On the Volume of Boolean Expressions of Large Congruent Balls

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Dedicated to Károly Bezdek and Egon Schulte on the occasion of their 60th birthdays.

Abstract We consider the volume of a Boolean expression of some congruent balls about a given system of centers in the d-dimensional Euclidean space. When the radius r of the balls is large, this volume can be approximated by a polynomial of r, which will be computed up to an $O(r^{d-3})$ error term. We study how the top coefficients of this polynomial depend on the set of the centers. It is known that in the case of the union of the balls, the top coefficients are some constant multiples of the intrinsic volumes of the convex hull of the centers. Thus, the coefficients in the general case lead to generalizations of the intrinsic volumes, in particular, to a generalization of the mean width of a set. Some known results on the mean width, along with the theorem on its monotonicity under contractions are extended to the "Boolean analogues" of the mean width.

Keywords Volume · Intrinsic volume · Quermassintegral · Unions and intersections of balls

1 Introduction

The long-standing conjecture of Kneser [10] and Poulsen [11] claims that if the points $\mathbf{p}_1, \ldots, \mathbf{p}_N$ and $\mathbf{q}_1, \ldots, \mathbf{q}_N$ of the d-dimensional Euclidean space $\mathbb{R}^d$ satisfy

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the inequalities $\mathrm{d}(\mathbf{p}_i, \mathbf{p}_j) \geq \mathrm{d}(\mathbf{q}_i, \mathbf{q}_j)$ for all $0 \leq i, j \leq N$, then

$$\mathrm{vol}_d \left(\bigcup_{i=1}^{N} B^d(\mathbf{p}_i, r) \right) \geq \mathrm{vol}_d \left(\bigcup_{i=1}^{N} B^d(\mathbf{q}_i, r) \right)$$

for any $r > 0$, where $B^d(\mathbf{p}, r)$ denotes the closed d-dimensional ball of radius r about the point $\mathbf{p}$ and vol_d is the d-dimensional volume. K. Bezdek and Connelly [2] proved the conjecture in the plane, but it is still open in dimensions $d \geq 3$.

Results of Gromov [9], Gordon and Meyer [8] and the author [4, 5] suggest that the Kneser–Poulsen conjecture could be true in a more general form, which we formulate below.

Let $\mathcal{B}_N$ be the free Boolean algebra generated by $N \geq 1$ symbols $x_1, \ldots, x_N$. We denote the greatest element of $\mathcal{B}_N$ by X, and the least element of $\mathcal{B}_N$ by $\emptyset$. Elements of $\mathcal{B}_N$ are equivalence classes of formal expressions built from the symbols $x_1, \ldots, x_N$, X and $\emptyset$, the binary operations $\cup, \cap$, and the unary operator $f \mapsto \bar{f}$. Two expressions are called equivalent if and only if we can prove their equality assuming that the operations satisfy the axioms of a Boolean algebra. We shall refer to an element of $\mathcal{B}_N$ by choosing a Boolean expression from its equivalence class, and we write "=" between two Boolean expressions if they are equivalent. We shall also use the derived operator $f \setminus g = f \cap \bar{g}$ and the partial ordering $f \subseteq g \overset{\text{def}}{\Longleftrightarrow} f \cup g = g$. We refer to [6] for more details on Boolean algebras.

Take a Boolean expression $f \in \mathcal{B}_N$ which can be represented by a formula built exclusively from the variables $x_1, \ldots, x_N$ and the operations $\cup, \cap, \setminus$ in such a way that each of the variables occurs in the formula exactly once. For any pair of indices $i \neq j$, $1 \leq i, j \leq N$, evaluate f replacing the variables x_k, $k \notin \{i, j\}$ by X or $\emptyset$ in all possible ways. It can be seen that the results of those evaluations that are not equal to X or $\emptyset$, are all equal to one another and to one of the expressions $x_i \cap x_j$, $x_i \setminus x_j$, $x_j \setminus x_i$, $x_i \cup x_j$. Let the sign ϵ_{ij}^f be -1 if the evaluations not equal to X or $\emptyset$ are equal to $x_i \cap x_j$, and set $\epsilon_{ij}^f = 1$ in the remaining three cases.

The generalization of the Kneser–Poulsen conjecture for Boolean expressions of balls claims that if the Boolean expression $f \in \mathcal{B}_N$ obeys the conditions of the previous paragraph, and the points $\mathbf{p}_1, \ldots, \mathbf{p}_N$ and $\mathbf{q}_1, \ldots, \mathbf{q}_N$ in $\mathbb{R}^d$ satisfy the inequalities $\epsilon_{ij}^f(\mathrm{d}(\mathbf{p}_i, \mathbf{p}_j) - \mathrm{d}(\mathbf{q}_i, \mathbf{q}_j)) \geq 0$ for all $0 \leq i, j \leq N$, then

$$\mathrm{vol}_d \left(f(B^d(\mathbf{p}_1, r_1), \ldots, B^d(\mathbf{p}_N, r_N)) \right) \geq \mathrm{vol}_d \left(f(B^d(\mathbf{q}_1, r_1), \ldots, B^d(\mathbf{q}_N, r_N)) \right) \tag{1}$$

for any choice of the radii $r_1, \ldots, r_N$.

A suitable modification of the arguments of Bezdek and Connelly [2] shows that this generalization of the Kneser–Poulsen conjecture is also true in the Euclidean plane (see [5]).

As it was pointed out by Capoyleas, Pach [3], and Gorbovickis [7], the original Kneser–Poulsen conjecture for large congruent balls is closely related to the monotonicity of the mean width of a set under contractions. The relation is based on the formula

$$\mathrm{vol}_d\left(\bigcup_{i=1}^{N} B^d(\mathbf{p}_i, r)\right) = \kappa_d r^d + \frac{d\kappa_d}{2}\boldsymbol{\omega}_d(\{\mathbf{p}_1, \dots, \mathbf{p}_N\})r^{d-1} + O(r^{d-2}), \quad (2)$$

where κ_d is the volume of the unit ball in $\mathbb{R}^d$, $\boldsymbol{\omega}_d(S)$ denotes the mean width of the bounded set $S \subset \mathbb{R}^d$. We remark that the mean width function $\boldsymbol{\omega}_d$ depends on the dimension d of the ambient space, but only up to a constant factor. More explicitly, if $\Phi\colon \mathbb{R}^d \to \mathbb{R}^{\tilde{d}}$ is an isometric embedding, then we have $\frac{d\kappa_d}{\kappa_{d-1}}\boldsymbol{\omega}_d(S) = \frac{\tilde{d}\kappa_{\tilde{d}}}{\kappa_{\tilde{d}-1}}\boldsymbol{\omega}_{\tilde{d}}(\Phi(S))$ for any bounded set $S \subset \mathbb{R}^d$. Applying Formula (2) and the fact that the Kneser–Poulsen conjecture is true if the dimension of the space is at least $N-1$ (see [9]), Capoyleas and Pach [3] proved that the mean width of a set cannot increase when the set is contracted. Using rigidity theory, Gorbovickis [7] sharpened this result by proving that if the d-dimensional configurations $(\mathbf{p}_1, \dots, \mathbf{p}_N)$ and $(\mathbf{q}_1, \dots, \mathbf{q}_N)$ are not congruent and satisfy the inequalities $\mathrm{d}(\mathbf{p}_i, \mathbf{p}_j) \geq \mathrm{d}(\mathbf{q}_i, \mathbf{q}_j)$ for all $0 \leq i, j \leq N$, then the strict inequality

$$\boldsymbol{\omega}_d(\{\mathbf{p}_1, \dots, \mathbf{p}_N\}) > \boldsymbol{\omega}_d(\{\mathbf{q}_1, \dots, \mathbf{q}_N\})$$

holds. This strict inequality, in return, implies that the Kneser–Poulsen conjecture is true if the radius of the balls is bigger than a constant depending on the configurations of the centers.

Gorbovickis [7] proved also that for the volume of the intersection of large congruent balls we have

$$\mathrm{vol}_d\left(\bigcap_{i=1}^{N} B^d(\mathbf{p}_i, r)\right) = \kappa_d r^d - \frac{d\kappa_d}{2}\boldsymbol{\omega}_d(\{\mathbf{p}_1, \dots, \mathbf{p}_N\})r^{d-1} + O(r^{d-2}), \quad (3)$$

thus, as a consequence of the strict monotonicity of the mean width, the above mentioned generalization of the Kneser–Poulsen conjecture is true also for the intersections of congruent balls if the radius of the balls is greater than a constant depending on the configurations of the centers.

In 2013 K. Bezdek [1] posed the problem of finding a suitable generalization of Eqs. (2) and (3) for the volume of an arbitrary Boolean expression of large congruent balls, and suggested to explore the interplay between the generalized Kneser–Poulsen conjecture and the monotonicity properties of the coefficient of r^{d-1} in the general formula. In the present paper, we summarize the results of the research initiated by these questions.

The outline of the paper is the following. In Sect. 2, we sharpen Eq. (2), expressing the volume of the union of some large congruent balls with an error term of order $O(r^{d-3})$. The coefficients appearing in the formula are some constant multiples of the intrinsic volumes V_0, V_1, V_2 of the convex hull of the centers. In Sect. 3, we show that if a Boolean expression $f(B_1, \dots, B_n)$ of some balls is bounded, then its volume can be obtained as a linear combination of the volumes of the unions of some of the balls. The coefficients of this inclusion-exclusion type formula, given in Proposition 5, depend purely on the Boolean expression f. These coefficients are used to define the Boolean analogues of the intrinsic volumes of the convex hull of a point set in Sect. 4. Theorem 1 gives a generalization of Eq. (2) for Boolean expressions of large balls using Boolean intrinsic volumes. In Sect. 5, some classical facts on intrinsic volumes are generalized for Boolean intrinsic volumes. For example, it is known that the kth intrinsic volume of a polytope can be expressed in terms of the volumes of the k-dimensional faces and the angular measures of the normal cones of these faces. This formula is generalized for Boolean intrinsic volumes in Theorem 2. As an application of Theorem 2, we prove that the kth Boolean intrinsic volumes corresponding to dual Boolean expressions differ only in a sign $(-1)^k$. This explains why the coefficients of r^{d-1} in the Eqs. (2) and (3) are opposite to one another. Theorem 3 provides a Boolean extension of the fact that the first intrinsic volume of a convex set is a constant multiple of the integral of its support function. Section 6 is devoted to the proof of Theorem 4 on the monotonicity of the Boolean analogue of the first intrinsic volume.

2 Comparison of the Volume of a Union of Balls and the Volume of Its Convex Hull

Every convex polytope $K \subset \mathbb{R}^d$ defines a decomposition of the space as follows. Denote by $\mathcal{F}(K)$ the set of all faces of K, including K, and by $\mathcal{F}_k(K)$ the set of its k-dimensional faces. Let $\pi : \mathbb{R}^d \to K$ be the map assigning to a point $\mathbf{x} \in \mathbb{R}^d$ the unique point of K that is closest to $\mathbf{x}$. For a face $L \in \mathcal{F}(K)$, denote by $V(L, K)$ the preimage $\pi^{-1}(\text{relint}\, L)$ of the relative interior of L. As K is the disjoint union of the relative interiors of its faces, $\mathbb{R}^d$ is the disjoint union of the sets $V(L, K)$, where L is running over $\mathcal{F}(K)$. If $L \in \mathcal{F}_k(K)$, then $V(L, K)$ is the Minkowski sum of the relative interior of L and the normal cone

$$N(L, K) = \{\mathbf{u} \in \mathbb{R}^d \mid \mathbf{u} \perp [L] \text{ and } \max_{\mathbf{x} \in K} \langle \mathbf{u}, \mathbf{x} \rangle \text{ is attained at a point } \mathbf{x} \in L\} \tag{4}$$

of K at L, where $[L]$ denotes the affine subspace spanned by L. Set $n(L, K) = N(L, K) \cap B^d(\mathbf{0}, 1)$ and $\nu(L, K) = \text{vol}_{d-k}(n(L, K))/\kappa_{d-k}$. Division by κ_{d-k} in the definition of $\nu(L, K)$ is advantageous because it makes the angle measure $\nu(L, K)$ of the normal cone $N(L, K)$ independent of the dimension d of the ambient space $\mathbb{R}^d$, though the normal cone itself changes if we embed K into a higher dimensional space.

Denote by $K_r = K + B^d(\mathbf{0}, r)$ the distance r parallel body of K. The decomposition

$$\mathbb{R}^d = \bigcup_{L \in \mathcal{F}(K)} N(L, K) \tag{5}$$

induces a decomposition of the parallel body K_r, which enables us to write the volume of K_r as a polynomial of r

$$\begin{aligned}\mathrm{vol}_d(K_r) &= \sum_{L\in\mathcal{F}(K)} \mathrm{vol}_d(K_r \cap N(L,K)) = \sum_{L\in\mathcal{F}(K)} \mathrm{vol}_d(L + r\,(n(L,K))) \\ &= \sum_{k=0}^{d} \kappa_{d-k} \left(\sum_{L\in\mathcal{F}_k(K)} \mathrm{vol}_k(L)\nu(L,K) \right) r^{d-k}.\end{aligned} \tag{6}$$

Equation (6) is a special case of Steiner's classical formula (see, e.g., [12, Eq. (4.2.27)])

$$\mathrm{vol}_d(K + B(\mathbf{0}, r)) = \sum_{k=0}^{d} \binom{d}{k} W_k^d(K) r^k = \sum_{k=0}^{d} \kappa_{d-k} V_k(K) r^{d-k}, \tag{7}$$

expressing the volume of the distance r parallel body of an arbitrary compact convex set K as a polynomial of r, in which the normalized coefficients $W_k^d(K)$ and $V_k(K)$ are the quermassintegrals and intrinsic volumes of K respectively. It is known that the intrinsic volumes are continuous functions on the space of compact convex sets endowed with the Hausdorff metric (see [12, Sect. 4.2]), and $V_0(K) \equiv 1$. Comparing (6) and (7) we obtain the formula

$$V_k(K) = \sum_{L\in\mathcal{F}_k(K)} \mathrm{vol}_k(L)\nu(L,K) \tag{8}$$

expressing the intrinsic volumes of a polytope K.

Proposition 1 *Let $\mathbf{p}_1, \ldots, \mathbf{p}_N$ be a fixed set of points in $\mathbb{R}^d$, $K = \mathrm{conv}(\{\mathbf{p}_1, \ldots, \mathbf{p}_N\})$ be the convex hull of the points. Denote by $B_i = B^d(\mathbf{p}_i, r)$ the ball of radius r centered at $\mathbf{p}_i$. Then we have*

$$\left| \mathrm{vol}_d(K_r) - \mathrm{vol}_d\Big(\bigcup_{i=1}^{N} B_i\Big) \right| = O(r^{d-3}) \tag{9}$$

for large values of r.

Proof Denote by Δ the diameter of K, and set $r' = r - \Delta^2/r$. It is easy to see that if $r \geq \Delta$, then $K_{r'} \subseteq \bigcup_{i=1}^{N} B_i \subseteq K_r$, (see [3]). Intersecting the decomposition (5) with the union of the balls, we get

$$\bigcup_{i=1}^{N} B_i = \bigcup_{L\in\mathcal{F}(K)} \Big(N(L,K) \cap \Big(\bigcup_{i=1}^{N} B_i \Big) \Big).$$

When $L \in \mathcal{F}_0(K)$ is a vertex, we have $N(L, K) \cap \left(\bigcup_{i=1}^N B_i\right) = N(L, K) \cap K_r$. Thus,

$$K_r \setminus \left(\bigcup_{i=1}^N B_i\right) \subseteq \bigcup_{k=1}^d \bigcup_{L \in \mathcal{F}_k(K)} N(L, K) \cap (K_r \setminus K_{r'}),$$

and

$$\left| \mathrm{vol}_d(K_r) - \mathrm{vol}_d\left(\bigcup_{i=1}^N B_i\right) \right| \leq \sum_{k=1}^d \mathrm{vol}_k(L)\kappa_{d-k}\nu(L, K)\left(r^{d-k} - \left(r - \frac{\Delta^2}{r}\right)^{d-k}\right) = O(r^{d-3}), \tag{10}$$

as claimed.

Corollary 1 *Using the notations of Proposition 1, we have*

$$\mathrm{vol}_d\left(\bigcup_{i=1}^N B_i\right) = \kappa_d r^d + \kappa_{d-1} V_1(K) r^{d-1} + \kappa_{d-2} V_2(K) r^{d-2} + O(r^{d-3}). \tag{11}$$

3 Combinatorics of Boolean Expressions

For a subset I of the set $[N] = \{1, \ldots, N\}$, define $a_I \in \mathcal{B}_N$ by $a_I = (\bigcap_{j \notin I} x_j) \setminus (\bigcup_{i \in I} x_i)$. The elements a_I, $(I \subseteq [N])$ are the *atomic elements* of $\mathcal{B}_N$. Any $f \in \mathcal{B}_N$ can be decomposed uniquely as $f = \bigcup_{a_I \subseteq f} a_I$. In particular, $\mathcal{B}_N$ has 2^{2^N} elements.

Definition 1 The *reduced Euler characteristic* $\tilde{\chi}_N(f)$ *of* $f \in \mathcal{B}_N$ is the integer $\tilde{\chi}_N(f) = \sum_{a_I \subseteq f} (-1)^{|I|+1}$.

Obviously, the reduced Euler characteristic of a Boolean expression is an integer number in the interval $[-2^{N-1}, 2^{N-1}]$.

Proposition 2 *If $f \in \mathcal{B}_N$ can be represented by a formal expression which does not contain all the variables $x_1, \ldots, x_N$, then $\tilde{\chi}_N(f) = 0$.*

Proof We may assume without loss of generality that f can be written as an expression not using the variable x_N. This means that if $\iota \colon \mathcal{B}_{N-1} \to \mathcal{B}_N$ is the natural embedding, then $f = \iota(g)$ for some $g \in \mathcal{B}_{N-1}$. If $I \subseteq [N-1]$, and $a_I \in \mathcal{B}_{N-1}$ is the corresponding atomic expression in $\mathcal{B}_{N-1}$, then $\iota(a_I) \cap x_N$ and $\iota(a_I) \cap \bar{x}_N$ are atomic expressions in $\mathcal{B}_N$ corresponding to the index sets $I \subseteq [N]$ and $I \cup \{N\} \subseteq [N]$ respectively, furthermore,

$$a_I \subseteq g \iff \iota(a_I) \cap x_N \subseteq f \iff \iota(a_I) \cap \bar{x}_N \subseteq f.$$

Thus,

$$\tilde{\chi}_N(f) = \sum_{a_I \subseteq g}((-1)^{|I|} + (-1)^{|I \cup \{N\}|}) = 0.$$

□

Proposition 3 *If $\bar{f}$ is the complement of $f \in \mathcal{B}_N$, then $\tilde{\chi}_N(\bar{f}) = -\tilde{\chi}_N(f)$.*

Proof It is clear from the definition of the reduced Euler characteristic that if $f \cap g = \emptyset$, then $\tilde{\chi}_N(f \cup g) = \tilde{\chi}_N(f) + \tilde{\chi}_N(g)$. We also have

$$\tilde{\chi}_N(X) = \sum_{i=0}^{N}(-1)^{i+1}\binom{N}{i} = 0,$$

so $\tilde{\chi}_N(f) + \tilde{\chi}_N(\bar{f}) = \tilde{\chi}_N(f \cup \bar{f}) = \tilde{\chi}_N(X) = 0$.

Recall that the contradual $f^{\bar{*}}$ of $f \in \mathcal{B}_N$ is formed by replacing each variable x_i by its complement $\bar{x}_i$, while the dual $f^* = \overline{f^{\bar{*}}}$ of f is the complement of the contradual of f.

Proposition 4 *For any $f \in \mathcal{B}_N$, we have*

$$-\tilde{\chi}_N(f^*) = \tilde{\chi}_N(f^{\bar{*}}) = (-1)^N \tilde{\chi}_N(f).$$

Proof The first equation is a corollary of Proposition 3, so it is enough to show the second one. The contradual operation preserves the ordering and maps the atom a_I to $a_{[N]\setminus I}$. Consequently,

$$\tilde{\chi}_N(f^{\bar{*}}) = \sum_{a_I \subseteq f^{\bar{*}}}(-1)^{|I|+1} = (-1)^N \sum_{a_{[N]\setminus I} \subseteq f}(-1)^{|[N]\setminus I|+1} = (-1)^N \tilde{\chi}_N(f).$$

□

Let $\mathcal{L}_N$ be the sublattice of $\mathcal{B}_N$ generated by the elements $x_1, \dots, x_N$ and the operations $\cup$ and $\cap$. An element $f \in \mathcal{B}_n$ belongs to $\mathcal{L}_n$ if and only if $f \neq \emptyset$ and whenever $a_I \subseteq f$ and $J \subseteq I$ we also have $a_J \subseteq f$. This means that we can associate to any element $f \in \mathcal{L}_N$ an abstract simplicial complex $P_f = \{I \subset [N] \mid a_I \subseteq f\}$. This assignment gives a bijection between $\mathcal{L}_n$ and abstract simplicial complexes on the vertex set $[N]$ different from the abstract $(N-1)$-dimensional simplex. In this special case, the reduced Euler characteristic of f is one less than the ordinary Euler characteristic of P_f. The difference is due to the fact that $\emptyset$ is not counted as a -1-dimensional face when we compute the Euler characteristic, but it is taken into account in the computation of $\tilde{\chi}_N(f)$. The number of elements of $\mathcal{L}_N$ is $M_N - 2$, where M_N is the Nth Dedekind number.

There is a sublattice $\mathcal{C}_N \supset \mathcal{L}_N$ of $\mathcal{B}_N$ consisting of expressions that can be built from the variables $x_1, \dots, x_N$ using only the operations $\cup$, $\cap$, and $\setminus$. The lattice $\mathcal{C}_N$

contains exactly those elements of $\mathcal{B}_N$ that do not contain the atomic expression $a_{[N]}$. This way, $\mathcal{C}_N$ has 2^{2^N-1} elements.

Denote by $\mathcal{M}_N$ the linear space of real valued functions $\mu\colon \mathcal{C}_N \to \mathbb{R}$ such that $\mu(f \cup g) = \mu(f) + \mu(g)$ if $f \cap g = \emptyset$. As $\mu \in \mathcal{C}_N$ is uniquely determined by its values on the atomic expressions a_I, $(I \subsetneq [N])$, $\dim \mathcal{M}_N = 2^N - 1$.

For $\emptyset \neq I \subseteq [N]$, let $u_I \in \mathcal{L}_N$ be the union $u_I = \bigcup_{i\in I} x_i$.

Proposition 5 *For any $f \in \mathcal{C}_N$, there is a unique collection of integers $m_{f,I} \in \mathbb{Z}$ for* $(\emptyset \neq I \subseteq [N])$ *such that for any $\mu \in \mathcal{M}_N$, we have*

$$\mu(f) = \sum_{\emptyset\neq I\subseteq[N]} m_{f,I}\,\mu(u_I). \tag{12}$$

Proof There is a natural embedding $\mathrm{ev}\colon \mathcal{C}_N \to \mathcal{M}_N^*$ of $\mathcal{C}_N$ into the dual space of $\mathcal{M}_N$ given by the evaluation map $\mathrm{ev}\colon f \mapsto \mathrm{ev}_f$, where $\mathrm{ev}_f(\mu) = \mu(f)$ for any $\mu \in \mathcal{M}_N$. The proposition claims that for any $f \in \mathcal{C}_N$, ev_f can be decomposed uniquely as an integer coefficient linear combination of the evaluations ev_{u_I}, $(\emptyset \neq I \subseteq [N])$.

Any $f \in \mathcal{C}_N$ has an atomic decomposition $f = \bigcup_{a_I\subseteq f} a_I$, showing that

$$\mathrm{ev}_f = \sum_{a_I\subseteq f} \mathrm{ev}_{a_I}. \tag{13}$$

Applying the inclusion–exclusion formula

$$\mu\left(\bigcap_{k\in K} A_k\right) = \sum_{\emptyset\neq J\subseteq K} (-1)^{|J|+1}\mu\left(\bigcup_{j\in J} A_j\right)$$

for the Boolean expressions $A_k = x_k \setminus u_I$, $k \in K = [N] \setminus I$, we obtain

$$\begin{aligned}\mu(a_I) &= \sum_{\emptyset\neq J\subseteq([N]\setminus I)} (-1)^{|J|+1}\mu(u_J \setminus u_I) = \sum_{\emptyset\neq J\subseteq([N]\setminus I)} (-1)^{|J|+1}(\mu(u_{I\cup J}) - \mu(u_I))\\ &= \sum_{I\subseteq K\subseteq[N]} (-1)^{|K\setminus I|+1}\mu(u_K),\end{aligned} \tag{14}$$

for any $\mu \in \mathcal{M}_N$ and $I \neq [N]$.

Equations (13) and (14) show that ev_f can be written as a linear combination of the evaluations ev_{u_I}, $(\emptyset \neq I \subseteq [N]))$ with integer coefficients.

To show uniqueness of the coefficients $m_{f,I}$, observe that the evaluations ev_{a_I}, $(\emptyset \neq I \subseteq [N])$ form a basis of $\mathcal{M}_N^*$, and as the linear space spanned by the $2^N - 1$ evaluations ev_{u_I}, $(\emptyset \neq I \subseteq [N])$ contains this basis, it is the whole space $\mathcal{M}_N^*$. Since $\dim \mathcal{M}_N^* = 2^N - 1$, the evaluations ev_{u_I}, $(\emptyset \neq I \subseteq [N])$ are linearly independent.

Proposition 6 *The sum* $\sum_{\emptyset \neq I \subseteq [N]} m_{f,I}$ *of the coefficients is* 1 *if* $a_\emptyset \subseteq f$ *and* 0 *otherwise.*

Proof Let $\mu \in \mathcal{M}_N$ be the additive function the values of which on atoms are given by

$$\mu(a_I) = \begin{cases} 0 & \text{if } I \neq \emptyset \\ 1 & \text{if } I = \emptyset. \end{cases}$$

Then $\mu(u_I) = 1$, for all $\emptyset \neq I \subseteq [N]$. Applying (12) to μ, we obtain

$$\sum_{\emptyset \neq I \subseteq [N]} m_{f,I} = \sum_{\emptyset \neq I \subseteq [N]} m_{f,I} \mu(u_I) = \mu(f) = \sum_{a_I \subseteq f} \mu(a_I) = \begin{cases} 0 & \text{if } a_\emptyset \nsubseteq f \\ 1 & \text{if } a_\emptyset \subseteq f. \end{cases}$$

□

Proposition 7 *For any* $\emptyset \neq I \subseteq [N]$, $m_{u_I,J} = \delta_{I,J}$ *holds, where* $\delta_{I,J}$ *is the Kronecker delta symbol.*

Proof It is clear that $\mu(u_I) = \sum_{\emptyset \neq J \subseteq [N]} \delta_{I,J} \mu(u_J)$. By the uniqueness of the coefficients $m_{u_I,J}$, this equation implies $m_{u_I,J} = \delta_{I,J}$.

Proposition 8 *If* $f, g \in \mathcal{C}_N$ *and* $f \cap g = \emptyset$, *then* $m_{f \cup g,I} = m_{f,I} + m_{g,I}$ *for every* $\emptyset \neq I \subseteq [N]$.

Proof Since for any $\mu \in \mathcal{M}_N$, equation

$$\sum_{\emptyset \neq I \subseteq [N]} m_{f \cup g,I} \mu(u_I) = \mu(f \cup g) = \mu(f) + \mu(g) = \sum_{\emptyset \neq I \subseteq [N]} (m_{f,I} + m_{g,I}) \mu(u_I)$$

holds, uniqueness of the coefficients $m_{f \cup g,I}$ implies the statement.

If $\mu \in \mathcal{M}_N$, then there are infinitely many ways to extend μ to a map $\mu \colon \mathcal{B}_N \to \mathbb{R}$ preserving the additivity property $\mu(f \cup g) = \mu(f) + \mu(g) - \mu(f \cap g)$. Since such a map is uniquely defined by its values on the atomic expressions a_I, and $\mu(a_I)$ is already given for $I \neq [N]$, the extension of μ is uniquely given if we prescribe the value $\mu(a_{[N]}) \in \mathbb{R}$. This value is uniquely determined if we require that $\mu(X) = 0$, since this equation holds if and only $\mu(a_{[N]}) = -\sum_{I \subsetneq [N]} \mu(a_I)$.

Definition 2 The unique extension of $\mu \in \mathcal{M}_N$ to a map $\mu \colon \mathcal{B}_N$ satisfying the conditions $\mu(f \cup g) = \mu(f) + \mu(g) - \mu(f \cap g)$ and $\mu(X) = 0$ will be called the *0-weight extension of* μ.

4 Asymptotics for the Volume of Boolean Expressions of Large Balls

Let $f \in \mathcal{C}_N$ be a Boolean expression built from the variables $x_1, \dots, x_N$ and the operations $\cup$, $\cap$ and $\setminus$. For a system of N points $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N) \in (\mathbb{R}^d)^N$ and a given radius $r > 0$, consider the body

$$B_f^d(\mathbf{p}, r) = f(B^d(\mathbf{p}_1, r), \dots, B^d(\mathbf{p}_N, r))$$

obtained by evaluating f on the balls $x_i = B^d(\mathbf{p}_i, r)$. We are interested in the asymptotic behaviour of the volume $\mathcal{V}_f^d(\mathbf{p}, r) = \mathrm{vol}_d(B_f^d(\mathbf{p}, r))$ of this body.

For a system of points $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N) \in (\mathbb{R}^d)^N$ and a set $I \subseteq [N]$, denote by $K_I(\mathbf{p})$ the convex hull of the points $\{\mathbf{p}_i \mid i \in I\}$.

Definition 3 For $f \in \mathcal{C}_N$ and a system of points $\mathbf{p} \in (\mathbb{R}^d)^N$, define the *Boolean quermassintegrals* $W_{f,k}^d(\mathbf{p})$ and *Boolean intrinsic volumes* $V_{f,k}(\mathbf{p})$ by the equations

$$W_{f,k}^d(\mathbf{p}) = \sum_{\emptyset \neq I \subseteq [N]} m_{f,I} W_k^d(K_I(\mathbf{p})) \quad \text{and} \quad V_{f,k}(\mathbf{p}) = \sum_{\emptyset \neq I \subseteq [N]} m_{f,I} V_k(K_I(\mathbf{p})).$$

By Proposition 8, for any k and $\mathbf{p} \in (\mathbb{R}^d)^N$, the maps $\mathcal{C}_N \ni f \mapsto W_{f,k}^d(\mathbf{p})$ and $\mathcal{C}_N \ni f \mapsto V_{f,k}(\mathbf{p})$ are in $\mathcal{M}_N$. We define $W_{f,k}^d(\mathbf{p})$ and $V_{f,k}(\mathbf{p})$ *for arbitrary* $f \in \mathcal{B}_N$ as the 0-weight extension of these maps, respectively.

Theorem 1 *For any Boolean expression $f \in \mathcal{C}_N$, and any fixed system of centers $\mathbf{p} \in (\mathbb{R}^d)^N$ we have*

$$\mathcal{V}_f^d(\mathbf{p}, r) = \sum_{k=d-2}^{d} \binom{d}{k} W_{f,k}^d(\mathbf{p}) r^k + O(r^{d-3}) = \sum_{k=0}^{2} \kappa_{d-k} V_{f,k}(\mathbf{p}) r^{d-k} + O(r^{d-3}).$$

Proof Let $\mu \colon \mathcal{C}_N \to \mathbb{R}$ be the additive function defined by $\mu(g) = \mathcal{V}_g^d(\mathbf{p}, r)$. Applying Eq. (12) for μ, we obtain

$$\mathcal{V}_f^d(\mathbf{p}, r) = \mu(f) = \sum_{\emptyset \neq I \subseteq [N]} m_{f,I} \mu(u_I) = \sum_{\emptyset \neq I \subseteq [N]} m_{f,I} \mathcal{V}_{u_I}^d(\mathbf{p}, r).$$

For each I, $\mathcal{V}_{u_I}^d$ is the volume of the union of some balls, to which we can apply Corollary 1. This gives

$$\mathcal{V}_{u_I}^d = \kappa_d r^d + \kappa_{d-1} V_1(K_I) r^{d-1} + \kappa_{d-2} V_2(K_I) r^{d-2} + O(r^{d-3}). \tag{15}$$

The last two equations together with the definition of the Boolean quermassintegrals and Boolean intrisic volumes imply the theorem.

Remark One of the main goals set in the introduction was to extend Eqs. (2), (3), and (11) for the volumes of Boolean expressions of large congruent balls, finding suitable generalizations of the intrinsic volumes V_0, V_1, V_2, appearing in (11). Theorem 1 gives the desired extension and justifies our definition of the Boolean intrinsic volumes.

5 Properties of Boolean Intrinsic Volumes

The following properties are straightforward corollaries of the analogous properties of intrinsic volumes of convex bodies and the definitions.

Proposition 9

(a) $V_{f,0}(\mathbf{p})$ *does not depend on* $\mathbf{p}$. *Its value* $V_{f,0} \equiv \sum_{\emptyset \neq I \subset [N]} m_{f,I}$ *is* 1 *if* $a_\emptyset \subseteq f$, *and* 0 *otherwise.*

(b) *The Boolean intrinsic volume* $V_{f,k}(\mathbf{p})$ *does not depend on the dimension* d. *In particular,*

$$W_{f,k}^d = \frac{\kappa_k}{\binom{d}{k}} V_{f,d-k} = \frac{\kappa_k}{\binom{d}{k}} V_{f,(d+s)-(k+s)} = \frac{\binom{d+s}{k+s}\kappa_k}{\binom{d}{k}\kappa_{k+s}} W_{f,k+s}^{d+s} = \frac{(d+1)\cdots(d+s)\kappa_k}{(k+1)\cdots(k+s)\kappa_{k+s}} W_{f,k+s}^{d+s}$$

for any $s \in \mathbb{N}$.

(c) $V_{f,k}$ *is a continuous function on* $(\mathbb{R}^d)^N$ *for every* $d > 0$.

(d) *If* $f, g \in \mathcal{B}_N$ *and* $f \cap g = \emptyset$, *then* $W_{f\cup g,k}^d = W_{f,k}^d + W_{g,k}^d$ *and* $V_{f\cup g,k} = V_{f,k} + V_{g,k}$.

(e) $W_{\bar{f},k}^d = -W_{f,k}^d$ *and* $V_{\bar{f},k} = -V_{f,k}$ *for any* $f \in \mathcal{B}_N$.

We are going to find a formula for the Boolean intrinsic volumes that generalizes Eq. (8). Assume that any $k+2$ points of the system $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N) \in (\mathbb{R}^d)^N$ are affinely independent. This can always be achieved by a small perturbation of the points if $d \geq k+1$. Choose a $k+1$ element index set $S = \{i_1, \ldots, i_{k+1}\} \subset [N]$ and denote by σ_S the convex hull of the points $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_{k+1}}$. By the general position assumption on $\mathbf{p}$, σ_S is a k-dimensional simplex and the affine subspace $[\sigma_S]$ spanned by it does not contain any of the points $\mathbf{p}_j$ for $j \notin S$.

Define an integer valued function $n_{f,S,\mathbf{p}} : \mathbb{S}_S^{d-k-1} \to \mathbb{Z}$ on the unit sphere $\mathbb{S}_S^{d-k-1} = \{\mathbf{u} \in \mathbb{S}^{d-1} \mid \mathbf{u} \perp [\sigma_S]\}$ as follows. Choose a vector $\mathbf{u} \in \mathbb{S}_S^{d-k-1}$. Split the index set $[N]$ into three parts depending on the position of the point $\mathbf{p}_i$ relative to the hyperplane orthogonal to $\mathbf{u}$, containing the simplex σ_S by setting

$$\begin{aligned}
\Pi_+ &= \{j \in [N] \mid \langle \mathbf{p}_j - \mathbf{p}_{i_1}, \mathbf{u}\rangle > 0\},\\
\Pi_0 &= \{j \in [N] \mid \langle \mathbf{p}_j - \mathbf{p}_{i_1}, \mathbf{u}\rangle = 0\},\\
\Pi_- &= \{j \in [N] \mid \langle \mathbf{p}_j - \mathbf{p}_{i_1}, \mathbf{u}\rangle < 0\}.
\end{aligned}$$

It is clear that $S \subseteq \Pi_0$ and $S = \Pi_0$ for almost all $\mathbf{u}$. Define the elements $y_1, \ldots, y_N \in \mathcal{B}_N$ by the rule

$$y_j = \begin{cases} X & \text{if } j \in \Pi_+ \cup (\Pi_0 \setminus S), \\ x_j & \text{if } j \in S, \\ \emptyset & \text{if } j \in \Pi_-. \end{cases}$$

Evaluating the Boolean expression f on the y_j's we obtain an element $f(y_1, \dots, y_N) \in \mathcal{B}_{k+1}(x_{i_1}, \dots, x_{i_{k+1}})$ in the free Boolean algebra generated by the elements $x_{i_1}, \dots, x_{i_{k+1}}$. Set $n_{f,S,\mathbf{p}}(\mathbf{u}) = (-1)^{k+1}\tilde{\chi}_{k+1}(f(y_1, \dots, y_N))$.

The values of $n_{f,S,\mathbf{p}}$ are integers in the interval $[-2^k, 2^k]$. Let

$$\nu_{f,S,\mathbf{p}} = \frac{1}{(d-k)\kappa_{d-k}} \int_{\mathbb{S}_S^{d-k-1}} n_{f,S,\mathbf{p}}(\mathbf{u})\mathrm{d}\mathbf{u}$$

be the average value of $n_{f,S,\mathbf{p}}$.

Theorem 2 *If $f \in \mathcal{B}_N$ and $\mathbf{p} \in (\mathbb{R}^d)^N$ satisfies that any $k+2$ points of $\mathbf{p}$ are affinely independent, then we have*

$$V_{f,k}(\mathbf{p}) = \sum_{\substack{S \subseteq [N] \\ |S|=k+1}} \nu_{f,S,\mathbf{p}} \mathrm{vol}_k(\sigma_S). \tag{16}$$

Proof If $f, g \in \mathcal{B}_N$ are disjoint, that is $f \cap g = \emptyset$, then $V_{f \cup g,k} = V_{f,k} + V_{g,k}$, furthermore, $f(y_1, \dots, y_N) \cap g(y_1, \dots, y_N) = \emptyset$ for any choice of the variables y_i, and since the reduced Euler characteristic is an additive function, $\nu_{f \cup g,k} = \nu_{f,k} + \nu_{g,k}$. Thus, both sides of Eq. (16) are additive functions of the Boolean expression f. Since both sides vanish for $f = X$, the two sides are equal for any $f \in \mathcal{B}_N$ if they are equal for any $f \in \mathcal{C}_N$. As it was shown in the proof of Proposition 5, the evaluations ev_{u_I}, for $\emptyset \neq I \subseteq [N]$, form a basis of $\mathcal{M}_N^*$, so it is enough to check the proposition for the unions u_I.

Assume $f = u_I$. Then $V_{f,k}(\mathbf{p}) = V_k(K_I(\mathbf{p}))$ by Proposition 7. Let $S = \{i_1, \dots, i_{k+1}\} \subseteq [N]$ be a set of $k+1$ indices. To understand the geometrical meaning of $n_{f,S,\mathbf{p}}(\mathbf{u})$, consider first the value of $f(y_1, \dots, y_N) = \bigcup_{j \in I} y_j$.

If $y_j = X$ for an index $j \in I$, then $f(y_1, \dots, y_N) = X$ and $\nu_{f,S}(\mathbf{u}) = \tilde{\chi}_{k+1}(X) = 0$. Hence $n_{f,S,\mathbf{p}}(\mathbf{u})$ vanishes if $I \not\subseteq \Pi_- \cup \Pi_0$. By Proposition 2, $n_{f,S,\mathbf{p}}(\mathbf{u})$ vanishes also in the case when one of the variables $x_{i_1}, \dots, x_{i_{k+1}}$ does not appear in $f(y_1, \dots, y_N)$. These variables appear in $f(y_1, \dots, y_N)$ if and only if $S \subseteq I \cap \Pi_0$. This means that if $n_{f,S,\mathbf{p}}(\mathbf{u}) \neq 0$, then $K_I(\mathbf{p})$ is contained in the halfspace $\{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{u}, \mathbf{x} - \mathbf{p}_{i_1} \rangle \leq 0\}$ and the boundary hyperplane of this halfspace intersects the polytope $K_I(\mathbf{p})$ in a face that contains the k-dimensional simplex σ_S. What is the value of $n_{f,S,\mathbf{p}}(\mathbf{u})$ in this case? If $I \subseteq \Pi_- \cup \Pi_0$ and $S \subseteq I \cap \Pi_0$, then

$$n_{f,S,\mathbf{p}}(\mathbf{u}) = (-1)^{k+1}\tilde{\chi}_{k+1}(x_{i_1} \cup \dots \cup x_{i_{k+1}}) = -\tilde{\chi}_{k+1}(x_{i_1} \cap \dots \cap x_{i_{k+1}}) = 1.$$

If the simplex σ_S is not a face of $K_I(\mathbf{p})$, then the smallest face of $K_I(\mathbf{p})$ that contains σ_S has dimension bigger than k because of the general position assumption on $\mathbf{p}$.

In this case, the support of the function $n_{f,S,\mathbf{p}}$ is contained in a great subsphere of $\mathbb{S}_S^{d-k-1}$, and $\nu_{f,S,\mathbf{p}} = 0$.

If σ_S is a face of $K_I(\mathbf{p})$, then $n_{f,S,\mathbf{p}}$ is the indicator function of the intersection of the cone $N(\sigma_S, K_I(\mathbf{p}))$ and the sphere $\mathbb{S}_S^{d-k-1}$, therefore

$$\nu_{f,S,\mathbf{p}} = \frac{1}{(d-k)\kappa_{d-k}} \int_{\mathbb{S}_S^{d-k-1}} n_{f,S,\mathbf{p}}(\mathbf{u})\mathrm{d}\mathbf{u} = \frac{\mathrm{vol}_{d-k}(n(\sigma_S, K_I(\mathbf{p})))}{\kappa_{d-k}} = \nu(\sigma_S, K_I(\mathbf{p})).$$

As all the k-dimensional faces of $K_I(\mathbf{p})$ are simplicies, we conclude that for $f = u_I$, we have

$$\sum_{\substack{S\subseteq[N]\\|S|=k+1}} \nu_{f,S,\mathbf{p}}\mathrm{vol}_k(\sigma_S) = \sum_{\sigma\in\mathcal{F}_k(K_I(\mathbf{p}))} \nu(\sigma, K_I(\mathbf{p}))\mathrm{vol}_k(L) = V_k(K_I(\mathbf{p})) = V_{f,k}(\mathbf{p}),$$

as desired.

Proposition 10 *If $f \in \mathcal{B}_N$, f^* and $f^{\bar{*}}$ are the dual and contradual of f respectively, then*

$$V_{f^*,k} = -V_{f^{\bar{*}},k} = (-1)^k V_{f,k} \textit{ and } W^d_{f^*,k} = -W^d_{f^{\bar{*}},k} = (-1)^{d-k} W^d_{f,k}.$$

Proof Due to Proposition 9 (e) and (b), it is enough to show the equality $V_{f^*,k} = (-1)^k V_{f,k}$. As $V_{f,k}(\mathbf{p})$ does not depend on the dimension of the ambient space $\mathbb{R}^d$, we may assume that $d > k$. Then the set of configurations $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N) \in (\mathbb{R}^d)^N$ satisfying that any $k+2$ of the points $\mathbf{p}_1, \dots, \mathbf{p}_N$ are affinely independent is dense in $(\mathbb{R}^d)^N$. Since $V_{f,k}$ is continuous on $(\mathbb{R}^d)^N$ for all $f \in \mathcal{B}_N$, it suffices to prove the equation $V_{f^*,k}(\mathbf{p}) = (-1)^k V_{f,k}(\mathbf{p})$ for configurations satisfying this general position condition. Under this assumption, Theorem 2 implies the statement if we show the equations $\nu_{f^*,S,\mathbf{p}} = (-1)^k \nu_{f,S,\mathbf{p}}$.

Consider the function $n_{f,S,\mathbf{p}}(\mathbf{u}) = (-1)^{k+1}\tilde{\chi}_{k+1}(f(y_1, \dots, y_N))$ in the definition of $\nu_{f,S,\mathbf{p}}$. It is not difficult to see that $f^{\bar{*}}(y_1, \dots, y_N)$ is the contradual of $f(y_1, \dots, y_N)$ and $f^*(y_1, \dots, y_N)$ is the dual of it, so applying Proposition 4, we obtain $n_{f^*,S,\mathbf{p}} = (-1)^k n_{f,S,\mathbf{p}}$. Taking the mean value of both sides over the unit sphere $\mathbb{S}_S^{d-k-1}$ we get the desired equation $\nu_{f^*,S,\mathbf{p}} = (-1)^k \nu_{f,S,\mathbf{p}}$.

Denote by $l_{\mathbf{u}} \colon \mathbb{R}^d \to \mathbb{R}$ the linear function $l_{\mathbf{u}} \colon \mathbf{x} \mapsto \langle \mathbf{u}, \mathbf{x}\rangle$. If $\mathbf{u}$ is a unit vector, and K is a bounded convex set, then the length of the interval $l_{\mathbf{u}}(K)$ is the width $w_K(\mathbf{u})$ of K in the direction of $\mathbf{u}$. It is known that $V_1(K)$ is proportional to the mean width of K, namely,

$$V_1(K) = \frac{1}{2\kappa_{d-1}} \int_{\mathbb{S}^{d-1}} w_K(\mathbf{u})\mathrm{d}\mathbf{u} = \frac{d\kappa_d}{2\kappa_{d-1}} \boldsymbol{\omega}_d(K).$$

The width and the mean width can be expressed with the help of the support function of K. Recall that the support function of a bounded set $X \subset \mathbb{R}^d$ is defined as the function $h_X\colon \mathbb{S}^{d-1} \to \mathbb{R}$, $h_X(\mathbf{u}) = \sup_{\mathbf{x}\in X}\langle \mathbf{x}, \mathbf{u}\rangle$. It is clear that $w_K(\mathbf{u}) = h_K(\mathbf{u}) + h_K(-\mathbf{u})$, and

$$V_1(K) = \frac{1}{2\kappa_{d-1}}\int_{\mathbb{S}^{d-1}} (h_K(\mathbf{u}) + h_K(-\mathbf{u}))\mathrm{d}\mathbf{u} = \frac{1}{\kappa_{d-1}}\int_{\mathbb{S}^{d-1}} h_K(\mathbf{u})\mathrm{d}\mathbf{u}.$$

We can extend this formula for the case when $f \in \mathcal{L}_N$. Then f can be evaluated on real numbers by setting $a \cup b = \max\{a, b\}$ and $a \cap b = \min\{a, b\}$ for $a, b \in \mathbb{R}$.

Theorem 3 *If $f \in \mathcal{L}_N$, then for any $\mathbf{p} \in (\mathbb{R}^d)^N$, we have*

$$V_{f,1}(\mathbf{p}) = \frac{1}{\kappa_{d-1}}\int_{\mathbb{S}^{d-1}} f(\langle \mathbf{u}, \mathbf{p}_1\rangle, \ldots, \langle \mathbf{u}, \mathbf{p}_N\rangle)\mathrm{d}\mathbf{u}.$$

Proof Suppose that the points $\mathbf{p}_i$ are all contained in the interior of the ball $B_R = B^d(\mathbf{0}, R)$. Since $f \in \mathcal{L}_N$, $a_\emptyset \subseteq f$, therefore $\sum_{\emptyset\neq I\subseteq[N]} m_{f,I} = 1$, and

$$V_{f,1}(\mathbf{p}) = \sum_{\emptyset\neq I\subseteq[N]} m_{f,I} V_1(K_I(\mathbf{p})) = \left(\sum_{\emptyset\neq I\subseteq[N]} m_{f,I} V_1(K_I(\mathbf{p}) + B_R)\right) - V_1(B_R).$$

Denote by $S_i(\mathbf{u})$ the interval $l_{\mathbf{u}}(\{\mathbf{p}_i\} + B_R) = [\langle \mathbf{u}, \mathbf{p}_i\rangle - R, \langle \mathbf{u}, \mathbf{p}_i\rangle + R]$. Then

$$V_1(K_I(\mathbf{p}) + B_R) = \frac{1}{2\kappa_{d-1}}\int_{\mathbb{S}^{d-1}} \mathrm{vol}_1(l_{\mathbf{u}}(K_I + B_R)\mathrm{d}\mathbf{u} = \frac{1}{2\kappa_{d-1}}\int_{\mathbb{S}^{d-1}} \mathrm{vol}_1\left(\bigcup_{i\in I} S_i(\mathbf{u})\right)\mathrm{d}\mathbf{u},$$

and

$$V_{f,1}(\mathbf{p}) = \frac{1}{2\kappa_{d-1}}\int_{\mathbb{S}^{d-1}}\Big(\sum_{\emptyset\neq I\subseteq[N]} m_{f,I}\mathrm{vol}_1\Big(\bigcup_{i\in I} S_i(\mathbf{u})\Big)\Big)\mathrm{d}\mathbf{u} - \frac{d\kappa_d R}{\kappa_{d-1}}.$$

For any fixed $\mathbf{u} \in \mathbb{S}^{d-1}$, the function $\mu\colon \mathcal{C}_N \to \mathbb{R}$ defined by $\mu(f) = \mathrm{vol}_1(f(S_1(\mathbf{u}), \ldots, S_N(\mathbf{u})))$ is in $\mathcal{M}_N$, therefore Proposition 5 yields

$$\sum_{\emptyset\neq I\subseteq[N]} m_{f,I}\mathrm{vol}_1\Big(\bigcup_{i\in I} S_i(\mathbf{u})\Big) = \sum_{\emptyset\neq I\subseteq[N]} m_{f,I}\mu(u_I) = \mu(f) = \mathrm{vol}_1(f(S_1(\mathbf{u}), \ldots, S_N(\mathbf{u}))).$$

By the choice of R, 0 is a common interior point of all the intervals $S_i(\mathbf{u})$. For this reason, all the sets that can be obtained from these intervals using the operations $\cup$ and $\cap$ are also intervals. In particular,

$$f(S_1(\mathbf{u}), \ldots, S_N(\mathbf{u})) = [-f(-\langle \mathbf{u}, \mathbf{p}_1\rangle, \ldots, -\langle \mathbf{u}, \mathbf{p}_N\rangle) - R, f(\langle \mathbf{u}, \mathbf{p}_1\rangle, \ldots, \langle \mathbf{u}, \mathbf{p}_N\rangle) + R],$$

and

$$\mathrm{vol}_1(f(S_1(\mathbf{u}), \ldots, S_N(\mathbf{u})) = f(\langle \mathbf{u}, \mathbf{p}_1\rangle, \ldots, \langle \mathbf{u}, \mathbf{p}_N\rangle) + f(\langle -\mathbf{u}, \mathbf{p}_1\rangle, \ldots, \langle -\mathbf{u}, \mathbf{p}_N\rangle) + 2R.$$

Using the fact that for any integrable function $h\colon \mathbb{S}^{d-1} \to \mathbb{R}$, we have $\int_{\mathbb{S}^{d-1}} h(\mathbf{u})\mathrm{d}\mathbf{u} = \int_{\mathbb{S}^{d-1}} h(-\mathbf{u})\mathrm{d}\mathbf{u}$, these equations give

$$\begin{aligned} V_{f,1}(\mathbf{p}) &= \frac{1}{2\kappa_{d-1}} \int_{\mathbb{S}^{d-1}} (f(\langle \mathbf{u}, \mathbf{p}_1\rangle, \dots, \langle \mathbf{u}, \mathbf{p}_N\rangle) + f(\langle -\mathbf{u}, \mathbf{p}_1\rangle, \dots, \langle -\mathbf{u}, \mathbf{p}_N\rangle) + 2R)\mathrm{d}\mathbf{u} - \frac{d\kappa_d R}{\kappa_{d-1}} \\ &= \frac{1}{\kappa_{d-1}} \int_{\mathbb{S}^{d-1}} f(\langle \mathbf{u}, \mathbf{p}_1\rangle, \dots, \langle \mathbf{u}, \mathbf{p}_N\rangle)\mathrm{d}\mathbf{u}, \end{aligned}$$

as we wanted to show.

6 Monotonocity of the Boolean Intrinsic Volume $V_{f,1}$

In this section, we prove the following result.

Theorem 4 *Assume that the Boolean expression $f \in \mathcal{C}_N$ can be represented by a formula in which each of the variables occurs exactly once. Define the signs ϵ_{ij}^f, for $1 \le i < j \le N$, as in the introduction. If the configurations $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N) \in (\mathbb{R}^d)^N$ satisfy the inequalities $\epsilon_{ij}^f(\mathrm{d}(\mathbf{p}_i, \mathbf{p}_j) - \mathrm{d}(\mathbf{q}_i, \mathbf{q}_j)) \ge 0$ for all $0 \le i < j \le N$, then we have*

$$V_{f,1}(\mathbf{p}) \ge V_{f,1}(\mathbf{q}). \tag{17}$$

Proof It is proved in [5], that if there exist piecewise analytic continuous maps $\mathbf{z}_i\colon [0,1] \to \mathbb{R}^d$ for $1 \le i \le N$, such that $\mathbf{z}_i(0) = \mathbf{p}_i$, $\mathbf{z}_i(1) = \mathbf{q}_i$, and the distances $\mathrm{d}(\mathbf{z}_i(t), \mathbf{z}_j(t))$ are weakly monotonous functions of t for all i and j, then ineqality (1) is true for any choice of the radii. It is not difficult to see that the analytic curves $\mathbf{z}_i\colon [0,1] \to \mathbb{R}^d \times \mathbb{R}^d$ defined by $\mathbf{z}_i(t) = (\cos(t\pi/2)\mathbf{p}_i, \sin(t\pi/2)\mathbf{q}_i)$ connect the points $(\mathbf{p}_i, \mathbf{0})$ to the points $(\mathbf{0}, \mathbf{q}_i)$ in the required way, but jumping into $\mathbb{R}^{2d}$. Thus, embedding the centers into $\mathbb{R}^{2d}$, our assumptions imply the inequality

$$\mathcal{V}_f^{2d}(\mathbf{p}, r) = \mathrm{vol}_{2d}\left(B_j^{2d}(\mathbf{p}, r)\right) \ge \mathrm{vol}_{2d}\left(f(B_f^{2d}(\mathbf{q}, r)\right) = \mathcal{V}_f^{2d}(\mathbf{q}, r) \tag{18}$$

for any choice of the radius r. By Proposition 9 (a), $V_{f,0}(\mathbf{p}) = V_{f,0}(\mathbf{q})$, therefore Theorem 1 gives

$$0 \le \mathcal{V}_f^{2d}(\mathbf{p}, r) - \mathcal{V}_f^{2d}(\mathbf{q}, r) = \kappa_{2d-1}(V_{f,1}(\mathbf{p}) - V_{f,1}(\mathbf{q}))r^{2d-1} + O(r^{2d-2}).$$

This inequality can hold for large r only if the coefficient of the dominant term is nonnegative, i.e., $V_{f,1}(\mathbf{p}) \ge V_{f,1}(\mathbf{q})$.

It seems to be an interesting question whether we can write strict inequality in (17) if, in addition to the assumptions of Theorem 4, we know that the configurations $\mathbf{p}$ and $\mathbf{q}$ are not congruent. An affirmative answer would imply that the generalized Kneser–Poulsen conjecture holds for Boolean expression of congruent balls if the radius of the balls is greater than a certain number depending on the system of the centers.

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