Chapter 7
Visual Structuring Processes of Children When Determining the Cardinality of Sets: The Contribution of Eye-Tracking

Priska Schöner and Christiane Benz

Abstract Research claims that perceiving structures in visual presentation of sets is an important ability for children's numerical development. However, it is not easy to investigate whether and how children perceive structures. In this article, we analyze theoretically the processes of perceiving sets and determining the cardinality of sets and discuss possible benefits of the eye-tracking tool to get some insights into these processes of preschool children.

Keywords Perceiving structures · Determining the cardinality of sets (structural) subitizing · Eye-tracking · Preschool education · Early mathematics education

Introduction

In children's lives, structures play an important role – not only for emotional security and emotional development but also for cognitive development. Perceiving, recognizing, and using structures are seen as fundamental abilities especially for mathematical development. The more children's own idea of structuring and "internal representational systems (…) [have] developed structurally, the more coherent, well organized, more mathematically competent the child will be" (Mulligan, Prescott, & Mitchelmore, 2004, p. 394). Therefore, Mulligan and Mitchelmore point out that structure is not only in the focus of research on children's progress with respect to the "development of spatial abilities," but it "has been also a growing theme in the past two decades of research on students' development of numerical concepts" (Mulligan & Mitchelmore, 2013, p. 31).

P. Schöner (✉) C. Benz
University of Education Karlsruhe, Institute of Mathematics & Computer Science, Karlsruhe, Germany
e-mail: schoener@ph-karlsruhe.de

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The Importance of Perceiving Structures for Numerical Development

Numbers and mathematical relations are abstract and not concrete. Yet, to illustrate the abstract concept of numbers, collections of concrete objects often are used to help children build mental conception of numbers. In mathematics education, there is broad consensus that next to an ordinal and cardinal understanding the part-whole understanding of numbers is a very important concept in numerical development (Benz, Peter-Koop, & Grüßing, 2015; Fritz, Ehlert, & Balzer, 2013; Krajewski, 2008) that also forms the foundation for later calculation strategies (Padberg & Benz, 2011). In the part-whole concept, numbers are seen as compositions of other numbers (Gerster, 2009, p. 267). Therefore, to illustrate the part-whole concept of numbers, visual presentations (e.g., sets of objects) with structures are regularly used as models for combinations of groups and not only single items. Söbbeke (2005) describes the act of perceiving and using structures in such visually noticeable illustrations of numbers (collections of concrete objects) as *visual structuring ability*. This can be assumed as a precondition for a part-whole concept of numbers. Next to an association between the visual structuring ability and part-whole understanding (Gaidoschik, 2010; Young-Loveridge, 2002), there is further empirical evidence for the relation of visual structuring ability and the numerical development. For instance, Hunting (2003) found that the ability to change the focus of every single item to perceiving and identifying structures of parts is important for numerical development. Moreover, van Nes (2009) observed a strong association between the numerical development and spatial structuring abilities of children aged 4–6 years, whereas Lüken (2012) found an association between an early structure sense and arithmetical competencies. These research results underline the importance of visual structuring abilities when children deal with visual presentations of numbers in the form of sets of objects. In order to describe and analyze visual structuring abilities in detail when children identify cardinality of sets, we distinguish theoretically between two different processes: the process of perceiving a set and the process of determining the cardinality.

Perceiving Structures and Determining Cardinality of Sets: Two Processes

Both the process of perceiving a set and the process of determining the cardinality can be distinguished into three subgroups. These two processes and their possible relationship are illustrated in Fig. 7.1. The model is developed by an inductive approach (cf. Benz, 2013; Benz et al., 2015, p. 134) and the result of a first evaluation. The two processes can run one after the other or coincide with each other. The blue boxes in Fig. 7.1 show the different possibilities of perceiving a set of objects.

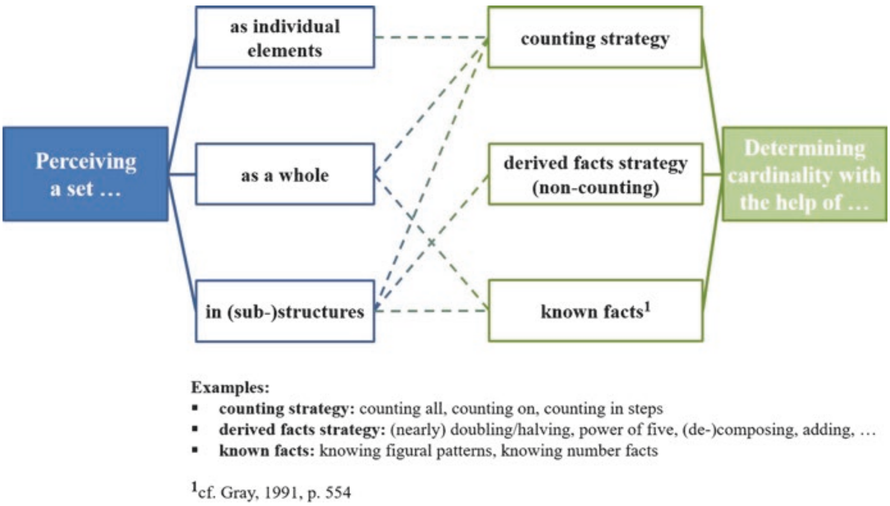


Fig. 7.1 Two processes: perception of sets and determining cardinality

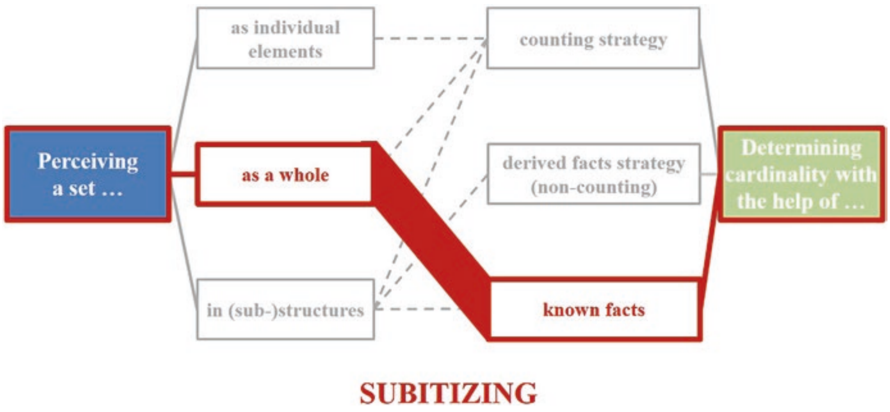


Fig. 7.2 Subitizing

The different possibilities to perceive a set allows various possibilities to determine the cardinality (cf. Fig. 7.1).

Perceiving a set as individual elements leads to the counting strategy *counting all* in order to determine the cardinality. If a set is perceived as a whole, there are two possibilities to determine the cardinality. In the determination process, it is again possible to use the counting strategy counting all or to apply *known facts* (cf. Gray, 1991, p. 554). In this last case, the two processes of perception and determination coincide (subitizing, cf. Fig. 7.2). When perceiving a set in (sub-)structures, there are various possibilities to determine the cardinality: using a counting strategy, a derived facts strategy (e.g., doubling/halving or (de-)composing), or to apply known

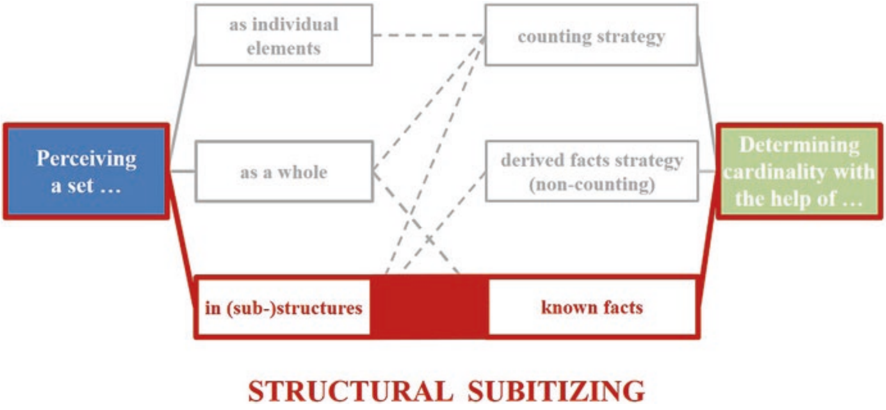


Fig. 7.3 Structural subitizing

facts. The two cases when the processes of perception and determination coincide (subitizing and structural subitizing cf. Figs. 7.2 and 7.3) are described in detail in the following.

The term subitizing was defined by Kaufman, Lord, Reese, and Volkmann (1949). It is derived from "the classical Latin adjective *subitus*, meaning *sudden*, and the medieval Latin verb *subitare*, meaning to *arrive suddenly*" (Kaufman et al., 1949, p. 520, emphasis in original). Subitizing in its original meaning describes that one can quickly and securely name the cardinality of a small set (Kaufman et al., 1949). There are two approaches. Gelman and Gallistel (1986) argue that subitizing is based on a fast counting process, while others claim that subitizing is a noncounting process (cf. Dornheim, 2008). In this paper, the term subitizing is used in its original definition: perceiving a (small) set and immediately naming the number. Two processes coincide, the process of perceiving a set as a whole and the application of known facts, how many elements there are (cf. Fig. 7.2).

Sarama and Clements (2009) also refer to Kaufman et al. (1949) in their definition of the term subitizing. They distinguish between perceptual and conceptual subitizing (Clements, 1999; Clements & Sarama, 2014; Sarama & Clements, 2009). "*Perceptual subitizing* […] is closest to the original definition of subitizing: recognizing a number without consciously using other mental or mathematical processes and then naming it" (Sarama & Clements, 2009, p. 44, emphasis in original). To recognize small numbers a preattentive quantitative process is used. For naming the cardinal number, a conscious numerical process is added (ibid.). With perceptual subitizing, it is therefore possible to just "see" how many objects there are and to name the cardinal number immediately. Here two processes can be identified which occur at the same time: on the one hand, the perception (just "see") and, on the other, the determination of the cardinality (name the cardinal number). Clements and Sarama (2014) assume that perceptual subitizing is possible up to a maximum of four elements (p. 18). A set of five elements can also be determined using perceptual subitizing if the image of the presented set has already been learned and recognized (ibid.).

If children perceive substructures in sets, they have different possibilities for determining the cardinality, see Fig. 7.1. Clements and Sarama (Clements & Sarama, 2014; Sarama & Clements, 2009) use the term *conceptual subitizing* if children perceive structures and use any of the possible different determination strategies. Conceptual subitizing is described as "Seeing the parts and putting together the whole" (Clements & Sarama, 2014, p. 10). The term refers to both the process of recognizing a structure of a set and to the conscious use of partitioning strategies like composing and decomposing for determining the cardinality of this set (Sarama & Clements, 2009, p. 45). Here, the two processes of perception and number determination are also described. Sarama and Clements (2009) say that the recognition of a structure is a necessary precondition for conceptual subitizing. The way in which the number is determined (determination process) plays a subordinate role. So the child can, for example, apply known facts that three and two results in five (Sarama & Clements, 2009) or *count in steps* (Sarama & Clements, 2009) but also use *counting on* to determine the cardinality (Clements & Sarama, 2014). These different descriptions of the determination processes as part of conceptual subitizing show that Clements and Sarama do not distinguish between different determination processes when using the term conceptual subitizing. For example, the described determination process "knowing that two and three result in five" (Sarama & Clements, 2009) is based on recognition of a structure and the use of known facts. This leads to naming the number of the whole set immediately. This description is consistent with the original definition of subitizing, because here the perception and the determination processes coincide. When counting on is the determination process of conceptual subitizing (Clements & Sarama, 2014), it is possible that only a part of the presented set is perceived in structures. The recognition of the structure is not sufficient to determine the cardinality quickly and securely. In this case, the perception and the determination processes do not coincide, which would be a prerequisite for subitizing. In order to clearly distinguish between perceiving the structure and different ways of determining the cardinality in this paper, conceptual subitizing is not used.

In the following, the term *structural subitizing* is defined and used as a logical continuation of the idea of subitizing. Structural subitizing also describes an application of known facts and an immediate determination of the cardinality of a set. The two processes of perceiving a set and determining the cardinality coincide as well (cf. Fig. 7.3). In the example "knowing that three and two result in five," the process of perceiving the set in substructures of three and two coincides with the process of the known facts that the cardinality of the set is five.

Looking on the studies above, it was shown that many children in preschool age are already able to perceive and use structures to identify the cardinality of sets. It is important to note that it is not easy to infer from mere observation whether children perceive structures in a set of objects. A major reason for this is that the process of perceiving structures is an invisible act. Therefore, we can only draw conclusions from the explanations of the children or from interpretations out of visible observations of their process of determining the cardinality of a set. When we observe that children count every single object (e.g., by pointing with the finger or uttering the respective number words), we cannot be sure what they perceive. What we know at least is that they do not use the structures of the arrangement of the respective set to determine the cardinality. To investigate visual structuring ability in most studies, children primarily have to reproduce structured visual sets or they are asked to determine the cardinality when the presentations were presented only for a short time to them. Out of these observations, the use of subitizing for determining the cardinality of parts or the whole is assumed. In order not to rely exclusively on external observations and explanations of the children in order to draw conclusions about whether and how children perceive structures and use them to determine the cardinality, it is helpful to observe the eye movements of the children. In this paper, we discuss the use of eye-tracking as a research tool allowing deeper insights into children's visual structuring abilities. In the long run, it may also be used as an evaluation instrument for intervention studies supporting visual structuring abilities.

Research Question

In this paper, we aim to answer the research question regarding the investigation of visual structuring abilities of preschool children:

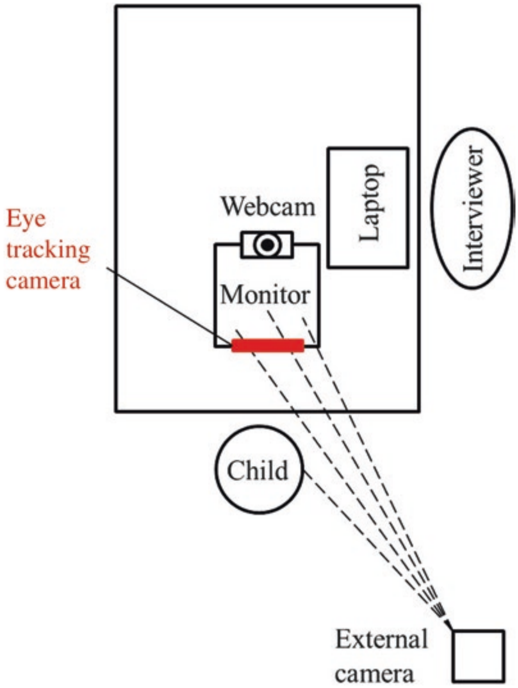
To what extent can eye-tracking contribute to gaining insights into children's perception and determination processes when identifying a set of objects?

Design

One hundred two children aged 5–6 years were interviewed individually to evaluate whether and how they perceive and use structures for determining the cardinality of a set of objects. Each interview consisted of different parts. In this paper, we focus only on the part that deals with sets of eggs in an egg carton for 10 eggs. This is the usual package for eggs children usually know from daily life. Also, its structure is analogue to the 10 frames, a typical didactical presentation used in primary school. Pictures of such egg cartons with different numbers of eggs (2, 3, 4, 5, 7, 9, and 10) were presented on a monitor allowing the recording of the eye movements of participating children. Before the pictures were presented, the child had been told that the interviewer would like to know how many eggs he or she could see. Children were instructed to say the number as soon as they knew it. There was no time limit for children to determine the cardinality of the eggs. Once they said a number, the interviewer asked how they came to the result.

In this study, a mobile eye-tracking system was used. The eyes can be tracked while the head is moved freely, promoting natural human behavior. Additionally,

Fig. 7.4 Interview setting



children sat at a child-sized table and chair supporting their natural position (cf. Fig. 7.4). The eye-tracking system tolerates large and fast head movements which was very important for these interviews with preschool children. The interviewer and the child sat at right angles to each other, thereby being able to talk directly to the child and see the monitor when the child pointed with his or her finger.

All pictures which were presented on the monitor and children's eye movements were recorded as long as the children looked at the screen. Additionally, an external camera was used to monitor other actions of the children, for example, activities with fingers. So it is possible to consider such actions when interpreting the processes of perceiving, determining, and explaining.

Task

Eleven photos of egg cartons were shown on the monitor to each child. The photos with different numbers of eggs were always presented at the same position on the monitor. Each item started with the presentation of a closed egg carton. Then the carton opened. After the child said a number and explained how he/she came to the result, the carton closed again. The screen was never empty because there was always a picture on it to ensure that the child knew where the photo would appear.

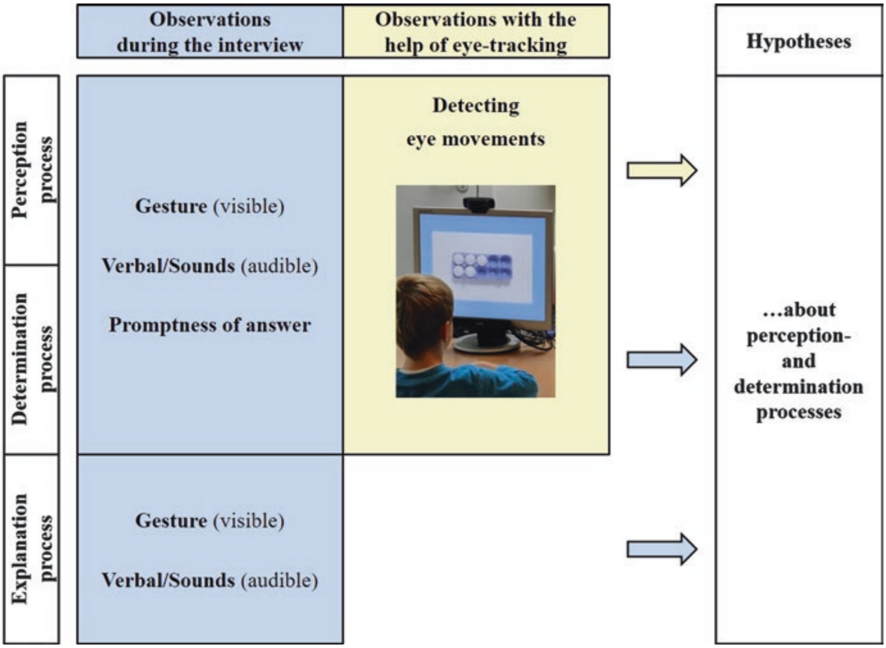


Fig. 7.5 Differentiation of aspects of analysis

Aspects for Analyzing the Data

Figure 7.5 provides an overview of different aspects of analysis. By videotaping, only different observations during the interview (blue column on the left) can be analyzed. Concerning the observation, we differentiate between observations during the two processes perception and determination on the one hand and the process of explanation on the other hand.

Next to visible and audible observations, aspects like gestures (e.g., movements of lips, fingers), verbal comments (e.g., whispering) and promptness of answers, eye-tracking provides additional relevant data (cf. Fig. 7.5, yellow column on the right). These additional data will be considered for the evaluation of the specific use of structures when perceiving sets. Furthermore, the eye-tracking data can also be used for reviewing hypotheses about perception and determination processes derived from visible and audible observations and for gaining additional insights, in identifying whether and how structures were used.

Eye-Tracking Data Analysis

In order to evaluate the eye-tracking data, the GazePlot-Graphic is used. In the GazePlot-Graphic, the order in which the child looked at the single objects is shown. Each colored dot reflects an eye fixation while the size of the dots indicates the duration of the fixation. The longer the child looked at a dot, the larger is the diameter of this dot. The dots can be displayed one after the other like a video sequence. Moreover, an illustration can be chosen where all dots are shown at once. This is called accumulate-graphic.

Still, some technical limitations have to be considered: In case children use gestures for determining the cardinality of sets or explaining, it can happen that their hands cover the camera of the eye-tracker. In that case, not all eye movements can be recorded. Eye movements and fixations can also not be recorded when a child looks at the interviewer when explaining the determination process. Due to these reasons, eye-tracking data reflect eye movements during the two processes of perceiving and determining.

Results and First Interpretations

The analysis of three examples shall illustrate in which situations eye-tracking helps to get additional insights into children's perception and determination processes. After presenting the examples, a first overview will be given summarizing in which cases eye-tracking is helpful (Fig. 7.18).

Interpretation Based on Only One Observation (Promptness) Can Be Confirmed

In the following, the three processes (perception, determination, and explanation) are separately analyzed and the resulting hypotheses are presented. The blue color indicates observations without an eye-tracker. The yellow color indicates observations with the eye-tracker (Fig. 7.5).

As Lisa named the cardinal number immediately (after 2 s), it can be hypothesized that Lisa used the structure of the arrangement of the eggs to derive the quantity (cf. Fig. 7.7). Research indicates that 2 s is too short for children at that age to count every single of the five eggs (cf. Fischer, Gebhardt, & Hartnegg, 2008). This hypothesis of a structural use leads to the assumption that Lisa perceived the set in structures. In Fig. 7.8, the explanation process is interpreted.

When Lisa was asked how she found out that there are five, she just answered: "Because just like that." Thus, the explanation process did not provide additional information about the way she has perceived and determined the presented set (cf. Fig. 7.8). With the help of eye-tracking data, it is possible to get some insights in Lisa's perception process (cf. Fig. 7.9).

On the GazePlot-Graphic of the eye-tracking data, it can be observed that Lisa focused her eyes on the middle egg of the top row and then looked right to the third egg in the top row (cf. Fig. 7.9). On the basis of these observations, it can be assumed that she perceived the set in (sub-)structures. Thereof, the hypothesis can be deduced

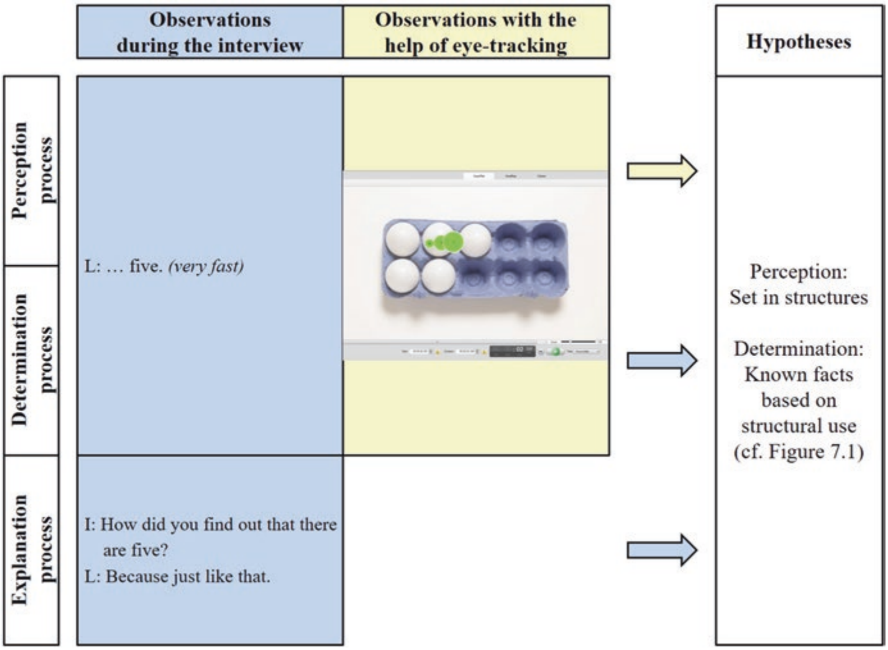


Fig. 7.6 First example – Lisa

that Lisa did not use the strategy counting all but she did use a structure to determine the cardinality of the set. Summing up all observations, Lisa seemed to use structural subitizing (cf. Fig. 7.3).

The example of Lisa shows, when interpreting the data without considering the eye-tracking results, the interpretation of the underlying processes might lead to the conclusion that Lisa used structures and therefore immediately knew the result (cf. Fig. 7.6, blue arrows). But only the component of the "promptness of the answer" would corroborate this hypothesis. No other data was observed. With the help of eye-tracking, it becomes visible that Lisa perceived structures. So the hypothesis of a strategy based on a structural use (cf. Fig. 7.1) can be confirmed (cf. Fig. 7.6, yellow arrow).

Two Inconsistent Observations: Confirming One of the Possible Hypotheses

The long duration of 12 s for the determination process of Tom leads to the assumption that he might have counted the eggs to derive the cardinality of the eggs. Also, small movements of fingers and lips were observed. This leads to the assumption that Tom perceived the set as individual elements. Still, Tom explained that he saw and used structures to determine the cardinality of the eggs (cf. Fig. 7.10).

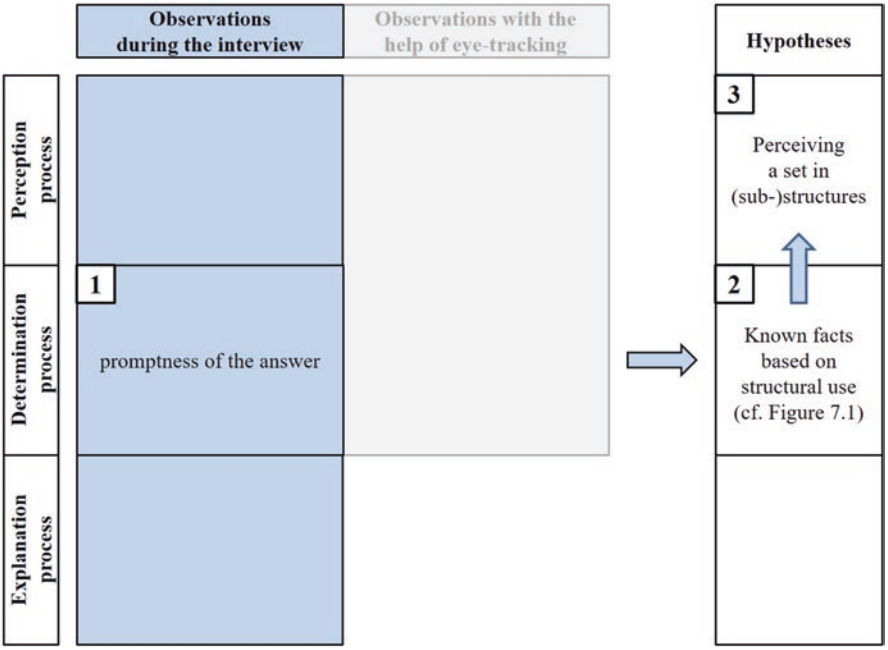


Fig. 7.7 Lisa: Observations and hypotheses *without* the help of eye-tracking data – 1

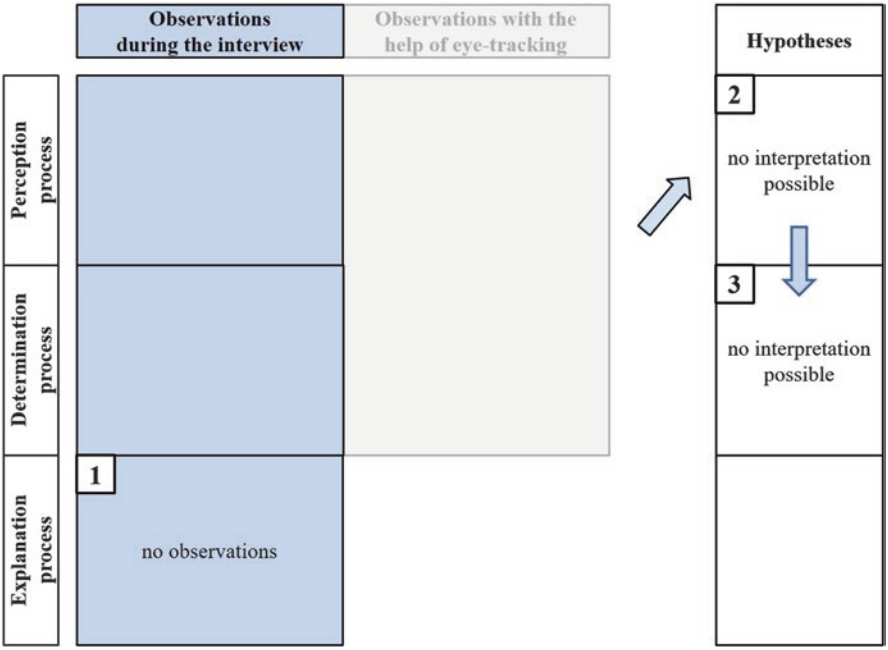


Fig. 7.8 Lisa: Observations and hypotheses *without* the help of eye-tracking data – 2

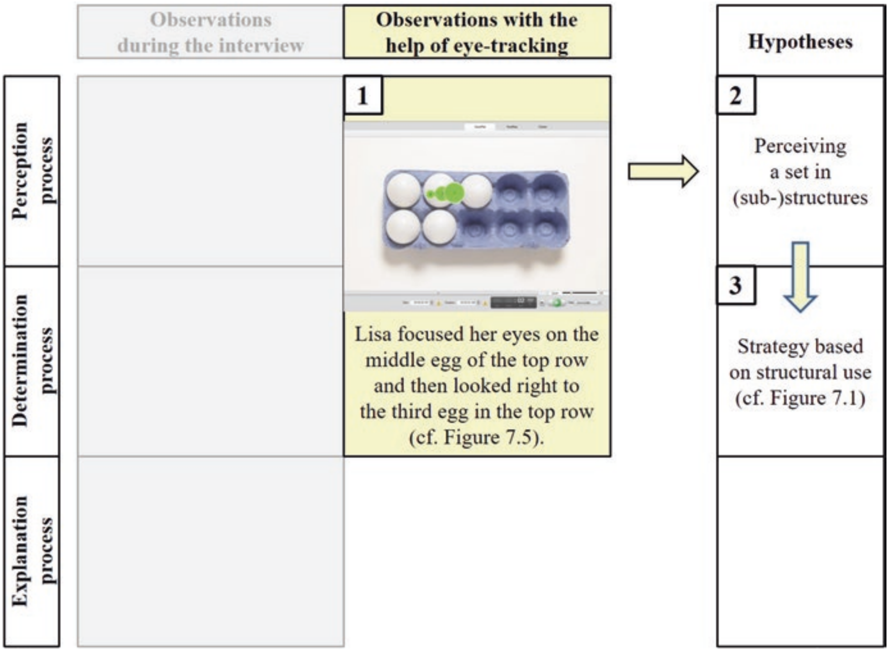


Fig. 7.9 Lisa: Observations and hypotheses *with* the help of eye-tracking data – 3

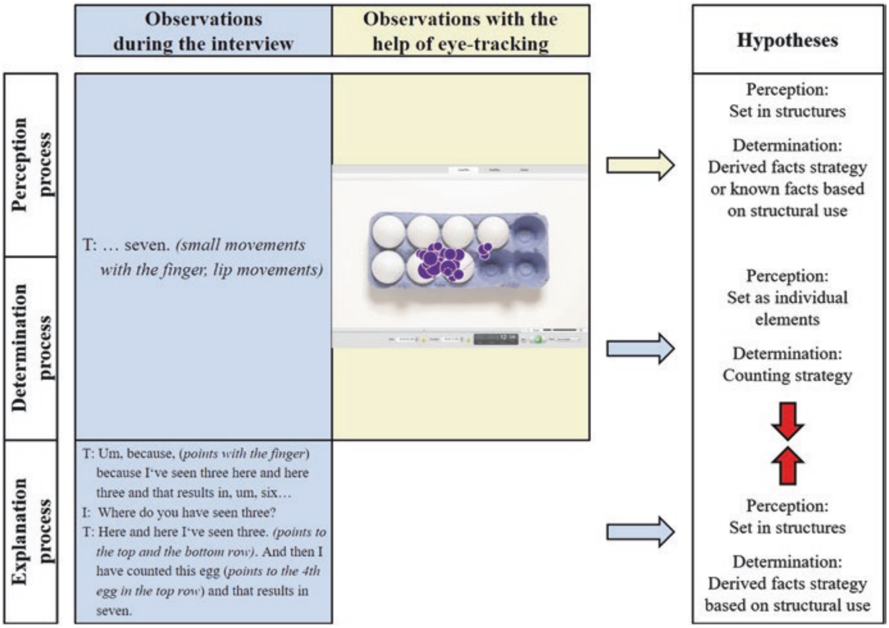


Fig. 7.10 Second example – Tom

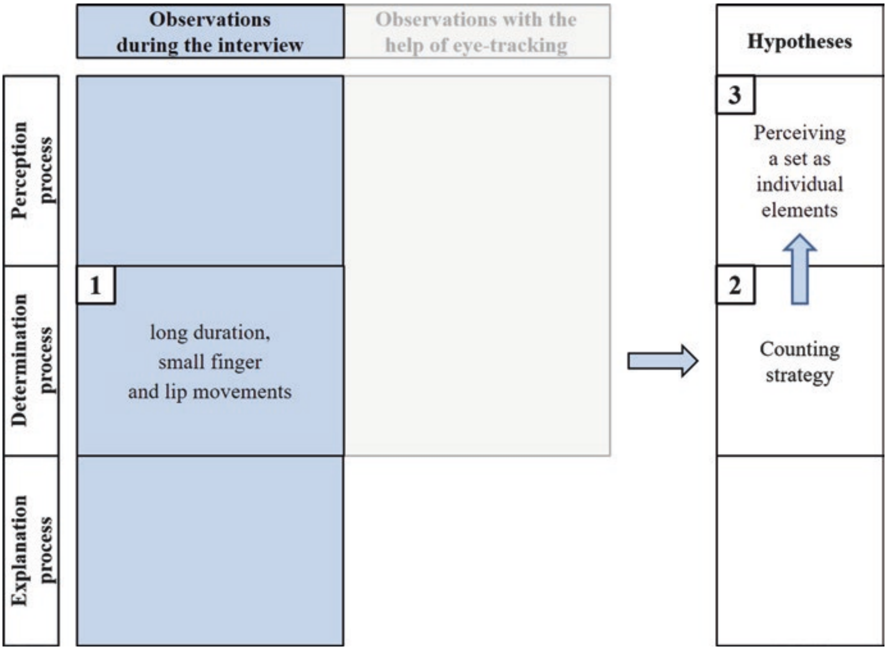


Fig. 7.11 Tom: Observations and hypotheses *without* the help of eye-tracking data – 1

It is now not easy to decide whether Tom just counted the eggs as one would conclude on the basis of the observations during the perception and determination processes (cf. Fig. 7.11) or whether Tom perceived and used structures as he explained (cf. Fig. 7.12). However, the eye-tracking data supported the hypothesis that Tom indeed perceived structures (cf. Fig. 7.13). Because of this observation, the hypothesis can be generated that he used the perceived structure to determine the cardinality of the set (cf. Fig. 7.13). So Tom could have used structural subitizing to determine the cardinality (cf. Fig. 7.3), a derived facts strategy or the counting strategy counting on. All these strategies are based on the use of structures.

At the first glance, the very long perception and determination processes as well as the interpretation of the visible movement of fingers and lips do indicate a counting process. Here, the observations of the perception, determination, and explanation processes did not match, so it is interesting that Tom obviously did not use a counting strategy. However, the eye-tracking data provided meaningful information to confirm one of the two contradictory interpretations by getting insights in the process of perceiving.

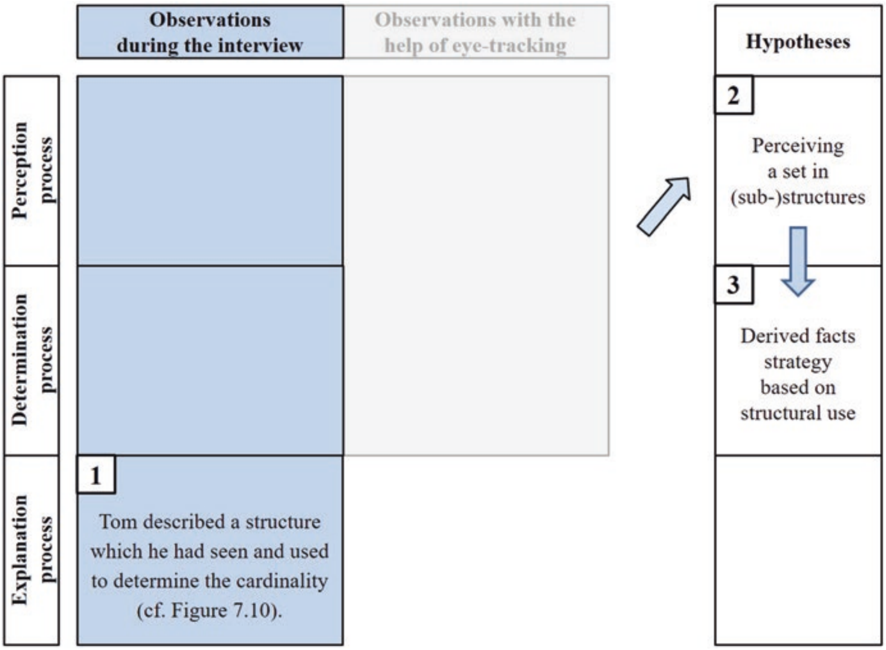


Fig. 7.12 Tom: Observations and hypotheses *without* the help of eye-tracking data – 2

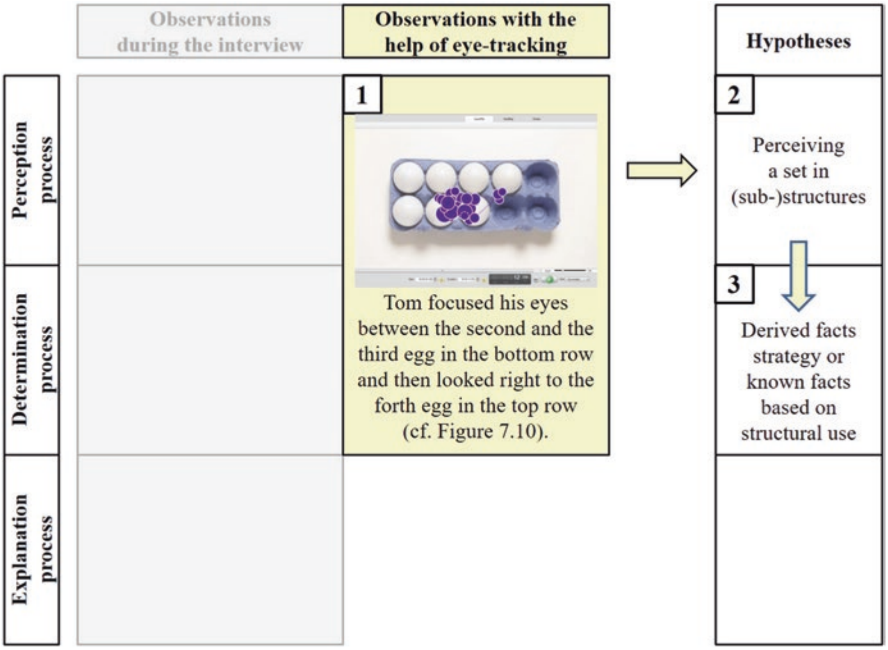


Fig. 7.13 Tom: Observations and hypotheses *with* the help of eye-tracking data – 3

Consistent Observations of the Determination and Explanation Processes Cannot Be Confirmed

The only observation during the determination process was that Emily needed a long time to determine the number (cf. Fig. 7.14). This observation suggests the hypothesis that she might have counted and thus perceived the set as individual elements (cf. Fig. 7.15). In the explanation process, Emily counted loudly every single egg and pointed with her finger on it. This observation also leads to the conclusion that she determined by means of counting all and therefore has perceived the set as individual elements (cf. Fig. 7.15). At this point, however, it cannot be ruled out that Emily could have perceived the set in structures but still used the familiar counting strategy for determining the cardinality determination (cf. Fig. 7.1).

The observations using the eye-tracker show that Emily has not fixed every single egg but that her gaze switched back and forth between the upper and lower row (cf. Fig. 7.17). Thus, the hypothesis is supported that she has perceived the set in structures and used structures for the determination of the cardinality of the whole presented set (cf. Fig. 7.17). So Emily used, based on the use of structures, a derived facts strategy or known facts (cf. Fig. 7.3) to determine the cardinality. To sum up: Due to the observations which are made without the eye-tracker, one might conclude

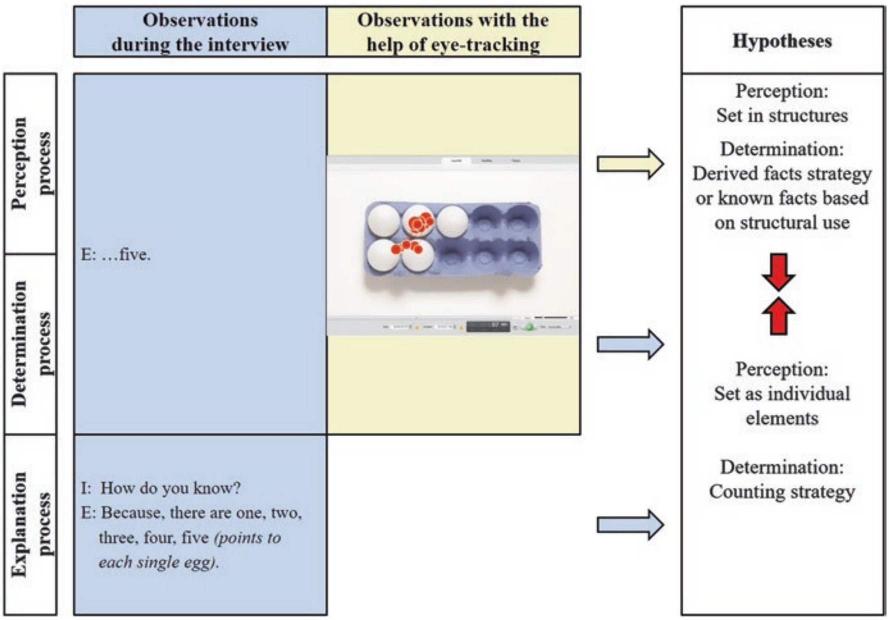


Fig. 7.14 Third example – Emily

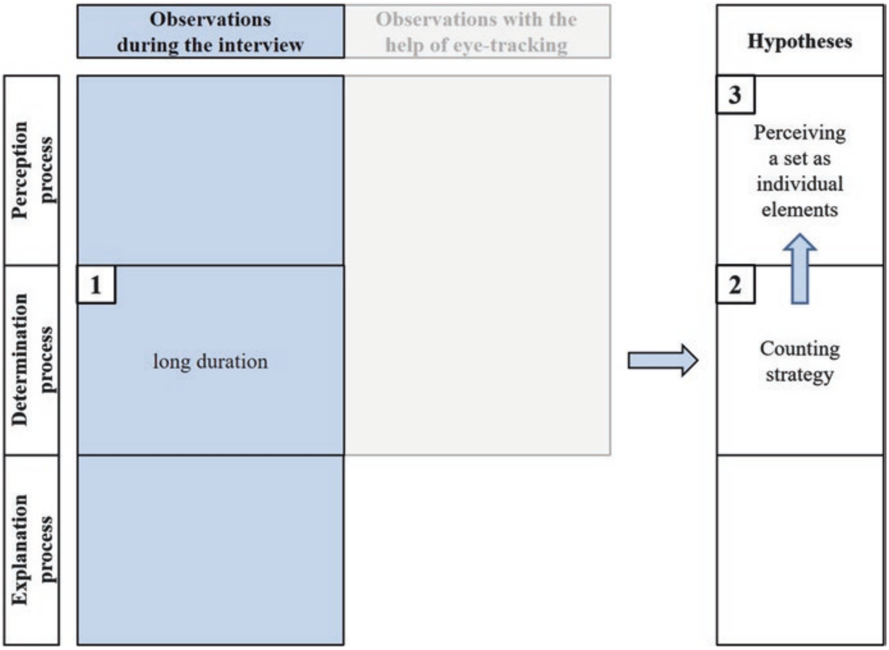


Fig. 7.15 Emily: Observations and hypotheses *without* the help of eye-tracking data – 1

that Emily counted all the eggs separately. The long duration for the determination process (7 s) and her gestures as well as the uttering of numbers supported that assumption (cf. Fig. 7.15 and 7.16). However, the GazePlot-Graphic clearly indicates that Emily perceived a structure. Her fixation switched back and forth between the upper and the lower row (cf. Fig. 7.17).

Here, without eye-tracking data, it would not be evident that Emily recognized and used structures, because the observations of all processes (perception, determination, and explanation) rather indicated counting. The idea of counting was probably just used for the explanation and was presumably not part of the perception processes.

Summary of the Results

In the research question, it was asked to what extent eye-tracking can contribute to gaining insights into children's perception and determination processes when identifying a set of objects. With the analysis of three children's eye-tracking data, this question can be of help to get new insights. We found evidence that eye-tracking data could be of added value for the interpretation of children's solution strategies in this task. In the example of Lisa, only one observation – that she named the

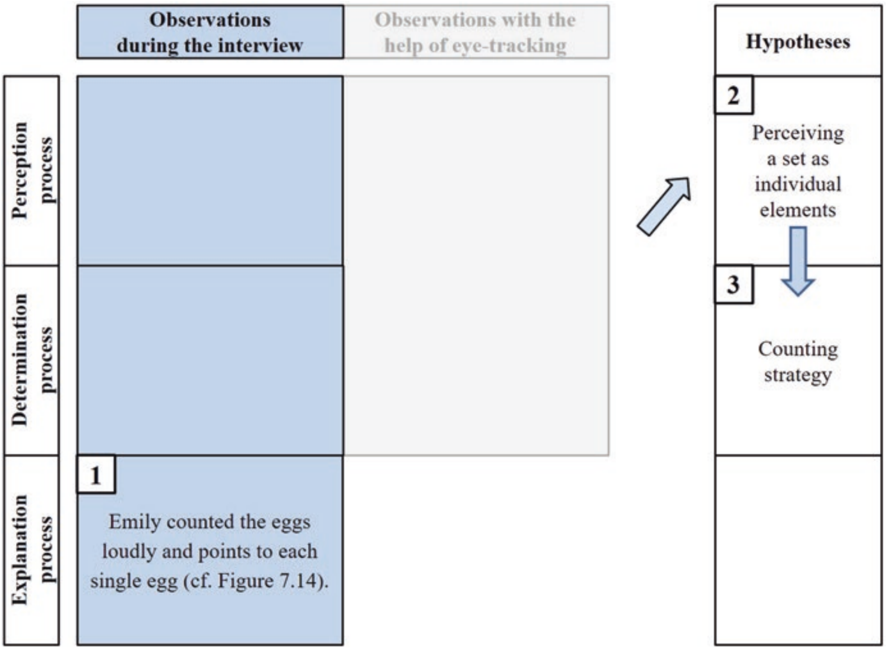


Fig. 7.16 Emily: Observations and hypotheses *without* the help of eye-tracking data – 2

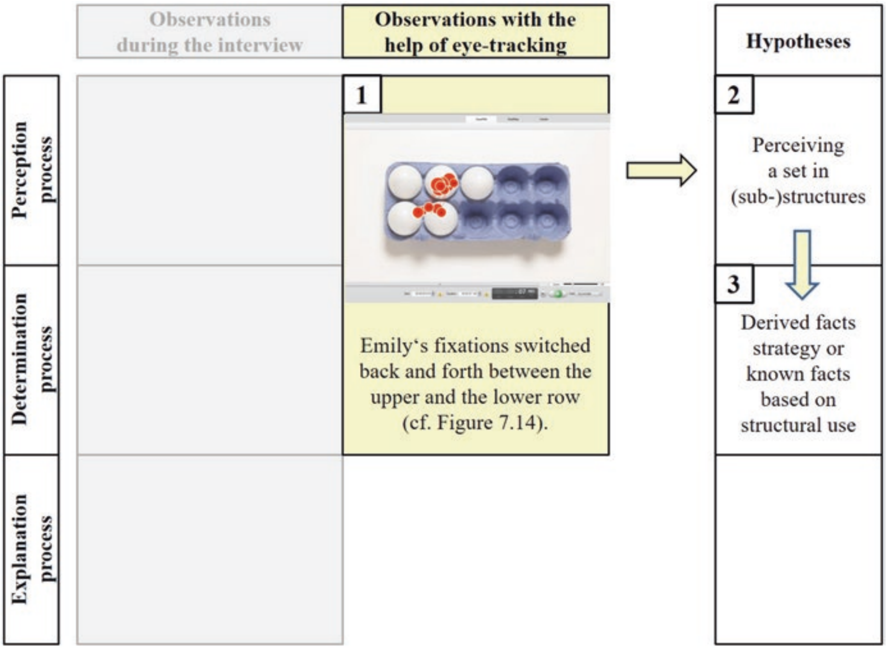


Fig. 7.17 Emily: Observations and hypotheses *with* the help of eye-tracking data – 3

cardinality of the set very quickly – was made during the perception and determination processes. Due to this observation, it can be stated that the child used structural subitizing because of the promptness of the answer. This was confirmed through the eye-tracking data. In the example of Tom, movements of fingers and lips were observed, indicating a counting process. However, in his explanations, he described the use of structures. These contradicting observations do not allow a clear conclusion. The interpretation is clearer after the consideration of the eye-tracking data. Thereby, the use of structures for the determination of the cardinality was confirmed. In the example of Emily, her explanations that she counted the eggs by pointing with the finger on each single egg indicate a counting process. Yet, the eye-tracking data contradicted this interpretation. The analysis indicated that Emily did use a strategy based on structures instead. The following figure (cf. Fig. 7.18) gives an overview of all possibilities of the analyzing processes. In the illustration, all described examples can be found.

The red marked fields in Fig. 7.18 highlight the cases in which additional relevant information can be provided by the eye-tracking data. If observations can be made either in the perception and determination processes or in the explanation process, then first interpretations and conclusions on solution strategies of the respective child are possible. The stated hypotheses can then be corroborated, refuted, or corrected by the help of the eye-tracking data. In the case where observations can be

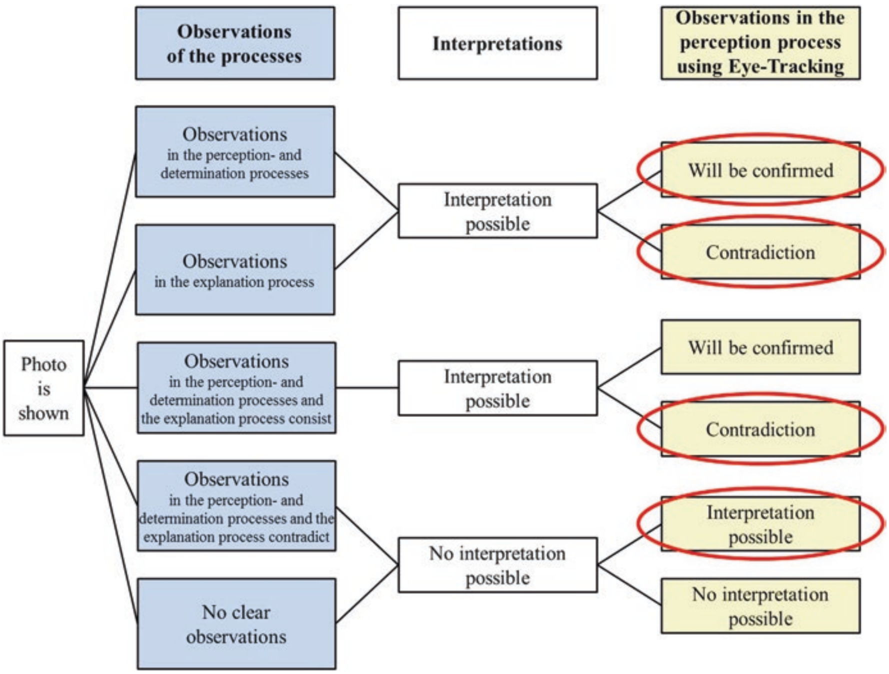


Fig. 7.18 Illustration of analyzing processes

made in all processes (perception, determination, and explanation) and they are consistent, an interpretation is also possible and a hypothesis can be derived. This hypothesis can again be confirmed or refuted by the eye-tracking data. In the latter case, one gets a new insight in the perception and determination processes of children. When observations made during these processes are contradicting or when no clear observations are possible, eye-tracking data may nevertheless be meaningful and allow for deriving a hypothesis on the underlying solution strategies. Thus, the additional observation level provided by eye-tracking data gives the possibility to gain insights into the perception process of children when asked to identify the cardinality of a set of objects. These insights, in turn, often provide opportunities to make statements about the determination process.

Discussion and Conclusion

The three presented interviews showed that with the help of eye-tracking, new insights into children's constructions can be gained while the children perceive visual sets and determine the cardinality. In the case of observations leading to inconsistent interpretations regarding the underlying processes, one of the possible interpretations could be confirmed through the eye-tracking data. In case no interpretation was possible from the observations, the analysis of the eye-tracking data provided new evidence to come to a new interpretation. Also, when interpretations were based on different observations, which seemed to be consistent, eye-tracking indicated another visual structuring strategy. In sum, this revealed that the consideration of children's eye fixation behavior is useful and promising. The visual structuring of a set of objects when determining the cardinality can be revealed through different analyses of the eye-tracking data, which indicate the perception and use of structures for quantification to be a foundation for acquiring the part-whole concept. Thus, the gained data and findings make an important contribution to the scientific discourse about the perception of structures of children not only in kindergarten. In addition to that, these insights into the visual structuring ability of children can be used for the choice and development of learning materials used in kindergarten that encourage playful discovering and exploring and in order to selectively facilitate the perception, recognition, and usage of structures in sets of objects. Possible stimuli in order to indicate perceiving and using structures could be, for example: "How did you see that? Can you present it in a way that you can see immediately that there are five? How did you know that there are seven?" Despite all advantages, it should be noted that the eye-tracking tool is complex and expensive. Therefore, it may not be a useful method for observations of daily life in kindergarten. In mathematics education, there is also a broad consensus that mathematics education in kindergarten should take place in meaningful and playful natural learning situations (cf. Benz et al., 2015; Gasteiger, 2015). One of the gained insights was that children often construct structures in the collection of objects, but they often lack the words to describe their constructions and approaches. When then asked to present an explanation for their approach, they often referred to descriptions of familiar strategies, as for example counting. This becomes evident when looking at the example of Emily. She did not mention that she is counting, but her explanation was a counting aloud process accompanied by pointing with the finger on some eggs. However, the analysis of her eye fixation behavior clearly indicated that she perceived the structures and did not focus on each single egg and probably did not count every single egg. This could not only be observed in the example of Emily. Often "counting" is the only way that children know as verbal explanation for determination processes, so for some children, counting is equalized with determining the cardinality. Therefore, next to giving stimuli and asking adequate questions for perceiving structures, a specific kind of language also has to be developed in kindergarten in order to help children explain their processes of perceiving and using structures when they determine the cardinality of sets. This is a particular challenge when designing mathematical learning opportunities and using opportunities for mathematical learning to support visual structuring abilities.

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