Self-inscribed Regular Hyperbolic Honeycombs

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Abstract This paper describes ways that certain regular honeycombs of non-finite type in d-dimensional hyperbolic space $\mathbb{H}^d$ for $d = 2, 3$ and 5 can be inscribed in others, in particular showing that some can be inscribed properly in copies of themselves.

Keywords Coxeter group · Simplex dissection · Hyperbolic space · Regular honeycomb · Self-inscribed · Compound

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1 Introduction

The purpose of this paper is to draw attention to some curious properties of certain families of regular hyperbolic honeycombs with ideal vertices. The existence of similarities in euclidean spaces enables some regular honeycombs to be inscribed in smaller copies of themselves, by which we mean that the vertices of one form a subset of the vertices of the other. The d-dimensional cubic tilings exemplify this property, in infinitely many different ways. We shall see here that the same behaviour is exhibited in four families of regular hyperbolic honeycombs, one in $\mathbb{H}^2$, two in $\mathbb{H}^3$ and one in $\mathbb{H}^5$.

At the instigation of one of the referees of an earlier version of the paper, we have shown that certain subgroups of Coxeter groups that we employ are themselves Coxeter groups; these connexions are closely related to simplex dissections of Debrunner [1] which formalized two folkloristic results. At the suggestion of the other, we have added two more families of honeycombs. As a consequence, the paper has been substantially rewritten.

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We have been told by Asia Weiss that Donald Coxeter had come up with similar ideas to those in this note, although nothing was ever published.

2 Regular Polytopes and Automorphism Groups

In this section, we briefly set the scene. As shown in, for example, [3, Chap. 2], a regular n-polytope $\mathcal{P}$ and its automorphism group $\boldsymbol{G}$ can be identified in a natural way. The group $\boldsymbol{G}$ of $\mathcal{P}$ has distinguished generators $\boldsymbol{r}_0, \dots, \boldsymbol{r}_{n-1}$ satisfying – possibly among others – the relations $(\boldsymbol{r}_j\boldsymbol{r}_k)^{p_{jk}} = \boldsymbol{e}$ for $0 \leqslant j \leqslant k \leqslant n-1$, where

$$p_{jk} = \begin{cases} 1, & \text{if } j = k, \\ p_k & \text{if } j = k-1, \\ 2, & \text{if } j \leqslant k-2. \end{cases} \tag{2.1}$$

We always assume here that $p_k \geqslant 3$ for each k, and that $p_k = \infty$ is allowed. In addition, the $\boldsymbol{r}_i$ also have the intersection property:

$$\langle \boldsymbol{r}_i \mid i \in \mathsf{J} \rangle \cap \langle \boldsymbol{r}_i \mid i \in \mathsf{K} \rangle = \langle \boldsymbol{r}_i \mid i \in \mathsf{J} \cap \mathsf{K} \rangle \tag{2.2}$$

for $\mathsf{J}, \mathsf{K} \subseteq \{0, \dots, n-1\}$. Conversely, such a group $\boldsymbol{G}$ is the automorphism group of a regular polytope $\mathcal{P}$, in which case $\boldsymbol{\{}p_1, \dots, p_{n-1}\boldsymbol{\}}$ is called the Schläfli type of $\mathcal{P}$. Combinatorially, for $0 \leqslant k \leqslant n-1$ a k-face of $\mathcal{P}$ is identified with a right coset of the distinguished subgroup $\boldsymbol{G}_k := \langle \boldsymbol{r}_i \mid i \neq k \rangle$, with the incidence relation given by

$$\boldsymbol{G}_j\boldsymbol{a} \leqslant \boldsymbol{G}_k\boldsymbol{b} \iff j \leqslant k \text{ and } \boldsymbol{G}_j\boldsymbol{a} \cap \boldsymbol{G}_k\boldsymbol{b} \neq \emptyset$$

for $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{G}$ turning $\mathcal{P}$ into a poset.

If $\boldsymbol{G}$ is specified solely by the relations implied by (2.1), then it is a (string) Coxeter group, and is denoted $[p_1, \dots, p_{n-1}]$. The corresponding polytope $\mathcal{P}$ is universal, by which we mean that any regular polytope of Schläfli type $\boldsymbol{\{}p_1, \dots, p_{n-1}\boldsymbol{\}}$ is a quotient of $\mathcal{P}$. It is this situation that prevails throughout the paper; $\boldsymbol{\{}p_1, \dots, p_{n-1}\boldsymbol{\}}$ will henceforth mean the universal polytope.

Associated with the Coxeter group $\boldsymbol{G}$ is its contragredient representation G, say. We need little from this, except to know that G acts faithfully on a certain convex cone – the Tits cone – in $\mathbb{R}^n$. Its generators R_k corresponding to the involutions $\boldsymbol{r}_k$ are linear reflexions in hyperplane mirrors, which bound a fundamental chamber C; copies of C under $\boldsymbol{G}$ fit together face-to-face to form the chamber complex, and their union is the Tits cone. The fact that $\boldsymbol{G}$ is a Coxeter group means that the local relations – how chambers fit together around their $(n-2)$-faces – determine the whole structure of the chamber complex. See [3, Sect. 3A] for a brief exposition, as well as further references.

Remark 2.3 We adopt the convention that heavy braces denote an abstract regular polytope, as in the Schläfli type $\boldsymbol{\{}p_1, \dots, p_{n-1}\boldsymbol{\}}$ of $\mathcal{P}$. Light braces indicate a geometric regular polytope in euclidean or hyperbolic space. Thus $\{4, 3\}$ is an abstract 3-cube; $\{4, 3\}$ is the ordinary 3-cube in $\mathbb{E}^4$. We similarly use R_k to denote a geometric reflexion corresponding to the involutory automorphism $\boldsymbol{r}_k$.

3 The Coxeter Group $[3^{n-2}, 2r]$

This section treats the first of the subgroup relationships among Coxeter groups. We begin with something that should be obvious to which we shall appeal twice.

Lemma 3.1 *For $k = 0, \ldots, n-2$ and $r \geqslant 3$, the mapping $\boldsymbol{r}_{n-1} \mapsto \boldsymbol{e}$ and $\boldsymbol{r}_j \mapsto \boldsymbol{r}_j$ for $j = 0, \ldots, n-2$ induces a homomorphism on $[3^{n-2}, 2r]$ with quotient $[3^{n-2}] \cong \mathcal{S}_n$, the symmetric group.*

The notation p^m stands for $p, \ldots, p$, with m occurrences of p.

We then have

Theorem 3.2 *For $k = 0, \ldots, n-2$ and $r \geqslant 3$, the Coxeter group $[3^{k-1}, 2r, r, 2r, 3^{n-k-3}]$ is a subgroup of $[3^{n-2}, 2r]$ of index $\binom{n}{k+1}$.*

Proof The conventions for extreme values of r should be obvious; just think of the block $2r, r, 2r$ as migrating through a sequence of 3s. The generators $\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{n-1}$ of the subgroup $\boldsymbol{G}_k$ (say) are given by

$$\boldsymbol{s}_j := \begin{cases} \boldsymbol{r}_j, & \text{if } j = 0, \ldots, k-1, \\ \boldsymbol{r}_k\boldsymbol{r}_{k+1}\cdots\boldsymbol{r}_{n-2}\boldsymbol{r}_{n-1}\boldsymbol{r}_{n-2}\cdots\boldsymbol{r}_{k+1}\boldsymbol{r}_k, & \text{if } j = k, \\ \boldsymbol{r}_{n+k-j}, & \text{if } j = k+1, \ldots, n-1. \end{cases} \tag{3.3}$$

In the language of [3, Chap. 7], this defines a mixing operation $\boldsymbol{\nu}_k \colon (\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1}) \mapsto (\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{n-1})$. The indexing of $\boldsymbol{\nu}_k$ is chosen to indicate that $\boldsymbol{r}_k$ is the only generator which changes, although the order of $\boldsymbol{r}_{k+1}, \ldots, \boldsymbol{r}_{n-1}$ is reversed. We can extend the range of k in a natural way by $\boldsymbol{\nu}_{n-1} = \boldsymbol{\iota}$ (the identity), and $\boldsymbol{\nu}_{-1} = \boldsymbol{\delta}$ (the duality operation – which reverses the order of all the $\boldsymbol{r}_j$).

It is a routine matter (which we leave to the reader) to verify that $\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{n-1}$ do generate a group satisfying the relations of $[3^{k-1}, 2r, r, 2r, 3^{n-k-3}]$. Bear in mind that, if $\boldsymbol{a}^2 = \boldsymbol{b}^2 = (\boldsymbol{ab})^3 = \boldsymbol{e}$, then $\boldsymbol{aba} = \boldsymbol{bab}$; then appeal to conjugacy. For example,

$$\begin{aligned} \boldsymbol{s}_{k-1}\boldsymbol{s}_k &= \boldsymbol{r}_{k-1} \cdot \boldsymbol{r}_k\boldsymbol{r}_{k+1}\cdots\boldsymbol{r}_{n-2}\boldsymbol{r}_{n-1}\boldsymbol{r}_{n-2}\cdots\boldsymbol{r}_{k+1}\boldsymbol{r}_k \\ &\sim \boldsymbol{r}_k\boldsymbol{r}_{k-1}\boldsymbol{r}_k \cdot \boldsymbol{r}_{k+1}\cdots\boldsymbol{r}_{n-2}\boldsymbol{r}_{n-1}\boldsymbol{r}_{n-2}\cdots\boldsymbol{r}_{k+1} \\ &= \boldsymbol{r}_{k-1}\boldsymbol{r}_k\boldsymbol{r}_{k-1} \cdot \boldsymbol{r}_{k+1}\cdots\boldsymbol{r}_{n-2}\boldsymbol{r}_{n-1}\boldsymbol{r}_{n-2}\cdots\boldsymbol{r}_{k+1} \\ &\sim \boldsymbol{r}_k \cdot \boldsymbol{r}_{k+1}\cdots\boldsymbol{r}_{n-2}\boldsymbol{r}_{n-1}\boldsymbol{r}_{n-2}\cdots\boldsymbol{r}_{k+1} \\ &\cdots \\ &\sim \boldsymbol{r}_{n-2}\boldsymbol{r}_{n-1}, \end{aligned}$$

and so on. We must therefore check that no additional relations are acquired.

To see what $\boldsymbol{v}_k$ does geometrically, we look at the contragredient representation G of $\boldsymbol{G}$. The fundamental chamber C is a cone over a simplex. If, as in Sect. 2, we let R_k be the linear reflexion corresponding to $\boldsymbol{r}_k$, then a consequence of Lemma 3.1 is that the conjugates of R_{n-1} under $\langle R_0, \ldots, R_{n-2}\rangle$ generate a subgroup of G whose fundamental region is the cone F corresponding to the initial simplex facet of $\mathcal{P} := \{3^{n-2}, 2r\}$.

There are $n!$ copies of C in F, which are the images of C under the subgroup $\mathrm{H} := \langle R_0, \ldots, R_{n-2}\rangle$ (this is a symmetric group). Regarding R_j interchangeably as a linear reflexion and its mirror, the hyperplane R_j slices F into two halves, which the reflexion R_j swaps. The fundamental cone C_k of the representation G_k corresponding to $\boldsymbol{G}_k$ is similarly cut out of F by the hyperplanes S_j with $j \neq k, n-1$. The images of C_k in F are those under the subgroup $\langle S_0, \ldots, S_{k-1}\rangle\langle S_{k+2}, \ldots, S_{n-1}\rangle = \langle R_0, \ldots, R_{k-1}\rangle\langle R_{k+1}, \ldots, R_{n-2}\rangle$, of order $(k+1)!(n-k-1)!$.

It should now be clear that the local geometric structure around C_k is inherited from that around C, and thus that this suffices to determine G_k, and hence $\boldsymbol{G}_k$. In other words, the latter is also a Coxeter group. □

As we have just pointed out, Lemma 3.1 says that $\mathcal{P} = \{3^{n-2}, 2r\}$ collapses onto its initial facet, which is an $(n-1)$-simplex. Consequently, lifting this collapse back into $\mathcal{P}$ implies the first part of

Theorem 3.4 *The vertices of the universal regular polytope $\mathcal{P} = \{3^{n-2}, 2r\}$ can be n-coloured. Moreover, the polytope $\mathcal{P}_k := \{3^{k-1}, 2r, r, 2r, 3^{n-k-3}\}$ can be inscribed in $\mathcal{P}$, using (any) $k+1$ of the colour-classes of its vertices.*

Proof If $\mathcal{P}$, $\mathcal{Q}$ are regular polytopes, we write $\mathcal{Q} \prec \mathcal{P}$ to mean that $\mathcal{Q}$ is inscribed in $\mathcal{P}$; that is, $\operatorname{vert}\mathcal{Q} \subset \operatorname{vert}\mathcal{P}$, with $\operatorname{vert}\mathcal{P}$ the vertex-set of $\mathcal{P}$. The crucial fact is the subgroup relationship between the groups of the vertex-figures: in the previous notation, $\langle \boldsymbol{s}_1, \ldots, \boldsymbol{s}_{n-1}\rangle \leqslant \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{n-1}\rangle$. This means that $\mathcal{P}_k$ has the same initial vertex v (say) as $\mathcal{P}$; indeed, it has the same initial j-face for $j = 0, \ldots, k$.

We now appeal to induction on n. Replacing n by $n-1$ implies replacing k by $k-1$, which means that we first have to establish the case $k = 0$. In this case, $\boldsymbol{s}_0$ swaps the initial facet $\{3^{n-2}\}$ of $\mathcal{P}$ with the one that shares the ridge opposite v; then $v\boldsymbol{s}_0$ has the same colour 1 as v. Moreover, since $\langle \boldsymbol{s}_1, \ldots, \boldsymbol{s}_{n-1}\rangle = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{n-1}\rangle$, all such antipodal vertices in facets through the initial vertex are vertices of $\mathcal{P}_0$, and this quickly leads to $\operatorname{vert}\mathcal{P}_0$ consisting of the whole of colour-class 1 of $\mathcal{P}$.

If $\mathcal{Q}$ is a regular polytope (or honeycomb), then we denote by $\mathcal{Q}^{\boldsymbol{v}}$ its broad vertex-figure; that is, the vertices of $\mathcal{Q}^{\boldsymbol{v}}$ consist of those vertices of $\mathcal{Q}$ that are joined to its initial vertex by an edge. For $k > 0$, we may now assume that $\mathcal{P}_k^{\boldsymbol{v}}$ consists of all vertices of $\mathcal{P}^{\boldsymbol{v}}$ in colour-classes $2, \ldots, k+1$, that is, those adjacent to the initial vertex coloured 1. The claim of the theorem quickly follows from the action of $\boldsymbol{G}_k$ and the symmetry of the colour-classes. □

We next observe that the polytopes $\mathcal{P}_k$ occur in dual pairs. More specifically, since we can freely permute colour-classes, we deduce

Theorem 3.5 *For each $n \geqslant 4$, $r \geqslant 3$ and $k = 0, \ldots, n-2$, $\mathcal{P}_k$ and $\mathcal{P}_{n-k-2}$ are dual polytopes. Moreover, they can be inscribed in $\mathcal{P}$ so that their vertex-sets are complementary colour-classes.*

The following illustrates Theorems 3.2 and 3.4.

Example 3.6 The first theorem yields subgroups [6, 3] and [3, 6] of index 3 in the Coxeter group [3, 6]. By Theorem 3.4, the vertices of the planar tessellation {3, 6} of $\mathbb{E}^3$ by triangles can be 3-coloured; we do not need the general theory to see this. Two out of the three colour classes form the vertices of an inscribed copy of a tessellation {6, 3}, while the third then forms the vertex-set of the dual copy of {3, 6}, now scaled up from the original by $\sqrt{3}$, we have

$$\{3, 6\} \succ \{6, 3\} \succ \{3, 6\};$$

we can iterate the process and extend it to

$$\cdots \succ \{3, 6\} \succ \{3, 6\} \succ \{3, 6\} \succ \cdots$$

with each copy having a third of the vertices of the one before; there is a similar infinite sequence with {6, 3} replacing {3, 6}.

4 The Tessellation {3, ∞}

The vertices of the tessellation $\{3, \infty\}$ in the hyperbolic plane $\mathbb{H}^2$ can be 3-coloured (again, we do not really need the general discussion to see this). Either two out of three, or one out of three of the colour classes yields a tessellation $\{\infty, \infty\}$, and so we can strictly inscribe one copy in another using half the vertices. This process can be repeated to inscribe a copy using a quarter of the vertices, and then an eighth, so on. We can clearly treat this easy case by hand.

5 Honeycombs Inscribed in {3, 3, 6}

We now apply the resuts of Sect. 3 to $\{3^{n-2}, 6\}$. We looked at the first case {3, 6} in Example 3.6, and so here we concentrate on the honeycomb {3, 3, 6} with ideal vertices in $\mathbb{H}^3$. We can take its symmetry group to have generators R_j for $j = 0, \ldots, 3$ given by

$$xR_j := \begin{cases} (\eta, \zeta_2, \zeta_1, \zeta_3), & \text{if } j = 0, \\ (\eta, \zeta_1, \zeta_3, \zeta_2), & \text{if } j = 1, \\ (\eta, -\zeta_2, -\zeta_1, \zeta_3), & \text{if } j = 2, \\ \frac{1}{4}(5\eta - \sqrt{3}\langle z, u\rangle, \sqrt{3}\eta u + 4z - 3\langle z, u\rangle u), & \text{if } j = 3, \end{cases} \tag{5.1}$$

where $u := (1, 1, 1)$. The initial vertex is $(\sqrt{3}, -1, 1, 1)$, and – after rationalization – the vertices in general can be written in the form $x = (\sqrt{3}\eta, z)$, with $\eta, \zeta_1, \zeta_2, \zeta_3 \in \mathbb{Z}$ having no common factor, and $\eta > 0$ such that $3\eta^2 = \zeta_1^2 + \zeta_2^2 + \zeta_3^2$. In fact, considering congruences modulo 8, it is easy to see that $\eta, \zeta_1, \zeta_2, \zeta_3$ must all be odd, if the expression for x is in lowest terms. Note that R_3 does not preserve such expressions, because of the factor $\frac{1}{4}$.

If we successively apply $R_0, R_1, R_2, R_3, R_2, R_1, R_0$ to $(\sqrt{3}, -1, 1, 1)$, then we obtain all $(\sqrt{3}, z)$ with

$$\begin{aligned} z = {} & (1, -1, 1), (1, 1, -1), (-1, -1, -1), (1, 1, 1), \\ & (-1, -1, 1), (-1, 1, -1), (1, -1, -1); \end{aligned}$$

in other words, we have all vertices of the form $(\sqrt{3}, \pm 1, \pm 1, \pm 1)$. This reflects the following fact.

Lemma 5.2 *The two honeycombs* $\{3, 3, 6\}$ *and* $\{4, 3, 6\}$ *have the same vertices.*

Indeed, $[3, 3, 6]$ and $[4, 3, 6]$ have a common subgroup, of index 5 in the first and 2 in the second; see [3, Sect. 11G] for the details. This further implies that all changes of sign of the coordinates of z are allowed, as well as all permutations, and so leads us to

Theorem 5.3 *The vertices of* $\{3, 3, 6\}$ *can be taken to be all points of the form* $(\sqrt{3}\eta, z)$ *just described.*

Proof It is clear that vertices of $\{3, 3, 6\}$ are all of the required form. The form of R_3 gets in the way of showing the converse immediately. However, let $(\sqrt{3}\eta, z)$ be of the given form in its lowest terms; permuting the coordinates and changing signs allows us to assume that $\zeta_1 \geqslant \zeta_2 \geqslant \zeta_3 > 0$, with at least one more strict inequality if $\eta > 1$; in particular, $\zeta_1 > \eta$ and $\langle z, u \rangle - \eta > 0$. If $\langle z, u \rangle - \eta \equiv 0 \mod 4$, then we see at once that $(\sqrt{3}\eta', z') = (\sqrt{3}\eta, z)R_3$ has integer entries $\eta', \ldots, \zeta_3'$ with $\eta' < \eta$. Otherwise, observe that $\zeta_1 + \zeta_2 - \zeta_3 - \eta \equiv 0 \mod 4$, since $2\zeta_3 \equiv 2 \mod 4$. We thus change the sign of ζ_3; since η remains the same, while the new $\langle z, u \rangle - \eta$ is still positive, we can apply R_3 as before, yielding a new integral $\eta' < \eta$. This completes the argument. □

Remark 5.4 Something that we cannot explain is that every odd positive η seems to occur in such a reduced expression $3\eta^2 = \|z\|^2$ (we have only checked this for $\eta \leqslant 19$)

As we have seen, the vertices of $\{3, 3, 6\}$ can be 4-coloured; its facets are tetrahedra. Inscribed in $\{3, 3, 6\}$, using 3, 2 and 1 of its colour classes in turn, we find

$$\{3, 3, 6\} \succ \{3, 6, 3\} \succ \{6, 3, 6\} \succ \{3, 6, 3\}.$$

From this, it follows that $\{3, 6, 3\}$ can be inscribed in itself, using just a third of its vertices. Of course, we have the same pattern as before; the two copies of $\{3, 6, 3\}$, and the two copies of {6, 3, 6}, can be inscribed in {3, 3, 6} using complementary colour classes.

This leads to families of compounds. For example, iterating {3, 6, 3}[*3*{3, 6, 3}] leads to {3, 6, 3}[3^k{3, 6, 3}] for each $k \geqslant 1$. But we actually have more: Lemma 5.2 and duality show that we have compounds like

$$\begin{aligned} &\{4, 3, 6\}[\mathit{2}\{6, 3, 6\}]\{6, 3, 4\}, \\ \mathit{3}&\{4, 3, 6\}[\mathit{4}\{3, 6, 3\}]\{6, 3, 4\}, \\ &\{4, 3, 6\}[\mathit{4}\{3, 6, 3\}]\mathit{3}\{6, 3, 4\}. \end{aligned}$$

Once again, we leave further details to the interested reader.

6 The Coxeter Group $[3^{n-3}, 4, q]$

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For the other families, we again begin with a subsidiary remark; compare Lemma 3.1.

Lemma 6.1 *For $k = 0, \ldots, n-2$ and $q \geqslant 3$, the mapping $\boldsymbol{r}_{n-2}, \boldsymbol{r}_{n-1} \mapsto \boldsymbol{e}$ and $\boldsymbol{r}_j \mapsto \boldsymbol{r}_j$ for $j = 0, \ldots, n-3$ induces a homomorphism on $[3^{n-3}, 4, q]$ with quotient $[3^{n-3}] \cong \mathcal{S}_{n-1}$, the symmetric group.*

The main result of the section is

Theorem 6.2 *For $k = 0, \ldots, n-2$ and $q \geqslant 3$, the Coxeter group $[3^{k-1}, 4, q, q, 4, 3^{n-k-4}]$ is a subgroup of $[3^{n-3}, 4, q]$ of index $\binom{n-1}{k+1}$.*

Proof Let $\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1}$ be the distinguished generators of $\boldsymbol{G} = [3^{n-3}, 4, q]$. For $k = 0, \ldots, n-3$, we define the mixing operation $\boldsymbol{\mu}_k \colon (\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1}) \mapsto (\boldsymbol{t}_0, \ldots, \boldsymbol{t}_{n-1})$ by

$$\boldsymbol{t}_j := \begin{cases} \boldsymbol{r}_j, & \text{if } j = 0, \ldots, k-1, \\ \boldsymbol{r}_k \boldsymbol{r}_{k+1} \cdots \boldsymbol{r}_{n-3} \boldsymbol{r}_{n-2} \boldsymbol{r}_{n-3} \cdots \boldsymbol{r}_{k+1} \boldsymbol{r}_k, & \text{if } j = k, \\ \boldsymbol{r}_{n+k-j}, & \text{if } j = k+1, \ldots, n-1. \end{cases} \tag{6.3}$$

Again, the indexing of $\boldsymbol{\mu}_k$ is chosen to indicate that $\boldsymbol{r}_k$ is the only generator which changes, although the order of $\boldsymbol{r}_{k+1}, \ldots, \boldsymbol{r}_{n-1}$ is reversed. As before, we can extend the range of k in a natural way by $\boldsymbol{\mu}_{n-2} = \boldsymbol{\iota}$ (the identity), and $\boldsymbol{\mu}_{-1} = \boldsymbol{\delta}$ (the duality operation).

The proof follows the lines of that of Theorem 3.2 quite closely, in particular in verifying that the required relations are satisfied. In the present case, $2^{k-1}(k-1)!$ copies of the fundamental cone of $\boldsymbol{G}$ in the contragredient representation fit together to form the cone over an $(n-1)$-cross-polytope. We perform the same construction as before, but in a facet of this cross-polytope; Lemma 6.1 ensures that the construction is compatible with the whole group. We leave it to the reader to fill in the details. □

Remark 6.4 The operation $\boldsymbol{\mu}_{n-3}$ coincides with the halving operation $\boldsymbol{\eta}$ applied to the 3-coface $\{4, q\}$; compare [3, (10E2)].

Corresponding to the group $\boldsymbol{G}_k$ of Theorem 6.2 is the (universal) abstract regular polytope

$$\mathcal{P}_k := \{3^{k-1}, 4, q, q, 4, 3^{n-k-4}\},$$

with the same conventions as in the proposition; in particular, $\mathcal{P} := \mathcal{P}_{n-2} = \{3^{n-3}, 4, q\}$. Exactly analogous to Theorem 3.4, we have

Theorem 6.5 *The vertices of the universal regular polytope $\mathcal{P} = \{3^{n-3}, 4, q\}$ can be $(n-1)$-coloured. Moreover, the polytope $\mathcal{P}_k := \{3^{k-1}, 4, q, q, 4, 3^{n-k-4}\}$ can be inscribed in $\mathcal{P}$ using (any) $k+1$ of the colour-classes of its vertices.*

Proof As before, we appeal to induction on n, noting that the initial vertex always stays the same. Replacing n by $n-1$ implies replacing k by $k-1$, which means that we first have to establish the case $k = 0$. In this case, $\boldsymbol{t}_0$ takes the initial vertex into the opposite vertex of the initial cross-polytopal facet of $\mathcal{P}$; this vertex has the same colour 1 as the initial one. Moreover, since $\langle \boldsymbol{t}_1, \ldots, \boldsymbol{t}_{n-1} \rangle = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{n-1} \rangle$, all such antipodal vertices in facets through the initial vertex are vertices of $\mathcal{P}_0$, and this quickly leads to vert $\mathcal{P}_0$ consisting of the whole colour-class 1 of $\mathcal{P}$.

For $k > 0$, we may now assume that $\mathcal{P}_k^{\boldsymbol{v}}$ consists of all vertices of $\mathcal{P}^{\boldsymbol{v}}$ in colour-classes $2, \ldots, k+1$, that is, those adjacent to the initial vertex coloured 1. The claim of the theorem quickly follows from the action of $\boldsymbol{G}_k$ and the symmetry of the colour-classes. □

Again as before, the polytopes $\mathcal{P}_k$ occur in dual pairs, and we deduce

Theorem 6.6 *For each $n \geqslant 4$, $q \geqslant 3$ and $k = 0, \ldots, n-3$, $\mathcal{P}_k$ and $\mathcal{P}_{n-k-3}$ are dual polytopes. Moreover, they can be inscribed in $\mathcal{P}$ so that their vertex-sets are complementary colour-classes.*

For the moment, we just illustrate Theorems 6.2 and 6.5 by two familiar cases. We have expressed them in terms of abstract polytopes, but of course they are isomorphic to the geometric ones in $\mathbb{E}^4$.

Example 6.7 When $n = 4$ and $q = 3$, we have

$$\{3, 4, 3\} \succ \{4, 3, 3\} \succ \{3, 3, 4\}.$$

Example 6.7 tells us that the vertices of the 24-cell $\{3, 4, 3\}$ are 3-colourable, and that the vertices of the 4-cross-polytope $\{3, 3, 4\}$ and 4-cube $\{4, 3, 3\}$ comprise one and two of the colour-classes, respectively. Moreover, they can be re-arranged so that $\{3, 3, 4\}$ and $\{4, 3, 3\}$ have complementary vertex-sets in vert $\{3, 4, 3\}$; they are then in dual position.

Example 6.8 When $n = 5$ and $q = 3$, we have

$$\{3, 3, 4, 3\} \succ \{3, 4, 3, 3\} \succ \{4, 3, 3, 4\} \succ \{3, 3, 4, 3\}.$$

We have the same pattern in Example 6.8. The vertices of the last copy of $\{3, 3, 4, 3\}$ form one out of four colour-classes of the first; the complementary three colour-classes make up the vertex-set of the dual $\{3, 4, 3, 3\}$. Similarly, we can inscribe two dual copies of the cubic tiling $\{4, 3, 3, 4\}$ in $\{3, 3, 4, 3\}$ with complementary vertex-sets (or colour-classes).

Familiar coordinates for the vertices of the geometric honeycombs graphically illustrate all this. For the original copy, we have

$$\text{vert}\{3, 3, 4, 3\} = \{(\xi_1, \ldots, \xi_4) \in \tfrac{1}{2}\mathbb{Z}^4 \mid \xi_1 \equiv \cdots \equiv \xi_4 \mod 1\}.$$

This actually identifies vert$\{3, 3, 4, 3\}$ with the integer quaternions $\xi_1 + \xi_2\mathbf{i} + \xi_3\mathbf{j} + \xi_4\mathbf{k}$.

The obvious splitting

$$\text{vert}\{3, 3, 4, 3\} = \mathbb{Z}^4 \cup \left(\mathbb{Z}^4 + \tfrac{1}{2}(1, 1, 1, 1)\right)$$

into two congruence classes modulo 1 gives the vertices of two dual copies of $\{4, 3, 3, 4\}$. Finally, the other copy of $\{3, 3, 4, 3\}$ has vertex-set

$$\{(\xi_1, \ldots, \xi_4) \in \mathbb{Z}^4 \mid \xi_1 + \cdots + \xi_4 \equiv 0 \mod 2\}.$$

7 Honeycombs Inscribed in {3, 4, 4}

We now have two applications of the results of Sect. 6 which yield information that we do not recall having seen before. For $n = q = 4$, our pattern is

$$\{3, 4, 4\} \succ \{4, 4, 4\} \succ \{4, 4, 4\}; \tag{7.1}$$

the vertex-sets of the two copies of $\{4, 4, 4\}$ form two or one of the three colour-classes of vertices of $\{3, 4, 4\}$, respectively. Indeed, the two copies can be regarded as duals, and so re-arranged to have complementary vertex-sets. However, a striking consequence is that one copy of $\{4, 4, 4\}$ can be inscribed in another using half its vertices; this leads to a doubly-infinite sequence

$$\cdots \succ \{4, 4, 4\} \succ \{4, 4, 4\} \succ \{4, 4, 4\} \succ \cdots,$$

with each copy having half the vertices of the one before.

The universal polytopes are realizable as regular honeycombs in hyperbolic space $\mathbb{H}^3$. For the latter, we adopt the standard model

$$\mathbb{H}^n = \{(\xi_0, \ldots, \xi_n) \in \mathbb{R}^{n+1} \mid \xi_0 > 0,\ \xi_0^2 = \xi_1^2 + \cdots + \xi_n^2 + 1\}.$$

The symmetry group of $\{3, 4, 4\}$ can be taken to have generators R_j (corresponding to $\boldsymbol{r}_j$) as follows. With $x = (\eta, z)$, where $z = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{E}^3$, we have

$$xR_j := \begin{cases} (\eta, \zeta_2, \zeta_1, \zeta_3), & \text{if } j = 0, \\ (\eta, \zeta_1, \zeta_3, \zeta_2), & \text{if } j = 1, \\ (\eta, \zeta_1, \zeta_2, -\zeta_3), & \text{if } j = 2, \\ \big(2\eta - \langle z, u\rangle, z + (\eta - \langle z, u\rangle)u\big), & \text{if } j = 3, \end{cases} \tag{7.2}$$

where $u = (1, 1, 1)$. Thus R_0, R_1, R_2 generate the symmetry group of the octahedron in a natural way. We write R_3 and points of $\mathbb{H}^3$ in this way for future computational convenience. Observe as well that each R_j preserves the set $\mathbb{Z}^4$ of integer vectors.

The vertices of $\{3, 4, 4\}$ are ideal, and so are to be thought of as rays $\{\lambda(\eta, z) \mid \lambda > 0\}$, with $\eta > 0$ and $\eta^2 = \|z\|^2$. We can normalize these in two ways, either by taking $\eta = 1$ and thus $z \in \mathbb{S}^2$ (the unit sphere), or $(\eta, z) \in \mathbb{Z}^4$ with $\gcd(\eta, \zeta_1, \zeta_2, \zeta_3) = 1$. With the latter representation, we have

Theorem 7.3 *The vertex-set of* $\{3, 4, 4\}$ *is*

$$\operatorname{vert}\{3, 4, 4\} = \{(\eta, z) \in \mathbb{Z}^4 \mid \eta > 0,\ \eta^2 = \|z\|^2\}.$$

Proof We first note that the assumed condition $\gcd(\eta, \zeta_1, \zeta_2, \zeta_3) = 1$ implies that η is odd since, if $\zeta \in \mathbb{Z}$, then $\zeta^2 \equiv 0$ or $1 \mod 4$; thus we cannot have η even and at least one of $\zeta_1, \zeta_2, \zeta_3$ odd. It follows that exactly one of $\zeta_1, \zeta_2, \zeta_3$ is odd; hence $\langle z, u\rangle$ must also be odd, and it is then easy to see that each R_j takes one vector of the given form into another.

To see that every vector of that form occurs, we begin by noting that the initial vertex of $\{3, 4, 4\}$ is $(1, 1, 0, 0)$. We next observe that R_0, R_1, R_2 allow us freedom to permute the coordinates of z and change their signs. If $\eta > 1$, then we change signs so that z is a non-negative vector. From $\gcd(\eta, \zeta_1, \zeta_2, \zeta_3) = 1$ we infer that $\eta < \langle z, u\rangle = \zeta_1 + \zeta_2 + \zeta_3$ (just compare η^2 and $\langle z, u\rangle^2$); if $(\eta, z)R_3 =: (\eta', z')$, then we deduce that $\eta' < \eta$. Induction on η leads at once to the claim of the theorem. □

Remark 7.4 In the alternative normalization, we can identify $\operatorname{vert}\{3, 4, 4\}$ with $\mathbb{S}^2 \cap \mathbb{Q}^3$.

In fact, we can say rather more. We have seen that exactly one of $\zeta_1, \zeta_2, \zeta_3$ is odd; moreover, which coordinate is odd is preserved by R_2 and R_3 (for the latter, note that $\eta - \langle z, u\rangle$ is even – of course, R_0 and R_1 permute the colour classes). As a consequence, we have

Proposition 7.5 *With the previous notation, the vertex* (η, z) *of* $\{3, 4, 4\}$ *is coloured* j *just when the* j*th coordinate of* z *is odd.*

Remark 7.6 As a matter of interest, in the given coordinate system the isometry Φ of $\mathbb{H}^3$ with matrix

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

is such that $\mathcal{P}_0 \prec \mathcal{P}_0\Phi = \mathcal{P}_1$, with the indexing introduced in Sect. 6; we do not give the details of the calculation. Thus the powers of Φ (positive and negative) induce the sequence of copies of $\{4, 4, 4\}$, each properly inscribed in the next.

As we have seen, we can inscribe $\{4, 4, 4\}$ in $\{3, 4, 4\}$, using either two or one of its three colour-classes. As a result, we obtain two, dual, regular compounds (on the abstract level as well)

$$\mathit{2}\{3, 4, 4\}[\mathit{3}\{4, 4, 4\}]\{4, 4, 3\}, \ \ \{3, 4, 4\}[\mathit{3}\{4, 4, 4\}]\mathit{2}\{4, 4, 3\}.$$

(The fact that the copies P_0 and P_1 of $\{4, 4, 4\}$ can be arranged to have complementary vertex-sets in vert$\{3, 4, 4\}$ accounts for the numbers *2* and *3*.) Thus, even though $\{4, 4, 4\}$ is self-dual, perhaps surprisingly its compounds in $\{3, 4, 4\}$ are not.

8 Honeycombs Inscribed in {3, 3, 3, 4, 3}

For $n = 6$ and $q = 3$, our pattern is

$$\{3, 3, 3, 4, 3\} \succ \{3, 3, 4, 3, 3\} \succ \{3, 4, 3, 3, 4\} \succ \{4, 3, 3, 4, 3\} \succ \{3, 3, 4, 3, 3\}. \tag{8.1}$$

There are five colour-classes of vertices of $\{3, 3, 3, 4, 3\}$, and the inscribed polytopes use four, three, two or one of these, respectively. The two copies of $\{3, 3, 4, 3, 3\}$ can be regarded as duals, with complementary vertex-sets in those of $\{3, 3, 3, 4, 3\}$; the dual polytopes $\{3, 4, 3, 3, 4\}$ and $\{4, 3, 3, 4, 3\}$ can be viewed similarly.

Remark 8.2 Note that we have an alternative picture of $\{3, 3, 4, 3, 3\} \prec \{4, 3, 3, 4, 3\}$, in the form

$$\{3, 3, 4, 3, 3\} = \left\{3, {3, 4, 3 \atop 3}\right\}$$

with vertices alternate vertices of $\{4, 3, 3, 4, 3\}$ (of course, as in the pattern here).

As in Sect. 7, we obtain a doubly-infinite sequence

$$\cdots \succ \{3, 3, 4, 3, 3\} \succ \{3, 3, 4, 3, 3\} \succ \{3, 3, 4, 3, 3\} \succ \cdots ;$$

in this case, each copy has a quarter of the vertices of its predecessor. Here, though, we can interpolate copies of **{**4, 3, 3, 4, 3**}** and **{**3, 4, 3, 3, 4**}** between successive ones of **{**3, 3, 4, 3, 3**}**, as in the first pattern.

Geometrically, we have a realization {3, 3, 3, 4, 3} as a regular honeycomb in $\mathbb{H}^5$. Of its symmetry group $\langle R_0, \dots, R_5 \rangle$ acting on vectors (η, z), $R_0, \dots, R_4$ fix η and act on $z = (\zeta_1, \dots, \zeta_5)$ in the standard way as symmetries of the 5-cross-polytope (that is, R_j interchanges ζ_{j+1} and ζ_{j+2} for $j = 0, \dots, 3$, while R_4 changes the sign of ζ_5 – compare (7.2)). Further,

$$(\eta, z)R_5 = \tfrac{1}{2}\big(3\eta - \langle z, u\rangle, 2z + (\eta - \langle z, u\rangle)u\big), \tag{8.3}$$

where $u = (1, 1, 1, 1, 1)$. In analogy to the case {3, 4, 4}, we can represent a vertex of {3, 3, 3, 4, 3} by a vector $(\eta, z) \in \mathbb{Z}^6$ with $\eta > 0$ and $\gcd(\eta, \zeta_1, \dots, \zeta_5) = 1$. Exactly the same arguments as deployed in Sect. 7 lead to

Theorem 8.4 *The vertex-set of* {3, 3, 3, 4, 3} *is*

$$\operatorname{vert}\{3, 3, 3, 4, 3\} = \{(\eta, z) \in \mathbb{Z}^6 \mid \eta > 0,\ \eta^2 = \|z\|^2\}.$$

Remark 8.5 In the alternative normalization, we can identify vert{3, 3, 3, 4, 3} with $\mathbb{S}^4 \cap \mathbb{Q}^5$.

In a similar way, we have

Proposition 8.6 *With the same convention as before, the vertex* (η, z) *of* {3, 3, 3, 4, 3} *is coloured* j *just when the* j*th coordinate* ζ_j *of* z *has the same parity as* η.

Proof If η is even, then the assumed condition that $\gcd(\eta, \zeta_1, \dots, \zeta_5) = 1$ implies that at least one of $\zeta_1, \dots, \zeta_5$ must be odd; since $\eta^2 \equiv 0 \mod 4$ and $\zeta^2 \equiv 1 \mod 4$ if $\zeta \in \mathbb{Z}$ is odd, we see that exactly four of them are odd. If η is odd, then $\eta^2,\ \zeta^2 \equiv 1 \mod 8$ (for odd ζ) similarly implies that exactly one of $\zeta_1, \dots, \zeta_5$ is odd. A final observation that R_4 and R_5 preserve the parity condition completes the proof; once again, the fact that $\eta - \langle z, u\rangle$ is even is the key for R_5. □

The discussion shows that, for example, we can inscribe four copies of {3, 3, 4, 3, 3} in itself (with disjoint vertex-sets); this leads to geometric vertex-regular compounds of the form

$$\{3, 3, 4, 3, 3\}[4^k\{3, 3, 4, 3, 3\}]$$

for each k. Since, as remarked earlier, we can interpolate copies of {4, 3, 3, 4, 3} and {3, 4, 3, 3, 4} between successive ones of {3, 3, 4, 3, 3}, a consequence is that completely classifying possible regular compounds of hyperbolic honeycombs may be far from straightforward. So, what we shall do is point out that even some simple compounds do not behave as one might expect.

As in the [3, 4, 4]-family, we have a pair of compounds

$$\mathit{4}\{3,3,3,4,3\}[\mathit{5}\{3,3,4,3,3\}]\{3,4,3,3,3\},$$
$$\{3,3,3,4,3\}[\mathit{5}\{3,3,4,3,3\}]\mathit{4}\{3,4,3,3,3\}$$

where a self-dual honeycomb is inscribed in non-self-dual compounds. Of course, we also have the dual pair

$$\mathit{3}\{3,3,3,4,3\}[\mathit{5}\{3,4,3,3,4\}]\mathit{2}\{3,4,3,3,3\},$$
$$\mathit{2}\{3,3,3,4,3\}[\mathit{5}\{4,3,3,4,3\}]\mathit{3}\{3,4,3,3,3\};$$

once again, the numbers are explained by the fact that $\{3,4,3,3,4\}$ and $\{4,3,3,4,3\}$ can be taken to have complementary subsets of vertices of $\{3,3,3,4,3\}$ (that is, counting colour-classes). Last, though, note that we have further compounds such as

$$\mathit{3}\{3,3,4,3,3\}[\mathit{4}\{3,4,3,3,4\}],\ \{3,3,4,3,3\}[\mathit{2}\{4,3,3,4,3\}],$$
$$\mathit{2}\{3,4,3,3,4\}[\mathit{3}\{4,3,3,4,3\}],$$

which are only vertex-regular; the interested reader will easily be able to derive many others.

9 Quotients

The regular hyperbolic honeycombs with ideal vertices have quotients which are locally toroidal, in that their facets and vertex-figures are either spherical or toroidal; these are discussed in considerable detail in [3, Chaps. 10–12]; we also mention [4]. But on passing to the quotients, it is usually the case that subgroup relationships are not preserved.

However, among the locally toroidal regular polytopes described in [3, Chap. 10] are

$$\{\{3,4\},\{4,4:2s\}\} \succ \{\{4,4:2s\},\{4,4\mid s\}\} \succ \{\{4,4\mid s\},\{4,4:2s\}\},$$

for each $s \geqslant 2$; recall that the torus components$\{4,4:2s\} = \{4,4\}_{(s,s)}$ and $\{4,4\mid s\} = \{4,4\}_{(s,0)}$ (in the notation of the monograph) are determined by their Petrie polygons $\{2s\}$ and holes $\{s\}$, respectively. Exactly the same pattern of colour-classes of vertices carries over to the quotients. Only $s = 2$ gives a finite case; for $s = 3$ the polytopes are naturally realizable in $\mathbb{E}^5$.

In [3, Chap. 11] other polytopes than those arising from quotients of $[3,3,6]$ and its subgroups are considered. In that family, the quotients do not preserve indices of subgroups; indeed, the same group may occur. In no case do the inscriptions carry over.

Though there are far from degenerate finite quotients of $\{3,3,3,4,3\}$, the discussion of [3, Chap. 12] shows that these do not induce nice inscriptions of locally toroidal regular polytopes like those in the previous family. However, the operation $\boldsymbol{\mu}_k$ (with different indices) was employed in a different context in [2] (see also [3, Sect. 14A]) to produce a family of locally projective regular polytopes.

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