Sphere-of-Influence Graphs in Normed Spaces

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Dedicated to Károly Bezdek and Egon Schulte on the occasion of their 60th birthdays

Abstract We show that any k-th closed sphere-of-influence graph in a d-dimensional normed space has a vertex of degree less than $5^d k$, thus obtaining a common generalization of results of Füredi and Loeb (Proc Am Math Soc 121(4):1063–1073, 1994 [1]) and Guibas et al. (Sphere-of-influence graphs in higher dimensions, Intuitive geometry [Szeged, 1991], 1994, pp. 131–137 [2]).

Toussaint [8] introduced the sphere-of-influence graph of a finite set of points in Euclidean space for applications in pattern analysis and image processing (see [7] for a recent survey). This notion was later generalized to so-called closed sphere-of-influence graphs [3] and to k-th closed sphere-of-influence graphs [4]. Our setting

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will be a d-dimensional normed space $\mathcal{N}$ with norm $\|\cdot\|$. We denote the ball with center $c \in \mathcal{N}$ and radius r by $B(c, r)$.

Definition 1 Let $k \in \mathbb{N}$ and let $V = \{c_i : i = 1, \ldots, m\}$ be a set of points in the d-dimensional normed space $\mathcal{N}$. For each $i \in \{1, \ldots, m\}$, let $r_i^{(k)}$ be the smallest r such that

$$\{j \in \mathbb{N} \colon j \neq i, \left\|c_i - c_j\right\| \leq r\}$$

has at least k elements. Define the *k-th closed sphere-of-influence graph on V* by setting $\{c_i, c_j\}$ an edge whenever $B(c_i, r_i^{(k)}) \cap B(c_j, r_j^{(k)}) \neq \emptyset$.

Füredi and Loeb [1] gave an upper bound for the minimum degree of any closed sphere-of-influence graph in $\mathcal{N}$ in terms of a certain packing quantity of the space (see also [5, 6].)

Definition 2 Let $\vartheta(\mathcal{N})$ denote the largest cardinality of a subset A of the ball $B(o, 2)$ of the normed space $\mathcal{N}$ such that any two points of A are at distance at least 1, and the origin o is in A.

Füredi and Loeb [1] showed that any closed sphere-of-influence graph (that is, in our terminology, a first closed sphere-of-influence graph) in $\mathcal{N}$ has a vertex of degree smaller than $\vartheta(\mathcal{N}) \leq 5^d$. (It is clear that $\vartheta(\mathcal{N})$ is bounded above by the number of balls of radius $1/2$ that can be packed into a ball of radius $5/2$, which is at most 5^d by volume considerations.)

Guibas, Pach and Sharir [2] showed that any k-th closed sphere-of-influence graph in d-dimensional Euclidean space has a vertex of degree at most $c^d k$, for some universal constant $c > 1$. In this note we show the following more precise result, valid for all norms, and generalizing the result of Füredi and Loeb [1] mentioned above.

Theorem 3 *Every k-th sphere-of-influence graph on at least two points in a normed space $\mathcal{N}$ has at least two vertices of degree smaller than $\vartheta(\mathcal{N})k \leq 5^d k$.*

We note that the theorem still holds when there are repeated elements.

Corollary 4 *A k-th sphere-of-influence graph on n points in $\mathcal{N}$ has at most $(\vartheta(\mathcal{N})k - 1)n \leq (5^d k - 1)n$ edges.*

Proof of Theorem 3 Let $V = \{c_1, c_2, \ldots, c_m\}$. Relabel the vertices $c_1, c_2, \ldots, c_m$ such that $r_1^{(k)} \leq r_2^{(k)} \leq \cdots \leq r_m^{(k)}$. We define an auxiliary graph H on V by joining c_i and c_j whenever $\left\|c_i - c_j\right\| < \max\{r_i^{(k)}, r_j^{(k)}\}$. Thus, if $\{c_i : i \in I\}$ is an independent set in H, then no ball in $\{B(c_i, r_i^{(k)}) \colon i \in I\}$ contains the center of another in its interior. We next bound the chromatic number of H.

Lemma 5 *The chromatic number of H does not exceed k.*

Proof Note that for each $i \in \{1, \ldots, m\}$, the set

$$\{j < i : c_i c_j \in E(H)\} = \{j < i : \left\|c_i - c_j\right\| < r_i^{(k)}\}$$

has less than k elements. Therefore, we can greedily color H in the order $c_1, c_2, \ldots, c_m$ by k colors. □

We next show that the degrees of c_1 and c_2 (corresponding to the two smallest values of $r_i^{(k)}$) are both at most $\vartheta(\mathcal{N})k$, which will complete the proof of Theorem 3. We first need the so-called "bow-and-arrow" inequality of [1]. For completeness, we include the proof from [1].

Lemma 6 *(Füredi–Loeb [1]) For any two non-zero elements a and b of a normed space,*

$$\left\| \frac{1}{\|a\|}a - \frac{1}{\|b\|}b \right\| \geq \frac{\|a-b\| - |\|a\| - \|b\||}{\|b\|}.$$

Proof Without loss of generality, we may assume that $\|a\| \geq \|b\| > 0$. Then

$$\begin{aligned}
\|a-b\| &= \left\| \|a\| \frac{1}{\|a\|}a - \|b\| \frac{1}{\|b\|}b \right\| \\
&= \left\| \|b\| \left(\frac{1}{\|a\|}a - \frac{1}{\|b\|}b\right) + (\|a\| - \|b\|)\frac{1}{\|a\|}a \right\| \\
&\leq \|b\| \left\| \frac{1}{\|a\|}a - \frac{1}{\|b\|}b \right\| + \|a\| - \|b\|.
\end{aligned}$$

□

The next lemma is abstracted with minimal hypotheses from [5, Proof of Theorem 6] (see also [1, Proof of Theorem 2.1]).

Lemma 7 *Consider the balls $B(v_1, \lambda_1)$ and $B(v_2, \lambda_2)$ in the normed space $\mathcal{N}$, such that* $\max\{\lambda_1, \lambda_2\} \geq 1$, $v_1 \notin \operatorname{int}(B(v_2, \lambda_2))$, $v_2 \notin \operatorname{int}(B(v_1, \lambda_1))$ *and* $B(v_i, \lambda_i) \cap B(o, 1) \neq \emptyset$ $(i = 1, 2)$. *Define* $\pi : \mathcal{N} \to B(o, 2)$ *by*

$$\pi(x) = \begin{cases} x & \text{if } \|x\| \leq 2, \\ \frac{2}{\|x\|}x & \text{if } \|x\| \geq 2. \end{cases}$$

Then $\|\pi(v_1) - \pi(v_2)\| \geq 1$.

Proof In terms of the norm, we are given that $\|v_1 - v_2\| \geq \max\{\lambda_1, \lambda_2\} \geq 1$, $\|v_1\| \leq \lambda_1 + 1$, and $\|v_2\| \leq \lambda_2 + 1$. Without loss of generality, we may assume that $\|v_2\| \leq \|v_1\|$.

If $v_1, v_2 \in B(o, 2)$ then $\|\pi(v_1) - \pi(v_2)\| = \|v_1 - v_2\| \geq 1$.

If $v_1 \notin B(o, 2)$ and $v_2 \in B(o, 2)$, then

$$\begin{aligned}
\|\pi(v_1) - \pi(v_2)\| &= \left\| 2\frac{1}{\|v_1\|}v_1 - v_2 \right\| \geq \|v_1 - v_2\| - \left\| v_1 - 2\frac{1}{\|v_1\|}v_1 \right\| \\
&= \|v_1 - v_2\| - (\|v_1\| - 2) \geq \lambda_1 - (\lambda_1 + 1) + 2 = 1.
\end{aligned}$$

If $v_1, v_2 \notin B(o, 2)$, then

$$\|\pi(v_1) - \pi(v_2)\| = \left\| 2\frac{1}{\|v_1\|}v_1 - 2\frac{1}{\|v_2\|}v_2 \right\| \geq 2\frac{\|v_1 - v_2\| - \|v_1\| + \|v_2\|}{\|v_2\|} \quad \text{by Lemma 6}$$
$$\geq 2\left(\frac{\lambda_1 - (\lambda_1 + 1)}{\|v_2\|} + 1\right) = \frac{-2}{\|v_2\|} + 2 \geq -1 + 2 = 1. \qquad \square$$

We can now finish the proof of Theorem 3. Let $i \in \{1, 2\}$, and let $c := c_i$, that is, the radius corresponding to c is the smallest, or second smallest. By Lemma 5 we can partition the set of neighbors of c in the k-th closed sphere-of-influence graph on V into k classes $N_1, \ldots, N_k$ so that each N_t is an independent set in H. We may assume that the radius $r_i^{(k)}$ corresponding to c is 1. Then for any $t \in \{1, \ldots, k\}$, each ball in $\{B(c_j, r_j^{(k)}) \colon c_j \in N_t\}$ intersects $B(c, 1)$, and the center of no ball is in the interior of another ball. By Lemma 7, $\{\pi(p - c) \colon p \in N_t\}$ is a set of points contained in $B(o, 2)$ with a distance of at least 1 between any two. That is, $|N_t \setminus \mathrm{int}(B(c, 1))| \leq \vartheta(\mathcal{N}) - 1$ for each $t = 1, \ldots, k$. Since there are at most $k - 1$ points in $V \cap \mathrm{int}(B(c, 1)) \setminus \{c\}$, it follows that the degree of c is at most $\sum_{t=1}^{k} |N_t \setminus \mathrm{int}(B(c, 1))| + k - 1 \leq (\vartheta(\mathcal{N}) - 1)k + k - 1 = \vartheta(\mathcal{N})k - 1$.

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