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Saeed Zakeri

Rotation Sets and Complex Dynamics

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Preface

For an integer $d \geq 2$, let $m_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the multiplication by d map of the circle defined by $m_d(t) = dt \pmod{\mathbb{Z}}$. A ***rotation set*** for m_d is a compact subset of $\mathbb{R}/\mathbb{Z}$ on which m_d acts in an order-preserving fashion and therefore has a well-defined rotation number. Rotation sets for the doubling map m_2 seem to have first appeared under the disguise of Sturmian sequences in a 1940 paper of Morse and Hedlund on symbolic dynamics [17] (the equivalence with the rotation set condition was later shown by Gambaudo et al. [10] and Veerman [28]). Fertile ground for their comeback was provided half a century later by the resurgence of the field of holomorphic dynamics. For example, in the early 1990s Goldberg [11] and Goldberg and Milnor [12] studied rational rotation sets in their work on fixed point portraits of complex polynomials. The main result of [11] was later extended by Goldberg and Tresser to irrational rotation sets [13]. Around the same time, Bullett and Sentenac investigated rotation sets for the doubling map and their connection with the Douady–Hubbard theory of the Mandelbrot set [7] (see Fig. 1 for an illustration of this link). Aspects of this work were generalized to arbitrary degrees a decade later by Blokh et al. who in particular gave recipes for constructing a rotation set for m_{d+1} from one for m_d and vice versa [2]. More recently, Bonifant, Buff, and Milnor used rotation sets for the tripling map m_3 in their work on antipode-preserving cubic rational maps [4]. In an entirely different context, rational rotation sets appear in McMullen's study of the space of proper holomorphic maps of the unit disk [19]; they play a role analogous to simple closed geodesics on compact hyperbolic surfaces.

This monograph presents the first systematic treatment of the theory of rotation sets for m_d in both rational and irrational cases. Our approach, partially inspired by the ideas in [4], has a rather geometric flavor and yields several new results on the structure of rotation sets, their gap dynamics, maximal and minimal rotation sets, rigidity, and continuous dependence on parameters. This "abstract" part is supplemented with a "concrete" part which explains how rotation sets arise in the dynamical plane of complex polynomial maps and how suitable parameter spaces of such polynomials provide a complete catalog of all rotation sets of a given degree.

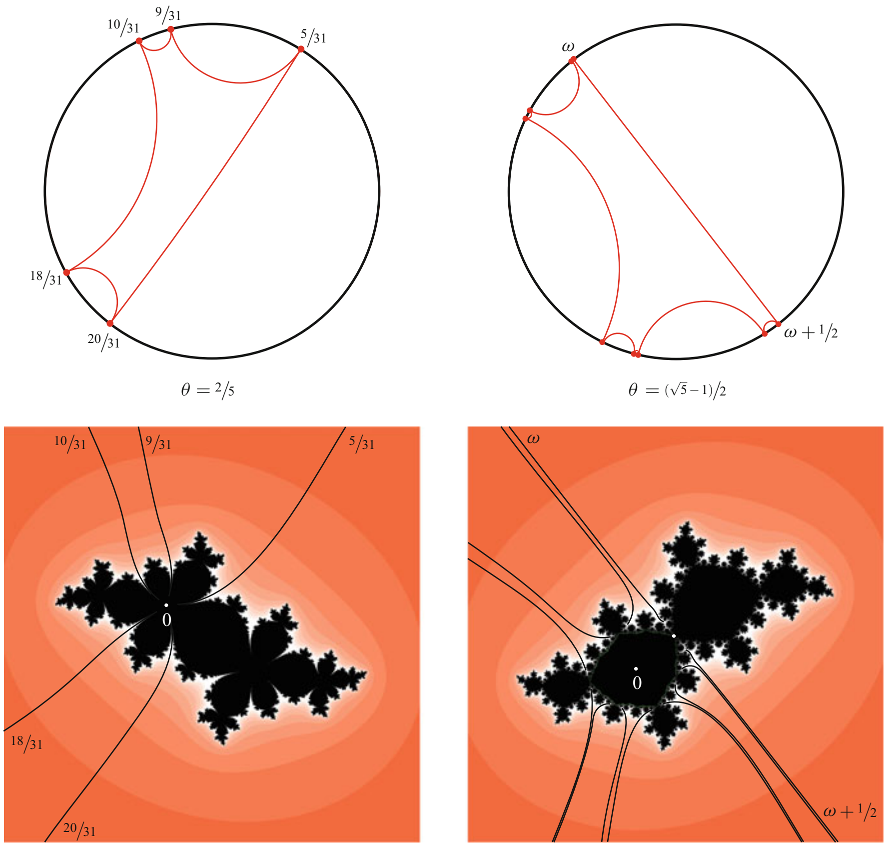

Fig. 1 For each $0 \leq \theta < 1$ the doubling map $t \mapsto 2t \pmod{\mathbb{Z}}$ has a unique minimal invariant set $X_\theta \subset \mathbb{R}/\mathbb{Z}$ of rotation number θ which is a period orbit if θ is rational and a Cantor set otherwise. Top left: The case $\theta = 2/5$ where X_θ is the 5-cycle $\frac{5}{31} \mapsto \frac{10}{31} \mapsto \frac{20}{31} \mapsto \frac{9}{31} \mapsto \frac{18}{31}$. Top right: The golden mean case $\theta = \frac{\sqrt{5}-1}{2}$ where the Cantor set X_θ is the closure of the orbit of $\omega \approx 0.35490172\ldots$. According to Douady and Hubbard, the "rotation set" X_θ is related to the external rays of the corresponding quadratic map $z \mapsto e^{2\pi i\theta}z + z^2$ (shown in the bottom row) as well as the parameter rays that land on the boundary of the main cardioid of the Mandelbrot set. See Sect. 5.3 for details

Here is an outline of the material presented in this monograph:

Chapter 1 provides background material on the dynamics of degree 1 monotone maps of the circle. Given such a map $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, its Poincaré ***rotation number*** $\rho(g)$ is constructed using a Dedekind cut approach that quickly leads to basic properties of the rotation number and how it essentially determines the asymptotic behavior of the orbits of g. These orbits converge to a cycle if $\rho(g)$ is rational and to a unique minimal Cantor set if $\rho(g)$ is irrational. A key tool in understanding this dichotomy is the semiconjugacy between g and the rigid rotation $r_\theta : t \mapsto t + \theta \pmod{\mathbb{Z}}$ by the angle $\theta = \rho(g)$. This semiconjugacy is also utilized in studying the existence and uniqueness of invariant probability measures for g: If $\rho(g)$ is rational, every such measure is a convex combination of Dirac measures supported on the cycles of g, while if $\rho(g)$ is irrational, there is a unique invariant measure supported on the minimal Cantor set of g.

Chapter 2 introduces rotation sets for the map m_d and develops their basic properties. A rotation set for m_d is a non-empty compact set $X \subset \mathbb{R}/\mathbb{Z}$, with $m_d(X) = X$, such that the restriction $m_d|_X$ extends to a degree 1 monotone map of the circle. The rotation number of X, denoted by $\rho(X)$, is defined as the rotation number of any such extension. We refer to X as a rational or irrational rotation set according as $\rho(X)$ is rational or irrational. Understanding X is facilitated by studying the dynamics of the complementary intervals of X called its ***gaps***. A gap I is labeled ***minor*** or ***major*** according as $m_d|_I : I \to m_d(I)$ is or is not a homeomorphism, and the ***multiplicity*** of I is the number of times the covering map m_d wraps I around the circle. Counting multiplicities, X has $d-1$ major gaps, a statement reminiscent of the fact that a degree d polynomial has $d-1$ critical points. Major gaps completely determine a rotation set and the pattern of how they are mapped around can be recorded in a combinatorial object called the ***gap graph***.

Next, we study maximal and minimal rotation sets. Maximal rotation sets for m_d are characterized as having $d-1$ distinct major gaps of length $1/d$. A rational rotation set may well be contained in infinitely many maximal rotation sets. By contrast, we show that an irrational rotation set for m_d is contained in at most $(d-1)!$ maximal rotation sets. Minimal rotation sets are cycles in the rational case and Cantor sets in the irrational case. We prove that a rational rotation set contains at most as many minimal rotation sets as the number of its distinct major gaps. As a special case, we recover Goldberg's result in [11] according to which a rational rotation set for m_d contains at most $d-1$ cycles. On the other hand, every irrational rotation set is easily shown to contain a unique minimal rotation set.

Chapter 3 offers a more in-depth study of minimal rotation sets by presenting a unified treatment of the deployment theorem of Goldberg and Tresser. Suppose X is a minimal rotation set for m_d with the rotation number $\rho(X) = \theta \neq 0$. Then X is a q-cycle (i.e., a cycle of length q) if $\theta = p/q$ in lowest terms and a Cantor set if θ is irrational. The ***natural measure*** on X is the unique m_d-invariant Borel probability measure μ supported on X. The ***canonical semiconjugacy*** associated with X is a degree 1 monotone map $\varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, whose plateaus are precisely the gaps of X, which satisfies $\varphi \circ m_d = r_\theta \circ \varphi$ on X. It is related to the natural measure by $\varphi(t) = \mu[0,t] \pmod{\mathbb{Z}}$. The covering map m_d has $d-1$ fixed points $u_i = i/(d-1) \pmod{\mathbb{Z}}$. The ***deployment vector*** of X is the probability vector $\delta(X) = (\delta_1, \ldots, \delta_{d-1})$ where $\delta_i = \mu[u_{i-1}, u_i)$. Note that $q\delta(X) \in \mathbb{Z}^{d-1}$ if θ is rational of the form p/q.

The deployment theorem asserts that given any θ and any probability vector $\delta \in \mathbb{R}^{d-1}$ that satisfies $q\delta \in \mathbb{Z}^{d-1}$ if $\theta = p/q$, there exists a unique minimal rotation set $X = X_{\theta,\delta}$ for m_d with $\rho(X) = \theta$ and $\delta(X) = \delta$. The rational case of this theorem that appears in [11] and its irrational case proved in [13] are treated using very different arguments. By contrast, we provide a proof that reveals the nearly identical nature of the two cases. The key tool in our unified treatment is the ***gap measure***

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - k\theta},$$

where $\sigma_i = \delta_1 + \cdots + \delta_i$ and $\mathbb{1}_x$ denotes the unit mass at x. This is an atomic measure supported on the union of at most $d-1$ backward orbits of the rotation r_θ. The general idea is that the gap measure can be used to construct the "inverse" of the canonical semiconjugacy of X and therefore X itself. This measure makes a brief appearance in an appendix of [13], but its real power is not nearly utilized there. In addition to its theoretical role, the gap measure turns out to be a highly effective computational gadget.

Chapter 3 also includes a fairly detailed discussion of finite rotation sets, namely, unions of cycles that have a well-defined rotation number. Let $\mathscr{C}_d(p/q)$ denote the collection of all q-cycles under m_d with rotation number p/q. According to the deployment theorem, $\mathscr{C}_d(p/q)$ can be identified with a finite subset of the simplex $\Delta^{d-2} = \{(x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} : x_i \geq 0 \text{ and } \sum_{i=1}^{d-1} x_i = 1\}$ with $\binom{q+d-2}{q}$ elements. A collection of cycles in $\mathscr{C}_d(p/q)$ are ***compatible*** if their union forms a rotation set. In [19], McMullen proposes that $\mathscr{C}_d(p/q)$ can be identified with the vertices of a simplicial subdivision Δ_q^{d-2} of Δ^{d-2}, where each collection of compatible cycles corresponds to the vertices of a simplex in Δ_q^{d-2}. We provide a justification for this geometric model; in particular, for each $x \in \Delta^{d-2}$ our proof gives an explicit algorithm for finding a simplex in Δ_q^{d-2} that contains x. The subdivision Δ_q^{d-2} is different from (and in a sense simpler than) the standard barycentric subdivision and could perhaps be of independent interest in applications outside dynamics.

In Chap. 4, we give sample applications of the results of Chaps. 2 and 3, especially the deployment theorem. For example, we show that every admissible graph without loops can be realized as the gap graph of an irrational rotation set. We also study the dependence of the minimal rotation set $X_{\theta,\delta}$ on the parameter (θ, δ). We prove that the map $(\theta, \delta) \mapsto X_{\theta,\delta}$ is lower semicontinuous in the Hausdorff topology, and it is continuous at some parameter (θ_0, δ_0) if and only if X_{θ_0,δ_0} is ***exact*** in the sense that it is both minimal and maximal. We provide a characterization of exactness which shows that the set of such parameters has full measure in $(\mathbb{R}/\mathbb{Z}) \times \Delta^{d-2}$.

As another application, we use the gap measure to compute the ***leading angle*** ω of $X = X_{\theta,\delta}$, that is, the smallest angle when X is identified with a subset of (0, 1):

$$\omega = \frac{1}{d-1}\, \nu(0, \theta] + \frac{N_0}{d-1} = \frac{1}{d-1} \sum_{i=1}^{d-1} \sum_{0<\sigma_i - k\theta \leq \theta} d^{-(k+1)} + \frac{N_0}{d-1}.$$

Here, $N_0 \geq 0$ is the number of indices i with $\sigma_i = 0$. The formula gives an explicit algorithm for computing the base-d expansion of the angle $(d-1)\omega$, which has an itinerary interpretation in the context of polynomial dynamics. We exploit the leading angle formula in the low-degree cases $d = 2$ and $d = 3$ to carry out a detailed analysis of the structure of minimal rotation sets under the doubling and tripling maps.

Chapter 5 explores the link between rotation sets and complex polynomial maps. After a brief review of the basic definitions in polynomial dynamics, we explain how an indifferent fixed point of a polynomial of degree d determines a rotation set under m_d. More precisely, the angles of the dynamic rays that land on a parabolic point or on the boundary of a "good" Siegel disk define a rotation set X with $\rho(X) = \theta$, where $e^{2\pi i\theta}$ is the multiplier of the parabolic point or the center of the Siegel disk. In the parabolic case, this statement is well known and goes back to the work of Goldberg and Milnor [12]. The Siegel case, while similar in spirit, is trickier because of the possibility of rays accumulating but not landing on the boundary. The "good" Siegel disk assumption refers to a limb decomposition hypothesis, similar to Milnor's in [22], which allows us to prove the required landing statements (this hypothesis is weaker than local connectivity of the Julia set and presumably holds for Lebesgue almost every θ). The deployment vector $\delta(X)$ can be recovered from the internal angles of the marked roots on the boundary of the Siegel disk, as seen from its center.

These general remarks are illustrated in greater detail in two low-degree families of polynomial maps. According to Douady and Hubbard, the combinatorial structure of the Mandelbrot set (specifically, the boundary of the main cardioid and the limbs growing from it) catalogs all rotation sets under the doubling map m_2 (see [9] and [20]). We give a brief account of this in a section on the quadratic family, setting the stage for the simplest higher degree example, namely, the family of cubic polynomials with an indifferent fixed point of a given rotation number. This one-dimensional slice was studied in [30] in the irrational case and has been the subject of investigations by others (see for example [6]). There are indeed intriguing connections between rotation sets under the tripling map m_3 and this cubic family.

Fix $0 < \theta < 1$ and consider the space of monic cubic polynomials with a fixed point of multiplier $e^{2\pi i\theta}$ at the origin. Each such cubic has the form $f_a : z \mapsto e^{2\pi i\theta} z + az^2 + z^3$ for some $a \in \mathbb{C}$, where f_a and f_{-a} are affinely conjugate under the involution $z \mapsto -z$. The ***connectedness locus***

$$\mathscr{M}_3(\theta) = \{a \in \mathbb{C} : \text{The Julia set } J(f_a) \text{ is connected}\}$$

is compact, connected, and full (compare Figs. 5.8 and 5.10). Outside $\mathscr{M}_3(\theta)$ exactly one critical point of f_a escapes to ∞ and the Böttcher coordinate of the escaping co-critical point gives a conformal isomorphism $\mathbb{C} \smallsetminus \mathscr{M}_3(\theta) \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ which can be used to define the ***parameter rays*** of $\mathscr{M}_3(\theta)$.

When θ is rational of the form p/q in lowest terms, the set X_a of angles of dynamic rays that land at the parabolic point 0 is a rotation set under tripling with $\rho(X_a) = p/q$. There are $2q + 1$ possibilities for X_a parametrized by their deployment vectors $(i/(2q), 1 - i/(2q))$ for $i = 0, \ldots, 2q$. Here, X_a is a q-cycle if i is even and the union of two compatible q-cycles if i is odd. We briefly describe the locus of each possibility in the a-plane as a piece of $\mathscr{M}_3(\theta)$ cut off by a pair of parameter rays.

Next we assume θ is an irrational of bounded type. In this case, each f_a has a Siegel disk $\varDelta_a$ centered at 0 whose topological boundary $\partial\varDelta_a$ is a quasicircle containing at least one critical point of f_a. According to [30], there is an embedded arc $\varGamma \subset \mathscr{M}_3(\theta)$ containing $a = 0$ and having endpoints at $a = \pm\sqrt{3e^{2\pi i\theta}}$ with the property that $a \in \varGamma$ if and only if $\partial\varDelta_a$ contains both critical points of f_a (Fig. 5.10). This arc is parametrized by a well-defined choice $\tau_a \in [0, 1]$ of the conformal angle between the two critical points, as seen from the center 0 of the disk $\varDelta_a$. For each $a \in \varGamma$, the set of angles of dynamic rays that land on the boundary $\partial\varDelta_a$ contains a unique minimal rotation set X_a under tripling with $\rho(X_a) = \theta$. If $(\delta_a, 1 - \delta_a)$ is the deployment vector of X_a, then $\delta_a = \tau_a$. Thus, as a moves along $\varGamma$ from one end to the other, δ_a assumes all values between 0 and 1 monotonically. In particular, every minimal rotation set for m_3 with rotation number θ occurs exactly once in the family $\{X_a\}_{a\in\varGamma}$.

The connectedness locus $\mathscr{M}_3(\theta)$ has a limb structure much like the Mandelbrot set, where the role of the boundary of the main cardioid is being played by $\varGamma$. The analysis of the rotation sets under tripling in Chap. 4 allows us to give a combinatorial description of the limbs growing from $\varGamma$ and the associated wakes $\pm\mathscr{W}_n$ corresponding to the parameters $\pm a_n \in \varGamma$ where $\delta_{\pm a_n} = \pm n\theta \pmod{\mathbb{Z}}$ for $n \geq 0$. We show that the angles of the parameter rays bounding these wakes are all transcendental but depend rationally on a single ***base angle***

$$\omega = \sum_{0<-k\theta\leq\theta} 3^{-(k+1)}$$

which is just the leading angle of the minimal rotation set X under tripling with $\rho(X) = \theta$ and $\delta(X) = (1, 0)$. Explicit computations are given for the golden mean $\theta = \frac{(\sqrt{5}-1)}{2}$, where $\omega \approx 0.12809959$. This description is combinatorial, as we do not address the issue of landing of these parameter rays.

To make sense of the rotation set X_a for $a \notin \varGamma$, one possibility is to verify the limb decomposition hypothesis for all Julia sets $J(f_a)$, but this has not yet been verified for every $a \in \mathscr{M}_3(\theta)$ (although it is known to hold for many parameters). We take an alternative route by approaching $\mathscr{M}_3(\theta)$ from outside, allowing disconnected Julia sets and bifurcated rays. Using the fact that outside $\mathscr{M}_3(\theta)$ the cubic f_a has a quadratic-like restriction hybrid equivalent to $z \mapsto e^{2\pi i\theta}z + z^2$, we define the rotation set X_a for $a \notin \mathscr{M}_3(\theta)$, describe its deployment vector in terms of the external angle of a, and show that it remains unchanged within each open set $\pm\mathscr{W}_n \smallsetminus \mathscr{M}_3(\theta)$. A holomorphic motion argument then allows extending this result to the entire wakes $\pm\mathscr{W}_n$.

Understanding the combinatorial structure of the limbs growing from $\varGamma$ is related to the question of whether $\mathscr{M}_3(\theta)$ contains a homeomorphic copy of the filled Julia set of the quadratic $z \mapsto e^{2\pi i\theta}z + z^2$ in which the Siegel disk is collapsed into an arc. A similar statement has been proved for the attracting perturbations of these maps in [24], and there are strong indications that the phenomenon persists in the indifferent case, at least when θ is of bounded type.

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Contents

Chapter 1
Monotone Maps of the Circle

Throughout this monograph the following conventions are adopted:

- The circle is represented as the quotient $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- $\pi : \mathbb{R} \to \mathbb{T}$ is the canonical projection.
- Three or more distinct points $t_1, t_2, \ldots, t_k \in \mathbb{T}$ are in ***positive cyclic order*** if there are representatives $x_i \in \pi^{-1}(t_i)$ such that $x_1 < x_2 < \cdots < x_k < x_1 + 1$.
- For a distinct pair $t_1, t_2 \in \mathbb{T}$, the interval $(t_1, t_2) \subset \mathbb{T}$ is defined as the set of all $t \in \mathbb{T}$ such that t_1, t, t_2 are in positive cyclic order. We define the intervals $(t_1, t_2]$, $[t_1, t_2)$, $[t_1, t_2]$ by adding the suitable endpoints to (t_1, t_2).
- The length of an interval $(t_1, t_2) \subset \mathbb{T}$ is always understood as its normalized Lebesgue measure, that is, the unique representative of $t_2 - t_1$ in $[0, 1)$.

Every continuous map $g : \mathbb{T} \to \mathbb{T}$ lifts under the canonical projection π to a continuous map $G : \mathbb{R} \to \mathbb{R}$, so $\pi \circ G = g \circ \pi$, and G is unique up to an additive integer. The lift G satisfies $G(x + 1) = G(x) + d$ for some integer d called the ***degree*** of g. We say that g is a ***monotone map*** if G is monotone in the usual sense (non-increasing or non-decreasing).

This chapter studies the dynamics of degree 1 monotone maps of the circle, which can be thought of as slight generalizations of orientation preserving homeomorphisms. It will be convenient to first work with lifts of such maps, i.e., continuous non-decreasing self-maps of the real line that commute with the unit translation.

1.1 The Translation Number

Suppose $G : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing map which satisfies

$$G(x + 1) = G(x) + 1 \quad \text{for all} \quad x \in \mathbb{R}. \tag{1.1}$$

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If $0 \le x < y < 1$, then

$$(G(x) - x) - (G(y) - y) \le (G(y) - x) - (G(y) - y) = y - x < 1$$

and

$$(G(x) - x) - (G(y) - y) \ge (G(0) - x) - (G(1) - y) = y - x - 1 > -1.$$

Since by (1.1) the function $G - \mathrm{id}_{\mathbb{R}}$ is 1-periodic, the inequality

$$|(G(x) - x) - (G(y) - y)| < 1$$

follows for all $x, y \in \mathbb{R}$. The same reasoning applied to the n-th iterate $G^{\circ n}$ shows that

$$|(G^{\circ n}(x) - x) - (G^{\circ n}(y) - y)| < 1 \quad \text{for all} \quad x, y \in \mathbb{R} \quad \text{and} \quad n \ge 1. \tag{1.2}$$

Lemma 1.1 *There exists at most one rational number p/q with $q > 0$ for which the equation $G^{\circ q}(x) = x + p$ has a solution in $x \in \mathbb{R}$.*

Proof Suppose $G^{\circ q}(x) = x + p$ and $G^{\circ n}(y) = y + m$. Then $G^{\circ nq}(x) = x + np$ and $G^{\circ nq}(y) = y + mq$. By (1.2),

$$|(G^{\circ nq}(x) - x) - (G^{\circ nq}(y) - y)| = |np - mq| < 1,$$

which implies $np = mq$. □

Consider the sets

$$\mathbb{Q}_G^- = \left\{ \frac{p}{q} : G^{\circ q}(x) > x + p \text{ for all } x \in \mathbb{R} \right\},$$
$$\mathbb{Q}_G^+ = \left\{ \frac{p}{q} : G^{\circ q}(x) < x + p \text{ for all } x \in \mathbb{R} \right\},$$

where p, q are integers with $q > 0$. Evidently $\mathbb{Q}_G^-$ and $\mathbb{Q}_G^+$ are non-empty disjoint subsets of the set $\mathbb{Q}$ of rational numbers. Furthermore,

1. If $p/q \notin \mathbb{Q}_G^- \cup \mathbb{Q}_G^+$, both equations $G^{\circ q}(x) > x + p$ and $G^{\circ q}(x) < x + p$ have solutions and so does $G^{\circ q}(x) = x + p$ by continuity. Applying Lemma 1.1, we see that the union $\mathbb{Q}_G^- \cup \mathbb{Q}_G^+$ can omit at most one rational number.
2. If $p/q \in \mathbb{Q}_G^-$ and $m/n \in \mathbb{Q}_G^+$, then $x + np < G^{\circ nq}(x) < x + mq$ for all x, so $p/q < m/n$.
3. If $p/q \in \mathbb{Q}_G^-$, since the function $G^{\circ q} - \mathrm{id}_{\mathbb{R}}$ is 1-periodic and $> p$, there is an $\varepsilon > 0$ such that $G^{\circ q}(x) > x + p + \varepsilon$ for all x. It follows by induction that $G^{\circ nq}(x) > x + np + n\varepsilon$ for all x and $n \ge 1$, which proves $(np + 1)/(nq) \in \mathbb{Q}_G^-$ as soon as $n > 1/\varepsilon$. This shows $\mathbb{Q}_G^-$ has no largest element. Similarly, $\mathbb{Q}_G^+$ has no smallest element.

Properties (1) and (2) show that the pair $(\mathbb{Q}_G^-, \mathbb{Q}_G^+)$ is a “Dedekind cut” of $\mathbb{Q}$ and

$$\sup \mathbb{Q}_G^- = \inf \mathbb{Q}_G^+.$$

We call this common value the ***translation number*** of G and denote it by $\tau(G)$. It follows from property (3) that

$$\mathbb{Q}_G^- \cup \mathbb{Q}_G^+ = \begin{cases} \mathbb{Q} \setminus \{\tau(G)\} & \text{if } \tau(G) \in \mathbb{Q} \\ \mathbb{Q} & \text{if } \tau(G) \notin \mathbb{Q}. \end{cases} \tag{1.3}$$

The terminology for $\tau(G)$ is justified by the following

Theorem 1.2 (Poincaré) *For every $x \in \mathbb{R}$,*

$$\tau(G) = \lim_{n\to\infty} \frac{G^{\circ n}(x) - x}{n}. \tag{1.4}$$

Thus, $\tau(G)$ measures the average translation per iterate that each point experiences under repeated applications of G.

Proof For any integer $n \geq 1$ we can find an integer m such that $(m-1)/n < \tau(G) < (m+1)/n$. Then $(m-1)/n \in \mathbb{Q}_G^-$ and $(m+1)/n \in \mathbb{Q}_G^+$, so

$$\frac{m-1}{n} < \frac{G^{\circ n}(x) - x}{n} < \frac{m+1}{n}$$

for all x. This gives the inequality

$$\left| \frac{G^{\circ n}(x) - x}{n} - \tau(G) \right| < \frac{2}{n} \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and} \quad n \geq 1. \tag{1.5}$$

The result follows by letting $n \to \infty$. □

Corollary 1.3 *The equation $G^{\circ q}(x) = x + p$ has a solution in $x \in \mathbb{R}$ if and only if $\tau(G) = p/q$.*

Proof Evidently $G^{\circ q}(x) = x + p$ for some x if and only if $p/q \notin \mathbb{Q}_G^- \cup \mathbb{Q}_G^+$. By (1.3), this is equivalent to $\tau(G) = p/q$. □

Corollary 1.4 *Suppose n_1, n_2, m_1, m_2 are integers with $n_1 \geq 0$ and $n_2 \geq 0$. Then*

$$n_1\tau(G) + m_1 < n_2\tau(G) + m_2 \tag{1.6}$$

if and only if

$$G^{\circ n_1}(x) + m_1 < G^{\circ n_2}(x) + m_2 \quad \textit{for all} \quad x \in \mathbb{R}. \tag{1.7}$$

Proof The case $n_1 = n_2$ is trivial, so let us assume $0 \leq n_1 < n_2$. In this case, the inequality (1.6) is equivalent to $(m_1 - m_2)/(n_2 - n_1) < \tau(G)$ which by (1.3) is equivalent to $(m_1 - m_2)/(n_2 - n_1) \in \mathbb{Q}_G^-$. The latter means $x + m_1 - m_2 < G^{\circ n_2 - n_1}(x)$ for all x, which is clearly equivalent to (1.7). The case $n_1 > n_2 \geq 0$ is treated similarly. □

1.2 The Rotation Number

Now consider a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$. By definition, this means g lifts to a continuous non-decreasing map $G : \mathbb{R} \to \mathbb{R}$ which commutes with the unit translation. All other lifts of g are of the form $G + k$ for some integer k, with the translation number $\tau(G + k) = \tau(G) + k$ by (1.4).

Definition 1.5 The ***rotation number*** $\rho(g)$ of a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ is the residue class modulo $\mathbb{Z}$ of the translation number $\tau(G)$, where $G : \mathbb{R} \to \mathbb{R}$ is any lift of g.

For convenience, we often identify $\rho(g)$ with its unique representative in $[0, 1)$. As a main example, for any $\theta \in [0, 1)$ the ***rigid rotation*** $r_\theta : \mathbb{T} \to \mathbb{T}$ defined by

$$r_\theta(t) = t + \theta \pmod{\mathbb{Z}}$$

has rotation number $\rho(r_\theta) = \theta$.

Theorem 1.6 *Let $g : \mathbb{T} \to \mathbb{T}$ be a degree 1 monotone map with $\rho(g) = \theta$. If the orbit points $r_\theta^{\circ i}(0), r_\theta^{\circ j}(0), r_\theta^{\circ k}(0)$ under the rigid rotation r_θ are in positive cyclic order, the same must be true of the orbit points $g^{\circ i}(t), g^{\circ j}(t), g^{\circ k}(t)$ for every $t \in \mathbb{T}$.*

(If θ is a fraction of the form p/q in lowest terms, we need to assume $q > 2$ in order for the theorem to have any content.)

Proof The assumption means that there are integers m_1, m_2 such that

$$i\theta < j\theta + m_1 < k\theta + m_2 < i\theta + 1.$$

If we choose a lift G of g so that $\tau(G) = \theta$, Corollary 1.4 shows that for all $x \in \mathbb{R}$,

$$G^{\circ i}(x) < G^{\circ j}(x) + m_1 < G^{\circ k}(x) + m_2 < G^{\circ i}(x) + 1.$$

Projecting down to the circle, it follows that $g^{\circ i}(t), g^{\circ j}(t), g^{\circ k}(t)$ are in positive cyclic order for every $t \in \mathbb{T}$. □

Theorem 1.7 *For every degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ and every integer $k \geq 0$,*

$$\rho(g^{\circ k}) = k\,\rho(g) \pmod{\mathbb{Z}}. \tag{1.8}$$

If g is a homeomorphism, the above formula holds for negative k as well.

Here $g^{\circ 0} = \mathrm{id}$ and $g^{\circ k}$ means the $(-k)$-th iterate of g^{-1} if $k < 0$.

Proof (1.8) is trivial for $k = 0$, so let us assume $k \geq 1$. For any lift G of g, the iterate $G^{\circ k}$ is a lift of $g^{\circ k}$ and

$$\tau(G^{\circ k}) = \lim_{n\to\infty} \frac{G^{\circ kn}(0)}{n} = k \lim_{n\to\infty} \frac{G^{\circ kn}(0)}{kn} = k\,\tau(G).$$

Taking residue classes modulo $\mathbb{Z}$ then proves (1.8). If g is a homeomorphism, so is G and the inverse G^{-1} is a lift of g^{-1}. The uniform estimate (1.5) applied to $x = (G^{-1})^{\circ n}(0)$ gives

$$\left| \frac{(G^{-1})^{\circ n}(0)}{n} + \tau(G) \right| < \frac{2}{n}$$

for all $n \geq 1$. Letting $n \to \infty$, we obtain $\tau(G^{-1}) = -\tau(G)$, which proves (1.8) for $k = -1$. The general case $k < 0$ follows from this by iteration. □

Theorem 1.8 *Either of the following assumptions on degree* 1 *monotone maps* $g, h : \mathbb{T} \to \mathbb{T}$ *implies* $\rho(g) = \rho(h)$*:*

(i) *g and h agree along some orbit, that is, there is a $t \in \mathbb{T}$ such that $g^{\circ n}(t) = h^{\circ n}(t)$ for all $n \geq 1$.*
(ii) *g and h are semiconjugate, that is, there is a degree* 1 *monotone map $\varphi : \mathbb{T} \to \mathbb{T}$ which satisfies the relation $\varphi \circ g = h \circ \varphi$.*

Proof

(i) Let $t_n = g^{\circ n}(t) = h^{\circ n}(t)$. We may assume $t_n \neq t_{n-1}$ for all n since otherwise both g, h have a fixed point and $\rho(g) = \rho(h) = 0$. Pick any $x_0 \in \pi^{-1}(t_0)$ and define x_n inductively as the smallest element of $\pi^{-1}(t_n)$ that is $> x_{n-1}$. Thus, $x_{n-1} < x_n < x_{n-1} + 1$ for all n. Take the unique lift G of g that sends x_0 to x_1 and let $y_n = G^{\circ n}(x_0)$. Applying G repeatedly on the inequalities $y_0 < y_1 < y_0 + 1$ then shows $y_{n-1} < y_n < y_{n-1} + 1$ for all n, where the inequalities remains strict by the assumption $t_n \neq t_{n-1}$. Since y_n is an integer translation of x_n and $y_0 = x_0$, it follows that $y_n = x_n$ for all n. Similarly, the unique lift H of h that sends x_0 to x_1 satisfies $x_n = H^{\circ n}(x_0)$ for all n. It follows from Theorem 1.2 that $\tau(G) = \lim_{n\to\infty}(x_n - x_0)/n = \tau(H)$, which proves $\rho(g) = \rho(h)$.

(ii) Choose lifts $G, H, \Phi : \mathbb{R} \to \mathbb{R}$ of g, h, φ such that $\Phi \circ G = H \circ \Phi$. Then $\Phi \circ G^{\circ n} = H^{\circ n} \circ \Phi$ for all n. Since Φ commutes with the unit translation, the function $\Phi - \mathrm{id}_{\mathbb{R}}$ is 1-periodic and therefore bounded on $\mathbb{R}$. It follows that

$$\lim_{n\to\infty} \frac{H^{\circ n}(\Phi(0)) - G^{\circ n}(0)}{n} = \lim_{n\to\infty} \frac{\Phi(G^{\circ n}(0)) - G^{\circ n}(0)}{n} = 0,$$

which shows $\tau(G) = \tau(H)$. □

The following can be thought of as an analog of Corollary 1.3 for arbitrary rotation numbers:

Lemma 1.9 *Let $g : \mathbb{T} \to \mathbb{T}$ be a degree 1 monotone map with $\rho(g) = \theta$. Then $g(t) = r_\theta(t)$ for some $t \in \mathbb{T}$.*

Proof Let $0 \le \theta < 1$ and choose a lift G of g with $\tau(G) = \theta$. Suppose $G(x) > x+\theta$ for all $x \in \mathbb{R}$. Then, since the function $G - \mathrm{id}_\mathbb{R}$ is 1-periodic, there is an $\varepsilon > 0$ such that $G(x) > x+\theta+\varepsilon$ for all x. It follows by induction that $G^{\circ n}(x) > x+n(\theta+\varepsilon)$ for all x and all $n \ge 1$. By (1.4) this would imply $\tau(G) \ge \theta+\varepsilon$, which is a contradiction. Similarly, the assumption $G(x) < x + \theta$ for all x leads to a contradiction. Thus, $G(x) = x + \theta$ for some $x \in \mathbb{R}$. □

Here is a consequence of the above lemma that will be used in Sect. 4.1:

Corollary 1.10 *For every orientation-preserving homeomorphism $g : \mathbb{T} \to \mathbb{T}$ and every rigid rotation r_θ, the commutator $[g, r_\theta] = g \circ r_\theta \circ g^{-1} \circ r_\theta^{-1}$ has rotation number zero.*

Proof By Theorem 1.8(ii), $\rho(g \circ r_\theta \circ g^{-1}) = \rho(r_\theta) = \theta$. By Lemma 1.9, there is a $t \in \mathbb{T}$ such that $(g \circ r_\theta \circ g^{-1})(t) = r_\theta(t)$. This means $r_\theta(t)$ is a fixed point of $[g, r_\theta]$, which proves $\rho([g, r_\theta]) = 0$. □

We end this section by showing that the rotation number $\rho(g)$ depends continuously and monotonically on g. Observe that the space of continuous non-decreasing functions $\mathbb{R} \to \mathbb{R}$ which commute with the unit translation is closed in the topology of uniform convergence on the real line. Hence the space of degree 1 monotone maps $\mathbb{T} \to \mathbb{T}$ is closed in the topology of uniform convergence on the circle.

Theorem 1.11 *The mapping $g \mapsto \rho(g)$ is continuous in the topology of uniform convergence on the circle.*

Proof It suffices to check that $G \mapsto \tau(G)$ is continuous in the topology of uniform convergence on the real line. This is easy because by (1.5) this mapping is the uniform limit of the sequence of continuous mappings $G \mapsto G^{\circ n}(0)/n$. □

Now suppose we have a family $\{g_\alpha\}$ of degree 1 monotone maps of the circle depending continuously on a parameter α which varies in some interval on the real line. We say that $\{g_\alpha\}$ is a ***monotone family*** if it lifts to a continuous family $\{G_\alpha\}$ of maps of the real line such that $G_\alpha \le G_\beta$ whenever $\alpha < \beta$. An easy induction then shows that $G_\alpha^{\circ n} \le G_\beta^{\circ n}$ for all n, so $\tau(G_\alpha) \le \tau(G_\beta)$. This proves

Theorem 1.12 *For every monotone family $\{g_\alpha\}$, the map $\alpha \mapsto \rho(g_\alpha)$ is monotone.*

Of course the rotation number of a monotone family can be constant. Suppose however that in the above situation $G_\beta = G_\alpha + 1$ for some $\alpha < \beta$, so $\tau(G_\beta) = \tau(G_\alpha) + 1$. Since the function $\alpha \mapsto \tau(G_\alpha)$ is continuous by (the proof of) Theorem 1.11, it assume all values in the interval $[\tau(G_\alpha), \tau(G_\alpha)+1]$ and it follows that the translation number is not constant.

Corollary 1.13 *Suppose $g_0 : \mathbb{T} \to \mathbb{T}$ is a degree 1 monotone map and*

$$g_\alpha(t) = g_0(t) + \alpha \pmod{\mathbb{Z}}$$

for $\alpha \in \mathbb{T}$. Then the assignment $\alpha \mapsto \rho(g_\alpha)$ itself is a degree 1 monotone map.

1.3 Dynamics in the Presence of Periodic Points

We continue assuming that $g : \mathbb{T} \to \mathbb{T}$ is a degree 1 monotone map. It is easy to see using Corollary 1.3 that $\rho(g) = p/q$ if and only if $g^{\circ q}(t) = t$ for some $t \in \mathbb{T}$. Here is a sharper statement:

Theorem 1.14 *Suppose $g : \mathbb{T} \to \mathbb{T}$ is a degree 1 monotone map with $\rho(g) = p/q$ in lowest terms. Then,*

(i) g has a periodic orbit of length q.
(ii) All periodic orbits of g have length q.
(iii) If the points of a periodic orbit are labeled in positive cyclic order as $t_1, t_2, \ldots, t_q$, then $g(t_j) = t_{j+p}$, where the subscripts are taken modulo q.

Proof By what we have seen, g has a periodic point whose period n divides q. This, in turn, implies that $\rho(g)$ is a fraction of the form $m/n \pmod{\mathbb{Z}}$. Since p and q are assumed to be relatively prime, it easily follows that $n = q$. This proves (i).

To see (ii), let t be a periodic point of g of period n. Take any $x \in \pi^{-1}(t)$ and a lift G of g with $\tau(G) = p/q$. Then $G^{\circ n}(x) = x + m$ for some integer m, where $m/n = p/q$ by Corollary 1.3. Since p and q are assumed relatively prime, we have $n = kq$ and $m = kp$ for some integer $k \geq 1$. If the minimal period n were greater than q, then either $G^{\circ q}(x) > x + p$ or $G^{\circ q}(x) < x + p$. Since G is monotone and commutes with the unit translation, it would follow inductively that $G^{\circ iq}(x) > x + ip$ or $G^{\circ iq}(x) < x + ip$ for all $i \geq 1$. This would contradict $G^{\circ kq}(x) = x + kp$. Thus $n = q$.

Finally, (iii) follows at once from Theorem 1.6 since if $a_j = j/q \pmod{\mathbb{Z}}$, the points $a_1, a_2, \cdots, a_q$ are in positive cyclic order and form the orbit of 0 under the rigid rotation $r_{p/q}$, which sends each a_j to a_{j+p}. □

For convenience we often use the term ***q*-cycle** for a periodic orbit of length q. Part (iii) of the above theorem can be expressed as a semiconjugacy relation as follows. Suppose we label the points of a q-cycle C of g as $t_1, \ldots, t_q$ in positive cyclic order. Define the piecewise constant map $\varphi : \mathbb{T} \to \mathbb{T}$ by sending each half-open interval $[t_j, t_{j+1})$ to the point $a_j = j/q \pmod{\mathbb{Z}}$. Then one has the relation

$$\varphi \circ g = r_{p/q} \circ \varphi \quad \text{on} \quad C. \tag{1.9}$$

Note that there are q different ways of labeling the points of C in positive cyclic order, giving rise to q such semiconjugacies which only differ by a rotation. In

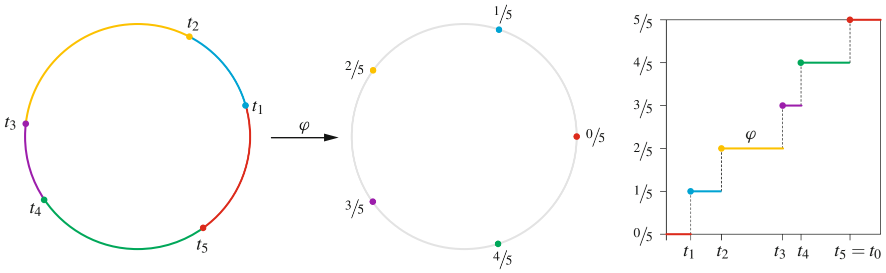

Fig. 1.1 Left: The combinatorial semiconjugacy φ associated with a 5-cycle $C = \{t_1, \ldots, t_5\}$. Right: The graph of φ. Observe that C is the complement of the union of plateaus of φ

particular, if we choose the labeling so that $0 \in [t_q, t_1)$, then $\varphi(0) = 0$. We call φ normalized this way the ***combinatorial semiconjugacy*** associated with the cycle C. To establish the analogy with the more interesting case of irrational rotation numbers to be discussed in the next section, let us comment that the cycle C can be described as the complement of the union of the "plateaus" of φ (by definition, a plateau is a maximal open interval on which the map is constant; see Sect. 1.4 and compare Fig. 1.1).

Remark 1.15 The relation (1.9) may not hold globally since g may well map a point in (t_j, t_{j+1}) to t_{j+p+1}. However, if g maps each $[t_j, t_{j+1})$ onto $[t_{j+p}, t_{j+p+1})$, then (1.9) holds on the whole circle.

The preceding discussion provides a simple characterization for the cycles that occur as periodic orbits of degree 1 monotone maps of the circle. Let C consist of q points $t_1, \ldots, t_q$ labeled in positive cyclic order and $g : C \to C$ be any transitive action. We say that C has ***combinatorial rotation number*** p/q under g if $g(t_j) = t_{j+p}$ for all j. In this case, we can extend g to an orientation-preserving homeomorphism of the circle by mapping each half-open interval $[t_j, t_{j+1})$ homeomorphically onto $[t_{j+p}, t_{j+p+1})$. Theorem 1.14(iii) then shows that $\rho(g) = p/q$.

Corollary 1.16 *A cycle can be realized as a periodic orbit of a degree* 1 *monotone map if and only if it has a well-defined combinatorial rotation number.*

See Fig. 1.2.

Recall that the ***omega limit set*** of a point $t \in \mathbb{T}$ under the action of g is the set of all accumulation points of the forward orbit of t:

$$\omega_g(t) = \bigcap_{n \geq 1} \overline{\{g^{\circ n}(t), g^{\circ n+1}(t), g^{\circ n+2}(t), \ldots\}}.$$

It is easy to see that $\omega_g(t)$ is non-empty and compact, and $g(\omega_g(t)) = \omega_g(t)$.

Theorem 1.17 *Suppose $g : \mathbb{T} \to \mathbb{T}$ is a degree* 1 *monotone map with $\rho(g) = p/q$ in lowest terms. Then $\omega_g(t)$ is a q-cycle for every $t \in \mathbb{T}$.*

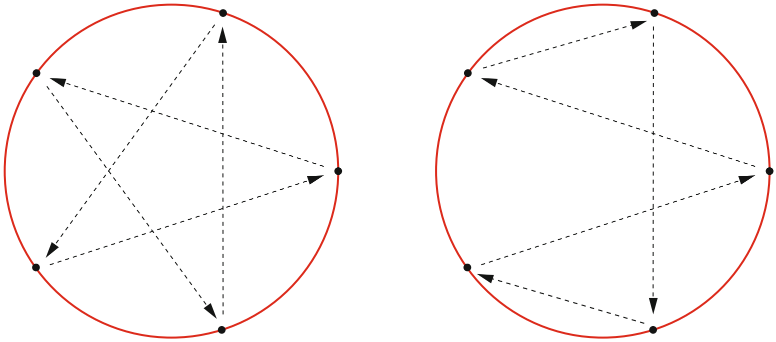

Fig. 1.2 Every q-cycle under a degree 1 monotone map of the circle has a well-defined combinatorial rotation number of the form p/q, where p and q are relatively prime. The 5-cycle on the left has combinatorial rotation number $\frac{2}{5}$, while the one on the right, having no combinatorial rotation number, cannot be realized as a periodic orbit of any degree 1 monotone map

Proof Let $E = \{t \in \mathbb{T} : g^{\circ q}(t) = t\}$. By Theorem 1.14, E is non-empty and every $t \in E$ has period q, so $\omega_g(t) = \{t, g(t), \ldots, g^{\circ q-1}(t)\}$. If $t \notin E$, then t belongs to a connected component J of the open set $\mathbb{T} \smallsetminus E$. The iterate $g^{\circ q}$ maps the interval J onto itself, keeping the endpoints fixed but moving all the interior points (note however that a point in J may map to an endpoint). An easy calculus exercise shows that one endpoint t' of J is attracting under $g^{\circ q}$ and the other is repelling. It follows that $g^{\circ nq}(t) \to t'$ as $n \to \infty$. But then $g^{\circ i+nq}(t) \to g^{\circ i}(t')$, which proves $\omega_g(t) = \{t', g(t'), \ldots, g^{\circ q-1}(t')\}$. □

1.4 Dynamics in the Absence of Periodic Points

We now turn to the case of irrational rotation numbers.

Theorem 1.18 (Poincaré) *Suppose $g : \mathbb{T} \to \mathbb{T}$ is a degree 1 monotone map with $\rho(g) = \theta$ irrational. Then there exists a degree 1 monotone map $\varphi : \mathbb{T} \to \mathbb{T}$ which satisfies $\varphi \circ g = r_\theta \circ \varphi$. Moreover, φ is unique up to postcomposition with a rigid rotation.*

We call the unique such φ normalized by $\varphi(0) = 0$ the ***Poincaré semiconjugacy*** between g and r_θ.

Proof Lift g to a map $G : \mathbb{R} \to \mathbb{R}$ with $\tau(G) = \theta$. We will construct a map $\Phi : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) Φ is continuous and non-decreasing;
(ii) $\Phi(x + 1) = \Phi(x) + 1$ for all x;
(iii) $\Phi(G(x)) = \Phi(x) + \theta$ for all x.

The quotient map $\varphi : \mathbb{T} \to \mathbb{T}$ will then have the desired property.

Consider the set

$$\Lambda = \{G^{\circ n}(0) + m : n, m \text{ are integers with } n \geq 0\}. \tag{1.10}$$

Since $\tau(G) = \theta$ is irrational, Corollary 1.4 shows that each element of Λ has a unique representation of this form. Define $\Phi : \Lambda \to \mathbb{R}$ by

$$\Phi(G^{\circ n}(0) + m) = n\theta + m.$$

The image $\Phi(\Lambda)$ is dense in $\mathbb{R}$ since θ is irrational, and Φ is strictly increasing on Λ by Corollary 1.4. Extend Φ to the real line by

$$\Phi(x) = \sup_{y \in \Lambda \cap (-\infty, x]} \Phi(y).$$

Clearly Φ is non-decreasing, so it has one-sided limits $\Phi(x^-) \leq \Phi(x^+)$ at every x. If the inequality were strict at some x, the image $\Phi(\Lambda)$ would omit all points in the interval $(\Phi(x^-), \Phi(x^+))$, with the possible exception of $\Phi(x)$ if $x \in \Lambda$, which contradicts density of $\Phi(\Lambda)$. Thus, Φ is continuous everywhere.

Properties (ii) and (iii) clearly hold when $x \in \Lambda$, and by continuity they hold when $x \in \overline{\Lambda}$. If (a, b) is a connected component of $\mathbb{R} \smallsetminus \overline{\Lambda}$, the definition of Φ shows that Φ is constant in (a, b). If $x \in (a, b)$, invariance of $\overline{\Lambda}$ under the unit translation gives

$$\Phi(x + 1) = \Phi(a + 1) = \Phi(a) + 1 = \Phi(x) + 1,$$

while monotonicity gives

$$\Phi(G(a)) \leq \Phi(G(x)) \leq \Phi(G(b)) \Longrightarrow \Phi(a) + \theta \leq \Phi(G(x)) \leq \Phi(b) + \theta.$$

Since $\Phi(a) = \Phi(b) = \Phi(x)$, we obtain $\Phi(G(x)) = \Phi(x) + \theta$. This proves that (ii) and (iii) hold for all $x \in \mathbb{R}$.

Uniqueness follows since Φ is uniquely determined by its values on Λ, which in turn are uniquely determined by $\Phi(0)$. □

Since the Poincaré semiconjugacy φ constructed above is a monotone map, each fiber $E_s = \varphi^{-1}(s)$ is either a point or a closed non-degenerate interval. It follows that the interior I_s of E_s is either empty or an open interval. In the latter case we call I_s a ***plateau*** of φ.[1] We can visualize a plateau as a maximal open interval on which the graph of φ is a horizontal line.

[1] Let us emphasize that our plateaus are *open* intervals, a convention that is not commonly adopted in the literature.

Lemma 1.19 *Let φ be the Poincaré semiconjugacy between g and r_θ, given by Theorem 1.18.*

(i) For every $s \in \mathbb{T}$, $g^{-1}(E_s) = E_{s-\theta}$.
(ii) If $I_s \neq \emptyset$ then $I_{s-\theta} \neq \emptyset$. Moreover, $I_{s-\theta}$ contains the open interval $g^{-1}(I_s)$.

By part (ii), the plateaus of φ are indexed by a countable union of *backward* orbits of r_θ. This turns out to be a characteristic property of Poincaré semiconjugacies (see Theorem 1.22).

Proof Statement (i) follows directly from the semiconjugacy relation $\varphi \circ g = r_\theta \circ \varphi$. For (ii), simply note that I_s being a plateau implies that E_s does not reduce to a point. By (i), the same must be true of $E_{s-\theta}$, which shows $I_{s-\theta}$ is a plateau. □

The following theorem is the analogue of Theorem 1.17 for monotone maps with irrational rotation number. Unlike the rational case, there are now two possible regimes for the asymptotic behavior of orbits.

Theorem 1.20 *Suppose $g : \mathbb{T} \to \mathbb{T}$ is a degree 1 monotone map with $\rho(g) = \theta$ irrational, and $\varphi : \mathbb{T} \to \mathbb{T}$ is the Poincaré semiconjugacy between g and r_θ.*

(i) If φ is a homeomorphism, then $\omega_g(t) = \mathbb{T}$ for all $t \in \mathbb{T}$.
(ii) If φ is not a homeomorphism, there exists a g-invariant Cantor set $K \subset \mathbb{T}$ with the property that $\omega_g(t) = K$ for every $t \in \mathbb{T}$.

The map g is called ***linearizable*** or ***non-linearizable*** according as case (i) or (ii) holds. We refer to K in (ii) as the ***Cantor attractor*** of g (see Fig. 1.3).

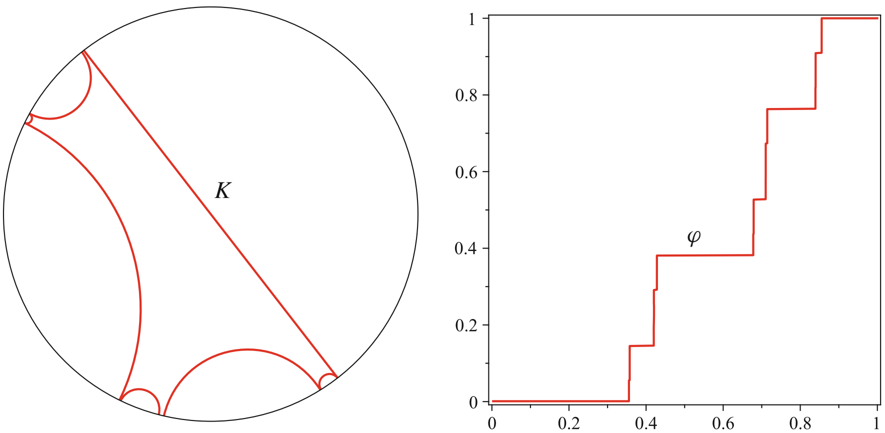

Fig. 1.3 The Cantor attractor K of some degree 1 monotone map with irrational rotation number, and the graph of the corresponding Poincaré semiconjugacy φ. (Here and elsewhere, we use hyperbolic convex hulls to make subsets of the circle more visible.) Similar to the rational case, K can be described as the complement of the union of plateaus of φ

Proof If φ is a homeomorphism, then g is conjugate to r_θ under which all orbits are dense, so (i) holds. Let us then assume that φ is not a homeomorphism and define K to be the complement of the union of all plateaus of φ. Evidently K is a compact proper subset of the circle. If $g(t)$ belongs to a plateau I_s, Lemma 1.19 shows that $I_{s-\theta}$ is a plateau containing t. This proves $g(K) \subset K$. To prove the reserve inclusion, suppose $t \in K$ and take any t' with $g(t') = t$. If $t' \in K$, then $t \in g(K)$. Otherwise t' belongs to a plateau I_s. By Lemma 1.19, $g(E_s) = E_{s+\theta}$ contains t. Thus, g maps E_s either to the single point t, or to the non-degenerate closed interval $E_{s+\theta}$ having t as a boundary point. In either case, monotonicity of g implies that some endpoint of I_s maps to t, proving $t \in g(K)$.

To check that K is a Cantor set, first observe that K has no isolated point since distinct plateaus of φ have disjoint closures. If K were not totally disconnected, it would necessarily contain a non-empty open interval J. As J does not meet any plateau, φ would be one-to-one in J, and the image $\varphi(J)$ would also be an open interval. We could then take any plateau I_s and an integer $n \geq 1$ such that $s - n\theta \in \varphi(J)$. Then, $I_{s-n\theta}$, a plateau by Lemma 1.19, would have to intersect J, contradicting $I_{s-n\theta} \cap K = \emptyset$.

Next, we show that K is strongly minimal in the sense that if X is non-empty, compact and g-invariant, then $K \subset X$. Let us verify that every $p \in K$ which is not an endpoint of a plateau belongs to X. Since such p are dense in K, this will prove $K \subset X$. Pick any $t \in X$ and an increasing sequence $\{n_i\}$ of positive integers such that $r_\theta^{\circ n_i}(\varphi(t)) = \varphi(g^{\circ n_i}(t)) \to \varphi(p)$. By passing to a subsequence, we may assume $g^{\circ n_i}(t) \to u \in X$, so $\varphi(u) = \varphi(p)$ by continuity. If $p \neq u$, the fiber $E_{\varphi(p)}$ would be non-degenerate, hence $I_{\varphi(p)}$ would be a plateau with p as an endpoint, contradicting our assumption. Hence, $p = u \in X$.

It is now easy to prove that $\omega_g(t) = K$ for every $t \in \mathbb{T}$. If $g^{\circ n}(t) \in K$ for some $n \geq 0$, then $\omega_g(t) = K$ follows immediately from minimality. Consider then the case where $g^{\circ n}(t) \notin K$ for every $n \geq 0$. If I_s is the plateau containing t, it follows from Lemma 1.19 that $I_{s+n\theta}$ is the plateau containing $g^{\circ n}(t)$. The $I_{s+n\theta}$ are disjoint with $\sum |I_{s+n\theta}| \leq 1$, so $|I_{s+n\theta}| \to 0$ as $n \to \infty$. Therefore the distance between $g^{\circ n}(t)$ and the endpoints of $I_{s+n\theta}$ tends to zero. It follows that $\omega_g(t) \subset K$, and again by minimality $\omega_g(t) = K$. □

Remark 1.21 The non-linearizable case can always be reduced to the linearizable case at the expense of working in a quotient dynamical system. Consider the equivalence relation $\sim$ on the circle where $t \sim t'$ if and only if $\varphi(t) = \varphi(t')$. Let $\tilde{\mathbb{T}}$ be the set of all equivalence classes $[t]$ of $\sim$. The map $\tilde{\varphi} : \tilde{\mathbb{T}} \to \mathbb{T}$ defined by $\tilde{\varphi}[t] = \varphi(t)$ is clearly a bijection, so it induces a topology on $\tilde{\mathbb{T}}$ with respect to which $\tilde{\varphi}$ is a homeomorphism. The induced action $\tilde{g} : \tilde{\mathbb{T}} \to \tilde{\mathbb{T}}$ given by $\tilde{g}([t]) = [g(t)]$ is easily seen to be well-defined and homeomorphic, and it is linearizable since $\tilde{\varphi} \circ \tilde{g} = r_\theta \circ \tilde{\varphi}$.

The next result characterizes the monotone maps that arise as Poincaré semi-conjugacies. It will be used later in Theorem 2.35. We will continue denoting the interior of the fiber $\varphi^{-1}(s)$ by I_s.

Theorem 1.22 *Let θ be irrational and $\varphi : \mathbb{T} \to \mathbb{T}$ be a degree 1 monotone map with the property that $I_s \neq \emptyset$ implies $I_{s-\theta} \neq \emptyset$. Then, there exists a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ which satisfies $\varphi \circ g = r_\theta \circ \varphi$.*

Observe that any such g has rotation number θ by Theorem 1.8(ii). The map g could be a homeomorphism even if φ has plateaus. This happens when the plateaus of φ are indexed by *full* orbits of r_θ.

Proof It will be convenient to work on the universal cover. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be any lift of φ and set $E_y = \Phi^{-1}(y)$ for every $y \in \mathbb{R}$. By the assumption, if E_y reduces to a point, so does $E_{y+\theta}$. Hence there is a unique map $G : \mathbb{R} \to \mathbb{R}$ which sends each fiber E_y affinely to $E_{y+\theta}$, preserving the orientation. The relations $G(x+1) = G(x)+1$ and $\Phi(G(x)) = \Phi(x)+\theta$ for all x follow immediately. It remains to show that G is non-decreasing and continuous.

Take any $x, x' \in \mathbb{R}$ with $x < x'$. If both x, x' belong to the same fiber of Φ, then clearly $G(x) \leq G(x')$ by the definition of G. Suppose then that $x \in E_y$ and $x' \in E_{y'}$, where necessarily $y < y'$ since Φ is non-decreasing. Then $\Phi(G(x)) = y + \theta < y' + \theta = \Phi(G(x'))$, which implies $G(x) \leq G(x')$. This shows G is non-decreasing. Moreover, every point of $\mathbb{R}$ belongs to some fiber E_y, which is contained in the image of G since $G(E_{y-\theta}) = E_y$. Thus G is surjective. Because of monotonicity, this proves that G is continuous. □

1.5 Invariant Measures

Let $\mathscr{M}(\mathbb{T})$ denote the space of all Borel probability measures on the circle. Every Borel measurable map $g : \mathbb{T} \to \mathbb{T}$ acts on $\mathscr{M}(\mathbb{T})$ by sending a measure μ to its ***push-forward*** $g_*\mu$ defined by $(g_*\mu)(E) = \mu(g^{-1}(E))$. A measure $\mu \in \mathscr{M}(\mathbb{T})$ is called g***-invariant*** if $g_*\mu = \mu$. According to Krylov and Bogolyubov, there is at least one g-invariant measure when g is continuous [14]. In fact, if we start with any $\mu_0 \in \mathscr{M}(\mathbb{T})$ and define the sequence $\mu_n \in \mathscr{M}(\mathbb{T})$ by

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} (g^{\circ i})_* \mu_0 \qquad n \geq 1,$$

then any weak* limit of the sequence $\{\mu_n\}$ will be g-invariant.

A g-invariant measure $\mu \in \mathscr{M}(\mathbb{T})$ is called ***ergodic*** if $g^{-1}(E) = E$ implies $\mu(E) = 0$ or $\mu(E) = 1$. In this case, it follows from Birkhoff's ergodic theorem that for every function $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(g^{\circ i}(t)) = \int_{\mathbb{T}} f \, d\mu$$

holds for μ-almost every $t \in \mathbb{T}$ [14]. If we choose for f the characteristic function of an interval $I \subset \mathbb{T}$, we deduce that

$$\mu(I) = \lim_{n\to\infty} \frac{1}{n} \#\big\{0 \leq i \leq n-1 : g^{\circ i}(t) \in I\big\}$$

for μ-almost every $t \in \mathbb{T}$. In particular, almost every orbit is dense in the support of μ.

It may happen that g has a unique invariant measure $\mu \in \mathscr{M}(\mathbb{T})$. In this case, μ is necessarily ergodic and the map g is called ***uniquely ergodic***. A sharper form of Birkhoff's theorem then shows that for every continuous function $f : \mathbb{T} \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(g^{\circ i}) = \int_{\mathbb{T}} f \, d\mu$$

uniformly on $\mathbb{T}$. If μ has no atoms, we can deduce by a standard approximation argument that for every interval $I \subset \mathbb{T}$ and *every* $t \in \mathbb{T}$,

$$\mu(I) = \lim_{n\to\infty} \frac{1}{n} \#\big\{0 \leq i \leq n-1 : g^{\circ i}(t) \in I\big\}. \tag{1.11}$$

Now suppose g is a degree 1 monotone map. If $\rho(g)$ is rational of the form p/q in lowest terms, then g has at least one q-cycle C by Theorem 1.14, and the Dirac measure μ_C which assigns a mass of $1/q$ to each point of C is clearly g-invariant (in fact, ergodic). Moreover, the combinatorial semiconjugacy φ associated with C (see the end of Sect. 1.3) is related to μ_C by the formula

$$\varphi(t) = \int_0^t d\mu_C = \mu_C[0,t] \pmod{\mathbb{Z}}.$$

It is not hard to see using Theorem 1.14 that the support of every g-invariant measure $\mu \in \mathscr{M}(\mathbb{T})$ is contained in the union of q-cycles of g. As the restriction of μ to each q-cycle is also g-invariant, it must give an equal mass (possibly zero) to each point of the cycle. In the special case where g has finitely many q-cycles $C_1, \ldots, C_n$, it follows that μ is a convex combination of the Dirac measures μ_{C_i}, that is,

$$\mu = \alpha_1 \, \mu_{C_1} + \cdots + \alpha_n \, \mu_{C_n}, \quad \text{where} \quad \alpha_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

In this case the space of all g-invariant measures is isomorphic to an $(n-1)$-dimensional simplex. The ergodic measures in this space are $\mu_{C_1}, \ldots, \mu_{C_n}$, corresponding to the n vertices of the simplex. Thus, g is uniquely ergodic if and only if it has a single periodic orbit.

The situation when $\rho(g) = \theta$ is irrational is quite different. It is well known that the rigid rotation r_θ is uniquely ergodic, with Lebesgue measure λ being its unique invariant measure. In the linearizable case where the Poincaré semiconjugacy φ between g and r_θ is a homeomorphism, it immediately follows that g is also uniquely ergodic, with the unique invariant measure $\varphi_*^{-1}\lambda$ supported on the full circle. In the non-linearizable case a similar construction gives a unique g-invariant measure μ, supported on the Cantor attractor K, with the property that $\varphi_*\mu = \lambda$. In fact, let $D \subset K$ be the countable set of the endpoints of plateaus of φ, and let S be the countable set of $s \in \mathbb{T}$ for which $I_s \neq \emptyset$. Then $\varphi : K \setminus D \to \mathbb{T} \setminus S$ is continuous and bijective, and the measure μ can be described as the push-forward under φ^{-1} of the restriction of λ to $\mathbb{T} \setminus S$. Similar to the rational case, the Poincaré semiconjugacy φ is related to the invariant measure μ by the formula

$$\varphi(t) = \mu[0, t] \pmod{\mathbb{Z}}.$$

In fact, $\varphi^{-1}[0, \varphi(t)] \supset [0, t]$ for every t by monotonicity of φ. Moreover, the difference $\varphi^{-1}[0, \varphi(t)] \setminus [0, t]$ is disjoint from $K \setminus D$, so its μ-measure is zero. Hence,

$$\varphi(t) = \lambda[0, \varphi(t)] = \mu\big(\varphi^{-1}[0, \varphi(t)]\big) = \mu[0, t] \pmod{\mathbb{Z}}.$$

Chapter 2
Rotation Sets

Throughout this chapter d will be a fixed integer ≥ 2. We study certain invariant sets for the multiplication by d map $m_d : \mathbb{T} \to \mathbb{T}$ defined by

$$m_d(t) = dt \qquad (\text{mod } \mathbb{Z}).$$

The low-degree cases m_2 and m_3 are often referred to as the ***doubling*** and ***tripling*** maps.

Definition 2.1 A non-empty compact set $X \subset \mathbb{T}$ is called a ***rotation set*** for m_d if

- X is m_d-invariant in the sense that $m_d(X) = X$,[1] and
- the restriction $m_d|_X$ can be extended to a degree 1 monotone map of the circle.

Roughly speaking, the latter condition means that m_d preserves the cyclic order of all triples in X, except that it may identify some pairs.

If X is a rotation set for m_d and g, h are degree 1 monotone extensions of $m_d|_X$, then $g = h$ on every orbit in X, so $\rho(g) = \rho(h)$ by Theorem 1.8. This quantity, which therefore depends on X only, is called the ***rotation number*** of X and is denoted by $\rho(X)$. We refer to X as a rational or irrational rotation set according as $\rho(X)$ is rational or irrational.

2.1 Basic Properties

Since multiplication by d commutes with the rigid rotation $r : t \mapsto t + 1/(d-1)$ (mod $\mathbb{Z}$), if X is a rotation set for m_d, so are its $d-2$ rotated copies

$$X + \frac{1}{d-1},\ X + \frac{2}{d-1},\ \ldots,\ X + \frac{d-2}{d-1} \quad (\text{mod } \mathbb{Z}).$$

[1]Thus, our notion of invariance is stronger than *forward invariance* $m_d(X) \subset X$ and weaker than *full invariance* $m_d^{-1}(X) = X$.

S. Zakeri, *Rotation Sets and Complex Dynamics*, Lecture Notes in Mathematics 2214, https://doi.org/10.1007/978-3-319-78810-4_2

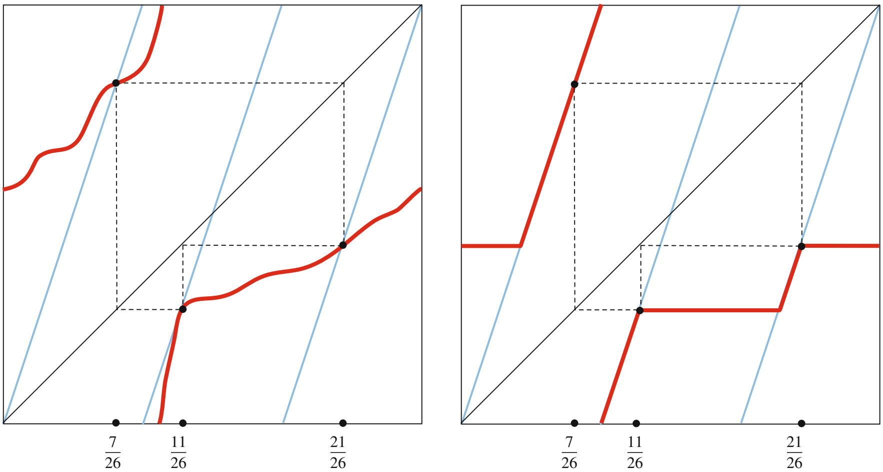

Fig. 2.1 The cycle $X : \frac{7}{26} \mapsto \frac{21}{26} \mapsto \frac{11}{26}$ under tripling is a rotation set with $\rho(X) = \frac{2}{3}$. Left: A generic monotone extension of $m_3|_X$. Right: The "standard" monotone extension of $m_3|_X$ (see the discussion leading to Eq. (2.1))

Moreover, all these sets have rotation number $\rho(X)$ since if g is a monotone extension of $m_d|_X$, then the conjugate map $r^{\circ i} \circ g \circ r^{-i}$ will be a monotone extension of the restriction of m_d to $X + i/(d-1)$.

Example 2.2 The 3-cycle $X : \frac{7}{26} \mapsto \frac{21}{26} \mapsto \frac{11}{26}$ under tripling is a rotation set with rotation number $\frac{2}{3}$. Two possible monotone extensions of m_3 restricted to this cycle are shown in Fig. 2.1. The 180°-rotation of X produces the new rotation set $X + \frac{1}{2} : \frac{8}{26} \mapsto \frac{24}{26} \mapsto \frac{20}{26}$ with the same rotation number. On the other hand, the 4-cycle $\frac{1}{5} \mapsto \frac{3}{5} \mapsto \frac{4}{5} \mapsto \frac{2}{5}$ under tripling is *not* a rotation set since it fails to have a combinatorial rotation number (compare Corollary 1.16).

A rotation set containing periodic orbits is clearly rational. Conversely, every orbit in a rational rotation set is eventually periodic. Here is a more precise statement:

Theorem 2.3 *Suppose X is a rational rotation set for m_d, with $\rho(X) = p/q$ in lowest terms. Then, every forward orbit in X under m_d is finite. More precisely, for every $t \in X$ there is an integer $i \geq 0$ such that $m_d^{\circ i}(t)$ is periodic of period q. In particular, X is at most countable.*

Proof Take any $t \in X$ and any degree 1 monotone extension g of $m_d|_X$. We know from Theorem 1.14 that the sequence $\{g^{\circ nq}(t) = m_{d^q}^{\circ n}(t)\}$ tends to a periodic point $t' \in X$ of period q as $n \to \infty$. Since the map m_{d^q} is uniformly expanding on the circle, its fixed point t' is repelling. Hence $m_{d^q}^{\circ n}(t)$ cannot converge to t' unless $m_{d^q}^{\circ n}(t) = t'$ for some n. □

Remark 2.4 Most periodic orbits of m_d do not define rotation sets. For each prime number q the equation $m^{\circ q}(t) = t$ has $d^q - 1$ solutions $t = i/(d^q - 1) \pmod{\mathbb{Z}}$. Discarding the $d - 1$ fixed points of m_d, it follows that the number $(d^q - d)/q$ of distinct q-cycles of m_d grows exponentially fast as $q \to \infty$. On the other hand, the number of q-cycles of m_d that form a rotation set is precisely $(q-1)\binom{q+d-2}{q}$, which grows like the power q^{d-1} as $q \to \infty$ (see Corollary 3.11).

Every rotation set is nowhere dense since any open interval on the circle eventually maps to the whole circle under the iterations of m_d. By contrast, Lebesgue measure on the circle is ergodic for m_d,[2] so a randomly chosen point on $\mathbb{T}$ has a dense orbit almost surely. This proves the following

Theorem 2.5 *The union $\mathcal{R}_d$ of all rotation sets for m_d has Lebesgue measure zero.*

McMullen [19] has proved the sharper statement that the Hausdorff dimension of $\mathcal{R}_d$ is zero.[3] For more on the set $\mathcal{R}_d$, see Sect. 4.3.

To study of the structure of a rotation set, we first look at its complement.

Definition 2.6 Let X be a rotation set for m_d. A connected component of the complement $\mathbb{T} \smallsetminus X$ is called a ***gap*** of X. A gap of length ℓ is ***minor*** if $\ell < 1/d$ and ***major*** otherwise. The ***multiplicity*** of a major gap is the integer part of $d\ell \geq 1$. A major gap is ***taut*** or ***loose*** according as $d\ell$ is or is not an integer.

Intuitively, a minor gap is short enough so it maps homeomorphically onto its image by m_d. On the other hand, a major gap is too long and wraps around the circle by m_d as many times as its multiplicity (see Lemma 2.8 below).

It will be convenient to work with a specific degree 1 monotone extension of $m_d|_X$ which can be defined whenever X has more than one point. This map, which we call the ***standard monotone map*** of X and denote by g_X, is defined as follows: On every minor gap, set $g_X = m_d$. On every major gap $(a, a+\ell)$ of length $0 < \ell < 1$ and multiplicity n, define

$$g_X(t) = \begin{cases} m_d(a) & t \in \left(a, a + \frac{n}{d}\right] \\ m_d(t) & t \in \left(a + \frac{n}{d}, a + \ell\right) \end{cases} \tag{2.1}$$

(see Figs. 2.1 and 2.2). The map g_X is piecewise affine with derivatives 0 and d, so the total length of the plateaus of g_X is $1 - 1/d = (d-1)/d$. Since by the

[2] Assuming $m_d^{-1}(E) = E$ for some measurable set E, the characteristic function χ_E satisfies $\chi_E \circ m_d = \chi_E$. Expanding χ_E into the Fourier series $\sum c_n e^{2\pi i n t}$, it follows that $\sum c_n e^{2\pi i d n t} = \sum c_n e^{2\pi i n t}$ which implies $c_n = c_{dn}$ for all n. Since $c_n \to 0$, this can hold only if $c_n = 0$ for all $n \neq 0$.

[3] He proves the statement for the closure of the union of all *finite* rotation sets for m_d, but an inspection of his proof shows that it also works for the a priori larger set $\mathcal{R}_d$. The zero dimension statement for individual rotation sets was known much earlier [29].

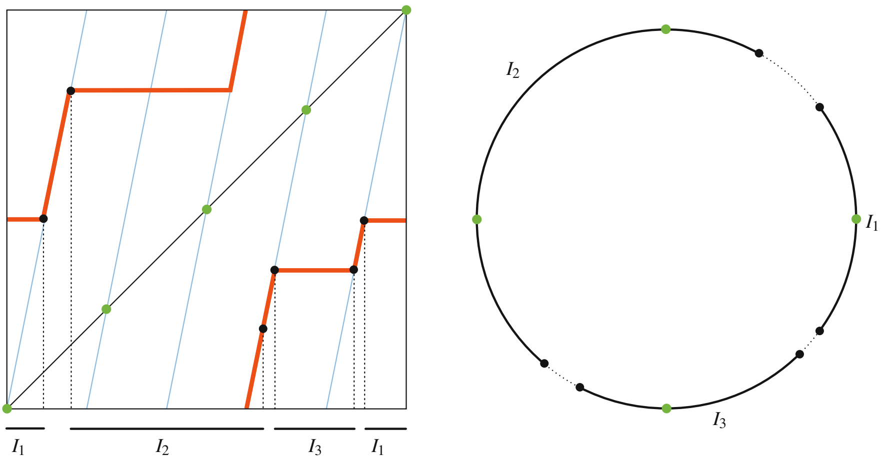

Fig. 2.2 Left: The standard monotone map g_X of some rotation set X for m_5. Counting multiplicities, X has four major gaps, two taut gaps I_1, I_3 of multiplicity 1 and a loose gap I_2 of multiplicity 2. Right: The position of major gaps around the circle. Notice that each major gap contains as many fixed points of m_d as its multiplicity, as asserted in Lemma 2.13

construction each major gap of multiplicity n contributes a plateau of length n/d, we arrive at the following fundamental fact (compare [2] and [4]):

Theorem 2.7 *Every rotation set for m_d containing more than one point has $d-1$ major gaps counting multiplicities.*

The following lemma summarizes the mapping properties of gaps:

Lemma 2.8 *Let X be a rotation set for m_d containing more than one point and $I=(a, a+\ell)$ be a gap of X. Take any degree 1 monotone extension g of $m_d|_X$.*

(i) If I is a minor gap, the interior J of $g(I)$ is a gap of length $d\ell$. Moreover, $m_d : I \to J$ is a homeomorphism.
(ii) If I is a taut gap of multiplicity n, the image $g(I)$ is the single point $m_d(a) \in X$. Under m_d, the point $m_d(a)$ has $n-1$ preimages in I, whereas every point in $\mathbb{T} \smallsetminus \{m_d(a)\}$ has n preimages in I.
(iii) If I is a loose gap of multiplicity n, the interior J of $g(I)$ is a gap of length $d\ell - n$. Under m_d, every point in J has $n+1$ preimages in I, whereas every point outside J has n preimages in I.

Proof For the standard monotone map g_X the statements follow immediately from the definition. For an arbitrary extension g, we can use the fact that g is monotone and takes the same values as g_X on the boundary of gaps to arrive at the same conclusions. The details are straightforward and will be left to the reader. □

The preceding lemma shows that the pattern of how gaps map around is independent of the choice of the monotone extension g. For any gap I, the image

$g(I)$ is either a point or a gap J modulo its endpoints. In practice, it is convenient to ignore the issue of endpoints and simply declare that I *maps to* J. With this convention in mind, we see from the above lemma that every minor gap eventually maps to a major gap I. If I is taut, it maps to a point and the process stops. If I is loose, it maps to a new gap and the process continues.

Let us collect some corollaries of this basic observation.

Theorem 2.9 *A rotation set is uniquely determined by its major gaps.*

Proof Let X, Y be rotation sets with the same collection of major gaps. We may assume neither of X, Y is a single point. Suppose there is some $t \in Y \smallsetminus X$. Then t must belong to a minor gap I of X. Take the smallest integer $i > 0$ such that $J = m_d^{\circ i}(I)$ is a major gap of X. Then $m_d^{\circ i} : I \to J$ is a homeomorphism, so $m_d^{\circ i}(t) \in J \cap Y$, which is impossible since J is a major gap of Y as well. This proves $Y \subset X$. Similarly, $X \subset Y$. □

Theorem 2.10 *Suppose X is a rotation set containing more than one point and I is a gap of X. Then either I is periodic or it eventually maps to a taut gap.*

Proof Let I_i denote the interior of $g_X^{\circ i}(I)$ and assume that I_i is not taut for any i. By Lemma 2.8 there is a sequence $i_1 < i_2 < i_3 < \cdots$ of positive integers for which I_{i_k} is loose. Since there are finitely many loose gaps, we must have $I_{i_j} = I_{i_k}$ for some $j < k$. This proves that I eventually maps to a periodic gap. Since by monotonicity of g_X every gap is the image of precisely one gap, it follows that I itself must be periodic. □

Corollary 2.11 *Every infinite rotation set has at least one taut gap.*

Conversely, all major gaps of a finite rotation set are loose since in this case m_d, being surjective, must also be injective on the rotation set.

Proof Otherwise every gap would be periodic by the previous theorem, so its endpoints would be periodic points in the rotation set. By Theorem 1.14 these infinitely many endpoints would have the same period $q > 0$ under m_d. This is impossible since m_d has only finitely many q-cycles. □

Remark 2.12 Here is an alternative approach to the above corollary (compare [2]): Lemma 2.8 applied to g_X shows that $m_d(t) = m_d(t')$ for a distinct pair $t, t' \in X$ precisely when t, t' form the endpoints of a taut gap or more generally when there is a chain $t = t_1, t_2, \ldots, t_k = t' \in X$ such that each pair t_i, t_{i+1} forms the endpoints of a taut gap. Thus, if X had no taut gap, the map $m_d : X \to X$ would be a homeomorphism. Since m_d is expanding, this would imply that X is finite [21, Lemma 18.8].

The next result establishes a connection between the major gaps of a rotation set and the $d - 1$ fixed points

$$u_i = \frac{i}{d-1} \pmod{\mathbb{Z}}$$

of the map m_d. This connection will play an important role in Sects. 3.2 and 3.3.

Lemma 2.13 *Suppose X is a rotation set for m_d with $\rho(X) \neq 0$. Then each major gap of X of multiplicity n contains exactly n fixed points of m_d.*

Compare Fig. 2.2.

Proof The assumption $\rho(X) \neq 0$ tells us that each fixed point of m_d belongs to a gap, which is necessarily major since a minor gap is disjoint from its image under m_d. Let I be a major gap of multiplicity n and assume that it contains $n+1$ adjacent fixed points $u_i, \ldots, u_{i+n}$. Since each open interval (u_j, u_{j+1}) contains precisely one preimage of every fixed point under m_d, it follows that u_i has at least $n+1$ preimages in I. By Lemma 2.8, I is loose and u_i belongs to the interior of $g_X(I)$. This implies that the closure of I maps onto itself by g_X, so the endpoints of I must be fixed by m_d, which contradicts the assumption $\rho(X) \neq 0$. Thus, I contains at most n fixed points of m_d.

Now let $\{I_i\}$ be the finite collection of major gaps of X of multiplicities $\{n_i\}$. We have shown that the number k_i of fixed points in I_i satisfies $0 \leq k_i \leq n_i$. Since $\sum k_i = \sum n_i = d-1$, we must have $k_i = n_i$ for all i. □

To each rotation set X for m_d we can assign a ***gap graph*** Γ_X which is a finite directed (not necessarily connected) graph having one vertex for each major gap of X, with an edge going from vertex I to vertex J whenever J is the first major gap in the forward orbit of I. We also assign to each vertex I a weight $w(I) \geq 1$ equal to its multiplicity. Thus, Γ_X has the following properties:

(i) $\displaystyle\sum_{\text{vertices } I} w(I) = d-1.$

(ii) The degree of every vertex is either 0 (no edge going out or coming in), or 1 (only one edge going out or coming in), or 2 (one edge going out and one coming in, possibly a loop).

If X has no loose gaps, Γ_X is a trivial graph consisting of at most $d-1$ vertices and no edges. If X is an irrational rotation set, Theorem 2.10 tells us that every directed path in Γ_X terminates at a taut vertex and in particular there are no closed paths (see Fig. 2.3).

Let us call a finite directed graph ***admissible*** of degree d if it satisfies the conditions (i) and (ii) above. It is natural to ask the following

Question 2.14 Given an admissible graph Γ of degree d, does there exist a rotation set X for m_d whose gap graph Γ_X is isomorphic to Γ?

In Sect. 4.2 we will provide the answer to this question in the case Γ has no closed paths (see Theorem 4.6).

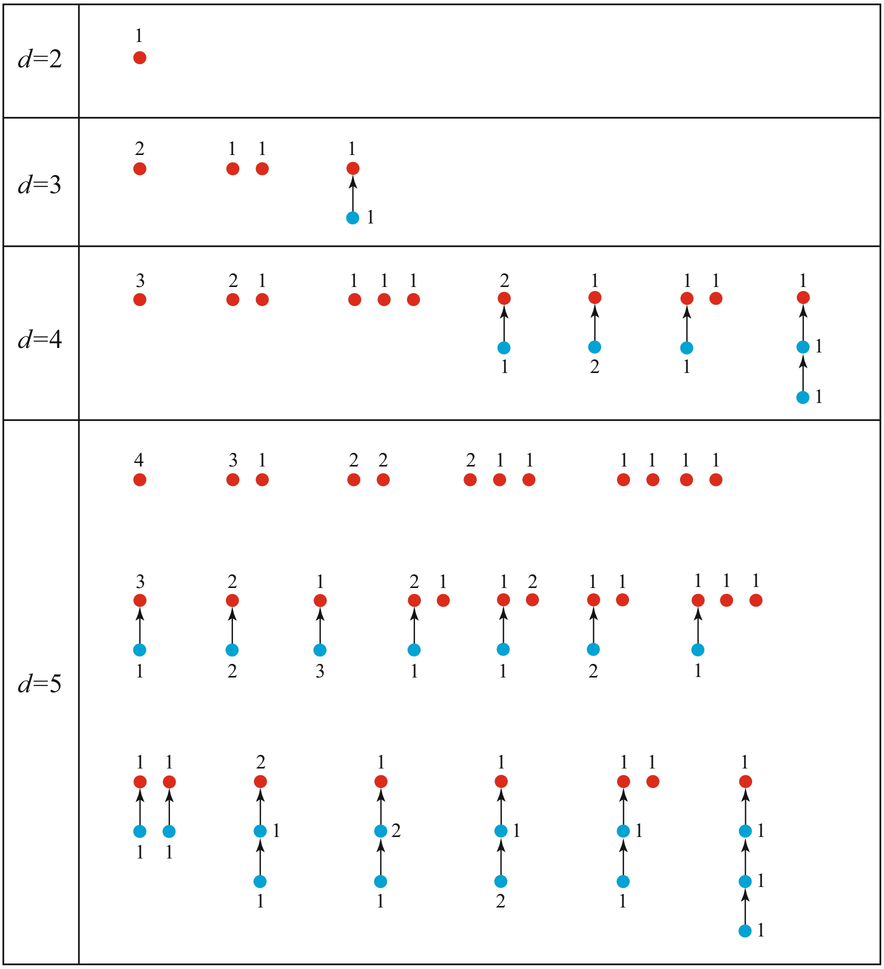

Fig. 2.3 Possible gap graphs for irrational rotation sets under m_d for $2 \leq d \leq 5$. The red and blue vertices correspond to taut and loose gaps respectively, and the weights denote multiplicities

2.2 Maximal Rotation Sets

Take any collection

$$\mathscr{I} = \{I_1, \ldots, I_{d-1}\}$$

of *disjoint* open intervals on the circle, each of length $1/d$. Consider the set

$$X_{\mathscr{I}} = \{t \in \mathbb{T} : m_d^{\circ n}(t) \notin I_1 \cup \cdots \cup I_{d-1} \text{ for all } n \geq 0\}.$$

Theorem 2.15 ([4]) *$X_{\mathscr{I}}$ is a rotation set for m_d.*

Proof First we check that $X_{\mathscr{I}} \neq \emptyset$. Denote by U the open set $I_1 \cup \cdots \cup I_{d-1}$. Under m_d, every $t_0 \in \mathbb{T}$ has d preimages which are a distance $1/d$ apart, hence at least one of these preimages, say t_1, must be outside U. It follows inductively that there is a backward orbit $\cdots \mapsto t_2 \mapsto t_1 \mapsto t_0$ such that $t_n \notin U$ for every $n \geq 1$. Evidently, any accumulation point of the sequence $\{t_n\}$ belongs to $X_{\mathscr{I}}$.

It is immediate from the definition that $X_{\mathscr{I}}$ is compact and maps into itself by m_d. Of the d preimages of any point in $X_{\mathscr{I}}$, at least one lies outside U and therefore belongs to $X_{\mathscr{I}}$. This proves $m_d(X_{\mathscr{I}}) = X_{\mathscr{I}}$. Finally, m_d restricted to $X_{\mathscr{I}}$ can be extended to a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ by setting $g = m_d$ outside U and mapping each interval I_i to a point. □

Corollary 2.16 *A non-empty compact m_d-invariant set X is a rotation set if and only if $\mathbb{T} \smallsetminus X$ contains $d-1$ disjoint open intervals, each of length $1/d$.*

Proof Necessity follows from Theorem 2.7. For sufficiency, let $\mathscr{I}$ be the collection of the $d-1$ disjoint intervals of length $1/d$ in $\mathbb{T} \smallsetminus X$. By the above theorem $X_{\mathscr{I}}$ is a rotation set that contains X. Hence X itself is a rotation set. □

If Y is a rotation set for m_d and if $X \subset Y$ is compact and m_d-invariant, then clearly X is also a rotation set for m_d, with $\rho(X) = \rho(Y)$. We record the following simple lemma for future reference:

Lemma 2.17 *Suppose X, Y are rotation sets for m_d containing more than one point, and assume $X \subset Y$. Then each major gap of X of multiplicity n contains n major gaps of Y counting multiplicities.*

Proof Evidently each major gap of Y is contained in a major gap of X. Let $\{I_i\}$ be the collection of major gaps of X of multiplicities $\{n_i\}$. The number k_i of major gaps of Y contained in I_i satisfies $0 \leq k_i \leq n_i$. Since $\sum k_i = \sum n_i = d-1$ by Theorem 2.7, we must have $k_i = n_i$ for all i. □

Let us call a rotation set ***maximal*** if it is not properly contained in another rotation set. Theorem 2.15 provides a convenient recipe for enlarging every rotation set to a maximal one.

Lemma 2.18 *Every rotation set is contained in a maximal rotation set.*

Proof Suppose X is a rotation set for m_d. For each major gap $(a, a+\ell)$ of X of multiplicity n, consider the n disjoint subintervals $(a + (j-1)/d, a + j/d)$ for $1 \leq j \leq n$. Let $\mathscr{I}$ denote the collection of the $d-1$ disjoint open intervals of length $1/d$ thus obtained. The rotation set $X_{\mathscr{I}}$ of Theorem 2.15 clearly contains X. Moreover, the endpoints of the intervals in $\mathscr{I}$ map to X under m_d, which shows they all belong to $X_{\mathscr{I}}$. Thus, $X_{\mathscr{I}}$ has $d-1$ taut gaps of multiplicity 1. By Theorem 2.9 and Lemma 2.17, $X_{\mathscr{I}}$ is maximal. □

Corollary 2.19 *A rotation set X for m_d is maximal if and only if it has $d-1$ distinct gaps of length $1/d$. In this case $X = X_{\mathscr{I}}$, where $\mathscr{I}$ is the collection of the major gaps of X.*

The proof of Lemma 2.18 in fact gives the following improved lower bound for the number $N_{\max}(X)$ of the maximal rotation sets containing X:

Corollary 2.20 *Suppose X is a rotation set for m_d with loose gaps $I_1, \ldots, I_k$ of multiplicities $n_1, \ldots, n_k$. Then*

$$N_{\max}(X) \geq \prod_{j=1}^{k} (n_j + 1).$$

In particular, X is contained in at least 2^k maximal rotation sets.

Proof For each loose gap $I = (a, a + \ell)$ of X with multiplicity n, there are $n + 1$ different ways of choosing n disjoint subintervals of length $1/d$ whose endpoints map to $m_d(a)$ or $m_d(a + \ell)$ (the one in the proof of Lemma 2.18 was one of these choices). This leads to $\prod_{j=1}^{k}(n_j + 1)$ different choices for the collection $\mathscr{I}$. □

Example 2.21 The 2-cycle $X = \{\frac{1}{3}, \frac{2}{3}\}$ under doubling is contained in precisely two maximal rotation sets

$$X_{\mathscr{I}_1} = \left\{\frac{1}{3}, \frac{2}{3}\right\} \cup \left\{\frac{1}{3} - \frac{1}{3 \cdot 2^{2n-1}}\right\}_{n \geq 1} \cup \left\{\frac{2}{3} - \frac{1}{3 \cdot 2^{2n}}\right\}_{n \geq 1}$$

and

$$X_{\mathscr{I}_2} = \left\{\frac{1}{3}, \frac{2}{3}\right\} \cup \left\{\frac{1}{3} + \frac{1}{3 \cdot 2^{2n}}\right\}_{n \geq 1} \cup \left\{\frac{2}{3} + \frac{1}{3 \cdot 2^{2n-1}}\right\}_{n \geq 1}$$

corresponding to the collections $\mathscr{I}_1 = \{(\frac{2}{3}, \frac{1}{6})\}$ and $\mathscr{I}_1 = \{(\frac{5}{6}, \frac{1}{3})\}$. Note that each orbit in $X_{\mathscr{I}_i}$ eventually hits the 2-cycle X, and the intersection of $X_{\mathscr{I}_i}$ with the major gap of X is countably infinite.

The above example is a special case of a count for $N_{\max}(X)$ that we will establish in the next section for certain rational rotation sets (see Theorem 2.30). These rotation sets, however, are not typical. In fact, *when $d > 2$ there are rational rotation sets for m_d that are contained in infinitely many maximal rotation sets*. Here is an example:

Example 2.22 Consider the 2-cycle $X = \{\frac{1}{4}, \frac{3}{4}\}$ under tripling. Define the sequences

$$t_n = \sum_{j=0}^{n} \frac{1}{3^{2j+1}} + \frac{1}{3^{2n+1} \cdot 12}$$

$$s_n = m_3(t_n) = \sum_{j=0}^{n} \frac{1}{3^{2j}} + \frac{1}{3^{2n} \cdot 12}$$

for $n \geq 0$. Then $\frac{1}{3} < t_0 < t_1 < t_2 < \cdots$ with $t_n \to \frac{3}{8}$ and $\frac{1}{12} = s_0 < s_1 < s_2 < \cdots$ with $s_n \to \frac{1}{8}$. For each $n \geq 0$ the collection

$$\mathscr{I}_n = \left\{ \left(t_n, t_n + \frac{1}{3}\right), \left(\frac{3}{4}, \frac{1}{12}\right) \right\}$$

produces a rotation set $X_{\mathscr{I}_n}$ which evidently contains the 2-cycle X. The endpoints $\frac{3}{4}, \frac{1}{12}$ map to $\frac{1}{4}$ under m_3, so they both belong to $X_{\mathscr{I}_n}$. The other endpoints $t_n, t_n + \frac{1}{3}$ have the m_3-orbit

$$t_n,\ t_n + \frac{1}{3} \mapsto s_n \mapsto t_{n-1} \mapsto s_{n-1} \mapsto \cdots \mapsto t_0 \mapsto s_0 = \frac{1}{12} \mapsto \frac{1}{4}$$

which, by monotonicity of $\{t_j\}$ and $\{s_j\}$, never meets the pair of intervals in $\mathscr{I}_n$. This shows that both $t_n, t_n + \frac{1}{3}$ belong to $X_{\mathscr{I}_n}$. Thus $X_{\mathscr{I}_n}$ has a pair of major gaps of length $\frac{1}{3}$ and therefore is maximal by Corollary 2.19.

The situation in the irrational case is different and in fact simpler:

Theorem 2.23 *Every irrational rotation set X for m_d is contained in finitely many maximal rotation sets. For any maximal rotation set $Y \supset X$ and any gap I of X, the intersection $Y \cap I$ is finite (possibly empty) and eventually maps into X under the iterations of m_d.*

Proof Take any maximal rotation set $Y \supset X$. First suppose I is a major gap of X of multiplicity n. By Lemma 2.17 and Corollary 2.19, Y has exactly n taut gaps of multiplicity 1 contained in I. We distinguish two cases:

- *Case 1: I is taut.* Then I has the form $(a, a + n/d)$ and

$$Y \cap \bar{I} = \left\{ a, a + \frac{1}{d}, \ldots, a + \frac{n}{d} \right\}.$$

  This condition uniquely determines the major gaps of Y that are contained in I. Notice that the inclusion $m_d(Y \cap \bar{I}) \subset X$ holds.
- *Case 2: I is loose.* Consider the standard monotone map g_Y which is also an extension of $m_d|_X$. By Theorem 2.10, there is an $i > 0$ such that the interior J of $g_Y^{\circ i}(I)$ is a taut gap of X (there can be no periodic loose gap of X since $\rho(X)$ is irrational). Note that $m_d^{\circ i}(Y \cap I) = g_Y^{\circ i}(Y \cap I)$ is contained in $Y \cap \bar{J}$ which is uniquely determined by *Case 1*. Hence the elements of $Y \cap I$ are among the finitely many $m_d^{\circ i}$-preimages of $Y \cap \bar{J}$. This gives finitely many choices for the major gaps of Y in I.

The two cases above show that there are only finitely many choices for the major gaps of Y, hence for Y itself by Theorem 2.9.

We have shown that for any major gap I of X, the intersection $Y \cap I$ is finite and eventually maps into X. Since every minor gap of X maps homeomorphically onto a major gap under some iterate of m_d, the result must also hold when I is minor. $\square$

The number $N_{\max}(X)$ of maximal rotation sets Y containing an irrational rotation set X depends on the structure of the gap graph Γ_X defined in the previous section. Suppose there is a maximal path in Γ_X of the form

$$I_k \to I_{k-1} \to \cdots \to I_1, \quad \text{with} \quad w(I_i) = n_i. \tag{2.2}$$

Since I_1 is taut, the major gaps of Y in I_1 are already determined. However, there are $\binom{n_1+n_2}{n_2}$ choices for the major gaps of Y in I_2. For each of these choices, there are $\binom{n_1+n_2+n_3}{n_3}$ choices for the major gaps of Y in I_3 and so on. This gives the count

$$N_{\max}(X) = \prod \binom{n_1+n_2}{n_2}\binom{n_1+n_2+n_3}{n_3} \cdots \binom{n_1+\cdots+n_k}{n_k} = \prod \frac{(n_1+\cdots+n_k)!}{n_1! \cdots n_k!}, \tag{2.3}$$

where the product is taken over all maximal paths in Γ_X of the form (2.2) (if there is no path in Γ_X, the product is taken over the empty set and is understood to be 1).

A quick inspection of Fig. 2.3 reveals that $N_{\max}(X) = 1$ for $d = 2$, $N_{\max}(X) \leq 2$ for $d = 3$, and $N_{\max}(X) \leq 6$ for $d = 4$, and $N_{\max}(X) \leq 24$ for $d = 5$. More generally, we have the following

Theorem 2.24 $N_{\max}(X) \leq (d-1)!$ *whenever X is an irrational rotation set for m_d.*

Proof If the gap graph Γ_X has no path, then $N_{\max}(X) = 1$ and there is nothing to prove. Otherwise, let Γ_X have $p \geq 1$ distinct maximal paths of the form (2.2), where the weights of the vertices in the i-th path add up to N_i, so $N_1 + \cdots + N_p \leq d - 1$. Then, by (2.3),

$$N_{\max}(X) \leq \prod_{i=1}^{p} N_i! \leq \Big(\sum_{i=1}^{p} N_i\Big)! \leq (d-1)!$$

as required. □

2.3 Minimal Rotation Sets

A rotation set is called ***minimal*** if it does not properly contain another rotation set. This section will study the question of existence and uniqueness of minimal rotation sets that are contained in a given rotation set, in both rational and irrational cases.

Before we begin, a quick comment on topological dynamics is in order. The simple proof that minimality is equivalent to having all orbits dense requires a slight modification here, as the closure of an orbit in a rotation set is only *forward* invariant and may not be a rotation set.[4] Similarly, the standard application of Zorn's lemma

[4] In fact, it will follow from the results of this section that for rotation sets minimality is equivalent to having a single dense orbit, a property that is often called ***point transitivity***.

to show that every rotation set contains a minimal rotation set needs some care because the intersection of a linearly ordered family of rotation sets is a priori *forward* invariant. This minor problem is addressed by observing that under m_d, every compact forward invariant set contains a compact invariant set. In fact, if Z is compact and satisfies $m_d(Z) \subset Z$, the nested intersection $K = \bigcap_{n \geq 0} m_d^{\circ n}(Z)$ is easily seen to satisfy $m_d(K) = K$.

Let us first consider the rational case, where minimal rotation sets are cycles. Let $C = \{t_1, \dots, t_q\}$ be a cycle of rotation number p/q under m_d, where the t_j are in positive cyclic order and their subscripts are taken modulo q (see Sect. 1.3). By Theorem 1.14, $dt_j = t_{j+p} \pmod{\mathbb{Z}}$ for every j. The q gaps $I_j = (t_j, t_{j+1})$ are permuted under any monotone extension g of $m_d|_C$, so $g(\bar{I}_j) = \bar{I}_{j+p}$. Recall that these gaps are either minor or loose: there can be no taut gap.

It follows from Theorem 2.3 that every rotation set X for m_d with $\rho(X) = p/q$ in lowest terms contains at least one q-cycle. But there could be several such minimal sets in X. For instance, under the tripling map m_3, the union $X = C_1 \cup C_2$ of the 3-cycles

$$C_1 : \frac{4}{26} \mapsto \frac{12}{26} \mapsto \frac{10}{26} \quad \text{and} \quad C_2 : \frac{7}{26} \mapsto \frac{21}{26} \mapsto \frac{11}{26}$$

is a rotation set with $\rho(X) = \frac{2}{3}$. This can be seen, for example, from Corollary 2.16 since $\mathbb{T} \smallsetminus X$ contains the intervals $(\frac{12}{26}, \frac{12}{26} + \frac{1}{3})$ and $(\frac{21}{26}, \frac{21}{26} + \frac{1}{3})$ on the circle. The general situation can be understood as follows.

We call a collection $C_1, \dots, C_N$ of distinct q-cycles under m_d with the same rotation number ***compatible*** if their union $C_1 \cup \cdots \cup C_N$ is a rotation set. We say that $C_1, \dots, C_N$ are ***superlinked*** if for every pair $i \neq j$, each gap of C_i meets C_j. Geometrically, this means that the points of C_i and C_j alternate as we go around the circle.

Lemma 2.25 *$C_1, \dots, C_N$ are compatible if and only if they are superlinked.*

In follows in particular that a collection of cycles are compatible if and only if they are pairwise compatible.

Proof First suppose $X = C_1 \cup \cdots \cup C_N$ is a rotation set. Consider the standard monotone map $g = g_X$, which is also a monotone extension of $m_d|_{C_i}$ for each i. Pick any pair C_i, C_j. Since these cycles are distinct, there is a gap I of C_i that meets C_j at some point t. Then for every $k \geq 0$, the interior J_k of $g^{\circ k}(I)$ meets C_j at $g^{\circ k}(t) = m_d^{\circ k}(t)$. Since $J_0 = I, J_1, \dots, J_{q-1}$ form all the gaps of C_i, we conclude that C_i, C_j are superlinked.

Conversely, suppose $C_1, \dots, C_N$ are superlinked and consider the standard monotone map $g = g_{C_1}$. Take a gap I of C_1 and let J be the interior of $g(I)$. For $2 \leq i \leq N$, let $C_i \cap I = \{a_i\}$ and $C_i \cap J = \{b_i\}$. Using the fact that the C_i have the same rotation number, it is easy to see that $b_i = m_d(a_i)$. As the C_i are superlinked, the points a_i appear in the same order in I as the points b_i in J, so there is an orientation-preserving homeomorphism $h : I \to J$ such that

$h(a_i) = b_i$ for $2 \leq i \leq N$. Repeating this process for every gap of C_1 and gluing together the resulting homeomorphisms will then yield an orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ which restricts to m_d on the union $C_1 \cup \cdots \cup C_N$. □

Theorem 2.26 *The number of distinct cycles in a rational rotation set is bounded above by the number of its distinct major gaps.*

In view of Theorem 2.7, we recover the following result of Goldberg as a special case (see [11] for the original combinatorial proof and [2] for an inductive argument reducing the problem down to $d = 2$):

Corollary 2.27 *A rational rotation set for m_d contains at most $d-1$ distinct cycles.*

The upper bound $d - 1$ can always be achieved; see Corollary 3.15.

Proof of Theorem 2.26 Let Y be a rational rotation set for m_d with $\rho(Y) = p/q$ in lowest terms. Suppose $C_1, \ldots, C_N$ are the distinct cycles in Y, all necessarily of length q. The union $X = C_1 \cup \cdots \cup C_N$ is an m_d-invariant subset of Y, so it is a rotation set. By Lemma 2.25, the C_i are superlinked. It follows that any gap I of C_1 contains precisely N gaps $J_1, \ldots, J_N$ of X. Each J_i is periodic of period q and its orbit contains at least one major gap of X. Moreover, the orbits of $J_1, \cdots, J_N$ are disjoint, so they cannot share any major gap of X. It follows that X, hence Y, has at least N distinct major gaps. □

Corollary 2.28 *Every rational rotation set under the doubling map contains a unique cycle.*

Example 2.29 Under the tripling map m_3 there are five 4-cycles of rotation number $\frac{1}{4}$:

$$\begin{aligned}
C_1 &: \frac{1}{80} \mapsto \frac{3}{80} \mapsto \frac{9}{80} \mapsto \frac{27}{80} \\
C_2 &: \frac{2}{80} \mapsto \frac{6}{80} \mapsto \frac{18}{80} \mapsto \frac{54}{80} \\
C_3 = C_3 + \frac{1}{2} &: \frac{5}{80} \mapsto \frac{15}{80} \mapsto \frac{45}{80} \mapsto \frac{55}{80} \\
C_4 = C_2 + \frac{1}{2} &: \frac{14}{80} \mapsto \frac{42}{80} \mapsto \frac{46}{80} \mapsto \frac{58}{80} \\
C_5 = C_1 + \frac{1}{2} &: \frac{41}{80} \mapsto \frac{43}{80} \mapsto \frac{49}{80} \mapsto \frac{67}{80}
\end{aligned}$$

By Corollary 2.27, at most two 4-cycles under tripling can be compatible. By Lemma 2.25, this happens precisely when the two 4-cycles are superlinked. Simple inspection shows that (C_1, C_2), (C_2, C_3), (C_3, C_4) and (C_4, C_5) are the only compatible pairs (compare Fig. 2.4).

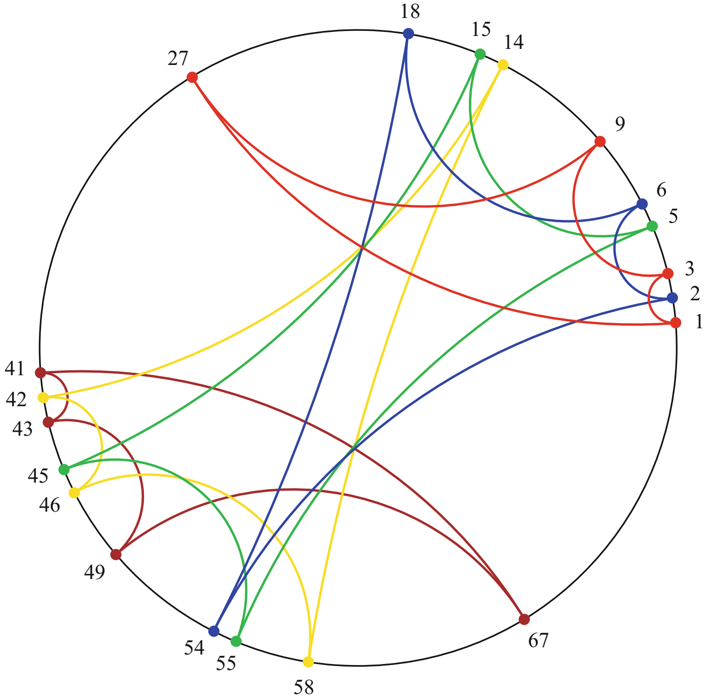

Fig. 2.4 The five 4-cycles of rotation number $\frac{1}{4}$ under tripling, shown in different colors (angles are given in multiples of $\frac{1}{80}$). Only the four superlinked pairs (red, blue), (blue, green), (green, yellow), and (yellow, brown) are compatible cycles

Before moving on to the irrational case, let us use the above ideas to show that for some rational rotation sets the lower bound of Corollary 2.20 is sharp:

Theorem 2.30 *Let X be a rational rotation set for m_d which is the union of $d-1$ distinct cycles. Then $N_{\max}(X) = 2^{d-1}$.*

Proof By Theorem 2.26 X has $d-1$ major gaps, all loose and of multiplicity 1. If $\rho(X) = p/q$, these major gaps have disjoint orbits which are periodic of period q. Let Y be any maximal rotation set containing X. Each major gap $I = (a, b)$ of X contains a single major gap J of Y of length $1/d$. We claim that $J = (a, a + 1/d)$ or $J = (b - 1/d, b)$. Otherwise $J = (t, t + 1/d)$, where $a < t < t + 1/d < b$. The standard monotone map $g = g_Y$ is also a monotone extension of $m_d|_X$, so $g^{\circ q}$ maps I onto itself fixing the endpoints a, b. Moreover, the gaps $g(I), \ldots, g^{\circ q-1}(I)$ of X are all minor, so they cannot contain major gaps of Y; as such, g acts homeomorphically on them. It follows that $g^{\circ q}$ is homeomorphic on $[a, t] \cup [t + 1/d, b]$ and collapses J to the single point $m_d^{\circ q}(t)$. This image point necessarily lies in J since $g^{\circ q} = m_d^{\circ q}$ is expanding on both $[a, t]$ and $[t + 1/d, b]$. This is a contradiction since $m_d^{\circ q}(t) \in Y$.

Thus, there are just two possibilities for each major gap of Y inside a given major gap of X, hence 2^{d-1} possibilities altogether for the major gaps of Y, and therefore for Y itself. This proves $N_{\max}(X) \leq 2^{d-1}$. The result now follows since $N_{\max}(X) \geq 2^{d-1}$ by Corollary 2.20. □

The following corollary immediately follows from the above theorem and its proof:

Corollary 2.31 *Every rotation cycle X under the doubling map is contained in exactly two maximal rotation sets. Moreover, if (a, b) is the major gap of X, then the intervals $(a, a + \frac{1}{2})$ and $(b - \frac{1}{2}, b)$ are the major gaps of these maximal rotation sets.*

Compare Example 2.21.

Example 2.32 Consider the 2-cycle $X = \{\frac{1}{4}, \frac{3}{4}\}$ under tripling. We showed in Example 2.22 that $N_{\max}(X) = \infty$. However, the enlarged rotation set $Y = \{\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{3}{4}\}$, a union of two 2-cycles under tripling, has $N_{\max}(Y) = 4$ by Theorem 2.30!

We now consider minimal rotation sets in the irrational case.

Theorem 2.33 *Every irrational rotation set X for m_d contains a unique minimal rotation set K. Moreover,*

(i) K is the Cantor attractor of any monotone extension of $m_d|_X$.
(ii) Each gap of K contains at most finitely many points of X, all of which eventually map to K under the iterations of m_d.

Proof Take a monotone extension g of $m_d|_X$ and let K be the Cantor attractor of g, as in Theorem 1.20. Let Z be any non-empty compact m_d-invariant subset of X. By Theorem 1.20, $K = \omega_g(t) \subset Z$ for every $t \in Z$. It follows that K is the unique minimal rotation set contained in X.

To verify the second statement, let Y be any maximal rotation set containing X (whose existence is guaranteed by Lemma 2.18). Since Y contains K, Theorem 2.23 shows that for each gap I of K, the intersection $Y \cap I$ is at most finite and maps into K under the iterations of m_d. Hence the same must be true of $X \cap I$. □

By (the proof of) Theorem 1.20, the gaps of the Cantor attractor of g are the plateaus of the Poincaré semiconjugacy φ between g and r_θ. Thus, we have the following

Corollary 2.34 *Suppose X is a minimal rotation set for m_d with $\rho(X) = \theta$ irrational. Then there exists a degree 1 monotone map $\varphi : \mathbb{T} \to \mathbb{T}$, whose plateaus are precisely the gaps of X, which satisfies $\varphi \circ m_d = r_\theta \circ \varphi$ on X.*

Here is the converse statement. Recall that for each point $s \in \mathbb{T}$, I_s denotes the interior of the fiber $E_s = \varphi^{-1}(s)$.

Theorem 2.35 *Let θ be irrational and $\varphi : \mathbb{T} \to \mathbb{T}$ be a degree 1 monotone map with the property that $I_s \neq \emptyset$ implies $I_{s-\theta} \neq \emptyset$. Denote by X the complement of the* *union of all plateaus of* φ. *If*

$$\varphi \circ m_d = r_\theta \circ \varphi \quad on \quad X, \tag{2.4}$$

then X *is a minimal rotation set for* m_d *with* $\rho(X) = \theta$.

Proof The assumptions imply that φ has plateaus; otherwise $X = \mathbb{T}$ and (2.4) would exhibit a global conjugacy between the degree $d \geq 2$ map m_d and the rotation r_θ, which is impossible.

We invoke Theorem 1.22 to find a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ such that $\varphi \circ g = r_\theta \circ \varphi$ on $\mathbb{T}$. Then X is the Cantor attractor of g. If $t \in X$ is not an endpoint of a plateau and $s = \varphi(t)$, then $E_s = \{t\}$, so by the assumption $E_{s+\theta}$ is a singleton $\{t'\}$. The semiconjugacy relation (2.4) for m_d and the one for g then show that $m_d(t) = t' = g(t)$. Since the set of such t is dense in X, we conclude that $g = m_d$ on X. As the Cantor attractor of g, X is minimal for g and hence for m_d, and $m_d(X) = g(X) = X$. This completes the proof that X is a minimal rotation set. □

We conclude this section with characterizations of minimal rotation sets, as well as those that are both minimal and maximal.

Theorem 2.36 *A rotation set for* m_d *is a Cantor set if and only if it is minimal and has irrational rotation number.*

Proof The "if" part follows from Theorem 2.33. For the "only if" part, suppose X is a Cantor set. Then $\rho(X)$ is irrational since a rational rotation set is at most countable (Theorem 2.3). Let K be the unique minimal rotation set contained in X. If $K \neq X$, some gap I of K would have to meet X. But then by Theorem 2.33 the intersection $X \cap I$ would be finite, consisting of isolated points of X. This would contradict the assumption that X is a Cantor set. □

Let us call a rotation set ***exact*** if it is both minimal and maximal.[5] Evidently a rational rotation set can never be exact. In the irrational case, the following criterion follows immediately from Corollary 2.19 and Theorem 2.36:

Theorem 2.37 *An irrational rotation set for* m_d *is exact if and only if it is a Cantor set with* $d - 1$ *distinct gaps of length* $1/d$.

Corollary 2.38 *Every irrational rotation set under the doubling map is exact.*

Proof Let X be an irrational rotation set under doubling. Then X has a single major gap I of multiplicity 1 which is necessarily taut by Corollary 2.11. If K is the unique minimal rotation set contained in X, then K is a Cantor set with a single taut gap of multiplicity 1 which can only be I. It follows from Theorem 2.9 that $K = X$, and then from Theorem 2.37 that X is exact. □

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[5]The terminology is meant to suggest that nothing can be added to or removed from such a set without losing the property of being a rotation set.

Remark 2.39 The above corollary is false in higher degrees. For example, there are minimal irrational rotation sets under tripling with a pair of major gaps of lengths $\frac{1}{3}$ and $\frac{4}{9}$ which therefore are not maximal (compare Theorem 4.31). However, every irrational rotation set under tripling is either minimal, or maximal, or both. In every degree > 3, there are irrational rotation sets that are neither minimal nor maximal.

For more on the role of exact rotation sets, see Sect. 4.3.

Chapter 3
The Deployment Theorem

The main result of this chapter is that a minimal rotation set for m_d is uniquely determined by its rotation number together with an invariant called the "deployment vector" which, roughly speaking, describes how the points of the rotation set are deployed relative to the $d-1$ fixed points of m_d. This was first proved in the rational case by Goldberg [11] and was later extended to the irrational case by Goldberg and Tresser [13] using a Farey tree machinery. By contrast, our presentation here builds upon the ideas developed in the previous chapter and treats both rational and irrational cases in a unified fashion. Various applications of this result will be discussed in the next chapter.

3.1 Preliminaries

To begin the discussion, consider a minimal rotation set X for m_d with $\rho(X) = \theta \neq 0$ and the standard monotone map g_X. Let $\varphi : \mathbb{T} \to \mathbb{T}$ be the combinatorial semiconjugacy between g_X and r_θ if θ is rational, or the Poincaré semiconjugacy between g_X and r_θ if θ is irrational. In either case, we have the semiconjugacy relation

$$\varphi \circ m_d = r_\theta \circ \varphi \quad \text{on} \quad X.$$

Recall that φ is normalized by $\varphi(0) = 0$ and its plateaus are precisely the gaps of X. We refer to φ as the ***canonical semiconjugacy*** associated with X.

It follows from the discussion in Sect. 1.5 that there is a unique m_d-invariant Borel probability measure μ supported on X. This measure, which henceforth will be called the ***natural measure*** of X, is related to the canonical semiconjugacy by

$$\varphi(t) = \mu[0, t] \pmod{\mathbb{Z}}.$$

S. Zakeri, *Rotation Sets and Complex Dynamics*, Lecture Notes in Mathematics 2214, https://doi.org/10.1007/978-3-319-78810-4_3

If $\theta = p/q$ in lowest terms so X is a q-cycle, then μ is just the uniform Dirac measure on X which assigns a mass of $1/q$ to each point of X. On the other hand, if θ is irrational so X is a Cantor set, then μ is the (well-defined) pull-back of Lebesgue measure under φ.

Recall that the $d-1$ fixed points of m_d are denoted by

$$u_i = \frac{i}{d-1} \pmod{\mathbb{Z}}.$$

Set

$$\delta_i = \mu[u_{i-1}, u_i) \qquad 1 \le i \le d-1.$$

Then $(\delta_1, \ldots, \delta_{d-1})$ is a probability vector, that is, it belongs to the $(d-2)$-dimensional simplex

$$\Delta^{d-2} = \Big\{(x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} : x_i \ge 0 \text{ and } \sum_{i=1}^{d-1} x_i = 1\Big\}.$$

Definition 3.1 The vector $\delta(X) = (\delta_1, \ldots, \delta_{d-1}) \in \Delta^{d-2}$ is called the ***deployment vector*** of the minimal rotation set X.

Here is a more explicit description for the components of $\delta(X)$. If $\rho(X) = p/q$ in lowest terms, the component δ_i is the fraction of points of X that fall between the fixed points u_{i-1} and u_i:

$$\delta_i = \frac{1}{q} \#\big\{t \in X : t \in [u_{i-1}, u_i)\big\}.$$

If $\rho(X)$ is irrational, it follows from unique ergodicity that δ_i is the fraction of time that the orbit of every $t \in X$ spends in $[u_{i-1}, u_i)$:

$$\delta_i = \lim_{n\to\infty} \frac{1}{n} \#\big\{0 \le k \le n-1 : m_d^{\circ k}(t) \in [u_{i-1}, u_i)\big\}$$

(compare (1.11)).

Observe that the deployment vectors of the rotation sets

$$X + \frac{1}{d-1},\ X + \frac{2}{d-1},\ \ldots,\ X + \frac{d-2}{d-1} \pmod{\mathbb{Z}}$$

are obtained by cyclically permuting the components of $\delta(X)$. For example, if X is a rotation set under m_4 with $\delta(X) = (\delta_1, \delta_2, \delta_3)$, then $\delta(X + \frac{1}{3}) = (\delta_3, \delta_1, \delta_2)$ and $\delta(X + \frac{2}{3}) = (\delta_2, \delta_3, \delta_1)$.

Closely related is the ***cumulative deployment vector*** $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1}) \in [0, 1]^{d-1}$ whose components are defined by

$$\sigma_i = \delta_1 + \cdots + \delta_i \qquad 1 \le i \le d-1 \tag{3.1}$$

and therefore satisfy $0 \le \sigma_1 \le \cdots \le \sigma_{d-1} = 1$. In terms of the natural measure μ, the number σ_i is just $\mu[u_0, u_i)$. Whether we use $\delta(X)$ or $\sigma(X)$ is solely a matter of preference, as each of these vectors determines the other uniquely.

Let

$$N_0 = \#\{1 \le i \le d-1 : \sigma_i = 0\}$$
$$N_1 = \#\{1 \le i \le d-1 : \sigma_i = 1\},$$

so the components of $\sigma(X)$ begin with $N_0 \ge 0$ zeros and end in $N_1 \ge 1$ ones. It is easy to check that the major gap I_0 of X containing the fixed point $u_0 = 0$ contains precisely the fixed points $u_{-N_1+1}, \dots, u_{N_0}$. It follows from Lemma 2.13 that $N_0 + N_1$ is the multiplicity of I_0.

Remark 3.2 We can assign a deployment vector to every rotation set X, even if it is not minimal: If X is rational, consider the finitely many cycles $C_1, \dots, C_N$ that are contained in X (Corollary 2.27) and define $\delta(X)$ to be the average $(1/N) \sum_{i=1}^{N} \delta(C_i)$. If X is irrational, define $\delta(X) = \delta(K)$, where K is the unique minimal rotation set contained in X (Theorem 2.33).

Lemma 3.3 *Let X be a minimal rotation set for m_d with $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1})$, and let $\varphi : \mathbb{T} \to \mathbb{T}$ be the canonical semiconjugacy associated with X. Then,*

$$\sigma_i = \varphi(u_i) \pmod{\mathbb{Z}} \quad \textit{for all} \quad 1 \le i \le d-1. \tag{3.2}$$

Proof Let μ be the natural measure of X, so $\varphi(t) = \mu[u_0, t] \pmod{\mathbb{Z}}$ for all $t \in \mathbb{T}$. Since $\rho(X) \neq 0$ by the assumption, X contains none of the fixed points u_i, so $\mu\{u_i\} = 0$ for every i. Hence $\varphi(u_i) = \mu[u_0, u_i] = \mu[u_0, u_i) = \sigma_i \pmod{\mathbb{Z}}$, as required. □

Remark 3.4 The congruences (3.2) allow us to determine $\sigma(X)$ from the knowledge of the $d-1$ points $\varphi(u_1), \dots, \varphi(u_{d-1})$ on $\mathbb{T}$ except when $\varphi(u_i) = 0 \pmod{\mathbb{Z}}$ for all i because in this case we cannot decide whether each σ_i is 0 or 1. For example, when $d = 4$, each of the vectors

$$\sigma(X) = (0, 0, 1) \quad \text{or} \quad (0, 1, 1) \quad \text{or} \quad (1, 1, 1)$$

would correspond to a minimal rotation set whose canonical semiconjugacy satisfies

$$\varphi(u_1) = \varphi(u_2) = \varphi(u_3) = 0 \pmod{\mathbb{Z}}.$$

This ambiguity can be dealt with, for example, by looking at a lift of φ. Alternatively, we can work with rotation sets for which $\sigma_1 \neq 0$ so every σ_i lies in $(0, 1]$. This condition can always be achieved by simply rotating the set: If the components of $\sigma(X)$ begin with a string of 0's of length N_0, replace X by its rotated copy $X - N_0/(d-1)$.

3.2 Deployment Theorem: The Rational Case

Throughout this section we assume that X is a minimal rational rotation set, that is, a q-cycle $\{t_1, \dots, t_q\}$ under m_d with $\rho(X) = p/q$ in lowest terms. As usual, we label the points of X so that $0, t_1, \dots, t_q$ are in positive cyclic order (in particular, $0 \in (t_q, t_1)$) and the subscripts are taken modulo q.

Lemma 3.5 *The interval $I_j = (t_j, t_{j+1})$ is a major gap of X of multiplicity n if and only if $j/q \pmod{\mathbb{Z}}$ appears exactly n times as a component of $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1})$.*

Note that since $0/q = q/q \pmod{\mathbb{Z}}$, this generalizes our previous observation that the multiplicity of $I_0 = I_q$ is $N_0 + N_1$.

Proof According to Lemma 2.13, I_j is a major gap of multiplicity n if and only if it contains exactly n fixed points. Under the canonical semiconjugacy associated with X, each such fixed point maps to j/q. The result now follows from Lemma 3.3. □

The main result of this section asserts that a minimal rational rotation set is uniquely determined by its rotation number and deployment vector. To motivate the main idea of the proof, we begin with an example.

Example 3.6 Suppose we want to find a 5-cycle $X = \{t_1, \cdots, t_5\}$ under m_4 with $\rho(X) = \frac{1}{5}$ and $\delta(X) = (\frac{3}{5}, 0, \frac{2}{5})$. Let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$. By Lemma 3.5, the knowledge of the cumulative deployment vector $\sigma(X) = (\frac{3}{5}, \frac{3}{5}, \frac{5}{5})$ tells us that I_3 is a major gap of multiplicity 2, $I_5 = I_0$ is a major gap of multiplicity 1, and the remaining I_j are minor (see Fig. 3.1). Since $\rho(X) = \frac{1}{5}$, we know that I_j maps to I_{j+1}. It follows from Lemma 2.8 that

$$\begin{aligned}
\ell_2 &= 4\ell_1 \\
\ell_3 &= 4\ell_2 = 4^2\ell_1 \\
\ell_4 &= 4\ell_3 - 2 = 4^3\ell_1 - 2 \\
\ell_5 &= 4\ell_4 = 4^4\ell_1 - 8 \\
\ell_1 &= 4\ell_5 - 1 = 4^5\ell_1 - 33.
\end{aligned}$$

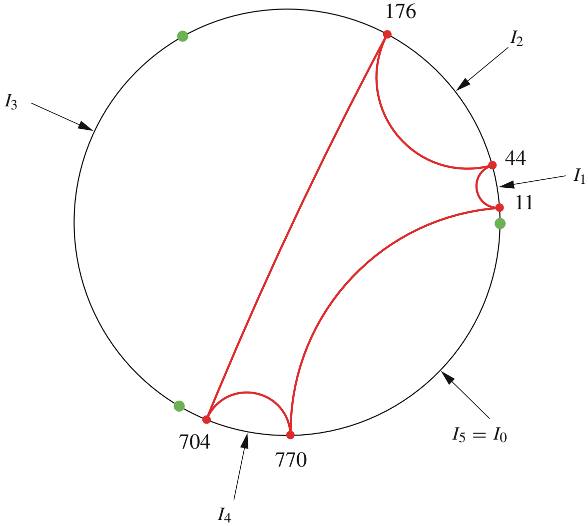

Fig. 3.1 The unique minimal rotation set X under m_4 with $\rho(X) = \frac{1}{5}$ and $\delta(X) = (\frac{3}{5}, \frac{0}{5}, \frac{2}{5})$ (angles in X are given in multiples of $\frac{1}{1023}$). Here X has cumulative deployment vector $\sigma(X) = (\frac{3}{5}, \frac{3}{5}, \frac{5}{5})$, and major gaps I_3 and $I_5 = I_0$ of multiplicities 2 and 1, respectively, which are also the number of fixed points of m_4 (shown as green dots) they contain

The last equation can be solved uniquely for ℓ_1, which in turn determines every ℓ_j:

$$\ell_1 = \frac{33}{1023}, \quad \ell_2 = \frac{132}{1023}, \quad \ell_3 = \frac{528}{1023}, \quad \ell_4 = \frac{66}{1023}, \quad \ell_5 = \frac{264}{1023}.$$

Since $\ell_1 = t_2 - t_1 = 4t_1 - t_1 = 3t_1$, we find t_1 and therefore every t_j:

$$t_1 = \frac{11}{1023}, \quad t_2 = \frac{44}{1023}, \quad t_3 = \frac{176}{1023}, \quad t_4 = \frac{704}{1023}, \quad t_5 = \frac{770}{1023}.$$

It is easily checked that this 5-cycle has the required rotation number and deployment vector. The uniqueness automatically follows from the above computation.

In general, the method of Example 3.6 can be described more formally as follows. Suppose we are looking for a minimal rotation set $X = \{t_1, \dots, t_q\}$ for m_d with $\rho(X) = p/q \neq 0$ and $\delta(X) = (\delta_1, \dots, \delta_{d-1})$. Let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$. Set n_j to be the multiplicity of I_j if I_j is major, and $n_j = 0$ otherwise. Then the relations $\ell_{j+p} = d\ell_j - n_j$ hold for every j (recall that all subscripts are taken modulo q). Introduce the vectors

$$\boldsymbol{\ell} = (\ell_1, \dots, \ell_q) \quad \text{and} \quad \boldsymbol{n} = (n_1, \dots, n_q)$$

in $\mathbb{R}^q$ and denote by $T : \mathbb{R}^q \to \mathbb{R}^q$ the isometry

$$T(x_1, x_2, \dots, x_q) = (x_{1+p}, x_{2+p}, \dots, x_{q+p}).$$

Notice that T is determined by the rotation number while $\boldsymbol{n}$ is determined by the deployment vector (Lemma 3.5). The q relations above can then be written as the non-homogeneous linear equation

$$T(\boldsymbol{\ell}) = d\boldsymbol{\ell} - \boldsymbol{n} \tag{3.3}$$

which can be easily solved for $\boldsymbol{\ell}$ by applying T repeatedly on each side and using the fact that $T^{\circ q} = \mathrm{id}$. The result is

$$\boldsymbol{\ell} = \frac{1}{d^q - 1} \sum_{i=0}^{q-1} d^{q-i-1}\, T^{\circ i}(\boldsymbol{n}). \tag{3.4}$$

Since $\boldsymbol{n} \neq \boldsymbol{0}$ and since the addition $j \mapsto j + p \pmod{q}$ acts transitively on $\mathbb{Z}_q$, the right hand sum has strictly positive components, so the above formula gives a unique solution $\boldsymbol{\ell}$ of (3.3) with $\ell_j > 0$ for all j. Once the gap lengths ℓ_j are known, we can find the t_j by noting that the counterclockwise distance from t_j to $t_{j+p} = dt_j \pmod{\mathbb{Z}}$ is the sum $\ell_j + \cdots + \ell_{j+p-1}$. The method produces a unique candidate q-cycle X, but one still needs to verify that this X has indeed the required rotation number and deployment vector.

There is an alternative way to solve (3.3) which, despite its appearance, will turn out more advantageous. Write (3.3) as

$$\boldsymbol{\ell} = \frac{1}{d} T(\boldsymbol{\ell}) + \frac{1}{d}\boldsymbol{n}$$

which can then be turned into

$$\boldsymbol{\ell} = \frac{1}{d}\Big(\frac{1}{d} T^{\circ 2}(\boldsymbol{\ell}) + \frac{1}{d} T(\boldsymbol{n})\Big) + \frac{1}{d}\boldsymbol{n} = \frac{1}{d^2} T^{\circ 2}(\boldsymbol{\ell}) + \frac{1}{d^2} T(\boldsymbol{n}) + \frac{1}{d}\boldsymbol{n}.$$

Continuing this way and using the fact that $T^{\circ k}(\boldsymbol{\ell})/d^k \to 0$ as $k \to \infty$, we obtain the series solution

$$\boldsymbol{\ell} = \sum_{k=0}^{\infty} d^{-(k+1)}\, T^{\circ k}(\boldsymbol{n}). \tag{3.5}$$

The vectors $\boldsymbol{\ell}$ and $\boldsymbol{n}$ can be thought of as positive measures supported on the subset

$$S = \left\{ \frac{j}{q} \pmod{\mathbb{Z}} : 0 \leq j \leq q-1 \right\} \cong \mathbb{Z}_q$$

of the circle by identifying ℓ_j with $\boldsymbol{\ell}\{j/q\}$ and n_j with $\boldsymbol{n}\{j/q\}$. Under this identification, $\boldsymbol{\ell}$ is just the push-forward of Lebesgue measure under the canonical semiconjugacy associated with X. Lemma 3.5 can then be translated into the statement that

$$\boldsymbol{n} = \sum_{i=1}^{d-1} \mathbb{1}_{\sigma_i},$$

where $\mathbb{1}_x$ is the unit mass at x. Thus, for each $k \geq 0$,

$$T^{\circ k}(\boldsymbol{n}) = \sum_{i=1}^{d-1} \mathbb{1}_{\sigma_i - kp/q}$$

and (3.5) can be written as

$$\boldsymbol{\ell} = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - kp/q}. \tag{3.6}$$

This means that to find $\boldsymbol{\ell}$ we start with a point mass $1/d$ at each σ_i and spread it around S by taking pull-backs under the rigid rotation $r_{p/q}$, each time dividing the mass by d. The measure $\ell_j = \boldsymbol{\ell}\{j/q\}$ is the sum of $d-1$ infinite series, each representing the contribution from the initial mass concentrated at one of the σ_i. This slightly disguised form of the solution (3.5) will be used in the proof of Theorem 3.7 below. Why do we use (3.6) instead of the simpler formula (3.4)? Because this formulation allows us to construct the cycle explicitly and to verify that it has the given rotation number and deployment vector. More importantly, it generalizes without any modification to the irrational case discussed in the next section, thus allowing a unified treatment of both rational and irrational cases of the deployment theorem.

Theorem 3.7 (Goldberg) *For every fraction $0 < p/q < 1$ in lowest terms and every vector $(\delta_1, \dots, \delta_{d-1}) \in \Delta^{d-2}$ with $q\delta_i \in \mathbb{Z}$ there is a unique minimal rotation set X for m_d such that $\rho(X) = p/q$ and $\delta(X) = (\delta_1, \dots, \delta_{d-1})$.*

Proof It will be convenient to use the notation $\equiv$ for congruence modulo $\mathbb{Z}$, so we write $m_d(t) \equiv dt$, $r_\theta(t) \equiv t + \theta$ and so on. We may also assume $\delta_1 \neq 0$; the general case will follow by cyclically permuting the components of $(\delta_1, \dots, \delta_{d-1})$ and rotating the corresponding rotation set. Define $\sigma_i = \delta_1 + \dots + \delta_i$ for $1 \leq i \leq d-1$. Then $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{d-1} = 1$ and each σ_i is congruent to some element of the set $S = \{j/q \pmod{\mathbb{Z}} : 0 \leq j \leq q-1\}$. Motivated by (3.6), we consider the atomic probability measure ν supported on S defined by

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - kp/q}. \tag{3.7}$$

Notice that $\nu\{\sigma_i\} > 1/d$. More precisely, if some $j/q \in S$ appears exactly n times as a σ_i, then $n/d < \nu\{j/q\} < (n+1)/d$. The lower bound is immediate from the definition. The upper bound holds since the contribution of the remaining terms of (3.7) to $\nu\{j/q\}$ is at most

$$(d-1)\sum_{k=0}^{\infty} d^{-(kq+2)} = \frac{d-1}{d^2}\cdot\frac{d^q}{(d^q-1)} < \frac{1}{d}.$$

The same argument also proves that $0 < \nu\{j/q\} < 1/d$ whenever j/q is not congruent to any of the σ_i.

Let, as before, $N_1 \geq 1$ be the number of indices $1 \leq i \leq d-1$ for which $\sigma_i = 1$. Define

$$\psi_j = \nu\Big[0, \frac{j}{q}\Big) = \nu\Big\{\frac{0}{q}\Big\} + \cdots + \nu\Big\{\frac{j-1}{q}\Big\} \qquad 1 \leq j \leq q, \tag{3.8}$$

so $N_1/d < \psi_1 < \cdots < \psi_{q-1} < \psi_q = 1$. Set

$$a = \frac{N_1 - \nu[0, p/q)}{d-1} = \frac{N_1 - \psi_p}{d-1} \tag{3.9}$$

and

$$t_j \equiv \psi_j - a \qquad 1 \leq j \leq q.$$

We show that $X = \{t_1, \ldots, t_q\}$ is the desired rotation set.

The relation

$$\nu\Big(B + \frac{p}{q}\Big) \equiv d\nu(B) \tag{3.10}$$

for every set $B \subset \mathbb{T}$ is easily verified from the definition of ν. It implies

$$\nu\Big[0, \frac{j+p}{q}\Big) \equiv \nu\Big[0, \frac{p}{q}\Big) + \nu\Big[\frac{p}{q}, \frac{j+p}{q}\Big) \equiv \nu\Big[0, \frac{p}{q}\Big) + d\nu\Big[0, \frac{j}{q}\Big),$$

which yields the relation

$$\psi_{j+p} \equiv d\psi_j + \psi_p$$

for all j. Thus,

$$\begin{aligned} t_{j+p} &\equiv \psi_{j+p} - a \equiv d\psi_j + \psi_p - a \\ &\equiv dt_j + (d-1)a + \psi_p \equiv dt_j + N_1 \equiv dt_j. \end{aligned} \tag{3.11}$$

Since $t_1, \ldots, t_q$ are in positive cyclic order, this proves that X is a q-cycle under m_d with combinatorial rotation number p/q. It follows from Corollary 1.16 that X is a rotation set with $\rho(X) = p/q$.

Next, we verify that $\delta(X) = (\delta_1, \ldots, \delta_{d-1})$ or equivalently $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$. First note that $\psi_p > N_1/d$, so $N_1 - \psi_p < N_1(d-1)/d$, so $0 < a < N_1/d < \psi_1$. This shows that $0 \in (t_q, t_1)$. Suppose there is an n-fold incidence of the form

$$\frac{j}{q} = \sigma_i = \sigma_{i+1} = \cdots = \sigma_{i+n-1}.$$

Then, by our earlier remark,

$$\frac{n}{d} < t_{j+1} - t_j \equiv \psi_{j+1} - \psi_j = \nu\left\{\frac{j}{q}\right\} < \frac{n+1}{d},$$

which implies (t_j, t_{j+1}) is a major gap of multiplicity n, and therefore contains n fixed points of m_d by Lemma 2.13. Under the canonical semiconjugacy φ associated with X, these n fixed points all map to j/q. Thus, φ maps the fixed point set $\{u_1, \ldots, u_{d-1}\}$ to the set $\{\sigma_1, \ldots, \sigma_{d-1}\}$, sending n of the u_i to the same point j/q if and only if n of the σ_i collide at j/q. Since $\varphi(0) \equiv 0$, it follows from monotonicity of φ that $\varphi(u_i) \equiv \sigma_i$ for every i. Since every σ_i lies in $(0, 1]$ by our assumption $\delta_1 \neq 0$, Lemma 3.3 proves that $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$.

It remains to prove uniqueness. Suppose $\hat{X} = \{\hat{t}_1, \ldots, \hat{t}_q\}$ is another rotation set for m_d with rotation number p/q and deployment vector $(\delta_1, \ldots, \delta_{d-1})$. As we have seen in the discussion leading to (3.4) or (3.5), for each j the gap $\hat{I}_j = (\hat{t}_j, \hat{t}_{j+1})$ of $\hat{X}$ has the same length as the gap $I_j = (t_j, t_{j+1})$ of X. Hence there is a rigid rotation r_α which maps t_j to $\hat{t}_j$ for all j. We must show that $\alpha \equiv 0$. The major gaps I_0 and $\hat{I}_0 = r_\alpha(I_0)$ contain the same set of fixed points of m_d since X and $\hat{X}$ have the same deployment vector. Since the fixed points of m_d are $1/(d-1)$ apart, it follows that the distance between α and 0 is less than $1/(d-1)$. On the other hand, $r_\alpha : X \to \hat{X}$ commutes with m_d, so $d(t_j + \alpha) \equiv dt_j + \alpha$ for every j, which implies $(d-1)\alpha \equiv 0$. The only solution of this equation whose distance to 0 is $< 1/(d-1)$ is $\alpha \equiv 0$, and the proof is complete. □

Remark 3.8 The $d-1$ solutions for a of the equation $(d-1)a + \psi_p \equiv 0$, which was key in (3.11), correspond to minimal rotation sets with rotation number p/q whose deployment vectors are cyclic permutations of $(\delta_1, \ldots, \delta_{d-1})$. The particular choice of a in (3.9) guarantees that this permutation is the identity.

Example 3.9 Let us revisit Example 3.6, this time using the idea of the measure ν in the proof of Theorem 3.7. Recall that we were looking for the unique 5-cycle $X = \{t_1, \cdots, t_5\}$ under m_4 with $\rho(X) = \frac{1}{5}$ and $\delta(X) = (\frac{3}{5}, 0, \frac{2}{5})$ or $\sigma(X) = (\frac{3}{5}, \frac{3}{5}, \frac{5}{5})$. We compute the atomic measure ν on the set $S = \{\frac{0}{5}, \ldots, \frac{4}{5}\}$, starting with a mass $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ at $\sigma_1 = \sigma_2 = \frac{3}{5}$ and a mass $\frac{1}{4}$ at $\sigma_3 = \frac{5}{5} \equiv \frac{0}{5}$. We then spread the measure around S by pulling back under the rotation $r_{1/5}$, each time dividing the mass by 4. Since $\nu\{t+\frac{1}{5}\} \equiv 4\nu\{t\}$ for every $t \in S$ by the transformation rule (3.10), it suffices to compute ν only at $\frac{0}{5}$:

$$\nu\left\{\frac{0}{5}\right\} = \left(\frac{1}{4} + \frac{1}{4^6} + \frac{1}{4^{11}} + \cdots\right) + \left(\frac{1}{2 \cdot 4^3} + \frac{1}{2 \cdot 4^8} + \cdots\right) = \frac{264}{1023}$$

It follows that

$$\begin{aligned}
4\nu\left\{\frac{0}{5}\right\} &= \frac{1056}{1023} \Longrightarrow \nu\left\{\frac{1}{5}\right\} = \frac{33}{1023} \\
4\nu\left\{\frac{1}{5}\right\} &= \frac{132}{1023} \Longrightarrow \nu\left\{\frac{2}{5}\right\} = \frac{132}{1023} \\
4\nu\left\{\frac{2}{5}\right\} &= \frac{528}{1023} \Longrightarrow \nu\left\{\frac{3}{5}\right\} = \frac{528}{1023} \\
4\nu\left\{\frac{3}{5}\right\} &= \frac{2112}{1023} \Longrightarrow \nu\left\{\frac{4}{5}\right\} = \frac{66}{1023}
\end{aligned}$$

(these are just the gap lengths ℓ_j computed in Example 3.6). Thus,

$$\begin{aligned}
\psi_1 &= \nu\left\{\frac{0}{5}\right\} = \frac{264}{1023} \\
\psi_2 &= \nu\left\{\frac{0}{5}\right\} + \nu\left\{\frac{1}{5}\right\} = \frac{297}{1023} \\
\psi_3 &= \nu\left\{\frac{0}{5}\right\} + \nu\left\{\frac{1}{5}\right\} + \nu\left\{\frac{2}{5}\right\} = \frac{429}{1023} \\
\psi_4 &= \nu\left\{\frac{0}{5}\right\} + \nu\left\{\frac{1}{5}\right\} + \nu\left\{\frac{2}{5}\right\} + \nu\left\{\frac{3}{5}\right\} = \frac{957}{1023} \\
\psi_5 &= \nu\left\{\frac{0}{5}\right\} + \nu\left\{\frac{1}{5}\right\} + \nu\left\{\frac{2}{5}\right\} + \nu\left\{\frac{3}{5}\right\} + \nu\left\{\frac{4}{5}\right\} = 1.
\end{aligned}$$

Now $t_j = \psi_j - a$, where $a = (1-\psi_1)/3 = \frac{253}{1023}$. We obtain

$$\begin{aligned}
t_1 &= \frac{264}{1023} - \frac{253}{1023} = \frac{11}{1023} & t_2 &= \frac{297}{1023} - \frac{253}{1023} = \frac{44}{1023} \\
t_3 &= \frac{429}{1023} - \frac{253}{1023} = \frac{176}{1023} & t_4 &= \frac{957}{1023} - \frac{253}{1023} = \frac{704}{1023} \\
t_5 &= \frac{1023}{1023} - \frac{253}{1023} = \frac{770}{1023},
\end{aligned}$$

which is of course the same cycle obtained by the method of Example 3.6.

Remark 3.10 A different approach to the rational case of the deployment theorem can be found in the recent work [27] which solves the general problem of realizing cyclic permutations of q objects as period q orbits of m_d. The idea is to reduce the problem to finding the stationary state of an associated Markov chain, which can then be tackled by classical Perron-Frobenius theory.

For each $q > 0$ the number of distinct vectors $(\delta_1, \dots, \delta_{d-1}) \in \Delta^{d-2}$ with $q\delta_i \in \mathbb{Z}$ can be computed as the number of ways to deploy q identical balls in $d-1$ labeled boxes. This, in view of Theorem 3.7, gives the following

Corollary 3.11 (Goldberg) *For every fraction $0 < p/q < 1$ in lowest terms, there are*

$$\binom{q+d-2}{q} = \frac{(q+d-2)!}{q!(d-2)!}$$

distinct minimal rotation sets X under m_d with $\rho(X) = p/q$.

For $d = 2$ this number reduces to 1, proving that *there is a unique minimal rotation set under doubling with a given rational rotation number*.

The deployment theorem can be generalized to unions of cycles as follows. Suppose X is a rotation set for m_d, with $\rho(X) = p/q \neq 0$ in lowest terms, consisting of distinct q-cycles $C_1, \dots, C_N$ (here $N \leq d-1$ by Corollary 2.27). As in Remark 3.2, we define the deployment vector and the cumulative deployment vector of X as the averages

$$\delta(X) = \frac{1}{N}\sum_{i=1}^{N} \delta(C_i) \quad \text{and} \quad \sigma(X) = \frac{1}{N}\sum_{i=1}^{N} \sigma(C_i).$$

Of course the ith components of $\delta(X)$ and $\sigma(X)$ are simply the fraction of points of X that fall within the intervals $[u_{i-1}, u_i)$ and $[u_0, u_i)$, respectively. Note that these components are now rational numbers with denominator dividing Nq.

Suppose we are looking for such a rotation set X with $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1})$. Let $X = \{t_1, \dots, t_{Nq}\}$, where the points are labeled so that $0, t_1, \dots, t_{Nq}$ are in positive cyclic order and the subscripts are taken modulo Nq. Since each cycle in X has combinatorial rotation number $p/q = Np/(Nq)$, the map m_d acts as $t_j \mapsto t_{j+Np}$ on X. As in the case $N = 1$, let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$ and n_j be the multiplicity of I_j if I_j is major, and $n_j = 0$ otherwise. Then the equations $\ell_{j+Np} = d\ell_j - n_j$ for $1 \leq j \leq Nq$ can be written in vector form as $T(\boldsymbol{\ell}) = d\boldsymbol{\ell} - \boldsymbol{n}$. Here $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{Nq})$ is unknown, $\boldsymbol{n} = (n_1, \dots, n_{Nq})$ is determined by the cumulative deployment vector $\sigma(X)$, and $T : \mathbb{R}^{Nq} \to \mathbb{R}^{Nq}$ is the isometry

$$T(x_1, x_2, \dots, x_{Nq}) = (x_{1+Np}, x_{2+Np}, \dots, x_{Nq+Np})$$

determined by the rotation number. Since $T^{\circ q} = \text{id}$, the same argument as in the minimal case gives a unique solution $\boldsymbol{\ell}$ of this equation which can be expressed in either of the forms (3.4) or (3.5) or (3.6). If every component of $\boldsymbol{\ell}$ obtained this way is strictly positive, then the gap lengths are uniquely determined and an argument similar to the minimal case shows that the desired rotation set X exists and is unique. On the other hand, if the solution $\boldsymbol{\ell}$ has a zero component, then no X with the given rotation number and deployment vector can exist. Using the form (3.6) of the solution, it follows that a necessary and sufficient condition for the existence of X is that the support of the atomic measure

$$\sum_{i=1}^{d-1}\sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - kp/q}$$

is the set $S = \{j/(Nq) \pmod{\mathbb{Z}} : 0 \leq j \leq Nq - 1\}$. In other words, each point of S must belong to the orbit of some σ_i under $r_{p/q}$. Using the fact that p, q are relatively prime, it is easy to see that $j/(Nq), j'/(Nq) \in S$ belong to the same orbit under $r_{p/q}$ if and only if $j = j' \pmod{N}$. Thus, S is the union of N disjoint q-cycles under $r_{p/q}$, indexed by the distinct residue classes modulo N. Consider the ***signature*** $s(X) = Nq\,\sigma(X)$, that is the integer vector $s(X) = (s_1, \ldots, s_{d-1})$, where s_i is the number of points of X in $[u_0, u_i)$.[1] Then the above condition is equivalent to every residue class modulo N being represented by some s_i. This proves

Theorem 3.12 (Goldberg) *Suppose* $0 < p/q < 1$ *is a fraction in lowest terms,* $N \geq 1$ *is an integer, and* $\{s_i\}_{1\leq i\leq d-1}$ *is an integer sequence such that* $0 \leq s_1 \leq \cdots \leq s_{d-1} = Nq$. *Then there is a rotation set* X *for* m_d *with rotation number* $\rho(X) = p/q$ *and signature* $s(X) = (s_1, \ldots, s_{d-1})$ *if and only if every residue class modulo* N *is represented by some* s_i. *Moreover,* X *subject to these conditions is unique.*

Notice that this result gives an alternative proof for the inequality $N \leq d - 1$ in Corollary 2.27.

Example 3.13 Consider finite rotation sets with rotation number $\frac{1}{4}$ under tripling. According to Theorem 3.12, such a rotation set is either a 4-cycle (where $N = 1$) or a union of two 4-cycles (where $N = 2$), and is uniquely determined by its signature. For $N = 1$, all five signatures $(s, 4)$ for $0 \leq s \leq 4$ can occur; they are realized by the following rotation sets that we already encountered in Example 2.29:

X	$s(X)$	$\delta(X)$
$C_1 : \frac{1}{80} \mapsto \frac{3}{80} \mapsto \frac{9}{80} \mapsto \frac{27}{80}$	$(4, 4)$	$(1, 0)$
$C_2 : \frac{2}{80} \mapsto \frac{6}{80} \mapsto \frac{18}{80} \mapsto \frac{54}{80}$	$(3, 4)$	$(\frac{3}{4}, \frac{1}{4})$
$C_3 : \frac{5}{80} \mapsto \frac{15}{80} \mapsto \frac{45}{80} \mapsto \frac{55}{80}$	$(2, 4)$	$(\frac{1}{2}, \frac{1}{2})$
$C_4 : \frac{14}{80} \mapsto \frac{42}{80} \mapsto \frac{46}{80} \mapsto \frac{58}{80}$	$(1, 4)$	$(\frac{1}{4}, \frac{3}{4})$
$C_5 : \frac{41}{80} \mapsto \frac{43}{80} \mapsto \frac{49}{80} \mapsto \frac{67}{80}$	$(0, 4)$	$(0, 1)$

[1] In the terminology of [11], the integers s_i define the *deployment sequence* of X.

However, for $N = 2$ only the signatures $(s, 8)$ with odd $0 \le s \le 8$ occur. These are realized by the following four rotation sets, also encountered in Example 2.29 as unions of compatible pairs:

X	$s(X)$	$\delta(X)$
$C_1 \cup C_2$	$(7, 8)$	$(\frac{7}{8}, \frac{1}{8})$
$C_2 \cup C_3$	$(5, 8)$	$(\frac{5}{8}, \frac{3}{8})$
$C_3 \cup C_4$	$(3, 8)$	$(\frac{3}{8}, \frac{5}{8})$
$C_4 \cup C_5$	$(1, 8)$	$(\frac{1}{8}, \frac{7}{8})$

Notice that the signatures $(0, 8), (2, 8), (4, 8), (6, 8)$ cannot occur for the rotation number $\frac{1}{4}$, although they can be realized by 8-cycles with any of the rotation numbers $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}$, or $\frac{7}{8}$.

The above example shows that the cycles C_i and the unions $C_i \cup C_{i+1}$ have distinct deployment sequences. This is a special case of the following stronger form of the uniqueness part of Theorem 3.12:

Corollary 3.14 *Suppose X, X' are finite rotation sets with the same rotation number and deployment sequence. Then $X = X'$.*

Proof Let $\rho(X) = \rho(X') = p/q$ and suppose X and X' are unions of N and N' distinct q-cycles respectively. Consider the signatures $s(X) = (s_1, \ldots, s_{d-1})$ and $s(X') = (s'_1, \ldots, s'_{d-1})$. The assumption $\delta(X) = \delta(X')$ shows that $s_i/N = s'_i/N'$ or

$$N' s_i = N s'_i \quad \text{for all} \quad 1 \le i \le d-1.$$

By Theorem 3.12, $s_j = 1 \pmod{N}$ for some j. It follows from the above equation that N divides N'. A similar reasoning shows that N' divides N, so $N = N'$. It now follows from the uniqueness statement of Theorem 3.12 that $X = X'$. □

Corollary 3.15 *For every fraction $0 < p/q < 1$ in lowest terms, there are q^{d-2} rotation sets X for m_d with $\rho(X) = p/q$, each consisting of the maximum number $d-1$ of distinct q-cycles.*

In particular, the upper bound in Corollary 2.27 is optimal.

Proof By Theorem 3.12 for $N = d-1$, such X are in one-to-one correspondence with signatures $s = (s_1, \ldots, s_{d-2}, (d-1)q)$ for which the unordered set $A = \{s_1, \ldots, s_{d-2}\}$ reduces to $\{1, \ldots, d-2\}$ modulo $d-1$. For each $1 \le k \le d-2$ such A contains exactly one element of the form $j(d-1)+k$ with $0 \le j \le q-1$. Evidently there are q^{d-2} choices for A, hence for the signature s. □

Another application of Theorem 3.12 is the following characterization of compatible cycles in terms of their signature (compare §2 of [19]). It will be convenient to use the notation $\mathscr{C}_d(p/q)$ for the collection of all q-cycles under m_d with rotation number p/q.

Theorem 3.16 *Two distinct cycles $C, C' \in \mathscr{C}_d(p/q)$ are compatible if and only if the non-zero components of $s(C) - s(C')$ are all* 1 *or all* -1.

Proof First suppose C, C' are compatible. By Lemma 2.25 C, C' are superlinked, so their points alternate as we go around the circle. If μ, μ' denote the natural measures of C, C', it follows that the function

$$\chi : t \mapsto q\Big(\mu[0, t) - \mu'[0, t)\Big)$$

takes values in $\{0, 1\}$ or in $\{0, -1\}$. Thus, the non-zero components $(\chi(u_1), \ldots, \chi(u_{d-1}))$ of $s(C) - s(C')$ are all 1 or all -1.

Conversely, and without loss of generality, assume that all non-zero components of $\boldsymbol{\varepsilon} = s(C) - s(C')$ are 1. The sum $s(C) + s(C')$ has both even and odd components, so by Theorem 3.12 there is a rotation set X of size $2q$ with $\rho(X) = \rho(C) = \rho(C')$ and $s(X) = s(C) + s(C')$. Decompose X into the union of two compatible q-cycles Y, Y', where $s(Y) + s(Y') = s(C) + s(C')$. By the previous paragraph and after relabeling these cycles if necessary, we may assume that all non-zero components of $\boldsymbol{\varepsilon}' = s(Y) - s(Y')$ are 1. The relation $2s(C) + \boldsymbol{\varepsilon}' = 2s(Y) + \boldsymbol{\varepsilon}$ shows that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ have the same support (that is, their non-zero components occur at the same places), so $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}'$. It follows that $s(C) = s(Y)$ and $s(C') = s(Y')$. The uniqueness part of Theorem 3.12 then shows $C = Y$ and $C' = Y'$, which proves C, C' are compatible. □

The arithmetical criterion for realizability of signatures in Theorem 3.12 has a geometric interpretation due to McMullen. He comments in [19] that $\mathscr{C}_d(p/q)$ can be identified with the vertices of a simplicial subdivision of a $(d-2)$-dimensional simplex, with compatible cycles corresponding to adjacent vertices (compare Fig. 3.2). Below we provide a justification for this statement; Lemma 3.18 below will also play a role in the proof of Theorem 4.12 in the next chapter.

In view of Theorem 3.16 we can define a relation $\prec$ between any two compatible cycles $C, C' \in \mathscr{C}_d(p/q)$ by declaring $C \prec C'$ if the non-zero components of $s(C') - s(C)$ are all 1. Evidently a collection $C_1, \ldots, C_n$ in $\mathscr{C}_d(p/q)$ are compatible if and only if they are linearly ordered by $\prec$.

Lemma 3.17 *Suppose $C_1, \ldots, C_n$ are distinct compatible cycles in $\mathscr{C}_d(p/q)$. Then the deployment vectors $\delta(C_1), \ldots, \delta(C_n) \in \mathbb{R}^{d-1}$ are affinely independent.*

Proof After relabeling the cycles we may assume $C_1 \prec C_2 \prec \cdots \prec C_n$. Let $\boldsymbol{\varepsilon}_{i,j} = s(C_j) - s(C_i)$. The cocycle relation

$$\boldsymbol{\varepsilon}_{i,j} + \boldsymbol{\varepsilon}_{j,k} = \boldsymbol{\varepsilon}_{i,k}$$

shows that the vectors $\boldsymbol{\varepsilon}_{1,2}, \boldsymbol{\varepsilon}_{2,3}, \ldots, \boldsymbol{\varepsilon}_{n-1,n}$ have disjoint supports and therefore are linearly independent in $\mathbb{R}^{d-1}$. It follows that the vectors

$$\boldsymbol{\varepsilon}_{1,2}, \quad \boldsymbol{\varepsilon}_{1,3} = \boldsymbol{\varepsilon}_{1,2} + \boldsymbol{\varepsilon}_{2,3}, \quad \ldots, \quad \boldsymbol{\varepsilon}_{1,n} = \boldsymbol{\varepsilon}_{1,2} + \boldsymbol{\varepsilon}_{2,3} + \cdots + \boldsymbol{\varepsilon}_{n-1,n}$$

are also linearly independent.

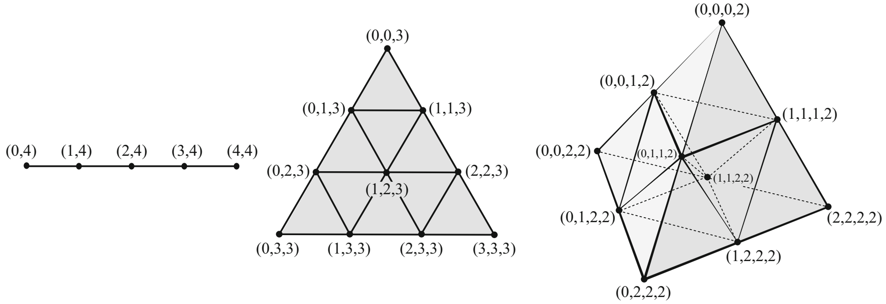

Fig. 3.2 Geometric representation of q-cycles as vertices of a subdivision Δ_q^{d-2} of the standard simplex Δ^{d-2}, following McMullen. Here each cycle is labeled by its signature and two cycles are compatible if and only if they are connected by an edge in Δ_q^{d-2}. Left: The five vertices of Δ_4^1 representing 4-cycles under m_3 with rotation number $\frac{1}{4}$ or $\frac{3}{4}$. Middle: The ten vertices of Δ_3^2 representing 3-cycles under m_4 with rotation number $\frac{1}{3}$ or $\frac{2}{3}$. Right: The ten vertices of Δ_2^3 representing 2-cycles under m_5 with rotation number $\frac{1}{2}$

To prove $\delta(C_1), \ldots, \delta(C_n)$ are affinely independent, it suffices to verify the linear independence of the vectors $\{\delta(C_i) - \delta(C_1)\}_{2 \leq i \leq n}$. If $\sum_{i=2}^{n} \alpha_i (\delta(C_i) - \delta(C_1)) = 0$ for some scalars $\alpha_i \in \mathbb{R}$, then $\sum_{i=2}^{n} \alpha_i (\sigma(C_i) - \sigma(C_1)) = 0$, so

$$\sum_{i=2}^{n} \alpha_i \, \boldsymbol{\varepsilon}_{1,i} = \sum_{i=2}^{n} \alpha_i (s(C_i) - s(C_1)) = q \sum_{i=2}^{n} \alpha_i (\sigma(C_i) - \sigma(C_1)) = 0.$$

It follows from the previous paragraph that $\alpha_i = 0$ for all i. □

Recall that Δ^{d-2} is the standard simplex $\{(x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} : x_i \geq 0$ and $\sum_{i=1}^{d-1} x_i = 1\}$. Fix a rotation number p/q and consider the finite set V consisting of vectors $(x_1, \ldots, x_{d-1}) \in \Delta^{d-2}$ such that $qx_i \in \mathbb{Z}$ for all i. By Theorem 3.7, the assignment $C \mapsto \delta(C)$ is a bijection between $\mathscr{C}_d(p/q)$ and V. Let Δ_q^{d-2} be the collection of all convex hulls

$$[\delta(C_1), \ldots, \delta(C_n)] = \left\{ \sum_{i=1}^{n} \alpha_i \, \delta(C_i) : 0 \leq \alpha_i \leq 1 \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\},$$

where $C_1, \ldots, C_n$ are distinct compatible cycles in $\mathscr{C}_d(p/q)$. Lemma 3.17 shows that $[\delta(C_1), \ldots, \delta(C_n)]$ is an $(n-1)$-simplex in Δ^{d-2}.

Lemma 3.18 *Suppose $C_1, \ldots, C_n$ are distinct compatible cycles in $\mathscr{C}_d(p/q)$, with $n > 1$. Then the interior of the $(n-1)$-simplex $[\delta(C_1), \ldots, \delta(C_n)]$ does not meet V.*

Proof We may assume again that $C_1 \prec C_2 \prec \cdots \prec C_n$. Suppose there is a cycle $C \in \mathscr{C}_d(p/q)$ and scalars $0 < \alpha_1, \ldots, \alpha_n < 1$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that

$\sum_{i=1}^{n} \alpha_i \, \delta(C_i) = \delta(C)$. Then $\sum_{i=1}^{n} \alpha_i \, s(C_i) = s(C)$, so

$$\sum_{i=2}^{n} \alpha_i \, \boldsymbol{\varepsilon}_{1,i} = \sum_{i=2}^{n} \alpha_i (s(C_i) - s(C_1)) = s(C) - s(C_1),$$

where $\boldsymbol{\varepsilon}_{i,j} = s(C_j) - s(C_i)$ as before. Using the relation

$$\boldsymbol{\varepsilon}_{1,i} = \boldsymbol{\varepsilon}_{1,2} + \boldsymbol{\varepsilon}_{2,3} + \cdots + \boldsymbol{\varepsilon}_{i-1,i}, \tag{3.12}$$

we can rewrite this as

$$\sum_{i=2}^{n} \beta_i \, \boldsymbol{\varepsilon}_{i-1,i} = s(C) - s(C_1),$$

where $0 < \beta_i = \alpha_i + \cdots + \alpha_n < 1$. Since the vectors $\{\boldsymbol{\varepsilon}_{i-1,i}\}_{2 \le i \le n}$ have disjoint supports, the components of $\sum_{i=2}^{n} \beta_i \, \boldsymbol{\varepsilon}_{i-1,i}$ consist of the β_i and possibly some 0's. This contradicts the fact that $s(C) - s(C_1)$ is a non-zero integer vector. □

Theorem 3.19 *Δ_q^{d-2} is a simplicial subdivision of Δ^{d-2}.*

By Corollaries 3.11 and 3.15, Δ_q^{d-2} has $\binom{q+d-2}{q}$ vertices and q^{d-2} top-dimensional cells. The cases $d = 3, 4$ produce regular linear and triangular subdivisions, but the situation for $d > 4$ is not as symmetric (see Fig. 3.2).

Proof To show Δ_q^{d-2} is a simplicial complex, it suffices to check that two simplices $[\delta(C_1), \ldots, \delta(C_n)]$ and $[\delta(C'_1), \ldots, \delta(C'_m)]$ in Δ_q^{d-2} whose interiors intersect must coincide. The case $n = m = 1$ is trivial and the cases $n = 1, m > 1$ or $n > 1, m = 1$ are already covered by Lemma 3.18, so we may assume $n, m > 1$. Label the cycles so that $C_1 \prec \cdots \prec C_n$ and $C'_1 \prec \cdots \prec C'_m$. By our hypothesis, there are scalars $0 < \alpha_1, \ldots, \alpha_n < 1$ and $0 < \alpha'_1, \ldots, \alpha'_m < 1$, with $\sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{m} \alpha'_j = 1$, such that $\sum_{i=1}^{n} \alpha_i \, \delta(C_i) = \sum_{j=1}^{m} \alpha'_j \, \delta(C'_j)$. Then $\sum_{i=1}^{n} \alpha_i \, s(C_i) = \sum_{j=1}^{m} \alpha'_j \, s(C'_j)$. Letting $\boldsymbol{\varepsilon}_{i,j} = s(C_j) - s(C_i)$ and $\boldsymbol{\varepsilon}'_{i,j} = s(C'_j) - s(C'_i)$, it follows that

$$s(C_1) + \sum_{i=2}^{n} \alpha_i \, \boldsymbol{\varepsilon}_{1,i} = \sum_{i=1}^{n} \alpha_i \, s(C_i) = \sum_{j=1}^{m} \alpha'_j \, s(C'_j) = s(C'_1) + \sum_{j=2}^{m} \alpha'_j \, \boldsymbol{\varepsilon}'_{1,j},$$

or

$$\sum_{i=2}^{n} \alpha_i \, \boldsymbol{\varepsilon}_{1,i} - \sum_{j=2}^{m} \alpha'_j \, \boldsymbol{\varepsilon}'_{1,j} = s(C'_1) - s(C_1)$$

Using (3.12) and the similar relation for the $\boldsymbol{\varepsilon}'_{1,j}$, we can rewrite the above equation as

$$\sum_{i=2}^{n} \beta_i \, \boldsymbol{\varepsilon}_{i-1,i} - \sum_{j=2}^{m} \beta'_j \, \boldsymbol{\varepsilon}'_{j-1,j} = s(C'_1) - s(C_1), \tag{3.13}$$

where $0 < \beta_i = \alpha_i + \cdots + \alpha_n < 1$ and $0 < \beta'_j = \alpha'_j + \cdots + \alpha'_m < 1$. Since the vectors $\{\boldsymbol{\varepsilon}_{i-1,i}\}_{2 \le i \le n}$ have disjoint supports, the non-zero components of $\sum_{i=2}^{n} \beta_i \, \boldsymbol{\varepsilon}_{i-1,i}$ are precisely the β_i. Similarly, the non-zero components of $\sum_{j=2}^{m} \beta'_j \, \boldsymbol{\varepsilon}'_{j-1,j}$ are the β'_j. It follows that the components of the left hand side of (3.13) lie strictly between -1 and 1. Since the right hand side of (3.13) is an integer vector, the two sides must vanish. Thus, $s(C_1) = s(C'_1)$ and the finite sequences

$$1 > \beta_2 > \cdots > \beta_n = \alpha_n > 0 \quad \text{and} \quad 1 > \beta'_2 > \cdots > \beta'_m = \alpha'_m > 0$$

coincide. This implies $n = m$, $\alpha_i = \alpha'_i$ and $s(C_i) = s(C'_i)$ for all $1 \le i \le n$.

To finish the proof of the theorem, it remains to show that every $x = (x_1, \ldots, x_{d-1}) \in \Delta^{d-2}$ belongs to a simplex in Δ_q^{d-2}. Let $y = (y_1, \ldots, y_{d-1})$, where $y_i = q(x_1 + \cdots + x_i)$. Then $0 \le y_1 \le \cdots \le y_{d-1} = q$. Let $t_i \in [0, 1)$ be the fractional part of y_i. If all the t_i are zero, then $x \in V$ and we are done. Otherwise, list the non-zero elements of $\{t_1, \ldots, t_{d-1}\}$ in decreasing order as

$$t_{i_1} \ge \ldots \ge t_{i_n}, \quad \text{where} \quad 1 \le n \le d-2.$$

Here we adopt the convention that if several t_i's are equal, we list them in the order of decreasing subscripts, that is, if $t_{i_k} = t_{i_{k+1}}$, then $i_k > i_{k+1}$. Let $e_1, \ldots, e_{d-1}$ denote the unit coordinate vectors in $\mathbb{R}^{d-1}$ and define

$$\begin{aligned} v_1 &= y - (t_1, \ldots, t_{d-1}) \\ v_{k+1} &= v_k + e_{i_k} \qquad 1 \le k \le n. \end{aligned} \tag{3.14}$$

It is not hard to check that the components of each v_k form a monotonic sequence of non-negative integers ending in q, and therefore there is a unique cycle $C_k \in \mathscr{C}_d(p/q)$ with $s(C_k) = v_k$. By Theorem 3.16, $C_1, \ldots, C_{n+1}$ are compatible. Define the scalars $\{\alpha_k\}_{1 \le k \le n+1}$ by

$$\alpha_k = t_{i_{k-1}} - t_{i_k},$$

where $t_{i_0} = 1$ and $t_{i_{n+1}} = 0$. Note that the α_k are non-negative and add up to 1. It follows from (3.14) that

$$y = v_1 + \sum_{k=1}^{n} t_{i_k} \, e_{i_k} = v_1 + \sum_{k=1}^{n} t_{i_k} (v_{k+1} - v_k)$$

$$= \sum_{k=1}^{n+1} (t_{i_{k-1}} - t_{i_k})\, v_k = \sum_{k=1}^{n+1} \alpha_k\, v_k = \sum_{k=1}^{n+1} \alpha_k\, s(C_k),$$

so $x = \sum_{k=1}^{n+1} \alpha_k\, \delta(C_k)$, as required. □

3.3 Deployment Theorem: The Irrational Case

We now proceed to the irrational case of the deployment theorem. Our approach closely parallels the one presented for the rational case in the proof of Theorem 3.7.

Theorem 3.20 (Goldberg-Tresser) *For every irrational number* $0 < \theta < 1$ *and every vector* $(\delta_1, \ldots, \delta_{d-1}) \in \Delta^{d-2}$ *there is a unique minimal rotation set* X *for* m_d *such that* $\rho(X) = \theta$ *and* $\delta(X) = (\delta_1, \ldots, \delta_{d-1})$.

Thus, the space of all minimal rotation sets for m_d of a given irrational rotation number can be identified with the simplex $\Delta^{d-2} \subset \mathbb{R}^{d-1}$. When $d = 2$, it follows from this and Corollary 2.38 that *there is a unique rotation set under doubling with a given irrational rotation number*.

Proof We continue using the notation $\equiv$ for congruence modulo $\mathbb{Z}$. As in the rational case, we may assume without loss of generality that $\delta_1 \neq 0$. Set $\sigma_i = \delta_1 + \ldots + \delta_i$ for $1 \leq i \leq d-1$, so $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{d-1} = 1$. We construct a degree 1 monotone map φ of the circle with the following properties:

(i) $I_s \neq \emptyset$ implies $I_{s-\theta} \neq \emptyset$, where I_s is the interior of the fiber $\varphi^{-1}(s)$;
(ii) $\varphi(dt) \equiv \varphi(t) + \theta$ whenever t is not in the closure of a plateau of φ; and
(iii) $\varphi(u_i) \equiv \sigma_i$ for $1 \leq i \leq d-1$.

Properties (i) and (ii) prove that the complement of the union of all plateaus of φ is a minimal rotation set X with $\rho(X) = \theta$ (Theorem 2.35), while property (iii) proves that $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$ (Lemma 3.3).

Let S be the union of the backward orbits of the σ_i under r_θ:

$$S = \{\sigma_i - k\theta \ (\mathrm{mod}\ \mathbb{Z}) : 1 \leq i \leq d-1 \text{ and } k \geq 0\}. \tag{3.15}$$

Consider the atomic probability measure ν supported on S defined by

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)}\, \mathbb{1}_{\sigma_i - k\theta}. \tag{3.16}$$

Observe that $\nu\{\sigma_i\} \geq 1/d$ for every i. More precisely, if some $s \in S$ appears exactly n times in the list $\{\sigma_1, \ldots, \sigma_{d-1}\}$, then $n/d \leq \nu\{s\} < (n+1)/d$. The lower bound follows from the definition, whereas the upper bound holds since the contribution

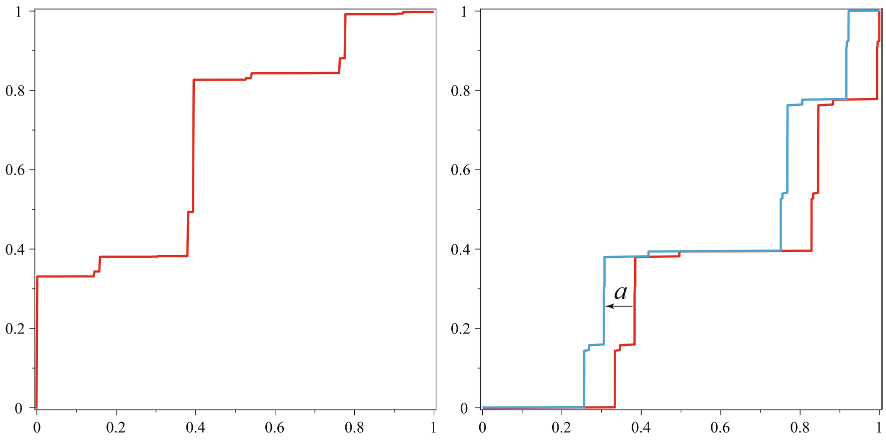

Fig. 3.3 Left: The graph of the map ψ obtained by integrating the atomic measure ν. Right: The graph of the left-inverse map ψ^{-1} along with its translation φ (in blue). In this example, $d = 3$, $\rho(X) = \frac{(\sqrt{5}-1)}{2}$, and $\delta(X) = (0.39475, 0.60525)$. Computation gives $a \approx 0.07713$

of the remaining terms of (3.16) to $\nu\{s\}$ is at most

$$(d-2)\sum_{k=1}^{\infty} d^{-(k+1)} = \frac{d-2}{d(d-1)} < \frac{1}{d}.$$

The same argument shows that $0 < \nu\{s\} < 1/d$ whenever $s \in S$ is not congruent to any of the σ_i.

The map $\psi : \mathbb{T} \to \mathbb{T}$ defined by $\psi(t) \equiv \nu[0, t)$ has degree 1, is strictly monotone, is continuous on $\mathbb{T} \smallsetminus S$ and is discontinuous at every $s \in S$ where it jumps by $\nu\{s\}$. The left-inverse ψ^{-1} extends to a *continuous* degree 1 monotone map of the circle, with a plateau I_s precisely when $s \in S$. Let $N_1 \geq 1$ be the number of indices $1 \leq j \leq d-2$ for which $\sigma_j = 1$. Set

$$a = \frac{N_1 - \nu[0, \theta)}{d-1}. \tag{3.17}$$

We show that the map $\varphi : \mathbb{T} \to \mathbb{T}$ defined by $\varphi(t) \equiv \psi^{-1}(t + a)$ has properties (i)–(iii) (see Fig. 3.3 for a typical graph of ψ and φ for the case $d = 3$).

Property (i) is immediate since $s \in S$ implies $s - \theta \in S$. The relation

$$\nu(B + \theta) \equiv d\nu(B) \tag{3.18}$$

for every Borel set B is easily verified from the definition of ν. It implies

$$\nu[0, t + \theta) \equiv \nu[0, \theta) + \nu[\theta, t + \theta) \equiv \psi(\theta) + d\nu[0, t),$$

which gives the functional equation

$$\psi(t+\theta) \equiv d\psi(t) + \psi(\theta).$$

Applying the left-inverse ψ^{-1} to both sides, we obtain

$$t+\theta \equiv \psi^{-1}(d\psi(t)+\psi(\theta)) \equiv \varphi(d\psi(t)+\psi(\theta)-a).$$

If t is not in the closure of a plateau of φ, then $\psi(\varphi(t)) = t + a$ and it follows that

$$\begin{aligned} \varphi(t)+\theta &\equiv \varphi(d(t+a)+\psi(\theta)-a) \\ &\equiv \varphi(dt+(d-1)a+\psi(\theta)) \\ &\equiv \varphi(dt+N_1) \equiv \varphi(dt). \end{aligned} \tag{3.19}$$

This proves property (ii).

To verify (iii), first note that $\psi^{-1}(t) \equiv 0$ for $t \in [0, \nu\{0\}]$. Since $\nu[0,\theta) > \nu\{0\} \geq N_1/d$, we have $N_1/d < \nu[0,\theta) < 1$, or $0 < a < N_1/d \leq \nu\{0\}$. In particular, $\varphi(0) \equiv \psi^{-1}(a) \equiv 0$. By what we have seen above, if there is an n-fold incidence $s = \sigma_i = \sigma_{i+1} = \cdots = \sigma_{i+n-1}$, the jump $\ell = \nu\{s\}$ of ψ at s satisfies the inequalities $n/d \leq \ell < (n+1)/d$. It follows that φ has a plateau of length ℓ on which it takes the constant value s. This plateau is a major gap of X, so it contains precisely n fixed points of m_d by Lemma 2.13. Thus, φ maps the fixed point set $\{u_1, \ldots, u_{d-1}\}$ to the set $\{\sigma_1, \ldots, \sigma_{d-1}\}$, sending n of the u_i to the same point s if and only if n of the σ_i collide at s. Since $\varphi(0) \equiv 0$, it follows from monotonicity of φ that $\varphi(u_i) \equiv \sigma_i$ for every i.

Finally, we prove uniqueness of X. Suppose $\hat{X}$ is any minimal rotation set with $\rho(\hat{X}) = \theta$ and $\delta(\hat{X}) = (\delta_1, \ldots, \delta_{d-1})$. Let $\hat{\varphi}$ be the canonical semiconjugacy associated with $\hat{X}$. By Lemma 3.3, $\hat{\varphi}(u_i) \equiv \sigma_i$, so $\hat{\varphi}$ takes the value σ_i on the major gap of $\hat{X}$ containing u_i. Moreover, if $\hat{X}$ has a major gap of multiplicity n, there will be an n-fold incidence between the σ_i. Since the gaps of $\hat{X}$ are precisely the plateaus of $\hat{\varphi}$, and since every gap eventually maps to a major gap, it follows that the values taken by $\hat{\varphi}$ on its plateaus form the set S in (3.15). It is now easy to see that the push-forward $\hat{\varphi}_*\lambda$ of Lebesgue measure is just the measure ν in (3.16). Since $\varphi_*\lambda = \nu$ also by the construction, the relation $\hat{\varphi}_*\lambda = \varphi_*\lambda$ must hold. Let $D \subset X$ be the countable set of the endpoints of gaps, and similarly define $\hat{D} \subset \hat{X}$. As the maps $\varphi : X \smallsetminus D \to \mathbb{T} \smallsetminus S$ and $\hat{\varphi} : \hat{X} \smallsetminus \hat{D} \to \mathbb{T} \smallsetminus S$ are bijective, the composition $\hat{\varphi}^{-1} \circ \varphi : X \smallsetminus D \to \hat{X} \smallsetminus \hat{D}$ defines a bijection $t \mapsto \hat{t}$ that preserves the cyclic order of all triples and commutes with m_d. Since for every $t_1, t_2 \in X \smallsetminus D$,

$$\lambda((t_1,t_2)) = \nu((\varphi(t_1), \varphi(t_2))) = \lambda((\hat{t}_1, \hat{t}_2)),$$

it follows that $t \mapsto \hat{t}$ is the restriction of some rigid rotation r_α to $X \smallsetminus D$. In other words, r_α maps $X \smallsetminus D$ onto $\hat{X} \smallsetminus \hat{D}$ and therefore X onto $\hat{X}$, and it commutes with m_d. To finish the proof, we must show that $\alpha \equiv 0$. The proof is identical to the rational case: Let I_0 be the major gap of X containing 0, so $r_\alpha(I_0)$ is the major gap of $\hat{X}$ containing 0 (this follows from the normalization $\varphi(0) \equiv \hat{\varphi}(0) \equiv 0$). By our construction, I_0 and $r_\alpha(I_0)$ contain the same set of fixed points of m_d, namely those which map under φ or $\hat{\varphi}$ to $\sigma_{d-1} = 1 \equiv 0$. Since the fixed points of m_d are $1/(d-1)$ apart, it follows that the distance between α and 0 must be $< 1/(d-1)$. On the other hand, r_α commutes with m_d, so $d(t+\alpha) \equiv dt + \alpha$ for every $t \in X$, which implies $(d-1)\alpha \equiv 0$. The only solution of this equation whose distance to 0 is $< 1/(d-1)$ is $\alpha \equiv 0$, and the proof is complete. □

Epilogue To conclude this chapter, let us briefly recap the main constructions related to a minimal rotation set and how they lead to the proofs of the deployment Theorems 3.7 and 3.20. Suppose X is a minimal rotation set for m_d with $\rho(X) = \theta \neq 0$, so X is a q-cycle if $\theta = p/q$ in lowest terms, and a Cantor set if θ is irrational.

- The ***canonical semiconjugacy*** associated with X is a degree 1 monotone map $\varphi : \mathbb{T} \to \mathbb{T}$, normalized by $\varphi(0) = 0$, which satisfies

$$\varphi \circ m_d = r_\theta \circ \varphi \quad \text{on} \quad X.$$

  The plateaus of φ are precisely the gaps of X.
- The ***natural measure*** of X is the unique m_d-invariant probability measure μ supported on X. It is related to the canonical semiconjugacy by

$$\varphi(t) = \int_0^t d\mu = \mu[0,t] \pmod{\mathbb{Z}}.$$

  If $\theta = p/q$ in lowest terms, then μ is the uniform Dirac measure on X:

$$\mu = \frac{1}{q} \sum_{x \in X} \mathbb{1}_x.$$

  If θ is irrational, then μ is the (well-defined) pull-back of Lebesgue measure λ under φ:

$$\lambda = \varphi_* \mu.$$

- The ***deployment vector*** of X is the probability vector $\delta(X) = (\delta_1, \ldots, \delta_{d-1}) \in \mathbb{R}^{d-1}$ defined by

$$\delta_i = \mu[u_{i-1}, u_i) \qquad 1 \leq i \leq d-1,$$

  where the $u_i = i/(d-1)$ are the fixed points of m_d.
- The ***cumulative deployment vector*** $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$ is defined by

$$\sigma_i = \mu[u_0, u_i) = \delta_1 + \cdots + \delta_i \qquad 1 \leq i \leq d-1.$$

- The ***gap measure*** of X is the push-forward ν of Lebesgue measure λ under φ:

$$\nu = \varphi_* \lambda.$$

  The terminology comes from the observation that each gap I of X maps under φ to a single point s with $\nu\{s\} = |I|$. The gap measure can be expressed in terms of $\rho(X) = \theta$ and $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$ by the explicit formula

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - k\theta}. \tag{3.20}$$

  In particular, ν is an atomic measure supported on the set

$$S = \{\sigma_i - k\theta \ (\mathrm{mod}\ \mathbb{Z}) : 1 \leq i \leq d-1 \text{ and } k \geq 0\},$$

  which is a union of at most $d-1$ backward orbits of the rotation r_θ. Thus, S consists of the qth roots of unity if $\theta = p/q$ in lowest terms, and is dense if θ is irrational.
- The minimal rotation set X can be recovered from its rotation number (whether rational or irrational) and deployment data as follows: Form the gap measure ν as above, and let $\psi(t) = \nu[0, t)$ for $t \in \mathbb{T}$ which has a well-defined left inverse ψ^{-1}. Define $\varphi : \mathbb{T} \to \mathbb{T}$ by

$$\varphi(t) = \psi^{-1}(t + a), \quad \text{where} \quad a = \frac{N_1 - \nu[0, \theta)}{d-1}.$$

  Here $N_1 \geq 1$ is the number of indices $1 \leq j \leq d-1$ for which $\sigma_j = 1$. Then φ is the canonical semiconjugacy associated with X, so X is the complement of the union of plateaus of φ.

Chapter 4
Applications and Computations

In this chapter we establish further properties of (minimal) rotation sets for m_d by exploiting the ideas and tools developed in the previous chapters, most notably the deployment theorem. We also study minimal rotation sets under doubling and tripling in some detail and carry out explicit computations. These computations will tie in with the dynamical study of quadratic and cubic polynomials in the next chapter.

4.1 Symmetries

It was already observed in Sect. 3.1 that if X is a minimal rotation set for m_d, the deployment vectors of the $d-2$ rotation sets

$$X + \frac{1}{d-1}, \; X + \frac{2}{d-1}, \; \ldots, \; X + \frac{d-2}{d-1} \pmod{\mathbb{Z}} \tag{4.1}$$

are obtained by cyclically permuting the components of $\delta(X)$. The uniqueness parts of the deployment Theorems 3.7 and 3.20 show at once that the converse statement is also true. In particular, if $\delta(X)$ is invariant under some cyclic permutation of its components, then X itself has a corresponding symmetry. Explicitly, suppose $\Pi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ is defined by

$$\Pi(x_1, x_2, \ldots, x_{d-1}) = (x_{d-1}, x_1, \ldots, x_{d-2}).$$

Theorem 4.1 *A minimal rotation set X for m_d has the symmetry $X = X + i/(d-1) \pmod{\mathbb{Z}}$ if and only if its deployment vector $\delta(X)$ is fixed by the iterate $\Pi^{\circ i}$.*

For example, a minimal rotation set X under tripling is ***self-antipodal*** in the sense $X = X + \frac{1}{2} \pmod{\mathbb{Z}}$ if and only if $\delta(X) = (\frac{1}{2}, \frac{1}{2})$. Moreover, there is a unique such

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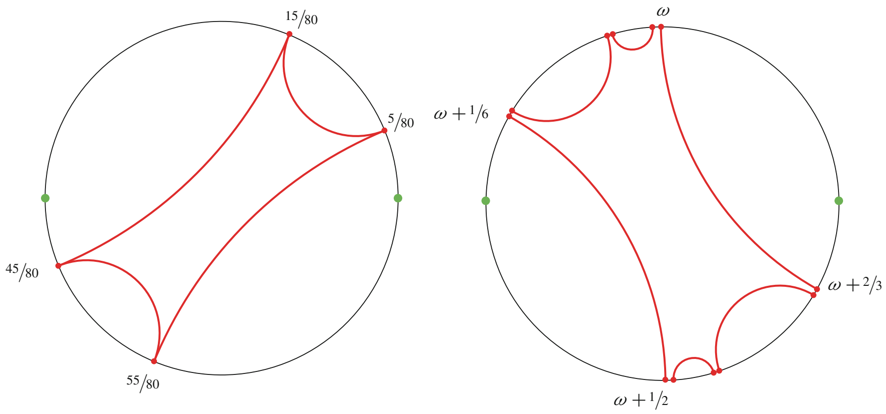

Fig. 4.1 If θ is irrational or rational with even denominator, there is a unique self-antipodal minimal rotation set X under tripling with $\rho(X) = \theta$. Left: The self-antipodal 4-cycle of rotation number $\frac{1}{4}$. Right: The self-antipodal Cantor set of the golden mean rotation number $\frac{(\sqrt{5}-1)}{2}$. Here $\omega \approx 0.25208333$ (see Sect. 4.6 for the method of such computations)

X with a given rotation number, which can only be irrational or rational with even denominator (compare Fig. 4.1).

It turns out that the sets (4.1) are the only copies of X that are rotation sets of the same rotation number:

Theorem 4.2 *Suppose both X and $X + \alpha \pmod{\mathbb{Z}}$ are rotation sets for m_d with $\rho(X) = \rho(X + \alpha)$. Then $\alpha = i/(d-1) \pmod{\mathbb{Z}}$ for some $0 \leq i \leq d-2$.*

Here the assumption $\rho(X) = \rho(X + \alpha)$ is necessary, as is illustrated by the rotation sets

$$X = \left\{\frac{5}{80}, \frac{15}{80}, \frac{45}{80}, \frac{55}{80}\right\} \quad \text{and} \quad X + \frac{1}{4} = \left\{\frac{25}{80}, \frac{35}{80}, \frac{65}{80}, \frac{75}{80}\right\}$$

under tripling for which $\rho(X) = \frac{1}{4}$ and $\rho(X + \frac{1}{4}) = \frac{3}{4}$.

Proof Denote the distinct major gaps of X by $I_1, \ldots, I_n$, so $I_1 + \alpha, \ldots, I_n + \alpha$ are the distinct major gaps of $X + \alpha$. For each $1 \leq i \leq n$, let J_i be the gap of X which maps to I_i and $\hat{J}_i$ be the gap of $X + \alpha$ which maps to $I_i + \alpha$. Evidently a gap of length ℓ for X or $X + \alpha$ belongs to $\{J_1, \ldots, J_n\}$ or $\{\hat{J}_1, \ldots, \hat{J}_n\}$ if and only if the fractional part of $d\ell$ is at least $1/d$. It follows that $\{\hat{J}_1, \ldots, \hat{J}_n\} = \{J_1 + \alpha, \ldots, J_n + \alpha\}$. We prove that in fact $\hat{J}_i = J_i + \alpha$ for every i.

Consider the standard monotone maps $g, \hat{g}$ associated with $X, X + \alpha$ and let $\varphi, \hat{\varphi}$ be the semiconjugacies between $g, \hat{g}$ and the rigid rotation r_θ, where $\theta = \rho(X) = \rho(X + \alpha)$. Recall that $\varphi, \hat{\varphi}$ map each gap of their respective rotation set to a single point. Let $\varphi(I_i) = \{t_i\}$ and $\hat{\varphi}(I_i + \alpha) = \{\hat{t}_i\}$. Then $\varphi(J_i) = \{t_i - \theta\}$ and $\hat{\varphi}(\hat{J}_i) = \{\hat{t}_i - \theta\}$. Since $X + \alpha$ is a rotation of X and since $\varphi, \hat{\varphi}$ are order-preserving, there is an orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ which maps t_i to $\hat{t}_i$ for every i and maps the set $\{t_1 - \theta, \ldots, t_n - \theta\}$ onto the set $\{\hat{t}_1 - \theta, \ldots, \hat{t}_n - \theta\}$. The claim $\hat{J}_i = J_i + \alpha$ is then equivalent to $h(t_i - \theta) = \hat{t}_i - \theta$. This is proved in the following

Lemma 4.3 *Suppose $t_1, \ldots, t_n \in \mathbb{T}$ are distinct and $h : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism which maps the set $\{t_1 - \theta, \ldots, t_n - \theta\}$ onto the set $\{h(t_1)-\theta, \ldots, h(t_n)-\theta\}$ for some θ. Then $h(t_i-\theta) = h(t_i)-\theta$ for every $1 \le i \le n$.*

Proof The assumption means that the commutator $[r_\theta, h^{-1}] = r_\theta \circ h^{-1} \circ r_\theta^{-1} \circ h$ preserves the finite set $\{t_1, \ldots, t_n\}$ and therefore has a well-defined combinatorial rotation number on it, which coincides with the Poincaré rotation number $\rho([r_\theta, h^{-1}])$. By Corollary 1.10, $\rho([r_\theta, h^{-1}]) = -\rho([h^{-1}, r_\theta]) = 0$. It follows that $[r_\theta, h^{-1}]$ acts as the identity on $\{t_1, \ldots, t_n\}$. □

Back to the proof of the theorem, we now know that $\hat{J}_i = J_i + \alpha$ for every i. Let $J_1 = (t, s)$. Then, on the one hand, $I_1 = (dt, ds)$ so $I_1 + \alpha = (dt + \alpha, ds + \alpha)$. On the other hand, $\hat{J}_1 = J_1 + \alpha = (t + \alpha, s + \alpha)$ so $I_1 + \alpha = (dt + d\alpha, ds + d\alpha)$. It follows that $d\alpha = \alpha \pmod{\mathbb{Z}}$, or $\alpha = i/(d-1)$ for some $0 \le i \le d-2$, as required. □

Remark 4.4 The crucial point in the above proof was to use the assumption $\rho(X) = \rho(X+\alpha)$ to show that $r_\alpha \circ m_d = m_d \circ r_\alpha$ holds at some point of X, hence everywhere on the circle.

The following is an immediate corollary of Theorem 4.2:

Corollary 4.5 *For every rotation set X for m_d, the symmetry group $\{\alpha \in \mathbb{T} : X = X + \alpha \pmod{\mathbb{Z}}\}$ is a subgroup of $\mathbb{Z}/(d-1)\mathbb{Z}$.*

4.2 Realizing Gap Graphs and Gap Lengths

As an application of Theorem 3.20, we give a partial answer to the question of realizing admissible graphs as gap graphs that was raised at the end of Sect. 2.1.

Theorem 4.6 *Given an irrational number θ and an admissible graph Γ of degree d without closed paths, there exists a (minimal) rotation set X for m_d with $\rho(X) = \theta$ whose gap graph Γ_X is isomorphic to Γ.*

Proof Suppose Γ consists of α degree 0 vertices of weights $n_1, \ldots, n_\alpha$ and β maximal paths $P_1, \ldots, P_\beta$ of total weights $n_{\alpha+1}, \ldots, n_{\alpha+\beta}$ (thus, for every $\alpha + 1 \le i \le \alpha + \beta$, the number n_i is the sum of the weights of the vertices in the path $P_{i-\alpha}$). Then $\sum_{i=1}^{\alpha+\beta} n_i = d - 1$.

Choose $\alpha + \beta$ distinct points $s_1 = 0, s_2, \ldots, s_{\alpha+\beta}$ on $\mathbb{T}$ subject only to the condition that their full orbits under the rotation r_θ are disjoint. We use the s_i to produce a list L of $d - 1$ not necessarily distinct points in $\mathbb{T}$ as follows: For each

$1 \leq i \leq \alpha$, let L include n_i copies of the point s_i. For each $\alpha + 1 \leq i \leq \alpha + \beta$, consider the maximal path $P_{i-\alpha}$ which has the form

$$I_k \to I_{k-1} \to \cdots \to I_1 \quad \text{with} \quad \sum_{j=1}^{k} w(I_j) = n_i, \tag{4.2}$$

and let L include $w(I_j)$ copies of the point $s_i - (j-1)\theta$ for every $1 \leq j \leq k$. Represent points of L by numbers $0 < \sigma_1 \leq \cdots \leq \sigma_{d-2} \leq \sigma_{d-1} = 1$ and let X be the minimal rotation set with $\rho(X) = \theta$ and $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$ given by Theorem 3.20. Recall that under the canonical semiconjugacy of X, each major gap of multiplicity n corresponds to an n-fold incidence $\sigma_i = \cdots = \sigma_{i+n-1}$. Using this and the selection of the list L, it is easy to see that Γ_X is isomorphic to Γ. □

Remark 4.7 The above proof shows that we have the freedom of arbitrarily prescribing the number of iterates it takes to go from each loose vertex of Γ_X to its adjacent vertex. To see this, suppose for each maximal path of Γ of the form (4.2) and each $2 \leq j \leq k$ we are given an integer N_j, which is to be the number of iterates it takes to map I_j to I_{j-1}. Set $N_1 = 0$, modify the list L by including $w(I_j)$ copies of the point $s_i - (N_1 + \cdots + N_j)\,\theta$ for every $1 \leq j \leq k$, and construct the rotation set X as before.

Remark 4.8 We would naturally want to know if every admissible graph can be realized as a gap graph even when it contains closed paths. It may seem at first glance that all closed paths of a realizable graph must have the same length, but this is not the case: Consider the rotation set

$$X = \left\{ \frac{17}{124}, \frac{18}{124}, \frac{23}{124}, \frac{53}{124}, \frac{78}{124}, \frac{79}{124}, \frac{85}{124}, \frac{90}{124}, \frac{115}{124} \right\}$$

under m_5 with $\rho(X) = \frac{2}{3}$, which is a union of three compatible 3-cycles. The cycle of gaps

$$\left(\frac{23}{124}, \frac{53}{124}\right) \mapsto \left(\frac{115}{124}, \frac{17}{124}\right) \mapsto \left(\frac{79}{124}, \frac{85}{124}\right)$$

has two major gaps of multiplicity 1, so it is represented by a closed path of length 2 in Γ_X. However, the cycle of gaps

$$\left(\frac{18}{124}, \frac{23}{124}\right) \mapsto \left(\frac{90}{124}, \frac{115}{124}\right) \mapsto \left(\frac{78}{124}, \frac{79}{124}\right)$$

has only one major gap of multiplicity 1, so it is represented by a closed path of length 1. This example also shows that the total weights around closed paths of Γ_X can be different.

We have already described possible gap lengths for rational rotation sets as the solution (3.4) of some linear equation. Using the above theorem and remark, we can provide a characterization of gap lengths in the irrational case (compare [2] where a similar result is sketched via an inductive argument):

Theorem 4.9 *A number $\ell > 0$ appears as the length of a major gap of an irrational rotation set for m_d if and only if it has the form*

$$\ell = \sum_{j=1}^{k} \frac{\alpha_j}{d^{\beta_j}}, \tag{4.3}$$

where $1 \leq k \leq d-1$ and $\{\alpha_j\}$, $\{\beta_j\}$ are sequences of positive integers which satisfy

$$\sum_{j=1}^{k} \alpha_j \leq d-1 \quad \textit{and} \quad 1 = \beta_1 < \beta_2 < \cdots < \beta_k.$$

Proof First suppose X is an irrational rotation set for m_d with a major gap I of length ℓ and multiplicity n. If I is taut, then $\ell = n/d$, which clearly has the form (4.3). If I is loose, it is represented by a vertex in the gap graph Γ_X that belongs to a path $I = I_k \to I_{k-1} \to \cdots \to I_1$ where I_j has length ℓ_j and multiplicity n_j. For each $2 \leq j \leq k$, there is an integer $N_j \geq 1$ such that $I_{j-1} = g_X^{\circ N_j}(I_j)$. Hence $d^{N_j - 1}(d\,\ell_j - n_j) = \ell_{j-1}$. Since I_1 is taut, $\ell_1 = n_1/d$. Using these relations, we can solve for ℓ_k to obtain

$$\ell = \ell_k = \frac{n_k}{d} + \frac{n_{k-1}}{d^{N_k+1}} + \cdots + \frac{n_1}{d^{N_2+\cdots+N_k+1}},$$

which has the form (4.3).

Conversely, suppose ℓ is a positive number of the form (4.3) for some choice of k, $\{\alpha_j\}$, and $\{\beta_j\}$. Consider the admissible graph Γ of degree d consisting of a single degree 0 vertex of weight $d - 1 - \sum_{j=1}^{k} \alpha_j$, together with a single maximal path of the form

$$I_k \to I_{k-1} \to \cdots \to I_1 \quad \text{with} \quad w(I_j) = \alpha_{k-j+1}.$$

Consider also the positive integers $N_j = \beta_{k-j+2} - \beta_{k-j+1}$ for $2 \leq j \leq k$. By Remark 4.7, there is a minimal irrational rotation set X, with Γ_X isomorphic to Γ, with N_j equal to the number of iterates it takes to map I_j to I_{j-1}. Then, the computation in the first part of the proof shows that the major gap I_k of X has length ℓ. □

Remark 4.10 When $d = 2$, the only possible values for the above integers are $k = \alpha_1 = 1$, confirming what we already know: An irrational rotation set under doubling has a single major gap of length $\frac{1}{2}$. For $d = 3$, there are more possibilities: If $k = 1$, then either $\alpha_1 = 1$ so $\ell = \frac{1}{3}$, or $\alpha_1 = 2$ so $\ell = \frac{2}{3}$. On the other hand, if $k = 2$, then necessarily $\alpha_1 = \alpha_2 = 1$ so $\ell = \frac{1}{3} + \frac{1}{3^{\beta_2}}$ for some $\beta_2 > 1$. Compare Theorem 4.31 for a more precise statement.

4.3 Dependence on Parameters

We begin with a preliminary observation on convergence of rotation sets:

Lemma 4.11 *Suppose $\{X_n\}$ is a sequence of rotation sets for m_d which converges in the Hausdorff metric to a compact set X. Then X is a rotation set with $\rho(X) = \lim_{n\to\infty} \rho(X_n)$. If every X_n is maximal, so is X.*

Proof Since each X_n is m_d-invariant and its complement $\mathbb{T} \smallsetminus X_n$ contains $d-1$ disjoint intervals of length $1/d$, the Hausdorff limit X must have the same properties. By Corollary 2.16, X is a rotation set. The family $\{g_n\}$ of the standard monotone maps of $\{X_n\}$ is equicontinuous since each g_n is piecewise affine with derivative bounded by d. After passing to a subsequence, we may assume that g_n converges uniformly to a degree 1 monotone map $g : \mathbb{T} \to \mathbb{T}$ which necessarily extends $m_d|_X$ (in fact, this shows that the entire sequence $\{g_n\}$ converges and its limit g is the standard monotone map of X). It follows from Theorem 1.11 that $\rho(X) = \rho(g) = \lim_{n\to\infty} \rho(g_n) = \lim_{n\to\infty} \rho(X_n)$.

The last assertion follows from Corollary 2.19: If the X_n are maximal, they all have $d-1$ major gaps of length $1/d$. This property persists under Hausdorff convergence, so X is maximal as well. □

Now, let $A \subset \mathbb{T} \times \Delta^{d-2}$ be the set of all pairs $\boldsymbol{a} = (\theta, \delta)$ subject to the restriction that if θ is rational of the form p/q in lowest terms, then $q\delta \in \mathbb{Z}^{d-1}$. For each $\boldsymbol{a} = (\theta, \delta) \in A$, let $X_{\boldsymbol{a}}$ denote the unique minimal rotation set for m_d with $\rho(X_{\boldsymbol{a}}) = \theta$ and $\delta(X_{\boldsymbol{a}}) = \delta$, given by the deployment theorem.

Theorem 4.12 *The assignment $\boldsymbol{a} \mapsto X_{\boldsymbol{a}}$ from A to the space of compact subsets of the circle (equipped with the Hausdorff metric) is lower semicontinuous.*

Proof Let $\boldsymbol{a}_n = (\theta_n, \delta_n) \in A$ tend to $\boldsymbol{a}_0 = (\theta_0, \delta_0) \in A$ as $n \to \infty$. Suppose $X_{\boldsymbol{a}_n}$ converges in the Hausdorff metric to a compact set $Y \subset \mathbb{T}$. We need to show that $X_{\boldsymbol{a}_0} \subset Y$. By Lemma 4.11, Y is a rotation set for m_d with $\rho(Y) = \theta_0$. Moreover, the proof of that lemma shows that the sequence $\{g_n\}$ of the standard monotone maps of $\{X_{\boldsymbol{a}_n}\}$ converges uniformly to the standard monotone map g of Y.

Let μ_n be the natural measure of $X_{\boldsymbol{a}_n}$, that is, the unique m_d-invariant probability measure supported on $X_{\boldsymbol{a}_n}$. After passing to a subsequence, we may assume that μ_n is weak* convergent to a probability measure μ. For every continuous test function $f : \mathbb{T} \to \mathbb{R}$,

$$\int_{\mathbb{T}} f\, d\mu = \lim_{n\to\infty} \int_{\mathbb{T}} f\, d\mu_n = \lim_{n\to\infty} \int_{\mathbb{T}} (f \circ g_n)\, d\mu_n = \lim_{n\to\infty} \int_{\mathbb{T}} (f \circ g)\, d\mu_n$$
$$= \int_{\mathbb{T}} (f \circ g)\, d\mu.$$

Here the first and forth equalities hold by the weak* convergence $\mu_n \to \mu$, the second equality follows from the g_n-invariance of μ_n, and the third equality holds since the uniform convergence $g_n \to g$ implies $\int (f \circ g_n)\, d\mu_n - \int (f \circ g)\, d\mu_n \to 0$ as $n \to \infty$. This proves that μ is g-invariant. For the rest of the argument, we distinguish two cases:

If $\rho(g) = \theta_0$ is irrational, it follows from the discussion in Sect. 1.5 that μ is the unique invariant probability measure supported on the Cantor attractor K of g. By Theorem 2.33, K is the unique minimal rotation set contained in Y. Let $\delta_n = (\delta_{n,1}, \dots, \delta_{n,d-1})$ and $\delta_0 = (\delta_{0,1}, \dots, \delta_{0,d-1})$. Since $\mu_n \to \mu$ and $\mu\{u_i\} = 0$ (recall that $u_i = i/(d-1)$ are the fixed points of m_d), it follows that

$$\mu[u_{i-1}, u_i) = \lim_{n\to\infty} \mu_n[u_{i-1}, u_i) = \lim_{n\to\infty} \delta_{n,i} = \delta_{0,i} \tag{4.4}$$

for every i, so $\delta(K) = \delta_0$. Since $\rho(K) = \rho(Y) = \theta_0$, the uniqueness part of Theorem 3.20 shows that $K = X_{\boldsymbol{a}_0}$. This proves $X_{\boldsymbol{a}_0} \subset Y$, as required.

In the case $\rho(g) = \theta_0$ is rational of the form p/q in lowest terms, we must modify the above argument. Let K be the support of μ. We know from Sect. 1.5 that K is a union of q-cycles of g. The Hausdorff convergence $\operatorname{supp}(\mu_n) = X_{\boldsymbol{a}_n} \to Y$ together with the weak* convergence $\mu_n \to \mu$ show that $K \subset Y$. It follows that K is a union $C_1 \cup \dots \cup C_n$ of q-cycles in Y and therefore is a finite rotation set with $\rho(K) = p/q$. The measure μ is a convex combination $\sum_{i=1}^n \alpha_i\, \mu_{C_i}$ of the Dirac measures along the C_i, where every α_i is positive and $\sum_{i=1}^n \alpha_i = 1$. Since the limit (4.4) still holds for $1 \le i \le d-1$, we have

$$\sum_{i=1}^n \alpha_i\, \delta(C_i) = \delta_0 = \delta(X_{\boldsymbol{a}_0}).$$

By Lemma 3.18, this can happen only if $n = 1$ and $X_{\boldsymbol{a}_0} = C_1 = K$. Again, this implies $X_{\boldsymbol{a}_0} \subset Y$. □

Recall from Sect. 2.3 that a rotation set is exact if it is both maximal and minimal. Such rotation sets are necessarily irrational. Topologically, they are Cantor sets with $d-1$ major gaps of length $1/d$ (Theorem 2.37). The following lemma characterizes exactness in terms of the cumulative deployment vector:

Lemma 4.13 *Suppose X is a minimal rotation set for m_d with $\rho(X) = \theta$ and $\sigma(X) = (\sigma_1, \dots, \sigma_{d-1})$. Then X is exact if and only if $\sigma_1, \dots, \sigma_{d-1}$ have disjoint full orbits under r_θ.*

Proof Recall from the proof of Theorem 3.20 that the lengths of the major gaps of X are the values $\nu\{\sigma_i\}$, where ν is the gap measure of X defined in (3.20). If the σ_i have disjoint orbits under r_θ, then θ is irrational and the definition of ν shows that $\nu\{\sigma_i\} = 1/d$ for each i. Conversely, a relation of the form $\sigma_i = \sigma_j - k\theta$ for $i \neq j$ and $k \ge 0$ would contribute a mass of $1/d^{k+1}$ to $\nu\{\sigma_i\}$, so the corresponding major gap would have length $\nu\{\sigma_i\} \ge 1/d + 1/d^{k+1} > 1/d$. □

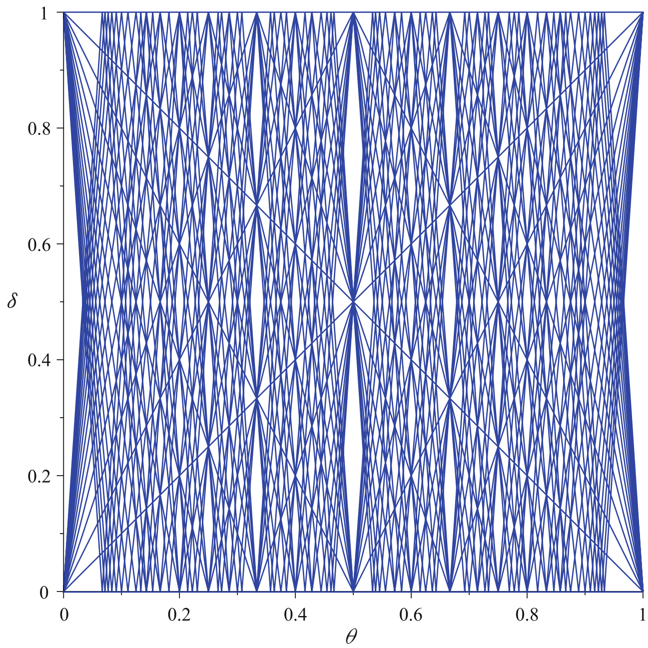

Fig. 4.2 An attempt to visualize the set of parameters $\boldsymbol{a} = (\theta, \delta)$ for which the minimal cubic rotation set $X_{\boldsymbol{a}}$ is exact. Here the deployment vector $(\delta, 1-\delta)$ is identified with its first component $\delta \in [0, 1]$. The set of exact parameters is the complement of the union of the lines $\delta + n\theta = 0 \pmod{\mathbb{Z}}$ over all $n \in \mathbb{Z}$

Lemma 4.14 *There is a full-measure set of parameters $\boldsymbol{a} \in A$ for which $X_{\boldsymbol{a}}$ is exact.*

Figure 4.2 is an attempt to visualize this set when $d = 3$.

Proof Take any $\delta = (\delta_1, \ldots, \delta_{d-1})$ in the interior of Δ^{d-2}. Then the δ_i are positive, so the numbers $\sigma_i = \delta_1 + \cdots + \delta_i$ for $1 \leq i \leq d-1$ are distinct. There are countably many θ for which the orbits of the σ_i under r_θ collide. Let H_δ be the complement of this countable set in $\mathbb{T}$. Then, the union

$$H = \bigcup_{\delta} \left(H_\delta \times \{\delta\} \right)$$

has full-measure and for every $\boldsymbol{a} \in H$ the rotation set $X_{\boldsymbol{a}}$ is exact by Lemma 4.13. □

The following theorem determines when a minimal rotation set depends continuously on its rotation number and deployment vector. The possibility of such characterization was suggested to me by J. Milnor:

Theorem 4.15 *The assignment $\boldsymbol{a} \mapsto X_{\boldsymbol{a}}$ is continuous at $\boldsymbol{a}_0 \in A$ if and only if $X_{\boldsymbol{a}_0}$ is exact.*

In particular, this assignment is discontinuous at every $\boldsymbol{a}_0$ for which $X_{\boldsymbol{a}_0}$ is rational.

Proof First assume $\boldsymbol{a} \mapsto X_{\boldsymbol{a}}$ is continuous at $\boldsymbol{a}_0 \in A$. By Lemma 4.14 we can choose a sequence $\boldsymbol{a}_n \in A$ converging to $\boldsymbol{a}_0$ such that $X_{\boldsymbol{a}_n}$ is exact for every n. Since $X_{\boldsymbol{a}_n} \to X_{\boldsymbol{a}_0}$ and each $X_{\boldsymbol{a}_n}$ is maximal, Lemma 4.11 shows that $X_{\boldsymbol{a}_0}$ is maximal. As $X_{\boldsymbol{a}_0}$ is minimal by definition, we conclude that $X_{\boldsymbol{a}_0}$ is exact.

Conversely, suppose $X_{\boldsymbol{a}_0}$ is exact and take any sequence $\boldsymbol{a}_n \in A$ converging to $\boldsymbol{a}_0$. After passing to a subsequence, we may assume that $X_{\boldsymbol{a}_n}$ converges to a compact set Y in the Hausdorff metric. Theorem 4.12 shows that $Y \supset X_{\boldsymbol{a}_0}$. Since Y is a rotation set by Lemma 4.11, it follows from exactness that $Y = X_{\boldsymbol{a}_0}$. □

Example 4.16 Minimal rotation sets under the doubling map m_2 are parametrized by their rotation number. The assignment $\theta \mapsto X_\theta$ is continuous at every irrational θ since such rotation sets are exact (Corollary 2.38). To get a feel for the nature of discontinuity at rational θ, consider the n-cycle

$$X_{1/n} : \frac{1}{2^n - 1} \mapsto \frac{2}{2^n - 1} \mapsto \cdots \mapsto \frac{2^{n-1}}{2^n - 1}.$$

As $n \to \infty$, $X_{1/n}$ does not converge to $X_0 = \{0\}$, but to the maximal rotation set $\{0\} \cup \{1/2^n\}_{n \geq 1}$.

Remark 4.17 Milnor has pointed out to me that one may also study the map from the union $\mathcal{R}_d$ of all rotation sets for m_d to the set A defined as follows: The forward m_d-orbit of every $t \in \mathcal{R}_d$ eventually lands in a well-defined minimal rotation set X_t (Theorems 2.3 and 2.33), so we can assign to t the parameter $(\rho(X_t), \delta(X_t)) \in A$. This map is surjective and clearly discontinuous since $\mathcal{R}_d$ is compact (see below) but A is not.

Let $\mathcal{C}_d \subset \mathcal{R}_d$ be the union of all cycles, and $\mathcal{E}_d \subset \mathcal{R}_d$ be the union of all exact rotation sets.

Theorem 4.18

(i) $\mathcal{R}_d$ is compact.
(ii) $\mathcal{C}_d$ and $\mathcal{E}_d$ are disjoint and non-compact, with $\overline{\mathcal{E}_d} \subset \overline{\mathcal{C}_d}$.
(iii) $\overline{\mathcal{E}_d}$ is a Cantor set.

Proof Let $t_n \in \mathcal{R}_d$ and $t_n \to t$. Take a rotation set X_n containing t_n. After passing to a subsequence, we may assume that X_n converges to a compact set X, which is a rotation set by Lemma 4.11. Hence $t \in X \subset \mathcal{R}_d$. This proves (i).

For (ii), first note that $\mathcal{C}_d$ and $\mathcal{E}_d$ are disjoint since rational rotation sets are never exact. To see $\mathcal{E}_d$ is non-compact, take any sequence $\{X_n\}$ of exact rotation sets with

$\rho(X_n)$ tending to some rational number p/q (for example, let $\boldsymbol{a}_n = (\theta_n, \delta_n)$ where θ_n are irrational tending to p/q and δ_n have rational components, and consider the rotation sets $X_{\boldsymbol{a}_n}$ which are exact by Lemma 4.13). Some subsequence of $\{X_n\}$ converges to a compact set X which, by Lemma 4.11, is a (maximal) rotation set with $\rho(X) = p/q$. Evidently $X \subset \overline{\mathcal{E}_d}$. However, $X \cap \mathcal{E}_d = \emptyset$ since the forward orbit of any $t \in X$ eventually hits a cycle, so t cannot belong to an exact rotation set.

Now suppose X is exact and choose cycles C_n such that $\rho(C_n) \to \rho(X)$ and $\delta(C_n) \to \delta(X)$. Theorem 4.15 then shows that $C_n \to X$, so $X \subset \overline{\mathcal{C}_d}$. This proves the inclusion $\overline{\mathcal{E}_d} \subset \overline{\mathcal{C}_d}$ and also shows that $\mathcal{C}_d$ is non-compact.

For (iii), simply note that $\overline{\mathcal{E}_d}$ has no isolated point since it is the closure of a union of Cantor sets, and it is totally disconnected since it is contained in the measure zero set $\mathcal{R}_d$ (Theorem 2.5). □

Question 4.19 Does the equality $\overline{\mathcal{E}_d} = \overline{\mathcal{C}_d} = \mathcal{R}_d$ hold?

The answer is affirmative when $d = 2$ (see Theorem 4.28) and is likely to be so for all d. Indeed, the following sharper statement seems plausible: Given any maximal rotation set X for m_d there is a sequence $\{X_n\}$ of exact rotation sets for m_d such that $X_n \to X$ in the Hausdorff metric.

4.4 The Leading Angle

A minimal rotation set X is uniquely determined by any of its elements: Simply iterate any angle in X under m_d and take the closure of the resulting orbit. This section will give a recipe for computing a canonical angle in every minimal rotation set from the knowledge of its rotation number and deployment vector. The particular choice of this angle is motivated by polynomial dynamics and plays a role in the representation of rotation sets in both dynamical and parameter planes, as outlined in the next chapter.

Definition 4.20 Let X be a minimal rotation set for m_d and $I_0 = (\omega', \omega)$ be its major gap containing the fixed point 0. We call the endpoint ω of I_0 the ***leading angle*** of X.

Thus, ω is the first point of X that is met when we start at 0 and go counter-clockwise around the circle. The closed intervals $[\omega', \omega]$ and $[m_d(\omega'), m_d(\omega)]$ can be described as the fibers $\varphi^{-1}(0)$ and $\varphi^{-1}(\theta)$ of the canonical semiconjugacy φ of X, where $\theta = \rho(X)$. For convenience we identify ω', ω and their images with the representatives which satisfy the order relations $-1 < \omega' < 0 < \omega < m_d(\omega') \leq m_d(\omega) < 1$ (see Fig. 4.3).

Suppose $\rho(X) = \theta \neq 0$ and $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$. Let ν be the gap measure of X as defined in (3.20) and $N_0 \geq 0$ be the number of indices $1 \leq j < d-1$ for which $\sigma_j = 0$.

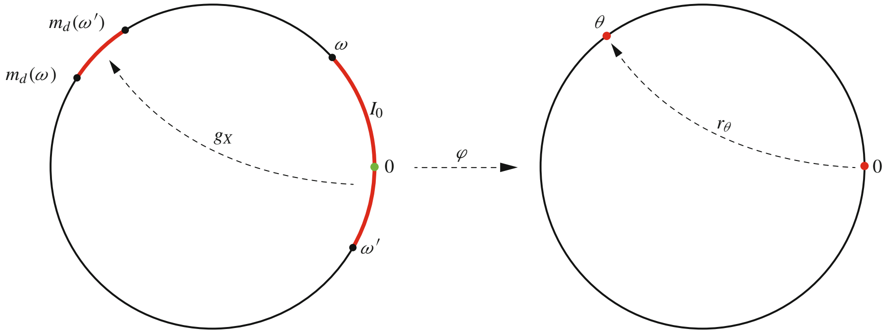

Fig. 4.3 The major gap $I_0 = (\omega', \omega)$ containing the fixed point $u_0 = 0$ and the leading angle ω of a minimal rotation set X. The closed intervals $[\omega', \omega]$ and $[m_d(\omega'), m_d(\omega)]$ are the fibers $\varphi^{-1}(0)$ and $\varphi^{-1}(\theta)$, respectively. Here φ is the canonical semiconjugacy of X and $\theta = \rho(X)$

Theorem 4.21 *The leading angle of X is given by*

$$\begin{aligned}\omega &= \frac{1}{d-1}\,\nu(0,\theta] + \frac{N_0}{d-1} \\ &= \frac{1}{d-1}\sum_{i=1}^{d-1}\sum_{0<\sigma_i-k\theta\leq\theta}\frac{1}{d^{k+1}} + \frac{N_0}{d-1}\end{aligned} \tag{4.5}$$

This formula gives an explicit algorithm for computing the base d expansion of the angle $(d-1)\omega$ (compare Lemma 4.24 below).

Proof As pointed out in the beginning of Sect. 3.2, if $N_1 \geq 1$ is the number of indices $1 \leq j \leq d-1$ for which $\sigma_j = 1$, then $n_0 = N_0 + N_1$ is the multiplicity of $I_0 = (\omega', \omega)$ as a major gap of X. Since

$$\omega' < \frac{-N_1+1}{d-1} \leq \frac{-N_1+1}{d} \leq 0 \leq \frac{N_0}{d} \leq \frac{N_0}{d-1} < \omega,$$

the gap I_0 already contains the n_0 points j/d for $j = -N_1+1, \ldots, N_0$. By Lemma 2.8, there could be no more preimages of 0 in I_0. In particular, $\omega < (N_0+1)/d$, which proves N_0 is the integer part of $d\,\omega$. Since $m_d(\omega) = d\omega \pmod{\mathbb{Z}}$, it follows that

$$m_d(\omega) = d\omega - N_0. \tag{4.6}$$

Now let $\varphi : \mathbb{T} \to \mathbb{T}$ be the canonical semiconjugacy of X and λ be Lebesgue measure on the circle. Since $\varphi_*\lambda = \nu$, we have

$$m_d(\omega) - \omega = \lambda(\omega, m_d(\omega)] = \lambda(\varphi^{-1}(0,\theta]) = \nu(0,\theta]. \tag{4.7}$$

The result follows by eliminating $m_d(\omega)$ from (4.6) and (4.7). $\square$

Remark 4.22 A similar argument gives the following formulas for the other angles involved in Fig. 4.3:

$$m_d(\omega) = \frac{d}{d-1}\,\nu(0,\theta] + \frac{N_0}{d-1}$$

$$\omega' = \frac{1}{d-1}\,\nu[0,\theta) - \frac{N_1}{d-1}$$

$$m_d(\omega') = \frac{d}{d-1}\,\nu[0,\theta) - \frac{N_1}{d-1}.$$

We point out that the above formulas for ω, ω' can be used to compute the endpoint angles of any major gap of X. For example, if I_i is the major gap of X containing the fixed point $u_i = i/(d-1) \pmod{\mathbb{Z}}$, consider the rotation set $X - u_i$ whose deployment vector is obtained by a cyclic permutation of the components of $\delta(X)$ (see Sect. 3.1), apply the above formulas to compute the endpoints of the major gap of $X - u_i$ containing 0, and rotate them back by r_{u_i} to find the endpoints of I_i.

4.5 Rotation Sets Under Doubling

In this section we focus on the basic case $d = 2$. Theorems 3.7 and 3.20 show that for every $0 < \theta < 1$ there is a unique minimal rotation set X_θ under doubling with rotation number θ, which is a periodic orbit if θ is rational and a Cantor set if θ is irrational. The structure of X_θ in either case can be explicitly described as follows.

Let us first consider the rational case. For every fraction p/q in lowest terms, $X_{p/q}$ is a q-cycle of the form $\{t_1, \ldots, t_q\}$, where as usual the points are labeled in positive cyclic order and $0 \in (t_q, t_1)$, and the subscripts are taken modulo q. Let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$. We can compute the ℓ_j explicitly using the general formulas we developed in Sect. 3.2. Recall that $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$ and $\boldsymbol{n} = (n_1, \ldots, n_q)$ are the gap length and gap multiplicity vectors of $X_{p/q}$, respectively. Since $I_q = I_0$ is the unique major gap of $X_{p/q}$ of multiplicity 1, we have $n_q = 1$ and $n_j = 0$ for $1 \leq j < q$. According to (3.4),

$$\boldsymbol{\ell} = \frac{1}{2^q - 1} \sum_{i=0}^{q-1} 2^{q-i-1}\, T^{\circ i}(\boldsymbol{n}),$$

where $T(x_1, x_2, \ldots, x_q) = (x_{1+p}, x_{2+p}, \ldots, x_{q+p})$. Since $T^{\circ i}(\boldsymbol{n}) = (n_{1+ip}, n_{2+ip}, \ldots, n_{q+ip})$, it follows that $\ell_j = 2^{q-i-1}/(2^q - 1)$, where $0 \leq i \leq q-1$ is the unique solution of $j + ip = 0 \pmod q$. If $1 \leq p^* \leq q-1$ is the multiplicative inverse of p modulo q, it follows that $q - i = jp^* \pmod q$. Thus,

$$\ell_j = \frac{2^{\langle jp^* \rangle - 1}}{2^q - 1}, \quad \text{where } 1 \leq \langle jp^* \rangle \leq q \text{ is the unique representative of } jp^* \pmod q.$$

In particular, I_p and $I_q = I_0$ are the shortest and longest gaps of lengths

$$\ell_p = \frac{1}{2^q - 1} \quad \text{and} \quad \ell_q = \frac{2^{q-1}}{2^q - 1}.$$

By (4.5), the leading angle $\omega = t_1$ is given by

$$\omega = \nu\Big(0, \frac{p}{q}\Big] = \ell_1 + \cdots + \ell_p = \sum_{j=1}^{p} \frac{2^{\langle jp^* \rangle - 1}}{2^q - 1}. \tag{4.8}$$

Example 4.23 Consider the 7-cycle $X_{\frac{3}{7}} = \{t_1, t_2, \ldots, t_7\}$ under doubling. Here $q = 7$, $p = 3$ and $p^* = 5$. By the above computation, the gap lengths are

$$\ell_1 = \frac{2^{\langle 5 \rangle - 1}}{127} = \frac{16}{127} \qquad \ell_2 = \frac{2^{\langle 10 \rangle - 1}}{127} = \frac{4}{127}$$
$$\ell_3 = \frac{2^{\langle 15 \rangle - 1}}{127} = \frac{1}{127} \qquad \ell_4 = \frac{2^{\langle 20 \rangle - 1}}{127} = \frac{32}{127}$$
$$\ell_5 = \frac{2^{\langle 25 \rangle - 1}}{127} = \frac{8}{127} \qquad \ell_6 = \frac{2^{\langle 30 \rangle - 1}}{127} = \frac{2}{127}$$
$$\ell_7 = \frac{2^{\langle 35 \rangle - 1}}{127} = \frac{64}{127}.$$

(Alternatively, we could start with the minimal gap length $\ell_3 = \frac{1}{127}$ and keep doubling it until all ℓ_j are found.) The leading angle t_1 is $\ell_1 + \ell_2 + \ell_3 = \frac{21}{127}$, which, in view of the relation $t_{j+3} = 2t_j \pmod{\mathbb{Z}}$, leads to the other angles t_j:

$$t_4 = \frac{42}{127}, \quad t_7 = \frac{84}{127}, \quad t_3 = \frac{41}{127}, \quad t_6 = \frac{82}{127}, \quad t_2 = \frac{37}{127}, \quad t_5 = \frac{74}{127}.$$

Thus,

$$X_{\frac{3}{7}} = \left\{ \frac{21}{127}, \frac{37}{127}, \frac{41}{127}, \frac{42}{127}, \frac{74}{127}, \frac{82}{127}, \frac{84}{127} \right\}.$$

When θ is irrational, the unique major gap I_0 of X_θ is taut, so it has length $\frac{1}{2}$. For every $n \geq 1$ there is a unique gap of length $\frac{1}{2^{n+1}}$ which maps to I_0 after n iterates. The rotation number θ determines the cyclic order of these gaps around the circle.

Now consider the leading angle $\omega(\theta)$ of X_θ as defined in the previous section. The cumulative deployment vector of X_θ is the trivial vector $(\sigma_1) = (1)$. Hence the formula (4.5) takes the form

$$\omega(\theta) = \nu(0, \theta] = \sum_{0 < -k\theta \leq \theta} \frac{1}{2^{k+1}}. \tag{4.9}$$

If θ is rational of the form p/q in lowest terms, this sum splits into p geometric series, each taken over all $k \geq 0$ for which $-kp/q = j/q \pmod{\mathbb{Z}}$ for a given $1 \leq j \leq p$. These p series in effect correspond to the p terms of the sum (4.8). Table 4.1 illustrates the computation of $\omega(p/q)$ using both formulas for all reduces fractions with denominators up to 8.

Equation (4.9) can be interpreted as a formula for the binary expansion of the leading angle $\omega(\theta)$. Consider the intervals

$$T_0 = [0, 1 - \theta) \qquad T_1 = [1 - \theta, 1)$$

on the circle. The binary expansion of $\omega(\theta)$ is obtained using the itinerary of the orbit of 0 under the rotation r_θ relative to the partition $T_0 \cup T_1$:

Lemma 4.24 *The binary expansion*

$$\omega(\theta) = 0.b_0 b_1 b_2 \cdots \qquad (\text{base } 2)$$

is determined by the condition $k\theta \in T_{b_k}$ for all $k \geq 0$.

Note in particular that always $b_0 = 0$.

Proof By (4.5), $b_k = 1$ if and only if $-k\theta \in (0, \theta]$, which is equivalent to $k\theta \in [1 - \theta, 1)$. □

We will see a dynamical interpretation of this lemma in the next chapter (see Sect. 5.3).

The following lemma provides yet another formula for the leading angle ω which already appears in Douady-Hubbard's work on the dynamics of the quadratic family and the Mandelbrot set. Although this formula is not computationally as efficient as (4.9), it greatly facilitates the study of the dependence of $\omega(\theta)$ on the rotation number θ:

Lemma 4.25 *The leading angle of X_θ satisfies*

$$\omega(\theta) = \frac{1}{2} \sum_{0 < p/q \leq \theta} \frac{1}{2^q - 1}, \tag{4.10}$$

where the fractions p/q in the sum are all reduced.

Table 4.1 The leading angle $\omega(p/q)$ of the cycle $X_{p/q}$ under the doubling map, for denominators $2 \le q \le 8$

p/q	Formula (4.8)	Formula (4.9)	$\omega(p/q)$
$\frac{1}{2}$	$\frac{2^0}{2^2-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{2j}}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{2^0}{2^3-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{3j}}$	$\frac{1}{7}$
$\frac{2}{3}$	$\frac{2^1+2^0}{2^3-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{3j-1}} + \frac{1}{2^{3j}}\right)$	$\frac{3}{7}$
$\frac{1}{4}$	$\frac{2^0}{2^4-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{4j}}$	$\frac{1}{15}$
$\frac{3}{4}$	$\frac{2^2+2^1+2^0}{2^4-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{4j-2}} + \frac{1}{2^{4j-1}} + \frac{1}{2^{4j}}\right)$	$\frac{7}{15}$
$\frac{1}{5}$	$\frac{2^0}{2^5-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{5j}}$	$\frac{1}{31}$
$\frac{2}{5}$	$\frac{2^2+2^0}{2^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{5j-2}} + \frac{1}{2^{5j}}\right)$	$\frac{5}{31}$
$\frac{3}{5}$	$\frac{2^1+2^3+2^0}{2^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{5j-1}} + \frac{1}{2^{5j-3}} + \frac{1}{2^{5j}}\right)$	$\frac{11}{31}$
$\frac{4}{5}$	$\frac{2^3+2^2+2^1+2^0}{2^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{5j-3}} + \frac{1}{2^{5j-2}} + \frac{1}{2^{5j-1}} + \frac{1}{2^{5j}}\right)$	$\frac{15}{31}$
$\frac{1}{6}$	$\frac{2^0}{2^6-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{6j}}$	$\frac{1}{63}$
$\frac{5}{6}$	$\frac{2^4+2^3+2^2+2^1+2^0}{2^6-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{6j-4}} + \frac{1}{2^{6j-3}} + \frac{1}{2^{6j-2}} + \frac{1}{2^{6j-1}} + \frac{1}{2^{6j}}\right)$	$\frac{31}{63}$
$\frac{1}{7}$	$\frac{2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{7j}}$	$\frac{1}{127}$
$\frac{2}{7}$	$\frac{2^3+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-3}} + \frac{1}{2^{7j}}\right)$	$\frac{9}{127}$
$\frac{3}{7}$	$\frac{2^4+2^2+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-4}} + \frac{1}{2^{7j-2}} + \frac{1}{2^{7j}}\right)$	$\frac{21}{127}$
$\frac{4}{7}$	$\frac{2^1+2^3+2^5+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-1}} + \frac{1}{2^{7j-3}} + \frac{1}{2^{7j-5}} + \frac{1}{2^{7j}}\right)$	$\frac{43}{127}$
$\frac{5}{7}$	$\frac{2^2+2^5+2^1+2^4+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-2}} + \frac{1}{2^{7j-5}} + \frac{1}{2^{7j-1}} + \frac{1}{2^{7j-4}} + \frac{1}{2^{7j}}\right)$	$\frac{55}{127}$
$\frac{6}{7}$	$\frac{2^5+2^4+2^3+2^2+2^1+2^0}{2^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{7j-5}} + \frac{1}{2^{7j-4}} + \frac{1}{2^{7j-3}} + \frac{1}{2^{7j-2}} + \frac{1}{2^{7j-1}} + \frac{1}{2^{7j}}\right)$	$\frac{63}{127}$
$\frac{1}{8}$	$\frac{2^0}{2^8-1}$	$\sum_{j=1}^{\infty} \frac{1}{2^{8j}}$	$\frac{1}{255}$
$\frac{3}{8}$	$\frac{2^2+2^5+2^0}{2^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{8j-2}} + \frac{1}{2^{8j-5}} + \frac{1}{2^{8j}}\right)$	$\frac{37}{255}$
$\frac{5}{8}$	$\frac{2^4+2^1+2^6+2^3+2^0}{2^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{8j-4}} + \frac{1}{2^{8j-1}} + \frac{1}{2^{8j-6}} + \frac{1}{2^{8j-3}} + \frac{1}{2^{8j}}\right)$	$\frac{91}{255}$
$\frac{7}{8}$	$\frac{2^6+2^5+2^4+2^3+2^2+2^1+2^0}{2^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{2^{8j-6}} + \frac{1}{2^{8j-5}} + \frac{1}{2^{8j-4}} + \frac{1}{2^{8j-3}} + \frac{1}{2^{8j-2}} + \frac{1}{2^{8j-1}} + \frac{1}{2^{8j}}\right)$	$\frac{127}{255}$

Proof For each integer $m \geq 1$, let k_m be the largest positive integer for which $m > k_m\theta$. Then $0 < m - k_m\theta \leq \theta$, so $-k_m\theta \pmod{\mathbb{Z}}$ is in the interval $(0, \theta]$. Conversely, if $-k\theta \pmod{\mathbb{Z}}$ belongs to $(0, \theta]$, there exists an integer $m \geq 1$ such that $0 < m - k\theta \leq \theta$, so $k\theta < m \leq (k+1)\theta$, which shows $k = k_m$. Thus, by (4.9),

$$\omega(\theta) = \sum_{m=1}^{\infty} \frac{1}{2^{k_m+1}}.$$

To relate this sum to (4.10), we use an idea of Douady (compare [12] and [7]). Assign to each pair (n, m) of positive integers the weight $W(n, m) = 1/2^n$. Let W be the total weight of all (n, m) for which $m/n \leq \theta$. On the one hand,

$$W = \sum_{m=1}^{\infty} \sum_{n=k_m+1}^{\infty} W(n, m) = \sum_{m=1}^{\infty} \sum_{n=k_m+1}^{\infty} \frac{1}{2^n} = \sum_{m=1}^{\infty} \frac{1}{2^{k_m}} = 2\omega(\theta).$$

On the other hand, computing the total weight along lines with rational slope gives

$$W = \sum_{0<p/q\leq\theta} \sum_{j=1}^{\infty} W(jq, jp) = \sum_{0<p/q\leq\theta} \sum_{j=1}^{\infty} \frac{1}{2^{jq}} = \sum_{0<p/q\leq\theta} \frac{1}{2^q - 1},$$

and the result follows. □

Corollary 4.26 *The leading angle $\omega(\theta)$ of X_θ is a strictly increasing function of $0 < \theta < 1$, with $\omega(0^+) = 0$ and $\omega(1^-) = \frac{1}{2}$. Moreover,*

(i) ω has a jump discontinuity at every rational value of θ. In fact, if $\theta = p/q$ in lowest terms, then

$$\omega(p/q) = \omega(p/q^+) = \omega(p/q^-) + \frac{1}{2(2^q - 1)}.$$

(ii) ω is continuous at every irrational value of θ.

(iii) For every $0 < \theta < 1$,

$$\omega(\theta^+) + \omega((1-\theta)^-) = \frac{1}{2}.$$

Compare Fig. 4.4. There is a well-known connection between the function $\theta \mapsto \omega(\theta)$ and the quadratic family $\{z \mapsto z^2 + c\}_{c\in\mathbb{C}}$ (see Sect. 5.3).

Proof Only (iii) needs a comment, as other properties follow at once from (4.10). For $0 < \theta < 1$,

$$\begin{aligned}\omega(\theta^+) = \omega(\theta) &= \sum_{0<p/q\leq\theta} \frac{1}{2^q - 1} \\ &= \frac{1}{2} - \sum_{\theta<p/q<1} \frac{1}{2^q - 1}\end{aligned}$$

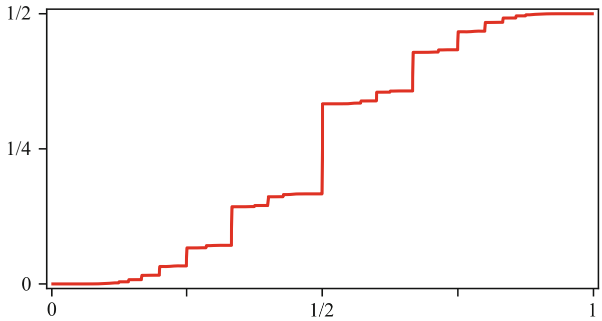

Fig. 4.4 The graph of the leading angle $\omega(\theta)$ of the minimal rotation set X_θ under doubling, as a function of the rotation number θ. Notice the jump discontinuities at every rational value of θ and the symmetry of the graph around the center point $(\frac{1}{2}, \frac{1}{4})$

$$= \frac{1}{2} - \sum_{0<(q-p)/q<1-\theta} \frac{1}{2^q - 1}$$

$$= \frac{1}{2} - \omega((1-\theta)^-),$$

as required. □

Remark 4.27 It follows from Corollary 4.26 that the map $\theta \mapsto \omega(\theta)$ has a left-inverse $\omega \mapsto \theta(\omega)$ which maps $(0, \frac{1}{2})$ monotonically onto $(0, 1)$ and has non-degenerate fibers over every rational value of θ. It is not hard to check that $\theta(\omega)$ is the rotation number of the rotation set consisting of all points in $\mathbb{T}$ whose forward orbit under doubling is contained in the closed half-circle $[\omega, \omega + \frac{1}{2}]$ (see Theorem 2.15).

The behavior of $\theta \mapsto \omega(\theta)$ makes it possible to answer Question 4.19 when $d = 2$:

Theorem 4.28 *For every maximal rotation set X under doubling there is a sequence $\{X_{\theta_n}\}$ of exact rotation sets such that $X_{\theta_n} \to X$ in the Hausdorff metric. In particular, $\overline{\mathcal{E}_2} = \overline{\mathcal{C}_2} = \mathcal{R}_2$.*

Proof If $\rho(X)$ is irrational, then X itself is exact (Corollary 2.38) and there is nothing to prove. If $\rho(X)$ is rational of the form p/q in lowest terms, then X contains the cycle $X_{p/q}$ with the major gap

$$\Big(\omega(p/q) - \frac{2^{q-1}}{2^q - 1}, \omega(p/q)\Big) = \Big(\omega(p/q^-) - \frac{1}{2}, \omega(p/q^+)\Big).$$

Corollary 2.31 then shows that the major gap of X is one of the intervals

$$I = \Big(\omega(p/q^+) - \frac{1}{2}, \omega(p/q^+)\Big) \quad \text{or} \quad J = \Big(\omega(p/q^-) - \frac{1}{2}, \omega(p/q^-)\Big).$$

Suppose the major gap of X is I. Take a decreasing sequence $\{\theta_n\}$ of irrational numbers with $\theta_n \to p/q$. The rotation sets X_{θ_n} are exact and their leading angles $\omega(\theta_n)$ tend to $\omega(p/q^+)$. By Lemma 4.11, any Hausdorff limit of $\{X_{\theta_n}\}$ is a maximal rotation set with rotation number p/q and major gap I, so it must be X. It follows that $X_{\theta_n} \to X$. If the major gap of X is J, take an increasing sequence $\{\theta_n\}$ of irrationals with $\theta_n \to p/q$, which now has the property $\omega(\theta_n) \to \omega(p/q^-)$, and conclude similarly that $X_{\theta_n} \to X$. □

4.6 Rotation Sets Under Tripling

We now consider the case $d = 3$. Theorems 3.7 and 3.20 show that for every $0 < \theta < 1$ and every $0 \leq \delta \leq 1$ there is a unique minimal rotation set $X_{\theta,\delta}$ under tripling with rotation number θ and deployment vector $(\delta, 1-\delta)$, which is a periodic orbit if θ is rational and a Cantor set if θ is irrational. Notice that changing δ to $1-\delta$ amounts to rotating $X_{\theta,\delta}$ by 180°:

$$X_{\theta,1-\delta} = X_{\theta,\delta} + \frac{1}{2}.$$

This means that to study the structure of $X_{\theta,\delta}$ we may restrict δ to either of the intervals $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$.

First suppose $\theta = p/q$ in lowest terms, so δ is of the form s/q for some $0 \leq s \leq q$. Then $X_{p/q,s/q}$ is a q-cycle of the form $\{t_1, \ldots, t_q\}$, where the points are labeled in positive cyclic order and $0 \in (t_q, t_1)$. As before, let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$, and let $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$ and $\boldsymbol{n} = (n_1, \ldots, n_q)$ be the gap length and gap multiplicity vectors of $X_{p/q,s/q}$, respectively. The two major gaps of $X_{p/q,s/q}$ are $I_q = I_0$ and I_s containing the fixed points 0 and $\frac{1}{2}$ of m_3, respectively. We distinguish three cases:

- *Case 1.* $s = q$. The cumulative deployment vector in this case is $(\sigma_1, \sigma_2) = (1, 1)$. Evidently $I_q = I_s$ is the unique major gap of multiplicity 2, so $n_q = 2$ and $n_j = 0$ for $1 \leq j < q$. This case turns out to be completely similar to the doubling case treated in the previous section. A similar computation gives

$$\ell_j = \frac{2 \cdot 3^{\langle jp^* \rangle - 1}}{3^q - 1}, \qquad \text{where } 1 \leq \langle jp^* \rangle \leq q \text{ is the unique representative of } jp^* \pmod{q}.$$

In particular, I_p and I_q are the shortest and longest gaps of lengths

$$\ell_p = \frac{2}{3^q - 1} \quad \text{and} \quad \ell_q = \frac{2 \cdot 3^{q-1}}{3^q - 1}.$$

By (4.5) the leading angle $\omega = t_1$ is given by

$$\omega = \frac{1}{2}\,\nu\Big(0, \frac{p}{q}\Big] = \frac{1}{2}(\ell_1 + \cdots + \ell_p) = \sum_{j=1}^{p} \frac{3^{\langle jp^* \rangle - 1}}{3^q - 1} \tag{4.11}$$

which is analogous to the formula (4.8) for the doubling case.

- *Case 2.* $s = 0$. The cumulative deployment vector in this case is $(\sigma_1, \sigma_2) = (0, 1)$. This is similar to *Case 1* and can be reduced to it by a 180° rotation. It easily follows that the gap lengths ℓ_j are given by the same formulas as above. However, the leading angle $\omega = t_1$ is

$$\omega = \frac{1}{2}\,\nu\Big(0, \frac{p}{q}\Big] + \frac{1}{2} = \frac{1}{2}(\ell_1 + \cdots + \ell_p) + \frac{1}{2} = \sum_{j=1}^{p} \frac{3^{\langle jp^* \rangle - 1}}{3^q - 1} + \frac{1}{2}.$$

- *Case 3.* $0 < s < q$. This time I_q and I_s are distinct major gaps of multiplicity 1, so $n_q = n_s = 1$ and $n_j = 0$ for $j \neq q, s$. In this case,

$$\ell_j = \frac{1}{3^q - 1}\Big[3^{\langle jp^* \rangle - 1} + 3^{\langle (j-s)p^* \rangle - 1}\Big].$$

Note that there are now two competing candidates I_p, I_{s+p} for the shortest gap and similarly two candidates I_q, I_s for the longest gap. The choice depends on the relative size of $\langle sp^* \rangle$ and $\langle -sp^* \rangle$. In fact, the above formula shows that if $\langle sp^* \rangle < \langle -sp^* \rangle$, then the minimum and maximum gap lengths are

$$\ell_{s+p} = \frac{3^{\langle sp^* \rangle} + 1}{3^q - 1} \quad \text{and} \quad \ell_q = \frac{3^{q-1} + 3^{\langle -sp^* \rangle - 1}}{3^q - 1},$$

while if $\langle sp^* \rangle > \langle -sp^* \rangle$, the minimum and maximum gap lengths are

$$\ell_p = \frac{3^{\langle -sp^* \rangle} + 1}{3^q - 1} \quad \text{and} \quad \ell_s = \frac{3^{q-1} + 3^{\langle sp^* \rangle - 1}}{3^q - 1}.$$

If $\langle sp^* \rangle = \langle -sp^* \rangle = q/2$ (so q is even), the minimum and maximum gap lengths are

$$\ell_p = \ell_{s+p} = \frac{1}{3^{q/2} - 1} \quad \text{and} \quad \ell_q = \ell_s = \frac{3^{q/2-1}}{3^{q/2} - 1}.$$

Whatever the case, the leading angle $\omega = t_1$ can still be computed as the sum $\omega = (\frac{1}{2})(\ell_1 + \cdots + \ell_p)$ which, in view of the relation $t_{j+p} = 3t_j \pmod{\mathbb{Z}}$, would determine every angle t_j.

Example 4.29 Consider the 5-cycle $X_{\frac{3}{5},\frac{5}{5}} = \{t_1, \ldots, t_5\}$ under tripling. Here $q = 5$, $p = 3$, $p^* = 2$ and $s = 5$. By the computation in *Case 1*, the gap lengths are

$$\ell_1 = \frac{2 \cdot 3^{\langle 2\rangle - 1}}{242} = \frac{6}{242} \quad \ell_2 = \frac{2 \cdot 3^{\langle 4\rangle - 1}}{242} = \frac{54}{242} \quad \ell_3 = \frac{2 \cdot 3^{\langle 6\rangle - 1}}{242} = \frac{2}{242}$$
$$\ell_4 = \frac{2 \cdot 3^{\langle 8\rangle - 1}}{242} = \frac{18}{242} \quad \ell_5 = \frac{2 \cdot 3^{\langle 10\rangle - 1}}{242} = \frac{162}{242}.$$

The leading angle t_1 is $(\frac{1}{2})(\ell_1 + \ell_2 + \ell_3) = \frac{31}{242}$. In view of $t_{j+3} = 3t_j \pmod{\mathbb{Z}}$, we obtain

$$t_4 = \frac{93}{242}, \quad t_2 = \frac{37}{242}, \quad t_5 = \frac{111}{242}, \quad t_3 = \frac{91}{242}.$$

Thus,

$$X_{\frac{3}{5},\frac{5}{5}} = \left\{\frac{31}{242}, \frac{37}{242}, \frac{91}{242}, \frac{93}{242}, \frac{111}{242}\right\}.$$

Example 4.30 Now let us determine the 5-cycle $X_{\frac{3}{5},\frac{2}{5}} = \{t_1, \ldots, t_5\}$ under tripling. Here $q = 5$, $p = 3$, $p^* = 2$ and $s = 2$. By the computation in *Case 3*, the gap lengths are

$$\ell_1 = \frac{3^{\langle 2\rangle - 1} + 3^{\langle -2\rangle - 1}}{242} = \frac{12}{242} \qquad \ell_2 = \frac{3^{\langle 4\rangle - 1} + 3^{\langle 0\rangle - 1}}{242} = \frac{108}{242}$$
$$\ell_3 = \frac{3^{\langle 6\rangle - 1} + 3^{\langle 2\rangle - 1}}{242} = \frac{4}{242} \qquad \ell_4 = \frac{3^{\langle 8\rangle - 1} + 3^{\langle 4\rangle - 1}}{242} = \frac{36}{242}$$
$$\ell_5 = \frac{3^{\langle 10\rangle - 1} + 3^{\langle 6\rangle - 1}}{242} = \frac{82}{242}.$$

The leading angle t_1 is $(\frac{1}{2})(\ell_1 + \ell_2 + \ell_3) = \frac{62}{242}$, which gives

$$t_4 = \frac{186}{242}, \quad t_2 = \frac{74}{242}, \quad t_5 = \frac{222}{242}, \quad t_3 = \frac{182}{242}.$$

Thus,

$$X_{\frac{3}{5},\frac{2}{5}} = \left\{\frac{62}{242}, \frac{74}{242}, \frac{182}{242}, \frac{186}{242}, \frac{222}{242}\right\}.$$

Unlike the case of the doubling map, irrational rotation numbers under tripling can have a wider variety of gap lengths depending on their deployment vector:

Theorem 4.31 *Suppose θ is irrational.*

(i) If $\delta = 0$ or 1, then $X_{\theta,\delta}$ has a single major gap of length $\frac{2}{3}$.
(ii) If $\delta = \pm n\theta \pmod{\mathbb{Z}}$ for some positive integer n, then $X_{\theta,\delta}$ has a pair of major gaps of lengths $\frac{1}{3}$ and $\frac{1}{3} + \frac{1}{3^{n+1}}$.
(iii) For all other choices of δ, $X_{\theta,\delta}$ has a pair of major gaps of length $\frac{1}{3}$.

Proof The major gaps of $X_{\theta,\delta}$ have lengths $\nu\{\delta\}$ and $\nu\{1\} = \nu\{0\}$, where

$$\nu = \sum_{k=0}^{\infty} 3^{-(k+1)} \mathbb{1}_{-k\theta} + \sum_{k=0}^{\infty} 3^{-(k+1)} \mathbb{1}_{\delta-k\theta}$$

is the gap measure of $X_{\theta,\delta}$ defined by (3.20). Since θ is irrational, the backward orbit $O_1 = \{-k\theta \pmod{\mathbb{Z}} : k \geq 0\}$ in the first sum and the backward orbit $O_2 = \{\delta - k\theta \pmod{\mathbb{Z}} : k \geq 0\}$ in the second sum consist of distinct points. However, for some values of δ the two orbits could collide. If $\delta = 0$ or 1, then $O_1 = O_2$ and there is a single major gap of length $\nu\{0\} = \frac{2}{3}$. If $\delta = n\theta \pmod{\mathbb{Z}}$ for some positive integer n, then $O_1 \subsetneq O_2$ and $\nu\{0\} = \frac{1}{3} + \frac{1}{3^{n+1}}$ and $\nu\{\delta\} = \frac{1}{3}$. Similarly, if $\delta = -n\theta \pmod{\mathbb{Z}}$ for some positive integer n, then $O_2 \subsetneq O_1$ and $\nu\{0\} = \frac{1}{3}$ and $\nu\{\theta\} = \frac{1}{3} + \frac{1}{3^{n+1}}$. For all other values of δ, $O_1 \cap O_2 = \emptyset$, so $\nu\{0\} = \nu\{\delta\} = \frac{1}{3}$. □

Let $\omega(\theta, \delta)$ denote the leading angle of $X_{\theta,\delta}$ as defined in Sect. 4.4. By the formula (4.5),

$$\omega(\theta, \delta) = \frac{1}{2} \left[\sum_{0<-k\theta\leq\theta} \frac{1}{3^{k+1}} + \sum_{0<\delta-k\theta\leq\theta} \frac{1}{3^{k+1}} \right] + \frac{N_0}{2}, \tag{4.12}$$

where $N_0 = 1$ if $\delta = 0$ and $N_0 = 0$ otherwise. One can study the function $(\theta, \delta) \mapsto \omega(\theta, \delta)$ by looking at the one-dimensional slices where θ or δ is kept fixed. The only values of δ for which $\theta \mapsto \omega(\theta, \delta)$ is defined for all $0 < \theta < 1$ are $\delta = 0$ and 1. As we have noticed before, these are similar to the doubling case. For example, when $\delta = 1$, the leading angle is given by

$$\omega(\theta, 1) = \sum_{0<-k\theta\leq\theta} \frac{1}{3^{k+1}} \tag{4.13}$$

which is similar to the formula (4.9) for the doubling map. Table 4.2 illustrates the computation of $\omega(p/q, 1)$ using formulas (4.11) and (4.13) for all reduces fractions with denominators up to 8.

Table 4.2 The leading angle $\omega(p/q, 1)$ of the cycle $X_{p/q,1}$ under the tripling map, for denominators $2 \le q \le 8$

p/q	Formula (4.11)	Formula (4.13)	$\omega(p/q, 1)$
$\frac{1}{2}$	$\frac{3^0}{3^2-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{2j}}$	$\frac{1}{8}$
$\frac{1}{3}$	$\frac{3^0}{3^3-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{3j}}$	$\frac{1}{26}$
$\frac{2}{3}$	$\frac{3^1+3^0}{3^3-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{3j-1}} + \frac{1}{3^{3j}}\right)$	$\frac{4}{26}$
$\frac{1}{4}$	$\frac{3^0}{3^4-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{4j}}$	$\frac{1}{80}$
$\frac{3}{4}$	$\frac{3^2+3^1+3^0}{3^4-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{4j-2}} + \frac{1}{3^{4j-1}} + \frac{1}{3^{4j}}\right)$	$\frac{13}{80}$
$\frac{1}{5}$	$\frac{3^0}{3^5-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{5j}}$	$\frac{1}{242}$
$\frac{2}{5}$	$\frac{3^2+3^0}{3^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{5j-2}} + \frac{1}{3^{5j}}\right)$	$\frac{10}{242}$
$\frac{3}{5}$	$\frac{3^1+3^3+3^0}{3^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{5j-1}} + \frac{1}{3^{5j-3}} + \frac{1}{3^{5j}}\right)$	$\frac{31}{242}$
$\frac{4}{5}$	$\frac{3^3+3^2+3^1+3^0}{3^5-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{5j-3}} + \frac{1}{3^{5j-2}} + \frac{1}{3^{5j-1}} + \frac{1}{3^{5j}}\right)$	$\frac{40}{242}$
$\frac{1}{6}$	$\frac{3^0}{3^6-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{6j}}$	$\frac{1}{728}$
$\frac{5}{6}$	$\frac{3^4+3^3+3^2+3^1+3^0}{3^6-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{6j-4}} + \frac{1}{3^{6j-3}} + \frac{1}{3^{6j-2}} + \frac{1}{3^{6j-1}} + \frac{1}{3^{6j}}\right)$	$\frac{121}{728}$
$\frac{1}{7}$	$\frac{3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{7j}}$	$\frac{1}{2186}$
$\frac{2}{7}$	$\frac{3^3+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-3}} + \frac{1}{3^{7j}}\right)$	$\frac{28}{2186}$
$\frac{3}{7}$	$\frac{3^4+3^2+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-4}} + \frac{1}{3^{7j-2}} + \frac{1}{3^{7j}}\right)$	$\frac{91}{2186}$
$\frac{4}{7}$	$\frac{3^1+3^3+3^5+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-1}} + \frac{1}{3^{7j-3}} + \frac{1}{3^{7j-5}} + \frac{1}{3^{7j}}\right)$	$\frac{274}{2186}$
$\frac{5}{7}$	$\frac{3^2+3^5+3^1+3^4+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-2}} + \frac{1}{3^{7j-5}} + \frac{1}{3^{7j-1}} + \frac{1}{3^{7j-4}} + \frac{1}{3^{7j}}\right)$	$\frac{337}{2186}$
$\frac{6}{7}$	$\frac{3^5+3^4+3^3+3^2+3^1+3^0}{3^7-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{7j-5}} + \frac{1}{3^{7j-4}} + \frac{1}{3^{7j-3}} + \frac{1}{3^{7j-2}} + \frac{1}{3^{7j-1}} + \frac{1}{3^{7j}}\right)$	$\frac{364}{2186}$
$\frac{1}{8}$	$\frac{3^0}{3^8-1}$	$\sum_{j=1}^{\infty} \frac{1}{3^{8j}}$	$\frac{1}{6560}$
$\frac{3}{8}$	$\frac{3^2+3^5+3^0}{3^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{8j-2}} + \frac{1}{3^{8j-5}} + \frac{1}{3^{8j}}\right)$	$\frac{253}{6560}$
$\frac{5}{8}$	$\frac{3^4+3^1+3^6+3^3+3^0}{3^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{8j-4}} + \frac{1}{3^{8j-1}} + \frac{1}{3^{8j-6}} + \frac{1}{3^{8j-3}} + \frac{1}{3^{8j}}\right)$	$\frac{841}{6560}$
$\frac{7}{8}$	$\frac{3^6+3^5+3^4+3^3+3^2+3^1+3^0}{3^8-1}$	$\sum_{j=1}^{\infty} \left(\frac{1}{3^{8j-6}} + \frac{1}{3^{8j-5}} + \frac{1}{3^{8j-4}} + \frac{1}{3^{8j-3}} + \frac{1}{3^{8j-2}} + \frac{1}{3^{8j-1}} + \frac{1}{3^{8j}}\right)$	$\frac{1093}{6560}$

The computations are identical to the doubling case in Table 4.1 once each power of 2 is replaced by the similar power of 3. In other words, the ternary expansion of $\omega(p/q, 1)$ is the same as the binary expansion of $\omega(p/q)$

An argument similar to the proof of Lemma 4.25 establishes the alternative formula

$$\omega(\theta, 1) = \frac{2}{3} \sum_{0<p/q\leq\theta} \frac{1}{3^q - 1},$$

which leads to the following analog of Corollary 4.26:

Corollary 4.32 *The leading angle $\omega(\theta, 1)$ of $X_{\theta,1}$ is a strictly increasing function of $0 < \theta < 1$, with $\omega(0^+, 1) = 0$ and $\omega(1^-, 1) = \frac{1}{6}$. Moreover,*

(i) $\omega(\theta, 1)$ has a jump discontinuity at every rational value of θ. In fact, if $\theta = p/q$ in lowest terms, then

$$\omega(p/q, 1) = \omega(p/q^+, 1) = \omega(p/q^-, 1) + \frac{2}{3(3^q - 1)}.$$

(ii) $\omega(\theta, 1)$ is continuous at every irrational value of θ.
(iii) For every $0 < \theta < 1$,

$$\omega(\theta^+, 1) + \omega((1-\theta)^-, 1) = \frac{1}{6}.$$

Compare Fig. 4.5. The function $\theta \mapsto \omega(\theta, 1)$ is related to the unicritical cubic family $\{z \mapsto z^3 + c\}_{c\in\mathbb{C}}$ (see Remark 5.14 at the end of Sect. 5.4).

Now let us fix some irrational $0 < \theta < 1$. For simplicity, let $\omega = \omega(\theta, 1)$.

Theorem 4.33 *The leading angle $\omega(\theta, \delta)$ of $X_{\theta,\delta}$ is a strictly decreasing function of $0 \leq \delta \leq 1$, with $\omega(\theta, 0) = \omega + \frac{1}{2}$ and $\omega(\theta, 1) = \omega$. Moreover,*

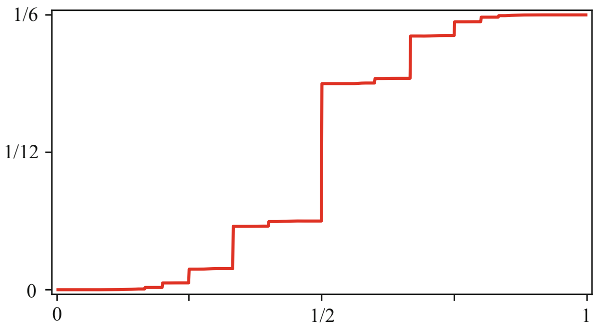

Fig. 4.5 The graph of the leading angle $\omega(\theta, 1)$ of the minimal rotation set $X_{\theta,1}$ under tripling, as a function of the rotation number θ. Notice the similarity with the graph of the leading angle $\omega(\theta)$ under doubling in Fig. 4.4

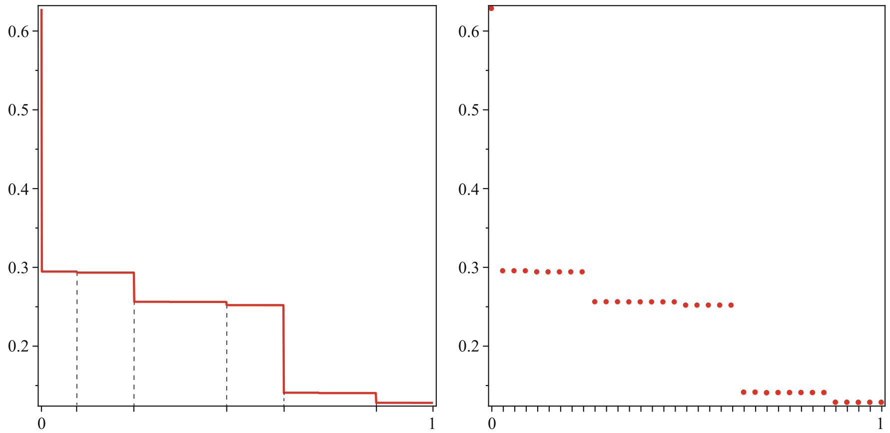

Fig. 4.6 Left: The graph of the leading angle $\omega(\theta, \delta)$ of the minimal rotation set $X_{\theta,\delta}$ under tripling, as a function of $0 \leq \delta \leq 1$. Here $\theta = \frac{(\sqrt{5}-1)}{2}$ is the golden mean. There is a jump of size $1/3^{n+1}$ at the parameter $\delta_n = n\theta \pmod{\mathbb{Z}}$ for every $n \geq 0$ (only six such jumps are visible in the figure). Right: The graph of the leading angle for the rational approximation $\frac{21}{34}$ of θ (see Remark 4.34)

(i) $\delta \mapsto \omega(\theta, \delta)$ *has a jump discontinuity at the points* $\delta_n = n\theta \pmod{\mathbb{Z}}$ *for integers* $n \geq 0$. *In fact,*

$$\omega(\theta, \delta_n) = \omega(\theta, \delta_n^-) = \omega(\theta, \delta_n^+) + \frac{1}{3^{n+1}}.$$

(ii) $\delta \mapsto \omega(\theta, \delta)$ *is continuous at every* $\delta \neq \delta_n$.

Compare Fig. 4.6.

Proof For each $0 < \delta \leq 1$ we have

$$\omega(\theta, \delta) = \frac{1}{2}\omega + \frac{1}{2}\sum_{k=0}^{\infty} \frac{\varepsilon_k(\theta, \delta)}{3^{k+1}},$$

where

$$\varepsilon_k(\theta, \delta) = \begin{cases} 1 & \text{if } \delta - k\theta \in (0, \theta] \\ 0 & \text{otherwise.} \end{cases}$$

Since $\varepsilon_k(\theta, \delta') \to \varepsilon_k(\theta, \delta)$ as $\delta' \to \delta^-$, it follows (say, from the dominated convergence theorem) that $\omega(\theta, \delta^-) = \omega(\theta, \delta)$, proving left-continuity at every δ. If $\delta \neq \delta_n$ for every $n \geq 0$, then $\delta - k\theta \in (0, \theta)$ or $(\theta, 1)$ for each $k \geq 0$. In either case, we have $\varepsilon_k(\theta, \delta') \to \varepsilon_k(\theta, \delta)$ as $\delta' \to \delta^+$ and right-continuity $\omega(\theta, \delta^+) = \omega(\theta, \delta)$

follows. However, suppose $\delta = \delta_n$ for some $n \geq 1$. Then the two orbit relations $\delta - n\theta = 0$ and $\delta - (n-1)\theta = \theta \pmod{\mathbb{Z}}$ show that

$$\begin{aligned} \varepsilon_n(\theta, \delta^+) &= 1 > 0 = \varepsilon_n(\theta, \delta), \\ \varepsilon_{n-1}(\theta, \delta^+) &= 0 < 1 = \varepsilon_{n-1}(\theta, \delta), \\ \varepsilon_k(\theta, \delta^+) &= \varepsilon_k(\theta, \delta) \quad \text{if} \quad k \neq n, n-1, \end{aligned}$$

where the third relation follows from the assumption that θ is irrational. It follows that

$$\omega(\theta, \delta) - \omega(\theta, \delta^+) = \frac{1}{2}\Big(- \frac{1}{3^{n+1}} + \frac{1}{3^n} \Big) = \frac{1}{3^{n+1}}.$$

Similarly, if $\delta = \delta_0 = 0$, then

$$\begin{aligned} \varepsilon_0(\theta, \delta^+) &= 1 > 0 = \varepsilon_0(\theta, \delta), \\ \varepsilon_k(\theta, \delta^+) &= \varepsilon_k(\theta, \delta) \quad \text{if} \quad k \neq 0, \end{aligned}$$

from which it follows that

$$\omega(\theta, 0) - \omega(\theta, 0^+) = \frac{1}{2} + \frac{1}{2}\Big(- \frac{1}{3} \Big) = \frac{1}{3}.$$

Finally, observe that for each n the sum $\delta \mapsto \sum_{k=0}^{n} \varepsilon_k(\theta, \delta)/3^{k+1}$ is a step function with discontinuities along $\{\delta_0, \dots, \delta_n\}$ where it jumps to a lower value, hence is decreasing in δ. Letting $n \to \infty$, it follows that the function $\delta \mapsto \omega(\theta, \delta)$ is decreasing as well. Since the set $\{\delta_n\}_{n \geq 0}$ is dense in $[0, 1]$, we conclude that this function must be strictly decreasing. □

Remark 4.34 The parameters δ_n are precisely the values of $\delta \in [0, 1)$ for which the major gap I_0 of $X_{\theta,\delta}$ containing 0 has length $> \frac{1}{3}$. A generic perturbation $\delta = \delta_n + \varepsilon$ will replace I_0 with a major gap of length $\frac{1}{3}$ together with a nearby minor gap of length $\frac{1}{3^{n+1}}$. This gives an intuitive explanation for the nature of discontinuity of the leading angle at every δ_n.

It is not hard to check that if θ is irrational and $\delta \neq \delta_n$ for all $n \geq 0$, and if $(p_i/q_i, s_i/q_i)$ is a sequence of rational parameters that converges to (θ, δ), then $\omega(p_i/q_i, s_i/q_i) \to \omega(\theta, \delta)$. In view of this, it is natural to expect the discrete graph of $\delta \mapsto \omega(p_i/q_i, \delta)$ (consisting of $q_i + 1$ points) to resemble the graph of $\delta \mapsto \omega(\theta, \delta)$ for large i; see Fig. 4.6.

The next result shows that the values of $\omega(\theta, \delta)$ at the discontinuity points $\delta_n = n\theta \pmod{\mathbb{Z}}$ depend rationally on the "base angle" $\omega = \omega(\theta, 1)$:

Theorem 4.35 *Let $\omega = \omega(\theta, 1)$. Then, for every $n \geq 1$,*

$$\omega(\theta, n\theta) = \frac{(3^n + 1)\,\omega + A_n}{2 \cdot 3^n} \tag{4.14}$$

$$\omega(\theta, -n\theta) = \frac{(3^n + 1)\,\omega - B_n}{2}, \tag{4.15}$$

where A_n, B_n are non-negative integers (in fact, sums of distinct non-negative powers of 3):

$$A_n = \sum_{\substack{1 \leq k \leq n \\ 0 < k\theta \leq \theta}} 3^{k-1} \quad \textit{and} \quad B_n = \sum_{\substack{1 \leq k \leq n \\ 0 < (k-n)\theta \leq \theta}} 3^{k-1}. \tag{4.16}$$

Proof For simplicity let Z denote the set of integers k such that $-k\theta \pmod{\mathbb{Z}}$ belongs to $(0, \theta]$. By the definition of $\omega(\theta, \delta)$ and (4.13),

$$\begin{aligned}
2\omega(\theta, n\theta) &= \sum_{k \in Z \cap [0,\infty)} 3^{-(k+1)} + \sum_{k-n \in Z \cap [-n,\infty)} 3^{-(k+1)} \\
&= \omega + \sum_{k \in Z \cap [-n,\infty)} 3^{-(k+n+1)} \\
&= \omega + \Bigg(\sum_{k \in Z \cap [0,\infty)} + \sum_{k \in Z \cap [-n,0)} \Bigg) 3^{-(k+n+1)} \\
&= (1 + 3^{-n})\,\omega + 3^{-n} \sum_{k \in Z \cap [-n,0)} 3^{-(k+1)},
\end{aligned}$$

which proves (4.14) with

$$A_n = \sum_{k \in Z \cap [-n,0)} 3^{-(k+1)} = \sum_{\substack{1 \leq k \leq n \\ 0 < k\theta \leq \theta}} 3^{k-1},$$

as in (4.16). Similarly,

$$\begin{aligned}
2\omega(\theta, -n\theta) &= \sum_{k \in Z \cap [0,\infty)} 3^{-(k+1)} + \sum_{k+n \in Z \cap [n,\infty)} 3^{-(k+1)} \\
&= \omega + \sum_{k \in Z \cap [n,\infty)} 3^{-(k-n+1)}
\end{aligned}$$

$$= \omega + \Big(\sum_{k \in Z \cap [0,\infty)} - \sum_{k \in Z \cap [0,n)} \Big) 3^{-(k-n+1)}$$

$$= (1 + 3^n)\omega - \sum_{k \in Z \cap [0,n)} 3^{-(k-n+1)},$$

which proves (4.15) with

$$B_n = \sum_{k \in Z \cap [0,n)} 3^{-(k-n+1)} = \sum_{\substack{1 \leq k \leq n \\ 0 < (k-n)\theta \leq \theta}} 3^{k-1},$$

as in (4.16). □

Remark 4.36 It can be shown that for every irrational θ the angle $\omega = \omega(\theta, 1)$ is transcendental (see [7] for the quadratic case and [1] for a more general result). It follows from the above theorem that all the leading angles $\omega(\theta, \pm n\theta)$ are also transcendental. These angles appear in the bifurcation loci of certain one-dimensional families of cubic polynomials (see Sect. 5.4).

Chapter 5
Relation to Complex Dynamics

In this chapter we outline how rotation sets occur in the dynamical study of complex polynomial maps. Special attention is paid to the relation with the dynamics of complex quadratic and cubic polynomials. This link provides a geometric realization of rotation sets under m_d, whose abstract theory was developed in the previous chapters.

5.1 Polynomials and Dynamic Rays

We assume the reader is familiar with the basic notions of complex dynamics, as in [21]. Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial map of degree $d \geq 2$. The ***filled Julia set*** $K(f)$ is the union of all bounded orbits of f, and the ***Julia set*** $J(f)$ is the topological boundary of $K(f)$. Both are compact non-empty subsets of the plane. The complement $\mathbb{C} \smallsetminus K(f)$ is connected and can be described as the ***basin of infinity*** for f, that is, the set of all points whose orbits under f tend to ∞. The ***Green's function*** of f is the continuous function $G : \mathbb{C} \to \mathbb{R}$ defined by

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^{\circ n}(z)|,$$

which describes the escape rate of z to ∞ under the iterations of f. It is easy to see that G satisfies the relation

$$G(f(z)) = d\, G(z)$$

with $G(z) = 0$ if and only if $z \in K(f)$. The Green's function is harmonic in the basin of ∞, with critical points at all precritical points of f. In other words, $\nabla G(z) = 0$ for some $z \in \mathbb{C} \smallsetminus K(f)$ if and only if $f^{\circ n}(z)$ is a critical point of f for some $n \geq 0$.

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There is a unique conformal isomorphism β, defined in some neighborhood of ∞, which is tangent to the identity at ∞ (in the sense that $\lim_{z\to\infty} \beta(z)/z = 1$) and conjugates the action of f to that of the power map $z \mapsto z^d$:

$$\beta(f(z)) = (\beta(z))^d \quad \text{for large} \quad |z|.$$

We call β the ***Böttcher coordinate*** of f near ∞. The modulus of β is related to the Green's function by the relation $|\beta(z)| = e^{G(z)}$ for large $|z|$. It is not hard to check that β is univalent in the domain $\{z \in \mathbb{C} : G(z) > G_0\}$, where

$$G_0 = \max\{G(c) : c \text{ is a critical point of } f\}.$$

In particular, if every critical point of f belongs to $K(f)$, then $G_0 = 0$ and β is a conformal isomorphism $\mathbb{C} \smallsetminus K(f) \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$. This happens precisely when $K(f)$ is connected.

In what follows and unless otherwise stated we assume that $K(f)$ is connected. In this case the inverse Böttcher coordinate $\psi = \beta^{-1} : \mathbb{C} \smallsetminus \overline{\mathbb{D}} \to \mathbb{C} \smallsetminus K(f)$ is a conformal isomorphism which satisfies

$$\psi(z^d) = f(\psi(z)) \quad \text{for} \quad |z| > 1. \tag{5.1}$$

By the ***(dynamic) ray*** of f at angle $t \in \mathbb{T}$ we mean the real-analytic curve

$$R(t) = \psi\big(\{re^{2\pi i t} : r > 1\}\big).$$

The functional equation (5.1) shows that

$$f(R(t)) = R(m_d(t)) \quad \text{for all} \quad t \in \mathbb{T}. \tag{5.2}$$

We say that $R(t)$ ***lands*** at $z \in J(f)$ if $\lim_{r\to 1} \psi(re^{2\pi i t}) = z$. It follows from (5.2) that if $R(t)$ lands at z, then $R(m_d(t))$ lands at $f(z)$. Similarly, if f has local degree k at $w \in f^{-1}(z)$, then there are k preimages $\{t_1, \dots t_k\}$ of t under m_d such that each $R(t_i)$ lands at w. A ray may or may not land, but the set of angles t for which $R(t)$ lands has full Lebesgue measure on the circle.

The ***impression*** $\hat{R}(t)$ of the ray $R(t)$ is the set of all $w \in \mathbb{C}$ for which there is a sequence $z_n \in \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ such that $z_n \to e^{2\pi i t}$ and $\psi(z_n) \to w$. It is not hard to check that $\hat{R}(t)$ is a non-empty compact connected subset of $J(f)$. Every point of the Julia set belongs to at least one impression. We say that the impression $\hat{R}(t)$ is ***trivial*** if it reduces to a single point $\{z\}$. In this case, $R(t)$ necessarily lands at z (a landing ray, however, may well have a non-trivial impression). Furthermore, it is easily seen that

$$\limsup_{n\to\infty} \hat{R}(t_n) \subset \hat{R}(t) \quad \text{whenever} \quad t_n \to t. \tag{5.3}$$

(As usual, the limsup on the left is the set of all $p \in \mathbb{C}$ such that every neighborhood of p meets infinitely many of the $\hat{R}(t_n)$.) We will also use the following separation property later on: Suppose the rays $R(t')$, $R(t'')$ land at z and W is one of the two connected components of $\mathbb{C} \smallsetminus (R(t') \cup R(t'') \cup \{z\})$. If a third ray $R(t)$ is contained in W, then $\hat{R}(t) \subset W \cup \{z\}$.

A point $z \in K(f)$ is the landing point of two or more rays if and only if $K(f) \smallsetminus \{z\}$ is disconnected. More precisely, z has $2 \leq n \leq \infty$ distinct rays landing on it if and only if $K(f) \smallsetminus \{z\}$ has n connected components [18]. If z has finite forward orbit under f, the number of rays landing on it can be arbitrarily large (see the case of a parabolic fixed point below). But if the forward orbit of z is infinite, there is an upper bound $C(d)$ for the number of rays that can land at z (one can take $C(d) = 2^d$, and the bound improves to $C(d) = d$ if z is not precritical [15]).

The ***multiplier*** of a fixed point $\zeta = f(\zeta)$ is the derivative $f'(\zeta)$. We call ζ ***attracting, repelling,*** or ***indifferent,*** according as the modulus $|f'(\zeta)|$ is less than, greater than, or equal to 1. An indifferent fixed point is called ***parabolic*** if its multiplier is a root of unity. The multiplier and type of a periodic point ζ of period n can be defined analogously by treating ζ as a fixed point of the iterate $f^{\circ n}$.

Suppose the angle $t \in \mathbb{T}$ is periodic of period $q \geq 1$ under m_d, so t is rational of the form $i/(d^q - 1)$. According to the Douady-Hubbard landing theorem [21], the ray $R(t)$ lands at a periodic point of f with period dividing q, and this periodic point is necessarily repelling or parabolic. Conversely, every repelling or parabolic periodic point of f is the landing point of finitely many rays whose angles are periodic under m_d of the same period.

As a special case, if $u_i = i/(d-1) \pmod{\mathbb{Z}}$, it follows that for each $0 \leq i \leq d-2$ the ***fixed ray*** $R(u_i)$ lands at a repelling or parabolic fixed point $\zeta_i = f(\zeta_i)$. When ζ_i is parabolic, the multiplier $f'(\zeta_i)$ is necessarily 1. Of course the fixed points $\zeta_0, \ldots, \zeta_{d-2}$ need not be distinct.

The study of dynamic rays when $K(f)$ is disconnected is a bit more complicated (an example of this case will be briefly discussed in Sect. 5.4). In this case at least one critical point of f escapes to ∞ and the Green's function G has infinitely many critical points outside $K(f)$. We can still define the dynamic rays $\{R(t)\}_{t \in \mathbb{T}}$ partially near ∞ by pulling back the radial lines under the Böttcher coordinate

$$\beta : \{z \in \mathbb{C} : G(z) > G_0\} \to \{z : |z| > e^{G_0}\}.$$

These partial rays are the trajectories of the gradient vector field ∇G near ∞, so they can be extended in backward time. Such an extended trajectory either avoids the critical points of G and tends to $K(f)$, or it eventually tends to such a critical point (namely an escaping precritical point of f). We call the ray ***smooth*** or ***bifurcated*** accordingly. For all but countably many $t \in \mathbb{T}$ the ray $R(t)$ is smooth. In this case $R(m_d(t))$ is also smooth and the relation (5.2) holds. On the other hand, for a countably infinite set of angles t the ray $R(t)$ is bifurcated. Under the iterations of f every bifurcated ray eventually maps to a smooth ray passing through a critical value of f.

5.2 Rotation Sets and Indifferent Fixed Points

This section will study polynomial maps of degree $d \geq 2$ with connected Julia set which have an indifferent fixed point of multiplier $e^{2\pi i\theta} \neq 1$. Every such map is affinely conjugate to a monic polynomial of the form

$$f : z \mapsto e^{2\pi i\theta} z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + z^d, \tag{5.4}$$

where the indifferent fixed point is placed at the origin. We consider two cases depending on the nature of the fixed point 0.

The parabolic case. First suppose 0 is a parabolic fixed point so θ is rational of the form p/q in lowest terms. Then there are finitely many rays landing at 0, each being periodic of period q. We can label these rays as

$$R(t_1), R(t_2), \ldots, R(t_{Nq})$$

where $N \geq 1$ and $0, t_1, \ldots, t_{Nq}$ are in positive cyclic order. Using the form of the multiplier, it is easily seen that $f(R(t_j)) = R(t_{j+Np})$, or $m_d(t_j) = t_{j+Np}$ for every j, where as usual the indices are taken modulo Nq. It follows that $\{t_1, \ldots, t_{Nq}\}$ is the union of N disjoint q-cycles under m_d, each with the combinatorial rotation number p/q.

The following lemma ties up the situation with rotation sets:

Lemma 5.1 *The set X of the angles $t \in \mathbb{T}$ for which the ray $R(t)$ lands at 0 is a rotation set under m_d with $\rho(X) = p/q$.*

Proof Label $X = \{t_1, \ldots, t_{Nq}\}$ as above. For $1 \leq i \leq N$, let C_i denote the q-cycle

$$t_i \mapsto t_{i+Np} \mapsto t_{i+2Np} \mapsto \ldots \mapsto t_{i+(q-1)Np}$$

under m_d. Evidently X is the disjoint union of $C_1, \ldots, C_N$ and these cycles are superlinked in the sense of Sect. 2.3. By Lemma 2.25, X is a rotation set with $\rho(X) = \rho(C_i) = p/q$. □

The deployment invariant of X can be described dynamically as follows. Two adjacent rays $R(t_j)$ and $R(t_{j+1})$ together with their common landing point 0 divide the plane into two open sectors. By definition, the ***(dynamic) wake*** W_j is the sector that contains the rays $R(t)$ with $t \in (t_j, t_{j+1})$ (thus, W_j is the sector defined by going counter-clockwise from $R(t_j)$ to $R(t_{j+1})$). The gap $I_j = (t_j, t_{j+1})$ of X corresponds to the part of the boundary of the wake W_j on the circle at ∞. By Lemma 2.13, the multiplicity n_j of I_j is the number of fixed rays that are contained in W_j. It is also the number of the critical points of f in W_j (see [12], where this invariant is called the "critical weight" of W_j, and compare Theorem 5.10 for a similar case). In particular, I_j is a major gap if and only if W_j contains a fixed point ζ_i, or equivalently a critical point. As there are $d-1$ fixed rays, there are at most $d-1$ indices $1 \leq j \leq Nq$ for which $n_j \neq 0$. Form the non-decreasing list of integers $0 \le s_1 \le s_2 \le \cdots \le s_{d-1} = Nq$ in which each index $1 \le j \le Nq$ appears n_j times. It then follows from Lemma 3.5 that $(s_1, \ldots, s_{d-1})$ is the signature $s(X)$ as defined in Sect. 3.2 and therefore $(s_1/(Nq), \ldots, s_{d-1}/(Nq))$ is the cumulative deployment vector $\sigma(X)$.

Since the multiplier of the fixed point 0 is a q-th root of unity, the q-th iterate of f has the local expansion

$$f^{\circ q}(z) = z + a\,z^m + O(z^{m+1}) \quad \text{for some} \quad a \neq 0 \quad \text{and} \quad m > 1.$$

The integer m, the algebraic multiplicity of 0 as the root of the equation $f^{\circ q}(z) - z = 0$, is necessarily of the form $kq + 1$ for some $1 \le k \le N$. According to Leau and Fatou [21], there are bounded Fatou components $U_1, \ldots, U_{kq}$ arranged as kq "petals" around the common boundary point 0. If we choose labeling counterclockwise, we have $f(U_j) = U_{j+kp}$ for every j, taking indices modulo kq, so the U_j are permuted with combinatorial rotation number p/q. Every point in the union $U_1 \cup \cdots \cup U_{kq}$ has an infinite orbit that tends to 0. Conversely, every infinite orbit converging to 0 must eventually enter this union. It follows from this local picture that the ***petal number*** kq of the parabolic fixed point is bounded above by the ***ray number*** Nq. The bound $N \le d - 1$ of Theorem 2.27 now shows that

$$q \le \text{petal number } kq \le \text{ray number } Nq \le (d-1)q.$$

In the quadratic case $d = 2$ it follows that the petal number and ray number are both q, while in the cubic case $d = 3$ these numbers can be q or $2q$ (see Fig. 5.1 for the case $(k, N) = (1, 1)$ and $(1, 2)$, and Fig. 5.9 for the case $(k, N) = (2, 2)$).

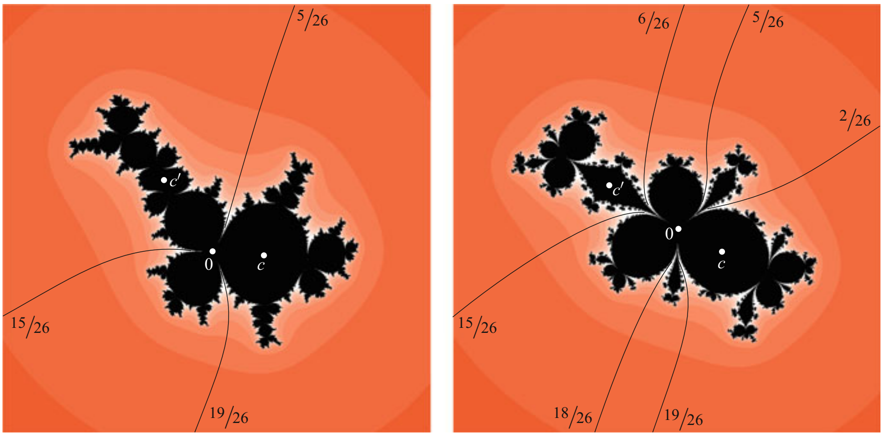

Fig. 5.1 Examples of parabolic points with multiplier $\lambda = e^{2\pi i/3}$ and petal number 3. Left: The cubic $z \mapsto \lambda z - (0.04 + 0.85i)z^2 + z^3$ with ray number 3. Right: The cubic $z \mapsto \lambda z + (0.23 - 0.20i)z^2 + z^3$ with ray number 6. The critical points c, c' are marked as white dots

The "good" Siegel case. Now suppose 0 is a linearizable fixed point, so it belongs to a bounded Fatou component Δ in which the action of f is conjugate to the irrational rotation $z \mapsto e^{2\pi i\theta}z$. The domain Δ is called the ***Siegel disk*** of f centered at 0. We will assume that the boundary $\partial\Delta$ is a Jordan curve containing at least one critical point of f. This is certainly the case if θ is an irrational number of bounded type, that is, if the partial quotients in the continued fraction expansion $\theta = [a_1, a_2, a_3, \ldots]$ form a bounded sequence (compare [8] and [31]).[1] To avoid topological complications and focus on the combinatorial aspects of the constructions, we further make the following assumption:

The Limb Decomposition Hypothesis There is a countable collection of disjoint non-trivial compact connected subsets of $K(f)$, called ***limbs***, such that

(LD1) $K(f)$ is $\overline{\Delta}$ union all the limbs,
(LD2) Each limb meets $\overline{\Delta}$ at a single point on $\partial\Delta$ called its ***root***,
(LD3) For each $\varepsilon > 0$ there are at most finitely many limbs with diameter $> \varepsilon$.[2]

We denote by $L(p)$ the limb with root $p \in \partial\Delta$.

Lemma 5.2 *A point $p \in \partial\Delta$ is a root if and only if $K(f) \smallsetminus \{p\}$ is disconnected.*

Proof For every root p the non-empty set $L(p) \smallsetminus \{p\}$, which is clearly closed in $K(f) \smallsetminus \{p\}$, is also open in there by the condition (LD3) above. It follows that $K(f) \smallsetminus \{p\}$ is disconnected. Conversely, if $K \smallsetminus \{p\}$ is disconnected for some $p \in \partial\Delta$, there are two distinct rays landing at p. These rays together with their landing point divide the plane into two open sectors, one containing Δ and the other containing a non-trivial subset of $K(f)$ which necessarily lies in a single limb. It easily follows that p is the root of this limb. □

Lemma 5.3 *The set of roots is backward-invariant and therefore everywhere dense on $\partial\Delta$.*

Proof Take a root p and let z be the unique point on $\partial\Delta$ such that $f(z) = p$. There are small neighborhoods U of z and U' of p such that $f : U \to U'$ acts as the power $w \mapsto w^k$ for some $k \geq 1$. Take two distinct rays landing at p, take their intersections with U' and pull them back under f to obtain $2k \geq 2$ arcs in U landing at z. Each such arc is necessarily contained in a ray because of the functional equation (5.1). It follows that $K \smallsetminus \{z\}$ is disconnected and therefore z is a root by Lemma 5.2. This proves backward-invariance of roots. Density of roots is now immediate since $f|_{\partial\Delta} : \partial\Delta \to \partial\Delta$ is conjugate to an irrational rotation. □

Every root p has infinite forward orbit since $f|_{\partial\Delta}$ is conjugate to an irrational rotation. It follows that there are at least 2 and at most 2^d rays landing at p. These

[1]It is conjectured that $\partial\Delta$ is a Jordan curve containing a critical point for almost every rotation number θ. This has been proved in the quadratic case in [25].

[2]The limb decomposition hypothesis is believed to hold for almost every rotation number θ (and at least for θ of bounded type), but so far this has been rigorously verified only for $d = 2$ where the whole Julia set is known to be locally connected; see [23] and [25].

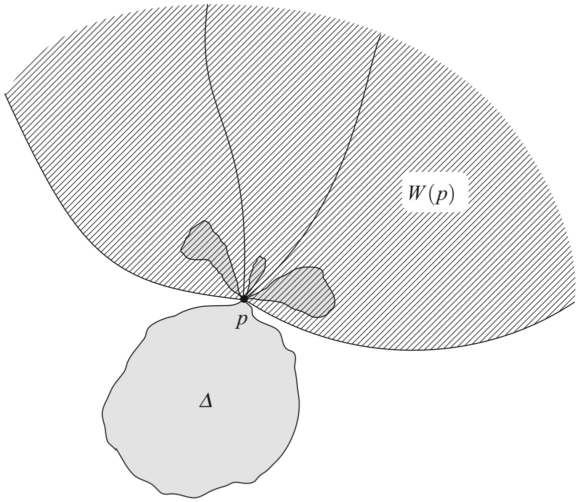

Fig. 5.2 The wake $W(p)$ with the root p on the boundary of the Siegel disk Δ

rays together with their landing point p divide the plane into finitely many open sectors. There is a unique sector that contains Δ which we call the ***co-wake*** with root p and denote by $V(p)$. The complement $W(p) = \mathbb{C} \smallsetminus \overline{V(p)}$ is called the ***(dynamic) wake*** with root p. Thus $W(p)$ is bounded by two rays landing at p and contains $L(p) \smallsetminus \{p\}$ (see Fig. 5.2). Notice that distinct wakes are disjoint. Every point in the plane is either in $\overline{\Delta}$, or in a unique wake, or else on a unique ray which is outside all wakes.

Lemma 5.4 *Every ray $R(t)$ that is outside all wakes lands at a point $z \in \partial\Delta$. Moreover,*

(i) If z is not a root, then $\hat{R}(t) = \{z\}$.
(ii) If z is a root, then $\hat{R}(t) \subset L(z)$ so $\hat{R}(t) \cap \partial\Delta = \{z\}$.

Proof Let us first make the extra assumption that the ray $R = R(t)$ is not a boundary ray of any wake. Suppose the impression $\hat{R}$ contains a point $z \notin \partial\Delta$. Then z belongs to a limb $L(p)$, and since $z \neq p$, we have $z \in W(p)$. Since by our assumption R is disjoint from $\overline{W(p)}$, it must be contained in the co-wake $V(p)$. But then $\hat{R} \subset V(p) \cup \{p\}$, which implies $z \in V(p)$, contradicting $z \in W(p)$. This proves $\hat{R} \subset \partial\Delta$. If the impression $\hat{R}$ is non-trivial, by connectivity it must contain an open subarc $T \subset \partial\Delta$. By Lemma 5.3, there are distinct roots $p, p' \in T$. The open set $\mathbb{C} \smallsetminus (\overline{W(p)} \cup \overline{W(p')} \cup \overline{\Delta})$ has two connected components and R is contained in one of them, say H. It follows that $T \subset \hat{R} \subset \overline{H} \cap \partial\Delta$. But the intersection $\overline{H} \cap \partial\Delta$ is one of the two closed subarcs of $\partial\Delta$ with endpoints p, p', neither of which contains the open arc T. The contradiction proves that $\hat{R}$ is a single point on $\partial\Delta$.

Now consider the case where R is one of the two boundary rays of a wake $W(z)$. An argument similar to the above paragraph shows that $\hat{R} \subset L(z) \cup \partial\Delta$. If $\hat{R}$ contained a point of $\partial\Delta$ other than z, it would have to contain a non-degenerate open arc in $\partial\Delta$. A similar argument as before would then yield a contradiction. This shows $\hat{R} \subset L(z)$ and completes the proof. □

Corollary 5.5 *Every non-root $z \in \partial\Delta$ belongs to the impression of a unique ray. This ray has trivial impression and therefore lands at z.*

Proof Let $R(t)$ be any ray whose impression contains z. Then $R(t)$ is outside all wakes since $R(t) \subset W(p)$ would imply $\hat{R}(t) \subset W(p) \cup \{p\}$ which in turn would imply $z = p$ is a root. It follows from the previous lemma that $\hat{R}(t) = \{z\}$. To see uniqueness, simply note that if $\hat{R}(s)$ also contained z for some $s \neq t$, then by the above observation $\hat{R}(s) = \{z\}$. As the landing point of two distinct rays, z would disconnect $K(f)$ and therefore would be a root by Lemma 5.2. □

Let $\iota : \mathbb{C} \to \overline{\Delta}$ be the map that is the identity on $\overline{\Delta}$, sends every wake to its root and sends every ray outside all wakes to its landing point (Lemma 5.4).

Lemma 5.6 *$\iota : \mathbb{C} \to \overline{\Delta}$ is a retraction.*

Proof We need only check continuity of ι at every point z that does not belong to Δ or any wake. First consider the easier case where $z \in \partial\Delta$. Take a sequence $z_n \notin \overline{\Delta}$ that tends to z. Each z_n belongs to a limb $L(p_n)$ and we may assume that these limbs are distinct. Since $\operatorname{diam}(L(p_n)) \to 0$ by (LD3), it easily follows that $\iota(z_n) = p_n \to z = \iota(z)$.

Now consider the case where z belongs to a ray $R(t)$ outside all wakes. Take any sequence $z_n \to z$. For large n, each z_n belongs to a unique ray $R(t_n)$, where $t_n \to t$. We distinguish two cases:

Case 1 After passing to a subsequence, every ray $R(t_n)$ is outside all wakes. Then, by (5.3) and Lemma 5.4,

$$\limsup_{n\to\infty}\{\iota(z_n)\} = \limsup_{n\to\infty} \hat{R}(t_n) \cap \partial\Delta \subset \hat{R}(t) \cap \partial\Delta = \{\iota(z)\}.$$

This proves $\iota(z_n) \to \iota(z)$.

Case 2 After passing to a subsequence, each $R(t_n)$ lies in some wake $W(p_n)$. Then the impression $\hat{R}(t_n)$ is contained in the limb $L(p_n)$ whose diameter tends to 0 as $n \to \infty$. Hence $\limsup_{n\to\infty} \hat{R}(t_n)$ coincides with the set of all accumulation points of the sequence of roots $\{p_n = \iota(z_n)\}$. Again, by (5.3) and Lemma 5.4, $\limsup_{n\to\infty} \hat{R}(t_n) \subset \hat{R}(t) = \{\iota(z)\}$, and we conclude that $\iota(z_n) \to \iota(z)$. □

Recall that for $0 \leq i \leq d-2$ the fixed point $\zeta_i \in J(f)$ is the landing point of the fixed ray $R(u_i)$. Let $w_i = \iota(\zeta_i) \in \partial\Delta$. Since the ζ_i do not belong to $\overline{\Delta}$, they lie in wakes, so every w_i must be a root. We call $\{w_0, \ldots, w_{d-2}\}$ the ***marked roots*** of f. Take the unique conformal isomorphism $h : \Delta \to \mathbb{D}$ which fixes 0 and sends w_0 to 1. According to Carathéodory, since $\partial\Delta$ is a Jordan curve, h extends to a homeomorphism between the closures [21]. Note that $h \circ f \circ h^{-1} : \mathbb{D} \to \mathbb{D}$ fixes 0 and has derivative $e^{2\pi i\theta}$ at the origin, so by the Schwarz lemma,

$$h(f(z)) = e^{2\pi i\theta} h(z) \quad \text{for all} \quad z \in \Delta.$$

We define the ***internal angle*** of a point $z \in \partial\Delta$ as the unique $\alpha \in \mathbb{T}$ such that $h(z) = e^{2\pi i\alpha}$. By the above conjugacy relation, the internal angle of $f(z)$ will then be $\alpha + \theta \pmod{\mathbb{Z}}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{d-1}$ denote the internal angles of the marked roots $w_1, w_2, \ldots, w_{d-1} = w_0$. The following is the analog of Lemma 5.1:

Theorem 5.7 *The set X' of all angles $t \in \mathbb{T}$ for which the ray $R(t)$ lands on $\partial\Delta$ contains a unique minimal rotation set X for m_d, with $\rho(X) = \theta$. Moreover, the cumulative deployment vector of X satisfies*

$$\sigma(X) = (\alpha_1, \ldots, \alpha_{d-1}) \qquad (\text{mod } \mathbb{Z}^{d-1}). \tag{5.5}$$

The proof will show that the difference $X' \smallsetminus X$ consists of at most countably many isolated points.

Proof For each root $p \in \partial\Delta$ let $I(p)$ be the open interval of angles $t \in \mathbb{T}$ for which $R(t) \subset W(p)$. Set $X = \mathbb{T} \smallsetminus \bigcup_p I(p)$. By Lemma 5.4 the compact set X is contained in X' and the difference $X' \smallsetminus X$ consists of the at most countable set of angles of rays within some wake that land at a root.

Let $\psi : \mathbb{C} \smallsetminus \overline{\mathbb{D}} \to \mathbb{C} \smallsetminus K(f)$ be the inverse Böttcher coordinate of f near ∞. Define $\varphi : \mathbb{T} \to \mathbb{T}$ by letting $\varphi(t)$ be the internal angle of the point $\iota(\psi(2e^{2\pi it})) \in \partial\Delta$. The map φ is continuous by the previous lemma, and is surjective by Corollary 5.5. Using the fact that distinct rays cannot cross, it is not hard to see that φ is monotone of degree 1, with the collection of intervals $\{I(p) : p \text{ is a root}\}$ as its plateaus. If $R(t)$ lands at $z \in \partial\Delta$ with internal angle α, then $R(m_d(t))$ lands at $f(z)$ with internal angle $\alpha + \theta$. This proves

$$\varphi \circ m_d = r_\theta \circ \varphi \quad \text{on} \quad X.$$

Furthermore, if the fiber $\varphi^{-1}(\alpha)$ is non-trivial, then $h^{-1}(e^{2\pi i\alpha})$ is a root, so its preimage $h^{-1}(e^{2\pi i(\alpha-\theta)})$ is also a root by Lemma 5.3, which proves the fiber $\varphi^{-1}(\alpha - \theta)$ is non-trivial as well. It now follows from Theorem 2.35 that X is a minimal rotation set for m_d with $\rho(X) = \theta$, and φ is the canonical semiconjugacy associated with X.

The claim (5.5) on $\sigma(X)$ follows from Lemma 3.3 since α_i, the internal angle of $w_i = \iota(\zeta_i) = \iota(\psi(2e^{2\pi i u_i}))$, is just the image $\varphi(u_i)$. □

Remark 5.8 The set X' of all rays landing on $\partial\Delta$ is closed and m_d-invariant, and every forward orbit in it has the combinatorial structure of an orbit under r_θ. Yet X' may fail to be a rotation set. For example, the cubic polynomial

$$f(z) = e^{\pi i(\sqrt{5}-1)} z + az^2 + z^3 \quad \text{with} \quad a \approx 0.44437107 - 0.35184284\, i$$

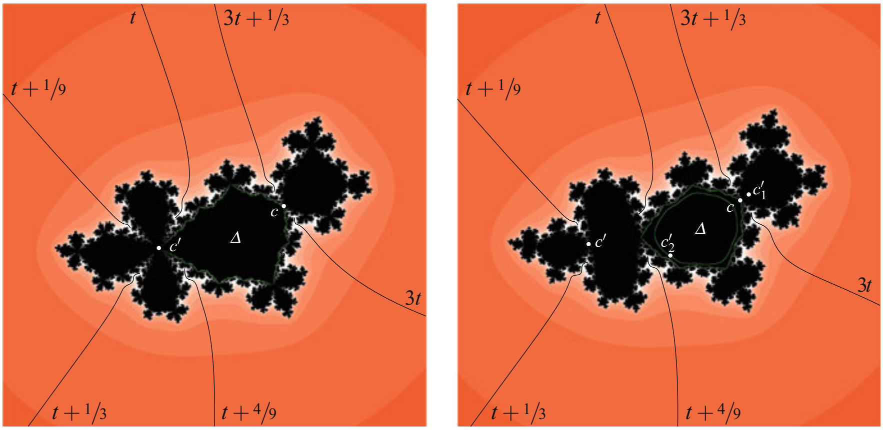

Fig. 5.3 Left: Filled Julia set of the cubic map f in Remark 5.8 with both critical points c, c' on the boundary of the Siegel disk Δ in the center of the picture, where $f(c') = c$. Right: A small perturbation of f in Remark 5.12 for which $c' \mapsto c'_1 = f(c') \mapsto c'_2 = f^{\circ 2}(c') \in \Delta$

has both critical points c, c' on $\partial\Delta$ with $f(c') = c$ as shown in Fig. 5.3 left. The critical point c' is the landing point of four rays at angles $t, t + \frac{1}{9}, t + \frac{1}{3}, t + \frac{4}{9}$ which map under f to the two rays at angles $3t, 3t + \frac{1}{3}$ landing at c_1. Here $t \approx 0.30762195$. The set X' in this example is not a rotation set since the complement of these six rays already fails to contain two disjoint open intervals of length $\frac{1}{3}$ (Corollary 2.16). However, removing $t + \frac{1}{9}, t + \frac{1}{3}$ and all their preimages from X' will yield a minimal rotation set X.

Remark 5.9 The congruences in (5.5) determine $\sigma(X)$ uniquely from the knowledge of the internal angles $\alpha_1, \ldots, \alpha_{d-1}$ except when $\alpha_i = 0 \pmod{\mathbb{Z}}$ for all i. This corresponds to the case where there is a single marked root $w_0 = \cdots = w_{d-2}$ which is necessarily a critical point of local degree d (compare Corollary 5.11 below). This type of ambiguity has already been pointed out in Remark 3.4 and can now be understood from the dynamical standpoint. For example, when $d = 4$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0 \pmod{\mathbb{Z}}$, we have the possible candidates

$$\sigma(X) = (0, 0, 1) \quad \text{or} \quad (0, 1, 1) \quad \text{or} \quad (1, 1, 1)$$

which correspond to quartic polynomials which are conjugate by the 120° rotation around the origin. Dynamically, these cases can be distinguished by the position of the Siegel disk Δ among the three fixed rays $R(0)$, $R(\frac{1}{3})$, $R(\frac{2}{3})$ (see Fig. 5.4).

Let us collect some corollaries of Theorem 5.7. As before, let $w_i = \iota(\zeta_i)$ $(0 \leq i \leq d-2)$ be the marked roots of f. To simplify the notation, we denote the limb

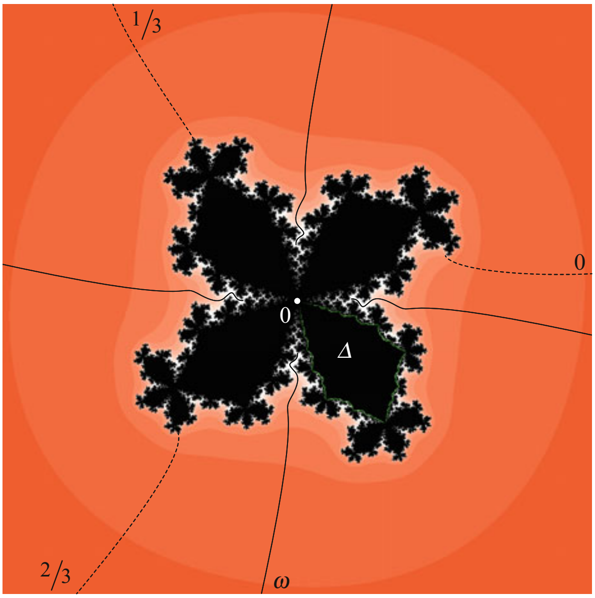

Fig. 5.4 Filled Julia set of a unicritical quartic polynomial $f(z) = z^4 + c$ with a Siegel disk Δ of the golden mean rotation number. Here the corresponding rotation set X has $\sigma(X) = (0, 0, 1)$. Conjugating f with the 120° and 240° rotations around the origin yields quartics with $\sigma(X) = (1, 1, 1)$ and $(0, 1, 1)$. In this example, $c \approx 0.59612528 - 0.46108628\,i$ and $\omega \approx 0.68914956$

$L(w_i)$ by L_i, the wake $W(w_i)$ by W_i and the gap $I(w_i)$ by I_i. The following can be thought of as the irrational counterpart of a result of Goldberg and Milnor in [12]:

Theorem 5.10 *Let X be the minimal rotation set of Theorem 5.7.*

(i) $I_0, \ldots, I_{d-2}$ are the major gaps of X.
(ii) The multiplicity n_i of I_i is the number of fixed rays in W_i. It is also the number of subscripts $0 \leq j \leq d-2$ for which $w_j = w_i$.
(iii) The limb $L_i = \overline{W_i} \cap K(f)$ contains n_i critical points of f counting multiplicities.

Proof By the proof of Theorem 5.7 every I_i is a gap of X. Since W_i contains the fixed ray $R(u_i)$, the gap I_i contains the fixed point u_i of m_d, so it must be major. By Lemma 2.13, the multiplicity n_i of I_i is the number of fixed rays in W_i or the number of times w_i appears in the list $w_0, \ldots, w_{d-2}$. Since there are $d-1$ fixed rays, the sum $\sum n_i$ over distinct I_i's is $d-1$ so $I_0, \ldots, I_{d-2}$ account for all major gaps of X by Theorem 2.7. This proves (i) and (ii).

The proof of (iii) is based on an idea of [12]. Let $I_i = (t, t')$, so W_i is bounded by the rays $R(t)$ and $R(t')$. Let η be a small loop around w_i which intersects each of $R(t)$ and $R(t')$ once, say at $\psi(r_1 e^{2\pi i t})$ and $\psi(r_1 e^{2\pi i t'})$. Fix a large radius r_2. Construct a positively oriented Jordan curve by going out along $R(t)$ from

$\psi(r_1 e^{2\pi it})$ to $\psi(r_2 e^{2\pi it})$, then following the equipotential curve $\{\psi(r_2 e^{2\pi is}) : t \leq s \leq t'\}$, then going down along $R(t')$ from $\psi(r_2 e^{2\pi it'})$ to $\psi(r_1 e^{2\pi it'})$, and finally going counter-clockwise along η from $\psi(r_1 e^{2\pi it'})$ back to $\psi(r_1 e^{2\pi it})$. Round off the four corners of this curve to obtain a smooth positively oriented Jordan curve γ. The number of the critical points of f in W_i is the number of roots of f' inside γ. By the argument principle, this is the winding number of the closed curve $f' \circ \gamma$ around 0, which is one less than the number of full counter-clockwise turns that the tangent vector to image curve $f \circ \gamma$ makes when γ is traversed once. By the construction of γ, this number is at least $n_i - k_i + 1$, where $k_i \geq 1$ is the local degree of f at w_i. Taking into account the fact that w_i itself is a critical point of multiplicity $k_i - 1$ if $k_i > 1$, it follows that the number N_i of the critical points of f in the limb L_i is at least $(n_i - k_i + 1) + (k_i - 1) = n_i$. Since the sums $\sum N_i$ and $\sum n_i$ over distinct I_i's are $d - 1$, it follows that $N_i = n_i$ for all i, as required. □

Corollary 5.11

(i) Every critical point $c \in \partial\Delta$ is a marked root. Moreover, the algebraic multiplicity of c (as a root of f') is at most the multiplicity of the corresponding gap $I(c)$.

(ii) Every marked root w_i whose corresponding gap $I_i = I(w_i)$ is taut must be a critical point.

(iii) A point on $\partial\Delta$ is a root if and only if it is pre-critical.

Proof First suppose $c \in \partial\Delta$ is a critical point. By Corollary 5.5 the critical value $f(c)$ is the landing point of at least one ray $R(t)$. As in the proof of Lemma 5.3, take small neighborhoods U of c and U' of $f(c)$ such that $f : U \to U'$ acts as the power $w \mapsto w^k$ for some $k \geq 2$. The intersection $R(t) \cap U'$ pulls back under f to the intersection of k rays $R(t_1), \dots, R(t_k)$ with U, all landing at c, where $t_1, \dots, t_k$ are among the d preimages of t under m_d. This proves that $K(f) \smallsetminus \{c\}$ is disconnected, hence c is a root by Lemma 5.2. Moreover, the wake $W(c)$ contains all $R(t_i)$'s in its closure, so $|I(c)| \geq (k-1)/d$. Hence $I(c)$ is a major gap of X, and the root c is marked by Theorem 5.10(i). The multiplicity n of $I(c)$ is the integer part of $d\,|I(c)|$, so $n \geq k - 1$. (Alternatively, we could invoke Theorem 5.10(iii) to conclude that $n \geq k - 1$.) This proves (i).

To verify (ii), suppose I_i is a taut gap of the form $(t, t' = t + n_i/d)$. Then w_i is the landing point of the rays $R(t)$, $R(t')$. Under f, these rays map to the same ray $R(m_d(t)) = R(m_d(t'))$ landing at $f(w_i)$. This shows f is not injective in any neighborhood of w_i, which proves w_i is a critical point.

For (iii), first note that by part (i) and the backward invariance in Lemma 5.3, all precritical points on $\partial\Delta$ are roots. Conversely, consider any root p so $I(p)$ is a gap of the minimal rotation set X of Theorem 5.7. Since $\rho(X)$ is irrational, Theorem 2.10 shows that there is a $k \geq 0$ such that $g_X^{\circ k}(I(p)) = I(f^{\circ k}(p))$ is a taut gap. By part (ii), $f^{\circ k}(p)$ is a critical point. □

Remark 5.12 Here are three comments related to various parts of the above corollary: (i) The algebraic multiplicity of a critical point $c \in \partial\Delta$ can be strictly less than the multiplicity of the gap $I(c)$. This happens precisely when the wake $W(c)$ contains a critical point of f. (ii) If a marked root w_i is critical, the gap I_i may be loose. For example, the cubic map f in Remark 5.8 has both critical points c, c' on $\partial\Delta$ with $f(c') = c$, where $I(c)$ is taut and $I(c')$ is loose (see Fig. 5.3 left). (iii) Marked roots can be non-critical. For example, one can perturb the above map to obtain a cubic with $c \in \partial\Delta$ and $f^{\circ 2}(c') \in \Delta$ (thus the critical point c' is "captured" by the Siegel disk Δ). Here the second marked root $f^{-1}(c) \cap \partial\Delta$ is non-critical. Figure 5.3 right shows one such perturbation where

$$f(z) = e^{\pi i(\sqrt{5}-1)} z + az^2 + z^3 \quad \text{with} \quad a \approx 0.54716981 - 0.31132075\, i.$$

The two examples before and after perturbation have identical minimal rotation sets X. We will discuss this phenomena in more detail in Sect. 5.4.

Corollary 5.13 *Suppose all critical points of f are on $\partial\Delta$. Then these critical points are precisely the marked roots $w_0, \ldots, w_{d-2}$, and the algebraic multiplicity of each w_i is equal to the multiplicity of its corresponding gap.*

Proof By Corollary 5.11 all critical points of f are marked roots. Let $c_1, \ldots, c_k$ be the distinct critical points of multiplicities $\alpha_1, \ldots, \alpha_k$. Let $n_1, \ldots, n_k$ be the multiplicities of the corresponding gaps. By Corollary 5.11(i), $\alpha_i \leq n_i$ for all i. Hence, by Theorem 2.7, $d - 1 = \sum \alpha_i \leq \sum n_i \leq d - 1$. It follows that $\alpha_i = n_i$ for all i and $\{c_1, \ldots, c_k\} = \{w_0, \ldots, w_{d-1}\}$. □

It would be interesting to investigate how the preceding constructions should be modified for indifferent fixed points with arbitrary irrational rotation numbers. The difficulty arises when the fixed point 0 is the center of a "wild" Siegel disk or is non-linearizable (a so-called "Ceremer point"). In this case, the natural candidate for the rotation set X would be the minimal set of angles of dynamic rays whose impressions meet $\partial\Delta$ in the Siegel case and the fixed point 0 in the Cremer case. But in the absence of some kind of control on the Julia set of such maps, proving analogous results seems out of reach even for quadratic polynomials.

5.3 The Quadratic Family

This section and the next illustrate the relation between indifferent fixed points and rotation sets in the low-degree cases $d = 2$ and $d = 3$, in both dynamical and parameter planes. The abstract analyses of these rotation sets, carried out in Sects. 4.5 and 4.6, come to life in these concrete realizations.

The case $d = 2$ is more straightforward and rather well-known. Consider the monic quadratic polynomial

$$P = P_\theta : z \mapsto e^{2\pi i\theta} z + z^2 \tag{5.6}$$

with an indifferent fixed point at the origin. When θ is rational of the form $p/q \neq 0$ in lowest terms, the parabolic fixed point 0 is the landing point of precisely q rays $R(t_1), \dots, R(t_q)$, where $X_{p/q} = \{t_1, \dots, t_q\}$ is the unique minimal rotation set under doubling with rotation number p/q. If as usual we assume $0, t_1, \dots, t_q$ are in positive cyclic order, it follows that the unique critical point $c = -e^{2\pi i\theta}/2$ lies in the wake bounded by $R(t_1), R(t_q)$, corresponding to the longest gap of $X_{p/q}$. Similarly, the critical value $v = P(c) = -e^{4\pi i\theta}/4$ lies in the wake bounded by $R(t_{1+p}), R(t_{q+p})$, corresponding to the shortest gap of $X_{p/q}$ (compare Fig. 5.5 left).

When θ is an irrational of bounded type (or more generally belongs to the full-measure set $\mathcal{E}$ in [25]), the Julia set $J(P)$ is locally connected. In this case the boundary of the Siegel disk Δ of P centered at 0 is a Jordan curve containing c, and the limb decomposition hypothesis automatically holds. It follows from the general results of the previous section that the set of angles of the rays that land on $\partial\Delta$ is precisely the minimal rotation set X_θ under doubling. Note that X_θ is a Cantor set with a single major gap of length $\frac{1}{2}$ bounded by the angles $\omega, \omega' = \omega - \frac{1}{2}$, where $0 < \omega = \omega(\theta) < \frac{1}{2}$ is the leading angle of X_θ as defined in Sect. 4.5. By Corollary 5.11, both rays $R(\omega), R(\omega')$ land at the critical point c which is the unique marked root. The precritical point $P^{-n}(c) \cap \partial\Delta$ is then the root whose corresponding wake defines the gap of X_θ of length $\frac{1}{2^n}$ (see Fig. 5.5 right).

The realization of rotation sets in the dynamical plane allows an alternative route to Lemma 4.24. The binary expansion $0.b_0b_1b_2\cdots$ of the leading angle $\omega = \omega(\theta)$ of X_θ is characterized by the condition $b_k = 1$ if and only if $2^k\omega \in (\frac{1}{2}, 1)$. If $\theta = p/q \neq 0$ and $X_{p/q} = \{t_1, \dots, t_q\}$ as above, then $0 \in (t_q, t_1)$ and

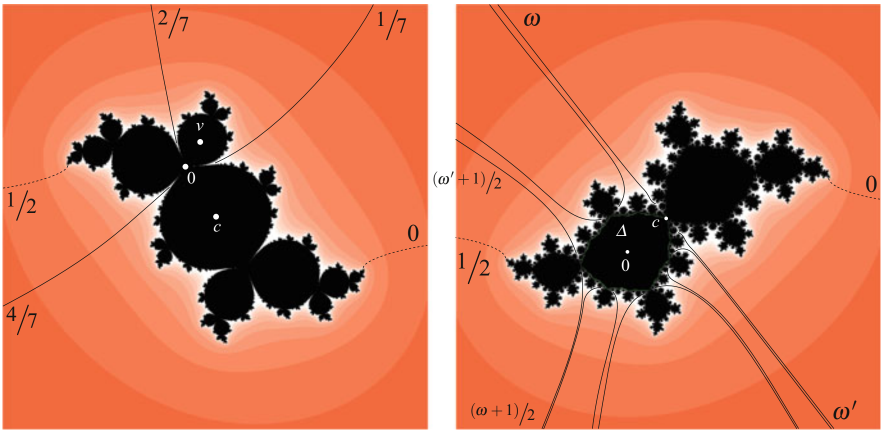

Fig. 5.5 Filled Julia set of the quadratic polynomial $z \mapsto e^{2\pi i\theta}z + z^2$ with the corresponding minimal rotation set X_θ under doubling. Left: The parabolic case $\theta = \frac{1}{3}$. Right: The Siegel case $\theta = \frac{(\sqrt{5}-1)}{2}$. Shown here are the wakes rooted at the critical point c and its first five preimages on $\partial\Delta$, which define the major gap (ω', ω) of X_θ and the five minor gaps of lengths 2^{-k} for $2 \leq k \leq 6$

$\frac{1}{2} \in (t_{q-p}, t_{q-p+1})$. Hence $t_1, \dots, t_{q-p} \in (0, \frac{1}{2})$ while $t_{q-p+1}, \dots, t_q \in (\frac{1}{2}, 1)$. Thus,

$$2^k \omega = t_{1+kp} \in \left(\frac{1}{2}, 1\right) \iff 1 + kp \ (\text{mod}\ q) \text{ is in } \{q - p + 1, \dots, q\}.$$

This is clearly equivalent to $k\theta \in [-\theta, 0)$.

A similar argument works when θ is an irrational and P has a "good" Siegel disk. In this case, $0 \in (\omega', \omega)$ and $\frac{1}{2} \in ((\omega' + 1)/2, (\omega + 1)/2)$, so $2^k\omega \in (\frac{1}{2}, 1)$ if and only if $2^k\omega \in ((\omega + 1)/2, \omega')$. But the pair $R(\omega)$, $R(\omega')$ land at c with the internal angle 0 and the pair $R((\omega+1)/2)$, $R((\omega'+1)/2)$ land at the preimage $P^{-1}(c) \cap \partial\Delta$ with the internal angle $-\theta$. It follows that $2^k\omega \in ((\omega+1)/2, \omega')$ precisely when $k\theta$, the internal angle of $P^{\circ k}(c)$, is in the interval $(-\theta, 0)$.

The parameter space of quadratic polynomials provides a complete catalog of all rotation sets under doubling. To see this, it will be convenient to represent our quadratics in the normal form $f_c(z) = z^2 + c$ where $c \in \mathbb{C}$. The connectedness locus

$$\mathscr{M}_2 = \{c \in \mathbb{C} : K(f_c) \text{ is connected}\},$$

commonly known as the ***Mandelbrot set***, is non-empty, compact, and full. If β_c denotes the Böttcher coordinate of f_c near ∞, the ***Douady-Hubbard map*** $\Phi : \mathbb{C} \smallsetminus \mathscr{M}_2 \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ which assigns to each c outside $\mathscr{M}_2$ the Böttcher coordinate $\beta_c(c)$ of the critical value $f_c(0) = c$, is a conformal isomorphism. By the ***parameter ray*** of $\mathscr{M}_2$ at angle $t \in \mathbb{T}$ we mean the real-analytic curve

$$\mathscr{R}(t) = \Phi^{-1}\big(\{re^{2\pi i t} : r > 1\}\big).$$

We say $\mathscr{R}(t)$ lands at $z \in \partial\mathscr{M}_2$ if $\lim_{r\to 1} \Phi^{-1}(re^{2\pi i t}) = z$.

Each quadratic P_θ in (5.6) is affinely conjugate to f_c with $c = c(\theta) = e^{2\pi i\theta}/2 - e^{4\pi i\theta}/4$. As θ varies in $[0, 1]$, the image $c(\theta)$ traces out a cardioid on the boundary of $\mathscr{M}_2$ that is prominently visible in Fig. 5.6. When $\theta \neq 0$ is rational, $c(\theta)$ is the landing point of the two parameter rays $\mathscr{R}(2\omega)$, $\mathscr{R}(2\omega')$. (Recall that (ω', ω) is the major gap of X_θ.) If θ is irrational, then $c(\theta)$ is the landing point of the unique parameter ray $\mathscr{R}(2\omega) = \mathscr{R}(2\omega')$. One may interpret this by saying that $c(\theta)$ is always the landing point of the parameter ray at angle $2\omega(\theta)$, which is a strictly increasing function of θ that jumps by $1/(2^q - 1)$ at every rational $\theta = p/q$ (Corollary 4.26). When θ is rational, the two parameter rays $\mathscr{R}(2\omega)$, $\mathscr{R}(2\omega')$ together with their landing point $c(\theta)$ define the ***parameter wake*** $\mathscr{W}(\theta)$, characterized by the property that the dynamic rays with angles in X_θ land at a fixed point of f_c if and only if $c \in \mathscr{W}(\theta) \cap \mathscr{M}_2$ (for a detailed treatment see [20] and compare Fig. 5.6).

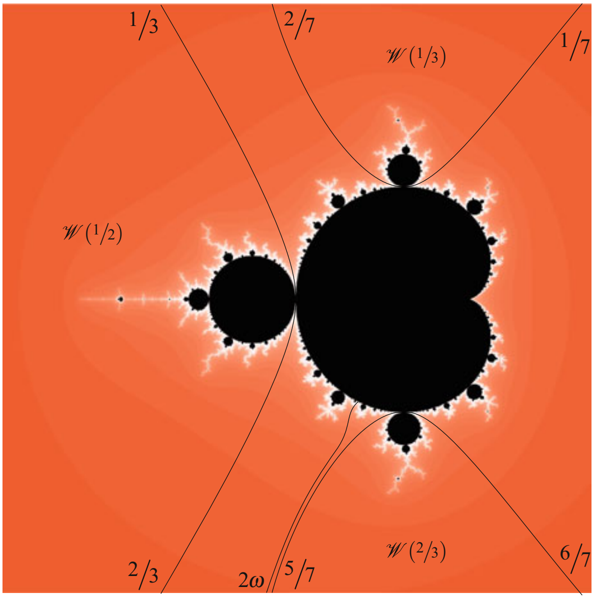

Fig. 5.6 The Mandelbrot set $\mathscr{M}_2$ and its parameter wakes $\mathscr{W}(\frac{1}{3})$, $\mathscr{W}(\frac{1}{2})$ and $\mathscr{W}(\frac{2}{3})$. Also shown is the parameter ray $\mathscr{R}(2\omega)$ landing at the quadratic that is affinely conjugate to $e^{2\pi i\theta}z + z^2$. Here $\theta = \frac{(\sqrt{5}-1)}{2}$ and $\omega = \omega(\theta) \approx 0.35490172$

Remark 5.14 The family of degree d unicritical polynomials $z \mapsto z^d + c$ exhibits very similar features in relation with rotation sets. As an example, the cubic map $f_c : z \mapsto z^3 + c$ has an indifferent fixed point of multiplier $e^{2\pi i\theta}$ if and only if

$$c = \pm c(\theta) \quad \text{where} \quad c(\theta) = -\frac{1}{3\sqrt{3}}\, e^{3\pi i\theta} + \frac{1}{\sqrt{3}}\, e^{\pi i\theta}.$$

The maps $f_{c(\theta)}$ and $f_{-c(\theta)}$ are conjugate by the 180° rotation $z \mapsto -z$. The angles of the dynamic rays of $f_{c(\theta)}$ that land on the indifferent fixed point when θ is rational, or on the boundary of the Siegel disk when θ is a suitable irrational, form the rotation set $X_{\theta,1}$ under tripling. The rotation set associated with the conjugate map $f_{-c(\theta)}$ is of course $X_{\theta,0}$. As θ varies in $[0, 1]$, the images $\pm c(\theta)$ trace out an algebraic curve (a nephroid) on the boundary of the corresponding connectedness locus $\mathscr{M}_3$ which bounds the central hyperbolic component containing $c = 0$. The analog of the Douady-Hubbard map is a conformal isomorphism $\mathbb{C} \smallsetminus \mathscr{M}_3 \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$, which can be used to define parameter rays in the c-plane. The boundary point $c(\theta)$ is the landing point of the parameter ray at angle $3\omega(\theta, 1)$, which strictly increases from 0 to $\frac{1}{2}$, jumping by $2/(3^q - 1)$ at every rational $\theta = p/q$ (Corollary 4.32). Similarly, $-c(\theta)$ is the landing point of the parameter ray at angle $3\omega(\theta, 0) = 3\omega(\theta, 1) + \frac{1}{2}$, which strictly increases from $\frac{1}{2}$ to 1 with similar jumps at every rational θ. As in the case of the Mandelbrot set, there is an analogous notion of parameter wakes for $\mathscr{M}_3$ and their dynamical characterization (see Fig. 5.7).

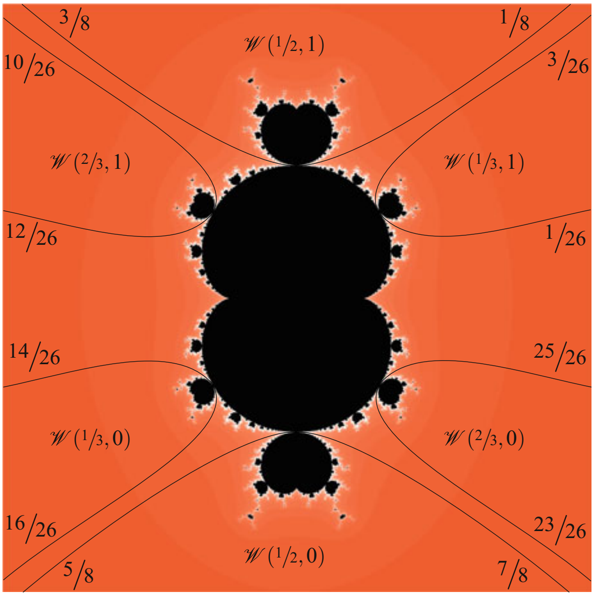

Fig. 5.7 The connectedness locus $\mathscr{M}_3$ of the unicritical cubic family $\{f_c : z \mapsto z^3 + c\}_{c \in \mathbb{C}}$, with selected parameter rays and wakes. Here $\mathscr{W}(p/q, \delta) \cap \mathscr{M}_3$ for $\delta = 0, 1$ is precisely the set of parameters c for which the dynamical rays at angles in $X_{p/q,\delta}$ land at a fixed point of f_c

5.4 The Cubic Family

This section is somewhat expository and contains outlines of the results. Consider the space of monic cubic polynomials with an indifferent fixed point of multiplier $e^{2\pi i\theta}$ at the origin. Each such cubic has the form

$$f_a : z \mapsto e^{2\pi i\theta} z + az^2 + z^3 \quad \text{for some} \quad a \in \mathbb{C}. \tag{5.7}$$

Note that f_a and f_{-a} are affinely conjugate by the involution $z \mapsto -z$. One could thus look at the quotient of the a-plane under $a \mapsto -a$ (equivalently, work with the parameter a^2). However, for our purposes in this section we prefer to treat f_a and f_{-a} as distinct cubics.

The connectedness locus of this cubic family is defined by

$$\mathscr{M}_3(\theta) = \{a \in \mathbb{C} : K(f_a) \text{ is connected}\}.$$

It is not hard to verify that $\mathscr{M}_3(\theta)$ is a compact, connected and full subset of $\mathbb{C}$ which is invariant under the involution $a \mapsto -a$ [30].

When $a \in \mathscr{M}_3(\theta)$, both critical points of f_a belong to the filled Julia set $K(f_a)$. When $a \notin \mathscr{M}_3(\theta)$, exactly one of the critical points, labeled c_a, belongs to $K(f_a)$ while the other, labeled e_a, escapes to ∞. The escaping critical value $v_a = f_a(e_a)$

has two preimages under f_a: the critical point e_a itself (with multiplicity 2) and a regular point $\hat{e}_a$ which we call the escaping ***co-critical point***. The Böttcher coordinate β_a of f_a near ∞ is defined and holomorphic in some neighborhood of $\hat{e}_a$. The analog of the Douady-Hubbard map $\Phi : \mathbb{C} \smallsetminus \mathscr{M}_3(\theta) \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ defined by

$$\Phi(a) = \beta_a(\hat{e}_a)$$

is a conformal isomorphism [6]. We define the ***parameter ray*** at angle $t \in \mathbb{T}$ by

$$\mathscr{R}(t) = \{\Phi^{-1}(re^{2\pi i t}) : r > 1\}.$$

We study the realization of rotation sets under m_3 in the dynamical plane of f_a as well as the parameter a-plane. The discussion is presented in two cases depending on whether θ is rational or an irrational of bounded type. We will outline the first case only briefly, as our main interest is the case of cubics with Siegel disks.

The parabolic case. Let us assume θ is rational of the form $p/q \neq 0$ in lowest terms. By the discussion of Sect. 5.2, the q-th iterate of f_a has the form

$$f_a^{\circ q}(z) = z + A(a)\, z^{q+1} + \cdots + z^{3^q}.$$

Here $A(a)$ is a polynomial of degree q in a with simple roots. Moreover, A is an even function if q is even, and odd function if q is odd. If $A(a) \neq 0$, the petal number of the parabolic point 0 is q and its ray number is q or $2q$. If, on the other hand, $A(a) = 0$, then the above expression reduces to

$$f_a^{\circ q}(z) = z + B(a)\, z^{2q+1} + \cdots + z^{3^q}.$$

where $B(a) \neq 0$, so the petal and ray numbers are both $2q$. In this case, we say f_a has a ***degenerate parabolic*** fixed point at 0.

By Lemma 5.1 the set X_a of angles of the dynamic rays of f_a that land at 0 is a rotation set under tripling with $\rho(X_a) = p/q$, which consists of one or two q-cycles. The deployment vector of X_a has the form $\delta(X_a) = (\delta_a, 1 - \delta_a)$, where $\delta_a \in [0, 1]$ is the ***deployment probability*** of f_a, i.e., the probability that a dynamic ray $R_a(t)$ of f_a landing on 0 has its angle t in $(0, \frac{1}{2})$. Note that by symmetry,

$$\delta_{-a} = 1 - \delta_a \qquad a \in \mathscr{M}_3(p/q).$$

First suppose the ray number is q, so X_a is a single q-cycle $\{t_1, \ldots, t_q\}$. Thus, in the notation of Sect. 4.6, $X_a = X_{p/q,i/q}$ for some $0 \leq i \leq q$. If we assume $0, t_1, \ldots, t_q$ are in positive cyclic order, it follows that one critical point of f_a lies in the wake bounded by the dynamic rays $R_a(t_q), R_a(t_1)$, the other in the wake bounded by $R_a(t_i), R_a(t_{i+1})$. Thus, the deployment probability $\delta_a = i/q$ is determined by the "combinatorial distance" i between the two critical points of f_a (that is, how many wakes they are apart). Figure 5.1 left illustrates this case with $p/q = i/q = \frac{1}{3}$.

Next consider the case where the ray number is $2q$, so $X_a = \{t_1, \dots, t_{2q}\}$. Under tripling, each t_j maps to t_{j+2p} so X_a splits into two q-cycles. As these q-cycles are compatible, Theorem 3.16 shows that

$$X_a = X_{p/q,i/q} \cup X_{p/q,(i+1)/q}$$

for some $0 \le i \le q-1$. Now one critical point of f_a lies in the wake bounded by $R_a(t_{2q})$, $R_a(t_1)$, the other in the wake bounded by $R_a(t_{2i+1})$, $R_a(t_{2i+2})$. Thus, similar to the above case, the deployment probability $\delta_a = (2i+1)/(2q)$ is determined by the combinatorial distance $2i+1$ between the two critical points of f_a. Figure 5.1 right illustrates this case with $p/q = i/q = \frac{1}{3}$.

Turning the attention to the parameter space, one can identify the following types of the interior components for $\mathscr{M}_3(p/q)$:

- ***adjacent***, where the two critical points belong to the same attracting petal at 0;
- ***bi-transitive***, where the two critical points belong to different attracting petals at 0 in the same cycle;
- ***capture***, where the orbit of one critical point eventually hits the cycle of attracting petals at 0;
- ***hyperbolic-like***, where the orbit of one critical point converges to an attracting cycle.

Conjecturally, every interior component of $\mathscr{M}_3(p/q)$ is of one of the above types. In fact, the only possibility to rule out is a "queer" component in a small copy of the Mandelbrot set in $\mathscr{M}_3(p/q)$ in which the interior of $K(f_a)$ is the basin of attraction of 0 but the Julia set $J(f_a)$ has positive measure and admits an invariant line field.

Let $a_0, \dots, a_{q-1}$ denote the degenerate parabolic parameters, i.e., simple roots of the equation $A(a) = 0$. There is a chain of interior components $C_0, C_1, \dots, C_q$ of $\mathscr{M}_3(p/q)$ such that $\partial C_{i-1} \cap \partial C_i = \{a_i\}$ for $1 \le i \le q$. Here $C_i = -C_{q-i}$, with C_0 and C_q of adjacent type and $C_1, \dots, C_{q-1}$ of bi-transitive type (see Fig. 5.8). For every parameter $a \in C_i$, we have $\delta_a = i/q$.

The deployment probability δ_a can be determined throughout the connectedness locus $\mathscr{M}_3(p/q)$. Each degenerate parabolic parameter a_i is the landing point of four parameter rays whose angles are those of the dynamic rays of f_{a_i} that bound the Fatou components containing its co-critical points. Using the general results of Sect. 4.6 it is not hard to find explicit formulas for these angles in terms of the leading angles $\omega(p/q, i/q)$ and $\omega(p/q, (i+1)/q)$. An example of this computation for $p/q = \frac{2}{3}$ and $i = 0$ is shown in Fig. 5.9.

These $4q$ parameter rays together with their landing points $\{a_0, \dots, a_{q-1}\}$ divide the a-plane into $3q+1$ ***parameter wakes*** $\mathscr{W}_0, \dots, \mathscr{W}_q, \Omega_0^{\pm}, \dots, \Omega_{q-1}^{\pm}$. Here $\mathscr{W}_i$ contains C_i and the pair $\Omega_i^{\pm}$ separate $\mathscr{W}_i$ from $\mathscr{W}_{i+1}$ (see Fig. 5.8). We have $X_a = X_{p/q,i/q}$ if $a \in \mathscr{W}_i \cap \mathscr{M}_3(p/q)$, and $X_a = X_{p/q,i/q} \cup X_{p/q,(i+1)/q}$ if $a \in \Omega_i^{\pm} \cap \mathscr{M}_3(p/q)$. Thus,

$$\delta_a = \begin{cases} \frac{i}{q} & \text{if } a \in \mathscr{W}_i \cap \mathscr{M}_3(p/q) \\ \frac{2i+1}{2q} & \text{if } a \in \Omega_i^{\pm} \cap \mathscr{M}_3(p/q). \end{cases}$$

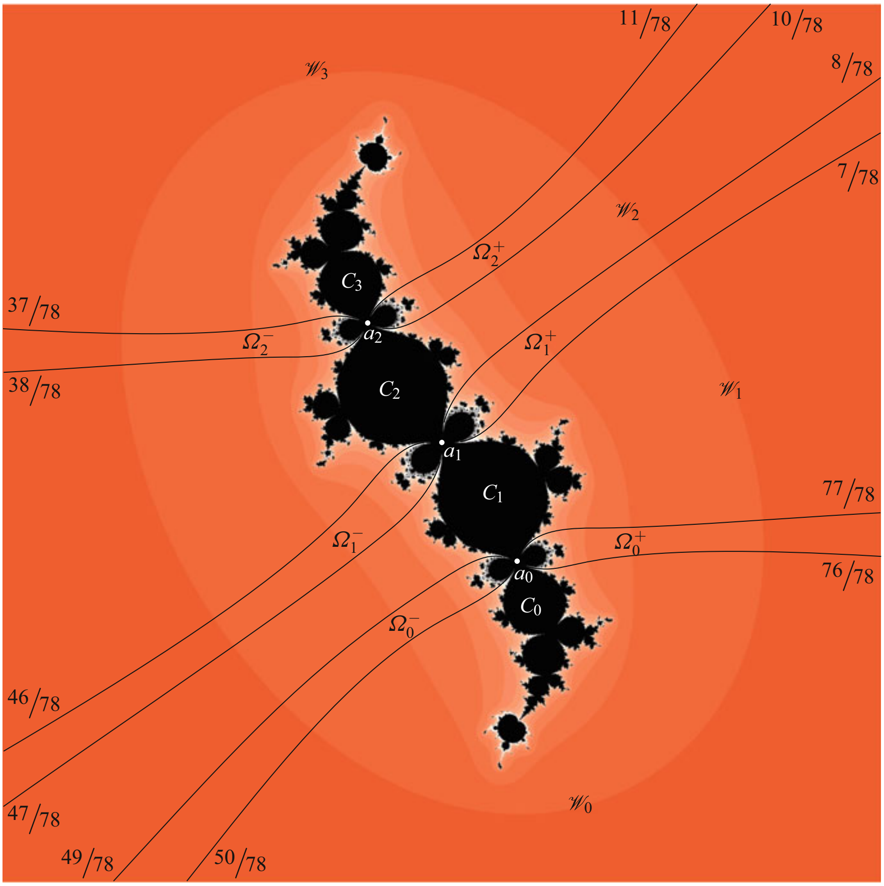

Fig. 5.8 The parabolic connectedness locus $\mathscr{M}_3(\frac{2}{3})$ and the chain of interior components C_0, C_1, C_2, C_3. The twelve parameter rays landing on the degenerate cubics a_0, a_1, a_2 define the ten wakes $\mathscr{W}_0, \mathscr{W}_1, \mathscr{W}_2 \mathscr{W}_3$ and $\Omega_0^{\pm}, \Omega_1^{\pm}, \Omega_2^{\pm}$. The deployment probability δ_a takes the value $i/3$ on $\mathscr{W}_i \cap \mathscr{M}_3(\frac{2}{3})$ and $(2i+1)/6$ on $\Omega_i^{\pm} \cap \mathscr{M}_3(\frac{2}{3})$

A detailed analysis of the landing properties of some of the parameter rays of $\mathscr{M}_3(p/q)$ can be found in [3].

The "good" Siegel case. Now suppose θ is an irrational of bounded type, so the fixed point 0 of f_a is the center of a Siegel disk Δ_a. The boundary $\partial \Delta_a$ is then a Jordan curve (in fact a quasicircle) passing through one or both critical points of f_a.

One can easily identify the following two types of interior components of the connectedness locus $\mathscr{M}_3(\theta)$:

- ***capture***, where the orbit of one critical point eventually hits the Siegel disk;
- ***hyperbolic-like***, where the orbit of one critical point converges to an attracting cycle.

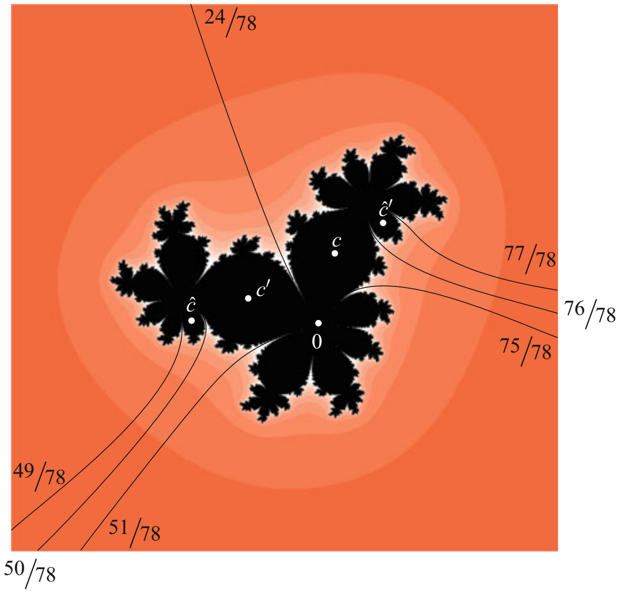

Fig. 5.9 Filled Julia set of the degenerate parabolic f_a in $\mathscr{M}_3(\frac{2}{3})$ with $X_a = X_{\frac{2}{3},\frac{0}{3}} \cup X_{\frac{2}{3},\frac{1}{3}} = \{\frac{24}{78}, \frac{51}{78}, \frac{60}{78}, \frac{69}{78}, \frac{72}{78}, \frac{75}{78}\}$ and $\delta_a = \frac{1}{6}$. Here $a \approx 0.68308975 - 1.08669099\,i$. The ray pairs at angles $(\frac{75}{78}, \frac{24}{78})$ and $(\frac{24}{78}, \frac{51}{78})$ bound the Fatou components containing the critical points c and c', respectively. It follows that the ray pairs at angles $(\frac{75}{78} - \frac{1}{3} = \frac{49}{78}, \frac{24}{78} + \frac{1}{3} = \frac{50}{78})$ and $(\frac{24}{78} + \frac{2}{3} = \frac{76}{78}, \frac{51}{78} + \frac{1}{3} = \frac{77}{78})$ bound the Fatou components containing the co-critical points $\hat{c}$ and $\hat{c}'$, respectively

As in the rational case, it is conjectured that every interior component of $\mathscr{M}_3(\theta)$ has one of these types. In Fig. 5.10 left the capture components are the blue bulbs, while the hyperbolic-like components are the grey bulbs that belong to a small copy of the Mandelbrot set.

The following is proved in [30]:

Theorem 5.15 *There is a closed embedded arc $\Gamma(\theta) \subset \mathscr{M}_3(\theta)$ with the property that $a \in \Gamma(\theta)$ if and only if $\partial \Delta_a$ contains both critical points of f_a.*

The arc $\Gamma(\theta)$ is clearly invariant under the involution $a \mapsto -a$. The endpoints of $\Gamma(\theta)$ are the parameters $\pm\sqrt{3e^{2\pi i\theta}}$ corresponding to the cubics with a double critical point. We denote by a_0 the endpoint in the lower half-plane, so $-a_0$ is the other endpoint in the upper half-plane. The midpoint of $\Gamma(\theta)$ is the parameter $a = 0$ corresponding to the cubic with centered critical points. See Fig. 5.10 right.[3]

[3]In [30] the cubics are given in the normal form

$$z \mapsto e^{2\pi i\theta} z\Big(1 - \frac{1}{2}\big(1 + \frac{1}{c}\big)z + \frac{1}{3c}z^2\Big) \qquad c \in \mathbb{C}^*,$$

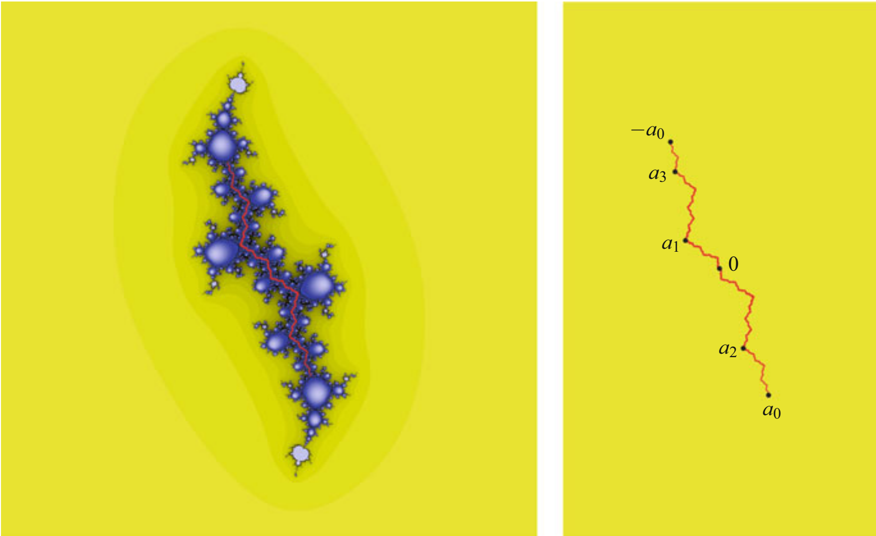

Fig. 5.10 Left: The cubic connectedness locus $\mathscr{M}_3(\theta) \subset \mathbb{C}$. Right: The arc $\Gamma(\theta) \subset \mathscr{M}_3(\theta)$. Here $\theta = \frac{(\sqrt{5}-1)}{2}$

The arc $\Gamma(\theta)$ is parametrized by the internal angle between the two critical points (as defined in Sect. 5.2). More precisely, if $a \in \Gamma(\theta)$ and if the internal angles of the critical points of f_a are 0 and $\tau_a \in [0, 1]$, where $\tau_{a_0} = 0$ and $\tau_{-a_0} = 1$, then the map $a \mapsto \tau_a$ is a homeomorphism $\Gamma(\theta) \to [0, 1]$.

Here are two alternative characterizations of $\Gamma(\theta)$:

- $\Gamma(\theta)$ is the set of parameters near which the boundary $\partial\Delta_a$ fails to move holomorphically. In fact, if U is a disk which does not intersect $\Gamma(\theta)$, then the critical point of f_a that lies on $\partial\Delta_a$ depends holomorphically on $a \in U$, so its forward orbit moves holomorphically over U. By the λ-lemma [16], this holomorphic motion extends to a holomorphic motion of the closure of this forward orbit, which is just $\partial\Delta_a$. On the other hand, if U is a disk that does intersect $\Gamma(\theta)$, the critical point on $\partial\Delta_a$ cannot be followed holomorphically in U, which shows $\partial\Delta_a$ does not move holomorphically over U (although it still moves continuously in the Hausdorff topology [30]).
- Let $\mathrm{rad}(a)$ denote the conformal radius of the Siegel disk Δ_a relative to its center 0. The function $a \mapsto \log \mathrm{rad}(a)$ is continuous and subharmonic in $\mathbb{C}$ and harmonic off $\Gamma(\theta)$ (see [5] and [32]). The arc $\Gamma(\theta)$ can be described as the support of the generalized Laplacian $4\partial\bar{\partial} \log \mathrm{rad}$. This has been proved by I. Zidane and independently by the author (unpublished).

with marked critical points at 1 and c. The punctured c-plane is a double-cover of the a^2-plane, branched at $c = \pm 1$. In this normalization, $\Gamma(\theta)$ appears as a Jordan curve passing through these branch points, and is invariant under the involution $c \mapsto 1/c$.

An adaptation of the work of Petersen in [23], using complex a priori bounds for critical circle maps, proves that for every $a \in \Gamma(\theta)$ the Julia set of f_a is locally connected and has measure zero. Thus, along $\Gamma(\theta)$ the Julia set is tame enough to allow the general constructions of Sect. 5.2 to go through. In particular, it follows from Theorem 5.7 that we can assign to each $a \in \Gamma(\theta)$ a minimal rotation set X_a under tripling with $\rho(X_a) = \theta$, consisting of angles of the dynamic rays of f_a which land on $\partial \Delta_a$. Notice the symmetry

$$X_{-a} = X_a + \frac{1}{2} \pmod{\mathbb{Z}}. \tag{5.8}$$

For each $a \in \Gamma(\theta)$ consider the deployment vector $\delta(X_a) = (\delta_a, 1 - \delta_a)$, where $\delta_a \in [0, 1]$ is the deployment probability of f_a, i.e., the probability that a dynamic ray $R_a(t)$ landing on $\partial \Delta_a$ has its angle t in $(0, \frac{1}{2})$. It follows from the symmetry relation (5.8) that

$$\delta_{-a} = 1 - \delta_a \qquad a \in \Gamma(\theta).$$

At the two endpoints $a = \pm a_0$ of $\Gamma(\theta)$ the cubic f_a has a double critical point whose wake contains both dynamic rays $R_a(0)$ and $R_a(\frac{1}{2})$. At any other $a \in \Gamma(\theta)$ the critical points of f_a are distinct and we label them as $*_a$ and $*'_a$ by requiring that the wake $W(*_a)$ contains $R_a(0)$ and the wake $W(*'_a)$ contains $R_a(\frac{1}{2})$. Under this labeling, the internal angle of $*_a$ will be 0 and that of $*'_a$ will be τ_a.

The following is an immediate corollary of Theorem 5.7:

Theorem 5.16 *For every parameter $a \in \Gamma(\theta)$, the deployment probability of X_a is the internal angle between the two critical points of f_a:*

$$\delta_a = \tau_a.$$

Thus, starting at the endpoint a_0 of $\Gamma(\theta)$ in the lower half-plane and moving to the other endpoint $-a_0$, the probability δ_a increases monotonically and takes all values between 0 and 1. In particular, the family $\{X_a\}_{a \in \Gamma(\theta)}$ spans all minimal rotation sets under tripling with $\rho(X_a) = \theta$.

For each integer $n \geq 1$, let a_n be the unique parameter on $\Gamma(\theta)$ for which $\delta_{a_n} = n\theta \pmod{\mathbb{Z}}$ (the first few a_n are shown in Fig. 5.10 right). Using Theorem 5.16, it is readily seen that $f_{a_n}^{\circ n}(*_{a_n}) = *'_{a_n}$. By Theorem 4.31, the rotation set X_{a_n} has a taut gap of length $\frac{1}{3}$ corresponding to the wake $W(*'_{a_n})$ and a loose gap of length $\frac{1}{3} + \frac{1}{3^{n+1}}$ corresponding to the wake $W(*_{a_n})$ (compare Fig. 5.12). Of course by symmetry the parameters $-a_n$ have similar dynamical description, with $*_a$ and $*'_a$ exchanged. Namely, $\delta_{-a_n} = -n\theta \pmod{\mathbb{Z}}$, $f_{-a_n}^{\circ n}(*'_{-a_n}) = *_{-a_n}$, and X_{-a_n} has a taut gap of length $\frac{1}{3}$ corresponding to $W(*_{-a_n})$ and a loose gap of length $\frac{1}{3} + \frac{1}{3^{n+1}}$ corresponding to $W(*'_{-a_n})$.[4]

[4]Each parameter $\pm a_n$ is the "root" of a capture component in which the $(n + 1)$-st iterate of one critical point hits the Siegel disk. We will not be using this fact in our presentation.

We can combinatorially describe $\Gamma(\theta)$ by specifying the angles of the candidate parameter rays that presumably land on it. This description is related to rotation sets under tripling, much like what we have seen in the case of the boundary of the main cardioid of the Mandelbrot set. It will be convenient to use Theorem 5.16 to parametrize $\Gamma(\theta)$ by the deployment probability. For each $\delta \in [0, 1]$, let $a(\delta) \in \Gamma(\theta)$ be the unique parameter with $\delta_{a(\delta)} = \delta$. Thus, $a(\frac{1}{2}) = 0$ and in terms of our previous notation, $a(0) = a_0$, $a(1) = -a_0$, and $a(\pm n\theta) = \pm a_n$ for all $n \geq 1$. If $\delta \neq n\theta \pmod{\mathbb{Z}}$ for all n, there are two angles $-\frac{1}{6} < s(\delta) < \frac{1}{6}$ and $\frac{1}{3} < t(\delta) < \frac{2}{3}$ such that the parameter rays $\mathscr{R}(s(\delta))$ and $\mathscr{R}(t(\delta))$ land at $a(\delta)$ (thus, in Fig. 5.14, $\mathscr{R}(s(\delta))$ lands at $a(\delta)$ from the right side of $\Gamma(\theta)$ while $\mathscr{R}(t(\delta))$ lands there from the left side). These angles can be expressed in terms of the leading angle $\omega(\theta, \delta)$ of $X_{a(\delta)} = X_{\theta,\delta}$ studied in Sect. 4.6:

$$t(\delta) = \omega(\theta, \delta) + \frac{1}{3}$$
$$s(\delta) = \omega(\theta, 1 - \delta) - \frac{1}{6}$$

This can be seen by examining Fig. 5.11 which illustrates the angles of the dynamic rays landing at the co-critical points of $f_{a(\delta)}$. Notice that by symmetry,

$$t(\delta) = s(1 - \delta) + \frac{1}{2}.$$

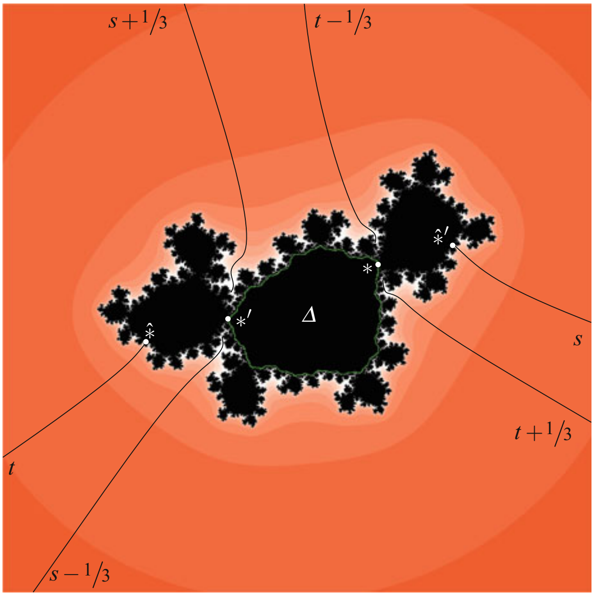

Fig. 5.11 Filled Julia set of a typical cubic map f_a with $a \in \Gamma(\theta)$, where the critical points $*, *'$ have disjoint orbits on $\partial\Delta$. Here the rays at angles $t \pm \frac{1}{3}$ land at $*$ and those at angles $s \pm \frac{1}{3}$ land at $*'$. If δ is the deployment probability of the associated rotation set X_a, we have $t - \frac{1}{3} = \omega(\theta, \delta)$ and $s - \frac{1}{3} = \omega(\theta, 1 - \delta) + \frac{1}{2}$. Thus, the rays landing at the co-critical points $\hat{*}, \hat{*}'$ have angles $t = \omega(\theta, \delta) + \frac{1}{3}$ and $s = \omega(\theta, 1 - \delta) - \frac{1}{6}$, respectively

Recall from Theorem 4.33 that the leading angle $\delta \mapsto \omega(\theta, \delta)$ is a decreasing, left-continuous function with a jump discontinuity of size $\frac{1}{3^{n+1}}$ at $\delta = n\theta \pmod{\mathbb{Z}}$ for each $n \geq 0$. Moreover,

$$\omega(\theta, 0) = \omega(\theta, 0^+) + \frac{1}{3} = \omega(\theta, 1) + \frac{1}{2}.$$

It follows from the above formulas that $s(\delta)$ is increasing and $t(\delta)$ is decreasing as a function of δ. For each $n \geq 1$ the angle $t(\delta)$ has a jump discontinuity of size $\frac{1}{3^{n+1}}$ at $\delta = n\theta \pmod{\mathbb{Z}}$, while $s(\delta)$ remains continuous there, and similarly, $s(\delta)$ has a jump discontinuity of size $\frac{1}{3^{n+1}}$ at $\delta = -n\theta \pmod{\mathbb{Z}}$, while $t(\delta)$ remains continuous there. These values of δ correspond to the parameters $\pm a_n$ along $\Gamma(\theta)$ and the aforementioned discontinuity suggests that every a_n with $n \geq 1$ is the landing point of three parameter rays at angles

$$\begin{aligned} t_n^- &= \omega(\theta, n\theta) + \frac{1}{3} - \frac{1}{3^{n+1}} \\ t_n^+ &= \omega(\theta, n\theta) + \frac{1}{3} \\ s_n &= \omega(\theta, -n\theta) - \frac{1}{6} \end{aligned}$$

while the parameter $-a_n$ is the landing point of the three parameter rays at angles

$$\begin{aligned} s_n^- &= \omega(\theta, n\theta) - \frac{1}{6} - \frac{1}{3^{n+1}} \\ s_n^+ &= \omega(\theta, n\theta) - \frac{1}{6} \\ t_n &= \omega(\theta, -n\theta) + \frac{1}{3}. \end{aligned}$$

These computations are illustrated in Fig. 5.12 which shows the angles of the dynamic rays that land at the co-critical points of f_{a_n}.

Finally, the endpoint a_0 of $\Gamma(\theta)$ is the landing point of the two parameter rays at angles

$$\begin{aligned} t_0^- &= \omega(\theta, 1) + \frac{1}{2} \\ t_0^+ &= \omega(\theta, 1) + \frac{5}{6}, \end{aligned}$$

while the other endpoint $-a_0$ is the landing point of the two parameter rays at angles

$$\begin{aligned} s_0^- &= \omega(\theta, 1) \\ s_0^+ &= \omega(\theta, 1) + \frac{1}{3}. \end{aligned}$$

Compare Fig. 5.13 which provides a justification for these formulas.

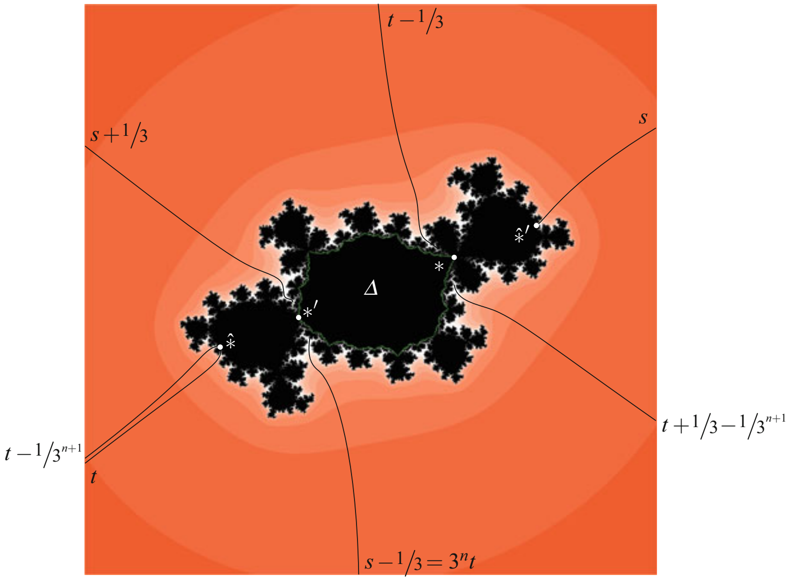

Fig. 5.12 Filled Julia set of the cubic map f_{a_n}, where the n-th iterate of the critical point $*$ hits the critical point $*'$. Here the rays at angles $s \pm \frac{1}{3}$ land at $*'$ and those at angles $t \pm \frac{1}{3}$ and $t \pm \frac{1}{3} - \frac{1}{3^{n+1}}$ land at $*$ (although only two of them, shown in the picture, are present in the rotation set X_{a_n}). We have $t - \frac{1}{3} = \omega(\theta, n\theta)$ and $s - \frac{1}{3} = \omega(\theta, -n\theta) + \frac{1}{2}$. Thus, the ray at angle $s = \omega(\theta, -n\theta) - \frac{1}{6}$ lands at the co-critical point $\hat{*}'$ and the rays at angles $t = \omega(\theta, n\theta) + \frac{1}{3}$ and $t - \frac{1}{3^{n+1}} = \omega(\theta, n\theta) + \frac{1}{3} - \frac{1}{3^{n+1}}$ land at the co-critical points $\hat{*}$

By Theorem 4.35, the above angles can be expressed rationally in terms of the (transcendental) base angle $\omega = \omega(\theta, 1)$. It follows that

$$t_n^+ = \frac{(3^n + 1)\omega + A_n}{2 \cdot 3^n} + \frac{1}{3}$$
$$s_n = \frac{(3^n + 1)\omega - B_n}{2} - \frac{1}{6},$$

where A_n, B_n are the integers defined by (4.16).

Example 5.17 For the golden mean $\theta = \frac{(\sqrt{5}-1)}{2}$, the base angle $\omega = \omega(\theta, 1)$ can be effectively computed with desired precision using the formula (4.13):

$$\omega \approx 0.128099593431 \cdots$$

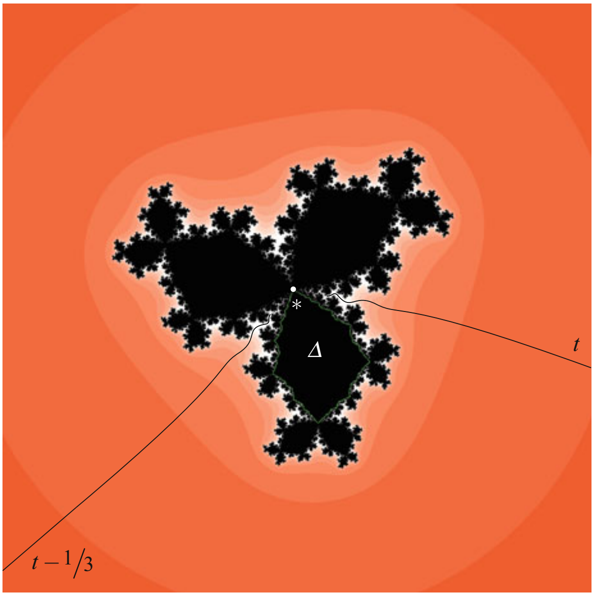

Fig. 5.13 Filled Julia set of the cubic map f_{a_0} with a double critical point $* = *'$ (which also coincides with the co-critical points $\hat{*} = \hat{*}'$). Here the rays at angles $t = \omega(\theta, 1) + \frac{5}{6}$ and $t - \frac{1}{3} = \omega(\theta, 1) + \frac{1}{2}$ land at $*$

Using the formula (4.16) it is easy to compute the integers A_n, B_n. Here are the results for $1 \le n \le 5$:

$$A_1 = 3^0 = 1 \qquad B_1 = 0$$

$$A_2 = 3^0 + 3^1 = 4 \qquad B_2 = 3^0 = 1$$

$$A_3 = 3^0 + 3^1 = 4 \qquad B_3 = 3^1 = 3$$

$$A_4 = 3^0 + 3^1 + 3^3 = 31 \qquad B_4 = 3^0 + 3^2 = 10$$

$$A_5 = 3^0 + 3^1 + 3^3 + 3^4 = 112 \qquad B_5 = 3^0 + 3^1 + 3^3 = 31.$$

The corresponding angles are listed in Table 5.1. Figure 5.14 shows selected parameter rays at these angles.

We can extend this picture to parameters outside the arc $\Gamma(\theta)$. One possible approach is to show that when θ is of bounded type, the filled Julia sets $K(f_a)$ for $a \in \mathscr{M}_3(\theta)$ satisfy the limb decomposition hypothesis in Sect. 5.2 so the rotation set X_a is well defined. This is already known for many parameters in $\mathscr{M}_3(\theta)$, including the hyperbolic-like ones, and is surely true for all capture parameters. An alternative route, which is outlined below, is to approach $\mathscr{M}_3(\theta)$ from outside, allowing disconnected Julia sets.

Outside the connectedness locus, the filled Julia set $K(f_a)$ consists of countably many homeomorphic copies of the filled Julia set of the quadratic polynomial

Table 5.1 Angles of some parameter rays which "land" on the arc $\Gamma(\theta)$ for $\theta = \frac{(\sqrt{5}-1)}{2}$

Angle	In terms of $\omega = \omega(\theta, 1)$	Approximate value
t_0^-	$\omega + \frac{1}{2}$	0.628099593431
t_0^+	$\omega + \frac{5}{6}$	0.961432926764
t_1^-	$\frac{2}{3}\,\omega + \frac{7}{18}$	0.474288617843
t_1^+	$\frac{2}{3}\,\omega + \frac{1}{2}$	0.585399728954
s_1	$2\omega - \frac{1}{6}$	0.089532520195
t_2^-	$\frac{5}{9}\,\omega + \frac{14}{27}$	0.589684959314
t_2^+	$\frac{5}{9}\,\omega + \frac{5}{9}$	0.626721996351
s_2	$5\omega + \frac{1}{3}$	0.973831300488
t_3^-	$\frac{14}{27}\,\omega + \frac{32}{81}$	0.461483739804
t_3^+	$\frac{14}{27}\,\omega + \frac{11}{27}$	0.473829418816
s_3	$14\omega - \frac{5}{3}$	0.126727641367
t_4^-	$\frac{41}{81}\,\omega + \frac{253}{486}$	0.585416666634
t_4^+	$\frac{41}{81}\,\omega + \frac{85}{162}$	0.589531892972
s_4	$41\omega - \frac{31}{6}$	0.085416664004
t_5^-	$\frac{122}{243}\,\omega + \frac{410}{729}$	0.626727642244
t_5^+	$\frac{122}{243}\,\omega + \frac{137}{243}$	0.628099384356
s_5	$122\omega - \frac{44}{3}$	0.961483731915

$P : z \mapsto e^{2\pi i\theta} z + z^2$ and uncountably many points. In particular, the connected component K_a of $K(f_a)$ containing the Siegel disk Δ_a, called the ***little filled Julia set***, is homeomorphic to $K(P)$. More precisely, let $G_a : \mathbb{C} \to \mathbb{R}$ be the Green's function of f_a as defined in Sect. 5.1, and U_a and V_a be the connected components of $G_a^{-1}[0, G_a(e_a))$ and $G_a^{-1}[0, G_a(e_a)/3)$ containing K_a, respectively (recall that e_a is the escaping critical point). Then U_a and V_a are Jordan domains with $K_a \subset V_a \subset \overline{V_a} \subset U_a$ and the restriction $f_a : V_a \to U_a$ is a degree 2 branched covering (see Fig. 5.15). According to Douady and Hubbard, this restriction is hybrid equivalent to the quadratic P, namely, there is a quasiconformal homeomorphism $\phi_a : U_a \to \phi_a(U_a)$ which satisfies $\phi_a \circ f_a = P \circ \phi_a$ in V_a, with $\phi_a(K_a) = K(P)$ and $\overline{\partial}\phi_a = 0$ a. e. on K_a (see for example [30] or [6]).

When a is outside $\mathscr{M}_3(\theta)$, it belongs to the parameter ray $\mathscr{R}(t)$ for a unique $t \in \mathbb{T}$ called the ***external angle*** of a. It follows that the dynamic rays $R_a(t \pm \frac{1}{3})$ are bifurcated and crash into the escaping critical point e_a. Let N_t be the countable dense set of angles whose forward m_3-orbit hit either of $t \pm \frac{1}{3}$. If $u \notin N_t$, the ray $R_a(u)$ is smooth. If $u \in N_t$, the ray $R_a(u)$ is bifurcated and crashes into an iterated preimage of e_a (only once if neither $t \pm \frac{1}{3}$ is periodic under m_3, infinitely many times

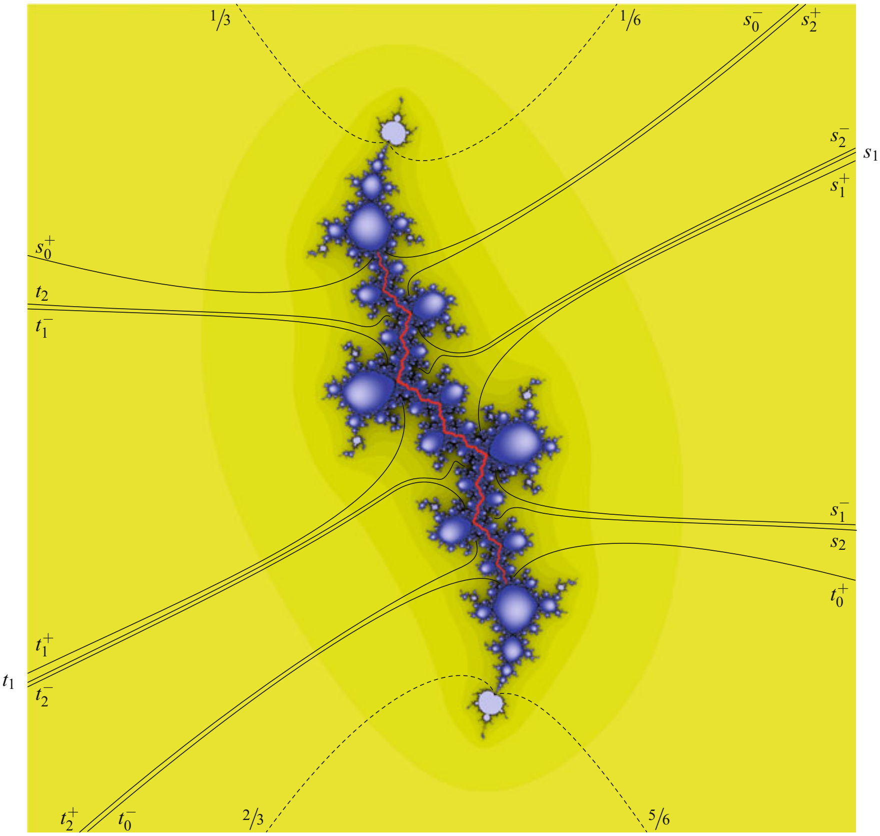

Fig. 5.14 Some parameter rays which "land" on the roots of capture components along the arc $\Gamma(\theta)$. Here $\theta = \frac{(\sqrt{5}-1)}{2}$

otherwise). For each $u \in N_t$ we can define the ***limit rays*** $R_a(u^\pm)$ as the pointwise limits

$$R_a(u^+) = \lim_{\substack{v \to u^+ \\ v \notin N_t}} R_a(v) \quad \text{and} \quad R_a(u^-) = \lim_{\substack{v \to u^- \\ v \notin N_t}} R_a(v),$$

with one always turning to the right at a bifurcation point, the other always turning to the left. Every point of the little filled Julia set K_a is accumulated by at least one smooth or limit ray. When $u \in N_t$, only one of $R_a(u^+)$ or $R_a(u^-)$ can accumulate on K_a and we agree to denote this simply by $R_a(u)$.

Consider the compact set

$$Y_t = \left\{ u \in \mathbb{T} : m_3^{\circ i}(u) \notin \left(t + \frac{1}{3}, t - \frac{1}{3} \right) \text{ for all } i \geq 0 \right\}.$$

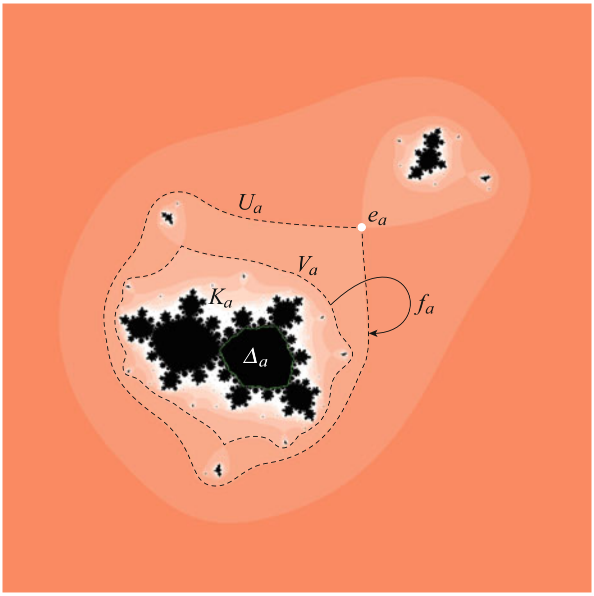

Fig. 5.15 Filled Julia set of a cubic f_a outside the connectedness locus $\mathscr{M}_3(\theta)$. The restriction $f_a : V_a \to U_a$ is a degree 2 branched covering hybrid equivalent to the quadratic $z \mapsto e^{2\pi i\theta} z + z^2$

It is not hard to show that Y_t contains a maximal m_3-invariant Cantor set A_t characterized by the property that $u \in A_t$ if and only if the (smooth or limit) ray $R_a(u)$ accumulates on K_a. Every endpoint of a gap of A_t belongs to N_t and the inclusion $A_t \supset Y_t \smallsetminus N_t$ holds. According to [2], there exists a degree 1 monotone map $h : \mathbb{T} \to \mathbb{T}$, with plateaus over the gaps of A_t, which satisfies

$$h \circ m_3 = m_2 \circ h \quad \text{on} \quad A_t. \tag{5.9}$$

The following is a special case of the main result of [26]:

Theorem 5.18 *The ray $R_a(u)$ with $u \in A_t$ lands at $z \in K_a$ if and only if the ray $R(h(u))$ of the quadratic P lands at $\phi_a(z) \in K(P)$.*

Since $K(P)$ is locally connected [23], it follows that all rays $R_a(u)$ with $u \in A_t$ land on K_a. In particular, since every point on the boundary of the Siegel disk of P is the landing point of one or two rays, and since $h|_{A_t}$ is at most 2-to-1, we see that every point of $\partial\Delta_a$ is the landing point of at most four (smooth or limit) rays. An argument similar to Sect. 5.2 for connected Julia sets then shows that the set of angles of rays landing on $\partial\Delta_a$ contains a minimal rotation set $X_a \subset A_t$ under tripling, with $\rho(X_a) = \theta$. Let us investigate the relation between the deployment probability $\delta_a \in [0, 1]$ of X_a and the external angle t of a.

We may assume without loss of generality that $s_0^+ = \omega + \frac{1}{3} < t \leq t_0^+ = \omega + \frac{5}{6}$ (the complementary case is treated by symmetry). Then the interval $(t + \frac{1}{3}, t - \frac{1}{3})$ of length $\frac{1}{3}$ is contained in the major gap I_0 of X_a that contains the fixed point 0. It will be convenient to first study the case where $X_a \cap N_t \neq \emptyset$, so at least one of the angles $t \pm \frac{1}{3}$ belongs to X_a. Since no angle in X_a is periodic under m_3, the rays $R_a(t \pm \frac{1}{3})$ crash at e_a and then join as a single smooth path to land at a point $w_a \in \partial \Delta_a$ which is characterized by the property that the internal angle from the non-escaping critical point $c_a \in \partial \Delta_a$ to w_a is δ_a. Here are the possibilities:

Case 1. $\delta_a = 0$. Then $w_a = c_a$. We either have $I_0 = (t, t - \frac{1}{3})$ where $t = \omega + \frac{5}{6} = t_0^+$, or $I_0 = (t + \frac{1}{3}, t)$ where $t = \omega + \frac{1}{2} = t_0^-$ (see Fig. 5.16a, b).

Case 2. $\delta_a = n\theta \pmod{\mathbb{Z}}$ for some $n \geq 1$. Then $c_a = f_a^{\circ n}(w_a)$. We either have

$$I_0 = \left(t + \frac{1}{3} - \frac{1}{3^{n+1}}, t - \frac{1}{3}\right), \quad \text{where} \quad t = \omega(\theta, n\theta) + \frac{1}{3} = t_n^+,$$

or

$$I_0 = \left(t + \frac{1}{3}, t - \frac{1}{3} + \frac{1}{3^{n+1}}\right), \quad \text{where} \quad t = \omega(\theta, n\theta) + \frac{1}{3} - \frac{1}{3^{n+1}} = t_n^-$$

(see Fig. 5.16c, d which show the case $n = 1$).

Case 3. $\delta_a = -n\theta \pmod{\mathbb{Z}}$ for some $n \geq 1$. Then $w_a = f_a^{\circ n}(c_a)$ and we have $I_0 = (t + \frac{1}{3}, t - \frac{1}{3})$ where $t = \omega(\theta, -n\theta) + \frac{1}{3} = t_n$ (see Fig. 5.16e which shows the case $n = 1$).

Case 4. $\delta_a \neq n\theta \pmod{\mathbb{Z}}$ for all integers n. In this case c_a and w_a have disjoint orbits on $\partial \Delta_a$, and we have $I_0 = (t + \frac{1}{3}, t - \frac{1}{3})$ where $t = t(\delta_a)$ (see Fig. 5.16f).

Using monotonicity of $\delta \mapsto \omega(\theta, \delta)$, it is easy to see that the above cases classify X_a for all external angles t except when $t \in (t_n^-, t_n^+)$ for some $n \geq 0$. As a corollary, we obtain

Corollary 5.19 *If the external angle t of $a \notin \mathscr{M}_3(\theta)$ lies in (t_n^-, t_n^+) for some $n \geq 0$, then X_a is contained in the set*

$$Y_t \smallsetminus N_t = \left\{ u \in \mathbb{T} : m_3^{\circ i}(u) \notin \left[t + \frac{1}{3}, t - \frac{1}{3}\right] \textit{for all } i \geq 0 \right\}.$$

In particular, every dynamic ray $R_a(u)$ with $u \in X_a$ is smooth.

It remains to determine X_a when t belongs to such an interval. We will need a preliminary observation:

Lemma 5.20 *Corollary 5.19 holds if we replace X_a with the rotation set $X_{\theta,n\theta}$.*

Proof We know that $X_{\theta,n\theta}$ has a loose gap $I_0 = (\alpha + \frac{1}{3} - \frac{1}{3^{n+1}}, \alpha - \frac{1}{3})$ containing 0 and a taut gap $(\beta + \frac{1}{3}, \beta - \frac{1}{3})$ containing $\frac{1}{2}$. Here

$$\alpha = \omega(\theta, n\theta) + \frac{1}{3} \quad \text{and} \quad \beta = \omega(\theta, -n\theta) - \frac{1}{6}$$

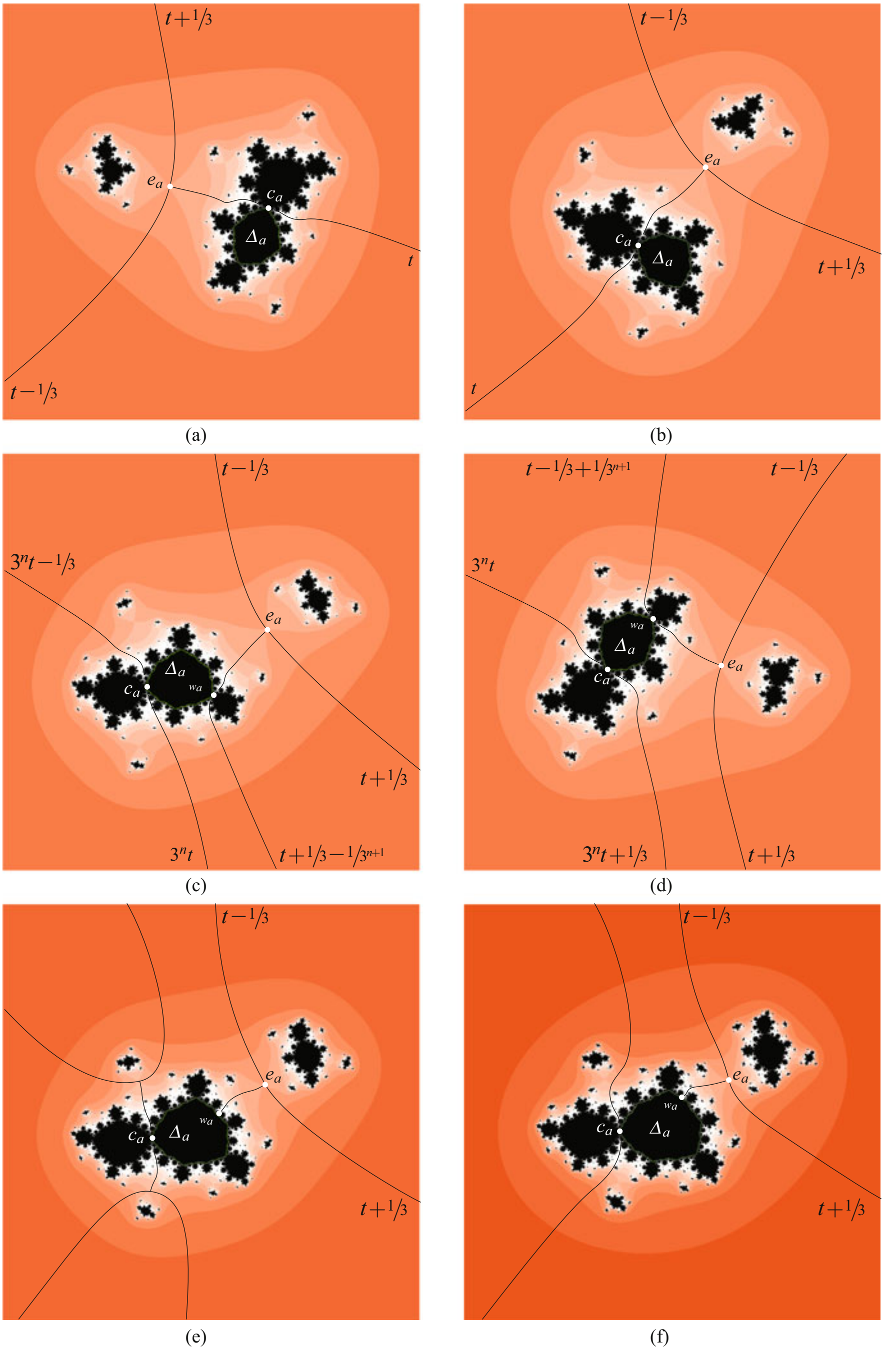

Fig. 5.16 Possible types of cubics f_a with $a \notin \mathscr{M}_3(\theta)$ which have a non-smooth ray landing on $\partial\Delta_a$. **(a)** $\delta_a = 0, t = t_0^+$. **(b)** $\delta_a = 0, t = t_0^-$. **(c)** $\delta_a = n\theta, t = t_n^+$. **(d)** $\delta_a = n\theta, t = t_n^-$. **(e)** $\delta_a = -n\theta, t = t_n$. **(f)** $\delta_a \neq n\theta, t = t(\delta_a)$

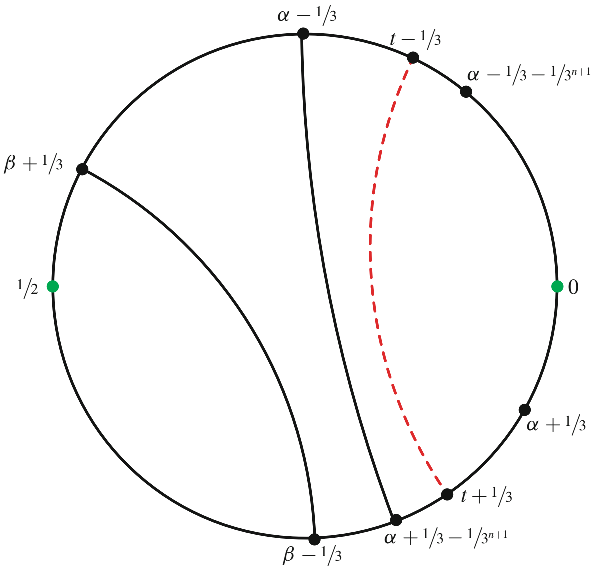

Fig. 5.17 Major gaps of $X_{\theta,n\theta}$ and the proof of Lemma 5.20

(see Fig. 5.17). We have

$$t_n^- = \omega(\theta, n\theta) + \frac{1}{3} - \frac{1}{3^{n+1}} = \alpha - \frac{1}{3^{n+1}} \quad \text{and} \quad t_n^+ = \omega(\theta, n\theta) + \frac{1}{3} = \alpha,$$

so the assumption $t_n^- < t < t_n^+$ implies $[t + \frac{1}{3}, t - \frac{1}{3}] \subset I_0$. Since the forward m_3-orbit of every $u \in X_{\theta,n\theta}$ avoids I_0, it must avoid the subinterval $[t + \frac{1}{3}, t - \frac{1}{3}]$, which implies $u \in Y_t \smallsetminus N_t$. □

Theorem 5.21 *If the external angle t of $a \notin \mathscr{M}_3(\theta)$ lies in (t_n^-, t_n^+) for some $n \geq 0$, then $X_a = X_{\theta,n\theta}$.*

Proof By Corollary 5.19, $X_a \subset Y_t \smallsetminus N_t \subset A_t$. The semiconjugacy h of (5.9) has plateaus over the gaps of A_t, so it is injective on X_a. Hence h maps X_a homeomorphically onto an m_2-invariant Cantor set $C = h(X_a)$. If φ is the canonical semiconjugacy associated with X_a, the composition $\varphi \circ h^{-1}$ is a well-defined degree 1 monotone map of the circle since each fiber of h maps to a single point under φ. Since $\varphi \circ h^{-1}$ semiconjugates $m_2|_C$ to the rotation r_θ, it follows that C is a rotation set for m_2 with $\rho(C) = \theta$. Similarly, by Lemma 5.20 $X_{\theta,n\theta} \subset Y_t \smallsetminus N_t \subset A_t$ and an identical argument shows that $C' = h(X_{\theta,n\theta})$ is also a rotation set for m_2 with $\rho(C') = \theta$. By the uniqueness of rotation sets under doubling, $C = C'$. It follows from injectivity of h that $X_a = X_{\theta,n\theta}$. □

Assuming that the rays $\mathscr{R}(t_n^\pm)$ in fact land at a_n, we can define the ***parameter wake*** $\mathscr{W}_n$ as the connected component of $\mathbb{C} \smallsetminus (\mathscr{R}(t_n^-) \cup \mathscr{R}(t_n^+) \cup \{a_n\})$ which does not meet $\Gamma(\theta)$. Using monotonicity of $\delta \mapsto \omega(\theta, \delta)$ it is easy to see that distinct parameter wakes are disjoint. Theorem 5.21 can be restated as saying that $X_a = X_{\theta,n\theta}$ whenever $a \in \mathscr{W}_n \setminus \mathscr{M}_3(\theta)$. We can show that this holds for every $a \in \mathscr{W}_n$ (this contains the claim that X_a is well defined for $a \in \mathscr{W}_n \cap \mathscr{M}_3(\theta)$). The argument uses holomorphic motions as follows.

A dynamic ray $R_a(u)$ moves holomorphically over the parameter $a \in \mathbb{C}$ as long as it remains smooth (see [6], Proposition 2). Lemma 5.20 shows that every ray $R_a(u)$ with $u \in X_{\theta,n\theta}$ is smooth for $a \in \mathscr{W}_n \setminus \mathscr{M}_3(\theta)$. Since $R_a(u)$ is trivially smooth for $a \in \mathscr{M}_3(\theta)$, it follows that this ray moves holomorphically over the entire parameter wake $\mathscr{W}_n$. By the λ-lemma, this motion extends to a holomorphic motion of the closure $\overline{R_a(u)}$ over $\mathscr{W}_n$. But for $a \in \mathscr{W}_n \setminus \mathscr{M}_3(\theta)$ this closure is $R_a(u)$ union its landing point on $\partial \Delta_a$. Since $\partial \Delta_a$ also moves holomorphically over $\mathscr{W}_n$, it follows that $R_a(u)$ lands on $\partial \Delta_a$ for every $a \in \mathscr{W}_n$, as required.

Away from the endpoints $\pm a_0$ of $\Gamma(\theta)$ the critical points of f_a can be continued analytically as a function of a (however, going around $\pm a_0$ will swap the two critical points, so the monodromy is non-trivial). In particular, the escaping and non-escaping critical points of f_a for $a \in \mathscr{W}_n \setminus \mathscr{M}_3(\theta)$ extend to holomorphic functions $a \mapsto e_a, c_a$ defined for all $a \in \mathscr{W}_n$. The preceding paragraph then shows that e_a belongs to the dynamical wake $W(f_a^{-n}(c_a))$ whenever $a \in \mathscr{W}_n$. It seems likely that this property is the dynamical characterization of the parameter wake $\mathscr{W}_n$.

To summarize, we have identified the dependence of δ_a on a in the following cases:

- If $a \in \overline{\mathscr{W}_0}$, then $\delta_a = 0$.
- If $a \in \overline{-\mathscr{W}_0}$, then $\delta_a = 1$.
- If $a \in \overline{\mathscr{W}_n \cup \mathscr{R}(s_n)}$ for some $n \geq 1$, then $\delta_a = n\theta \pmod{\mathbb{Z}}$.
- If $a \in \overline{-\mathscr{W}_n \cup \mathscr{R}(t_n)}$ for some $n \geq 1$, then $\delta_a = -n\theta \pmod{\mathbb{Z}}$.
- If $a \in \overline{\mathscr{R}(t(\delta)) \cup \mathscr{R}(s(\delta))}$ where $\delta \neq n\theta \pmod{\mathbb{Z}}$ for all n, then $\delta_a = \delta$.

It is conjectured that an analog of the limb decomposition hypothesis in Sect. 5.1 holds in this cubic parameter space, in the sense that the ***parameter limbs*** $\mathscr{L}_n = \mathscr{M}_3(\theta) \cap \overline{\mathscr{W}_n}$ have shrinking diameters as $n \to \infty$. Under this assumption, the connectedness locus $\mathscr{M}_3(\theta)$ would be the union of the arc $\Gamma(\theta)$ together with the parameter limbs $\pm\mathscr{L}_n$ for all $n \geq 0$, and the five cases above would describe δ_a (hence X_a) for every $a \in \mathbb{C}$.

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1. Lecture Notes aim to report new developments in all areas of mathematics and their applications – quickly, informally and at a high level. Mathematical texts analysing new developments in modelling and numerical simulation are welcome.

   Manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They may be based on specialised lecture courses. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for doctoral theses to be accepted for the Lecture Notes series, though habilitation theses may be appropriate.

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