1.18.*

(i) For every $s \in \mathbb{T}$, $g^{-1}(E_s) = E_{s-\theta}$.
(ii) If $I_s \neq \emptyset$ then $I_{s-\theta} \neq \emptyset$. Moreover, $I_{s-\theta}$ contains the open interval $g^{-1}(I_s)$.

By part (ii), the plateaus of φ are indexed by a countable union of *backward* orbits of r_θ. This turns out to be a characteristic property of Poincaré semiconjugacies (see Theorem 1.22).

Proof Statement (i) follows directly from the semiconjugacy relation $\varphi \circ g = r_\theta \circ \varphi$. For (ii), simply note that I_s being a plateau implies that E_s does not reduce to a point. By (i), the same must be true of $E_{s-\theta}$, which shows $I_{s-\theta}$ is a plateau. □

The following theorem is the analogue of Theorem 1.17 for monotone maps with irrational rotation number. Unlike the rational case, there are now two possible regimes for the asymptotic behavior of orbits.

Theorem 1.20 *Suppose $g : \mathbb{T} \to \mathbb{T}$ is a degree 1 monotone map with $\rho(g) = \theta$ irrational, and $\varphi : \mathbb{T} \to \mathbb{T}$ is the Poincaré semiconjugacy between g and r_θ.*

(i) If φ is a homeomorphism, then $\omega_g(t) = \mathbb{T}$ for all $t \in \mathbb{T}$.
(ii) If φ is not a homeomorphism, there exists a g-invariant Cantor set $K \subset \mathbb{T}$ with the property that $\omega_g(t) = K$ for every $t \in \mathbb{T}$.

The map g is called ***linearizable*** or ***non-linearizable*** according as case (i) or (ii) holds. We refer to K in (ii) as the ***Cantor attractor*** of g (see Fig. 1.3).

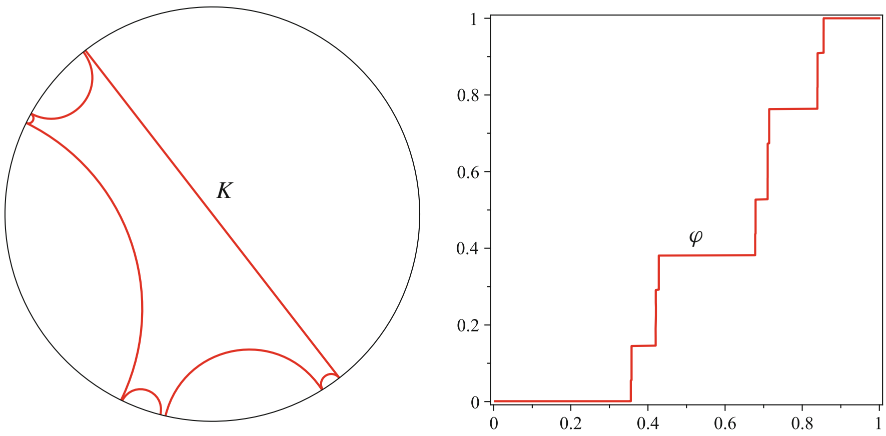

Fig. 1.3 The Cantor attractor K of some degree 1 monotone map with irrational rotation number, and the graph of the corresponding Poincaré semiconjugacy φ. (Here and elsewhere, we use hyperbolic convex hulls to make subsets of the circle more visible.) Similar to the rational case, K can be described as the complement of the union of plateaus of φ

Proof If φ is a homeomorphism, then g is conjugate to r_θ under which all orbits are dense, so (i) holds. Let us then assume that φ is not a homeomorphism and define K to be the complement of the union of all plateaus of φ. Evidently K is a compact proper subset of the circle. If $g(t)$ belongs to a plateau I_s, Lemma 1.19 shows that $I_{s-\theta}$ is a plateau containing t. This proves $g(K) \subset K$. To prove the reserve inclusion, suppose $t \in K$ and take any t' with $g(t') = t$. If $t' \in K$, then $t \in g(K)$. Otherwise t' belongs to a plateau I_s. By Lemma 1.19, $g(E_s) = E_{s+\theta}$ contains t. Thus, g maps E_s either to the single point t, or to the non-degenerate closed interval $E_{s+\theta}$ having t as a boundary point. In either case, monotonicity of g implies that some endpoint of I_s maps to t, proving $t \in g(K)$.

To check that K is a Cantor set, first observe that K has no isolated point since distinct plateaus of φ have disjoint closures. If K were not totally disconnected, it would necessarily contain a non-empty open interval J. As J does not meet any plateau, φ would be one-to-one in J, and the image $\varphi(J)$ would also be an open interval. We could then take any plateau I_s and an integer $n \geq 1$ such that $s - n\theta \in \varphi(J)$. Then, $I_{s-n\theta}$, a plateau by Lemma 1.19, would have to intersect J, contradicting $I_{s-n\theta} \cap K = \emptyset$.

Next, we show that K is strongly minimal in the sense that if X is non-empty, compact and g-invariant, then $K \subset X$. Let us verify that every $p \in K$ which is not an endpoint of a plateau belongs to X. Since such p are dense in K, this will prove $K \subset X$. Pick any $t \in X$ and an increasing sequence $\{n_i\}$ of positive integers such that $r_\theta^{\circ n_i}(\varphi(t)) = \varphi(g^{\circ n_i}(t)) \to \varphi(p)$. By passing to a subsequence, we may assume $g^{\circ n_i}(t) \to u \in X$, so $\varphi(u) = \varphi(p)$ by continuity. If $p \neq u$, the fiber $E_{\varphi(p)}$ would be non-degenerate, hence $I_{\varphi(p)}$ would be a plateau with p as an endpoint, contradicting our assumption. Hence, $p = u \in X$.

It is now easy to prove that $\omega_g(t) = K$ for every $t \in \mathbb{T}$. If $g^{\circ n}(t) \in K$ for some $n \geq 0$, then $\omega_g(t) = K$ follows immediately from minimality. Consider then the case where $g^{\circ n}(t) \notin K$ for every $n \geq 0$. If I_s is the plateau containing t, it follows from Lemma 1.19 that $I_{s+n\theta}$ is the plateau containing $g^{\circ n}(t)$. The $I_{s+n\theta}$ are disjoint with $\sum |I_{s+n\theta}| \leq 1$, so $|I_{s+n\theta}| \to 0$ as $n \to \infty$. Therefore the distance between $g^{\circ n}(t)$ and the endpoints of $I_{s+n\theta}$ tends to zero. It follows that $\omega_g(t) \subset K$, and again by minimality $\omega_g(t) = K$. □

Remark 1.21 The non-linearizable case can always be reduced to the linearizable case at the expense of working in a quotient dynamical system. Consider the equivalence relation $\sim$ on the circle where $t \sim t'$ if and only if $\varphi(t) = \varphi(t')$. Let $\tilde{\mathbb{T}}$ be the set of all equivalence classes $[t]$ of $\sim$. The map $\tilde{\varphi} : \tilde{\mathbb{T}} \to \mathbb{T}$ defined by $\tilde{\varphi}[t] = \varphi(t)$ is clearly a bijection, so it induces a topology on $\tilde{\mathbb{T}}$ with respect to which $\tilde{\varphi}$ is a homeomorphism. The induced action $\tilde{g} : \tilde{\mathbb{T}} \to \tilde{\mathbb{T}}$ given by $\tilde{g}([t]) = [g(t)]$ is easily seen to be well-defined and homeomorphic, and it is linearizable since $\tilde{\varphi} \circ \tilde{g} = r_\theta \circ \tilde{\varphi}$.

The next result characterizes the monotone maps that arise as Poincaré semi-conjugacies. It will be used later in Theorem 2.35. We will continue denoting the interior of the fiber $\varphi^{-1}(s)$ by I_s.

Theorem 1.22 *Let θ be irrational and $\varphi : \mathbb{T} \to \mathbb{T}$ be a degree* 1 *monotone map with the property that $I_s \neq \emptyset$ implies $I_{s-\theta} \neq \emptyset$. Then, there exists a degree* 1 *monotone map $g : \mathbb{T} \to \mathbb{T}$ which satisfies $\varphi \circ g = r_\theta \circ \varphi$.*

Observe that any such g has rotation number θ by Theorem 1.8(ii). The map g could be a homeomorphism even if φ has plateaus. This happens when the plateaus of φ are indexed by *full* orbits of r_θ.

Proof It will be convenient to work on the universal cover. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be any lift of φ and set $E_y = \Phi^{-1}(y)$ for every $y \in \mathbb{R}$. By the assumption, if E_y reduces to a point, so does $E_{y+\theta}$. Hence there is a unique map $G : \mathbb{R} \to \mathbb{R}$ which sends each fiber E_y affinely to $E_{y+\theta}$, preserving the orientation. The relations $G(x+1) = G(x)+1$ and $\Phi(G(x)) = \Phi(x)+\theta$ for all x follow immediately. It remains to show that G is non-decreasing and continuous.

Take any $x, x' \in \mathbb{R}$ with $x < x'$. If both x, x' belong to the same fiber of Φ, then clearly $G(x) \leq G(x')$ by the definition of G. Suppose then that $x \in E_y$ and $x' \in E_{y'}$, where necessarily $y < y'$ since Φ is non-decreasing. Then $\Phi(G(x)) = y + \theta < y' + \theta = \Phi(G(x'))$, which implies $G(x) \leq G(x')$. This shows G is non-decreasing. Moreover, every point of $\mathbb{R}$ belongs to some fiber E_y, which is contained in the image of G since $G(E_{y-\theta}) = E_y$. Thus G is surjective. Because of monotonicity, this proves that G is continuous. □

1.5 Invariant Measures

Let $\mathscr{M}(\mathbb{T})$ denote the space of all Borel probability measures on the circle. Every Borel measurable map $g : \mathbb{T} \to \mathbb{T}$ acts on $\mathscr{M}(\mathbb{T})$ by sending a measure μ to its ***push-forward*** $g_*\mu$ defined by $(g_*\mu)(E) = \mu(g^{-1}(E))$. A measure $\mu \in \mathscr{M}(\mathbb{T})$ is called g***-invariant*** if $g_*\mu = \mu$. According to Krylov and Bogolyubov, there is at least one g-invariant measure when g is continuous [14]. In fact, if we start with any $\mu_0 \in \mathscr{M}(\mathbb{T})$ and define the sequence $\mu_n \in \mathscr{M}(\mathbb{T})$ by

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}(g^{\circ i})_*\mu_0 \qquad n \geq 1,$$

then any weak* limit of the sequence $\{\mu_n\}$ will be g-invariant.

A g-invariant measure $\mu \in \mathscr{M}(\mathbb{T})$ is called ***ergodic*** if $g^{-1}(E) = E$ implies $\mu(E) = 0$ or $\mu(E) = 1$. In this case, it follows from Birkhoff's ergodic theorem that for every function $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(g^{\circ i}(t)) = \int_{\mathbb{T}} f\, d\mu$$

holds for μ-almost every $t \in \mathbb{T}$ [14]. If we choose for f the characteristic function of an interval $I \subset \mathbb{T}$, we deduce that

$$\mu(I) = \lim_{n\to\infty} \frac{1}{n} \#\big\{0 \le i \le n-1 : g^{\circ i}(t) \in I\big\}$$

for μ-almost every $t \in \mathbb{T}$. In particular, almost every orbit is dense in the support of μ.

It may happen that g has a unique invariant measure $\mu \in \mathscr{M}(\mathbb{T})$. In this case, μ is necessarily ergodic and the map g is called ***uniquely ergodic***. A sharper form of Birkhoff's theorem then shows that for every continuous function $f : \mathbb{T} \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(g^{\circ i}) = \int_{\mathbb{T}} f \, d\mu$$

uniformly on $\mathbb{T}$. If μ has no atoms, we can deduce by a standard approximation argument that for every interval $I \subset \mathbb{T}$ and *every* $t \in \mathbb{T}$,

$$\mu(I) = \lim_{n\to\infty} \frac{1}{n} \#\big\{0 \le i \le n-1 : g^{\circ i}(t) \in I\big\}. \tag{1.11}$$

Now suppose g is a degree 1 monotone map. If $\rho(g)$ is rational of the form p/q in lowest terms, then g has at least one q-cycle C by Theorem 1.14, and the Dirac measure μ_C which assigns a mass of $1/q$ to each point of C is clearly g-invariant (in fact, ergodic). Moreover, the combinatorial semiconjugacy φ associated with C (see the end of Sect. 1.3) is related to μ_C by the formula

$$\varphi(t) = \int_0^t d\mu_C = \mu_C[0,t] \pmod{\mathbb{Z}}.$$

It is not hard to see using Theorem 1.14 that the support of every g-invariant measure $\mu \in \mathscr{M}(\mathbb{T})$ is contained in the union of q-cycles of g. As the restriction of μ to each q-cycle is also g-invariant, it must give an equal mass (possibly zero) to each point of the cycle. In the special case where g has finitely many q-cycles $C_1, \ldots, C_n$, it follows that μ is a convex combination of the Dirac measures μ_{C_i}, that is,

$$\mu = \alpha_1 \, \mu_{C_1} + \cdots + \alpha_n \, \mu_{C_n}, \quad \text{where} \quad \alpha_i \ge 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

In this case the space of all g-invariant measures is isomorphic to an $(n-1)$-dimensional simplex. The ergodic measures in this space are $\mu_{C_1}, \ldots, \mu_{C_n}$, corresponding to the n vertices of the simplex. Thus, g is uniquely ergodic if and only if it has a single periodic orbit.

The situation when $\rho(g) = \theta$ is irrational is quite different. It is well known that the rigid rotation r_θ is uniquely ergodic, with Lebesgue measure λ being its unique invariant measure. In the linearizable case where the Poincaré semiconjugacy φ between g and r_θ is a homeomorphism, it immediately follows that g is also uniquely ergodic, with the unique invariant measure $\varphi_*^{-1}\lambda$ supported on the full circle. In the non-linearizable case a similar construction gives a unique g-invariant measure μ, supported on the Cantor attractor K, with the property that $\varphi_*\mu = \lambda$. In fact, let $D \subset K$ be the countable set of the endpoints of plateaus of φ, and let S be the countable set of $s \in \mathbb{T}$ for which $I_s \neq \emptyset$. Then $\varphi : K \setminus D \to \mathbb{T} \setminus S$ is continuous and bijective, and the measure μ can be described as the push-forward under φ^{-1} of the restriction of λ to $\mathbb{T} \setminus S$. Similar to the rational case, the Poincaré semiconjugacy φ is related to the invariant measure μ by the formula

$$\varphi(t) = \mu[0, t] \pmod{\mathbb{Z}}.$$

In fact, $\varphi^{-1}[0, \varphi(t)] \supset [0, t]$ for every t by monotonicity of φ. Moreover, the difference $\varphi^{-1}[0, \varphi(t)] \setminus [0, t]$ is disjoint from $K \setminus D$, so its μ-measure is zero. Hence,

$$\varphi(t) = \lambda[0, \varphi(t)] = \mu\big(\varphi^{-1}[0, \varphi(t)]\big) = \mu[0, t] \pmod{\mathbb{Z}}.$$