The Twist Operator on Maniplexes

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Abstract Maniplexes are combinatorial objects that generalize, simultaneously, maps on surfaces and abstract polytopes. We are interested on studying highly symmetric maniplexes, particularly those having maximal 'rotational' symmetry. This paper introduces an operation on polytopes and maniplexes which, in its simplest form, can be interpreted as twisting the connection between facets. This is first described in detail in dimension 4 and then generalized to higher dimensions. Since the twist on a maniplex preserves all the orientation preserving symmetries of the original maniplex, we apply the operation to reflexible maniplexes, to attack the problem of finding chiral polytopes in higher dimensions.

Keywords Graph · Automorphism group · Symmetry · Polytope · Maniplex
Map · Flag · Transitivity · Rotary · Reflexible · Chiral

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1 Introduction

We have been struck by the beauty of the Platonic solids for thousands of years.

We saw them first when we asked this question: How can we make a polyhedron in such a way that the faces are identical regular polygons and there are the same number of them meeting at every vertex? In answer, a simple argument that the sum of the angles around a vertex must be less than 360° shows that there are exactly 5 possibilities:

(1) triangles meeting three around a vertex (the tetrahedron);
(2) triangles meeting four around a vertex (the octahedron);
(3) triangles meeting five around a vertex (the icosahedron);
(4) squares meeting three around a vertex (the cube);
(5) pentagons meeting three around a vertex (the dodecahedron).

These are often given the name of their Schläfli symbol: $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 3\}$, and $\{5, 3\}$, respectively.

It is worth noting that these five objects, besides having the requested local niceness, also have a global niceness, *symmetry*. They have *rotational* symmetry; we can rotate any of these objects about any of their faces and about any of their vertices. Moreover, they have *reflectional* symmetry; we can reflect about planes through face-centers, through vertices, across edges and along edges.

The discovery and proof that there are five and only five regular convex polyhedra was an interesting bit of reasoning. But it was over so soon, we hardly had a chance to enjoy it. How can we work in a more general but similar field?

There are at least three viable generalizations:

1. by regarding the cube, for instance, not as a solid hewn from stone, but as an assemblage of squares connected by hinges;
2. by regarding the cube as the convex hull of a finite set of points in 3-space;
3. by regarding the cube as having *faces* of many kinds: 2-faces (the squares), 1-faces (the edges) and 0-faces(the vertices);

The first of these viewpoints generalizes to *maps* on a surface and the second generalizes to *convex polytopes* in higher dimensions. The third generalizes to the idea of an *abstract polytope*. The first and third have *maniplexes* as their common generalization. All of these we define in the next section.

Then in Sect. 3 we discuss symmetry of maniplexes, with emphasis on chirality, meaning the property of having maximal rotational symmetry but no reflections. The aim we pursue is to devise a technique to construct higher rank chiral maniplexes, a task that has proved very difficult (see [15]).

2 Polyhedra, Maps, Maniplexes and Polytopes

A map is often defined as an embedding of a graph on a (compact, connected) surface so that components of the complement of the embedding (called faces) are topologically open discs. In some contexts graphs are allowed to have multiple edges, loops and semi-edges. We can, for example, regard the cube as an embedding of the graph Q_3 on the sphere.

To look more closely at the structure of a map, we find the following subdivision useful: choose a point in the interior of each face to be its *center* and a point in the relative interior of each edge to be its *midpoint*. Draw dotted lines to connect each face-center with each incidence of the surrounding vertices and edge-midpoints. The original edges and these dotted lines divide the surface into triangles called *flags*. Figure 1 shows the subdivision of the cube into flags.

Each flag corresponds to a mutual incidence of face, edge, and vertex, though several different flags may correspond to the same triple. For instance, consider the map shown in Fig. 2.

The map has one face A, an octagon, with opposite edges identified orientably. As a result, it has exactly one vertex v as well. The dotted lines divide it into its 16 flags. Each of the four flags marked with a dot correspond to the same triple (vertex, edge, face), namely, their vertex is v, their edge is 1 and their face is A.

Let Ω be the set of flags of a map $\mathcal{M}$. Then let r_0, r_1, r_2 be the permutations on Ω which match each flag f with its three immediate neighbors, as in Fig. 3. Define C to be the *connection group*, i.e., the group $\langle r_0, r_1, r_2 \rangle$.

In this paper, we will write elements of the connection group on the left: the image of the flag f under the connection r_2 is written $r_2 f$.

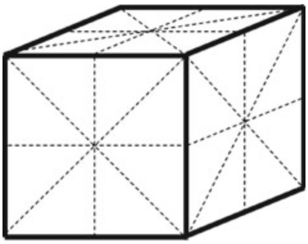

Fig. 1 The cube divided into flags

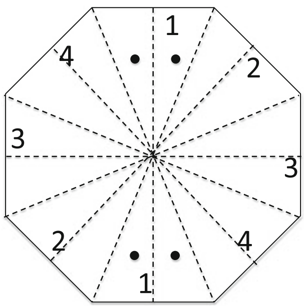

Fig. 2 A map with one face

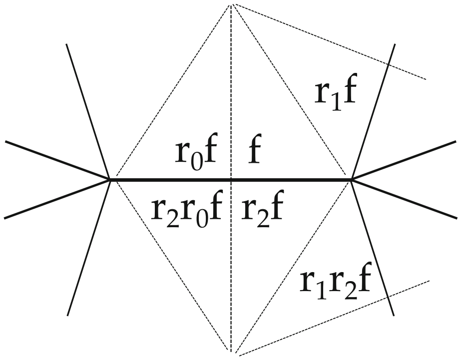

Fig. 3 Flags in a map

In Fig. 3, we see that f and $r_0 f$ are adjacent along a face-center-to-edge-midpoint line. Thus f and $r_0 f$ are incident to the same face and edge; they differ, if at all, in their incidences to a vertex, a 0-dimensional face of $\mathcal{M}$. We say these two flags are r_0-*adjacent*, or just *0-adjacent*. Similarly, f and $r_1 f$ meet the same 2-face and 0-face, while f and $r_2 f$ meet the same 0-face and 1-face. Notice from Fig. 3 that the flag r_2-adjacent to $r_0 f$ is also r_0-adjacent to $r_2 f$. In other words, as permutations on Ω, r_0 and r_2 commute.

We next take a slightly more abstract point of view by defining a map $\mathcal{M}$ to be a pair $(\Omega, [r_0, r_1, r_2])$ where Ω is a set of things called *flags*, the r_i's are permutations of order 2 on Ω, the *connection group* $C(\mathcal{M}) = \langle r_0, r_1, r_2 \rangle$ is transitive on Ω, and r_0 and r_2 commute. This $C(\mathcal{M})$ is often called the *monodromy group* of the map (see for example [9], and for higher ranks see also [13]). We can then think of vertices in $\mathcal{M}$ as orbits of $\langle r_1, r_2 \rangle$ in Ω. Similarly, edges correspond to orbits of $\langle r_0, r_2 \rangle$ and faces to orbits of $\langle r_0, r_1 \rangle$.

2.1 Maniplexes

This leads to the notion, introduced in [19], of a *maniplex*. An *(n+1)-dimensional maniplex* $\mathcal{M}$ is a pair $(\Omega, [r_0, r_1, \ldots, r_n])$, where Ω is a set of things called *flags* and each r_i is an involutory permutation on Ω such that (1) the *connection group* $C = \langle r_0, r_1, \ldots, r_n \rangle$ acts transitively on Ω, and (2) for all $0 \leq i < j - 1 < n - 1$, we have that $(r_i r_j)^2 = I$, where I is the identity in C. One can easily verify that every map on a surface is a 3-maniplex with Ω being its set of (triangular) flags. Furthermore, every 3-maniplex can be realised as a map on a surface. When we desire to avoid degeneracies, such as semi-edges or maps on a surface with boundary, we often also require that (3) each r_i and $r_i r_j$ are fixed-point-free, whenever $i \neq j$.

The *type* of a maniplex is the sequence $\{p_1, p_2, \ldots, p_n\}$, where each p_i is the order of $r_{i-1} r_i$ in C. The cube, then, is of type $\{4, 3\}$, the simplex is of type $\{3, 3, \ldots, 3\}$, and the 600-cell is of type $\{3, 3, 5\}$ (see [1, Chap. VII]). Type is well-defined even if not all faces have the same size. For example, the cuboctahedron, which is also the medial map of the cube, has triangles and squares, two of each meeting at each vertex. We say, then, that this map is of type $\{12, 4\}$.

Let C_i be the subgroup of C generated by all of the r_j's except r_i. Then an orbit of flags under C_i is called an *i-face*. A 0-face is a *vertex,* a 1-face is an *edge*, a 2-face is a *face*, an n-face is a *facet*. A facet of a facet is a *subfacet*; this is an orbit under $\langle r_0, r_1, \ldots, r_{n-2}\rangle$. The restriction to a subfacet of the permutation r_n acts as an isomorphism from that subfacet to some subfacet.

We wish to assign colors, red and white, to flags so that for any given two i-adjacent flags, either one is colored red (and not white) and one is colored white (and not red), or both flags are colored both red and white. Choose a *root* flag (sometimes called also *base flag*) and call it I. Let $\mathcal{R}_0 = \{I\}$. Recursively let $\mathcal{W}_i$ be the set of all flags adjacent to flags in $\mathcal{R}_i$, and let $\mathcal{R}_{i+1}$ be the set of all flags adjacent to flags in $\mathcal{W}_i$. Finally, let $\mathcal{R}$ be the union of all $\mathcal{R}_i$'s and similarly let $\mathcal{W}$ be the union of all $\mathcal{W}_i$'s. We often say this another way: let C^+ be the subgroup of C generated by all products of the form $r_i r_j$. Then $\mathcal{R}$ is the orbit of I under C^+ and $\mathcal{W}$ is the orbit of $r_0 I$ under C^+. Consider these as assignments of the colors red and white, respectively to the flags. There are two possibilities for the result:

1. it could happen that $\mathcal{R}$ and $\mathcal{W}$ are disjoint; in this case we say that $\mathcal{M}$ is *orientable*;
2. otherwise it must happen that $\mathcal{R} = \mathcal{W} = \Omega$, and in this case we say that $\mathcal{M}$ is *non-orientable*.

See [10] for more information about bi-colorings of flags.

The idea of having one flag designated as a 'root' flag helps us in several constructions and theorems. Henceforward, we will assume that any maniplex does have a root flag chosen, and that isomorphisms and projections are required to send root flag to root flag. Notice that the choice of root affects the colors of flags. In particular, let $\mathcal{M}'$ be the maniplex identical to $\mathcal{M}$ except that $I' = r_0 I$ is chosen as its root. We will refer to $\mathcal{M}'$ as the *mirror-image* of $\mathcal{M}$. The red flags of $\mathcal{M}$ are the white flags of $\mathcal{M}'$ and vice versa.

If $\mathcal{M} = (\Omega, [r_0, r_1, \ldots, r_{n-1}])$ is any n-maniplex, we can make an $(n+1)$-maniplex, called the *trivial* maniplex over $\mathcal{M}$, by using $\Omega \times \mathbb{Z}_2$ as a flag set (though we will write f_i instead of (f, i)), and connections $[s_0, s_1, \ldots, s_{n-1}, s_n]$, where $s_j f_i = (r_j f)_i$ for all $j = 0, 1, 2, \ldots, n-1$, $f \in \Omega, i \in \mathbb{Z}_2$, and $s_n f_i = f_{1-i}$. For example, the trivial maniplex over an n-gon has only two n-gonal faces over the same vertices and edges, and can be realised as a map on the sphere in such a way that the n vertices and the n edges lie on the equator. Note that if a maniplex $\mathcal{M}$ has type $\{p_1, p_2, \ldots, p_{n-1}\}$, then the trivial maniplex over $\mathcal{M}$ has type $\{p_1, p_2, \ldots, p_{n-1}, 2\}$. In particular, the trivial maniplex over an n-gon has type $\{n, 2\}$.

2.2 Polytopes

A convex polytope $\mathcal{P}$ is the convex hull of a finite set S of points in some Euclidean space. A *face* in $\mathcal{P}$ is the intersection of $\mathcal{P}$ with some hyperplane which does not separate S; it is an i-face if its affine hull has dimension i. The set of faces of $\mathcal{P}$ (of all dimensions) is partially ordered by inclusion. This partial ordering has certain properties, and these form the axiomatics for abstract polytopes. An *abstract polytope* is a partially ordered set $(\mathcal{P}, \leq)$ (whose elements are called *faces*) satisfying the following axioms:

(1) $\mathcal{P}$ contains a unique maximal and a unique minimal element.

(2) All maximal chains (these are called $flags$) have the same length. This allows us to assign a "rank" or "dimension" to each face. The unique minimal face (usually called "$\emptyset$") is given rank -1.

(3) If $f < g < h$ are consecutive in some flag, then there exists exactly one $g' \neq g$ such that $f < g' < h$. This axiom is usually called the *diamond condition*.

(4) For any $f \leq h$, the *section* $[f, h]$ is the sub-poset consisting of all faces g such that $f \leq g \leq h$. We require it to be true in any section that if Φ_1 and Φ_2 are any two flags of the section, then there is a sequence of flags of the section, beginning at Φ_1 and ending at Φ_2, such that any two consecutive flags differ in exactly one rank. This condition is called *strong flag-connectivity*.

See [2, 12, 16, 17] for illuminating work on polytopes and their symmetry.

In particular, if the rank of the maximal element is n, we call $\mathcal{P}$ an n-polytope. If f is any flag, let f_i be its face of rank i, let f_i' be the unique face of rank i other than f_i such that $f_{i-1} \leq f_i' \leq f_{i+1}$, and let f^i be the flag identical to f except that the face of rank i is f_i'. From a given n-polytope, we can form its *flag graph* in the following way: the vertex set is Ω, the set of all flags (maximal chains) in $\mathcal{P}$. It has edges of colors $0, 1, 2, \ldots, n-1$. The edges of color i are all $\{f, f^i\}$ for $f \in \Omega$. Thus, two vertices of the flag graph are joined (by an edge colored i) if they are flags which are identical except at rank i. Let r_i be the set of all edges colored i. Because all flags have the same entry at rank -1 and at rank n, r_i will be defined only for $i = 0, 1, \ldots, n-1$.

Thus, for every abstract polytope $\mathcal{P}$, the flag graph of $\mathcal{P}$ is a maniplex. The converse does not hold. Briefly, and very loosely, the flag graphs of polytopes are those maniplexes in which no contact between a facet and itself is permitted. We refer the reader to [7] for examples of non-polytopal maniplexes, as well as for necessary and sufficient conditions on a maniplex to be polytopal.

3 Symmetry

We define a *symmetry* of a maniplex $\mathcal{M}$ as a permutation of the flags which preserves the connections. We write symmetries on the right, so that the image of the flag f under the symmetry α is $f\alpha$. We denote the group of symmetries of $\mathcal{M}$ by $\mathrm{Aut}(\mathcal{M})$, and the notation gives the nice statement that for all $i \in \{0, 1, 2, \ldots, n\}$ and all $\alpha \in \mathrm{Aut}(\mathcal{M})$, we have that

$$(r_i f)\alpha = r_i(f\alpha).$$

There are two levels of symmetry that are particularly interesting in maps and maniplexes. First, we say that $\mathcal{M}$ is *rotary* provided that $\mathrm{Aut}(\mathcal{M})$ acts transitively on $\mathcal{R}$, the set of red flags. Also, $\mathcal{M}$ is *reflexible* provided that $\mathrm{Aut}(\mathcal{M})$ acts transitively on Ω. It follows trivially, then, that if $\mathcal{M}$ is rotary and non-orientable, then it is reflexible. If $\mathcal{M}$ is rotary but not reflexible, we say it is *chiral*. If $\mathcal{M}$ is orientable, it is often useful to consider $\mathrm{Aut}^+(\mathcal{M})$; this is the group of all symmetries which send $\mathcal{R}$ (the set of red flags) to itself (and so send $\mathcal{W}$ to itself).

A reflexible maniplex is nice in several ways. First, $C = C(\mathcal{M})$ is isomorphic to $\mathrm{Aut}(\mathcal{M})$ [19]. Further, each of these groups acts regularly on Ω and so has the same cardinality as Ω. These correspondences allow us to label each flag with the element g of C which sends the root flag I there. In short, "g" is the label of the flag gI. Then, we claim, elements of C can act on the right as symmetries. For each $h \in C$, and for each pair of i-adjacent flags g and $r_i g$, the element h sends them to gh and $r_i gh$, respectively, and these two are also i-adjacent. Thus h, acting on the right, acts as a symmetry of $\mathcal{M}$.

Consider, for instance, Fig. 4, and observe how multiplication on the right by r_2 acts as a reflection about the horizontal edge, while multiplication on the right by r_0 acts as a reflection about the vertical axis.

Thus when the maniplex is reflexible, we can use the same names for the elements of $C(\mathcal{M})$, Ω and $\mathrm{Aut}(\mathcal{M})$. We must, though, be aware that multiplication by, say, r_0 on the left is a different permutation of the group elements than multiplication on the right.

When we have a reflexible maniplex $\mathcal{M}$ with $C(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{M})$ expressed as permutations of some neutral set Ω, we can still talk about the symmetry corresponding to an element of the connection set by referring to the root flag; specifically, we

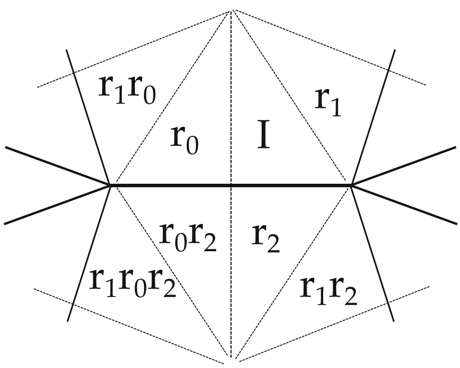

Fig. 4 Connections as symmetries

say that α is the symmetry *corresponding to* the connection x when xI is the same as $I\alpha$. This is a one-to-one correspondance and is operation preserving, and this is an isomorphism from $C(\mathcal{M})$ to $\mathrm{Aut}(\mathcal{M})$.

A note on language: Map theorists, starting with Brahana, have used the word *regular* to describe maps with rotational symmetries. Polytope theorists, though, use the word 'regular' to describe polytopes that we would call reflexible. In this paper, we remove the perhaps overused word 'regular' and instead use words that mean what they say. Also, we recognize that the English word 'chiral' simply means 'without reflections'. In our context, where generally only rotary maniplexes and polytopes are of interest, we will permit ourselves to use it to mean 'rotary but not reflexible'. And yes, we do recognize the contradictory flavors of these two preferences.

4 The Twist

We begin this section by presenting a very interesting maniplex which has only two facets, but is chiral.

4.1 The Krughoff Cubes

Consider the cube shown on the left in Fig. 5. The edges have been colored in such a way that each of the six possible circular orderings of the four colors appears exactly once clockwise about some face. Notice that this coloring is chiral; i.e., every rotation of the cube permutes the colors, while any reflection sends edges of any one fixed color to edges of different colors.

It is not obvious but a careful examination of the cube on the right shows that, ignoring the letter face-labels, the arrangement of colors is the same as on the left. Then each face of the left cube matches a face of the right with orientation reversed; these matching faces have the same letter. For instance face A-left has colors blue-yellow-red-green in order clockwise, while face A-right has colors blue-yellow-red-green in order counterclockwise. Thus, when we identify matching faces and colors, the result is a chiral 4-maniplex $\mathcal{K}$ with two cubical facets and four edges, one for each color. It was first discovered by Krughoff [11].

We introduce $\mathcal{K}$ in this paper because it can be formed from the trivial maniplex over the cube in a simple but very interesting way: separate each pair of attached squares and re-attach them after making a twist (locally) clockwise. The purpose of this paper is to generalize and re-generalize this operation, investigating the resulting chiral maniplexes.

4.2 The Twist in 4 dimensions

Let $\mathcal{M}$ be any *orientable* 4-maniplex. Recall that this means that $C = \langle r_0, r_1, r_2, r_3 \rangle$ is its connection group, that its facets are maps, that r_3 connects a face of each facet to some face (of the same size) in some facet, and that $\mathcal{R}$ and $\mathcal{W}$ are disjoint.

We construct the maniplex $T_j(\mathcal{M})$ to be $(\Omega, [s_0, s_1, s_2, s_3])$, where $s_0 = r_0$, $s_1 = r_1$, $s_2 = r_2$ and

$$s_3 f = \begin{cases} (r_0 r_1)^j r_3 f & \text{if } f \in \mathcal{R} \\ (r_1 r_0)^j r_3 f & \text{if } f \in \mathcal{W} \end{cases}$$

for all $f \in \Omega$. The index j indicates how much twist is performed to the faces of $\mathcal{M}$, after being separated, before gluing them back. This construction first appeared in [4].

Theorem 4.1 *For any orientable 4-maniplex $\mathcal{M}$ and any integer j, $T_j(\mathcal{M})$ is a maniplex.*

Proof We need to show that s_3 is an involution and that it commutes with r_0 and r_1. Suppose that $f \in \mathcal{R}$. Then $s_3^2 f = s_3(s_3 f) = s_3((r_0 r_1)^j r_3 f)$. Since $(r_0 r_1)^j r_3 f \in \mathcal{W}$, this is equal to $(r_1 r_0)^j r_3 (r_0 r_1)^j r_3 f$. Since r_3 commutes with r_0 and r_1, and $r_0 r_1$ is the inverse of $r_1 r_0$, this evaluates to f. Also, since $f \in \mathcal{R}$, $r_0 f \in \mathcal{W}$ and $s_3 r_0 f = (r_1 r_0)^j r_3 r_0 f = (r_1 r_0)^{j-1} r_1 r_3 f = r_0 (r_0 r_1)^j r_3 f = r_0 s_3 f$. Similar computations for r_1 and for $f \in \mathcal{W}$ show that the result holds. ■

Theorem 4.2 *If $\mathcal{M}$ is an orientable 4-maniplex and j is any integer, then every α in* $\text{Aut}^+(\mathcal{M})$ *is also in* $\text{Aut}^+(T_j(\mathcal{M}))$.

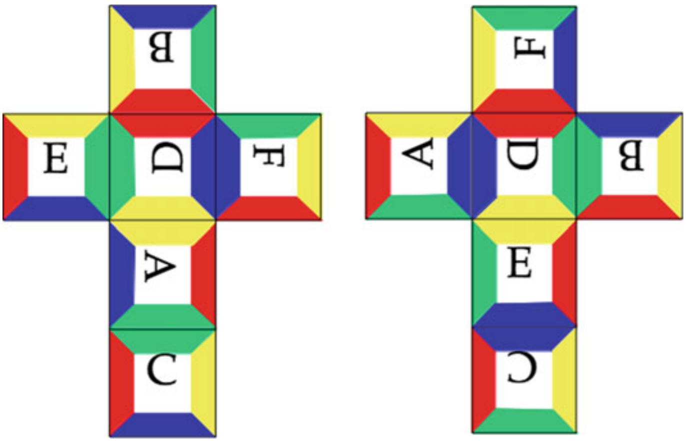

Fig. 5 Krughoff's two-cube maniplex

Proof Since the 0, 1, and 2-connections are the same in both maps, we only need to show that α preserves the 3-connections. Consider any red flag f and its neighbor $s_3 f = (r_0 r_1)^j r_3 f$. Then $f\alpha$ is also red, and its s_3-neighbor is $(r_0 r_1)^j r_3(f\alpha) = ((r_0 r_1)^j r_3 f)\alpha = (s_3 f)\alpha$. Thus α preserves all connections, and so is a symmetry of $T_j(\mathcal{M})$. ■

Because $r_0 r_1$ is color-preserving, the bicoloring of flags which results from the orientability of $\mathcal{M}$ shows that each $T_j(\mathcal{M})$ is orientable as well. Thus we have:

Corollary 4.3 *If a 4-maniplex $\mathcal{M}$ is orientable and rotary, then so is every $T_j(\mathcal{M})$.*

Corollary 4.4 *If a 4-maniplex $\mathcal{M}$ is reflexible, then $T_{-j}(\mathcal{M})$ is the mirror image of $T_j(\mathcal{M})$.*

There are examples of reflexible maniplexes for which some, all or none of the T_j's result in chiral maniplexes. It is most common, though, that the result of the Twist operation on a reflexible maniplex is chiral. For example, T_1 of the 4-dimensional cube is a chiral maniplex of type {4, 3, 8}, while the 4-cube itself has type {4, 3, 3}. There are also examples of chiral maniplexes for which every $T_j(\mathcal{M})$ is chiral.

We will address the question of the chirality or reflexibility of a Twist of a reflexible maniplex after we have introduced a more general form of the definition.

4.3 The General Twist

Let $\mathcal{M} = (\Omega, [r_0, r_1, \ldots, r_n])$ be an orientable $(n+1)$-maniplex of dimension at least 4. Let $B = \langle r_0, r_1, \ldots, r_{n-2}\rangle$; this is the connection group of the root subfacet. Further, let $B^+ = \langle r_0 r_1, r_1 r_2, \ldots, r_{n-3} r_{n-2}\rangle$; this is the subgroup of B which preserves $\mathcal{R}$. Let w be an element of B^+, such that for $i = 0, 1, 2, \ldots, n-2$ we have $(w r_i)^2 = I$; i.e., that the conjugate of w by r_i is w^{-1}. Call such a w *sub-invertible* because it is invertible within the sub-facet group B. Define the twist $T_w(\mathcal{M})$ of $\mathcal{M}$ to be $(\Omega, [s_0, s_1, \ldots, s_n])$, where $s_i = r_i$ for $i < n$ and

$$s_n f = \begin{cases} w r_n f & \text{if } f \in \mathcal{R}, \\ w^{-1} r_n f & \text{if } f \in \mathcal{W}, \end{cases}$$

for all $f \in \Omega$. Note that, since r_n commutes with all r_i with $i \le n-2$, $r_n w = w r_n$. Imitating the proofs of Theorems 4.1 and 4.2 yields these results:

Theorem 4.5 *For any orientable maniplex $\mathcal{M}$ and any sub-invertible w, $T_w(\mathcal{M})$ is a maniplex.*

Note that if $w \in B^+$ is not sub-invertible, then $T_w(\mathcal{M})$ is not a maniplex, but *is* a complex in the sense of [19] (or a combinatorial map in the sense of [18]). Moreover, in this case, some $T_w(\mathcal{M})$ could be a chiral hypertope, in the sense of [6].

Theorem 4.6 *If $\mathcal{M}$ is an orientable maniplex and w is sub-invertible, then every α in* $\mathrm{Aut}^+(\mathcal{M})$ *is also in* $\mathrm{Aut}^+(T_w(\mathcal{M}))$.

Corollary 4.7 *If $\mathcal{M}$ is orientable and rotary, then so is every $T_w(\mathcal{M})$).*

For a 4-maniplex, the subfacets are polygons, and so the only candidates for sub-invertible elements are the powers of $r_0 r_1$, and these are sub-invertible. For higher dimensions, there are no obvious non-trivial candidates for w, and indeed, some sub-facets have no such elements. We claim that the simplex is one such maniplex. To see that, first notice that if w is sub-invertible, then w is central in B^+. Thus if the sub-facet of some maniplex $\mathcal{M}$ is a simplex of dimension $n-2$ for n greater than 4, then B^+ is A_{n-1}. This has a trivial center, and hence no viable w.

Contrast this with the cube of dimension $n-2$. Here, when n is even, the central inversion is orientation-preserving and is central and thus sub-invertible.

In a maniplex $\mathcal{M}$ of any rank, if its symmetry group has k orbits on flags, then:

1. If $\mathrm{Aut}(\mathcal{M})$ contains an orientation-reversing element then $T_w(\mathcal{M})$ has either k or $2k$ orbits on flags.
2. If $\mathrm{Aut}(\mathcal{M})$ does not contain an orientation-reversing element then $T_w(\mathcal{M})$ has either k or $\frac{k}{2}$ orbits on flags.

5 Chirality

In the paper [14], the third author demonstrated the existence of a series of chiral polytopes of all dimensions. By using the twist operator, we hope to produce such maniples in a simpler way. In this section we address the following question: What are the conditions on an orientable reflexible maniplex $\mathcal{M}$ and a sub-invertible element w that would force $T_w(\mathcal{M})$ to be reflexible?

So, suppose that $\mathcal{M}$ is an orientable and reflexible $(n+1)$-maniplex; suppose that w is sub-invertible in $\mathcal{M}$; finally suppose that $T_w(\mathcal{M})$ is reflexible. Since $\mathcal{M}$ is reflexible, its set of flags is (or can be considered as) the group $C(\mathcal{M})$, which we will call G for convenience. Remember how nice this is: elements of G are the flags, elements of G are the connections (acting by multiplication on the left), and elements of G are symmetries (acting by multiplication on the right). Hence "$r_0 r_1$" is the name of a flag. It is 0-adjacent to the flag r_1. It is the image of r_0 under the symmetry sending each flag g to the flag gr_1.

Because $\mathcal{M}$ is orientable, its flags come in two colors, red and white, and the identification gives us that $\mathcal{R} = C^+$, the subgroup consisting of products of even lengths in the generators.

In $T_w(\mathcal{M}) = (G, [s_0, s_1, \ldots, s_n])$, all of the connections are in G except perhaps s_n. Since $T_w(\mathcal{M})$ is reflexible, it must have a symmetry α_0 which sends the flag I to the flag s_0. This α_0 is probably not in G. Let $H = \langle r_0, r_1, \ldots, r_{n-1}\rangle$. This group is the stabilizer (in $\mathcal{M}$ and in $T_w(\mathcal{M})$) of the 'central' facet; i.e., the facet containing the root flag I. Hence, on one hand, we can regard the elements of H as the flags of the central facet. On the other hand, we can think of the elements in H as paths of the graph with colours in $\{0, 1, \ldots, n-1\}$. This means that if we have $h \in H$, for every flag f, the flags f and hf are in the same facet (of both $\mathcal{M}$ and $T_w(\mathcal{M})$).

We will deduce the action of α_0 first in the central facet and then in facets farther and farther away.

Remembering that the identity I of G is assigned to the root flag I of Ω, we have that for $h \in H$, i.e. for flags in the central facet, the action of α_0 must be the same as in $\mathcal{M}$: thus, $h\alpha_0 = (hI)\alpha_0 = h(I\alpha_0) = h(r_0 I) = hr_0$.

Given $h \in H$, $r_n h$ is a flag in one of the facets adjacent to the central facet in $\mathcal{M}$. Since the twist operator preserves facets, and facet-adjacency, each flag $r_n h$ is in a facet adjacent to the central facet in $T_w(\mathcal{M})$ as well.

Let $g = r_n h$ be such a flag, for some $h \in H$, and suppose that g is red. Then it is n-adjacent in $T_w(\mathcal{M})$ to $s_n r_n h = w r_n r_n h = wh$, which is a white flag in H since $T_w(\mathcal{M})$ is orientable, and so its image under α_0 is whr_0, a red flag. Thus the flag $g\alpha_0$ must be n-adjacent to that red flag and so must be $wr_n whr_0 = w^2 r_n hr_0$. Similarly, if g is white then $g\alpha_0 = w^{-2} r_n hr_0$.

Now, every flag in a facet adjacent to the central facet is of the form $g = h_1 r_n h_0$, where the h_i's are from H. Then $g\alpha_0 = (h_1 r_n h_0)\alpha_0 = h_1(r_n h_0 \alpha_0) = h_1(w^{\pm 2} r_n h_0 r_0)$, where the exponent is $+2$ if $r_n h_0$ is red (i.e., if a product of generators equalling $r_n h_0$ has even length) and -2 otherwise.

Thus, we know the effect of α_0 on the central facet and on each facet adjacent to it. Next, consider a flag g in the layer of facets two steps away from the central facet, but n-adjacent to a flag in a facet adjacent to the central facet. Then $g = r_n h_1 r_n h_0$, where the h_i's are from H. If g is red, then g is n-adjacent to $wr_n r_n h_1 r_n h_0 = wh_1 r_n h_0$, a white flag. Then the image of this white flag under α_0 is the red flag $wh_1 w^{\pm 2} r_n h_0 r_0$; again, the exponent depends on the color of $r_n h_0$. Then $g\alpha_0$ is the flag n-adjacent to this one, which is $wr_n wh_1 w^{\pm 2} r_n h_0 r_0 = w^2 r_n h_1 w^{\pm 2} r_n h_0 r_0$; similarly, if g is white, then $g\alpha_0 = w^{-2} r_n h_1 w^{\pm 2} r_n h_0 r_0$.

In general, then, if $g = h_{k+1}(r_n h_k)(r_n h_{k-1})(r_n h_{k-2}) \ldots (r_n h_0)$, define

$$P(g) = h_{k+1}(t_k r_n h_k)(t_{k-1} r_n h_{k-1})(t_{k-2} r_n h_{k-2}) \ldots (t_0 r_n h_0),$$

where each t_j is w^2 if $(r_n h_j) \ldots (r_n h_0)$ is red and w^{-2} if it is white. Then it must be that $g\alpha_0 = P(g)r_0$, and a similar argument shows that, if α_i is the symmetry which sends I to s_i, then

$$g\alpha_i = \begin{cases} P(g)r_i & \text{if } i = 0, 1, \ldots, n-1; \\ P(g)wr_i & \text{if } i = n. \end{cases} \tag{1}$$

To recap: if $\mathcal{M}$ is reflexible and orientable and *if* we know that $T_w(\mathcal{M})$ is reflexible, then we have that for $i = 0, 1, \ldots, n-1$, $g\alpha_i = P(g)r_i$, and $g\alpha_n = P(g)wr_n$. This implies that P is well-defined. Well-definedness *is* an issue, for we see that $P(g)$ is defined in terms of a product p of generators which evaluates to g. The well-definedness of P means that if p_1, p_2 are two products of generators which both equal g then $P(p_1) = P(p_2)$. Moreover, this must hold even if the generator r_n appears in the words p_1 and p_2 with different multiplicity.

We claim that this is equivalent to saying that if some product p evaluates to I, then $P(p)$ must also evaluate to I. To see that, one uses the following Lemma:

Lemma 5.1 *For any word p let $Q(p)$ be obtained from $P(p)$ by replacing each t_i by t_i^{-1}. Then:*

1. *If p_2 is white, then $P(p_1p_2) = Q(p_1)P(p_2)$, while if p_2 is red, $P(p_1p_2) = P(p_1)P(p_2)$.*
2. *$(P(p^{-1}))^{-1} = P(p)$ if p is red and $(P(p^{-1}))^{-1} = Q(p)$ if p is white.*

On the other hand, if P is well-defined, it is clear that the equations in (1) serve as definitions for reflective symmetries, making $T_w(\mathcal{M})$ reflexible.

At first glance, the process of checking, for a given $\mathcal{M}$ and w, that the set of words which evaluate to I is closed under P may seem to be a daunting task. Our hearts need not seize up in fear, however. When we consider a reflexible maniplex, we are quite often given the generator-and-relator form of G. In this case, the only products we need to check are the relators, since every other word evaluating to I is a consequence of those.

Theorem 5.2 *Suppose that $\mathcal{M}$ is an orientable reflexible n-maniplex for n at least 4, and $C(\mathcal{M})$ has presentation $C = \langle r_0, r_2, \ldots, r_{n-1} | I = W_1, W_2, W_3, \ldots, W_k\rangle$, where each W_j is a word in the r_i's. If w is a sub-invertible element then $T_w(\mathcal{M})$ is reflexible if and only if $P(W_j)$ evaluates to the identity in C for all j.*

For example, consider the trivial maniplex over the cube. Its generator-relator form (abbreviating 'r_i' by just 'i') is $G = \langle 0, 1, 2, 3 | I = 0^2 = 1^2 = 2^2 = 3^2 = (02)^2 = (30)^2 = (31)^2 = (01)^4 = (12)^3 = (32)^2\rangle$. We will use $w = 01$. Because w^2 commutes with 0 and 1, it is clear that $P(3030) = I$ and $P(3131) = I$, and any g which includes no 3 has $P(g) = g$. Thus the only word we need to check is $(32)^2$.

Consider $P(3232) = 0101\ 32\ 0101\ 32 = 0101\ 323\ 0101\ 2 = 0101\ 2\ 0101\ 2$. This is a motion of the cube $\mathcal{Q}$, and it can be seen as a 2-step rotation about a face. It is certainly not the identity and so $T_{01}(\mathcal{Q})$ is not reflexible. Therefore the Krughoff maniplex $\mathcal{K} = T_{01}(\mathcal{Q})$ is chiral, as claimed.

Remark 5.3 If w is its own inverse, so that $w^2 = I$, then for all g we have $P(g) = g$, and so P is well-defined and $T_w(\mathcal{M})$ is reflexible.

6 The Maniplex $\hat{\mathbf{2}}^{\mathcal{M}}$

In this section, we describe a construction of an $(n + 1)$-maniplex whose facets are all isomorphic to a given n-maniplex. Our motivation is this: we will show that if $\mathcal{M}$ is an orientable n-maniplex such that $C(\mathcal{M})$ contains an element w such that $(wr_i)^2 = I$ for $i = 0, 1, \ldots, n-1$, then, applying this construction twice, the constructed maniplex is one on which we can perform a twist and get a chiral maniplex as a result.

Definition 6.1 Let $\mathcal{M} = (\Omega, [r_0, r_1, \ldots, r_{n-1}])$ be an n-maniplex which has $m \geq 2$ facets named $F_1, \ldots, F_m$. Define $\hat{2}^{\mathcal{M}}$ to be the $(n+1)$-maniplex $(\Omega \times \mathbb{Z}_2^m,$ $[s_0, s_1, \ldots, s_{n-1}, s_n])$, where, for $f \in F_j$, $x \in \mathbb{Z}_2^m$, we have

$$s_i(f, x) = \begin{cases} (r_i f, x) & \text{if } i = 0, 1, \ldots, n-1; \\ (f, x^j) & \text{if } i = n. \end{cases}$$

Here, x^j stands for the bitstring which differs from x in the j-th place and there only. If I is the root flag for $\mathcal{M}$, let $\hat{I} = (I, 000\ldots0)$ be the root flag for $\hat{2}^{\mathcal{M}}$.

Notice that if $\mathcal{M}$ has only one facet, then the above construction only yields the trivial maniplex over $\mathcal{M}$. In general $\hat{2}^{\mathcal{M}}$ is a 2^{m-1} fold cover of the trivial maniplex over $\mathcal{M}$. In what follows we are mainly interested in maniplexes with more than one facet.

This construction very slightly generalizes one of Danzer (see [5]) and sets it in maniplex form. Here $\hat{2}^{\mathcal{M}}$ is the same as Danzer's $D(2^{D(\mathcal{M})})$, where D stands for the usual dual of a polytope or maniplex.

Proposition 6.2 *Let $\mathcal{M}$ be any n-maniplex with at least two facets. Then*

1. *$\hat{2}^{\mathcal{M}}$ is an $(n+1)$-maniplex,*
2. *all facets of $\hat{2}^{\mathcal{M}}$ are isomorphic to $\mathcal{M}$;*
3. *if $\mathcal{M}$ has type $\{p_1, \ldots, p_{n-1}\}$ then $\hat{2}^{\mathcal{M}}$ has type $\{p_1, \ldots, p_{n-1}, 4\}$.*

Proof For $i = 0, 1, \ldots, n-1$, s_i is an involution because r_i is, and s_n is an involution because $(x^j)^j = x$. For $i = 0, 1, \ldots, n-2$, each f and $r_i f$ are in the same F_j, and so $s_i s_n s_i s_n(f, x) = s_i s_n s_i(f, x^j) = s_i s_n(r_i f, x^j) = s_i(r_i f, (x^j)^j) = s_i(r_i f, x) = (r_i r_i f, x) = (f, x)$. Thus, s_i and s_n commute, and so $\hat{2}^{\mathcal{M}}$ is a maniplex.

For a fixed x, the set of flags of the form (f, x) for $f \in \Omega$ is a facet of $\hat{2}^{\mathcal{M}}$, and every facet of $\hat{2}^{\mathcal{M}}$ is of this kind. Then the function sending f to (f, x) is an isomorphism of $\mathcal{M}$ to that facet of $\hat{2}^{\mathcal{M}}$.

Finally, suppose that some flag f of $\mathcal{M}$ is in F_j and that $r_{n-1} f$ is in F_k. Then repeatedly applying s_n and then s_{n-1} to (f, x) yields:

$$(f, x) \to (f, x^j) \to (r_{n-1} f, x^j) \to (r_{n-1} f, (x^j)^k) \to (f, (x^j)^k)$$

$$= (f, (x^k)^j) \to (f, x^k) \to (r_{n-1} f, x^k) \to (r_{n-1} f, x) \to (f, x).$$

This shows that $(s_{n-1} s_n)$ has order 4, as claimed. ■

Let us now consider symmetries of $\hat{2}^{\mathcal{M}}$. First, suppose that σ is a symmetry of $\mathcal{M}$ and that it acts on the facets of $\mathcal{M}$ as a permutation also called σ; i.e., that for all i, we define $\sigma(i)$ to be the index of $F_i\sigma$. Thus $F_i\sigma = F_{\sigma(i)}$. And denote by $\sigma(x)$ the vector $(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(m)})$. Then define $\hat{\sigma}$ acting on $\hat{2}^{\mathcal{M}}$ by

$$(f, x)\hat{\sigma} = (f\sigma, \sigma(x)).$$

Noting that $\sigma(x^j) = [\sigma(x)]^{\sigma(j)}$, it is easy to check that this is a symmetry of $\hat{2}^{\mathcal{M}}$ fixing the facet consisting of all flags of the form $(f, (0, 0, \ldots, 0))$. Thus, all of $\mathrm{Aut}(\mathcal{M})$ appears in $\mathrm{Aut}(\hat{2}^{\mathcal{M}})$, with $\hat{\alpha}_i$ playing the role of α_i for $i = 0, 1, 2, \ldots, n-1$.

For any $y \in \mathbb{Z}_2^m$, the function τ_y defined by $(f, x)\tau_y = (f, x + y)$ is clearly a symmetry of $\hat{2}^{\mathcal{M}}$. Assuming the root flag is in F_1, then the symmetry $\tau_{(1,0,0,\ldots,0)}$ sends the root flag $\hat{I}$ to $s_n\hat{I}$, its n-adjacent flag.

This shows that

Proposition 6.3 *Let $\mathcal{M}$ be a reflexible n-maniplex with at least two facets. Then $\hat{2}^{\mathcal{M}}$ is reflexible. The stabilizer of the central facet is isomorphic to* $\mathrm{Aut}(\mathcal{M})$*, by an isomorphism sending α_i to $\hat{\alpha}_i$.*

Notice that even if $\mathcal{M}$ has no particular symmetry, the maniplex $\hat{2}^{\mathcal{M}}$ has the symmetry $\tau_{(1,0,0,\ldots,0)} = \hat{\alpha}_n$, which is a reflection. Hence, $\hat{2}^{\mathcal{M}}$ can never be a chiral maniplex. Moreover, since $\mathrm{Aut}(\mathcal{M})$ can be regarded as a subgroup of $\mathrm{Aut}(\hat{2}^{\mathcal{M}})$ and for all $y \in \mathbb{Z}_2^m$, $\tau_y \in \mathrm{Aut}(\hat{2}^{\mathcal{M}})$, if $\mathcal{M}$ is a k-orbit maniplex, then so is $\hat{2}^{\mathcal{M}}$. In particular, if $\mathcal{M}$ is a 2-orbit maniplex in class 2_J, $J \subset \{0, 1, 2, \ldots, n-1\}$ (in the sense of [3]), then $\hat{2}^{\mathcal{M}}$ is a 2-orbit maniplex in class $2_{J\cup\{n\}}$.

6.1 Color-Coded Extensions

Suppose that $\mathcal{M} = (\Omega, [r_0, r_1, \ldots, r_{n-1}])$ is an n-maniplex, and let $\mathcal{C}$ be a partition of its facets into k classes called 'colors'. Define a new $(n+1)$-maniplex called $2^{(\mathcal{M},\mathcal{C})} = (\Omega \times \mathbb{Z}_2^k, [s_0, s_1, \ldots, s_n])$, where for $i = 0, 1, \ldots, n-1$, $s_i(f, x) = (r_i f, x)$, and $s_n(f, x) = (f, x^j)$, where f is in a facet of color j and x^j is the bitstring which differs from x in place j and there only. This generalizes previous notions:

1. If each class in $\mathcal{C}$ contains just one facet, then $2^{(\mathcal{M},\mathcal{C})}$ is exactly $\hat{2}^{\mathcal{M}}$.
2. If $\mathcal{C}$ consists of just one class containing all facets, then $2^{(\mathcal{M},\mathcal{C})}$ is the trivial maniplex over $\mathcal{M}$.

Moreover, proofs similar to those about $\hat{2}^{\mathcal{M}}$ show that if $\mathcal{M}$ is an n-maniplex with at least two facets then:

1. $2^{(\mathcal{M},\mathcal{C})}$ is an $(n+1)$-maniplex,
2. all facets of $2^{(\mathcal{M},\mathcal{C})}$ are isomorphic to $\mathcal{M}$;
3. if $\mathcal{M}$ has type $\{p_1, \ldots, p_{n-1}\}$ then $2^{(\mathcal{M},\mathcal{C})}$ has type $\{p_1, \ldots, p_{n-1}, 4\}$.
4. If $\mathcal{M}$ is reflexible and $\mathcal{C}$ is $\mathrm{Aut}(\mathcal{M})-$invariant, then $2^{(\mathcal{M},\mathcal{C})}$ is also reflexible.

7 Example of Twist on Rank 5

7.1 The Map $n\mathcal{M}$

We first give a general construction for covering of maps:

Definition 7.1 Let $\mathcal{M} = (\Omega, [r_0, r_1, r_2])$ be a map, a 3-maniplex, which is orientable, and let n be any integer greater than 2. Define $n\mathcal{M}$ to be the map $(\Omega \times \mathbb{Z}_n, [t_0, t_1, t_2])$, where for each flag (f, i) of $n\mathcal{M}$,

$$t_0(f, i) = (r_0 f, i),$$
$$t_2(f, i) = (r_2 f, i),$$

and

$$t_1(f, i) = \begin{cases} (r_1 f, i + 1) & \text{if } f \text{ is red,} \\ (r_1 f, i - 1) & \text{if } f \text{ is white.} \end{cases}$$

It is easy to see that each t_i is an involution and that t_0 commutes with t_2, so $n\mathcal{M}$ is a map, whenever it is connected. Observe that $n\mathcal{M}$ is an n-fold cover of $\mathcal{M}$, and if $\mathcal{M}$ is of type $\{p, q\}$, then $n\mathcal{M}$ is of type $\{LCM(p, n), LCM(q, n)\}$ whenever it is a map.

The second entry i of a flag (f, i) of $n\mathcal{M}$ is preserved by r_0 and r_2, but changed by r_1, according of the color of the flag f. We next define a function h that counts, for a given word W on the elements r_0, r_1, r_2, the difference between the number of appearances of a factor r_1 in odd and even places of W. If W is any word in r_0, r_1, r_2, and $\hat{W}$ is the corresponding word in t_0, t_1, t_2, define $h(W)$ recursively by using $h(I) = 0$ and

$$h(r_i W) = \begin{cases} h(W) + 1 & \text{if } i = 1 \text{ and } W \text{ has even length} \\ h(W) - 1 & \text{if } i = 1 \text{ and } W \text{ has odd length} \\ h(W) & \text{otherwise.} \end{cases}$$

Then, for all f, $\hat{W}(f, i) = (Wf, j)$, where $j = \begin{cases} i + h(W) & \text{if } f \text{ is red,} \\ i - h(W) & \text{if } f \text{ is white} \end{cases}$.

Note that this fact holds because W and $\hat{W}$ are considered as *words* in their respective sets of generators.

If D is the greatest common divisor of n and all values of $h(W)$ for which W evaluates to the identity in $\mathcal{M}$, then $n\mathcal{M}$ has exactly D connected components.

If $\mathcal{M}$ is reflexible, and, as before, we denote by α_i the symmetry exchanging the root flag I with $r_i I$, and by $\hat{\alpha}_i$ the symmetry exchanging $\hat{I}$ with $t_i \hat{I}$, then these functions are the corresponding symmetries in $n\mathcal{M}$:

$$(f, i)\hat{\alpha}_0 = (f\alpha_0, -i)$$

$$(f,i)\hat{\alpha}_1 = (f\alpha_1, 1-i)$$

$$(f,i)\hat{\alpha}_2 = (f\alpha_2, -i).$$

Thus, $n\mathcal{M}$ is reflexible as well. Among the symmetries of $n\mathcal{M}$ is the function β, whichs sends each (f,i) to $(f,i+1)$; direct computation verifies that β is a symmetry. Further, a simple computation will show that for each $i = 0, 1, 2$, the relation $(\beta\hat{\alpha}_i)^2 = I$ holds.

7.2 A Series of 5-Maniplex Examples

We use this construction to produce a series of examples. Each starts with an orientable map $\mathcal{M}$ of type $\{p, p\}$, with p odd, having a word W such that $h(W)$ is relatively prime to p. We study the effect of the twist operation on the 5-maniplex $\mathcal{M}' = \hat{2}^{\hat{2}^{p\mathcal{M}}}$.

Consider an orientable 3-maniplex $\mathcal{M}$ with type $\{p, p\}$ and some $n \geq 5$. We then construct the n-fold cover $n\mathcal{M}$ of $\mathcal{M}$.

As long as the greatest common divisor D of p and all $h(W)$ for words evaluating to the identity, is 1, the map is connected.

As an example, consider the great dodecahedron $\mathcal{M} = \mathcal{P}_0$, a polyhedron and orientable map of type $\{5, 5\}$ with 12 vertices and 12 pentagonal faces, where every vertex is surrounded by 5 faces. It can be constructed from the icosahedron by disregarding the triangles and considering as faces the 2-*holes*, that is, the convex polygons (pentagons in this case) determined by the neighbours of some vertex. The triangles of the icosahedron can be recovered as the 2-holes of the great dodecahedron (see [1, Chap. VI]). The great dodecahedron is reflexible, and as before, we consider its symmetry group, its connection group and its flag set to all be the same group G. Its connection group satisfies the relation $(r_0r_1r_2r_1)^3 = I$, since this indicates that the 2-holes are triangles. Then $h((r_0r_1r_2r_1)^3) = 6 \equiv 1$ modulo 5, and hence $h((r_0r_1r_2r_1)^3)$ is relatively prime to 5. The polyhedral map $5\mathcal{P}_0 = (\Omega \times \mathbb{Z}_5, [t_0, t_1, t_2])$ is then connected, has type $\{5, 5\}$, 60 vertices and 60 facets. This polyhedron is denoted by $\{5, 5\} * 600$ in the atlas of Hartley [8].

Naturally, the element w of $C(5\mathcal{P}_0)$ corresponding to $\beta \in \mathrm{Aut}(5\mathcal{P}_0)$ has order 5. Furthermore, because $(\beta\hat{\alpha}_i)^2 = I$ we have that $(wt_i)^2 = I$ for $i \in \{0, 1, 2\}$.

Now, the element $(r_0r_1r_2r_1)^3$ acts trivially on Ω. Therefore $(t_0t_1t_2t_1)^3$ sends $(I, 0)$ to $(I, 1)$, and so it must be equal to w.

Let $\mathcal{M}'$ be the 5-maniplex (5-polytope) $\hat{2}^{\hat{2}^{5\mathcal{P}_0}}$ of type $\{5, 5, 4, 4\}$. The subfacets of $\mathcal{M}'$ are isomorphic to $5\mathcal{P}_0$ and w satisfies the desired properties in Sect. 4, so we can construct $T_w(\mathcal{M}')$. In what follows we prove that $T_w(\mathcal{M}')$ is chiral.

In $C(\mathcal{M}') = \langle s_0, s_1, s_2, s_3, s_4\rangle$, the relation $(s_3s_4)^4 = I$ holds. Assuming that $T_w(\mathcal{M}')$ is reflexible we have that $P((s_3s_4)^4)$ also equals I. But $P((s_3s_4)^4) = (s_3w^2s_4)^4$, since all flags $(s_3s_4)^k$ are red. Conjugating by s_4 we get

$$I = s_4(s_3 w^2 s_4)^4 s_4 = (s_4 s_3 w^2)^4.$$

Now, every flag in $\mathcal{M}'$ is of the form $(((f, i), x), y)$, where f is a flag of $\mathcal{P}_0$, $i \in \mathbb{Z}_5$, and x and y are bitstrings of the appropriate lengths. The connection s_4 changes only y, and s_3 changes only x.

Thus the image of $(((I, 0), x), y)$ under $s_4 s_3 w^2$ is $(((I, 2), x'), y')$ for some x', y'. Thus $(((I, 0), x), y)(s_4 s_3 w^2)^4 = (((I, 8), x''), y'')$ for some x'', y''. Since 5 is odd, $P((s_3 s_4)^4)$ is *not* the identity and so $T_w(\mathcal{M}')$ is chiral.

This example generalizes: If $\mathcal{M}$ is any map of type $\{p, p\}$ for some $p \geq 5$ and has a defining word W such that $h(W)$ is relatively prime to p, then it must have some defining word w such that $h(w) = 1$ and corresponds to β in $p\mathcal{M}$. If, in addition, p is odd, then in $\mathcal{M}' = \hat{2}^{\hat{2}^{p\mathcal{M}}}$, the word $(s_3 s_4)^4$ evaluates to the identity, and a computation similar to the previous paragraph shows that $P((s_3 s_4)^4) = (s_3 w^2 s_4)^4$ sends any flag $(((f, i), x), y)$ to $(((f, i + 8), x''), y'')$ for some x'', y'' and since p is odd, this is not the identity and so $T_w(\mathcal{M}')$ is chiral.

Remark Now, the maniplex $\hat{2}^{\hat{2}^{5\mathcal{P}_0}}$ is huge: $5\mathcal{P}_0$ is 5 times as large as the Great Dodecahedron and so has $5 * 120 = 600$ flags in 60 facets of 10 flags each. The 4-maniplex $\hat{2}^{5\mathcal{P}_0}$ has 2^{60} such facets. Then $\hat{2}^{\hat{2}^{5\mathcal{P}_0}}$ has $2^{2^{60}}$ facets. Then 2^{60} is an 19-digit number and thus $2^{2^{60}}$ is simply too large to contemplate.

We can reduce the scale of our constructions by using the color-coded extensions: $\mathcal{P}_0$, the great dodecahedron, has a coloring in 6 colors, opposite faces having the same colors. Then $5\mathcal{P}_0$, as a covering of $\mathcal{P}_0$, inherits this same 6-coloring. Call that coloring S. Then the 4-maniplex $\mathcal{P}_1 = 2^{(5\mathcal{P}_0, S)}$ has only $2^6 = 64$ facets and has a 2-coloring B, and so the 5-maniplex $\mathcal{P}_2 = 2^{(\mathcal{P}_1, B)}$ has only 4 facets, for a total of $600 * 64 * 4 = 153{,}600$ flags.

8 Open Questions

There are many interesting open questions regarding the twist operator on maniplexes. Here are just a few of them:

1. In general, if $\mathcal{M}$ is polytopal, what are the conditions on w for the maniplex $T_w(\mathcal{M})$ to be polytopal? Are there even any special cases in which this question can be answered?
2. Our original intent was to use the twist construction to produce chiral maniplexes and hopefully polytopes of all possible dimensions in an elegant way. Our main difficulty is that we seem to have no examples of maniplexes of any rank above five with sub-invertible elements that are not involutions, and thus we can display no k-maniplexes for $k \geq 6$ for which the twist operator is defined. We are trying not to conjecture that there are none.

3. Is there a way to generalize the construction of the map $n\mathcal{M}$ from the map $\mathcal{M}$ to apply to any maniplex $\mathcal{P}$? If there were, then we could have examples of maniplexes $\hat{2}^{\hat{2}^{\mathcal{P}}}$ of all dimensions to which we could apply the Twist operation.
4. Given a chiral maniplex $\mathcal{M}$, what are the criteria for $\mathcal{M}$ to be a twist of some reflexible maniplex?
5. If $T_w(\mathcal{M})$ is isomorphic to the mirror image of $\mathcal{M}$, is some $T_{w'}(\mathcal{M})$ reflexible?

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