Christiane Benz · Anna S. Steinweg
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Helene Vollmuth · Johanna Zöllner *Editors*

Mathematics Education in the Early Years

Results from the POEM3 Conference, 2016

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Introduction

The conference trade under the name *POEM – A Mathematics Education Perspective on Early Mathematics Learning Between the Poles of Instruction and Construction* has a vivid tradition since 2012. Every other year an invited researcher group working on early childhood mathematics education meets in order to exchange current research findings and ideas. Once again in Karlsruhe, Germany in May 2016, the productive and fruitful working atmosphere allowed the participants starting and intensifying research networks in early mathematics learning. The interactions, talks, and presentations paved the way for further joined studies and co-operations. A selection of the papers presented is shared in the book.

The discussed themes in Early Mathematics Learning are manifold and reflect the importance of this comparably new research field. The topics and contents appropriate for the youngest learners and in line with the idea of life-long-learning mathematical processes are set in frameworks and curricula in many countries. Even though, the setting of priorities of topics is discussed still. Moreover, the search for an appropriate and effective method of interaction or teaching is not over yet. Still the question in which way – and how much – children should be 'educated' in mathematics before school beginning is one of the issues addressed. Of great importance – and suitable for the age of children considered – are mathematical games and mathematical play. The particular role of learning environments is a key feature in many research approaches.

Currently deeper analysis of adult-child interaction in mathematical learning situations is increasingly focused on. This interaction in spontaneous or especially designed situations very much depends on the adult involved. The competencies of kindergarten teachers, educators, and parents actually are explored in some projects. The question, which competencies they essentially need, is addressed in other research studies.

This tendency in foci does obviously not mean to fall children and their thinking development into oblivion. On the contrary, adults competencies are always regarded as intertwined with possible support of learning opportunities for the children. The variety of children's strategies and mathematical ideas solving tasks or acting with material and game ideas is overwhelming. Research projects need to focus even more on very detailed processes in order to get to grips with typical or individual mathematical learning developments.

The structure of the book tries to guide the reader through the different research aims and issues. Four greater parts with a special common denominator among the arranged papers are identified. The first two concentrate on two particular kinds of development, teachers' professional and children's learning development. The third part pools research studies creating and evaluating designed learning situation. Last but not least, the fourth part bridges back to adults-child interaction focused on in the first part by a closer look on parent-child interaction. Of course the research subjects of each chapter are interweaving and there are natural overlaps between the parts.

Development of Kindergarten Teachers' Professional Mathematical Competencies

The first part of this book focuses on kindergarten and pre-school teacher's professional development. Each of the four chapters arranged here has a particular perspective on this multifaceted research area in mathematics education.

Inge Hauge, Suela Kacerja, Troels Lange, Johan Lie, Tamsin Meaney, and Elena Severina open the research field to teacher educators at universities in the chapter *Young children's engagement with mathematics: Expanding teacher educators' views*. Teacher education in Kindergarten or early mathematics is in most countries newly implemented at university level. Research in this field is therefore at the very beginning. In addition, teacher educators themselves often cannot draw from their own expertise because they usually have minor or no experience working as kindergarten teachers. The authors of the chapter propose one idea to combine theory and practice by analyzing pictures and notes of young children's activities led by Bishop's (1988) activities and Franzen's (2015) metaphors for pedagogical roles.

The flipped side of the coin of university education is considered by Dorota Lembrer, Suela Kacerja, and Tamsin Meaney. The authors of the chapter *Preservice Teachers Recognising and Responding to Young Children's Engagement with Mathematics* like the students and preservice teachers gain expertise in recognizing and responding to young children engaging in situations involving mathematical ideas. These situations might be both spontaneous and planned. Again a photograph is used in a survey as trigger to express the own understanding. The authors identify huge potential in the idea of utterance to a particularly and purposefully chosen picture, especially in the beginning of mathematics education courses.

Pre- and in-service teacher's knowledge concerning a specific mathematical topic is focused on in the chapter *Using Children's Patterning Tasks During Professional Development for Preschool Teachers* by Dina Tirosh, Pessia Tsamir, Ruthi Barkai, and Esther Levenson. The authors give some insight on a training program framed by a professional development program. The so-called cognitive affective mathematics teacher education framework spans between four major aspects, which are subject-matter and pedagogical content on the one hand and knowledge and self-efficacy on the other hand. Referring to the worked on subject, the authors stress the importance of precise mathematical language and the discussion of possibly occurring strategies used by children.

The very different aspects of teacher education indicated the interweaving aspects of various competencies need by educators and teachers working with young children in mathematics. Hedwig Gasteiger and Christiane Benz give a brief overview of essential existing models of competencies in the beginning of their chapter *Mathematics Education Competence of Professionals in Early Childhood Education: A Theory-Based Competence Model*. The aim is to design a valid and merged model, which allows measuring the effects of developmental programs or approaches. The authors integrate four structural facets – namely knowledge, situational observing and perceiving, pedagogical-didactical action and evaluation – in their model of mathematical and didactical competence of professional in early childhood education. The model is grounded by a differentiated analysis of domain-specific requirement and accompanied by empirical findings.

The particular reflective and subject-based competencies teachers need to identify mathematics in play situations are focused on in the chapter *Stories Neglected About Children's Mathematics Learning in Play* of Trude Fosse, Maria L. Johansson, Magni Hope Lossius, Anita Wager, and Anna Wernberg. The perspective to accept the specific role of play as fruitful cannot be taken for granted. The authors trace the ideas what counts as mathematical learning in early childhood in three different countries. A somewhat political statement frames the findings. The tendencies of "schoolification," i.e., school-like forms of teaching in early education, can be counteracted by narratives about children's playing and reflections on the mathematics involved. Moreover, the authors stress the importance to develop a language to talk about the mathematics in play activities.

Development of Children's Mathematical Competencies

In the second part of this book, four chapters are working on children's competence development in general and in particular content areas.

Götz Krummheuer shows in *The Genesis of Children's Mathematical Thinking in Their Early Years* by one possible trajectory to enable children to develop their mathematical thinking. This kind of thinking is specified as particular explanations given by children while activated by a mathematical concept or object. Mathematical thinking is referred to argumentative practice. Therefore formal discourse is often identified as mathematical one. The author puts emphasis on narrative discourse, i.e., a sequence of statements that resembles a narrative structure. Narrative or "narratory" discourse enables children chances to participate in mathematical discourses and may pave the way to more formal or paradigmatic thinking.

Contents children encounter in early childhood are manifold. One of an especially important one is cardinal numbers and perception of quantities. This special focus is worked on by Priska Schöner and Christiane Benz in *Visual Structuring Processes of Children When Determining the Cardinality of Sets: The Contribution of Eye-Tracking*. The authors evolve a nuanced vocabulary in their theory in order to differentiate processes of perceiving structured amounts of objects from the ones determining the amount. To capture the different processes, an eye-tracking tool is used. The filigree findings may come into effect in daily kindergarten, if teachers become aware of the important processes in the concept known as subitizing.

Children's competences concerning recognizing patterns and especially structured amounts of objects are regarded as essentially important in many studies. Simone Dunekacke, Meike Grüßing, and Aiso Heinze question this importance in *Is Considering Numerical Competence Sufficient? The Structure of 6-Year-Old Preschool Children's Mathematical Competence* from a theoretical research point of view. Besides quantities, children encounter various contents, e.g., shapes, space, and relations in daily life. The authors strive to find or construct a model of mathematical competence, which includes different content areas. In this holistic view about mathematical competencies, one possible instrument to measure these competencies is presented.

Just in line with the call for a broader perspective on competencies, Rebecca Klose and Christof Schreiber focus on two-dimensional shapes in their project *TellMEE: Telling Mathematics in Elementary Education*. The described approach is theoretically embedded in the differentiation of concept image and concept definition. A concept image – as an individual cognitive structure – is not accessible and therefore not assessable, whereas individual concept definitions as verbal articulation and actions concerning the concept image are. The authors present an idea to enhance the utterance by a four-stage process with the aim to produce an audio record about certain contents (shape).

Design and Evaluation of Mathematical Learning Settings

In the third part of the book, the chapters designing mathematical activities are centered. Moreover, the chapters evaluate the specially designed activities and give insight into the effects. The evaluation however addresses different subjects, i.e., teacher's abilities while re-designing designed activities, teacher's behavior and reactions within the design activities, and children's learning evoked by the designed activities.

Svanhild Breive, Martin Carlsen, Ingvald Erfjord, and Per Sigurd Hundeland follow a twofold aim in the chapter *Designing Playful Inquiry-Based Mathematical Learning Activities for Kindergarten*. The main idea of the project is designing activities by the researchers using play as starting points on the one hand. On the other hand, these designed activities serve as initial starting points for kindergarten teachers to adopt and vary the ideas and orchestrate the setting. In the end, the children need to profit from the offered activities. In particular, their willingness to ask questions (inquiry aspect) and their options to construct mathematical ideas depend on the activities. The authors conclude that the guided play as mediate form between free play and instruction may serve both demands.

Mathematical thinking and learning is intertwined with mathematical language. Fostering this special language is the focus of *Talking About Measuring in the Kindergarten: – Linguistic Means in Small Group Interaction* by Birgit Brandt and Sarah Keuch. Like in the chapter before the starting point are purposefully designed situations contributed by kindergarten teachers. The authors stress the fact that missing corrections on the linguistic level are an obstacle for children's mathematical development in the content area measurement.

Learning situations occur in various settings. Nowadays digital situations complement real-life situations almost naturally. In the chapter *Early Maths Via App Use: Some Insights in the EfEKt Project* by Laura Birklein and Anna Susanne Steinweg, the evaluation of implementing an especially designed early mathematics app in different settings is set out. The digital being of the offered learning environment may have special effects. The project combines quantitative data, gained from tests and log files, with qualitative analysis of video-recorded interactions. In the limelight of the chapter, two case studies are described concerning differences in competencies performed in the material- and paper-pencil-based test versus the digital environment. Moreover, particular habits exploiting digital features are outlined.

Mathematical Learning in Family Settings

The fourth part assorts two chapters with a special focus on noninstitutional, i.e., home and family, learning in the early years.

Ergi Acar Bayraktar aims to answer the question *How Can a Father Be Supportive for the Mathematics Learning Process of a Child? – The Relationship Between Scaffolding and the Interactional Niche in the Development of Mathematical Learning in the Familial Context.* She invites the reader to accompany a father playing with his son an especially designed geometrical play. The theory used to analyze the case study focuses on the three components: content, cooperation, and pedagogy and education. These are described referring to the aspects of allocation, situation, and child's contribution. These factors, the author concludes, may vary in natural learning situations.

Ann and Jim Anderson draw data from a longitudinal study (2 years) in *Instruction and Construction of Mathematics at Home: An Exploratory Study*. The mathematical experiences at home children can make alongside institutional setting or in other cases as standalone experiences are highlighted as important. The authors describe the various activities identified as mathematical activities by the voluntarily participating six parents. The adult-child activities are videotaped by either the parents or a research assistant. Categorizing the activities it becomes apparent that those labelled as instructive have mathematics in the core whether in the constructive activities mathematics happens more or less incidental. The authors find a balance of both types of interaction in the overall view. In contrast, the individual families can be aligned to tend to one or the other type. The dance between instruction and construction will and should be a major research focus in early mathematical learning still.

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Part I
Development of Kindergarten Teachers' Professional Mathematical Competencies

Chapter 1
Young Children's Engagement with Mathematics: Expanding Teacher Educators' Views

Inge Hauge, Suela Kacerja, Troels Lange, Johan Lie, Tamsin Meaney, and Elena Severina

Abstract In this paper, teacher educators' stories about young children engaging in mathematical activities are discussed. There has been little previous research about teacher educators, without experiences as kindergarten teachers, reflecting on their own understandings about how to engage children in mathematical situations. The teacher educators' stories were categorised in regard to the mathematics and the pedagogical practices that were evident. Bishop's six mathematical activities, which are Explaining, Playing, Designing, Locating, Measuring and Counting, were used to identify the mathematics that children engaged in. The stories were also analysed in regard to pedagogical practices, related to construction and instruction. Travel guide, travel agent and travel companion were used as metaphors to clarify the pedagogical roles that the teacher educators adopted when they interacted with children. Understanding our own practices with young children has implications for how to support kindergarten student teachers to connect their theoretical understandings about mathematics education in kindergarten with practical experiences from working with children.

Keywords Bishop's six mathematical activities · Learning through play · Mathematics teacher educators · Metaphors about adults interacting with young children · Pedagogical mathematical knowledge · Photostory interview · Reflection about own role · Young children parenting experiences · Young children's mathematical activities

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Mathematics Teacher Educators and Kindergarten Mathematics

Teacher educators are expected to influence student teachers' understandings about mathematics education so that when they are employed in classrooms and kindergartens, they can plan engaging learning activities and make use of spontaneous possibilities. Nevertheless, research consistently shows that the influence is often negated once teachers take up professional appointments (Brouwer & Korthagen, 2005). Brouwer and Korthagen (2005) proposed that part of the reason for the lack of sustainable influence could be because of how the theory and practice elements are combined, or not, within the teacher education programmes. Nolan's (2012) analysis of student teachers' discourses about their internship in secondary schools points towards incompatibilities and misrecognitions between the field of practice in secondary schools and the field of teacher education in which they participate. She further identified the dominance of the first field in that its rules were familiar and comfortable for the student teachers, while those of teacher education with its inquiry-teaching focus were challenging. She discussed this as different discourses that compete for the student teachers' attention.

A lack of sustainable influence on student teacher practices is likely to be compounded when the teacher education programmes are new and teacher educators have little experience of the milieu in which their student teachers will work. This is the case for mathematics education in kindergarten teacher education in Norway, which only became part of the teacher education programmes in the last two decades (Erfjord, Hundeland, & Carlsen, 2012). As with most kindergarten teacher education programmes, almost all of the mathematics teacher educators at our campus do not have experience as kindergarten teachers. Norway's situation is not unique as this has been noted as the case elsewhere, including the USA, where mathematics education courses for preservice kindergarten are generally run by mathematics education faculty who may have limited experiences of working in kindergartens (Parks & Wager, 2015). The possibilities for making connections between theory and practice for preservice teachers are restricted when the teacher educators have no practical experiences to draw upon.

In the last few years, there has been a flurry of activity to identify the pedagogical mathematical knowledge of kindergarten teachers and kindergarten teacher education students (see, e.g., Anders & Rossbach, 2015; Benz, 2014; Dunekacke, Jenßen, Eilerts, & Blömeke, 2016; Lee, 2010; McCray & Chen, 2012; Mosvold, Bjuland, Fauskanger, & Jakobsen, 2011). Two of the authors in this paper are also authors on another chapter in this book (Lembrér, Kacerja & Meaney, in this volume), which explores a method for evaluating Norwegian and Swedish kindergarten teacher education students' knowledge. Our interest in this area stems from the belief that teacher educators are influential in improving the likelihood that student teachers will adopt innovative practices as graduate teachers. For example, Lunenberg, Korthagen, and Swennen (2007) stated:

> We conclude that in order to improve the impact of teacher education, and especially the potential of teacher education to develop new visions of learning and the related practices in their graduates, one aspect that we have to look at carefully is the role of the teacher educator and educational practices within teacher education itself. (p. 588)

In working on the other chapter, it became clear that there was a need to consider the knowledge that we, as teacher educators, brought to our practice. Parks and Wager (2015) suggested that while government and research bodies promote mathematics education in kindergarten, "these same bodies of research may be guiding teacher educators to take up the preparation of early childhood teachers in ways that fail to acknowledge the particular knowledge, skills, and dispositions that teachers of young children need" (p. 124). It, therefore, is important that we better understand the complexity of linking theoretical and practical knowledge for student teachers. By examining our own interactions with young children about mathematical ideas from a theoretical perspective, we considered that we would gain understandings of how theory and practice could be better connected in our work as teacher educators.

Teacher educators' reflections on their practices are not new (Lunenberg & Willemse, 2006). Some of this work has looked at the influence of experiences outside of teacher education, which affected the transitioning into becoming teacher educators (Dinkelman, Margolis, & Sikkenga, 2006; Nicol, 1997). Generally, this has focused on the experiences of transitioning from being a teacher to being a teacher educator. For example, Swennen, Jones, and Volman (2010) discussed the sub-identities of teacher educators in terms of teacher or researcher. In contrast, Ainley (1999) in a powerful piece described the tensions between her roles of researcher, teacher, parent and perhaps mathematician. Her reflections suggest that her different roles coexisted, rather than superseded each other, with different circumstances bringing one or another to the fore. In investigating our own interactions with young children, it seems important that we also consider how our different roles affect those interactions and the sorts of circumstances that allow for their coexistence in any specific instance.

As our own experiences of being around young children came not from being kindergarten teachers but from other roles, we decided that we should examine these experiences as a basis for understanding their influence on our practices as teacher educators. In doing so, we acknowledge that parents and others have different kinds of relationships with young children than kindergarten teachers.[1] However, we considered that our experiences of these other roles were likely to provide us with insights that we might share with our student teachers in lieu of kindergarten teaching experiences, especially if we examined them theoretically.

Given that there seems to be no research in which teacher educators reflect on their own practice in regard to kindergarten mathematics teacher education, this project needs to be seen as an initial investigation. We, as teacher educators, reflect on our experiences, as either parents with young children or as a visiting kindergarten

[1] We thank Götz Krummheuer for this point.

assistant. By examining them theoretically, we come to better understanding the mathematics that young children engage with in both practice and theory.

We recognised that such an investigation may be difficult, as earlier research indicated that kindergarten teachers had difficulty identifying and making use of children's experiences from outside the kindergarten. For example, Wager and Whyte (2013) found that kindergarten teachers viewed home practices as valuable in two ways. The first, and most common, was when the practices were recognised by teachers as similar to their own, thus validating what they already did, while the second way involved finding out what the children did at home and incorporating this knowledge into kindergarten activities. In an Australian study on the mathematics that children from low-socioeconomic areas engaged in at home (Clarke & Robbins, 2004), some kindergarten teachers expressed surprise at the information provided by families, partly because they held deficit views of those families, based on their socioeconomic status. In order not to fall into the trap of seeing what we wanted to see, we realised that we needed to find innovative ways to explore how our experiences affected our understanding of the mathematics that young children engaged in. In the methodology section, we describe both our data collection and analysis in some detail in order to show how we grappled with this issue.

Pedagogical Mathematical Knowledge

In order to problematise our own understandings about the mathematics education needs of young children, we decided to investigate the pedagogical content knowledge that we seemed to draw on when interacting with young children. Shulman (1986) introduced the term "pedagogical content knowledge" to discuss the knowledge that prospective school teachers need to gain during their teacher education. Others have used the term to describe what they considered kindergarten teachers need for preparing children for school (see, e.g., Anders & Rossbach, 2015; Lee, 2010; McCray & Chen, 2012). However, there are some differences in how the term is used and the implications that are subsumed in it. For example, although Ginsburg and Amit (2008) determined in their research that teaching mathematics to young children was basically the same as teaching it to older children, we considered that this is not the case in Norway. Norway's kindergartens follow the social policy pedagogy tradition of respecting children's agency and inherent learning strategies (Bennett, 2005) and using children's play and own interests, in order to develop children's curiosity about mathematics. Consequently, the definition of pedagogical mathematical knowledge that we use needed to reflect this tradition. As Mosvold et al. (2011) stated in regard to kindergarten mathematics teacher education in Norway:

> Although it can be described as a similar kind of challenge, the way a kindergarten teacher has to use play situations and everyday activities in order to facilitate children's informal experiences with mathematical ideas is quite different from when mathematics teachers in school attempt to present mathematical ideas to their pupils. (p. 1809)

We consequently based our definition of pedagogical mathematical knowledge on the Norwegian Kindergarten Framework Plan (Kunnskapsdepartementet, 2011), both in regard to mathematical goals and to pedagogical approaches. The goals for kindergartens in regard to the mathematics education experiences that they should provide to children are based on Alan Bishop's (1988a) description of six universal mathematical activities, which he described as a response to the question "Do all cultures develop mathematics?" and [his] "search … for the activities and processes which lead to the development of mathematics" (Bishop, 1988a, p. 22). The activities were:

> Counting. The use of a systematic way to compare and order discrete phenomena. It may involve tallying, or using objects or string to record, or special number words or names.
>
> Locating. Exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means.
>
> Measuring. Quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'.
>
> Designing. Creating a shape or design for an object or for any part of one's spatial environment. It may involve making the object, as a 'mental template', or symbolising it in some conventionalised way.
>
> Playing. Devising, and engaging in, games and pastimes, with more or less formalised rules that all players must abide by.
>
> Explaining. Finding ways to account for the existence of phenomena, be they religious, animistic or scientific. (Adapted from, Bishop, 1988b, p. 182)

Bishop (1988a) argued that the six activities are universal – they are found in all cultures – and that they develop in response to environmental needs:

> All these activities are motivated by, and in their turn help to motivate, some environmental need. All of them stimulate, and are stimulated by, various cognitive processes, and I shall argue that all of them are significant, both separately and in interaction, for the development of mathematical ideas in any culture. Moreover all of them involve special kinds of language and representation. They all help to develop the *symbolic technology* which we call 'mathematics'. (Bishop, 1988a, p. 26)

In previous research we have found the six mathematical activities productive as they provided us with an opportunity to identify the problem solving nature of much of the interactions that young children had with mathematics, rather than be bound by comparisons with school mathematics (Johansson, Lange, Meaney, Riesbeck, & Wernberg, 2012). Using Bishop's (1988a) six activities to categorise potential learning situations provided us with a way of sharing our own understandings and experiences about young children's mathematics, which focused on what they could do, rather than what they could not yet do.

In the Norwegian Kindergarten Framework Plan (Kunnskapsdepartementet, 2011), pedagogical approaches highlight children's learning through play. By supporting children to engage in situations in which they are interested, the adults' role becomes one of developing that interest. The framework plan states that "through play, experimentation and everyday activities, children develop their mathematical skills. Kindergartens have a responsibility for encouraging children in their own investigations, and for facilitating early and good stimulation" (p. 41).

To discuss the pedagogical approaches used in interacting with young children, we built on the metaphors used by Franzén (2015) in her research in Sweden. Franzén introduced the metaphor of being a tour guide to describe the teacher who knows what children have to learn and sets goals and organises activities to guide that learning. Alternatively, the metaphor of being a travel companion describes teachers who use children's play, activities and interests as the starting point for creating learning situations and who are attentive to the opportunities they can offer for learning. Although not discussed by Franzén (2015), work on socialisation in kindergartens suggest that both metaphors include cultural expectations around the role of children in learning and the importance of specific knowledge to be learnt (see James & Prout, 2001). The metaphor of tour guide indicates that there is particular cultural knowledge which children need to learn and it is the teachers' role to provide that learning. On the other hand, the travel companion situates the child as being an active learner of the cultural knowledge needed to investigate the things that interest them. Franzén considered a travel companion to be more in alignment with the role given to teachers in the Swedish preschool curriculum, which has a similar philosophy to that of the Norwegian Kindergarten Framework Plan, while the tour guide more closely resembles teachers' work in schools.

Methodology

In order to investigate our understanding of pedagogical mathematical knowledge, both from a theoretical and practical manner, we collected data in two ways. This was partly due to circumstance and partly due to choice. One of us (A1) had the opportunity to be attached to a kindergarten as an assistant in November–December 2015. She[2] wrote notes for the 15 days she worked in the kindergarten about what the children did and about her role. It was these notes which indicated that learning to be a kindergarten assistant was connected to her role as a mathematics teacher educator, and this made her reflect on her role as a teacher educator.

Four of the other authors had small children, aged between 16 months and 5 years. However, as they had constant contact with their children, it was decided that a different method could be employed to gain insights into their pedagogical mathematical knowledge. We, therefore, decided to have the parents photograph their children engaged in mathematical activities which had been an effective method of gathering parental perspectives (Clarke & Robbins, 2004). Aarsand (2012), in discussing studies connected to families videoing their children, wrote:

> There seem to be several advantages to using members of the community to produce practice-reported data. First, the *access* argument: as members of the community, they are most likely to get access to the practices of interest to the researcher. Second, the *ethical*

[2] In order to focus on the pedagogical mathematical knowledge and not on the participants, we have chosen to describe all adults as females and children as males and to give each an identification number. A1 stands for Adult 1 and C3 stands for Child 3.

> argument: the members are able to decide what, when, where and how to record, which makes it easier for them to erase episodes they do not want the researchers to use. Third, the *knowledge* argument: members of the community often know what the key aspects of the practices are; they are the ones who know what information is relevant for understanding important aspects of that practice. (p. 186–187)

We considered that having parents photograph situations where they thought that their children were engaging with mathematics would allow us to take advantage of the advantages outlined by Aarsand (2012). In doing so, it would also provide a distance from the specific engagement for the parents so that the focus of the interview could be on highlighting the aspects of the interaction that they considered valuable mathematically. In work with graduate early childhood teachers, Meier and Stremmel (2010) found that taking photos and telling stories about young children's activities "sharpens the inquiry eye and mind and helps students link important points (observation, recording, representation, reflection) in the inquiry process" (p. 255). We anticipated that a similar approach would support our theoretical and practical reflections on our pedagogical mathematical knowledge. Consequently, we decided that the parents would take a few photos (between 5 and 12) of their children engaged with mathematics and then discuss these with A1. The parents then chose several of their photos (between four and seven) in the interview to talk about. The discussion of the photos was based on the following questions:

1. Which of these photos is the one that you find most interesting?
2. Can you tell me the story behind the photo?
3. Why did you take this photo? What was it that your child was doing to make you think that it was mathematics?
4. Do you think that it is valuable for your child to engage in these sorts of activities? Are there other things than just engaging in mathematics that makes you think that this was a worthwhile activity?
5. Do you think that having these sorts of experiences helps you as a teacher educator? Why is that the case?
6. Can you choose another photo where the mathematical ideas are different to the first photo and tell me about what was happening with it?

The interviews lasted between 27 and 53 min and were transcribed. All of the teacher educators provided examples where they described their observations of the children engaged with mathematics, sometimes with further reflections.

The analysis was done in two parts. In each set of data, interviews or logbooks, references to Bishop's six activities were identified by one of the authors. As the focus was on our own interpretations, classifying instances of the six mathematical activities was generally made at story level, when an author was describing what had occurred. Mostly, only one activity was identified in a story; however, in some instances two activities were identified. Justifications of the coding are provided in the Results section.

The coded data sets were then sent to the individuals who had been interviewed or wrote the logbooks to see if they agreed. In this way, we began our conversations about our understandings about how young children engaged with mathematics and what the role of the adult was in these potential learning situations. We considered these conversations as important in clarifying how theoretical and practical knowledge could be combined to better understand the pedagogical mathematical knowledge needed as kindergarten teachers. In this way, we were both interacting participants with often our own children and researchers reflecting on these interactions. Comparing these reflections with other teacher educators expanded our understandings of how we interacted with young children. These conversations often produced comments about the categorisation or more information about the interaction, leading sometimes to some stories being reclassified. Following these checks, the data sets for each of Bishop's six activities were combined.

The second analysis determined whether the interaction seemed to suggest that the adult was acting as a tour guide or travel companion, based on Franzén's (2015) descriptions. However, as the analysis progressed, it was apparent that the two possibilities were insufficient for describing the adult's role. The tour guide[3] was adapted so that it defined an adult that provided learning opportunities about specific mathematical ideas which were not based on what the child was already engaged in. On the other hand, the travel companion observed the child and then reflected on what the child's actions showed. If the adult who had this role asked any questions, it was to find out more information, not to make suggestions about possible alternative tasks. As the two metaphors seemed to be at either end of a continuum of how the adult might act, a third metaphor, the travel agent, was identified. A travel agent was considered to offer possibilities for learning mathematics based on the child's interests, through providing specific concrete material to play with, by asking questions or by suggesting tasks, but which the child had the possibility to reject.

Table 1.1 shows the matrix used for analysing the data sets. As the teacher educators did not provide the same number of stories about children engaged in mathematical learning situations and the stories varied in length and detail, no numerical comparison across cells in the matrix can be made, except to provide an indication of trends.

Table 1.1 Matrix used for analysing the combined data sets

Bishop's 6 activities	Travel companion	Travel agent	Tour guide
Measuring	16	5	2
Counting	16	5	11
Designing	11	13	4
Locating	11	3	2
Explaining	17	3	6
Playing	11	2	0

[3] When used as an analytical category, we capitalise the names of the roles.

The results in the next sections provide information about how we enacted our own pedagogical content knowledge when interacting with young children around mathematical ideas. As is discussed in these sections, the categorisation both in regard to Bishop's six activities and in regard to the three metaphors was not always straightforward. We present the material in relationship to the three metaphors and discuss how these were connected to the six activities. Quotes from the interviews and logbooks are given in English, although originally the material was presented in Danish, English and several dialects of Norwegian.

Results

Generally, the adults' interactions seemed more often to indicate that they were fulfilling the role of travel companion than either of the other two metaphors. It was only in regard to the mathematical activities of Counting and Designing (each mathematical activity begins with a capital letter to distinguish it from everyday usage of these terms) that a comparable number of stories were connected to the tour guide and travel agent, respectively. The role of travel companion is closest to the role that the Norwegian Kindergarten Framework Plan (Kunnskapsdepartementet, 2011) indicates that kindergarten teachers should adopt in their work with children.

Travel Companion

One example of a travel companion came from A2 discussing a photo that she took of her 2-year-old child eating from a bowl:

> He eats. And he wants to have more. 'But you've got a lot!', 'Yes, but I want more'. Size, what is much, what is little, he is, relatively, heading toward a volume concept.

This comment was categorised as being about the mathematical activity, Measuring, and having to do with the child's developing understanding about the word "more". From A2's perspective, her child was on his way to developing an understanding about volume. A2 is considered to have acted as a travel companion because her discussion with her child about having "a lot" can be seen as a clarification about his wish to have "more". A2 paid attention to the child's contribution to the interaction. However, she did not try to formally discuss volume as a concept but, instead, informally raised considerations about the relationship between "a lot" and "more". Although not noted by A2, this could be considered as an introduction to the mathematical activity, Explaining, as the child was not given more as A2 explained that he already had "a lot".

As travel companions, the teacher educators used their knowledge about mathematics to initially identify what the children were doing as mathematics and also to reflect on what this told them about the child's potential development in

Fig. 1.1 Collecting pine cones

mathematics or other areas. For example, A3 in discussing Fig. 1.1 in which her 2-year-old child was collecting pine cones stated:

> Then he was looking around and found another one. And he said 'one more now, one more pine cone' in Norwegian and I repeated in my native language hoping that he will get it. I always try to keep talking in my native language to him even though he answers in Norwegian so that at least he gets the content. I don't know if he has a clear concept of 'one more', but he always answers 'one more' when he gets one more.

In this quote, A3 reflected on her child's understanding about what it means mathematically to have "one more", an aspect of Counting, but also on his learning to talk about this in both Norwegian and the adult's native language. A3's offering of an alternative way of saying "one more" could have been classified as being connected to the travel agent metaphor, as her child did not have to take up this alternative way of talking about "one more". However, A3's suggestion was not considered to be about offering a different mathematical insight or extending her child's understanding about Counting and, therefore, was considered to be more about observing and so was classified as a travel companion contribution.

Uncertainty about what the children were learning and whether it was mathematics occurred in several of the photostory interviews and was a major point of our reflections. A4, in discussing the mathematics that her child might use when playing a computer game, stated that "it's a pretty fertile situation but the games themselves are perhaps not as mathematical". Rather she considered that it was in the discussion between herself and the child that the mathematics became prominent. Similarly, she was not sure about the transferring of mathematical content knowledge to different situations.

> When he was almost finished, his older sibling said, 'Now you have close to ninety-five percent, because now the background has turned grey'. But it's the older sibling that says it. Unless he manages to connect it later that it is greater than 95 percent …

Fig. 1.2 Pushing the car down a ramp

In these reflections, A4 is acting as a travel companion who accompanies the child in his exploration of different activities but does not try to change the interaction, so it focuses more on the mathematics. This is the case even though she determined that it was the discussion about the computer game, rather than the playing of the game which contributed to her child engaging with mathematics.

Stories that were categorised as the mathematical activity, Playing, endorsed the importance of exploring for young children. Exploring is related to Playing because of the implicit "what-if" stance which is reflected in Bishop's (1988a) question "Could playing represent the first stage of distancing oneself from reality in order to reflect on and perhaps to imagine modifying that reality?" (p. 43).[4]

Most instances of interactions that were categorised as Playing saw the adult in the role of travel companion. An example is A5 discussing the photos of her 1-year-old child pushing cars down a ramp (Fig. 1.2 is one of these).

> Here is another exploration of space (refers to 2 photos). He pushes the car to see how far it goes.

This interaction was classified as Playing because the child was testing out different ways of pushing the car down the ramp and seeing where it ended up. In this way, he was determining the rules of the game with his parent and exploring different possibilities both for getting the car down the ramp and for these being acceptable to his parent. In this way, the child was exploring different "what-if" scenarios. The situation could also have been classified as Locating because it was about moving the car to different situations. However, the engagement in exploring

[4] After citing a list of characterisations of play, Bishop (1988a, p. 43) remarked:

Clearly playing is a form of social activity which is different in character from any other kind of social intercourse which has been mentioned so far – playing takes place in the context of a game, and people become players. The real/not real boundary is well established and players can only play with other players if everyone agrees not to behave 'normally'.

Could these characteristics be at the root of hypothetical thinking? Could playing represent the first stage of distancing oneself from reality in order to reflect on and perhaps to imagine modifying that reality? Certainly, Vygotsky (1978) argued that 'the influence of play on a child's development is enormous' (p. 96) in that action and meaning can become separated and abstract thinking can thereby begin.

different options and determining the rules of the game with the parent made it clear to us that this was an instance of Playing.

Sometimes the adult could take an active part in the Playing, perhaps by putting words on what the children were doing or by contributing to the situation, such as when A1 worked with some adults in a kindergarten to put boxes together for the children to play in:

> Put the pieces of a large cardboard box together and three seconds afterwards, it was made use of by the little children. They were fighting for space so another teacher found a different, smaller box and cut windows in it. We taped the big box together. It was then taken over by the big children.

By putting the initial box together, A1's role could be considered to be that of travel agent as it offered a possibility with a learning situation to the children that they could reject. However, the children determined what to do with the box. From observing them interacting with it, the adults/teachers responded by putting together other boxes to provide opportunities for more children to develop "what-if" scenarios, an essential element of Playing (Helenius et al., 2016), by determining what the boxes could be and how they would interact with them. Hence, we consider A1's role in this episode to be that of a travel companion.

Although Franzén (2015) considered the travel companion to be more in alignment with the Swedish curriculum goals, the passive nature of the adult's role in the interaction seemed to reduce the children's opportunities to develop their mathematical curiosity because the children become responsible for identifying the potential mathematical aspects themselves. Therefore, it was interesting to find that we, as adults, took on more active roles.

Travel Agent

As a travel agent, the adult offered a potential learning opportunity to the child, generally based on an interest that the child had already shown, but in a way which allowed the child to reject the offer. In A4's interaction with her 5-year-old child who was playing an app on a tablet, she asked questions which the child did not always attend to because playing the app took all his concentration. A4 understood and respected the child's choice not to answer. At other times, the child responded to A4's questions as showing a genuine interest in what happened in the game. The following is A4's description of a successful interaction which was classified as the mathematical activity, explaining:

C4 I must have 60 diamonds
A4: What happens when you get 60 diamonds?
C4: I need 21 more, so I can get to a new level

The question was about what the child was interested in and requested clarification of what was occurring. By responding to his parent's question, the child described his aim for interacting with the game which seemed to facilitate him providing an example of the mathematical activity, Explaining. The question was based on what he showed interest in but offered him the possibility to change mathematical focus to Explaining. Without the question, the mathematical focus would have stayed on the numbers and been classified as being about the mathematical activity, Counting.

All the teacher educators felt that it was good that the children had the possibility to reject suggestions. If a suggestion was rejected, they often followed up with another suggestion. For example, A3 in describing her child's interest in pine cones, which were thrown under a table, stated "since he was not interested in talking about numbers, and that is fair enough, then I used the occasion to talk about under the table and over the table". A3 followed up on her child's changing interest to focus on other mathematical aspects of the situation to do with Locating. A3 was able to use her knowledge about the different mathematical activities to change what she offered to her child to match his new interest.

Sometimes the adult, acting as a travel agent, provided mathematical learning possibilities to the children by setting up situations with different resources. In these circumstances, the adult often seemed to model an aspect of Designing, while the children would engage in the situation by focusing on an aspect of Locating. With this switch in mathematical activity, there was often a switch in the adult's role, from being a travel agent, when involved in Designing, to being a travel companion when the children took up the offer but refocused it onto an aspect of Locating. This can be seen in the following example, from A1's logbook of her time as a kindergarten assistant, when it was raining outside and the sandbox filled with water.

> I made islands in the sandbox. The children hopped from island to island so I made some more. Quite a lot of children were engaged with this.

As was the case with making the boxes in the travel companion section, A1 made islands for the children to interact with because the sandbox had filled up with rain. In designing these, she offered the children possibilities for playing together that were not otherwise available. However, when they used the islands to navigate themselves from one to another, over the water, they could be considered as engaging in an aspect of Locating, and A1's role became that of travel companion, observing what the children did and modifying the islands, accordingly.

Sometimes as a travel agent, the adult needed to support the child's interest in the mathematical activities and not be insistent that he learnt or used specific mathematical content. For example, A2 described her child counting rings on his fingers (see Fig. 1.3).

A2 stated, "He can do fine one-to-one matching at least up to seven, eight, nine or like that. And then - how many are there? One two, one two, one two, one two. It's just to fool around". Playing with counting was a legitimate action in interactions when A2 saw her role as being a travel agent.

Fig. 1.3 One-to-one counting

Enjoyment in a situation generally contributed to the child wanting to continue with an activity set up by the adult. Often the child would focus on a specific aspect or adapt her interactions around that aspect. For example, A5 related the story of her toddler's location of round things.

> He points at the round holes in the hub caps and puts a finger in the hole. And he points at another wheel.

In reflecting on this story, A5 mentioned "he was also interested that it was a hole and he could place his finger through it, he repeated the gesture a few times". The repeated action indicated to A5 that the focus was on the hole, rather than exclusively its roundness, but his repetition of the movement also suggested that the child enjoyed the feeling that putting his finger in the hole provided. A5 considered that her child's focus could also be on Locating, because of repeating the gesture of putting his finger in and out of the hole.

The role of travel agent seemed to allow the adult to have an active part in contributing to the interaction by offering suggestions, describing in words what the child had done and supporting children to play with different aspects of mathematics. We considered that this role was still in alignment with the social pedagogy tradition (Bennett, 2005), emphasised in the Norwegian Kindergarten Framework Plan (Kunnskapsdepartementet, 2011), although the adult's part was more active than that suggested as being optimal by Franzén (2015).

Tour Guide

When the adult acted as a tour guide, she made available specific mathematical learning opportunities considered important for the children to engage in. In so doing, the tour guide enculturates young children into valued aspects of mathematics. Taking the term from anthropology, Bishop (2002) described enculturation as "the induction by the cultural group, of young people into the culture" (p. 194). Kindergarten teachers in a range of different countries have been noted for focusing on number and shapes in their work with mathematics (Anthony, McLachlan, & Poh, 2015; Björklund & Barendregt, 2016; Fosse & Lossius, 2015), suggesting that these aspects of mathematics have been identified as the valuable knowledge that a society wants children to be enculturated into. In our research, we found that stories connected to being a tour guide occurred more frequently in situations to do with the mathematical activity, Counting.

However, in the stories about being a tour guide and providing valuable mathematical knowledge to their children, there was often a tension in whether the children could actually learn or take in this knowledge. The following example, from A3, illustrates this tension:

> So I asked him how many pine cones do you have. He said two and he had two at that moment. Then I started using my fingers to help him a bit to go further with the counting and represent in another way, give him a bit more experience with counting and number. I was using my fingers and he tried to do the same. …
>
> He was trying to do what I did. I was at the time speaking and saying that you have two and counted with my fingers one, two. …
>
> I don't know how much he learned that day but I see there is a possibility for him to connect together different representation forms. He can count especially when he sings - he has a special song and before that he always asks - and I guess that is how they do it in the kindergarten - so I count first and he counts from one to ten and then he starts singing. But of course, to give content to his counting is important now.

A3 used the interaction about the pine cones with her child to reinforce that the number names represented a specific amount, an understanding of cardinality, by using different representations – "to give content to his counting is important now". To do this, she adopted the role of a tour guide, by showing the amount on her fingers as well as with the pine cones. Therefore, although this situation is a response to her child's own interest in collecting pine cones, A3 used her knowledge from being a mathematics educator to provide number understandings and different representations which could lay the foundation for further learning. However, she was uncertain about what her child learnt from imitating her actions. When the adult adopted the role of tour guide, the children did not have to show that they had learnt specific mathematical knowledge, only that they had engaged in some way with the knowledge, identified by the adult as culturally important.

As was also noted in the section on travel companion, A4 was uncertain about the depth of the mathematical knowledge being learnt in her interactions with her child and its transferability to other situations. This uncertainty was also clear when A4 had the role of tour guide. A4 had recorded a dialogue between herself and her child while the child was playing an app (see Fig. 1.4):

A4: 67%. Is this much?
C4: Yes, it is more than halfway. It will come to a hundred.

Fig. 1.4 Percentage bar on the app, in this case showing 77%

The following comes from the interview between A1 and A4 about the conversation:

A4: Yes. If you look at the end of the conversation.
A1: "It is more than half, it is 67"
A4: Yes, then he has an indicator and it shows how far you have come in your path. It (the app) writes it and displays it graphically. It provides an indication that the percentage is that part of a whole.
A1: Does that mean that he can comfortably read the numbers, that it says 67?
A4: Yes, and then he must lift up his eyes but he does not until he has finished with what is happening…
A1: It's interesting that he actually read those double-digit numbers.
A4: At the same time, if we go to the clock, when I asked him what's the time on the microwave, he says one, nine, one, nine. He does not see them in groups, he does not put them as nineteen, nineteen, which may be natural.
A1: Do you think that it is a relatively specific, related to the game, but not transferred to
A4: Yes, before he starts, he has gone onto the iPad and seen "29 percent" (left on the battery). He certainly understands that "ok at least I can play a little while, maybe a half an hour, or perhaps one hour".
A1: It's very functional knowledge.

In the interaction with the child, A4 has the role of tour guide in that she asks about her child's knowledge of percentages, which is not directly related to the child's playing of the app. In reflecting on what the interaction showed about her child's knowledge about counting, A4 suggests that the knowledge of percentages is connected to the graphical representation of the percentages and that reading double digit numbers may not be transferable to other situations. This understanding could be used in setting up later learning opportunities.

A5, whose family came from outside Norway, was very aware of family expectations around her child's developmental trajectory but also rejected these expectations as not being something that she was overly concerned about:

> I compare all the time with the expectation in my culture. At particular age you ought to be able to do these things. But I don't care that much, like my parents do. They have much more concrete expectations of the child, and our perception of deviations between the 'expected' and actual development is also different.

The importance of children meeting or exceeding expectations about what they can do at different ages was different for A5 and her extended family. However, it may be that whatever cultural context an adult is from, there will be cultural expectations about how young children learn, and these expectations could affect an adult's willingness to formally "teach" mathematical knowledge to a child.

Taking on the role of being a tour guide meant that the teacher educators had accepted a particular set of cultural considerations about what knowledge their children needed to be enculturated into, and it was likely that these considerations would change as the children became older. Number knowledge seemed to be highly valued by all the teacher educators, although our reflections and discussions about this have not resulted in any clear understanding about why this might be the case. For some, it had to do with being mathematics teacher educators, which caused them to value Counting as important cultural knowledge for children. This may unconsciously support them to more formally introduce their children to this knowledge by taking on the role of tour guide.

Teacher educators also took on the role of tour guide in regard to the mathematical activities Measuring, Designing and Locating. Sometimes, the children had already begun to value a particular task the family had enculturated them into. This contributed to the child taking on the responsibility for requesting that the task occur. For example, A2 discussed her child's requirement that his height be measured every time they visited the family's summer cottage.

> We measure him every time we're at the cottage as he is concerned that he shall be measured. We have a doorpost where we put a dash. 'Now you are higher. The last time you were there and now you're there'

The adult had the role of tour guide in setting up the task. Marking the height of a child against a specific measure is a common one in many households around the world. It can be considered as providing learning opportunities connected to the mathematical activity, Measuring, as it involved discussion of height comparisons – "now you are higher". However, it also enculturates children into the understanding that they grow taller as they get older (or at least while they are children) which is an important biological, but also cultural, understanding as it is also associated with what a child is allowed or not allowed to do. The vested interest that children might have in documenting that growth means that they can insist that their parents perform the task and thus continue to take on the role of being tour guides.

It is interesting to note that Playing was not connected to stories in which the adults took on the role of being a tour guide. This could be for a range of reasons, but more reflection is needed to understand why this was the case.

Our original definition of the tour guide's role connected it to the provision of formal instruction. However, in the examples provided by the teacher educators, it seemed to be more about setting up learning opportunities that the adults saw as being connected to valuable mathematical understandings but which the children might not highlight in their interactions with an adult. It was interesting to see that although we all expressed the view that one should build on children's own interests, aspects of the mathematical activities, particularly Counting, was made explicit to children, even when children did not show an initial interest in them. It seemed that we, perhaps reinforced by our roles as teacher educators, had come to accept that it was important that children become enculturated into this knowledge. Sometimes, children colluded in this formalising of how they should engage in learning some mathematical knowledge because participating in those tasks brought access to other knowledge, such as C4's knowledge of percentage provided important information about playing an app or A2's child's request to be measured against the door post of the family's summer cottage. In these circumstances, the child situated their parents as tour guides who must provide formal mathematical learning opportunities.

Discussion

In regard to the impact of being a researcher and a teacher on how she acted, Ainley (1999) stated "to be an effective researcher (and perhaps also an effective teacher) I believe that I need to be aware of the attractions and constraints of both roles" (p. 47). Similarly, our reflections on what we learnt from this research are of three kinds. The first is to do with bringing into our interactions with children our pedagogical mathematical knowledge from being teacher educators. The second set of reflections is to do with using our experiences with young children in our interactions with student teachers, particularly in how we could combine theoretical and practical understandings. The final set of reflections is about the impact of researching our own interactions and how this has contributed to our meta-awareness of our pedagogical mathematical knowledge.

The analysis of the data indicates that in our interactions we are able to use the knowledge that we have about mathematics education for young children from being teacher educators. A3, who has had the least amount of contact with the kindergarten teacher education, perhaps expressed best the contribution that her pedagogical mathematical knowledge had on her interactions with her child:

> So, reading about Bishop's six activities has been like, 'Wow, yes, that is mathematics really', and doing these things with my child. In the beginning when I started doing these things with my child I was not very aware but after also reading about Bishop's six activities I am much more aware and I see him doing things from another point of view and I am more aware and more attentive towards looking at what possibilities are here that I can teach him or that I can help him to learn new things both in mathematics and in language.

Updating her pedagogical mathematical knowledge had included reading about Bishop's (1988a) six mathematical activities. However, the impact of reading of theory was not just on her role as a teacher educator but also affected how she viewed her interactions with her child. It had a practical impact. She used these theoretical understandings in her daily interactions with her child.

At the same time, our reflections made us aware that the practical experiences from being a parent affected the role of being a teacher educator. A2 reflected on her teaching of kindergarten student teachers, before she had her child, and how, after watching him develop, it affected her teaching of student teachers who would teach much older children, in Years 5–10:

> I did teach in 2010 in preschool teacher education, before it became kindergarten teacher education. I did it for one year, but I did not have children, and I felt myself a little misplaced, I had no knowledge about it. So if I had been placed there now after I have a child and recognised more of the theory, then I would not have been like a fish on land. But the fact that I had a start and was involved in teaching in preschool (education), has marked my thinking when I teach [Year] 5–10 [teacher education students] and in the old mathematics teacher education. The preschool teacher education influenced the thinking that was there, and that I have children of my own and get closer to all the things that he does at home influences what I think about mathematics and how I assess the students' ways of expressing themselves in regards to mathematics and what mathematics and understanding of mathematics are and what creates understanding of mathematics and mathematics education concepts, constructivism, socio-cultural, more conscious relationship to those things.

Therefore, having interactions with young children was identified as a way of connecting theoretical and practical understanding. Reading theoretical papers and interacting with young children on a regular basis supported us as teacher educators to improve our pedagogical mathematical knowledge. This can be seen in the reply A4 gave to a question on how she used her experiences with her children as a teacher educator, "when I confirm what the textbooks say, all the time with my own children, so I feel it even more natural, I am more confident in my case. It becomes an anchor to reality, to the other children [that the student teachers meet]". When we, as teacher educators, do not have experiences as kindergarten teachers, then our interactions with young children become the way to share stories and make links to the theories described in their teacher education textbooks.

In a similar way that Ainley (1999) had reflected on the backgrounding of her role as a mathematician in her research, A2 reflected on how young children's play resembled working with mathematics:

> When I try to sit down to understand some mathematics, begin to figure it out, then it's a kind of play. And if I had been deprived of it, or if a child had been deprived of it, then I think that we would have limited so much, we would have created so, we would not have managed to curb it entirely, but we had not opened up for that potential, that play.

Playing, in the sense of Bishop's (1988a) mathematical activity, was seen as important for doing and learning mathematics, and this was something important that teacher educators needed to recognise in their work with student teachers.

The process of reflecting on our interactions with young children and how we connected them to our pedagogical mathematical knowledge was seen as very valuable. A4 summed this up by stating:

> I feel that when I have to document something, then I feel that one begins to think about a whole different way than when it just happens. So it is very exciting to do such a thing here.

One aspect of this research that we need to consider in our future work as teacher educators is how to make use of the three metaphors. The metaphors helped us to see how we were acting as adults in interactions with children around mathematics. In some ways, it surprised us that we adopted all three roles at different times, because the travel companion was the role suggested as being most in alignment with what a kindergarten teacher should do, according to the Norwegian Curriculum Framework. It may be that introducing these metaphors to student teachers may also support them to envision different scenarios in their practice in kindergartens and, thus, make stronger connections between theoretical and practical knowledge. Discussing the stories about the photos and in the logbook provided opportunities for us to consider how our different roles influenced what we did but also the metaphorical roles that we took on when interacting with young children.

The metaphors need to be considered in relationship to Bishop's (1988a) six mathematical activities. For example, stories that were connected to Bishop's (1988a) mathematical activity, Playing, never had the adult acting as tour guides. In earlier research, Helenius et al. (2015) found that teachers in Sweden rarely told stories about their experiences in preschools that were connected to Playing. It would seem that we, as teacher educators, do not see Playing as something that we should formally teach children to engage in, in the way that we do with Counting activities. A1's story about setting up the islands in the flooded sandpit was done to stimulate the children's imagination, but in telling the story, this aspect became invisible. As teacher educators, we need to discuss why this is the case. As well, the stories that we told, which were connected to Playing, were not as frequent as other activities such as Designing or Locating. This suggests that we need to think more about what it means to have children engage in aspects of Playing, as a mathematical activity.

It is also clear that societal and cultural considerations affect why we take up the different metaphorical roles in how we interact with children. Again, it would be useful to have discussions with our student teachers about whether the high valuing of counting knowledge in regard to young children influences us to more often take on the metaphor of tour guide. Explicit discussions of this type may not just be helpful to us as teacher educators but also for our student teachers to gain more awareness about how they can use their theoretical understandings in their work in kindergartens.

Conclusion

There is little research in which kindergarten mathematics teacher educators reflect on their work, particularly when they have never worked as kindergarten teachers. This chapter is a start to identifying what can be learnt from doing such research and how understandings about theory can be connected to the practice of interacting with young children as they engage with mathematical ideas. Reflecting on practical experiences and how they relate to theoretical understandings about mathematics, through Bishop's (1988a) six activities and Franzén's (2015) metaphors for pedagogical roles, has provided us with broader understandings of the relationships. This provides us not only with stories to support our student teachers make connections between theory and practice, but could help them reflect on the connections that they build when they are working in kindergartens.

The three metaphors provide a useful shorthand for discussing the different roles that adults have when interacting with children. However, these are metaphors which may not reflect the reality of the variety of interactions which are possible. It is also unlikely that in any situation a teacher will continuously have just one kind of role. However, like Bishop's (1988a) six mathematical activities, they provide a way to discuss interactions and what other alternatives could have been enacted. Thus, there is a need for further research to better understand whether our increased understanding of the relationship between theory and practice results in more informed mathematics teacher education courses for our kindergarten student teachers.

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Chapter 2
Preservice Teachers Recognising and Responding to Young Children's Engagement with Mathematics

Dorota Lembrér, Suela Kacerja, and Tamsin Meaney

Abstract In this paper, a methodology is proposed for gaining insights into preservice teachers' understandings about young children's mathematics learning. Using data from a Swedish and Norwegian pilot study, it is possible to see how a set of questions about a stimulus photo of children playing with some glass jars provided insights into the preservice teachers' mathematical and pedagogical understandings. Although the preservice teachers seemed to be able to recognise a range of mathematical activities and respond to children engaging in them, they often gave only implicit, general information. This raises questions about teacher educators' expectations about whether preservice teachers, at the end of their courses, should be able to provide more explicit descriptions of what children are doing and suggestions for how to develop their mathematical understandings. Information of this kind can inform teacher educators about what could be improved in future mathematics education courses in early years programmes.

Keywords Pedagogical mathematical knowledge · Photo stories (or photo-based survey) · Recognise and respond · Bishop's six mathematical activities · Student teacher assessment · Mathematics teacher education · Comparative studies · Preservice teachers education · Questionnaire approach (survey)

Introduction

Although the importance of determining the pedagogical mathematical knowledge of preservice school teachers has been noted for some time (Ponte & Chapman, 2008), there has not been the same attention to the pedagogical mathematical knowledge of preservice early years teachers (known as kindergarten teachers in

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Norway and preschool teachers in Sweden) (Dunekacke, Jenßen, Eilerts, & Blömeke, 2016). This may be because early years teacher education programmes have a short history when compared to school teacher education (Benz, 2012).

Nevertheless, some concerns have been noted in research about early years teachers' work with young children around mathematics. For example, in Norway (Fosse & Lossius, 2015) and in Sweden (Björklund & Barendregt, 2016), research has shown that early years teachers generally restrict their provision of mathematical learning opportunities to those connected to number concepts and geometrical shapes. Other mathematical ideas identified in the respective curricula (Kunnskapsdepartementet, 2011; Skolverket, 2016), such as patterning and reasoning, are often underrepresented in mathematical learning opportunities. Similarly, in New Zealand, Anthony, McLachlan, and Foh (2015) in evaluating portfolios of young children's learning stories, documented by early years teachers, found that the teachers mostly focused on counting and shapes. They also found that generally the teachers only considered what the children had done, not what they might be encouraged to do.

In both Norway and Sweden (Kunnskapsdepartementet, 2011; Skolverket, 2016), the mathematical goals of the kindergarten and preschools are broader than counting and shapes and are based on Bishop's six activities. Bishop identified the six mathematical activities as universal for any culture and labelled them as mathematics, with a small "m". The discipline of Mathematics, which he capitalised, includes specific versions of the six activities. The activities are:

> Counting. The use of a systematic way to compare and order discrete phenomena. It may involve tallying, or using objects or string to record, or special number words or names.
>
> Locating. Exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means.
>
> Measuring. Quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or "measure words".
>
> Designing. Creating a shape or design for an object or for any part of one's spatial environment. It may involve making the object, as a "mental template", or symbolising it in some conventionalised way.
>
> Playing. Devising, and engaging in, games and pastimes, with more or less formalised rules that all players must abide by.
>
> Explaining. Finding ways to account for the existence of phenomena, be they religious, animistic or scientific (adapted from Bishop, 1988a, p. 182)

As "the educational and cultural context has a big influence in teachers' or educators' beliefs" (Benz, 2012, p. 251), we are interested in whether it is possible to develop an instrument that could provide valid and reliable results for early years teacher education programmes situated in different countries, but which have a similar approach. Therefore, to consider how early years teacher education programmes can be improved so that graduates are able to recognise children engaging in a wider range of mathematics activities, we proposed and trialled a methodology for gaining insights into what preservice, early years teachers know and can do with young children. Consequently, in this paper, we explore the use of a survey, which included a photo of children playing with glass jars, to evaluate the pedagogical mathematical knowledge of preservice teachers in Sweden and Norway. Our aim is to develop a methodology for gaining insights into preservice teachers' pedagogical mathematical knowledge, which would enable us, as teacher educators, to improve relevant teacher education courses, cross-nationally.

Understanding Preservice, Early Years Teachers' Pedagogical Mathematical Knowledge

Norway and Sweden have a distinct early years education based on the "social policy pedagogical tradition" (Bennett, 2005) where children's own interests are to be the basis for providing learning opportunities through play. This is different to the readiness-for-school tradition connected to early years education in many English-speaking countries (Bennett, 2005). Consequently, understanding preservice teachers' pedagogical mathematical knowledge needs to be respectful of the Scandinavian tradition (Mosvold, Bjuland, Fauskanger, & Jakobsen, 2011).

In a review of research on assessing school teachers' pedagogical mathematical knowledge, Stahnke, Schueler, and Roesken-Winter (2016) identified two approaches: cognitive and situated. They considered the work of Deborah Ball and colleagues (e.g. Hill, Ball, & Schilling, 2008) who "have pursued a cognitive perspective and emphasized the significance of teachers' profound subject–specific knowledge base for the quality of instruction" (Stahnke et al., 2016, p. 1). Alternatively, situated perspectives compare novice and expert teachers and "use proximal measures of teachers' abilities such as classroom videos, video vignettes or exemplary student work" (p. 2). They considered that a combination of the cognitive and situated approaches would support research efforts to understand how teachers react to specific situations in classrooms.

Stahnke et al. (2016) also noted that research showed that the preservice teachers often struggled with recognising children's errors or misconceptions and knowing how to respond to them. Similarly, Dockett and Goff (2013) indicated that early years teachers' noticing of young children engaging with mathematics required the teachers to both recognise that the interaction involved mathematical ideas and to respond to the children's engagement in the learning situation by providing "options to pursue the mathematical thinking" (p. 773). The knowledge needed for this combines both pedagogical and mathematical knowledge, in order to respect the spontaneous nature of young children's interactions with mathematics (Dunekacke et al., 2016). Anthony et al. (2015) showed that the early years teachers in their study were more comfortable describing easily identifiable mathematical tasks than when the mathematics occurred in free play. As they stated, "given that most of the learning experiences in our kindergartens involve well–planned, free–choice play, it is critical that teachers are able to utilise free–choice play to support mathematics learning" (p. 398).

Thus, we consider pedagogical mathematical knowledge to be the knowledge that early years teachers use to both recognise and respond to young children engaging in both spontaneous and planned situations, involving mathematical ideas. Such a definition allows us to combine the cognitive and situated approaches for researching preservice teachers' professional practices as suggested by Stahnke et al. (2016), but also to respect the Scandinavian tradition of children learning through play (Mosvold et al., 2011).

Gaining Insights into Pedagogical Mathematical Knowledge

Other researchers had developed instruments for this or a similar purpose, and, in this section, we discuss some of their advantages and disadvantages. For example, Stahnke et al. (2016) found that most studies that assessed teacher mathematical content knowledge used either videoed teaching sequences or student-written documents as stimuli. Given that young children are not yet literate, the written material about mathematics that they produced may not be easily interpretable out of the context of its production.

Working in the USA, McCray and Chen (2012) developed an interview to assess early years teachers' pedagogical mathematical knowledge. They presented the teachers with two written scenarios about classroom-based, free-play situations. The teachers were then asked questions about the mathematics they saw in the scenarios, as well as how they would proceed in interactions with the children. The scenarios included possibilities for recognising and responding to mathematical concepts mentioned in curriculum guidelines, such as measurement, geometric thinking and one-to-one correspondence. It was assumed that teachers would be able to show how they were "following the thinking of children as they interact with materials, recognizing the mathematical potential in their activities, and knowing how to comment on and extend their mathematics related thinking" (p. 297). In their assessment of preservice, early years teachers, Dunekacke et al. (2016) showed video clips of young children engaged in number, geometry and measurement tasks and asked the preservice teachers to describe the mathematics that they saw and the subsequent actions that they would take. Although this provided valuable information about the preservice teachers pedagogical mathematical knowledge, they acknowledged the time-consuming nature of having participants react to video clips.

Developing the Survey

Taking these points into consideration, it was decided to survey preservice teachers who would go on to work in kindergartens or preschools, using as a stimulus a photo of young children placing their feet into glass jars (see Fig. 2.1). This photo had come from a video taken in a Swedish preschool, used in previous research (Lange, Meaney, Riesbeck, & Wernberg, 2014).

Fig. 2.1 Heels in jars

We consider that the photo has similarities to the written scenarios used by McCray and Chen (2012). The situation depicted in the photo was not immediately recognisable as a mathematics task, but would likely be accepted by the preservice teachers as typical of interactions that might occur in young children's play.

As teacher educators, we anticipated that the preservice teachers would identify the children engaging in different aspects of measuring. For example, the preservice teachers could recognise the children engaging in comparing the volume of their feet with the volume of the jars through direct comparison. However, as the children are unable to fit their heels into the glass jars, the photo might also prompt discussions about the circumference of the holes at the top of the jars. The possibility for different interpretations was one reason for choosing this photo, as we wanted to provide the preservice teachers with the opportunity to respond as they would to the spontaneous interactions that occur in actual early years institutions.

For ethical reasons connected to the researchers being teacher educators, it was important that the preservice teachers responded to the surveys after completing their courses. However, as they were no longer attending our classes, to motivate them to complete the surveys, we made them short. Dunekacke et al.'s (2016) experiences had indicated that having student teachers respond to video extracts could be time-consuming and difficult to conduct, once they were no longer attending mathematics education courses.

The survey began by asking for demographic information, such as age and gender, and if the preservice teachers had worked in early years institutions before beginning their studies. The remaining five questions were about recognising the mathematics potential in the activity and about their responses to the children. Thus, they were in alignment with our definition of pedagogical mathematical knowledge. These questions were:

- In the picture what do you see the children doing?
- Why might it be valuable for them to do this?
- If you were their preschool teacher, what questions could you ask these children?
- Why would you ask those questions?
- What is it that makes you "see" something in the photo? (What do you think from your previous experiences makes you aware of what the children are doing?)

The first two questions were to find out what mathematics the preservice teachers recognised that the children were engaging with in the photo, while the next two questions were to find out how they would respond to the children. None of the questions specifically mentioned mathematics, but given that the preservice teachers knew that they were being asked to complete the surveys by mathematics educators, then it was predictable that they would look for ways to make connections to mathematics. The final question that they were asked was to find out if they were aware of what influenced their interpretations. However, the preservice teachers' responses indicated that they found it difficult to answer this question and the results were not informative. This question has been excluded from the analysis.

Conducting the Survey

The surveys were completed by preservice teachers at universities in Norway and Sweden in January 2016. The Norwegian preservice teachers were in their final semester of a 3-year degree. They answered the questions as part of a lecture on methodology that they attended as preparation for writing a bachelor thesis on a small research topic. As they left the lecture, they were asked to hand in their responses so that they could be part of a research study. Twelve of the 150 students handed in the survey. The preservice teachers had different majors for their subject specialisations and had completed the compulsory mathematics education course at least 1 year previously. None of the preservice teachers with mathematics as part of their major handed in the survey.

The Swedish preservice teachers were in their fourth semester of seven-semester degree. They were asked to complete the survey at an introduction lecture to a new course that they attended 3 days after the final exam in the course "Childhood and Education: Mathematics". This was a full-time, mandatory 10-week course and was the only mathematics education course in their degree. As the preservice teachers left the lecture, they were asked to hand in their surveys or to put it into the researcher's work mailbox. Twenty-eight of the 42 students handed in the survey.

Analysing the Survey

The analysis was done in two ways. The first was to identify when the preservice teachers mentioned something that we, as the researchers, considered to be related to one of Bishop's (1988b) mathematical activities. Although Bishop's six activities have been used in research on young children's mathematics (Johansson, Lange, Meaney, Riesbeck & Wernberg, 2012; MacMillian, 1995, 1998; Wernet & Nurnberger-Haag, 2015), they had not previously been used for gaining insight into preservice teachers' pedagogical mathematical knowledge. Therefore, all three researchers read and categorised the responses of the 12 Norwegian preservice teachers. The results were compared and any discrepancies discussed and resolved.

Similar to what occurred when Johansson et al. (2012) undertook their research, we found that some comments could be classified simultaneously as several activities. For example, Norwegian preservice teacher 10 (HiB10 – all preservice teachers are numbered and named according to the institution; MAH numbers refer to the Swedish students) proposed that the following questions could be asked of the children in the picture: "How can one find space, up into the glass jar? Is there room for your feet? One toe or more?" (Hva kan man få plass til oppi glasset? Får dere plass til føttene? En tå eller flere?). The first question was identified as Designing as it seemed to be about adjusting part of the body to the shape of the jar. As discussed below, the connection to Designing was often less explicit than to other activities. In this case, the following questions, which seemed to be about other ways of fitting specific parts of the body into the jars, reinforced our perception that the first question was connected to Designing. The second question, about room for the feet, was considered to also be about Measuring, as we considered that the Norwegian term "plass" was an implicit reference to volume. The third question was classified as Counting as well as Measuring, because it referred to specific amounts, one toe or more, to describe the amount of space, or volume, in the glass jar. The set of questions were also classified as being about Playing as they required the children to hypothesise about possibilities.

After we had all worked with the Norwegian data, the Swedish preservice teacher data was analysed individually by two of the authors. These categorisations were then checked and any queries resolved.

While doing this classification, we noted interesting points, which were not related to mathematical understandings. These we considered to be pedagogical points, and identification of them formed the second analysis. To do this analysis, we used an approach inspired by grounded theory (Strauss & Corbin, 1990). As these comments were identified, we noted them and grouped them. For example, HiB5 wrote in response to the question about why they would ask children their questions: "To motivate children to explore and experience" (For å motivere barna til å ville utforske og erfare). This comment was grouped with other, similar points about inciting children's curiosity, fantasy and motivation. We considered that these comments were about what the child might gain from participating in the glass jar play or in other activities suggested by the preservice teacher in their responses to the children. As the analysis continued, the categories for the groups of comments were refined and strengthened. When the analysis was completed, two of the authors then checked that the groupings made sense and discussions about the points were compared with what had been noted in other research.

In the following section, we describe the results before returning to the discussion of the usefulness of this survey for gaining insights into preservice teachers' pedagogical mathematical knowledge so that teacher educators can improve their practices. The results are presented in two parts; the first discusses the categorisations into the six mathematical activities. The second part considers the pedagogical categories.

Results

In this section, we describe what the preservice teachers recognised and responded to as mathematics in the photo using Bishop's six mathematical activities. Generally, preservice teachers' recognition of examples of mathematical activities appeared in their answers about what they observed the children doing and the follow-up question about why the activity was valuable for children. Preservice teachers' indications that they were responding to the mathematics that children were engaged with appeared in the questions they would pose to work further with the children and reasons for their choice of questions.

As noted in the methodology section, it was rare for preservice teachers' responses to only be about one of the mathematical activities.

Counting

There were only three preservice teachers (one in Norway and two in Sweden) who made comments that were identified as instances of Counting. MAH27 in response to the question about what the children were engaged in and why this was worthwhile recognised the one-to-one principle, in relationship to the children matching the glass jars to their heels. MAH27 went further to respond to the children by asking them questions about how many jars were left after they had put their feet in them. The preservice teacher reinforced the connection when she justified asking such a question to the children by stating: "The one–to–one principle is an important part of knowing how to count – one of Bishop's activities" (ett–till–ett princip är en viktig del av att kunna kunskapsområde räkna – en av Bishops aktiviteter).

Two of the preservice teachers HiB10 and MAH28 did not recognise the children as engaging in Counting but indicated that they would respond to the children by suggesting Counting activities, which required them to quantify amounts. HiB10 asked about a specific amount of toes. Similarly, both MAH27 and MAH28 would ask "how many" questions about the glass jars. In this way the questions did not seem to build on what the children were doing, but moved them into a different investigation about "how many".

The lack of responses about Counting is not surprising given the stimulus photo. It is perhaps more surprising that the preservice teachers would consider that there was a connection to determining specific amounts. Regardless that there were three teachers who did make connections to Counting, the results suggest that, unlike the situations noted as common for early years teachers (Björklund & Barendregt, 2016; Fosse & Lossius, 2015), the preservice teachers in both programmes were able to focus on other mathematical activities.

Measuring

As we had predicted, there were many responses from the preservice teachers, which were classified as being about Measuring. The preservice teachers noted how children made comparisons between objects such as feet, heels and other body parts and the glass jars, using both direct and indirect comparison methods.

MAH13's response was a fairly typical example of recognising measuring. The preservice teacher stated: "They are trying to see if their heels fit into the jars´ holes" (De försöker se om deras häl får plats i glasburkarnas hål). The preservice teachers frequently mentioned that the children were exploring the size of their own body, compared with another object. MAH21 wrote: "compare similarities, differences by size of feet, glass jars and its opening" (jämföra likheter, skillnader genom storlek på fot, glasburk och dess öppning. förstå och kunna uppskatta storlek). Similarly, in justifying why it was worthwhile for the children to be interacting with the glass jars, HiB6 wrote: "The children will gain an understanding of big and small sizes relative to something else" (Barna vil få en forståelse av stor og liten størrelse i forhold til noe annet").

Responding to the children's engagement with Measuring, many of the preservice teachers indicated that they would ask the children closed questions about ordering the jars by height. For example, HiB5 wrote: "Which jar is the biggest?" (Hvilke glass er størst?), and MAH2 wrote: "Are all the jars equal? Do you all have the same size feet?" (Är alla burkar lika stora? Har ni lika stora fötter allihopa?). Although we considered these closed questions, the preservice teachers justified them as providing opportunities for children to explore.

As in the previous examples, the attribute for the ordering was often unspecified, with a generic term, such as "size", being used rather than "height" or "volume". Similarly, in Zöllner and Benz' (2016) study, comparison and ordering were associated with size (magnitude) and volume (space). This lack of specificity in preschool teacher and children's talk has been noted previously (Lembrér, 2013). There were also some responses, which explicitly mentioned that the children were engaged in measuring volume. The following example from HiB12 shows both explicit and implicit use of mathematical terms for Measuring: "Children try out the volume of the jam jars. They experience volume, what does fit and what doesn't fit into it. The photo shows that they are trying to put their feet into the jar. Mathematics. Explore their own size in regard to other things" (Barna prøver ut volumet i syltetøyglass. De erfarer volum, hva som er plass oppi og hva som ikke er. Bildet viser at de prøver å få foten oppi glasset. Matematikk. Utforsker sin egen størrelse i forhold til andre ting). HiB12 has elaborated in the three first sentences how children can experience and explore the jar's volume by putting other things into it. This reflects a direct comparison and is therefore categorised as Measuring. In the last sentence, HiB12 used the generic term "size", but it is in connection to comparing with other objects, which made us continue to classify it as Measuring. The HiB12's response does not show coherence in using mathematical terms, but this would not limit children's possibilities to engage in Measuring activities.

Locating

Three Norwegian and seven Swedish preservice teachers had responses that were categorised as Locating. Of these, almost all were about recognition of spatial perception, body orientation and exploration of the environment, by referring to the physical relationship between objects, such as their own bodies and the glass jars: "experience inside/outside, a form of spatial perception and body mind" (erfara innanför/utanför, en form av rumsuppfattning och kroppsuppfattning). Another preservice teacher (MAH25) suggested that exploration of environment with one's own body contributed to spatial understanding. Thorpe (1995) had stressed that the development of spatial representations and concepts takes place through visual, physical contact and exploration, which included interacting with, for example, toys and other objects in the environment. The preservice teachers who noted connections to Locating showed an awareness of how children used their bodies in connection to the glass jars to develop spatial awareness.

Two preservice teachers wrote about having the children learn prepositions for Locating. For example, MAH3 stated: "test different prepositions, like, for example, on and in" (pröva lika lägesord som på och i). As well, location indicators were used as adjectives, such as in "right heel" (höger häl) (MAH11). Bowerman (1996) indicates that everyday use of spatial terms such as "in", "out" and "under" by teachers can contribute to their appropriate use by young children.

Unlike the case for Counting, in Locating preservice teachers' responses recognised and responded to what the children were doing, rather than moving them towards different kinds of investigations.

Designing

There were fewer comments, compared with Measuring and Locating, in the preservice teachers' responses, which were classified as Designing. Of the responses that we considered as Designing, almost all were about recognising the shape or texture of the glass jars. For example, in response to the question about the worthwhileness of the children's interactions, HiB5 stated: "gives children experiences of texture and shape of the jam jars" (Gir barna erfaringer av teksturen og formen på syltetøyglasset). One preservice teacher (MAH24) specifically mentioned geometry by saying: "It is about geometric shapes" (Det kan handlar om geometriska former). It would seem that, generally, aspects of Designing were connected to recognising of what the children were already doing with the glass jars.

There were also a few points made about recognising the relationship between the shape of the foot and the shape of the glass jar. However, the description of the relationship was often unclear, and it was only after discussion between the researchers that we agreed that the preservice teachers' comments should be classified as Designing. Very often, these responses were also classified as Measuring. For example, MAH9 stated: "Shall we try and see if our feet fit into some other material?" (Ska vi pröva och se om era fötter får plats i något annat material?). "Fit" suggested that this was about Measuring, while "material" indicated that the situation also included aspects of Designing.

The points that were classified as Designing were connected to both recognising what the children were doing and responding to them. The suggestions for responding generally built on what the children were doing with the glass jars. Norwegian preservice teacher (HiB11) would respond to the children's actions by asking questions and wanted them to think about the properties of glass; HiB11 wrote: "[I] would ask if it is possible to use glass for shoes and why/why not?" (ville spurt om det er mulig å bruke glasset som sko og hvorfor/hvorfor ikke?). There were two preservice teachers, one in Norway and one in Sweden, who discussed the interactions in regard to the sounds that the children could make with the jars, either by using the heel to make a popping sound or by blowing into them. These suggestions built on the children's experiences of the attributes of the glass jars, but seemed to be moving the children's interactions in a different direction to the one shown in the photo.

Playing

Playing is often a mathematical activity that is unrecognised by preschool teachers (Helenius et al., 2015). Therefore, it was interesting to see that the preservice teachers both recognised and responded to the children's engagement with the glass jars in ways, which we classified as reflecting the mathematical activity, Playing.

Many of the responses, particularly about what kind of questions the preservice teachers would ask the children, were about having children develop reasoning about their actions. The preservice teachers' justifications of why they would ask these questions were classified as Playing, as they suggested that it was important for children to test out different possibilities. This can be seen in MAH1's response to why she would ask her questions: "For the children themselves must get to think and make hypotheses, open questions, see larger perspectives and together in interactions problematize, try different solutions" (För att barnen själva ska få tänka och ställa hypoteser, öppna frågor, ser större svar att tillsammans i samspel problematisera, pröva olika lösningar). We considered that the comments and questions were to support children to make and test hypotheses and in this way expand their possibilities for future actions. Many of these questions were also categorised as other activities. For example: “Is there any way to get your foot in? Other body parts?” (Går det på något sätt att få in foten, andra kroppspdelar?). MAH22 was also classified as being about Designing and Explaining.

It would seem that most of the Playing examples were connected to the preservice teachers responding to the children’s mathematical investigations, by offering related problems for investigation through asking open-ended questions. As Edo, Planas, and Badillo (2009) stated, children’s own knowledge can be a starting point for imitating social interaction in play and promoting construction of subject knowledge. The preservice teachers seemed to be using the questions to challenge the children to ask questions, reflect and discuss.

As well as responding to children’s engagement with the glass jars, there were a few responses that indicated that the preservice teachers recognised what the children were doing in the photo as Playing. HiB6 wrote: “Children play with the jam jars, test if their own feet fit/go down into” (Barna leker med sultetøyglass, tester om deres egne føtter får plass/passer nedi.). In this way, children’s playing activity was recognised as valuable in terms of possibilities for exploring and testing, aspects of Playing as a mathematical activity (Helenius et al., 2016).

Explaining

Responses classified as Explaining were mostly connected to the questions that preservice teachers would pose to the children. Given that many of these questions were “why” questions, such as: “why do you think that the whole foot does not fit in the glass jar?” (Hvorfor tror dere at hele foten ikke får plass i glasset?) (HiB4), then it is not surprising that they are classified as showing aspects of Explaining. We considered that in order for children to answer them, they would have to explain or justify what they were doing or thinking. Sometimes the teachers used a similar justification for asking their questions. For example, MAH26 wrote: “so the children will be able to explain a mathematical reason” (För att få barnen att föra ett matematisk resonemang). This response is in alignment with a mathematical goal in the Swedish preschool curriculum (Skolverket, 2016), which states that preschools need to provide opportunities for children to engage in reasoning. The preservice teachers’ questions can be considered attempts to support children to make links between the mathematics they had already encountered and what they continued to engage with. By interacting with the children, the teacher had opportunities to challenge their ideas and to have them describe experiences or thoughts (Clark & Statham, 2005). Although in interactions children have opportunities to explore their own and other’s ways of understanding a mathematical phenomenon (Björklund, 2010), it is necessary for the teacher to have appropriate knowledge and to encourage and challenge young children's mathematical awareness. Thus, recognising mathematics in what the children are engaging in is not enough; teachers, including preservice teachers, must also be able to respond appropriately to these situations so that children's possibilities for learning mathematics are extended.

As a photo, rather than a video, was used as stimulus, it is not surprising that the preservice teachers' connections to Explaining were related to their responding to children's actions, rather than recognising it. If it was necessary to check that preservice teachers could recognise the children giving mathematical explanations, then another kind of evaluative tool would be necessary.

Pedagogical Categories

There were some responses to the survey, which we considered referred to how children learnt or how the early childhood teachers should teach. Therefore, we considered the responses about teaching and learning to be in alignment with, albeit adapted to the early years situation, Shulman's (1986) notion of general pedagogical knowledge. He defined this as "those broad principles and strategies of classroom management and organization that appear to transcend subject matter" (p. 8). Pedagogical awareness was evident in answers to questions about what the preservice teachers saw the children doing in the picture, why there was value in doing this, and why they would ask their questions to the children. Most comments were about the conditions that the preservice teachers considered children needed for learning. It was generally only when they responded to the question about why they would ask the children specific questions that the preservice teachers described what they saw as being part of the teacher's role.

Investigating and Exploring

Both Norwegian and Swedish preservice teachers valued children investigating and exploring and seemed to consider these approaches as important for learning. This is similar to the views of some researchers. In longitudinal research, Gervasoni and Perry (2016) found that "informally exploring and discussing the mathematics encountered as part of everyday life is effective in facilitating mathematics learning and more in keeping with preschool children's development" (p. 133–134). Echoes of this can be heard in preservice teachers' responses to the children in the photo and the value for them to play with the jars: "Valuable because they can investigate things on their own/at their own initiative" (Verdifullt fordi de får eksperimentere på egen hånd/eget initiative) (HIB3). In the Norwegian Framework Plan (Kunnskapsdepartementet, 2011), learning is connected to children investigating their wonderings about their experiences, with the role of the adult being to support this investigation. Therefore, although not explicit, it is likely that the preservice teachers considered that investigation and exploration led to learning.

The value of children investigating by themselves also appeared in the reasons that preservice teachers gave for asking specific questions of the children. For example: “It is about exploring and investigating” (det handlar om att utforska och undersöka) (MAH4). The Swedish curriculum for preschool also emphasises that the adults in the preschool should support the children to develop their curiosity and enjoyment (Skolverket, 2016). As was the case with the Norwegian Framework, learning and teaching are connected to exploring and investigating in the Swedish curriculum and thus may be implicit in preservice teachers’ acknowledgement of their value. Therefore in both countries, many of the preservice teachers were able to notice children exploring and provide a response to the children by encouraging them to continue their learning through their own endeavours to make sense of their world.

The Body in the Activity

Very often when discussing the importance of investigating and exploring, the preservice teachers also mentioned that it was valuable for children to use their bodies. The picture of the children using their feet may have triggered the preservice teachers mentioning this. However, its frequency in both sets of data suggests that it was a pedagogical approach that preservice teachers had accepted as valuable. For example, “They explore if one can put the foot into the jar that way. There is a value in exploring things with their own bodies” (dei utforskar om ein kan fa plass til foten nedi glasset den vegen. Det er ein egen verdi i det utforske ting med sin egen kropp) (HIB2). This category was mostly found in preservice teachers’ answers about the value of children’s activity presented in the picture.

In the Norwegian Framework Plan (Kunnskapsdepartmentet, 2011), the body is discussed in relationship to communicating ideas by children who are still developing verbal language, whereas in the Swedish curriculum (Skolverket, 2016), it is deemed desirable that children should learn about their bodies. Therefore, unlike investigating and exploring, the preservice teachers’ attention to the body as providing opportunities for learning cannot be connected to their knowledge of relevant curricula. Franzén (2015) noted that research has paid little attention to very young children’s use of their bodies for learning mathematics. Although the preservice teachers valued the children using their bodies to investigate, they did not specifically connect it to mathematics learning. This can be seen in the general nature of their responses: “Create their own perceptions, body mind. They themselves try with their own body” (Skapa sina egna uppfattningar, body mind. Själva pröva med sin egen kropp) (MAH8). However, many of them referred to parts of the body in regard to the questions they would ask children, and so we considered that they were related to one of the mathematical activities. For example, HiB6 wrote: “Is there room for the foot? Hand?” (Er det plass til foten? Hånda?) which we categorised as referring to Measuring. Thus, it may be that the specific parts of the body seen in the photo prompted the preservice teachers to note how the whole body can be used for learning. The value that the preservice teachers gave to children using their bodies to learn needs further investigation.

Being in Interaction with Each Other

The preservice teachers also considered that it was valuable for children to interact with each other. This can be seen in the large number of responses to the survey questions about what children are doing and why it is valuable. These responses included words such as cooperate, communicate, investigate and reason together. However, the importance they saw in children interacting was often implicit in the responses HiB2 wrote: "They do it together with others, they can share experiences and communicate about what they experience" (dei gjer det saman med andre, kan dele erfaringar og kommunisere om kva dei opplever). Responses were more explicit if they were also categorised as Explaining: "Because it creates a discussion and possibilities for argumentation with each other" (För att det skapar till att diskutera och resonera med varandra) (MAH14).

In both the Norwegian and Swedish early years curricula (Kunnskapsdepartementet, 2011; Skolverket, 2016), interactions both between the teacher and the children and between children are emphasised as important for children's development. It is, therefore, perhaps not surprising to find that the preservice teachers valued learning how to interact, but rarely mentioned that they valued children's interactions.

Inciting Curiosity and Fantasy and Motivating Children

Another teaching/learning aspect that some preservice teachers commented on was the role of the teacher in inciting children's curiosity, fantasy and motivation to wonder about the things around them. These comments mostly appeared in the justifications for the questions that the preservice teachers would ask the children and were often connected to the value of children exploring. As a Norwegian preservice teacher puts it: "The reason why I would ask those questions is to create curiosity and an urge to explore" (Grunnen til at jeg ville spurt de spørsmålene er for å skape undring og en trang til utforskning) (HiB11). Similarly, another Norwegian preservice teacher, HiB5, stated: "To motivate children to want to explore and experience; recognition of children" (For å motivere barna til å ville utforske og erfare; anerkjennelse av barna). The Swedish preservice teachers also made comments about it being the role of the teacher to incite curiosity and motivate children. It seemed that as teachers, they saw it as their responsibility to support children to want to engage in different situations, if this was to contribute to children's learning.

Importance of Helping Children in Their Development or Developing Their Thinking

There were also a few Swedish preservice teachers' justifications for the questions they would ask which highlighted the importance of supporting children's development: "To develop their understanding of their environment. Help each other to develop" (För att utveckla sin förståelse för sin omvärld. Hjälpa varanda i utveckling) (MAH3). Like the aim to have children learn to interact together, the importance of helping children to develop seems to be for their holistic well-being, rather than in respect to their learning about the mathematical activities.

The Norwegian preservice teachers did not discuss development in this general sense. Nonetheless, a few talked about developing children's reflection processes. For example, HiB5 stated: "to initiate thought processes and reflections in children" (for å sette i gang tanke processer og refleksjoner hos barna). These comments were connected to the teacher's role. However, across the whole sample, there were very few that fitted comments that fitted this category.

Pedagogical Mathematical Knowledge: Insights

Our results show that the preservice teachers were able to recognise and respond to the mathematics that children were engaging with through answering survey questions about a photo of young children engaged in free play. Different aspects of Bishop's (1988b) six activities were identified in the preservice teachers' responses. Although we had predicted that Measuring would feature, it was interesting to note that there were some responses that could be classified for each of the mathematical activities. The preservice teachers' questions about what they would ask the children also provided information about how they would support the children to continue their mathematical explorations. The data set provided information about the preservice teachers' pedagogical beliefs, such as valuing the children's investigations as ways to learn, especially if they did this through using their bodies. Consequently, we consider that using such a survey would enable teacher educators to gain insights into the pedagogical mathematical knowledge of early years preservice teachers. However, there are some issues connected to the outcomes from the survey which need further discussion both for their own sake but also because they suggest ways to improve the survey.

As discussed in justifying our choice of method for the data collection, having preservice teachers respond to a photo of children engaged in free play was a compromise between in-depth knowledge of early years, preservice teachers' pedagogical mathematical knowledge and the time and effort required from preservice teachers to provide that information. The percentage of preservice teachers who handed in their surveys, particular in Norway, was quite small. It may be that this was because of the circumstances in which the surveys were completed. However, the small numbers do suggest that a longer survey was unlikely to be done by more preservice teachers, and so the amount of information that can be obtained in this way is quite limited.

Inspired by McCray and Chen's (2012) choices in developing their interview, we chose a photo as a stimulus, which was not recognisable as related to school mathematics. However, the context of a free-play situation did have consequences for the kind of information provided by the preservice teachers. Unlike specific school situations, free-play situations are open both to interpretation for what is occurring and also how the situation could be developed. This can be seen in the variety of responses given to the photo by the preservice teachers in this pilot study. This means that if specific information is being asked for about how children, for example, develop their counting knowledge, it is unlikely that a photo of free play would ensure that preservice teachers would show their knowledge of this because the situation is likely to allow for other interpretations. Nevertheless, given that most learning in Scandinavian early years institutions is supposed to occur within play situations, using a photo of a free play did provide an opportunity for the preservice teachers to show how they would interact with the children.

Although the photo did provide responses connected to all six of Bishop's (1988b) activities, the choice of photo facilitates some activities being highlighted more than others. From this photo, the most common activities identified were Measuring, Playing and Explaining. Some preservice teachers made comments, which we classified as Designing and Locating, while three teachers made reference to potential Counting tasks that the children could engage in. This would suggest that if we had wanted the preservice teachers to provide more information about their pedagogical mathematical knowledge related to Designing, Locating and Counting, we would need to provide at least one other photo as stimulus. It would be inappropriate to assume that simply because mentioning of these activities was limited in our data set, many preservice teachers could not recognise and respond to them in children's free play. For such a conclusion, more information from the preservice teachers would be required. However, given that most earlier research commented on early years teachers being able to recognise and respond mostly to number and geometric shapes tasks (Anthony et al., 2015; Björklund & Barendregt, 2016; Lee, 2010), if only one photo with questions was to be used in a survey, then perhaps a photo such as the one used in this survey is the most useful. It may be that if preservice teachers can recognise and respond to the other activities, then they would also be able to recognise and respond to children engaging with Counting and Designing. Nevertheless, this type of survey may be better able to provide insights into what a teacher education course is doing well in regard to preparing teachers to be flexible in recognising and responding to children in free play, rather than providing more specific information into what needs to be improved.

Still some areas for improvement in the teacher education programmes were identified. In this pilot study, the preservice teachers' references to activities in their suggestions for how the children could continue their investigations were generally in alignment with what the children could be seen doing in the photo. There were some exceptions, such as the case for two of the preservice teachers who suggested that the children engage in counting the jars or matching the jars to the children's collective amount of heels. These examples indicated that the preservice teachers were aware of important understandings about one-to-one matching, but by suggesting that children should focus on them, they were perhaps moving them away from children's own interests to more formal instructional practices. Given the social policy pedagogy tradition in the Scandinavian countries (Bennett, 2005), it is valuable for teacher educators to have ways to identify that preservice teachers may struggle with knowing how to develop curiosity about mathematics from children's own interests, and it is this aspect rather than mathematical knowledge which needs development.

Although only a pilot study, for teacher educators wishing to understand the pedagogical content knowledge of their preservice teachers, it would seem that such a survey would provide valuable information. Although we did identify some differences in the ways that the preservice teachers responded to the survey in the two groups, the samples are too small to investigate what the basis for those differences might be. For example, the Swedish preservice teachers were more likely to use explicit mathematical terms to describe the mathematics that they saw the children engaging in or to justify the sorts of questions they would ask them. This may be because they had just completed their mathematics education studies. However, the most useful time for administering this survey would be when the preservice teachers began their mathematics education courses to support teacher educators in adapting the course to suit the needs of their students. This would then enable teacher educators at the two institutions to work together on their mathematics education coursework to improve what is still a relatively new requirement to provide in teacher education programmes for early years teachers.

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Chapter 3
Using Children's Patterning Tasks During Professional Development for Preschool Teachers

Dina Tirosh, Pessia Tsamir, Ruthi Barkai, and Esther Levenson

Abstract Patterning activities in preschool are considered one way for enhancing young children's appreciation for structure. Preschool teachers, however, are not always aware of the mathematics behind these activities. This paper describes one part of a professional development program that employs the use of tasks for children to promote preschool teachers' knowledge for teaching patterns. Segments of the program reflect how the refined Cognitive Affective Mathematics Teacher Education framework helped to ensure that while engaging in pattern tasks for children, teachers enhanced their mathematics knowledge, knowledge of students, and knowledge of tasks.

Keywords Repeating patterns · Preschool teachers · Unit of repeat · Professional development · Pattern tasks · The CAMTE framework

Introduction

In Israel, the preschool curriculum encourages teachers to engage children with pattern activities with the aims of having children identify, draw, and continue repeating patterns as well as use mathematical language to describe these patterns (Israel National Mathematics Preschool Curriculum [INMPC], 2008). Yet prospective preschool teachers receive little, if any, preparation for teaching patterning in preschool. This paper describes a professional development program aimed at increasing preschool teachers' knowledge for teaching patterning. Preschool, in this paper, will relate to children ages 4–6, 1 and 2 years prior to first grade. In the next section, we offer some background on research related to children and

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patterning activities. Following that, we introduce the framework we used to investigate preschool teachers' knowledge for teaching and how that framework may be used when planning professional development for teachers.

Research Related to Patterning and Young Children

Why Engage with Patterning Tasks?

Several national curricula have recognized the potential of pattern activities in promoting early algebraic thinking among young children. For example, the National Council of Teachers of Mathematics' (2000) Algebra Standard for Pre-K-2 states that "algebraic concepts can evolve and… develop… through work with classifications, patterns, and relations…" (p. 91). Exploring patterns during the elementary years may enhance the meaning of algebra during the secondary years. Algebraic thinking relates to finding and using generalizations. "Every pattern is a type of generalization in that it involves a relationship that is 'everywhere the same'" (Papic, Mulligan, & Mitchelmore, 2011, p. 240). Thus, working with patterns can promote this aspect of algebraic thinking. At the preschool level, educators have specifically noted that exploring repeating patterns may promote children's appreciation of underlying structures (Starkey, Klein, & Wakeley, 2004).

Repeating patterns are patterns with a cyclical repetition of an identifiable "unit of repeat" (Zazkis & Liljedahl, 2006). For example, the pattern ABBAB… may have a minimal unit of repeat of length three (ABB) and ends with an incomplete unit of repeat. However, without specifically stating what the minimal unit of repeat is, one may claim that the minimal unit of repeat in the above sequence is ABBAB. In general, sequences may be generated in an infinite number of ways. For example, the sequence 1, 2, 4, 7 may continue with 11, with 12, or with 13, depending on the respective rules: $x_n = n(n - 1)/2 + 1$, $x_n = x_{n-1} + x_{n-2} + 1$, or $x_n = x_{n-1} + x_{n-2} + x_{n-3}$. According to the Israel National Mathematics Preschool Curriculum (2008), "patterning activities provide the basis for high-order thinking, requiring the child to generalize, to proceed from a given 'unit', to a pattern in which the unit is repeated in a precise way" (p. 23).

Children's Engagement with Various Patterning Tasks

Young children naturally engage in pattern activities such as building block towers with an ABAB pattern (Seo & Ginsburg, 2004). However, while most children by the end of kindergarten will be able to copy a repeating color pattern, few will be able to extend or explain it (Clarke & Clarke, 2004). Being able to copy a pattern may not necessarily indicate that the child recognizes the structure of the pattern. Papic et al. (2011) found that some preschool children may be able to draw an

ABABAB pattern from memory by recalling the pattern as single alternating colors of red, blue, red, blue, basically recalling that after red came blue and after blue came red. This strategy is sometimes called the "matching one item at a time" strategy, or the "alternation strategy," especially successful with simple AB patterns, and less so in patterns such as ABCD that have more elements. When shown a more complicated pattern such as ABBC, they could not replicate the pattern. Rittle-Johnson, Fyfe, McLean, and McEldoon (2013) found that when young children were asked to duplicate or extend an ABB pattern, some children could not produce more than one unit of repeat correctly, while some reverted to producing an ABAB pattern.

Recently, Tsamir, Tirosh, Barkai, Levenson, and Tabach (2015) found that when children were requested to choose possible ways to continue repeating patterns, more children were able to continue a pattern which ended with a complete unit of repeat than a pattern which ended with a partial unit. When deciding whether or not to choose some continuation, some children merely seemed to guess, while others exhibited some strategy. One strategy was to physically move each continuation to the end, trying it out before deciding whether or not it was appropriate. Another strategy was aligning up each continuation with the beginning of the pattern to see if it matched. One child chose continuations based on the last element of the pattern, claiming that the next element cannot be the same as the last element of the given patterns. They suggested that in addition to promoting children's recognition of the unit of repeat, we should encourage children to recognize the sequencing aspect of the pattern and how to continue a pattern from any point.

In addition to duplication and extension tasks, there are other patterning tasks which focus more on the pattern structure. For example, one could request the child to directly identify the smallest unit of the pattern by either circling it or placing a string around the unit (Papic et al., 2011). Similarly, one could build a tower with a repeating pattern and request the child to build the smallest tower that still keeps the same pattern as the one already built (Rittle-Johnson et al., 2013). An activity which calls for more abstraction on the part of the child is to request the child to construct (or draw) the "same kind of pattern" as a given pattern but with different materials (Rittle-Johnson et al., 2013). For example, if an AABB pattern is constructed from red and blue cubes, then the child is given triangles and circles to construct a similar pattern. Sarama and Clements (2009), in their description of children's developmental progression for patterns and structure, state that being able to translate patterns into new media is a more advanced stage than being able to duplicate, extend, or fix a pattern. Rittle-Johnson et al. (2013) also found that abstraction tasks are more difficult than duplication and extension tasks and children often turn to building random sequences when solving abstraction tasks.

While in the above activities children are requested to act, other tasks focus on verbalization. According to the NCTM (2000), describing how two patterns, such as "red, red, blue, red, red, blue" and "step, step, clap, step, step, clap," are the same and how they are different encourages children to focus on underlying structures and sets the foundation for recognizing that seemingly different mathematical expressions, such as $2x + y$ and $2a + b$, have the same algebraic structure, $ax + b$.

While this specific activity was not implemented in any of the studies reviewed here, Papic et al. (2011) did note that comparing two patterns may occur spontaneously among children. They describe an incident where a child claimed that a blocks pattern he created was similar to a flower pattern because one is "blue, yellow, yellow, blue, yellow, yellow" and the other is "curved, spiky, spiky, curved, spiky, spiky." When asked to elaborate on their similarity, the child responded that "There is one curved and one blue, and then there's two spiky and two yellow, that's the same pattern" (p. 255). Papic et al. took this claim as evidence of the child's emergent recognition of an ABB pattern and the child's readiness to consider structure.

In the above studies, children were observed without adult intervention. However, when given proper assistance, young children are capable of recognizing the unit of repeat in a repeating pattern and come to comprehend the underlying structure of the pattern (Papic et al., 2011). In other words, for children to achieve the benefit of engaging in pattern activities, adult guidance is advisable. Yet, teachers may not always provide worthwhile patterning opportunities for children, and when children engage spontaneously in patterning, teachers sometimes fail to capitalize on the child's interest, missing out on opportunities to extend children's interest and knowledge in patterning (Fox, 2005). One possible reason for these missed opportunities might be teachers' lack of focus or partial knowledge regarding some structural aspects of repeating patterns. Elements of structure include the minimal unit of repeat, the length of the unit of repeat and the number of times it is repeated, and whether or not the pattern ends in a complete unit.

The Cognitive Affective Mathematics Teacher Education Framework

Describing the Framework

It is widely accepted that the knowledge necessary for teaching a subject goes beyond knowing the subject matter and that knowledge of subject matter may also have various elements (e.g., Ball, Thames, & Phelps, 2008). It is also recognized that teachers' self-efficacy beliefs may have an impact on their instruction (Allinder, 1994). Bandura defined self-efficacy as "people's judgments of their capabilities to organize and execute a course of action required to attain designated types of performances" (1986, p. 391). The Cognitive Affective Mathematics Teacher Education (CAMTE) framework takes into consideration teachers' knowledge for teaching mathematics as well as their self-efficacy for teaching mathematics in preschool. Like our previous studies concerning professional development for preschool teachers (e.g., Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2014), the program described in this study was planned using this framework.

This paper focuses on the knowledge elements of the framework. These elements draw on the works of Ball and her colleagues (Ball et al., 2008) who differentiated between two aspects of pedagogical content knowledge (PCK) (Shulman, 1986): knowledge of content and students and knowledge of content and teaching. As before, we differentiated teachers' subject-matter knowledge (SMK) into knowledge for producing solutions and knowledge of evaluating given solutions. In this study, however, we refined our previous framework by dividing teachers' knowledge of students to include teachers' knowledge of ways in which students produce solutions and teachers' knowledge of students' abilities to evaluate others' solutions. As was shown in the previous section, most studies regarding children's patterning activities describe "production activities" (i.e., tasks where children have to produce something, such as building, copying, or extending a repeating pattern). However, it is also valuable for students to be given opportunities to engage in evaluation tasks (NCTM, 2000), tasks which require the learner to evaluate a given situation or solution. Likewise, teachers' knowledge of tasks was refined to include teachers' knowledge of designing and evaluating different tasks, specifically tasks that require students to produce solutions and tasks that require children to evaluate given solutions. Table 3.1 presents the framework and offers examples of knowledge elements with respect to each cell within the context of patterning.

Using the CAMTE Framework

Recently, we began investigating elements of preschool teachers' knowledge for teaching patterns (Tirosh, Tsamir, Levenson, Barkai, & Tabach, 2015). Related to Cell 1, we studied teachers' definitions for repeating patterns and their ways of drawing and continuing repeating patterns. Results indicated that participants found it difficult to write a definition for the notion of a repeating pattern yet were able to draw and extend a repeating pattern. In addition, although teachers correctly extended repeating patterns, there was a strong tendency on the part of the teachers to end patterns with a complete unit of repeat. That is, if the structure of the pattern is ABC, and they are shown the beginning of a pattern, for example, ABCABCABCA…, teachers tend to add BC or BCABC, and not just add a B, or BCA. Yet, repeating patterns, such as repeating decimals, do not always present themselves by ending in a complete unit. When dividing one by seven on a calculator, students might receive a solution of 0.142857142857142. Students need to recognize the pattern and surmise that after the two comes an eight, etc. Thus, it was suggested that the issue of ending or not ending a pattern in a complete cycle might be an aspect of pattern knowledge in need of more attention. Regarding cells 4a and 4b of the framework (see Table 3.1), another study found that most preschool teachers prefer production pattern tasks (e.g., extend the pattern) rather than evaluation tasks (e.g., is this a pattern?) (Tirosh, Tsamir, Levenson, Barkai, & Tabach, 2016). Taking into consideration the

Table 3.1 The refined CAMTE framework

	Subject matter		Pedagogical content			
			Students		Tasks	
	Producing	Evaluating	Producing	Evaluating	Producing	Evaluating
Knowledge	Cell 1 Identifying, describing, and creating repeated patterns, continuing a repeating pattern, identifying the unit of repeat	Cell 2 Evaluating correct and incorrect solutions to pattern tasks	Cell 3a Knowing children's strategies for solving patterning tasks, knowing correct and incorrect ways in which children will continue repeating patterns	Cell 3b Knowing examples and non-examples of patterns that children will easily identify as patterns or non-patterns	Cell 4a Knowing to design "producing" tasks	Cell 4b Knowing to design "evaluating" tasks
Self-efficacy	Cells 5 and 6 Mathematics self-efficacy related to cells 1 and 2		Cells 7a, 7b, 8a, and 8b Pedagogical-mathematics self-efficacy related to cells 3a, 3b, 4a, and 4b, respectively			

benefits of promoting preschool teachers' knowledge for teaching repeating patterns, including their knowledge of patterns as well as their knowledge of patterning tasks, this paper describes a professional development program that takes into consideration the necessity to promote preschool teachers' SMK and PCK for teaching repeating patterns.

Although this chapter does not focus on the affective side of the framework, the professional development program was designed to promote teachers' knowledge in a non-threatening way. Instead of explicitly stressing mathematics knowledge, the program was designed to take into account what Watson and Sullivan (2008) called teachers' obvious interest in planning and teaching lessons or, in the case of preschool teachers, their interest in activities that can be realistically implemented in classrooms with young children. As such, we designed patterning tasks that teachers could implement with children, but at the same time, we used those tasks to engage the teachers with the mathematics involved in patterning and to promote their knowledge of patterning tasks and children's ways of solving patterning tasks. The aims of this chapter are (1) to illustrate some elements of a professional development course for preschool teachers focusing on repeating patterns and (2) to investigate the affordances and constraints of using various pattern activities to promote preschool teachers' SMK and PCK related to teaching patterns.

Setting

The Program

Twenty-three preschool teachers participated in the program described in this study. All had a first degree in education and between 1 and 38 years of teaching experience in preschools. Many prospective preschool teachers in Israel attend only two mathematics education courses during their 4-year education degree. These courses sometimes include one semester for learning about the development of number concepts and one semester for the development of geometrical concepts. Thus, providing ongoing professional development focused on mathematics preschool education is imperative. Yet, while professional development is strongly recommended, and teachers are given credit for courses taken, the choice between programs is varied, and teachers are not necessarily mandated to specifically enroll in mathematics programs.

The program described in this study was planned for 21 h. The teachers met seven times over a period of about 4 months in the local professional development center in their area. Approximately five of the seven sessions were devoted to patterning with the other two focusing on number concepts. The main themes of each of the five sessions were as follows: (1) identifying repeating patterns, mathematical language, focusing on unit of repeat; (2) analyzing repeating pattern tasks, action and verbalization, using concrete materials, pictures, etc.; (3) choosing tasks for children, how to implement them, and how to use video as a tool; and (4 and 5) watching videos of the teachers in the program engaging children with repeating pattern tasks and analyzing the videos together. All lessons and tasks were planned by the four authors of this paper. The third author did the actual teaching and will be called in this paper the teacher educator (TE). All sessions were videotaped and transcribed.

The Tasks

Four main patterning tasks were used throughout the program (see Figs. 3.1, 3.2, 3.3, and 3.4). The first two were pictorial extension tasks (i.e., the patterns were presented as pictures on cards). However, the first task (see Fig. 3.1) was a production task, where one had to choose an element from a bank of elements and extend the given pattern. The second task (see Fig. 3.2) was an evaluation task where one had to evaluate different ways of extending various patterns and choose which ways were correct.

Note that for Task 1 the patterns presented have essentially three different pattern structures: AB, ABC, and ABB. In each case, the minimal unit of repeat is repeated at least three times. Taking this view, the first three patterns end with a complete unit of repeat; the last three do not.

Present the child with one pattern at a time. For each pattern prepare two or three separate containers, each container containing cut outs of triangles, squares, or circles. For example, when presenting the first pattern, place before the child two containers, one with blue squares and one with red triangles. For each pattern ask: What comes next? This question is repeated three times so that in the end, the child will have added three elements to the pattern.

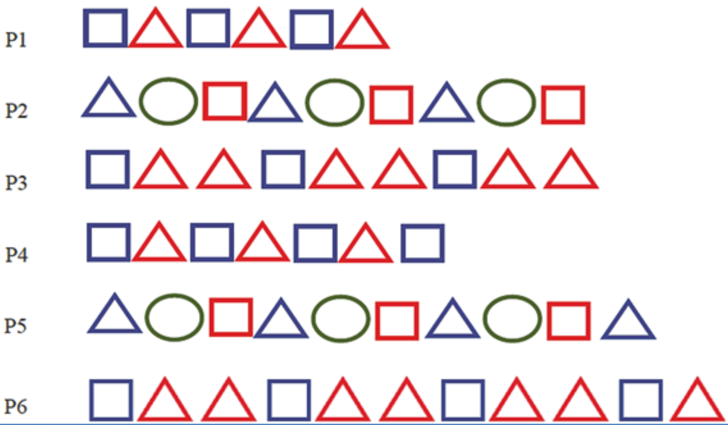

Fig. 3.1 Task 1 – what comes next?

For Task 2 (see Fig. 3.2) two patterns are used, both with an ABB structure. However, the first pattern ends with a complete unit of repeat, while the second does not. Furthermore, some of the possible correct extensions will end the pattern with a complete unit of repeat, and some will not. A main difference between Task 1 (see Fig. 3.1) and Task 2 (see Fig. 3.2) is that in the first task, extending the pattern is done one element at a time, while in the second task, the child has to look ahead and extend the pattern by two, three, or four elements at a time.

The third and fourth tasks involve concrete tangible items (colored beads) and are both production tasks in the sense that children are required to produce solutions as opposed to evaluate possible solutions. For Task 3, children are presented with two different pattern pairs and are requested to say how each two patterns are similar and how they are different (see Figs. 3.3 and 3.4). The first pattern pair consisted of two actual strands of colored beads. The first strand, S1, had three repetitions of the unit of repeat AAB, making a total of nine beads. The second strand, S2, had the same structure, but the colors of the beads in S2 were different from the colors of the beads in S1. The second pair of strands, S3 and S2, contained strands of the same colored beads and the same total amount of beads as S1, but the unit of repeat in each pair was different.

The fourth task consisted of having children construct a strand of beads with the same structure as one presented to them but with different colored beads (see Fig. 3.5). A key difference between Tasks 3 and 4 is that Task 3 calls for verbalization while in Task 4, children are requested to act.

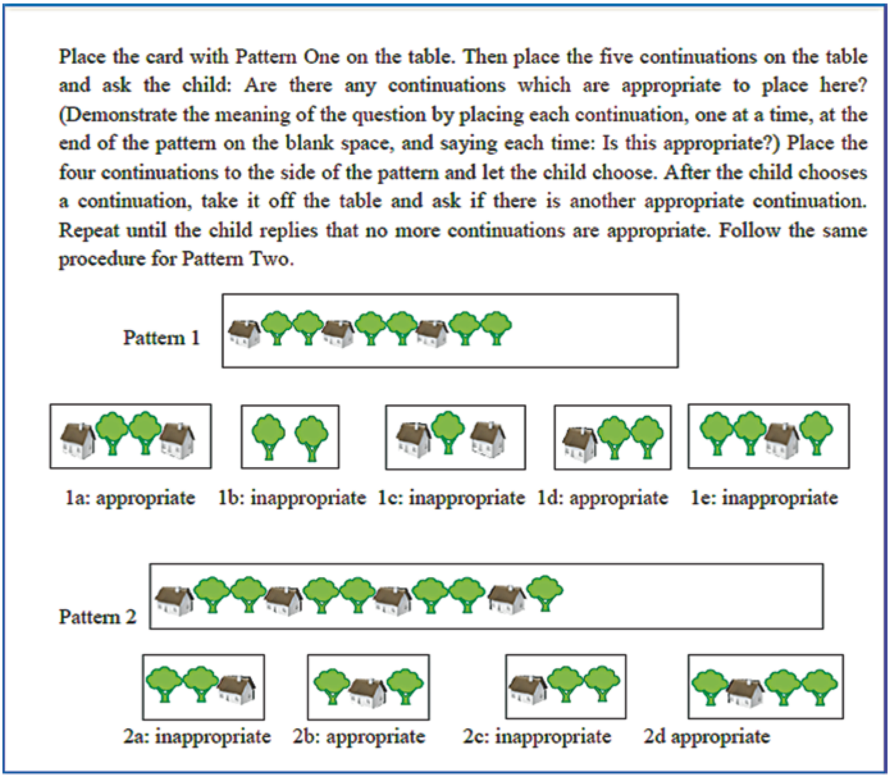

Fig. 3.2 Task 2 – which continuation is appropriate?

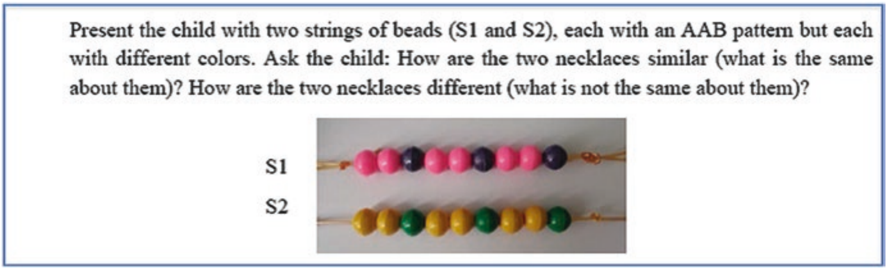

Fig. 3.3 Task 3, first pattern pair – what is similar and what is different?

A summary of the task characteristics, apart from the difference between the minimal units of repeat of the patterns, is offered in Table 3.2. These characteristics were inherent to the task. However, as will be seen in the next section, teachers were encouraged to discuss what may be varied within this framework. An essential element of learning about tasks (Cell 4 of the CAMTE framework) is understanding

Present the child with two strings of beads of the same colors, but one with an AAB pattern and one with an AB pattern. Ask the same questions as stated previously.

S3

S2

Fig. 3.4 Task 3, second pattern pair – what is similar and what is different?

Present the child with a string of beads with an AB structure (see S3 in Figure 4b above). Place on the table a string and two different colored beads (different colors from those on the presented strand). Ask the child to make a strand with the same pattern as the one you showed him. Repeat this activity but now present the child with a string of beads with an AAB structure (see S2 in Figures 4a and 4b above).

Fig. 3.5 Task 4 – construct a strand of beads

Table 3.2 A summary of the four tasks presented to preschool teachers

	Task 1	Task 2	Task 3	Task 4
Instructions	Extend the pattern one element at a time	Extend the pattern a few elements at a time	Compare two patterns	Build a pattern with a similar structure as a given pattern
Pictorial/ tangible	Pictorial	Pictorial	Tangible	Tangible
Production/ evaluation	Production	Evaluation	Production	Production
Verbal/action	Action	Action	Verbal	Action
Given pattern elements	3 full repeats of the minimal unit	(a) 3 full repeats of the minimal unit (b) 3 full repeats of the minimal unit and 2 additional elements	(a) 3 full repeats of the minimal unit (b) 4 1/2 repeats of the minimal unit	3 full repeats of the minimal unit

what may be varied in the task without changing the aim of the task. For example, if Task 4 used tangible items, and the TE presented this task with green and yellow beads, the teacher in the classroom might use red and blue beads, or yellow and green blocks. However, changing a task that was meant to use tangible items to one that would be pictorial was considered an inherent change to the task.

The variations in the tasks present different kinds of challenges to children. First, we regard the instructions. Extending a pattern, whether by one element or a few elements, is considered easier than tasks which require abstracting the pattern structure, such as Tasks 3 and 4 (Sarama & Clements, 2009). Regarding the use of pictorial representations versus tangible items, in general, one might think that the use of tangible items is more appropriate for young children. However, when it comes to repeating patterns, tangible items might be a distraction because they can be moved. For the first two tasks, it was important that children not be distracted by different spacing between the shapes. In the last two tasks, the string on which the beads were placed acted as an anchor for the elements. Thus, in this case presenting patterns using static pictures did not necessarily challenge the children more than presenting patterns with tangible items. Regarding production versus evaluation tasks, on the one hand, it might be easier to evaluate a solution that is presented rather than come up with a solution of one's own. However, in Task 2, which was an evaluation task, not only did children have to look ahead beyond the very next element, they had to consider that there might be more than one possible solution to the problem. In other words, production and evaluation tasks each present their own set of challenges. Regarding the issue of verbalization, one study reported an episode with a young girl who created a necklace out of game materials and described her necklace as "diamond, funny shape, diamond, funny shape" (Waters, 2004, p. 326). The challenge in this case is verbalizing not only what one sees but finding a way to express the abstractness of structure. Finally, as stated previously, children find it easier to extend a pattern when it ends with a complete unit (Tsamir et al., 2015). When a pattern is presented with an incomplete unit of repeat, the minimal unit of repeat might be more difficult to identify. For example, if the pattern ABAAB is a repeating pattern, is the minimal unit ABA or ABAAB? In either case, the next element would be A. But after that? In an attempt to make this minimal unit stand out a bit more, in our tasks, we always presented at least three repeats of what we considered to be the minimal unit of repeat.

Results: Program Segments

The following program segments were taken from the second and third sessions of the program because it was during these sessions that the above pattern tasks were introduced to the teachers. The teachers were told that these activities could be used with children in preschool and that later on in the program they would be asked to implement these activities with children, video the activity, and analyze together, in the course, the children's solutions. We analyze the segments according to the cells of the framework, pointing out how the tasks served to stimulate discussion revolving around the different knowledge cells of the refined CAMTE framework (see Fig. 3.1). The segments also illustrate how promoting different aspects of knowledge is intertwined.

Segment 1: To end or not to end the pattern with a complete unit of repeat

The participants, along with the TE, discuss the first three patterns of the first task and the differences between AB, ABC, and ABB patterns. They then examine the fourth pattern (P4).

1	TE:	What is the difference between P1 and P4?
2	Maya:	It (P4) has another cycle.
3	TE:	What do you mean by another cycle? I still have three repetitions of the minimal unit of repeat.
4	Sophie:	ABA.
5	TE:	Ok. The first ended with a complete unit and this one doesn't. It has the first element of the next cycle. When you video your children, we will see that some children choose which element comes next in the pattern by going back to the first element [of the given pattern] and adding that one. If the pattern ends in a complete unit, then that strategy works. But if they use that strategy here, what will they place?
6	Sherry:	A square.
7	TE:	But what really should come next is a triangle. That happens because most of the pattern tasks presented to children have patterns that end with a complete unit of repeat.

The minimal unit of repeat for both P1 and P4 is AB or, more specifically,

. P1 has three cycles of the minimal unit of repeat, while P4 has 3 1/2 cycles. In the above segment, the TE begins by promoting teachers' mathematical knowledge of patterns. This knowledge includes using precise mathematical terminology such as minimal unit of repeat and pointing out that the word cycle is not appropriate if only one element of the unit of repeat is given. In Line 5, the TE goes on to describe one strategy children use when asked to find the next element of a pattern. This promotes teachers' knowledge of students' ways of solving problems (Cell 3a of the CAMTE framework, see Fig. 3.1). The TE ends (in Line 7) by explaining to the teachers that many children do not realize that this strategy does not always work because nearly all of the patterns they engage with are patterns which end in a complete unit of repeat. In that case, the strategy of extending a pattern by adding the first element of that pattern works. In other words, sometimes a task can be successfully completed without a child fully understanding the underlying concept. Knowing this about tasks, how to analyze a task by taking into consideration children's ways of thinking, is necessary for choosing tasks. The TE is promoting teachers' knowledge of tasks which call for children to solve mathematical problems (Cell 4a of the framework) and the need for engaging children with various patterns as well as various tasks.

Segment 2: Focusing on the minimal unit of repeat

In the following segment, there is some disagreement as to the minimal unit of P1.

8	Shena:	I have a question. You said that the unit of repeat (in P1) is square triangle, but I see it as square triangle square and then a triangle and then square triangle square, and then a triangle. Could it be that a child will see it my way?
9	TE:	Let's see. What Shena is saying is that the unit of repeat could be square triangle square and then a triangle.
10	Shena:	The triangle is between the square triangle square.
11	TE:	And what would you put here (pointing to the end of P1)?
12	Shena:	A square and then a triangle. It comes out the same, but I see it differently.
13	TE:	Ok. But what's the big difference between the ways we each see it. The way I see it, P1 ends in a complete unit of repeat, but in your eyes, the pattern ends with a partial unit. And that is why it is very important to ask the child to explain how and why he chooses to continue the pattern in a certain way. We need to be able to evaluate the child's solution. The next element will be the same either way, but the child may see it differently. He may have in his mind a different minimal unit of repeat than we do.

The above segment focuses on the intertwining of Cell 2 (being able to evaluate mathematical solutions) and Cell 3b (knowledge of students' ways of evaluating solutions) of the CAMTE framework. First, teachers must themselves be able to evaluate the correctness of their students' mathematical solutions. Can one say that the minimal unit of P1 is ABAB and not just AB? In addition, it is important to recognize that children see things in their own way and that their way of evaluating patterns may be different than ours, but their way of thinking is not necessarily apparent from their solutions. In this case, a child may complete the task successfully by adding the correct elements, but still may not recognize the minimal unit as AB.

Segment 3: Focusing on the task instructions

In the following segment, the TE presents to the teachers the second task and has them say what is similar and what is different about the two strings of beads (S1 and S2). After they discuss different ways in which the patterns are similar and different, the following interaction occurs:

14	TE:	Maybe it would have been better to first ask the children what is different, and then ask what is the same.
15	Osher:	It's hard to say what is more difficult, saying what is similar or saying what is different.

16 TE: And after the children answer, you should ask if there is anything else similar, anything else different. Keep on asking till the children have nothing to add…. Now look at what would happen if I turn this one (S2) around (the TE flips S2 over so the left most bead is now green).
17 Osher: It's confusing.

When promoting knowledge of tasks, there are several elements to consider. First, there are the instructions, what the child is asked to do. But, there is also the sequence of instructions. What do we ask the child to do first and what do we ask the child to do second? The sequencing of the steps in the task may have an impact on the child's ability to complete the task. Thus, knowledge of children's patterning abilities (Cell 3) may impact on how the task is set up (Cell 4). Furthermore, if we request the child to complete a task only once, the extant of that child's knowledge may not be evident. Thus, the TE suggests asking the child over and over again to say what is the same and what is different. In other words, how we implement a task may impact on the knowledge we, as teachers, gain of our children's patterning conceptions.

Segment 4: Discussing task materials and characteristics
During the third session, the TE reviewed all four tasks which had been presented previously, this time drawing the teachers' attention not to the patterns but to the task features. Although during the professional development program we had supplied the task materials, it was understood that preschool teachers would use materials and supplies found in their own classes. Thus, discussing the actual material and how they might impact on the students' engagement with the task was important.

18 TE: Now, the materials (used in the tasks) are all different. There is what is called pictorial, drawings that I show them (the children), like the stickers that you use (stickers with pictures) on paper because it's hard for children to draw. There is also a tangible pattern, where I place a blue bottle cap on the table, then a red, and so on. Within those types there is still a wide variety – geometric shapes as opposed to abstract symbols. In addition, there are movement patterns (the TE demonstrates by patting her shoulders, raising her arms, and repeating three times) and sound patterns.
19 Rachel: What are pictorial patterns?
20 TE: Like these that I showed you (pointing to the patterns used in Task 1). The necklace (used in Task 4) uses real beads so that's tangible. The child actually strings the beads.
21 Rachel: But what is a sound pattern?

22 TE: Like a rhythm you hear that repeats. Or if I say Shena, Rachel, Osher, Shena, Rachel, Osher, Shena, Rachel, Osher. Now with a sound and movement pattern, you only see or hear the last element in the pattern. Right? If I do this (makes a pattern with hand movements), as soon as I do the second movement, the first is gone. And when I make the third movement the second is gone. With a drawing or with tangible items, I see the whole pattern.

In the above segment, the TE notes four types of pattern presentations. The first is a pictorial presentation, which (in Line 18) the TE says may be stickers. The stickers in this case are not used in the pattern as tangible items such as the beads are used, but are stickers with pictures on them. In addition, stickers are not mentioned merely because they are fun and available. Previously, the participants had discussed a pattern which only had triangles, but triangles of different sizes. The teachers and the TE discussed how difficult it was for young children to accurately draw these different size triangles, and so stickers with pictures of triangles could be used instead.

The various types of pattern representations allow the teachers to encourage children to use their sight, hearing, and touch senses. While this is especially important for young children, the different representations have different impacts on children's ability to extend a pattern. This is pointed out by the TE in Line 22.

In addition to the modes of representations, one of the teachers brings up the use of color in the beads when discussing Task 4.

23 TE: For the fourth task, you bring beads. Here is my strand (the TE holds up a strand of yellow and green beads with an AB structure). See what a nice pattern I have here. And then you show the boy or girl different color beads (the TE holds up a bowl with purple beads and another bowl with pink beads). It doesn't have to be these colors. Then ask the child to make a strand of beads with the same pattern as the one you have but with different colors.

24 Shena: Why a strand with different colors?

25 TE: Because otherwise, (if the child is given the exact same color beads as the one presented to him) it will just be a simple duplication task. One to one, he (the child) can say, first there is a yellow bead, so I will take a yellow bead. In our case, we can see if the child understands the concept of a pattern.

In the above segment, we see how a discussion of materials and a seemingly innocent question of color led to a more in-depth analysis of the task. Sometimes, the issue of color is unimportant. In Line 23, the TE says that the teachers can use any colors they wish. However, at times, the issue of color is important and can lead to very different types of tasks. Duplicating a pattern is simpler than abstracting a pattern from one medium and creating it in another medium. In this case, the hard material (beads) remains the same, but using different colors adds complexity.

The above segments specifically dealt with promoting teachers' knowledge of production tasks (Cell 4a). However, in the refined CAMTE framework (Fig. 3.1) we differentiated between two types of tasks – solving and evaluating tasks. This is pointed out in the following interaction:

26	TE:	The first task and the second task are different types of tasks. The first task is a producing task. The child has to extend the pattern. The second task is different. I show the child a pattern, and I say this is a pattern. Then I take this strip of paper with trees and houses on it, and I ask the child if this can be the continuation of the pattern. So, what kind of task is this – a production task or an evaluation task?
27	Many teachers:	An evaluation task.
28	Gale:	I don't understand why it's an evaluation task?
29	TE:	Because the child doesn't have to choose an element to continue the pattern. Instead, when I give him the strip of paper (with the drawings), he has to decide if this is a correct way to continue the pattern.

To summarize, several elements of patterning tasks, such as materials, are important to analyze for all patterning tasks, and thus all three vignettes above relate to Cell 4 of the CAMTE framework, promoting teachers' knowledge of patterning tasks. Specifically, the last few lines illustrate the difference between tasks which require the child to produce solutions and tasks which require the child to evaluate others' solutions.

Summary

The above segments represent only a sample of what took place during the entire program. Yet, several different aspects of teachers' knowledge for patterning were mentioned. Regarding teachers' SMK, we saw the emphasis on using precise mathematical language, recognizing the minimal unit of repeat and evaluating solutions. (In other studies, we focused to a greater extent on teachers' knowledge of solving patterning tasks and their knowledge of evaluating patterning tasks solutions (cf., Tirosh et al., 2015).) Regarding teachers' knowledge of students, the main issues discussed with participants while working on the tasks were different strategies children may use when extending a pattern and which tasks are simpler or more difficult for children. These and other aspects of knowledge of children's patterning abilities were again brought up later in the program when teachers viewed the videos of children engaging with the pattern tasks. Most of all, however, the above segments illustrate the promotion of teachers' knowledge for teaching patterning. Among the various points discussed were task instructions, modes of pattern representations, materials, and types of tasks. Recognizing that attention to detail is important, the TE and participants also related to the sequencing of patterns and the sequencing of instructions, colors, and how many times to repeat a question.

Epilogue

It is beyond the scope of this paper to convey all the different elements of the program. Instead, we jump right to the end. During the program, each teacher was required to video themselves implementing one pattern production task and one pattern evaluation task with at least one child in their preschool class. As a final project, the teacher was required to analyze and summarize what she learned from implementing the activity in terms of the mathematics involved, children's patterning concepts, and patterning tasks and summarize the experience. Here are excerpts from what three teachers wrote:

> T1: I sat with a five-year-old child, but in my opinion, the activity is relevant for all ages. The materials used were simple and appropriate. The patterns were not too difficult. Also, the way the activity is implemented is important, the way the task is presented, the instructions, the activity.
>
> T2: I greatly enjoyed watching the video with the other program participants. I felt proud of Shelly (the girl with whom she implemented the tasks) and the way she so nicely cooperated.
>
> T3: I was pleasantly surprised by the activity because I sat with a child who is 3 years and 10 months old and he knew how to identify patterns, continue a pattern which ended in a complete unit and one which did not end with a complete unit. Beforehand, I never worked with preschool children on patterns because I thought they were too little…. This experience caused me to understand that I can begin to work on patterning even with young children.

T1 stresses the activity, mentioning the patterns, materials, and the way it is implemented. T2 and T3 focus on the participants, the child they sat with, as well as themselves. T2 notes the child's cooperation, and T3 mentions the child's correct performance. Both write about their own enjoyment as teachers. T3 also points at that she has gained new knowledge about children's patterning abilities. Although we only bring three short excerpts, taken together, they reflect an overall positive experience with the professional development program, which can raise their self-efficacy for teaching patterning in preschool.

Discussion

Watson and Sullivan (2008) suggested that tasks for teachers have multiple purposes: to inform teachers about the variety and purpose of classroom tasks, to provide opportunities to learn more mathematics, to provide insight into the nature of mathematical activity, and to stimulate teachers' theorizing about students' learning. In this paper, we described a program that used classroom tasks to do just that.

At the start of the program, most teachers claimed that in their classroom, children engage in patterning tasks when they draw boarders or frames for pictures. It could be that the teachers were not aware of various patterning activities that can develop children's appreciation for pattern structure. This is in line with Zazkis and Liljedahl (2006) who found that although the topic of patterns may be found in curriculum objectives, pattern activities are often relegated to enrichment activities and not dealt with as seriously as intended. Reflecting on the program segments described above, it might be said that the instructor took a leading role during the sessions, introducing mathematical terms and posing questions, while the preschool teachers responded and reacted. One reason for this stance was the necessity of introducing mathematical language which would allow the teachers to engage with patterns on a mathematical level. In addition, as can be seen in several instances, the instructor's question often led one of the teachers to ask a question, leading to a deeper analysis of the pattern or of the task.

Our professional development program introduced teachers to various mathematically engaging patterning activities. The tasks also provided opportunities for the teachers to learn mathematics. In a previous study, it was found that when writing a definition for a repeating pattern, some preschool teachers wrote statements that mentioned the content of the pattern and that there is repetition, but did not mention structure (Tirosh et al., 2015). Building on that study, during the program described here, we discussed with the teachers critical attributes of a pattern as we analyzed tasks (e.g., that there is a specific core unit made up of a string of elements, the string of elements are not randomly laid out but have a fixed structure, and the unit is repeated.) The TE also used the first preschool patterning task to discuss with the teachers the broader issue of sequences, asking them, for example, what element of the pattern would appear in the 18th place. Affording preschool teachers a glimpse of how patterning in preschool will be developed in later school years may also increase their motivation to engage children with patterning activities during the early years.

In addition to promoting teachers' specific content knowledge, focusing on the unit of repeat and structure may enhance teachers' appreciation for the nature of mathematics. According to Schoenfeld, mathematics is "a living subject which seeks to understand patterns that permeate both the world around us and the mind within us" (Schoenfeld, 1992, p. 334). Thus, by engaging in patterning activities and, specifically, the comparison task, the preschool teachers were able to gain insight into the nature of mathematical activity and to see that mathematics is more than number concepts. This is especially significant in light of studies which found that most preschool teachers believe preschool mathematics consists of mastering arithmetic (e.g., one-to-one correspondence, counting, writing numerals) (Lee & Ginsburg, 2007). When asked to describe mathematical activities that take place in their preschool, the activity most often mentioned was counting (Benz, 2010). Finally, as the teachers engaged with the activities, they began to theorize about their young students' learning. This was evident in the above segments as teachers began to contemplate how the setup of an activity might affect children's performance.

Beyond listing the affordances of using classroom tasks during professional development, we also note a few constraints. While engaging teachers with classroom tasks (meant for children) might increase teachers' engagement, it may sometimes deflect their attention from other aspects. For example, some teachers were so enthusiastic about receiving from the TE ready-made activities that they tried out some of the activities in their class before it was fully analyzed in the program. A few teachers did not use the materials as were presented to them. For example, regarding Task 1 (see Fig. 3.2), teachers were told to prepare five separate containers for each possible cutout, and only present to the child those containers which contained cutouts for that pattern. Some teachers kept all of the containers on the table, no matter which pattern was being extended, while other teachers separated the elements into only three containers according to shape (e.g., putting blue and red squares in one container) and then placed all three containers on the table, no matter which pattern was being extended. A few teachers did not give the instructions as they were meant to be given. For example, some teachers asked the children to say out loud each element from the beginning of the pattern to the end, before asking them continue the pattern. Reflecting back on the instructional stance of the course, one concern might be for teachers' autonomy. These results demonstrate that despite the instructive stance of the instructor, the preschool teachers retained their independence and flexibility and varied aspects of the task that had been "instructed" to them. While a certain amount of variety is welcome, and teachers are certainly autonomous to enact activities in the way they see fit for their class, studies have shown that the way a task is implemented will affect the level of cognitive challenge felt by the student (Henningsen & Stein, 1997). Thus, it is important to understand how to implement a task in order to understand how each element of the task affects another element.

In addition, like T2 in the epilogue, many teachers were satisfied with the children's cooperation and apparent enjoyment of the activity. While this is commendable, we would also like to see teachers enthusiastic about the mathematical learning that is going on. Perhaps the focus on activities that were specifically designed for children, as opposed for teachers, took away from the fact that the activities were designed to promote mathematical learning. Yet, is this truly a constraint? There is a folk saying which says, "That which is done by rote will eventually come to be done with meaning." In our case, we believe that teachers, who are enthusiastic about implementing mathematical activities in their early childhood classrooms, even if it is "just" because they are fun, will come to see, with the guidance of professional development, the mathematical value of such activities. Furthermore, the refined CAMTE framework can help TEs, as well as teachers, pay attention to the different aspects of mathematical knowledge needed for teaching while enjoying the mathematical activities.

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Chapter 4
Mathematics Education Competence of Professionals in Early Childhood Education: A Theory-Based Competence Model

Hedwig Gasteiger and Christiane Benz

Abstract Early mathematics learning has become a topic of growing importance in early childhood education. Professionals in early childhood education therefore need domain-specific skills and knowledge in order to support children's mathematical learning. So far there are several theoretical and empirical research results in the field of teacher competence in mathematics education. But mathematics learning in pre-school settings differs from formal schooling. Therefore professionals in early childhood need different skills and knowledge to create mathematical learning opportunities than teachers need. This must be taken into account when discussing and investigating the competence of professionals in early childhood concerning mathematical learning. Based on a theoretical analysis of the requirements on professionals in early mathematics education and on a differentiated analysis of research results of professional competence, a theory-based domain-specific model of mathematics education competence for professionals in early childhood is presented.

Keywords In-service education · Pre-service education · Competence model · Teacher knowledge · Early mathematics education

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Introduction

Children can acquire many mathematical skills at already pre-school age. We know that these early mathematical skills are predictors for success on mathematical learning at school (Krajewski, Renner, Nieding, & Schneider, 2008). For this reason it is a serious challenge for professionals working in pre-school settings to support children at pre-school age so that they are enabled to learn mathematics coherently without major gaps. To face this challenge, expertise in mathematics education *and* in pre-school education is required. German pre-school education has a strong tradition in focusing on social pedagogical contents. Social pedagogical contents significantly determine the training of kindergarten teachers which is rather an exception on the European scale (Kucharz et al., 2014), and differently from other counties, professionals working in pre-school in Germany are often called educators – not teachers. The lack of focus on mathematics education in pre-service education probably leads to the situation that German educators in early childhood education feel rather insecure in the area of mathematics (Beher & Walter, 2012). A rather big proportion of educators working in kindergarten feel confused by mathematics or do not understand it (Benz, 2012). Mathematics is on the top of the list of requested on-the-job-trainings (Fuchs-Rechlin, 2007). This shows the need for an adequate support in implementing early mathematics education into practice – however, this is widely missing (Fthenakis, 2007).

These facts show that mathematics education in pre- and in-service education of professionals in early childhood is an important topic. Therefore it is important to get a general idea about the professional competence of early childhood educators who need to care for a high-qualitative early mathematics education. Concerning professional competence, there are domain-specific competence models for teaching in school, in particular related to teaching mathematics (e.g. Ball, Thames, & Phelps, 2008; Lindmeier, 2011). Moreover, models of general competence of professionals in early childhood education exist (Fröhlich-Gildhoff, Weltzien, Kirstein, Pietsch, & Rauh, 2014), but these models are not domain specific. The existing models represent either the mathematics education aspect or the pre-school education aspect. Both sorts of models cannot describe the professional *mathematics* educations skills of professionals in *early childhood*. The requirements on professionals in early childhood education differ significantly from those of school teachers because mathematical learning in pre-school settings takes place much more often in everyday situations and is much more informal than in school settings. And if educators are required to provide and design mathematical or other domain-specific learning opportunities, it is not sufficient to have general pedagogical competence concerning a holistic early childhood education.

National and international empirical approaches already exist for mathematics-specific knowledge and skills of professionals in early childhood education (Anders & Roßbach, 2015; Dunekacke et al. 2015a, b; McCray, 2008). But still, early childhood educators' professional action in mathematics education regarding the special requirements of pre-school settings is widely unexplored as yet (Kucharz et al., 2014). These special requirements can be described exemplarily: the professional has, for example, to encourage spontaneously well-grounded mathematical learning processes in situational contexts. Therefore he or she has to be sensitive in workaday life to identify mathematical learning situations, to plan learning situations when appropriate, to have an eye for all children and to react individually in adequate supportive ways. These requirements suggest a very complex arrangement of knowledge, skills and other facets of competence. These facets of competence need to be profoundly established and analysed.

In this chapter a competence model is presented. It is based on theoretical considerations, broadly accepted normative statements concerning early mathematics education and research results on competence of teachers and professionals in early childhood. In the model, professional requirements for early mathematics education and specific features of early childhood education are integrated. Moreover, the model brings the complex process of children's individual learning into focus.

To support mathematics education competence of professionals in early childhood education and to measure this competence, a precise definition of this competence is indispensable. Only then it is possible to specify in which situation individual differences in different facets of competence are shown in which way (cf. Klieme & Hartig, 2007, p. 24).

Key Points of Early Mathematics Education

Defining the structure of a domain-specific competence affords to analyse the requirements someone has to fulfil in this domain (Blömeke, Gustafsson, & Shavelson, 2015; Weinert, 2001a). Therefore, a theory-based analysis of competence required by professionals organising and stimulating early mathematics learning must be based on a well-founded analysis of relevant mathematical contents for early mathematics education and of adequate methods for children at an early age to learn these contents. Professionals need different skills if they teach in a traditional way than if they support children to learn in informal situations, if they use learning or training programmes with strict guidelines or if they use everyday situations for mathematical learning, where they need to recognise and use the potential of the situation for mathematical learning. The following section describes key points of early mathematics education. These key points are based on policy documents, theoretical findings and empirical research. They are broadly accepted in Germany – in mathematics education and likewise in early childhood education. These key points can be seen as the basis of the competence model (Sect. 5).

Early mathematics education must allow successful further mathematical learning. To achieve this, learning opportunities must be designed coherent (Benz et al., 2017, p. 39) and mathematically correct, though it is often necessary to make mathematical contents available in a more elementary way (Gasteiger, 2010). Moreover learning should be grounded on central, fundamental ideas of mathematics (Brownell et al., 2014; Gasteiger & Benz, 2012; Kaufmann, 2010; Lorenz, 2012; Rathgeb-Schnierer, 2012; Sarama & Clements, 2009; Wittmann & Müller, 2009).

Even though different fundamental ideas are named by different authors (Gasteiger, 2015), they all agree that early mathematics education must be based on concepts that are central for mathematics, that enable continuous learning and that build a broad understanding of mathematics.

There exists consensus that process-orientated skills like problem-solving, argumentation, etc. play an important role in mathematics in general and consequently in early childhood education. Sufficient space for creativity is necessary as well as enabling the children to solve problems on their own or to discuss mathematical ideas. This is more important than the teaching of procedures (Benz et al., 2017, p. 51). As a consequence, tight and instructive courses seem inappropriate for the field of early childhood mathematics. Instead, learning mathematics in natural learning situations (Gasteiger, 2012, 2014) is seen as crucial in early childhood education. Activities that are suitable for children, such as playing (free playing, board games, construction, role playing), everyday situations (e.g. preparing a breakfast, counting all children in the morning, planning the day) or looking at picture books serve as a basis to initiate and support early mathematics learning (Gasteiger, 2010; Hirsh-Pasek, Golnikoff, Berk, & Singer, 2009; Kaufmann, 2010; Van den Heuvel-Panhuizen, Elia, & Robitzsch, 2014; Van Oers, 2010).

In addition to the contents and organisation of learning opportunities, observation and documentation play an important role in the context of early childhood education. It is of particular importance to recognise, support, strengthen and promote the specific potential of each individual child (Roux, 2008, p. 9). Concerning early mathematics education, two key aspects of diagnostic efforts are mentioned: determining the individual learning basis and learning conditions in order to support successful further mathematical learning (Benz et al., 2017; Gasteiger, 2010; Peter-Koop & Grüßing, 2011; Steinweg, 2006) and identifying delayed development of individual children at an early stage in order to prevent persistent mathematical learning difficulties in school (Krajewski et al., 2008; Lorenz, 2012; Peter-Koop & Grüßing, 2014).

Professional Competence in the Context of Education

In the context of education, there are different theoretical descriptions and empirically analysed models of professional competence which describe knowledge, skills or beliefs required for successful teaching. In general, competence is attributed to someone who has knowledge-based skills in specific culture- and life-related domains and who can apply them in current learning and problem-solving situations (Oelkers & Reusser, 2008, p. 24). This description of competence shows that certain dispositions such as knowledge and skills as well as motivation and volition are seen as prerequisites for competent actions (Weinert, 2001b). However, someone can be seen as competent only if his dispositions can be applied in real-life situations. This successful acting in real-life situations is usually referred to as performance. Blömeke et al. (2015) suggest not to regard competence and performance as two separate notions but to regard competence as a continuum arising from specific dispositions to the visible actions, which are called performance. Dispositions are seen as important for teaching, but they do not automatically lead to the desired performance. Obviously, there are no technical transformation rules from dispositions to successful performance in the context of teaching (Baumert & Kunter, 2006, p. 476). According to Blömeke et al. (2015), detecting and interpreting relevant aspects in a concrete situation and deciding competently what to do are crucial abilities to translate disposition in performance. It seems to be clear that it is impossible to standardise teachers' actions. Each situation is different, and it cannot be clearly defined beforehand. Moreover, in educational contexts, interaction plays an important role, and interaction is normally influenced by characteristics of different personalities and therefore unforeseeable (Baumert & Kunter, 2006, p. 478).

This has to be considered when referring to different models of professional competence in the context of education. Frequently, these models are structural. Structural competence models can be used to describe competence, particularly if it can be assumed that different facets of competence or different dispositions interact in order to face the situational challenges that are relevant for a successful performance (Klieme, Maag-Merki, & Hartig, 2007, p. 12). Structural models describe and develop hierarchies of different facets of competence, e.g. knowledge, beliefs or skills (Anders, 2012, p. 13).

In contrast, process models focus on the process of professional actions. This means that they refer to a specific situation where the professional must act. They describe the requirements from analysing the situation via planning and concrete action through evaluation (Anders, 2012; Fröhlich-Gildhoff et al., 2014). Structural models analytically split up competence in several elements, whereas process models describe competence in its entirety – including performance (see continuum perspective of Blömeke et al., 2015).

The following section provides a short overview of research on teacher competence in mathematics education because often in the context of professional competence of early childhood educators, it is referred to these models (e.g. Hepberger et al., 2014). Subsequently, the status of research on the competence of professionals in early childhood education will be described in more detail.

Teacher Competence

Shulman's description of a competent teacher differentiates different knowledge facets and the interaction between these facets (Shulman, 1986, 1987). It has been used as a basis for several models of teacher competence. According to Shulman, content knowledge (CK) and pedagogical content knowledge (PCK) are essential parts of teacher knowledge. If not only pedagogy but mainly domain-specific education is seen as a teacher's task, Shulman (1986) assigns an important role to pedagogical content knowledge (knowledge of suitable representations, examples, knowledge of possibilities to present and explain and of pupils' prior knowledge and misconceptions) (Shulman, 1986). Within a model of general professional competence that was used in the context of the COACTIV study (Kunter, Klusmann, & Baumert, 2009), Shulman's knowledge facets (Shulman, 1986, 1987) are complemented by three additional aspects of professional competence. Furthermore regarding professional knowledge, self-regulation and motivational orientations as well as beliefs, values and objectives are seen as significant influences on educational activities.

Teacher's Competence in Mathematics Education

The model of professional competence of Shulman is rather general on the first glance, but it has been modified in many studies concerning mathematics education – for example, in the COACTIV study (Krauss et al., 2004). In this study pedagogical content knowledge (PCK) was differentiated as "diagnostic knowledge" and "explanation knowledge". Content knowledge (CK) was specified concerning mathematics going beyond mathematical content in school curricula (Krauss et al., 2008, p. 717).

Ball, Thames and Phelbs (Ball et al., 2008) focused on mathematics education in their model, and they split content knowledge in a general and a specific content-related knowledge. The latter is important for teaching. With this kind of content knowledge, a teacher may judge a child's solution process as successful or promising. A third area of content knowledge is labelled as "horizon content knowledge" (Ball et al., 2008, p. 403). This knowledge allows a teacher to guarantee continuity of mathematical learning, because he or she knows which content aspects are necessary for further learning. Ball et al. (2008) split pedagogical content knowledge as well. They differentiate between three areas of pedagogical content knowledge: (1) "knowledge of content and students" comprising diagnostic knowledge about misconceptions as well as the anticipation of different ways of thinking and acting; (2) "knowledge of content and teaching" comprising the knowledge needed to plan or sequence lessons, as well as all kinds of decisions that need to be made during lessons (e.g. choosing appropriate presentations); and (3) "mathematics-specific curricular knowledge".

All these models indicate mathematical knowledge as well as pedagogical content knowledge as key facets of professional teacher competence. This is the case for the model used within the TEDS-M study on the professional competence of mathematics teachers in primary education as well. These facets of knowledge serve as a basis for planning lessons in a way that is relevant for the subject matter or supporting children individually respecting their ways of thinking. In the TEDS-M study next to mathematical knowledge, pedagogical content knowledge and pedagogical knowledge, attitudes and beliefs about mathematics and about learning and teaching mathematics were investigated (see Blömeke, Kaiser, & Lehmann, 2010, p. 14).

The competence structure model described by Lindmeier (2011) shows a different approach for structuring mathematics-specific teacher competence. This model splits teacher competence in three components: basic knowledge, reflective competencies and action-related competencies. Basic knowledge comprises content knowledge (CK) and pedagogical content knowledge (PCK). Reflective competencies include pre-instructional reflections, decisions during planning processes and considerations for the further work arising from the analysis of pupils' learning processes. Action-related competencies refer to spontaneous decisions that are required during lessons – for example, due to incorrect or mathematically challenging statements made by pupils or when teachers need to find promptly an appropriate example or a question to test whether pupils have achieved the expected learning objective (Lindmeier, 2011, p. 105–110). Thus, this model combines the facet of knowledge and the facets of competencies required to implement the knowledge during teaching.

As requirements in pre-schools differ from requirements in schools (cf. Sects. 1 and 2), these models cannot be directly and uncritically adopted. It is therefore necessary to analyse if there are facets of competence, which are explicitly necessary and specific for early mathematics education, and if so, to describe them in greater detail.

Competence of Professionals in Early Childhood Education

There are general and non-domain-specific models describing facets of competence of professionals in early childhood in national and international contexts (Siraj-Blatchford, Sylva, Muttock, Gilden, & Bell, 2002). Fröhlich-Gildhoff et al. (2014, p. 13) point out a process model that seems appropriate to describe the particular requirements professionals face in early childhood education: the ability to act in a responsible, self-organised and reflective way in a concrete situation. According to Fröhlich-Gildhoff et al. (2014, p. 13), professionalism of educators in early childhood education is manifested by their ability to use their theoretical, domain-specific knowledge and their reflective experiential knowledge in a way that enables them to adjust their actions to the conditions and requirements of a concrete situation. These situations are complex, ambiguous and not entirely predictable and differ from time to time (Fröhlich-Gildhoff et al., 2014, p. 13). On this background, Fröhlich-Gildhoff, Nentwig-Gesemann, and Pietsch (2011) developed a general process model (Nentwig-Gesemann & Fröhlich-Gildhoff, 2015, p. 48) describing the competence of professionals in early childhood education and combining principles for action (disposition), preparedness for action, realisation of actions and the concrete action in early childhood education (performance) (cf. Fröhlich-Gildhoff et al., 2011, p. 17; Fröhlich-Gildhoff et al., 2014) (cf. Fig. 4.1).

According to Fröhlich-Gildhoff et al. (2014), knowledge subdivided into domain-specific theoretical knowledge as well as habitual, reflective experiential knowledge is a part of disposition. Disposition encloses as well special preconditions for action, consisting of procedurally methodical knowledge (related to

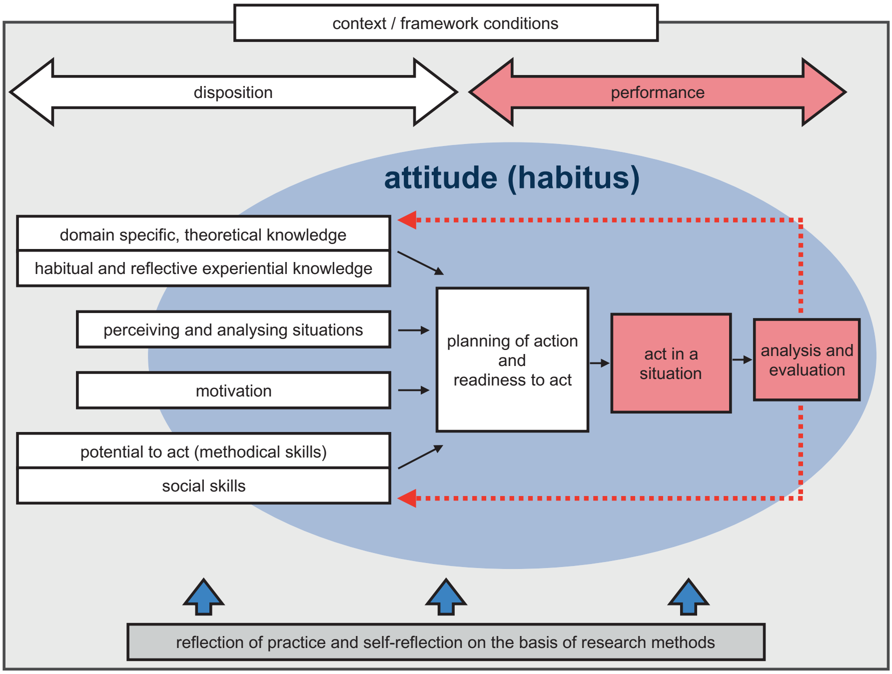

Fig. 4.1 Model of the competence of professionals in early childhood education (Fröhlich-Gildhoff et al., 2011, p. 17; revised version: Fröhlich-Gildhoff et al., 2014, p. 14, translated by the authors)

diagnostics, observation, etc.) and social skills (e.g. adoption of perspectives and empathy) (cf. Nentwig-Gesemann & Fröhlich-Gildhoff, 2015, p. 50; Fröhlich-Gildhoff et al., 2014, p. 14). Moreover, motivation and the ability to perceive and analyse specific situations are part of disposition. These four components of disposition (knowledge, motivation, perceiving and analysing situations) are supposed to have an influence on performance (Fröhlich-Gildhoff et al., 2014). In contrast to models describing teacher competence, the part of disposition combines not only various knowledge facets but also values and motivation. In addition, this model has a special knowledge facet: Fröhlich-Gildhoff et al. (2011, p. 17–18) assume that the acting of professionals in early childhood education is not only influenced by explicit, scientific-theoretical knowledge but also by implicit (habitual) experiential knowledge. It is seen as a part of the required knowledge. Furthermore, this model emphasises the competence to perceive and analyse situations, which could be seen as another difference to the teacher competence models of Ball et al. (2008), Krauss et al. (2004) and Lindmeier (2011).

Professionals in early childhood education show performance if they act in the specific situation. Competent actions are seen as tightly connected with reflective processes (Fröhlich-Gildhoff et al., 2014, p. 14). Therefore, the execution of actions is followed by analysis and evaluation as another aspect of performance. As situational, spontaneous actions are usually performed under pressure and are not always supported by explicit reasons (Fröhlich-Gildhoff et al., 2014, p. 15), the ability to analyse and reason retrospectively is a key component of professionalism. However, analysis and reflection are not exclusively performed after completing the action: sometimes, professionals not only act spontaneously but plan their actions in advance. In this case, they are able to factually justify their actions in advance based on their theoretical knowledge and reflective experiential knowledge. The aspects Fröhlich-Gildhoff et al. (2014) described within performance are similar to the aspects of the components of reflective and action-related competence described by Lindmeier (2011).

Similar to the COACTIV model, Fröhlich-Gildhoff et al. (2014) assume in their model that a habitus shaped by a professional's life is behind a professional performance of actions and is thus influential on disposition and performance. In this model, attitude or habitus is independent of the situation. This means that based on personal convictions, it is the primarily individual disposition that influences specific situational actions and the preceding or subsequent reflexive analysis. Furthermore, professional's ability to self-reflection and contextual factors and circumstances have an impact on the relations between dispositions and performance (cf. Fig. 4.1).

Mathematics Education Competence of Professionals in Early Childhood Education

In addition to studies specifying general competencies of professionals in early childhood education, there are empirical studies about individual facets of early childhood teachers' competence in mathematics education. These studies provide an important basis for considerations about the structure of mathematics education competence of professionals in early childhood education. Since different studies investigate very different competence facets or rather influencing factors of mathematical didactical competencies, the studies are described briefly. In paragraph 4 the different insights are summarised in a critical synopsis.

Based on the knowledge facets described in the different teacher competence models, (Hill, Schilling, & Ball, 2004; Shulman, 1987), McCray and Chen investigated knowledge of mathematics education (PCK) of professionals of early childhood education in the USA (McCray, 2008; McCray & Chen, 2012). They adjusted for their research tasks described by Hill, Schilling and Ball (2004). McCray (2008) was able to distinguish two different factors: elaborative and evaluative knowledge. Elaborative knowledge on mathematics education was defined as knowledge that helps professionals to identify, name and expand mathematical aspects in everyday activities initiated by children. Evaluative mathematics education knowledge was described as knowledge, which professionals need in order to evaluate the performance of children during targeted mathematical activities initiated by professionals. It was investigated by analysing early childhood teachers descriptions of children's activities and their descriptions of required next steps in diagnostic tasks. McCray (2008) could prove a positive correlation between the children's mathematical educational success and the professionals' elaborative knowledge. Concerning evaluative knowledge, however, no positive correlation could be found. In addition, McCray (2008) examined the correlation between PCK and everyday practice. To analyse everyday practice, she referred to the professional's mathematics-related language in everyday life. In group activities a positive correlation between the professional's elaborative knowledge of mathematics education and the use of mathematics-related language could be stated. In individual activities and unplanned activities, this correlation was also shown. However, in individual activities and unplanned activities, a negative correlation was shown with the evaluative mathematics education knowledge of the professional. The better the professionals performed in the tasks concerning evaluative mathematics education knowledge, the more rarely did they use mathematics-related language outside of planned activities. According to McCray (2008, p. 113), the operationalisation of the evaluative knowledge could be a possible explanation for the missing correlation between an evaluative knowledge of the professionals and the mathematical achievements of the children as well as for the negative correlation between evaluative knowledge of the professionals and the use of mathematics-related language. McCray critically remarks that not only knowledge and experience but also subjective attitudes of how mathematical learning should be supported could play an important role in the response to the questions concerning evaluative knowledge. This means that some tasks probably did not measure knowledge but attitudes or beliefs. Therefore, the construct of evaluative knowledge could possibly not be grasped selectively.

Anders and Roßbach (2015) asserted that there is a huge demand for professional development concerning early childhood teachers' abilities to recognise mathematical aspects in children's play. They used an elaborative mathematics education knowledge task from the study of McCray (2008) in order to investigate "sensitivity to mathematical content in children's play" (Anders & Roßbach, 2015, p. 20), and they collected data on mathematical knowledge and attitudes concerning mathematics. Their findings support the results of previous studies, which report a low competence level of professionals in early childhood to recognise mathematics in children's play (Lee, 2010; McCray & Chen, 2012). Early childhood teacher's mathematics-related experiences in school and their current attitudes concerning mathematics were described by Anders and Roßbach (2015) as neutral and not – as often expected – dismissive or negative. These authors revealed several connections between the different influencing factors or facets of mathematics education competence: The assumption could not be confirmed that negative experiences in school related to mathematics led to a less positive attitude towards mathematics later on and thus to a decreased valuing of mathematics. Still, emotions in connection with own mathematics-related experiences in school predict the present joy in doing mathematics or rather the interest in mathematics. However, the present joy and the interest in mathematics are closely dependent on the values professionals of early childhood education attach to mathematics. The present joy in doing mathematics and the interest in mathematics are predictors for the ability to recognise mathematics in children's play. Although these are first results, we have reason to believe that different competence facets can be distinguished and furthermore that these competence facets partly interact.

Dunekacke et al. (2015a, b), Dunekacke, Jenßen, Grassmann and Blömeke (2014), Dunekacke et al. (2013) also collected data on early childhood teachers' ability to perceive mathematics in informal situations. Here, correlations of this ability with mathematical knowledge and planning of activities were investigated. In the different partial studies, modified competence facets of the structure models of the TEDS-M study or rather the COACTIV study were used. Thus, in the KomMa project (Dunekacke et al., 2013), four facets were consulted for describing the competence of professionals in early childhood: subject-related mathematical knowledge, pedagogical content knowledge, pedagogical knowledge and beliefs. The design of planned and situational mathematical learning activities is seen as a subdimension of pedagogical content knowledge. Later, Dunekacke et al. (2015a) used a pre-version of the process model of Fröhlich-Gildhoff et al. (2011) (cf. Fig. 4.1) to investigate with pre-service students the correlations between pre-service students' mathematical knowledge (MCK), their ability to perceive mathematics in informal learning situations (PERC) and their planning of actions to support the development of mathematical abilities of children (ACT). A paper-pencil test (MCK) was used, and video vignettes with scenes of pre-school settings, showing informal learning situations. Items were linked to each video vignette concerning perceiving mathematics in informal learning situations (PERC) and items concerning the planning of activities (ACT), which referred to a specific section of the presented video. The planning of action (ACT) was taken as indicator for the real performance, whereas it was hypothesised that the ability to perceive mathematics in informal situations (PERC) mediates between the mathematical knowledge (MCK) and the planning of activities (ACT). Dunekacke et al. (2015a) state that the pre-service students could answer only half of the tasks concerning mathematical knowledge and the perception of mathematics in informal situations. Concerning the planning of actions, only about one third of the pre-service students succeeded in adequately answering the questions. Statistical analyses with structural equation model showed no significant direct effect of mathematical knowledge on the planning of actions. But mathematical knowledge proved to be a predictor with moderate effect for the ability to perceive mathematics in informal situations, whereas this ability again can be described as strong predictor for the planning of activities. When interpreting these results, it must be considered that the study is about pre-service students in early childhood education, who do not have experiences they can refer to.

Summary

In Sect. 3, different models for professionals' competence in the context of learning and education were exemplified. Moreover, empirical findings concerning different facets of competence or rather influencing factors on the skills and abilities of professionals in early childhood education were presented. In order to describe mathematics education competence of professionals in early childhood education in a theory-based and comprehensive way, a critical summary of the existing findings of competence of teachers and professionals in early childhood education on the basis of specific requirements for early mathematics education (cf. Sect. 2) is required.

If the concept "competence" is understood in the sense of Oelkers and Reusser (2008) or Weinert (2001a, 2001b) (cf. Sect. 3), it is more than different dispositions or prerequisites for competent acting. Using the term "competence" includes applying these prerequisites successfully in the respective situation. Especially for the description of the professionals' competence in early childhood education, it is central to be focused on the application in the respective situation. Because of vocational training situation (cf. Sect. 1), at present, the professionals' possibilities to gain differentiated knowledge are scarce – still they show efforts of mathematics education activities in their daily working routine. Therefore, next to the requested dispositions, the level of performance should be considered in a model concerning the mathematics education competence of professionals in early childhood education. In order to describe the competence of professionals in early childhood education – from a general pedagogical view as well as from a mathematics education view – a model seems to be adequate that does not only illustrate the *structure* of the professional competence (models by Ball et al., 2008; Kunter et al., 2009) but also considers the *process* of the domain-specific pedagogical activities up to the performance. This perspective is coherent with Blömeke et al. (2015) who see competence as a continuum. The reflexive as well as the action-oriented component in the model by Lindmeier (2011) points to the level of performance already but less to the *process* of the dispositions up to the performances. Especially for early childhood education, where mathematics is not taught in a classical sense and where professionals seem to have a heterogeneous background, it seems necessary to analyse precisely what is really required to turn dispositions into performance. Also an analysis of how mediators (Blömeke et al., 2015, p. 7) between disposition and performance can be characterised is needed.

The model by Fröhlich-Gildhoff et al. (2014) (see Fig. 4.1) indeed illustrates dispositions and performances, but – especially from an early childhood education perspective – there is some evidence that with a view to mathematics education single facets should be developed further, complemented and if necessary restructured. Thus, a further differentiation is required concerning the "domain-specific theoretical knowledge". Moreover, in the aspect of diagnosis and individual fostering, which has a huge relevance for early mathematical learning (Krajewski et al., 2008; Lorenz, 2012; Peter-Koop & Grüßing, 2011; Peter-Koop & Grüßing, 2014; Steinweg, 2006), specific competencies are required which are not explicitly illustrated in the model by Fröhlich-Gildhoff et al. (2014).

Furthermore, a couple of arguments lead to the necessity to critically reflect the classification of the perception of the situation as a disposition (as in the model of Fröhlich-Gildhoff et al., 2014). In the domain of early childhood education where activities are not very predictable and therefore not easy to plan (cf. 3.3), the perception of the situation is seen as a central requirement for acting competently (Anders & Roßbach, 2015; Blömeke et al., 2015; Dunekacke et al. 2015a; Fröhlich-Gildhoff et al., 2014). Emerging “natural learning situations” (Gasteiger, 2012, 2014) in the everyday life in kindergarten can only be used effectively for the learning of mathematics if they are recognised by the professionals as mathematically substantial in the concrete situation and if they are then commented, elaborated and discussed with the children. Professionals in early mathematics education show their didactical and pedagogical acting skills especially in the productive and mainly spontaneous use of these everyday situations and less in the implementation of previously planned activities.

In some studies concerning mathematics education competence of professionals in early childhood education, perceiving and using natural learning situations for mathematical learning are already considered. Still, there is disagreement where this ability is to be placed structurally in the entity of mathematics education competence of professionals in early childhood education. For Fröhlich-Gildhoff et al. (2014), the perception of situations is a disposition. Smith (2000), McCray (2008), McCray and Chen (2012) and Anders and Roßbach (2015) relate the ability to recognise mathematics in playing situations of children to the aspect of knowledge. Dunekacke et al. (2013) also mention “designing intended and situational mathematical educational processes” (p. 281) in their competence structure model as an example of pedagogical content *knowledge*. However, in their contribution from 2015a, Dunekacke et al. relate to the process model of Fröhlich-Gildhoff et al. (2011) and treat the competence facet to perceive mathematics in informal situations (PERC) as an independent component of the disposition and the competence facet of planning activities (ACT) both as disposition and as performance (Dunekacke et al. 2015a). Connections to knowledge are no more illustrated, whereas the perception of a situation and the planning of an activity are not explicitly separated from knowledge. This discordance in structuring the mathematics education competence of professionals in early childhood education demonstrates the necessity to illustrate different aspects as well as possible coherences and dependences of the facets of the competence – from disposition up to the performance – in a theory-based way and also in relation to the specific requirements of early childhood education (cf. Sect. 2).

Next to recognise, create and take advantage of learning possibilities, a further specific challenge for professionals in early childhood education is to perceive children’s individual learning prerequisites as well as to recognise special developmental deficits (cf. Sect. 2). Only based on individual prerequisites, children can be supported adequately. In early childhood education, mathematical support is required in a compensatory and preventive way – not least in order to enable coherent mathematical learning processes. Insofar, diagnostic abilities are seen as an important part of the mathematics education competence of professionals in early childhood education. These abilities are necessary, even if they are not explicitly investigated in the empirical researches mentioned above and also if they are not explicitly worked out in the so far existing models concerning the competence of professionals in early childhood education. Only McCray (2008) considers this aspect in her competence facet of the evaluative mathematics education knowledge. This competence facet is not differentiated further because she could not prove a correlation between this knowledge and the children's learning success. The critical question remains whether it has to be investigated anew and if there is really no correlation, or if – as McCray (2008) mentions as well (c.f. 3.4) – her investigation of the evaluative mathematics education knowledge was eventually not yet broad and mature enough.

It is noticeable that so far existing considerations on mathematics education competence of professionals in early childhood education do hardly consider reflexive aspects as a competence facet. In teacher competence models, the importance of a reflexive component is illustrated (Lindmeier, 2011), and also in the process model of a general competence of early childhood education, the ability of analysing and evaluating is regarded as important (Fröhlich-Gildhoff et al., 2014). Reflexive aspects of teacher's competence are shown both pre- and post-instructionally (Lindmeier, 2011, p. 107) – this means planning of lessons requires reflection (pre-instructional), and it is necessary to reflect after the mathematics lessons (post-instructional). In early childhood education, the spontaneous use of situations for mathematical learning is mainly emphasised. Though, sometimes small sequences are planned, which are consciously arranged, e.g. for circle time. The pre-instructional reflection (Lindmeier, 2011) – including all considerations concerning planning learning arrangements in early childhood education – is found in the facet of planning activities in the model by Fröhlich-Gildhoff et al. (2014). Especially due to the fact that spontaneous acting in a situation is very often required in early childhood education, the post-instructional reflection or rather the analysis and the evaluation are very important. A substantiated reflection about successful or less successful mathematical learning situations allows the professionals to gain important insights for further activities and is insofar also in a mathematics education view, explicitly relevant. Because planning concrete mathematical learning opportunities in early childhood education is not the main way of mathematical education, it can be argued that the post-analysis and evaluation of concrete situations should be emphasised, and the pre-instructional reflection which is a central facet of the competence of teachers should be regarded separately.

A Theory-Based Model Describing Mathematics Education Competence of Professionals in Early Childhood Education

In the context of emphasising mathematical learning in early childhood education, the mathematics education competence of the professionals comes into focus of research and of pre- and in-service education. A broad theoretical analysis of the structure of this competence as well as its empirical validation is thus required and so far not yet done (cf. Sect. 1). Regarding the competence models at hand, or rather the insights into the skills and knowledge of professionals in early childhood education, some aspects remain unexplained or rather unconnected, as it is shown in the summary (Sect. 4). Especially for early childhood education, it is necessary particularly to consider the process to the point of performance. This is not the case in most of the models of teacher competence of mathematics education. Thereby, situational but also planned learning processes (Dunekacke et al. 2015a; Fröhlich-Gildhoff et al., 2014) as well as individual fostering should be considered. Especially due to the fact that many professionals in childhood education do not have the opportunity to acquire mathematical pedagogical content knowledge and skills during their pre-service education, it can be assumed that the implementation of mathematical learning in kindergarten is often based on experiences. Therefore, the facet of *knowledge* has to be considered differently than in the case of teachers, and the ability to analyse and evaluate one's own action is of particular importance. Especially because action sometimes is less based on knowledge and more on experiences, the evaluation facet has to be seen as specific for early childhood education. Not least, early childhood education in the mathematics education context demands a distinctive diagnostic view and the ability to react supportively to individuals. The ability to diagnose and support an individual – especially related to specific difficulties with learning mathematics – has to be considered as significant part of the mathematics education competence of professionals in early childhood education.

Therefore, a competence model is elaborated on the basis of previous findings of three areas: (1) different aspects of teacher competence concerning mathematics education, (2) aspects of a general pedagogical competence model of professionals in early childhood and (3) empirical results about relations of different facets of competence of professionals in early childhood education concerning mathematics education (cf. Fig. 4.2).

This model has to be seen as a structure-process model. It consists of four structural facets: knowledge, situational observing and perceiving, pedagogical-didactical action and evaluation. Potential connections between these facets are also shown in the model. So theoretical assumptions are illustrated, relating to the early mathematics education process – starting from professionals' knowledge, to their perceiving and analysing of the situation, to the realisation of spontaneous or intended learning opportunities and finally to the evaluation.

The facet of *explicit knowledge* (EK) can be found in all models and in many studies. In the model at hand, explicit knowledge describes a kind of knowledge that is implicitly present in concrete mathematical activities in kindergarten, and not that kind of knowledge that refers to general mathematical school knowledge or to mathematical background knowledge (MCK) as in many other models. One characteristic of explicit knowledge in early mathematics education is knowledge of basic ideas concerning the domain of early childhood education (e.g. numbers, quantities, shapes). Furthermore, in the sense of "horizon content knowledge" (Ball et al., 2008) another characteristic of explicit knowledge is understanding the background of the mathematical concepts (e.g. classification of plane shapes) as well as knowing the mathematical competencies beyond early childhood education in order to design successful and coherent mathematical learning (c.f. Sect. 2). Essential mathematical knowledge is integrated in the knowledge of mathematical concepts and fundamental mathematical ideas. Explicit knowledge also comprises knowing appropriate materials for mathematical learning in early childhood education, as well as suitable criteria for a choice of these materials. Appropriate materials can be,

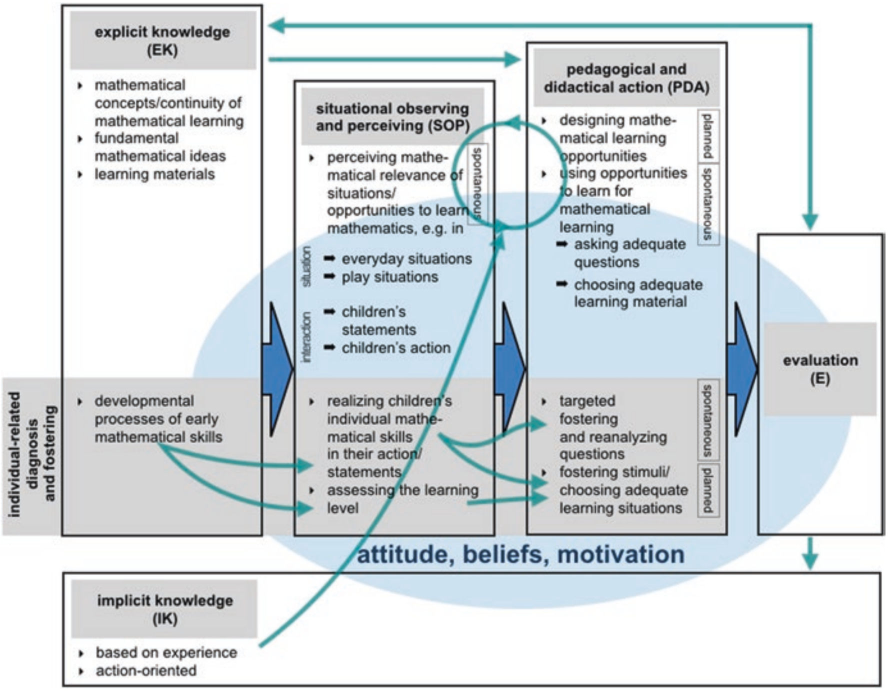

Fig. 4.2 Model concerning the mathematical didactical competence of professionals in early childhood education (Gasteiger & Benz, 2016, p. 280)

for example, counting objects, mirrors and building materials, but also parlour games or picture books. In order to be able to act supportively, in addition, diagnostic knowledge is required, as Ball et al. (2008) describe with “knowledge of content and students” (marked in the model within the grey section *individual-related diagnosis and fostering*). Professionals need to know developmental processes of mathematical abilities, such as the development of counting or comparing. In other models (COACTIV, Krauss et al., 2004), diagnostic knowledge is described as part of pedagogical content knowledge. As formal assessments are not part of the usual work in early childhood education, observing and analysing the expressions and activities of children are essential for a successful individual diagnosis. Therefore, in the model, competent diagnosis and fostering comprises more than just knowledge (see below).

Especially for professionals in early childhood education, it can be assumed that they possess an implicit action- and experience-oriented knowledge, because they often have had little mathematics education in their pre-service education. Therefore, the aspect of *implicit knowledge* (IK) was included in the model. If professionals of early childhood education have little opportunity to acquire explicit knowledge, they get their knowledge from the reflection on their experience (cf. Ruthven, 2000). Ruthven (2000) mentions this kind of knowledge for teachers. He talks about “craft knowledge” (p. 122) as knowledge that cannot be easily articulated by them and of which they might not be aware. Also Fröhlich-Gildhoff et al. (2014) point in their model to experiential-orientated knowledge as part of the disposition. In our model we assume that professionals of early childhood education can design mathematical learning opportunities or can perceive and use them situationally on the basis of their implicit knowledge (IK). A further assumption is that implicit knowledge can develop into explicit knowledge through a careful and conscious reflection and evaluation of carried out learning situations. This assumption is illustrated through the arrows from the pedagogical-didactical action, via evaluation to the explicit knowledge.

Perceiving mathematics in informal learning situations is regarded as a central ability of professionals in early childhood education in several studies (Sect. 3.4) and mostly assigned to the facet of pedagogical content knowledge (Anders & Roßbach, 2015; McCray, 2008; McCray & Chen, 2012; Smith, 2000). We regard the *situational observing and perceiving ability* (SOP) as a separate competence facet. It may be informed by knowledge (cf. Van Es & Sherin, 2008) but is not completely included by it. It can rather be considered as mediator between the required dispositions for a competent acting and the actual performance (Blömeke et al., 2015). The emphasis of this ability as a discrete facet regards the special requirement of mathematics education in early childhood education (cf. Sect. 2). Perceiving the mathematical relevance in play activities and everyday situations is a central prerequisite in order to use situations spontaneously for mathematical learning (Gasteiger, 2010). Additionally, the language and action of children often can be an ideal base for mathematical learning. However, these have to be perceived in the moment by the professionals and then have to be consciously picked up, in order to enable and enhance the children's mathematical learning and thinking. Still, the ability of a situational observing and perceiving is not only shown in recognising mathematical learning situations but is especially essential in the context of diagnosis and fostering and thus has an individual-related component. In order to nurture individual mathematical learning processes in a preventative and compensatory way, it is necessary that professionals are able to realise and evaluate the individual children's skills and knowledge in their expressions and activities (cf. Sect. 2). This competence is most likely significantly dependent on the professionals' knowledge of the way young children's mathematical conceptual growth progresses generally.

Action-related abilities constitute a further competence facet in models of mathematics teacher's competence (Baumert & Kunter, 2006; Lindmeier, 2011) as well as in the competence model for early childhood education (Fröhlich-Gildhoff et al., 2014). We specify this competence facet with the description of *pedagogical-didactical action* (PDA) (cf. also Gasteiger, 2010). The ability of mathematics-related pedagogical-didactical acting reveals itself in everyday situations of kindergarten both in designing intended learning situations and in the spontaneous use of suitable situations for learning mathematics. That is, if the learning processes of the children are encouraged and supported through adequate questions, suggestions and materials. We assume that pedagogical-didactical actions are strongly influenced by situational observing and perceiving (SOP). On the one hand, situational observing and perceiving is the precondition for which natural learning situations can be used for mathematical learning. On the other hand, it is a central element of pedagogical-didactical actions *within* each intended and emerging situationally used learning situation: Only if professionals in early childhood education perceive in the concrete learning situation both the mathematics and mathematical skills of the children and interpret them accordingly, then they can act appropriately. The influence of the situational observing and perceiving is illustrated in our model through a cycle (see the circle). This circle includes the assumption that successful learning situations, which can be influenced by implicit knowledge, in turn can contribute to the recognition of situations as mathematically relevant learning situations. A pedagogical-didactical action should thus not be seen as isolated from situational observing and perceiving. Based on this theoretical consideration, there is particular value attached to the competence facet of situational perception and observation. Dunekacke et al. (2015a) could demonstrate connections between the perception of a situation and pedagogical-didactical action in their study with prospective kindergarten teachers (see Sect. 3.4). However, in that study the same video vignette was used for the investigation of both competence facets. This might explain part of the investigated correlation. Further studies on possible relations between situational observing and perceiving and pedagogical-didactical action are necessary in order to support the thesis. These studies should analyse competencies of experienced professionals, and not only prospective kindergarten teachers and the instruments analysing SOP and PDA should respect selectivity. An individual-related diagnosis and fostering component also exists as part of the facet of pedagogical-didactical action just like in the situational observing and perceiving facet. The situational observing and perceiving of individual abilities (diagnostic aspect of SOP) represents the prerequisite for knowing how to foster individual children. The ability to diagnose and foster as part of the pedagogical-didactical action can manifest itself on the one hand in spontaneous purposeful interventional-diagnostic questions and stimulations (Steinweg, 2006) and on the other hand in the intended choice of learning stimulations, games or materials that adequately foster children's mathematical learning process.

Next to knowledge, situational observing and perceiving and pedagogical-didactical actions, the *evaluation* (E) is also seen as an important reflexive facet in our structure-process model of mathematics education competence of professionals in early childhood education. A reflexive component is grounded both in the mathematics education competence model of Lindmeier (2011) and in the general model of competence in early childhood education by Fröhlich-Gildhoff et al. (2014) as part of the performance (cf. Sect. 4). Although – as already mentioned several times – the intentional planning of mathematical learning opportunities in early childhood education happens less than in mathematics learning for older children, it is still not to be completely neglected. If learning situations are purposely planned on the basis of explicit mathematics-related knowledge, a so-called pre-instructional reflection (Lindmeier, 2011) is required. This reflexive component is illustrated in Fig. 4.2 through an arrow, which directly leads from the component of knowledge to the pedagogical-didactical actions. Knowledge about mathematical concepts and fundamental ideas will be used in the planning of games, situations in circle times or projects for coherent mathematical learning processes. Afterwards it will be implemented in these intended learning situations. For the mathematics education in early childhood education, a greater importance is attached to the evaluation of concrete situations and activities – as it is shown in the process model by Fröhlich-Gildhoff et al. (2014). If professionals evaluate their own actions regarding the progress in the child's learning and if professional analyses which factors contributed to the success or failure of the learning processes, important conclusions for future activities can be drawn. Ideally, as a result, new insights are gained, which can be either described explicitly or which at least enrich the experiences and insofar influence concrete action in early mathematics education as implicit knowledge.

The extent to which professionals succeed in using their particular abilities in their everyday work is dependent on their attitudes, their value systems and their own motivation. These connections are seen as meaningful both in the research of teacher competence (e.g. Baumert & Kunter, 2006) and also in the analysis of the competence of professionals in early childhood education (Anders & Roßbach, 2015; Fröhlich-Gildhoff et al., 2011).

Conclusion

Although there exist already studies concerning single facets of mathematics education competence of professionals in early childhood education, it must be stated that these findings are not yet sufficient for (further) development of pre- and in-service education of professionals in early childhood education. Before competence will be described for educational standards, the construct has to be theoretically justified, empirically proved and graduated, and it should possess a practical relevance (Oser, 2001, cited in Baumert & Kunter, 2006, p. 478).

The theory-based structure-process model presented here concerning the mathematics education competence of professionals in early childhood education is grounded on a differentiated analysis of the specific requirements for this kind of professionals, as requested by Weinert (2001a, S.62). In addition to that, theoretical and empirical findings from teacher competence research and research concerning competence of professionals of early childhood education form the basis of the model and are also included in the model. Thus, the criterion of a theoretical foundation (Baumert & Kunter, 2006, S. 478) is fulfilled. To what extent this model is empirically valid has to be shown yet. Klieme and Hartig (2007) demand precise descriptions of parts of the competence as the basis of an empirical measurement (p. 24). These descriptions were presented in Sect. 5 together with hypotheses about possible relations between the single facets of the theoretically based competence model. The empirical evaluation of the model is a next step that can be achieved in further studies. Here proven methods can be used (e.g. Dunekacke et al. 2015a, b; Lindmeier, 2011; McCray, 2008). However valid instruments have yet to be developed further for other facets of the competence model. This applies, for example, for the facet of explicit knowledge. Here knowledge, which is relevant for the work in kindergarten, should be examined rather than common mathematical content knowledge. Also for the individual-related facet of situational observation and perception or pedagogical-didactical action, there is need for further refinement of methods. The theoretical-based model can serve as a starting point for further research.

Note The theory-based competence model was published first in German in the Journal für Mathematikdidaktik (Gasteiger & Benz, 2016).

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Chapter 5
Stories Neglected About Children's Mathematics Learning in Play

Trude Fosse, Maria L. Johansson, Magni Hope Lossius, Anita Wager, and Anna Wernberg

Abstract In this paper we describe stories of mathematics learning in play that are often neglected in this era of schoolification and discussions of what counts as learning in early childhood. Drawing on theories of early childhood teaching and learning that emphasize the importance of teachers' (a) content knowledge, pedagogical content knowledge, and knowledge of children's development, (b) action competencies, and (c) attitudes and beliefs, we explore three stories of child-teacher interactions in play. We found that, despite different political and public perceptions of what counts as learning in three different countries, preschool teachers evidenced competencies in similar ways – each illustrating a neglected story of children's mathematics learning.

Keywords Mathematics learning · Play · Early childhood · Kindergarten · Preschool · Schoolification · Global neglected stories · Narratives · Counter-narrative · Narrative repair

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Introduction

Schoolification, what the Organisation for Economic Co-operation and Development (OECD, 2006) and the United Nations Educational, Scientific and Cultural Organization (UNESCO, 2010) use to describe the increasing pressure to make preschools more academic, is the subject of ongoing global debates. The debates taking place in the media and in academia raise philosophical, political, and ethical questions about schooling for young children. In the USA, this is particularly true for children whose skin color, language, cultural practices, and economic background differ from the white middle and upper class families who have access to well-resourced play-based preschool and to a lesser extent in Sweden and Norway. The abundance of international comparative data on education has had an adverse effect that is "fueling a competitive 'global race' where governments become increasingly concerned with national rating" (Ang, 2014, p. 188). The debates about schoolification are taking place in public and academic circles as evidenced by the proliferation of articles in popular press and in academic journals where it is argued that formal curriculum for young children devalues the early years' experience (Faulkner & Coates, 2013; Pugh, 2010; Rose & Rogers, 2012). The debates about academic push down can be particularly evident in mathematics. It is not our intent to engage in a debate on "false dichotomies" about play versus didactic instruction (Fuson, Clements, & Sarama, 2015) but to explore stories of children's learning of mathematics in play that are often neglected in public and policy arenas.

In this paper, we address this issue by examining ways in which mathematics in play or in play-like activities is getting taken up in different contexts. In particular, we are interested in preschool teachers' practices across contexts in an environment that is increasingly academic for our youngest children. We provide stories of how teachers engage with children in three preschools in Sweden, the USA, and Norway. These examples are situated in varying political and public narratives about mathematics in play and learning. We have found that despite varying political or policy climates in each country, preschool teachers share similar knowledge, competencies, and attitudes, and this is reflected in their practice. This has emerged in several ways; we have put forward three stories to demonstrate the neglected narratives from practice that do not find their way into public perception and political action.

Theoretical Perspective

Our study is at its' core grounded in a perspective that considers play not only a human right of childhood (OECD, 2006) but a critical space for children to learn and grow (cite). We use the definition of play recently espoused in the EECERA (2017) position statement where "children actively participate in constructing their play world based on their own interests and needs" rather than the interests and needs of adults (p. 2). With respect to mathematics, we are in particular interested in mathematics that is embedded in (Ginsburg, 2006) or emerges through (Wager & Parks, 2016) play, not activities that are planned by the teacher to engage children in mathematics play. In order to support and reflect on the stories of teacher-child interactions in play that are often neglected, we need a framework that gives us an understanding of the competencies needed for preschool teachers. The stories and the neglected stories that we as researchers and the teachers can see in particular in play situations is a way of highlighting the teachers' competences and also showing where it is important to support the teachers. To do this we draw on three ways of examining teachers work with children: Benz's (2016) professional competencies; Graue, Delaney, and White's (2014) improvisation; and shared thinking (Doverborg & Samuelsson, 2011; Siraj-Blatchford & Sylva, 2004).

Benz (2016) synthesized the professional competences needed for supporting children's early mathematical thinking. She found three categories of competences: (a) content knowledge, pedagogical content knowledge, and knowledge of children's development, (b) action competencies, and (c) attitudes and beliefs. According to Benz (2016) content knowledge, pedagogical content knowledge, and knowledge of children's development are the knowledge that supports the teachers in noticing children's mathematical competencies in their activities.

With respect to the first competency, teachers need to know the mathematics content which is the teacher's knowledge about understanding of what and why (Shulman, 1986, p. 9). Early childhood educators "have to see the relations between mathematics in the early years and later on to guarantee coherent mathematical learning" (Gasteiger, 2014, p. 278). Based on Shulman (1986) and Gasteiger (2014), teachers also need to have pedagogical content knowledge, in this case how mathematics content might be evident in play and how to support it. Further, they must understand and support children's development and the kinds of interactions, content, and questions that are appropriate for young children.

For the second competency, the teacher needs to not only know how to get the children to reflect on their own thinking but also "how to ask questions and communicate in order to strengthen children's understanding" (Doverborg & Samuelsson, 2011, p. 60). Benz (2016) bases the idea of action competencies on Ginsburg, Lee, and Boyd (2008) where the focus is on identifying "teachable moments." According to Ginsburg et al. (2008), this is quite challenging for teachers and, as seen in Wager and Parks (2014), this is especially challenging in children's play. One way of supporting this is to think about teachers' improvisational acts (Graue et al., 2014) wherein they respond in the moment to the play that children lead.

The third point about attitudes and beliefs is also important when it comes to noticing mathematics in children's play. Teachers need a broad view of what counts as mathematics in order to actually notice it. If the teachers do not notice the mathematics, they will not be able to tell the stories about all the mathematics that children engage in. In order for the teachers to notice or tell the stories of developing children's mathematical thinking, they need the abovementioned competencies and knowledge.

The Data Resources and the Methodology

This is a case study of three teachers, one each in Sweden, Norway, and the USA. To provide some context for these settings, a brief overview of the political and public perceptions with regard to mathematics in play and the terms used for early childhood settings is described in Table 5.1.

As is the case in many countries around the world, in Swedish preschools, play is considered the foundation for children's learning experiences (Skolverket, 2011). This is reflected in the curriculum. "Play is important for the child's development and learning. Conscious use of play to promote the development and learning of each individual child should always be present in preschool activities. Play and enjoyment in learning in all its various forms stimulate the imagination, insight, communication and the ability to think symbolically, as well as the ability to co-operate and solve problems" (Skolverket, 2011, p. 6). Connecting play with enjoyment assumes that learning will produce more easily "imagination, insight, communication and the ability to think symbolically, as well as the ability to co-operate and solve problems."

In the USA, the very notion of play in early childhood has become contested. In the not so distant past, early childhood classrooms were child (not content)-centered spaces (Elkind, 2009), but schoolification has taken hold and kindergarten classrooms have become "glorified first graders" that are increasingly standardized with limited time for play and driven by assessment (Graue, 2009). The US early childhood education system is becoming more aligned with practices in older grades that are heavily influenced by state and federal standards-based accountability movements (Brown, 2015).

The Norwegian framework plan for the content and tasks of kindergartens (Ministry of Education and Research, 2011) has play-oriented guidelines with focus on children's participation and interest. Still, there is a strong push among politicians

Table 5.1 Mathematics play and learning across contexts

Country Grades/age	Policy	Media
Sweden: Preschool (1–5) Preschool class (6) First grade (7)	Play-based goals for the preschools	Mixed message based on play but talks about school results (TIMSS and PISA)
USA: Preschool (0–5) Prekindergarten (4–5) Kindergarten (5–6) First grade (6–7)	Universal preschool Schoolification	Mixed messages about play
Norway: Preschool/ kindergarten (1–6) First grade (6)	Play-based goals for the preschools	Mixed message based on play but talks about school results (TIMSS and PISA)

and in media that children should engage in mathematics to get better school results. For instance, the Norwegian Minister of Education, Torbjörn Röe Isaksen, stated that: "Jeg mener at en enda sterkere vektlegging av matematikk kan være et godt tiltak for å snu trenden med dårlige matteresultater i skolen" (I believe that an even stronger emphasis of mathematics [in preschool] could be a good step to reverse the trend of poor math performance in school.) (Isaksen, 2014).

We draw on the idea of the counter-narrative used in critical race methodology (Solorzano & Yosso, 2002) and narrative repair (Nelson, 2001) to provide stories that counter and repair the notion that children do not have opportunities to learn and preschool teachers do not teach mathematics in play. Much as scholars use critical race methodology to study those at the margins of society (Solorzano and Yosso, p. 23), we are studying the youngest most vulnerable children who experience a different form of marginalization; who, because of their age, are not able to tell their own stories; and who have instruction done "to" them rather than "for" them (Wood, 2010). We use narrative repair (Nelson) to repair, write, and rewrite the stories that get told about preschool teachers who support children's learning of mathematics in play. We knew anecdotally there were numerous opportunities for mathematics in play learning and that these could provide evidence countering the schoolification discourse. As such, we approached our work by examining data from our studies to identify stories that provided evidence of opportunities children have to learn and teachers have to support mathematics learning in play. But beyond identifying the stories, we also aim to explore the themes evident in the stories.

To construct new narratives that counter and repair existing stories in each of our countries – what we are referring to as stories neglected – we first met to unearth themes we found common in studies of professional development that we have been involved in. Each of the authors has participated in research studies of professional development programs to support early childhood teachers to notice mathematics learning and teaching opportunities in play. We explored a subset of the data from these studies that included classroom video, teacher reflection on course work, teacher reflection on children's play, and researcher reflection. Through our analysis of the data across the projects in each country, we identified four "tropes" that emerged in all of our work with teachers: (a) the conflicting teacher-researcher narrative, (b) the congruent child-teacher narrative, (c) the conflicting child-teacher narrative, and (d) the shifting teacher narrative. We then selected a representative story of mathematics learning in play from the larger studies and examined those through the lens of Benz's (2016) categories of competencies. These stories are told through the experiences of the teacher, the child, and the researcher.

The Tropes

In all three setting, we found similar ways in which teachers involved in professional development engaged with children in mathematics during play. The four themes, or tropes, are summarized here and then exemplified in the stories below.

Conflicting teacher-researcher narratives Not surprisingly, we found that as researchers analyzing data after the fact, and even in the moment, there were instances when we saw evidence of and opportunities for teachers to engage children in mathematical thinking in their play or activities. We refer to these as conflicting teacher-researcher narratives to make evident the difference in what we see as researchers and what teachers notice in the moment. We do not intend this to be an opportunity to highlight what teachers miss but rather what is possible as we continue to work with teachers in professional development to recognize mathematics learning opportunities. These conflicting narratives tended to include two areas of conflict: what "counts" as mathematics and "where" we see mathematics in play and in children's everyday activities.

Child-teacher congruent narratives We found that those situations in which the child(ren) and teacher were in agreement met the following criteria: the activity was play oriented wherein the mathematics emerged in the play initiated by the child; both were engaged in the mathematics in the activity; and both were involved in the play. Further, there was a shifting back and forth between who was leading the activity – in other words neither the teacher nor the child was solely responsible for the direction of the play or the mathematics. And finally, communication acts were necessary to provide evidence of the interaction (but they were not always verbal). For example, there were times when there was evidence of mathematical thinking, such as a child nodding their head as they counted, that did not include a verbal exchange between the teacher and child.

Child-teacher conflicting narratives In some situations, we found that the goals of the child and teacher differed. In these cases, the teacher may have been trying to infuse the mathematics into an activity initiated by the child, and the child resisted the change to their play/activity. Teachers handled these situations by either walking away from the child's play, dropping the mathematics, or continuing to try to engage in mathematics.

Teachers' shifting narratives We think about shifting narratives as the ways teachers' stories change over time in response to curriculum, policy, public (media, parents, and community), other teachers, children, and professional development. In this manuscript, we provide examples of how teachers' narratives shift as a result of professional development and engagement with children. In all of our research doing professional development, we found that teachers' experience shifts toward recognizing the role of play in teaching and learning mathematics. And, as teachers respond to children's mathematics as they engage in play, the teachers' narratives about what counts as mathematics also shifts.

The Stories

Walking Along the Bench: Conflicting Teacher-Researcher Narrative

Looking at what counts as mathematics depends on the researchers' or the preschool teacher's attitudes and beliefs about what counts as mathematics. This view will also affect the teacher's possibility of supporting or promoting children's learning in mathematics. The following example can be found in Helenius et al. (2015) where it is analyzed using didactic space to see how the foci for the child and for the teacher changes during the course of a very short event occurring in a free play situation.

This situation takes place outside; a toddler is walking back and forth along a bench. The child gets to the end of the bench and looks down, and then the child turns and walks to the other side of the bench where the teacher is standing looking at the child. The girl's exploration could be seen as exploring space, locating in space, and learning about spatial relations such as being up on the bench above the ground, walking along the bench, back and forth, and looking down to the ground and hence is seen by the researchers as being mathematical (Helenius et al., 2015). The situation then continues with the toddler raising her arms toward the teachers; this could be interpreted as the toddler wanting assistance to get down. The teacher in this case puts her arms in a gesture that could be interpreted as the teacher wanting the child to find a way of getting down herself. Here the teacher has a goal of actually wanting the child to find a way of getting down. This could be interpreted as the teacher having action competence, she changes the situation by not acting on the child's intention but rather challenge the child in the learning situation.

The teacher might not see this activity as a mathematical situation but rather a situation where the teacher is encouraging the child to explore her motoric skills of climbing down the bench herself, so the teacher's pedagogical content knowledge might affect her actions. One reason for this could be the lack of mathematical language connected to other parts of the mathematics not only numbers and shapes. On the other hand, the researchers as seen in Helenius et al. (2015) see this as a mathematical situation, and it is categorized as such using Bishop's (1988) categories of mathematical activities. Seen from the teacher's perspective, it is not clear that she sees the situation as mathematical or that her actions actually have the goal of challenging the child's mathematical learning.

This video has also been used in different professional development courses, and in the discussions of the video, most of the preschool teachers attending this course do not at first see this as a mathematical situation; instead, it is framed as being about the child learning to get down from the bench or to develop her motor skills in climbing. So the preschool teacher's pedagogical content knowledge and their attitudes and beliefs will affect the situation, and this is why we see this story as a neglected story where the child is in fact challenged in the play situation but not necessarily in way of challenging the child's mathematical competences.

Water Table Fill and Spill: Both a Congruent and Conflicting Teacher-Child Narrative

In the following, we see within one brief exchange an example of both a child-teacher congruent story and a conflicting story. This play-oriented interaction (in other words, the mathematics emerges in play) is congruent as both the teacher and the children engage in the mathematics and the play in the activity; and there is shift back and forth between who is leading the activity. But it is also conflicting as there are times when the goals of the teacher differ from those of the children.

Nick and Ramone were playing at the water table. They had plastic buckets, cups, and funnels of varying sizes, cylinders, and one bucket with a bottom that popped off when it was too full. Nick was filling the big bucket so that the bottom would fall off and water would spill. He asked Ms. D to hold the big bucket so he could use a cup to fill it with water. Ramone joined in and held the big bucket; after the bucket was about half full, the bottom fell off and water spilled into the table. The boys found this hilarious and they put the bottom back on to try again. Ms. D asked the boys "so I'm wondering how many cups of water it takes before it spills."

Nick pours cups of water into it the big bucket Ms. D is holding. She counts each cup as he pours it in. Ramone is watching and nodding slightly each time Ms. D counts and eventually she stopped counting to have Ramone take over.

Ms. D:	*[counts each cup as Nick pours, getting louder and more enthusiastic with each cup poured in]* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *[whispers and nods to Ramon]* 13
Ramone:	14, 15, 16, 17, 18, 19 *[the boys are clearly getting excited that the water is probably about to spill, at 19 cups the bottom comes out and water pour out and all three jump back and laugh]*
Ramone:	19!
Nick:	19!
Ms. D:	19, oh my goodness *[holding up a cylinder]* What if you tried one of these?
Nick:	No *[now Ramone is holding the big bucket and Nick is filling it using two cups at a time, one in each hand]*
Ms. D:	Are you counting?
Ramone:	No, I don't want to count because I'm holding *[the bucket]*
Ms. D:	oh, okay *[Ms. D walks to another part of the classroom and the boys continue but start to disagree about what they are doing. Ms. D returns to the water table to moderate the disagreement and the boys happily continue]*
Ms. D:	Now you have two cups and are trying to fill it.
Ramone:	It will go faster.

Ms. D: Faster, you are right. I'm still wondering about… it took 19 cups, of these cups to fill it *[holding up the original cup]*, I wonder how many of these cups it would take *[holds up a cylinder]*. Do you think it will be 19 or do you think it will be a different number?

Ramone: I think, I think, I think it will be 16.

Ms. D: You think it will be 16. After Nick is done here maybe, you can try that out Ramone because it is hard to count when he is going by twos. Oh… *[The bucket spills again and now Ms. D encourages the boys to use the cylinder. Nick starts out by using two cylinders]* Let's see, let's count them and see if we get 16.

Ms. D: It will be hard to count using two, one at a time.

Ramone: *[to Nick]* One at a time! *[Ms. D starts the counting again and Ramone is mouthing the words for the count, Ms. D counts silently until they get to 16.]*

Ms. D: We are getting close, 16, did it pop off?

Ramone: Nope

Ms. D: 17, 18, 19, 20, 21 *[the bucket is full at this point and Nick pops the bottom off so the water will spill]*

Ms. D: So I think you had 21 before it flowed over the top.

Nick: I did it. I did it. I beat the record.

It is difficult in text to explain the excitement and enthusiasm the children expressed throughout this activity. In this example, we see evidence of the content knowledge, pedagogical content knowledge, and knowledge (Benz, 2016; Shulman, 1986) of children's development Ms. D possessed as she engaged the boys in counting during their play activity. And, importantly, she understood where each boy was in his mathematical understanding and build on that knowledge (Carpenter, Franke, Johnson, Torrou, & Wager, 2016). Ms. D was aware of Ramone's silent counting as she said the number sequence aloud. She also moved gently in and out of the play introducing possible ways to use mathematics – counting the number of cups and comparing different-sized cups – but did not push too far and interrupt the boys' ideas about the play. She took advantage of several teachable moments in this interaction demonstrating her action competencies or what Graue, Delaney, and Whyte (Graue et al., 2014) refer to as improvisation. With respect to Benz's (2016) third category, Ms. D clearly sees opportunities for mathematics in multiple ways in this brief interaction at the water table. There are certainly other things Ms. D could have done, such as count by two when Nick was using two cups at a time, but we can't know why she did not make that choice. Perhaps she knew she was pushing the boys as far as she could already, and perhaps she knew they were not yet counting by twos and wanted to reinforce the counting sequence. But we do know that Ms. D made some mathematically sound decisions as she asked questions during the fill and spill activity.

This story shifts between teacher-child congruent and teacher-child conflicting narratives as the teacher negotiates her place (and mathematics place) in the play. It starts with shared (congruent) engagement in the activity between Ramone and Ms. D, as they are both interested in counting the cups to determine how many it will take to spill out. They are also interested in knowing if the number of cups would differ if they use a different size cup. The story shifts to a conflicting narrative as Ms. D asks Ramone if he is counting and he replies, “No, I don’t want to count because I’m holding.” At this point Ramone has moved from an interest in the mathematics of the activity to the fun of holding the bucket. The narrative shifts back toward congruence when Ms. D returns to the table and again engages Ramone in thinking about the mathematics.

The Stone Story: The Shifting Teacher Narrative

In the following, we present a play-oriented activity that is in opposition to the schoolification process. Because it shows how children engage from an adult initiated activity to a child initiated activity. The spontaneous conversation shifts from an adult-guided counting activity to a child-oriented measuring activity.

Below are excerpts from a Norwegian preschool teacher and her reflection over her ability to support children’s learning. She tells about an incident where she had planned an outdoor activity with counting and sets with use of one die.

Excerpt 1
[The die showed four dots and all the children ran around finding four objects. Then two children started arguing.]

Child 1:	My stone is bigger than yours.
Child 2:	Is it?
Teacher:	*[What now, what with all my plans!]*
Teacher:	How could we work it out?
Child 1:	We must measure. We can hold the stones next to each other.
Teacher:	Yes that was smart.
Child 2:	They are the same length.
Child 3:	But, how long in numbers are they?
Teacher:	How can we work it out then?
Child 3:	We must find something to measure them.

[The children measure the length of the stones using a folding ruler]

In the reflections after, the teacher said: “I thought, what now, what with all my plans? It was so difficult not to interrupt the children, but I managed to follow the children’s interests.” Her goal and plan was to look at the children’s competencies in counting and sets, but she was able to support children’s activity when it shifted to measuring discussed as improvisation by Graue, Delaney, and Whyte (Graue et al., 2014).

Shulman (1986) defined “pedagogical content knowledge” as knowledge about teaching and not just knowledge about content. In this excerpt the preschool teacher managed to support children’s curiosity and the children’s desire to explore mathematical connections. Her ability to modify her plans to encourage the children’s participation is included in the concept pedagogical content knowledge. Such flexibility is part of being a preschool teacher.

The incident continues:

Excerpt 2

Child 1: My stone is thicker than your stone.
Child 2: We have to measure.
Child 1: Yes, but it is impossible for the folding rule to bend.
Teacher: But, how can we measure the thickness of the stones?

[The children struggle to measure the circumference using a ruler.]

Child 3: We can take two blades of grass and put them around the stones and then we can see which one are the longest. *[The children measure the circumference of the stones with blades of grass.]*
Teacher: That was smart.
Child 1: But we cannot exactly see numbers on the blade of grass.
Teacher: No, you're absolutely right. Can you look in the bag if there is anything we can use to measure the thickness of the stones?

[Having thrown everything on the ground and examined several of the objects, at least for 10 min, I thought the children were distracted and the measurement activity forgotten.]

Child 5: Here is something with numbers that are soft and we can bend *[looking at a measuring tape.]*

By following the children's initiative, the preschool teacher's goal for the activity shifts from hers to the children's. For example, the preschool teacher is asking them questions like "How could we work it out?" and "But, how can we measure the thickness of the stones?" These questions are supportive to the children's initiative. Instead of trying to guide them back toward her own goal, she let them take charge in the new activity. She is taking the role as a supporter for the children. In the dialog, we find the preschool teacher's utterances like "That was smart" and "No, you're absolutely right. Can you look in the bag if there are anything we can use to measure the thickness of the stones?" Not only is she supporting but she is also guiding them toward their goal. The preschool teacher shows us that she has pedagogical content knowledge and content knowledge (Benz, 2016; Shulman, 1986) about mathematics. She shows content knowledge when she supports children related to the subject numbers and measuring in this incident.

Doverborg and Samuelsson (2011, p. 60) emphasize both to know what early mathematics can be and know how to communicate and challenge children as important aspects of teacher knowledge. In this incident, the preschool teacher invites to "shared thinking." Siraj-Blatchford and Sylva (2004) define sustained shared thinking as "an interaction where two or more individuals 'work together' in an intellectual way to solve a problem, clarify a concept, evaluate activities, or extend a narrative" (p. 718). The communication between the preschool teacher and the children show us that the preschool teacher presents the children with problems like "But, how can we measure the thickness of the stones?" and the children responds.

Further:

Excerpt 3

Child 2:	My stone is so thick, *[the child holds its finger at digit 9 on the tape measure].*
Teacher:	Yes 9 centimeter thick.
Child 4:	How thick is your stone?
Child 1:	Mine is so thick, *[the child explains and keeps both hands around the stone].*
Child 4:	Yes but how many meters is it?
Child 1:	*[Put the tape measure around the stone and asked]* How thick is this?
Teacher:	It is 12 centimeter. Which stone is than thickest?
Children 1–5:	This *[everyone is pointing to the stone that is 12 centimeter].*
Teacher:	Which stone is the heaviest, do you think?
Child 1:	The one that is thickest is the heaviest because that how it is with humans.

In this excerpt, the teacher gives oral response to the child's experience when the teacher answers, "Yes 9 centimeter thick." Both the digit "9" and the unit "centimeter" are new for the child. This experience may contribute to the child's interests for numbers and measurements. By reading the numbers for the children, the preschool teacher identifies a teachable moment (Benz, 2016; Ginsburg et al., 2008, p. 7), a situation that might promote learning.

The preschool teacher helps the children to be on track by referring to the first problem: "Which stone is the heaviest, do you think?" She connects the children's previous and current experience in order to solve the original problem from the child 1: "My stone is bigger than yours."

Preschool teachers reflect on her experience:

> "Earlier I thought of the goal for the activity, but it was my thought about the goal. In my head, I focused on my written plans for activities and my aim with the activities. Now I realize that there is so much learning for the children if I listen and pay attention to the mathematical ideas that they express through play and conversation. I think there might be even more learning for the children if I pay more attention to them and their interest even when it is in conflict with my goal for the activity."

Here the preschool teacher reflects on her ability to facilitate learning opportunities and her knowledge for teaching. When she tells about her reflection, she develops her content knowledge about mathematics but also her pedagogical content knowledge (Benz, 2016; Shulman, 1986). "Noticing children's mathematics can be a way of respecting children and engaging with them to promote greater and deeper understandings" (Dockett & Goff, 2013, p. 774).

According to Benz (2016) the preschool teacher's attitude to children and their learning is an important part of being a preschool teacher. In the preschool teacher's self-reflection, there is a positive attitude toward the learning possibilities in the children's initiated activity. In her reflection, she acknowledges children's own exploration in play activities.

Discussion

The push for schoolification in the media and academia may affect the kinds of interactions, support, and engagement preschool teachers have in children's play and everyday activities. Schoolification, often characterized by teaching for the future, is in opposition to our examples that focus on teaching in here and now situations. The three findings across these stories are about teacher knowledge, teachable moments, and teachers' attitudes with respect to engaging children's learning of mathematics in play. These stories and findings are global – they come from different countries but each could have happened in any of these countries.

Preschool teachers' knowledge about how to engage children in mathematics in play is sometimes evident and sometimes it is not. For example, in the bench story, we as researchers see the math in the situation but it might be that the teachers does not see it or notice it because they may not have the language to describe what they are doing or we do not ask them about it. In the water table example, we are making assumptions about what the teacher knows based on what she does, and in the stone example the teacher reflects about how to engage children when she is telling about it. In our three stories, we have highlighted the value of flexible preschool teachers that have knowledge about what is mathematics for children discussed as content knowledge. In the bench story, the teacher challenged the child in the learning situation but not necessarily in the mathematical situation, maybe because of lack in pedagogical content knowledge. Also in other studies (Svensson, 2016) using Bishops 6 mathematical activities, the most common activity is counting and measuring. In the water table story, the preschool teacher understood that to encourage Ramone to count, she had to start first – she had knowledge of Ramone's counting skills and also his hesitancy to demonstrate those skills. In the stone story, the preschool teacher listens to the children and supports their exploration of measurement even if her original idea was to facilitate their learning in numbers and sets. This supports the argument that she has content knowledge about different topics in mathematics and is able to switch between these topics. The preschool teacher shows pedagogical content knowledge by listening to the children and challenging them with questions.

Teachable moments or actions are evidenced across the three stories in the ways that teachers respond in the moment to children's play. In the bench story we can see that the teacher notices the child's action and interacts with the child in a learning situation, even though it is not clear from the example that the teacher sees this as mathematics but still handles the situation as a learning situation. In some cases, the teachers support the child's thoughts by responding through oral response and in some cases with actions. In the bench story, the teacher challenged the child in changing the way of acting, i.e., not following the child initiative. So the teacher is initiating a new direction for the situation where the child is challenged. In the water table story, the teacher seizes on an opportunity to engage children in counting in an activity they have started. In the stone story, the teacher gives oral response "Yes, 9 centimeter thick" when the child holds her finger at digit 9 on the tape measure. Perhaps the child did not know how to read the symbol "9," and the teacher grabs the opportunity to extend the child's understanding of the concept of "9."

The three stories provide evidence of teachers' attitudes about what counts as mathematics in play and when children should be left to play or encouraged to engage in mathematics. In the bench story the teacher's attitude about what counts as mathematics will affect her actions and the way she challenges the child. In the water table story, the teacher wants the children to count the cups of water. That is not their initial intention but they willingly engage in the counting when she makes it about a challenge – how many cups will it take? But she also stops encouraging the counting during the times the children don't take it up. In the stone story, the preschool teacher supports the children's curiosity when they wonder who has the biggest stone. She pays attention to children's interests and their motivation for learning.

Conclusion

The goal in this article is to give language to neglected stories of mathematics teaching and learning in play to counter and repair those stories told by parents, politicians, and the media. These parties are telling early childhood teachers what to do, and the best stories are probably the ones they already do. These stories of repair empower the preschool teacher and the child by "reclaiming" their agency as teachers and learning of mathematics (Nelson, 2001).

Our aim is to highlight a different aspect of mathematics in early childhood and to develop language among preschool teachers to talk about mathematics in play activities. We describe these as neglected stories in the larger narrative that children do not have enough mathematics learning opportunities in play, which would suggest a need for schoolification. The teacher in the *bench story* tells a story of a play situation, which could be seen as a mathematical situation by the researcher, but the child is still challenged in her learning even though the teachers might not have seen this as a mathematical situation. If the teachers do not notice the mathematics, they will not be able to tell these kinds of stories. In the *water table story*, the preschool teacher sees mathematical learning opportunities in multiple spaces in her play-based classroom. She does engage in more didactic activities that she plans but also supports children in rich mathematical discussions during play. Stories such as this where the teacher spontaneously follows and leads the children's play to support their mathematics learning are not often shared in public spaces. The preschool teacher in the *stone story* tells us a narrative about a planned play activity that changed after children's engagements. This is a neglected story because stories of mathematical activities are usually about those that are well planned, organized, and instruction based. The measurement activity deviates from the original plan. Those stories we tend to tell are stories that are well planned, and the aim of the activities are fulfilled.

The stories, coming from three different countries, indicate that preschool teachers share similar knowledge, competencies, and attitudes despite varying political or policy climates in each country. Furthermore this state of affairs rubs off on teachers' practices regardless of different preschool situations occurring in different countries.

The stories we have shared provide an important counter-narrative and repair to narratives encouraging schoolification by demonstrating the possibilities of mathematics learning that can happen in play. In each example we show how neglected narratives of children's learning need to be shared not only among scholars but also the public and policy makers that drive what "counts" as mathematics learning. In this way, we may empower the preschool teachers' knowledge and children's learning of mathematics. Maybe if these stories of what happens in preschool were not neglected, the various public and policy approaches to schoolification may happen to a lower degree. Then policy and public approaches could change to match what teachers see as best for children related to learning of mathematics in play.

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Part II
Development of Children's Mathematical Competencies

Chapter 6
The Genesis of Children's Mathematical Thinking in Their Early Years

Götz Krummheuer

Abstract In a longitudinal study about the development of mathematical thinking of children ages 4–6, a first comparative analysis of the participation patterns of one child over this period of time in different peer situations of mathematical play and exploration has been completed. The theoretical background and the accomplished results will be presented.

Keywords Development of mathematical thinking · Longitudinal study · Interactionism · Situational perspective · Collective argumentation · Mathematics learning support system (MLSS) · Framing · Early Steps in Mathematical Thinking (erStMaL) · Narrative discourse · Narratory discourse · Narrative mode of thinking

Introduction: Comments on Some Common Sense Assumptions About Early Mathematics Learning and the Structure of the Paper

My focus is on the genesis of mathematical thinking of children ages 4–6. A widely spread opinion says that our children grow up in a culture where mathematics is everywhere. Thus, for the children to learn mathematics, it is rather a kind of gathering of this mathematics found in their environment and making sense of it. In the discussion in mathematics education, there is no doubt that these gathering and making-sense procedures are deeply affected by the culture the children are socialized in. In short, the children benefit from the culturally shaped representation of mathematics in their everyday life.

The discussion about this *common sense* in the community of mathematics education emphasizes the importance of the language in which the all-present mathematics is represented by concepts and statements about the relationship among

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them. Thus, focusing on mathematics learning in the early years of interest, a close theoretical connection is apparent to the child's acquisition of its mother tongue. The children do not contact all the mathematics directly and cannot somehow make sense of it. They need an emotionally warm and cognitively challenging social environment in which they can ask questions, formulate hypotheses and can argue for their ideas concerning mathematics, and receive supportive responses. With reference to Bruner's concept of "Language Acquisition Support System" (LASS) describing the social conditions for language acquisition (Bruner, 1982, 1983, 1985), we speak with regard to early mathematics learning processes of a "Mathematics Learning Support System" (MLSS; Krummheuer, 2012, 2013, 2014b).

My following delineation is concerned with questions, how such a MLSS is accomplished in the process of interaction in which the children are supposed to deal with a given mathematical task and how, in a longitudinal perspective of 2–3 years in preschool and kindergarten age, the genesis of mathematical thinking of children proceeds. The paper has the following structure:

1. The concept of "mathematical thinking"
2. The research context
3. The momentarily reached theoretical insights
4. Some deeper theoretical reflections on these insights
5. Further information about the research project

This structure allows the reader to gain an easy understanding of this research approach at the end of the third section. Deepening arguments follow in section "Momentarily Attained Theoretical Insights" with respect to the interesting theoretical aspects and in section "Some Theoretical Reflections" the underlying research design.

Mathematical Thinking

The theoretical approach that is referred to here as interactionist is based on three basic assumptions:

1. The subject matter to be learned and the learning conditions that are necessary to its acquisition are *situationally*[1] bound in interactive exchange between the participants in the process of the negotiation of meaning.

[1] By the term "situational,"I refer to a differentiation that stems from Goffman (1963): "Work tasks that an individual performs while others are present he can sometimes perform equally well when alone. This aspect of activity may occur *in* situations but it is not *of* situations, characteristically occurring at other times outside situations. This unblushing part of reality I will refer to as the *merely-situated* aspect of a situated activity. … my only interest in such matters will be to be able to segregate them analytically from the component of situated activity that will concern us here; namely, the part that could not occur outside situations, being intrinsically dependent on the condi-

2. The constitutive social condition of the possibility of learning of a mathematical content, concept, or procedure is the *participation in a collective argumentation* concerning the content, terms, or other procedures.
3. The expression of a successful process of learning of a child or a pupil is the *increased autonomous participation* in such collective argumentation in the process of a current interaction and/or in the following interaction that is thematically imbedded in the actual situation.

I do not want to deepen these assumptions too much here (Krummheuer, 1995, 2009, 2013, 2014a). However, I want to stress the second topic of argumentation that seems to me the most crucial issue with regard to our studies about children's mathematical thinking. I mentioned Bruner's concept of LASS and the adaption of this approach for mathematics learning, the MLSS. Bruner emphasizes that the acquisition of one's language is not sufficiently comprehensible if one only looks at sole linguistic aspects like the increase of vocabulary and the incremental use of correct grammar. Language acquisition is not restricted to such a "cracking of a linguistic code" (Bruner, 1982, p. 14), but it is widely related to learning to cope with the "demands of the culture" (Bruner, 1983, p. 103). Taking over this argumentation for early years mathematics learning, one can differentiate between the acquisition of early mathematics concepts and procedures in the sense of cracking the mathematical code and the development of mathematical thinking in the sense of an examination with the specific features of mathematical discourse, their rationalizing practice, and the mathematical culture lying beyond it.

An example of a mathematical concept might be the cuboid. When we talk of mathematical thinking, we associate the reasoning, the specific kind of explanation that a child connects with its activities concerning a mathematical concept. This we describe in Goffman's terms as a "framing" process which is a stabilized way of defining a situation (Goffman, 1974; Krummheuer, 1995, 2007). For example, a child might frame a situation together with other children in which it is supposed to build a construction of cuboids according to a given picture as an arithmetical situation. If it focusses on the amount of bricks that it still needs to complete its construction by counting, let's say, two more to the six it already used for its cuboid-building, then we would ascribe to the child's thinking an arithmetical framing. If, however, the child is framing the situation in a geometrical way, it also might need two more bricks. However, it would argue according to the geometrical purpose, for example: if I put these two bricks horizontally on top of the already built vertical columns, I would have a bridge like on the picture.[2]

By means of the concept of framing, one can differentiate between:

- The *content*, which is ascribed to a certain mathematical domain, like arithmetic, geometry, etc. and its related operations

tions that prevail therein. This part will be referred to as the situational aspect of situated activity" (p. 21 f.).

[2] This example stems from Brandt and Krummheuer (2015).

- The *culture of argumentation*, which is the content specific way of explaining and justifying one's actions

  Based on this differentiation, we use the concept of:

- "Acquisition" when we speak of the mathematical concepts and procedures at hand
- "Learning mathematics" when we point at the process of the cognitive construction of a new framing
- "Mathematical thinking" when we refer to the emerging argumentative practice in which the "logical" derivation and application of these concepts and operations take place.

By mathematical thinking we mean the transsituational, argumentatory elements of a child's framing of a mathematical situation.

The Research Context

This research is based on the longitudinal study "early Steps in Mathematics Learning" (erStMaL). This project is concerned with the development of mathematical thinking in preschool, kindergarten, and early school years in which we follow children in 12 daycare centers over a period of 4 years in which they are observed every 6 months.[3] The funding period ended after 6 years in 2014.[4] For these analyses Marcus Schütte and I selected several children for deeper scrutiny. Currently, we are working on our longitudinal comparisons concerning the first child we chose. We call her Ayse.

Ayse is the only daughter of Turkish parents who were born and went to school in Germany. Both parents work. The grandparents, who have immigrated to Germany, care for the child during the day. Ayse is 4;02 years old at the first time of observation and 6;04 years old at the latest episode. In accordance with the design of the erStMal study, Ayse participated several times in varying settings of play, designed as discovery situations dealing with the content areas of number and operations, geometry and spatial thinking, measuring and size, and data and probability. In both analyzed episodes, the content areas can be classified as "data and probability" and "geometry and spatial thinking." Here, I refer to three episodes in which Ayse is a participating child. They deal with permutation, measuring, and elementary topology. More information about the research design is found in section "Some Theoretical Reflections"

[3] For more details see Acar Bayraktar, Hümmer, Huth, Münz, and Reimann (2011).

[4] A group of two other colleagues and several research assistants conducted this project. For the part of the study that I especially was engaged in, I depended very much on the cooperation of Birgit Brandt, Rose Vogel, Anne Hümmer, Ergi Acar Bayraktar, Melanie Beck, Melanie Huth, and, specifically for recent comparative analyses, Marcus Schütte.

Momentarily Attained Theoretical Insights

Looking through the results of our analyses of these episodes, we reconstruct different kinds of discourses in which the children participate to different degrees. The most clearly recognizable forms are the:

- "Narrative discourse"
- "Formal discourse"

As a *narrative discourse*, we understand sections of the conversation in which the participants accomplish a collective argumentation that in its entirety constitutes a narrative or at least generates a sequence of statements that resembles a narrative structure. In former presentations and publications, I mentioned this phenomenon by referring to the notion of "narrative argumentation" (Krummheuer, 1999, 2009, 2013). It is the explanatory potential of a story that convinces the participants of the verisimilitude[5] of the presented result. In a *formal discourse*, in contrast, the conversation refers to the concretely present objects, like animal figures, wooden bricks, a toy train, etc., and to attempts to act with these objects according to extrinsic relationships.

The following example might help clarify this differentiation (Fig. 6.1).[6]

Fig. 6.1 Seating arrangement

[5] I refer here to Bruner's concept of "narrativity," which he does not only see as an "expository act" but also as a rhetorical one (Bruner, 1990, p. 87). In this context he also introduces the notion "verisimilitude," when he, for example, formulates ".. when reasons are used in this way, they must be made to seem not only logical but life like as well, for the requirements of narrative still dominate. This is the critical intersection where verifiability and verisimilitude seem to come together" (ibid, p. 94; also Bruner, 1996).

[6] For this example see also Vogel (2014) and Vogel and Huth (2010).

The situation of play and exploration has to do with a question from combinatorics, which concerns the different order of three animal figures when they walk across a platform. The adult introduces the episode by talking about the platform from a circus. Without being asked, Ayse mentions that she has already seen a circus (possibly only in television) and that there were elephants and clowns. Also, Kai mentions some of his experiences with a circus. The adult B shows three, until now concealed, toy animals: an elephant, a monkey, and a white tiger. Hardly had the elephant been placed on the carpet, Ayse takes it and gives it back only after the adult intervenes. Perhaps to her own relief, Ayse offers Kai the monkey.

At this point in the scene, we can say that a narrative discourse has been accomplished. This changes as B, after she placed the white tiger on their carpet, explains that it is a "baby tiger," as the other two are supposed to be too. It appears as if the redefinition of the animals to baby animals introduces a change in a rather formal discourse. In this discourse the specific qualities of the animals do not play a role anymore, and their sameness, in the sense of arbitrary objects of a set, is highlighted as a central theme: they are all babies and are thus not dangerous – just "mathematical objects."

Looking through our analyses, we can show that Ayse is more active in phases of a narrative discourse, while she more silently observes in phases of a formal discourse.

When looking at our analyses from a longitudinal perspective, we were able to reconstruct a third form of discourse, which we call "narratory" or "narrational." Narrative means "having the nature of a narration" (Webster, 1983). It is a particular way of explaining or understanding an event (Cambridge Dictionary online): by "narratory" we want to point to the act that someone refers to a narration by giving an account on certain aspects of a narration

Again, a second example might help to understand this differentiation between narrative and narratory (Fig. 6.2).

For this initiated play and exploration situation, four children, Ayse (6;04), Barbara (6;02), Elias (6;10), and Norbert (6;01), as well as an adult B are sitting on the floor. This scene takes place about 2 years after the episode discussed above. The participants have a wooden train as play material in front of them. B initiates a formal discourse by asking Ayse to place a red figure of a man "inside the railroad system." Ayse refuses to try. Barbara offers to try and places the figure on top of the track so that it fits somehow between the two rails. B asks several questions which none of the children answer directly. B then continues: "so now it lies on the track. I meant that the … stands within this circle (so that) the train always runs around him." At the same time, B makes clear circular movements with his right hand above the oval circle.

Here B enunciates a connection between the train and the topological feature "inside": "standing in the circle" can be recognized by the fact that the train can always run around it. This is not only expressed in relationship to the material in that he speaks of the oval track as a "circle" and no longer speaks in a more formal language of the "railroad system" but also in a flow of activity describing that the train

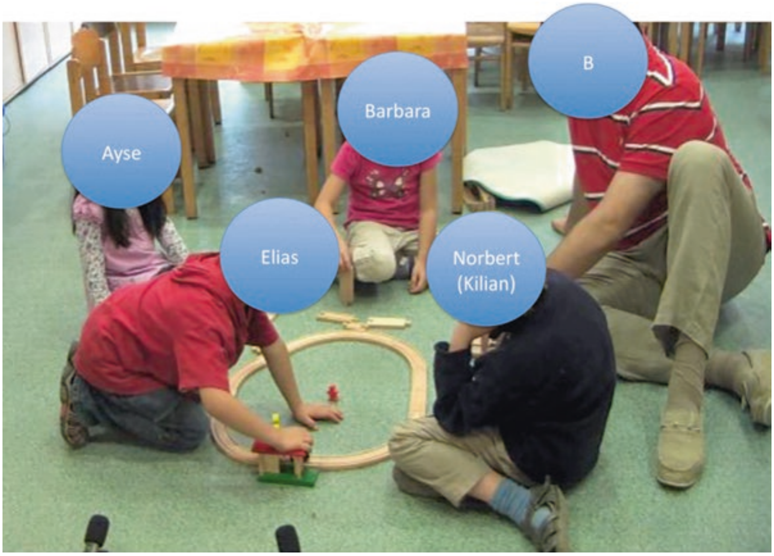

Fig. 6.2 Seating arrangement

is always running around the figure B is not accomplishing a narrative but referring to an account of events that would happen in a story about a train that circles around the given tracks. This is the beginning of narratory discourse.

In our analyses, we find that Ayse is not only active in narrative discourses but also in narratory ones. Whereas she acts in the narrative ones relatively autonomous by imposing her own, original idea, she participates in the narratory discourse rather in closer cooperation with her cohorts. We can characterize this way of participation as "co-active." Thus, we have in Ayse a child who, over a period of about 2 years, constantly takes the participation status of a "legitimate peripheral participant" (Krummheuer, 2011a; Lave & Wenger, 1991) as soon as the discourse turns into a formal one and is more actively participating in narrative and narratory discourses.

Some Theoretical Reflections

In the following I will clarify some theoretical thoughts regarding the results of our analyses.

We, as a research team, started with the implicit assumption that mathematical thinking is inevitably connected with a successful participation in formal discourses and that the development of mathematical thinking would best proceed when a child participates with increased autonomy in such discourses. The ongoing limitations of Ayse's participation in formal discourse made us more cautious about this implicit assumption. It could be that:

(a) The adaption of a formal discourse can take years or might even never happen for some children.
(b) The genesis of mathematical thinking in children's early years passes through a variety of transitory ways of thinking.

Bruner (1986) talks about "two modes of thought," an idea that seems very promising in critically reflecting and deepening our own thinking about the issue of the development of mathematical thinking. He introduces two different modes of "cognitive functioning":

- Narrative mode
- Paradigmatic mode

The narrative mode refers to "good stories, gripping drama, believable ... historical accounts. ... It strives to put its timeless miracles into the particulars of experience, and to locate the experience in time and place. ... The paradigmatic mode, by contrast, seeks to transcend the particular by higher and higher reaching for abstraction, and in the end disclaims in principle any explanatory value at all where the particular is concerned" (ibid., p. 13). Bruner proposes that these are two qualitatively different modes which do not coincide and which hold two different kinds of causality. He explains this with the term "then." Going back to the example with the platform of a circus above, it means the following: in the narrative mode, a child would argue, "first I let the elephant run over the platform, *then* the monkey, and finally the tiger." In a paradigmatic mode, it could argue: "If the elephant is in the first position, and the monkey at second, *then* the tiger is in the third position." The causality or rationality – as we would say within our theoretical framework – of a *narrative account* is its "lifelikeness," the *verisimilitude* of that what is told could really have happened. The rationality of a *paradigmatic account*, in contrast, is based on *logically deduced consistency and noncontradiction*.

Obviously, the paradigmatic mode of thinking seems to be the most appropriate one for successful participation in a formal discourse. Stringent mathematical arguments might work the best for MLSS, and the options for participation appear to be positive. Similarly, we assume that a child that thinks according to the narrative mode has positive options for participation in a narrative and a narratory discourse, but a formal discourse might not function as well for this child as a MLSS.

Thinking of a child like Ayse, one inevitably asks oneself, what might be a reasonable course allowing her to increase her options of active participation in formal discourses? When we look at our analyses, we find over the time span of 2 years, as we reconstructed the situations of play and exploration that Ayse was involved in, the amount of narratory discourses increases and Ayse's contributions in these discourses are co-actively integrated in joint action with her mates. This led us to the formulation of the following hypotheses:

A possible trajectory enabling a child to shift its narrative mode of thinking into a paradigmatic mode of thinking about mathematics might proceed by giving her an increased chance of participation in narratory discourses. We assume that its mode of thinking is still a narrative one, though there is already a process of conversion from the concrete action mentioned in a narrative to some formal aspects that are characteristic for this story. Thus, an awareness of how a formal discourse is going to proceed might develop.

A quote from Bruner (1986) might support this hypothesis:

> We all know by now that many scientific and mathematical hypotheses start their lives as little stories or metaphors, but they reach their scientific maturity by a process of conversion into verifiability, formal or empirical, and their power at maturity does not rest upon their dramatic origins (p. 12).

For me in Bruner's quote the most important information is that he speaks of a "process of conversion." This is what a child has to cognitively develop and situationally test and modify. From the *perspective of a paradigmatic mode* of thinking, the events that are picked out as themes in a narratory and a narrative discourse are just concretizations of mathematical concepts. From the *perspective of a narrative mode of thinking*, the events thematized in a narratory discourse and/or a formal discourse are only comprehensible by cognitive processes of conversions. Possibly, for a child still bound to a narrative mode of thinking, these conceptual conversions for a successful participation in a narratory discourse are easier to perform than those for a successful participation in formal discourse. Possibly, we have to face the fact that there are some children who will not perform the final conversion to step into a formal discourse as full participants. Perhaps only the encouragement through a MLSS might open this process.

Further Information About the Research Project

The two examples stem from the longitudinal study "early Steps in Mathematics Learning" (erStMaL). This research project of the research center ***I**ndividual **De**velopment and **A**daptive Education of Children at Risk* (IDeA; www.idea-frankfurt.eu) is concerned with the development of mathematical thinking in the preschool, kindergarten, and early primary school age. Empirically, we followed 144 chosen children in 12 daycare centers over a period of 4 years with fixed times of observation every 6 months. The daycare centers represented the whole social economic spectrum.

According to Goffman's (1963) conception of "situational," we always let the children interact in social group settings, usually accompanied by an adult person, "B," fluctuating over time in the episodes within the framework of the project. From the 144 children engaged in this project, we selected at the beginning 72 children, who were supposed to be constantly integrated over the period of 4 years in these group settings. One half of the children were chosen with German as their first language (L1) and the other half as children with German as their second language (L2). We also were careful that girls and boys were equally represented. All group sessions were videotaped. Finally, there were only 14 children who took part in all sessions over the whole period of 4 years, due either to sickness of the child on the day of recording and/or that the families had moved.

For the initiated group sessions, the research team developed mathematical learning environments that were called "situations of play and exploration." They were consistently designed according to a previously defined "design pattern" (Vogel, 2014).[7]

The whole project embraced both quantitative analyses and qualitative analyses.[8] For the specific research interest of gaining insights in the interactional components of the development of mathematical thinking as formulated in this article, we conducted a qualitative research approach and choose those children of the mentioned 14 children, who were relatively active in the video-recorded sessions. These sessions were transcribed – the protocols serving as a basis for our qualitative analyses.

One of these children was Ayse. Over the years, we observed her in eight group settings during preschool and kindergarten. Our research interest is focused on the changes in her participation in these settings. Methodologically, we refer here to the analysis of interaction, analysis of argumentation, analysis of the production design, and the analysis of the recipient design (Krummheuer, 2007, 2011a, 2011b, 2014a, 2015).

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[7] Here, the notion of "situation" " refers to a common terminology of design research and differs from the concept of a "social situation" as used in Goffman's definition of "situational" (see above).

[8] As an overview of all research activities, see Brandt, Krummheuer, and Vogel (2011) and the special issue "Alternative perspectives on learning mathematics in the early years" in the Educational Studies in Mathematics Volume 84, No. 2, October 2013.

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Chapter 7
Visual Structuring Processes of Children When Determining the Cardinality of Sets: The Contribution of Eye-Tracking

Priska Schöner and Christiane Benz

Abstract Research claims that perceiving structures in visual presentation of sets is an important ability for children's numerical development. However, it is not easy to investigate whether and how children perceive structures. In this article, we analyze theoretically the processes of perceiving sets and determining the cardinality of sets and discuss possible benefits of the eye-tracking tool to get some insights into these processes of preschool children.

Keywords Perceiving structures · Determining the cardinality of sets (structural) subitizing · Eye-tracking · Preschool education · Early mathematics education

Introduction

In children's lives, structures play an important role – not only for emotional security and emotional development but also for cognitive development. Perceiving, recognizing, and using structures are seen as fundamental abilities especially for mathematical development. The more children's own idea of structuring and "internal representational systems (…) [have] developed structurally, the more coherent, well organized, more mathematically competent the child will be" (Mulligan, Prescott, & Mitchelmore, 2004, p. 394). Therefore, Mulligan and Mitchelmore point out that structure is not only in the focus of research on children's progress with respect to the "development of spatial abilities," but it "has been also a growing theme in the past two decades of research on students' development of numerical concepts" (Mulligan & Mitchelmore, 2013, p. 31).

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The Importance of Perceiving Structures for Numerical Development

Numbers and mathematical relations are abstract and not concrete. Yet, to illustrate the abstract concept of numbers, collections of concrete objects often are used to help children build mental conception of numbers. In mathematics education, there is broad consensus that next to an ordinal and cardinal understanding the part-whole understanding of numbers is a very important concept in numerical development (Benz, Peter-Koop, & Grüßing, 2015; Fritz, Ehlert, & Balzer, 2013; Krajewski, 2008) that also forms the foundation for later calculation strategies (Padberg & Benz, 2011). In the part-whole concept, numbers are seen as compositions of other numbers (Gerster, 2009, p. 267). Therefore, to illustrate the part-whole concept of numbers, visual presentations (e.g., sets of objects) with structures are regularly used as models for combinations of groups and not only single items. Söbbeke (2005) describes the act of perceiving and using structures in such visually noticeable illustrations of numbers (collections of concrete objects) as *visual structuring ability*. This can be assumed as a precondition for a part-whole concept of numbers. Next to an association between the visual structuring ability and part-whole understanding (Gaidoschik, 2010; Young-Loveridge, 2002), there is further empirical evidence for the relation of visual structuring ability and the numerical development. For instance, Hunting (2003) found that the ability to change the focus of every single item to perceiving and identifying structures of parts is important for numerical development. Moreover, van Nes (2009) observed a strong association between the numerical development and spatial structuring abilities of children aged 4–6 years, whereas Lüken (2012) found an association between an early structure sense and arithmetical competencies. These research results underline the importance of visual structuring abilities when children deal with visual presentations of numbers in the form of sets of objects. In order to describe and analyze visual structuring abilities in detail when children identify cardinality of sets, we distinguish theoretically between two different processes: the process of perceiving a set and the process of determining the cardinality.

Perceiving Structures and Determining Cardinality of Sets: Two Processes

Both the process of perceiving a set and the process of determining the cardinality can be distinguished into three subgroups. These two processes and their possible relationship are illustrated in Fig. 7.1. The model is developed by an inductive approach (cf. Benz, 2013; Benz et al., 2015, p. 134) and the result of a first evaluation. The two processes can run one after the other or coincide with each other. The blue boxes in Fig. 7.1 show the different possibilities of perceiving a set of objects.

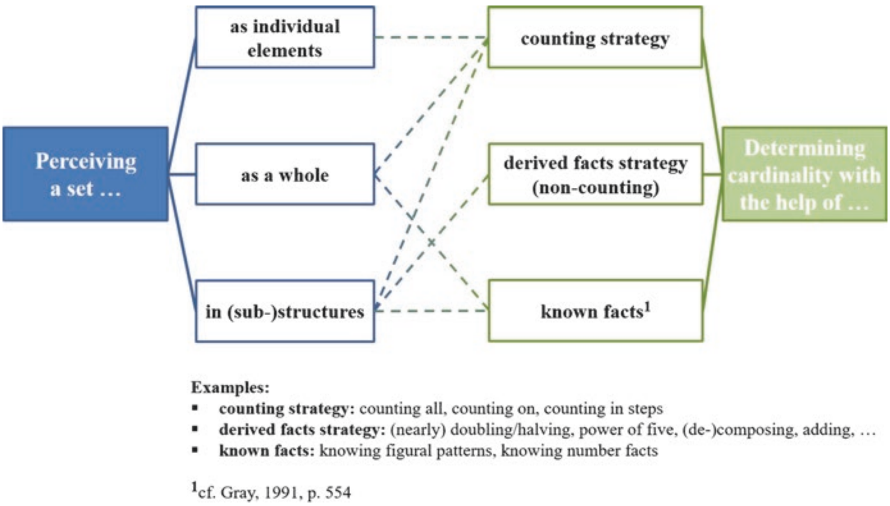


Fig. 7.1 Two processes: perception of sets and determining cardinality

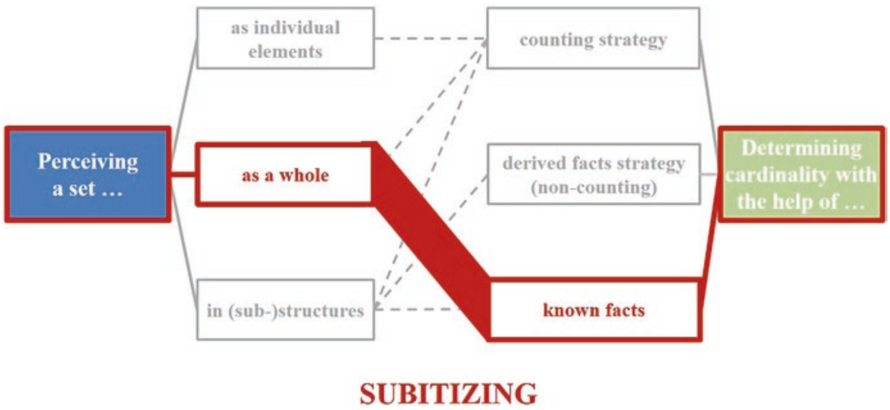


Fig. 7.2 Subitizing

The different possibilities to perceive a set allows various possibilities to determine the cardinality (cf. Fig. 7.1).

Perceiving a set as individual elements leads to the counting strategy *counting all* in order to determine the cardinality. If a set is perceived as a whole, there are two possibilities to determine the cardinality. In the determination process, it is again possible to use the counting strategy counting all or to apply *known facts* (cf. Gray, 1991, p. 554). In this last case, the two processes of perception and determination coincide (subitizing, cf. Fig. 7.2). When perceiving a set in (sub-)structures, there are various possibilities to determine the cardinality: using a counting strategy, a derived facts strategy (e.g., doubling/halving or (de-)composing), or to apply known

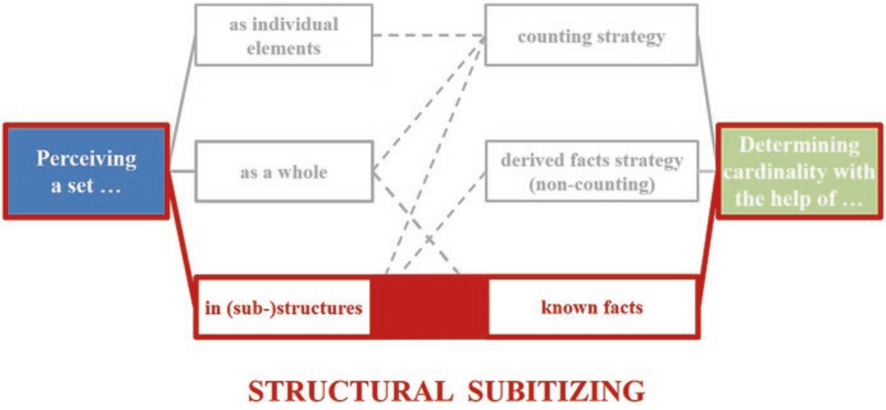


Fig. 7.3 Structural subitizing

facts. The two cases when the processes of perception and determination coincide (subitizing and structural subitizing cf. Figs. 7.2 and 7.3) are described in detail in the following.

The term subitizing was defined by Kaufman, Lord, Reese, and Volkmann (1949). It is derived from "the classical Latin adjective *subitus*, meaning *sudden*, and the medieval Latin verb *subitare*, meaning to *arrive suddenly*" (Kaufman et al., 1949, p. 520, emphasis in original). Subitizing in its original meaning describes that one can quickly and securely name the cardinality of a small set (Kaufman et al., 1949). There are two approaches. Gelman and Gallistel (1986) argue that subitizing is based on a fast counting process, while others claim that subitizing is a noncounting process (cf. Dornheim, 2008). In this paper, the term subitizing is used in its original definition: perceiving a (small) set and immediately naming the number. Two processes coincide, the process of perceiving a set as a whole and the application of known facts, how many elements there are (cf. Fig. 7.2).

Sarama and Clements (2009) also refer to Kaufman et al. (1949) in their definition of the term subitizing. They distinguish between perceptual and conceptual subitizing (Clements, 1999; Clements & Sarama, 2014; Sarama & Clements, 2009). "*Perceptual subitizing* […] is closest to the original definition of subitizing: recognizing a number without consciously using other mental or mathematical processes and then naming it" (Sarama & Clements, 2009, p. 44, emphasis in original). To recognize small numbers a preattentive quantitative process is used. For naming the cardinal number, a conscious numerical process is added (ibid.). With perceptual subitizing, it is therefore possible to just "see" how many objects there are and to name the cardinal number immediately. Here two processes can be identified which occur at the same time: on the one hand, the perception (just "see") and, on the other, the determination of the cardinality (name the cardinal number). Clements and Sarama (2014) assume that perceptual subitizing is possible up to a maximum of four elements (p. 18). A set of five elements can also be determined using perceptual subitizing if the image of the presented set has already been learned and recognized (ibid.).

If children perceive substructures in sets, they have different possibilities for determining the cardinality, see Fig. 7.1. Clements and Sarama (Clements & Sarama, 2014; Sarama & Clements, 2009) use the term *conceptual subitizing* if children perceive structures and use any of the possible different determination strategies. Conceptual subitizing is described as "Seeing the parts and putting together the whole" (Clements & Sarama, 2014, p. 10). The term refers to both the process of recognizing a structure of a set and to the conscious use of partitioning strategies like composing and decomposing for determining the cardinality of this set (Sarama & Clements, 2009, p. 45). Here, the two processes of perception and number determination are also described. Sarama and Clements (2009) say that the recognition of a structure is a necessary precondition for conceptual subitizing. The way in which the number is determined (determination process) plays a subordinate role. So the child can, for example, apply known facts that three and two results in five (Sarama & Clements, 2009) or *count in steps* (Sarama & Clements, 2009) but also use *counting on* to determine the cardinality (Clements & Sarama, 2014). These different descriptions of the determination processes as part of conceptual subitizing show that Clements and Sarama do not distinguish between different determination processes when using the term conceptual subitizing. For example, the described determination process "knowing that two and three result in five" (Sarama & Clements, 2009) is based on recognition of a structure and the use of known facts. This leads to naming the number of the whole set immediately. This description is consistent with the original definition of subitizing, because here the perception and the determination processes coincide. When counting on is the determination process of conceptual subitizing (Clements & Sarama, 2014), it is possible that only a part of the presented set is perceived in structures. The recognition of the structure is not sufficient to determine the cardinality quickly and securely. In this case, the perception and the determination processes do not coincide, which would be a prerequisite for subitizing. In order to clearly distinguish between perceiving the structure and different ways of determining the cardinality in this paper, conceptual subitizing is not used.

In the following, the term *structural subitizing* is defined and used as a logical continuation of the idea of subitizing. Structural subitizing also describes an application of known facts and an immediate determination of the cardinality of a set. The two processes of perceiving a set and determining the cardinality coincide as well (cf. Fig. 7.3). In the example "knowing that three and two result in five," the process of perceiving the set in substructures of three and two coincides with the process of the known facts that the cardinality of the set is five.

Looking on the studies above, it was shown that many children in preschool age are already able to perceive and use structures to identify the cardinality of sets. It is important to note that it is not easy to infer from mere observation whether children perceive structures in a set of objects. A major reason for this is that the process of perceiving structures is an invisible act. Therefore, we can only draw conclusions from the explanations of the children or from interpretations out of visible observations of their process of determining the cardinality of a set. When we observe that children count every single object (e.g., by pointing with the finger or uttering the respective number words), we cannot be sure what they perceive. What we know at least is that they do not use the structures of the arrangement of the respective set to determine the cardinality. To investigate visual structuring ability in most studies, children primarily have to reproduce structured visual sets or they are asked to determine the cardinality when the presentations were presented only for a short time to them. Out of these observations, the use of subitizing for determining the cardinality of parts or the whole is assumed. In order not to rely exclusively on external observations and explanations of the children in order to draw conclusions about whether and how children perceive structures and use them to determine the cardinality, it is helpful to observe the eye movements of the children. In this paper, we discuss the use of eye-tracking as a research tool allowing deeper insights into children's visual structuring abilities. In the long run, it may also be used as an evaluation instrument for intervention studies supporting visual structuring abilities.

Research Question

In this paper, we aim to answer the research question regarding the investigation of visual structuring abilities of preschool children:

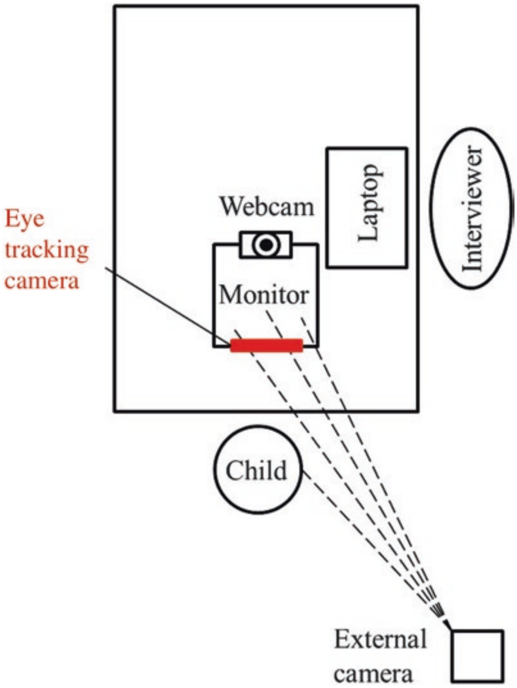
To what extent can eye-tracking contribute to gaining insights into children's perception and determination processes when identifying a set of objects?

Design

One hundred two children aged 5–6 years were interviewed individually to evaluate whether and how they perceive and use structures for determining the cardinality of a set of objects. Each interview consisted of different parts. In this paper, we focus only on the part that deals with sets of eggs in an egg carton for 10 eggs. This is the usual package for eggs children usually know from daily life. Also, its structure is analogue to the 10 frames, a typical didactical presentation used in primary school. Pictures of such egg cartons with different numbers of eggs (2, 3, 4, 5, 7, 9, and 10) were presented on a monitor allowing the recording of the eye movements of participating children. Before the pictures were presented, the child had been told that the interviewer would like to know how many eggs he or she could see. Children were instructed to say the number as soon as they knew it. There was no time limit for children to determine the cardinality of the eggs. Once they said a number, the interviewer asked how they came to the result.

In this study, a mobile eye-tracking system was used. The eyes can be tracked while the head is moved freely, promoting natural human behavior. Additionally,

Fig. 7.4 Interview setting



children sat at a child-sized table and chair supporting their natural position (cf. Fig. 7.4). The eye-tracking system tolerates large and fast head movements which was very important for these interviews with preschool children. The interviewer and the child sat at right angles to each other, thereby being able to talk directly to the child and see the monitor when the child pointed with his or her finger.

All pictures which were presented on the monitor and children's eye movements were recorded as long as the children looked at the screen. Additionally, an external camera was used to monitor other actions of the children, for example, activities with fingers. So it is possible to consider such actions when interpreting the processes of perceiving, determining, and explaining.

Task

Eleven photos of egg cartons were shown on the monitor to each child. The photos with different numbers of eggs were always presented at the same position on the monitor. Each item started with the presentation of a closed egg carton. Then the carton opened. After the child said a number and explained how he/she came to the result, the carton closed again. The screen was never empty because there was always a picture on it to ensure that the child knew where the photo would appear.

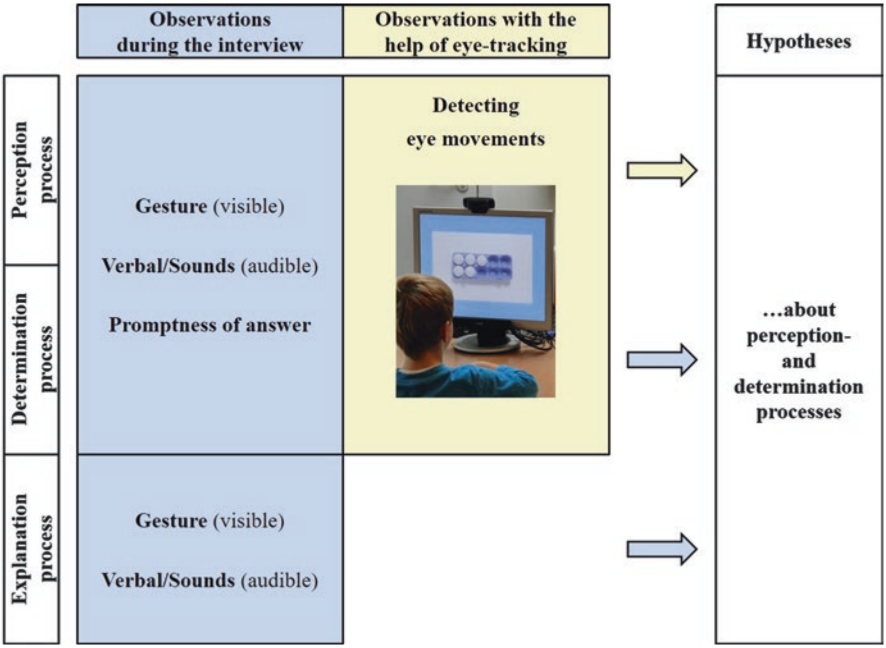


Fig. 7.5 Differentiation of aspects of analysis

Aspects for Analyzing the Data

Figure 7.5 provides an overview of different aspects of analysis. By videotaping, only different observations during the interview (blue column on the left) can be analyzed. Concerning the observation, we differentiate between observations during the two processes perception and determination on the one hand and the process of explanation on the other hand.

Next to visible and audible observations, aspects like gestures (e.g., movements of lips, fingers), verbal comments (e.g., whispering) and promptness of answers, eye-tracking provides additional relevant data (cf. Fig. 7.5, yellow column on the right). These additional data will be considered for the evaluation of the specific use of structures when perceiving sets. Furthermore, the eye-tracking data can also be used for reviewing hypotheses about perception and determination processes derived from visible and audible observations and for gaining additional insights, in identifying whether and how structures were used.

Eye-Tracking Data Analysis

In order to evaluate the eye-tracking data, the GazePlot-Graphic is used. In the GazePlot-Graphic, the order in which the child looked at the single objects is shown. Each colored dot reflects an eye fixation while the size of the dots indicates the duration of the fixation. The longer the child looked at a dot, the larger is the diameter of this dot. The dots can be displayed one after the other like a video sequence. Moreover, an illustration can be chosen where all dots are shown at once. This is called accumulate-graphic.

Still, some technical limitations have to be considered: In case children use gestures for determining the cardinality of sets or explaining, it can happen that their hands cover the camera of the eye-tracker. In that case, not all eye movements can be recorded. Eye movements and fixations can also not be recorded when a child looks at the interviewer when explaining the determination process. Due to these reasons, eye-tracking data reflect eye movements during the two processes of perceiving and determining.

Results and First Interpretations

The analysis of three examples shall illustrate in which situations eye-tracking helps to get additional insights into children's perception and determination processes. After presenting the examples, a first overview will be given summarizing in which cases eye-tracking is helpful (Fig. 7.18).

Interpretation Based on Only One Observation (Promptness) Can Be Confirmed

In the following, the three processes (perception, determination, and explanation) are separately analyzed and the resulting hypotheses are presented. The blue color indicates observations without an eye-tracker. The yellow color indicates observations with the eye-tracker (Fig. 7.5).

As Lisa named the cardinal number immediately (after 2 s), it can be hypothesized that Lisa used the structure of the arrangement of the eggs to derive the quantity (cf. Fig. 7.7). Research indicates that 2 s is too short for children at that age to count every single of the five eggs (cf. Fischer, Gebhardt, & Hartnegg, 2008). This hypothesis of a structural use leads to the assumption that Lisa perceived the set in structures. In Fig. 7.8, the explanation process is interpreted.

When Lisa was asked how she found out that there are five, she just answered: "Because just like that." Thus, the explanation process did not provide additional information about the way she has perceived and determined the presented set (cf. Fig. 7.8). With the help of eye-tracking data, it is possible to get some insights in Lisa's perception process (cf. Fig. 7.9).

On the GazePlot-Graphic of the eye-tracking data, it can be observed that Lisa focused her eyes on the middle egg of the top row and then looked right to the third egg in the top row (cf. Fig. 7.9). On the basis of these observations, it can be assumed that she perceived the set in (sub-)structures. Thereof, the hypothesis can be deduced

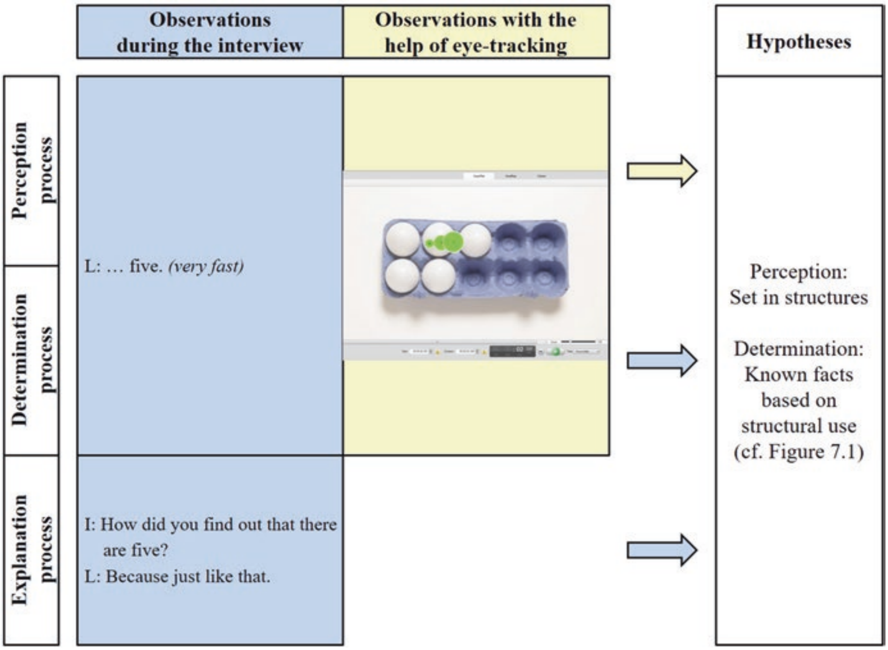


Fig. 7.6 First example – Lisa

that Lisa did not use the strategy counting all but she did use a structure to determine the cardinality of the set. Summing up all observations, Lisa seemed to use structural subitizing (cf. Fig. 7.3).

The example of Lisa shows, when interpreting the data without considering the eye-tracking results, the interpretation of the underlying processes might lead to the conclusion that Lisa used structures and therefore immediately knew the result (cf. Fig. 7.6, blue arrows). But only the component of the "promptness of the answer" would corroborate this hypothesis. No other data was observed. With the help of eye-tracking, it becomes visible that Lisa perceived structures. So the hypothesis of a strategy based on a structural use (cf. Fig. 7.1) can be confirmed (cf. Fig. 7.6, yellow arrow).

Two Inconsistent Observations: Confirming One of the Possible Hypotheses

The long duration of 12 s for the determination process of Tom leads to the assumption that he might have counted the eggs to derive the cardinality of the eggs. Also, small movements of fingers and lips were observed. This leads to the assumption that Tom perceived the set as individual elements. Still, Tom explained that he saw and used structures to determine the cardinality of the eggs (cf. Fig. 7.10).

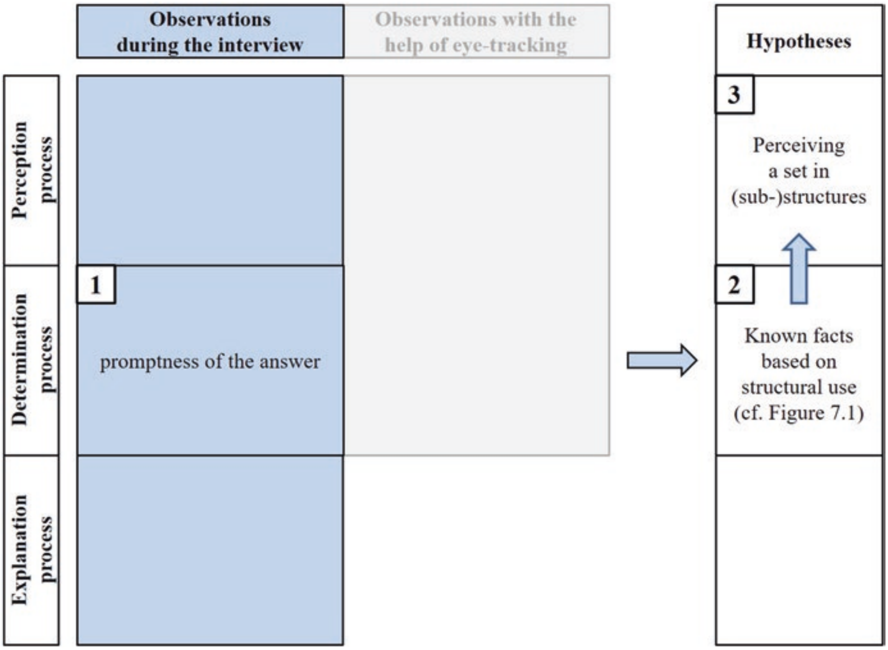


Fig. 7.7 Lisa: Observations and hypotheses *without* the help of eye-tracking data – 1

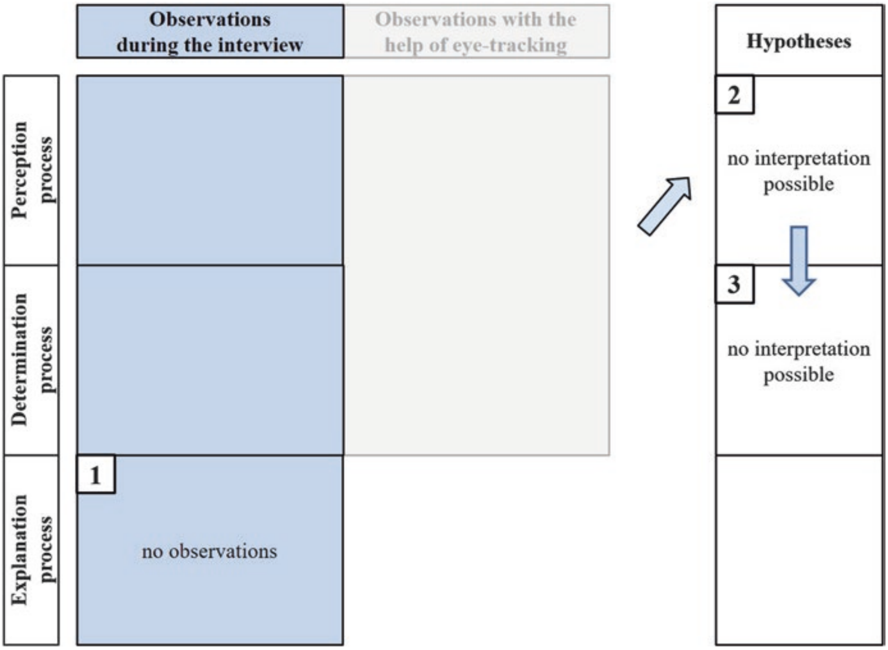


Fig. 7.8 Lisa: Observations and hypotheses *without* the help of eye-tracking data – 2

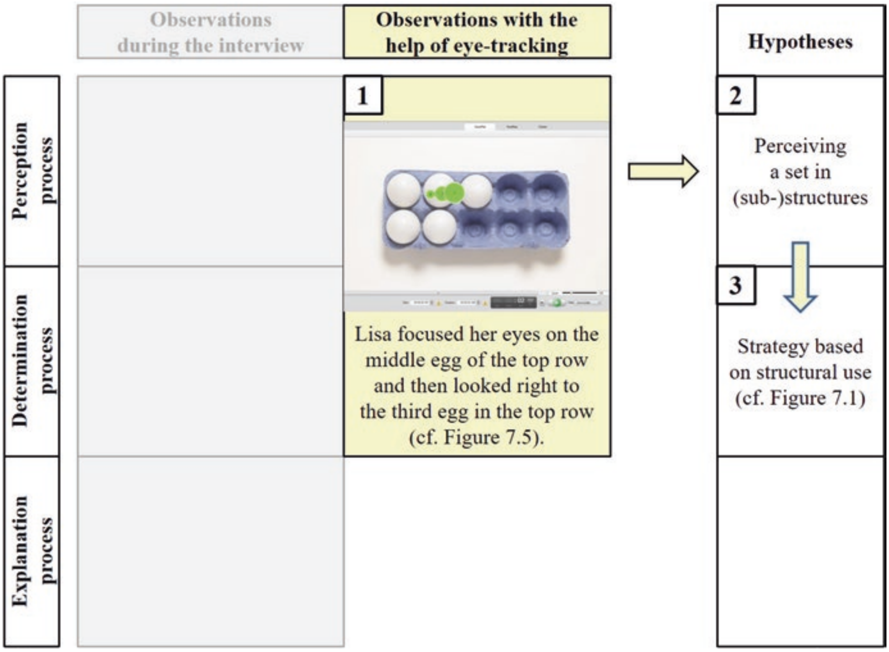


Fig. 7.9 Lisa: Observations and hypotheses *with* the help of eye-tracking data – 3

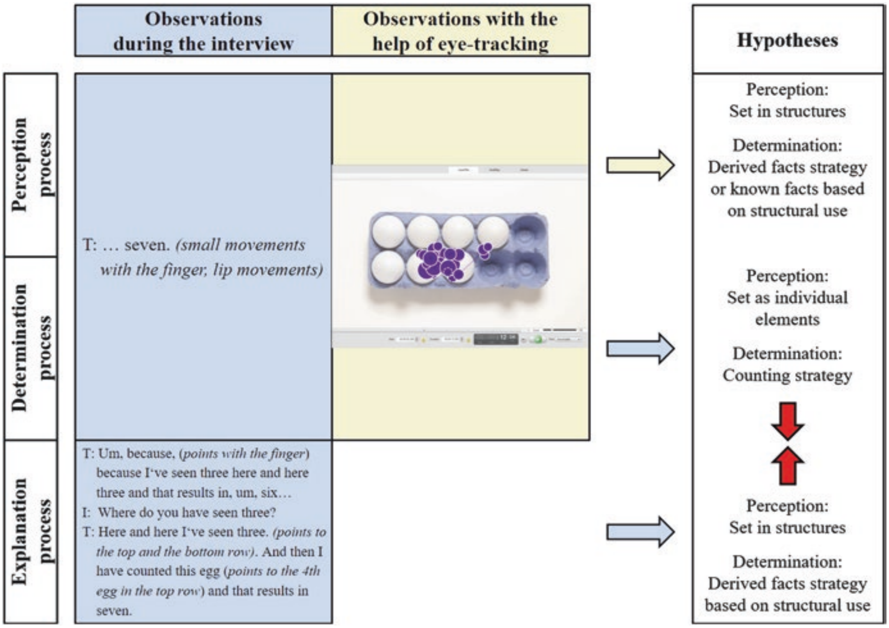


Fig. 7.10 Second example – Tom

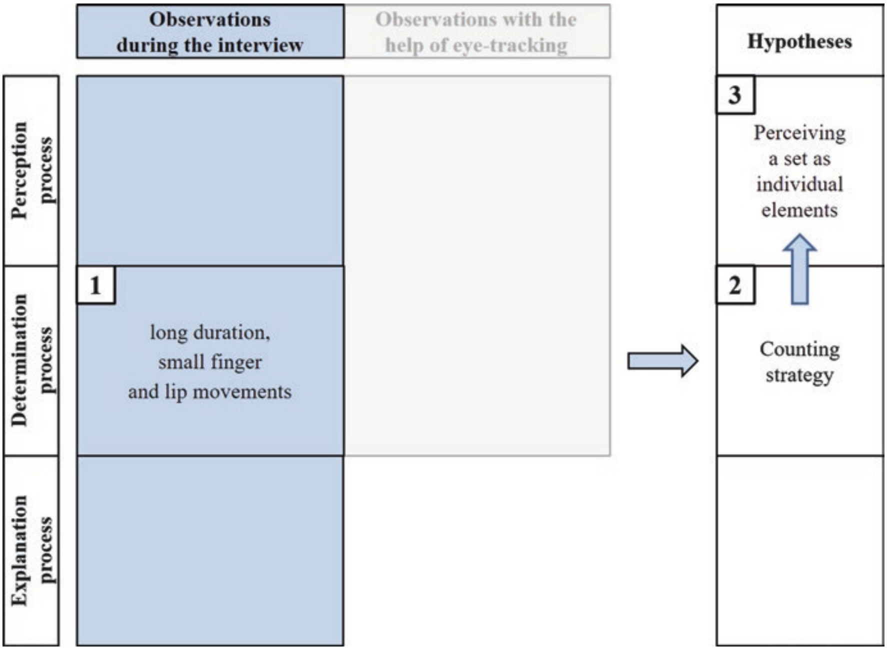


Fig. 7.11 Tom: Observations and hypotheses *without* the help of eye-tracking data – 1

It is now not easy to decide whether Tom just counted the eggs as one would conclude on the basis of the observations during the perception and determination processes (cf. Fig. 7.11) or whether Tom perceived and used structures as he explained (cf. Fig. 7.12). However, the eye-tracking data supported the hypothesis that Tom indeed perceived structures (cf. Fig. 7.13). Because of this observation, the hypothesis can be generated that he used the perceived structure to determine the cardinality of the set (cf. Fig. 7.13). So Tom could have used structural subitizing to determine the cardinality (cf. Fig. 7.3), a derived facts strategy or the counting strategy counting on. All these strategies are based on the use of structures.

At the first glance, the very long perception and determination processes as well as the interpretation of the visible movement of fingers and lips do indicate a counting process. Here, the observations of the perception, determination, and explanation processes did not match, so it is interesting that Tom obviously did not use a counting strategy. However, the eye-tracking data provided meaningful information to confirm one of the two contradictory interpretations by getting insights in the process of perceiving.

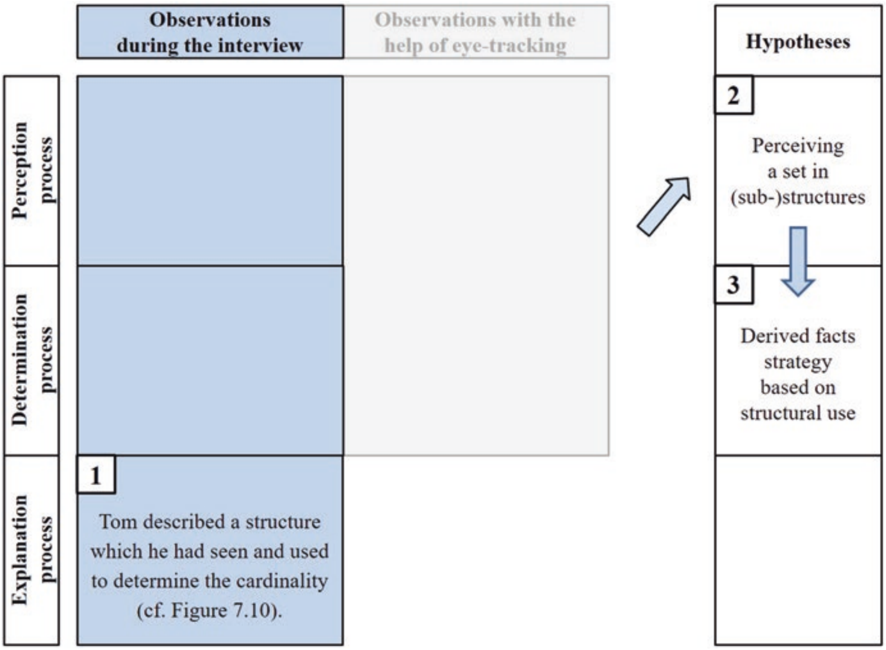


Fig. 7.12 Tom: Observations and hypotheses *without* the help of eye-tracking data – 2

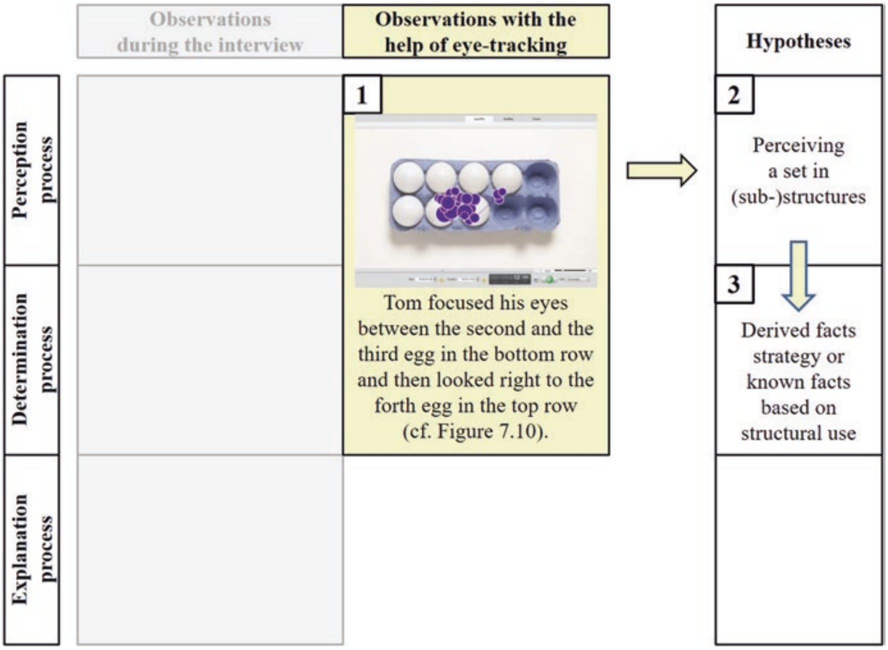


Fig. 7.13 Tom: Observations and hypotheses *with* the help of eye-tracking data – 3

Consistent Observations of the Determination and Explanation Processes Cannot Be Confirmed

The only observation during the determination process was that Emily needed a long time to determine the number (cf. Fig. 7.14). This observation suggests the hypothesis that she might have counted and thus perceived the set as individual elements (cf. Fig. 7.15). In the explanation process, Emily counted loudly every single egg and pointed with her finger on it. This observation also leads to the conclusion that she determined by means of counting all and therefore has perceived the set as individual elements (cf. Fig. 7.15). At this point, however, it cannot be ruled out that Emily could have perceived the set in structures but still used the familiar counting strategy for determining the cardinality determination (cf. Fig. 7.1).

The observations using the eye-tracker show that Emily has not fixed every single egg but that her gaze switched back and forth between the upper and lower row (cf. Fig. 7.17). Thus, the hypothesis is supported that she has perceived the set in structures and used structures for the determination of the cardinality of the whole presented set (cf. Fig. 7.17). So Emily used, based on the use of structures, a derived facts strategy or known facts (cf. Fig. 7.3) to determine the cardinality. To sum up: Due to the observations which are made without the eye-tracker, one might conclude

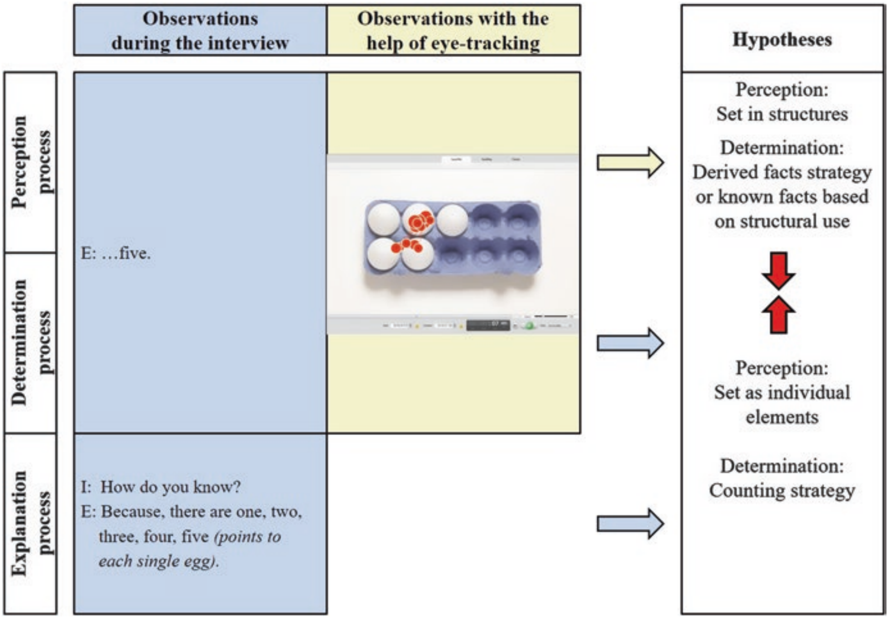


Fig. 7.14 Third example – Emily

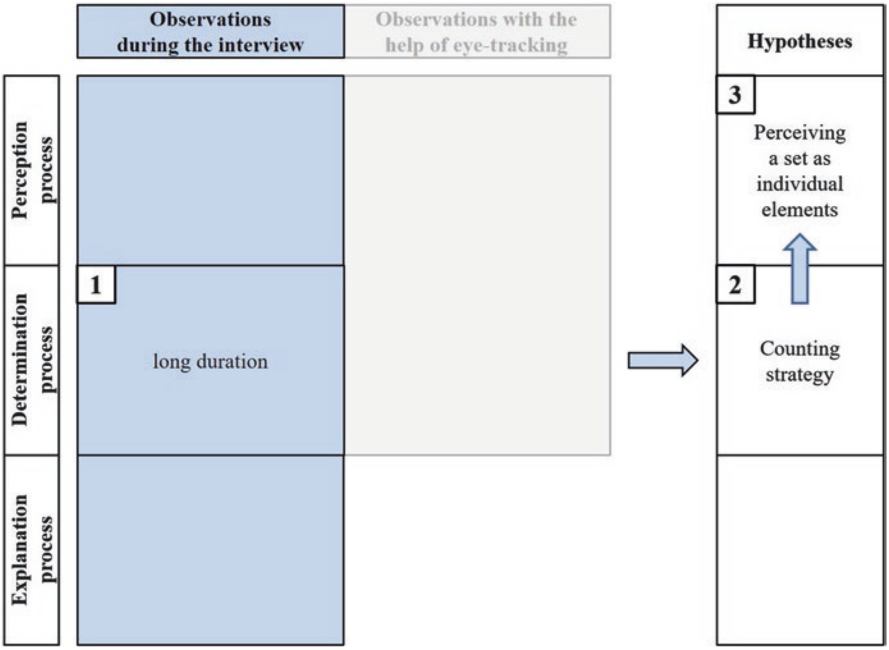


Fig. 7.15 Emily: Observations and hypotheses *without* the help of eye-tracking data – 1

that Emily counted all the eggs separately. The long duration for the determination process (7 s) and her gestures as well as the uttering of numbers supported that assumption (cf. Fig. 7.15 and 7.16). However, the GazePlot-Graphic clearly indicates that Emily perceived a structure. Her fixation switched back and forth between the upper and the lower row (cf. Fig. 7.17).

Here, without eye-tracking data, it would not be evident that Emily recognized and used structures, because the observations of all processes (perception, determination, and explanation) rather indicated counting. The idea of counting was probably just used for the explanation and was presumably not part of the perception processes.

Summary of the Results

In the research question, it was asked to what extent eye-tracking can contribute to gaining insights into children's perception and determination processes when identifying a set of objects. With the analysis of three children's eye-tracking data, this question can be of help to get new insights. We found evidence that eye-tracking data could be of added value for the interpretation of children's solution strategies in this task. In the example of Lisa, only one observation – that she named the

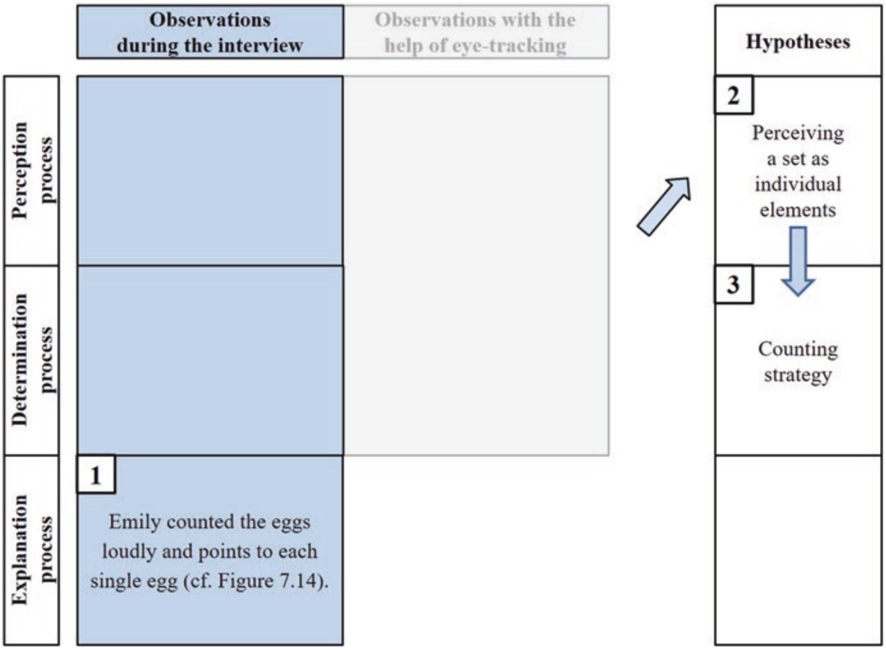


Fig. 7.16 Emily: Observations and hypotheses *without* the help of eye-tracking data – 2

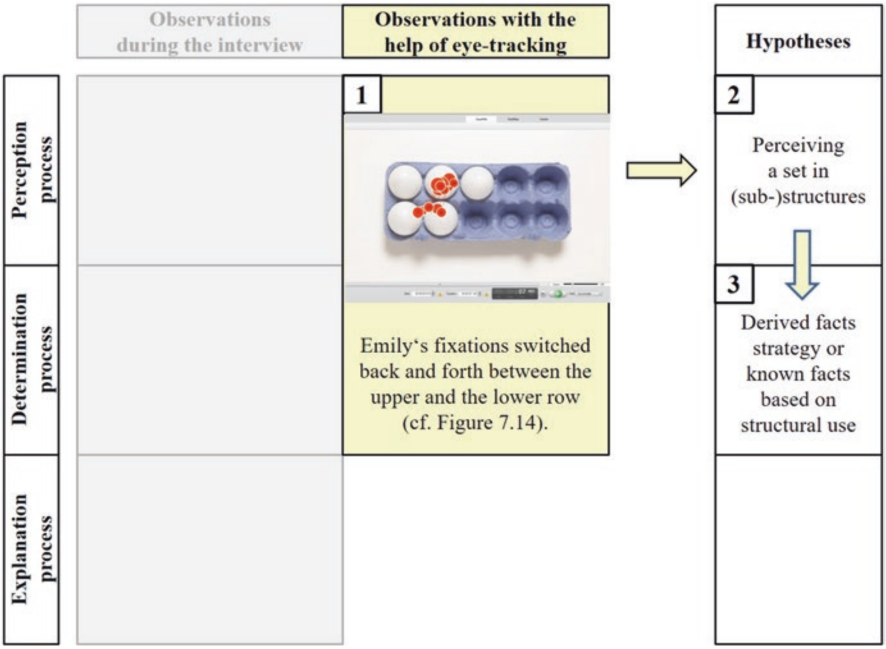


Fig. 7.17 Emily: Observations and hypotheses *with* the help of eye-tracking data – 3

cardinality of the set very quickly – was made during the perception and determination processes. Due to this observation, it can be stated that the child used structural subitizing because of the promptness of the answer. This was confirmed through the eye-tracking data. In the example of Tom, movements of fingers and lips were observed, indicating a counting process. However, in his explanations, he described the use of structures. These contradicting observations do not allow a clear conclusion. The interpretation is clearer after the consideration of the eye-tracking data. Thereby, the use of structures for the determination of the cardinality was confirmed. In the example of Emily, her explanations that she counted the eggs by pointing with the finger on each single egg indicate a counting process. Yet, the eye-tracking data contradicted this interpretation. The analysis indicated that Emily did use a strategy based on structures instead. The following figure (cf. Fig. 7.18) gives an overview of all possibilities of the analyzing processes. In the illustration, all described examples can be found.

The red marked fields in Fig. 7.18 highlight the cases in which additional relevant information can be provided by the eye-tracking data. If observations can be made either in the perception and determination processes or in the explanation process, then first interpretations and conclusions on solution strategies of the respective child are possible. The stated hypotheses can then be corroborated, refuted, or corrected by the help of the eye-tracking data. In the case where observations can be

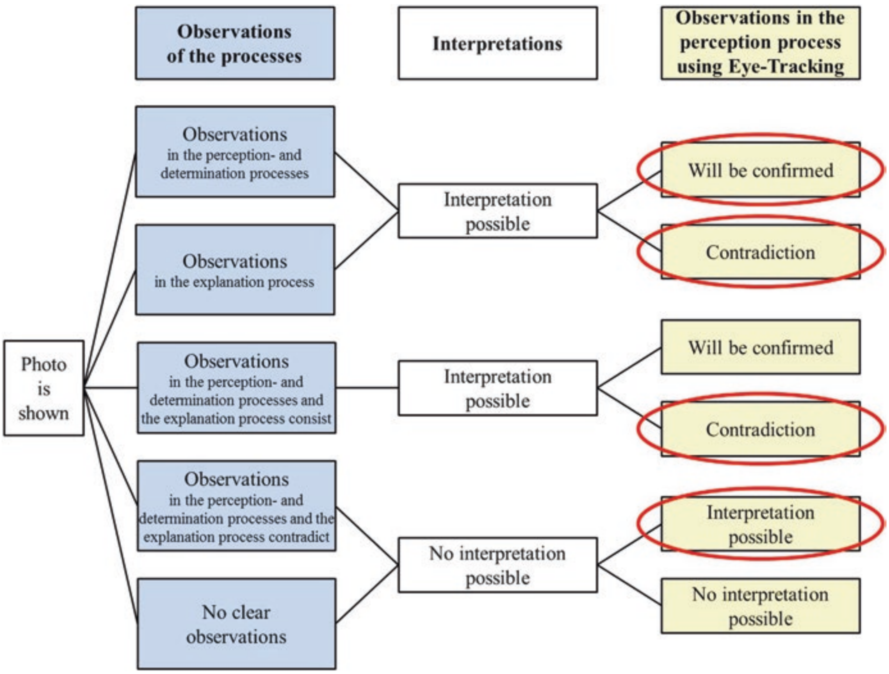


Fig. 7.18 Illustration of analyzing processes

made in all processes (perception, determination, and explanation) and they are consistent, an interpretation is also possible and a hypothesis can be derived. This hypothesis can again be confirmed or refuted by the eye-tracking data. In the latter case, one gets a new insight in the perception and determination processes of children. When observations made during these processes are contradicting or when no clear observations are possible, eye-tracking data may nevertheless be meaningful and allow for deriving a hypothesis on the underlying solution strategies. Thus, the additional observation level provided by eye-tracking data gives the possibility to gain insights into the perception process of children when asked to identify the cardinality of a set of objects. These insights, in turn, often provide opportunities to make statements about the determination process.

Discussion and Conclusion

The three presented interviews showed that with the help of eye-tracking, new insights into children's constructions can be gained while the children perceive visual sets and determine the cardinality. In the case of observations leading to inconsistent interpretations regarding the underlying processes, one of the possible interpretations could be confirmed through the eye-tracking data. In case no interpretation was possible from the observations, the analysis of the eye-tracking data provided new evidence to come to a new interpretation. Also, when interpretations were based on different observations, which seemed to be consistent, eye-tracking indicated another visual structuring strategy. In sum, this revealed that the consideration of children's eye fixation behavior is useful and promising. The visual structuring of a set of objects when determining the cardinality can be revealed through different analyses of the eye-tracking data, which indicate the perception and use of structures for quantification to be a foundation for acquiring the part-whole concept. Thus, the gained data and findings make an important contribution to the scientific discourse about the perception of structures of children not only in kindergarten. In addition to that, these insights into the visual structuring ability of children can be used for the choice and development of learning materials used in kindergarten that encourage playful discovering and exploring and in order to selectively facilitate the perception, recognition, and usage of structures in sets of objects. Possible stimuli in order to indicate perceiving and using structures could be, for example: "How did you see that? Can you present it in a way that you can see immediately that there are five? How did you know that there are seven?" Despite all advantages, it should be noted that the eye-tracking tool is complex and expensive. Therefore, it may not be a useful method for observations of daily life in kindergarten. In mathematics education, there is also a broad consensus that mathematics education in kindergarten should take place in meaningful and playful natural learning situations (cf. Benz et al., 2015; Gasteiger, 2015). One of the gained insights was that children often construct structures in the collection of objects, but they often lack the words to describe their constructions and approaches. When then asked to present an explanation for their approach, they often referred to descriptions of familiar strategies, as for example counting. This becomes evident when looking at the example of Emily. She did not mention that she is counting, but her explanation was a counting aloud process accompanied by pointing with the finger on some eggs. However, the analysis of her eye fixation behavior clearly indicated that she perceived the structures and did not focus on each single egg and probably did not count every single egg. This could not only be observed in the example of Emily. Often "counting" is the only way that children know as verbal explanation for determination processes, so for some children, counting is equalized with determining the cardinality. Therefore, next to giving stimuli and asking adequate questions for perceiving structures, a specific kind of language also has to be developed in kindergarten in order to help children explain their processes of perceiving and using structures when they determine the cardinality of sets. This is a particular challenge when designing mathematical learning opportunities and using opportunities for mathematical learning to support visual structuring abilities.

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Chapter 8
Is Considering Numerical Competence Sufficient? The Structure of 6-Year-Old Preschool Children's Mathematical Competence

Simone Dunekacke, Meike Grüßing, and Aiso Heinze

Abstract Studies investigating mathematical competence of children aged 3–6 mostly focus on children's knowledge in the content area of quantity. However, researchers and educators agree that young children also develop mathematical competence in other mathematical content areas, e.g., space and shape. Up to now, there are only few instruments to measure mathematical competence of young children as a broad construct: one is the Kieler Kindergartentest (KiKi). There is a lack of evidence whether children's competence in different mathematical content areas are empirically distinguishable. We collected data from 335 children at the end of their last preschool year with the KiKi and analyzed the structure of their mathematical competence. Our results indicate that even for young children, mathematical competence can be considered as a multidimensional construct structured by the three domains of (1) quantity; (2) space, shape, change, relationship; and (3) data and chance. These empirical findings can give a hint that also on the preschool level, different aspects of mathematical competence should be addressed.

Keywords Early mathematical competence · Assessment · Competence structure · Pattern · Space and shape · Quantity · Change and relationship

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Theoretical Background

Mathematical Competence in Different Age Cohorts

In the last decades, there was much effort to conceptualize and investigate mathematical competence of individuals at different age levels. Common to many approaches is that content areas as well as cognitive components are distinguished (Neumann et al., 2013). Apart from different labels, conceptualizations of mathematical competence often cover five content areas: (1) *quantity*, (2) *change and relationship*, (3) *space and shape*, (4) *data and chance*, and (5) *units and measuring* (KMK, 2004; Mullis, Martin, Foy, & Arora, 2012; Neumann et al., 2013; OECD, 2013). Sometimes the aspects of *units and measuring* are included in *quantity* and/or *space and shape* (Mullis et al., 2012; Neumann et al., 2013; OECD, 2013). For younger children or students, *change and relationship* is often considered under the perspective of patterns, and sometimes related aspects are distributed among the other content areas (KMK, 2004; Mullis et al., 2012). Cognitive components are also labeled differently; mostly five or six cognitive components are distinguished: (1) *mathematical communication*, (2) *mathematical argumentation*, (3) *modeling*, (4) *using representational forms*, (5) *mathematical problem solving*, and (6) *technical abilities and skills* (Neumann et al., 2013; OECD, 2013). Sometimes these are integrated to three broader cognitive components (e.g., Mullis et al., 2012).

As mentioned before, most conceptualizations of mathematical competence describe mathematical competence for specific age cohorts, for instance, the OECD framework for 15-year-olds (OECD, 2013) or the KMK (2004) for children at the end of grade 4. Competence is defined as domain-specific and learnable (Hartig & Klieme, 2006); it can be assumed that children acquire mathematical competence already before entering school. From a life span perspective, it is useful to describe mathematical competence for preschool children as well as for students as a coherent model.

Research in mathematics education and psychology has already considered the mathematical competence of young children (aged 3–6) for a long time. Up to today, most studies focused on children's knowledge and skills in the content area of *quantity*. Competence in the content area of *quantity* can be characterized by skills in verbal and object counting; pre-numeric skills as subitizing, comparing, and ordering; as well as first arithmetic experiences in everyday situations by the counting-all or counting-on strategy (Clements & Sarama, 2007, 469ff). During the preschool years, a first understanding of part-whole relationships is developed and the children develop skills to read and write Arabic numbers (ebd.). For competence in the content area of *quantity*, empirical research describes a large heterogeneity for children at the end of preschool (e.g., Gervasoni & Perry, 2015). Furthermore, Krajewski and Schneider (2009) identified levels of specific early mathematical skills in children's development of a number concept (e.g., quantity discrimination, counting). Research has also shown that such specific skills in the content area of *quantity* predict school achievement in mathematics and other domains (e.g., Duncan et al., 2007; Krajewski & Schneider, 2009). All in all, research in the last years has figured out how competence in the mathematical content area of *quantity* can be described for young children and how predictive it is for later mathematics achievement.

When observing young children in their everyday live, it becomes apparent that they are faced with informal mathematical learning opportunities which are beyond the content area *quantity* (e.g., dealing with shapes, observing patterns, playing with dice). Hence, mathematics educators agree that young children develop mathematical competence from other content areas than *quantity* (e.g., Benz, Peter-Koop, & Grüßing, 2015; Clements & Sarama, 2007). Based on this assumption, it is interesting to examine how children's competences in content areas beyond quantity develop and whether the development of competences in different content areas is independent. If the competence developments in different content areas are independent, then further questions arise, e.g., concerning the predictive power of competences in content areas beyond quantity for later school development and, subsequently, whether it is useful to develop specific instructional material for all mathematical content areas. Clements and Sarama (2007) give an overview about the state of research concerning different mathematical content areas for preschool children. The content area of *space and shape* includes aspects of spatial thinking, i.e., spatial orientation, spatial perspective taking, and first experiences with maps and two-dimensional coordinates (Clements & Sarama, 2007, 488ff). Moreover, two- and three-dimensional shapes are part of children's daily environment, and they are able to identify these kinds of shapes (ebd., 507). Empirical findings of Maier and Benz (2014) show that preschool children are able to deal with two-dimensional shapes in different ways (naming, drawing, and identifying). The content area of *units and measurement* covers initial ideas of length and measurement as well as everyday experiences with weights and time or time periods (Benz et al., 2015, 250ff). Clements and Sarama (2007, 523) figure out that particularly measurement can bridge between *quantity* and *space and shape* because it makes continuous quantities as length and areas "countable." Little is known about how deep children's understanding of measuring is and whether (and even how) it can be fostered (ebd.). The content area of *change and relationship* is characterized by dealing with different kinds of patterns as well as basic numerical relationships (Clements & Sarama, 2007, 524). Dealing with patterns includes copying or continuing a pattern as well as explaining the internal relations in a pattern (Lüken, Peter-Koop, & Kollhoff, 2014). For the content area of *data and chance*, Clements and Sarama (2007, 525) mention that preschool children experience situations with *data*, for example, by classifying and sorting different objects and analyzing how many objects of each group they have. Benz et al. (2015, 281) describe furthermore that preschool children also make experiences with the idea of *chance* (e.g., when playing dice games) and develop a subjective but not necessarily mathematical understanding of concepts like probable, impossible, or safe events.

Assessment of Preschool Children's Mathematical Competence

Meanwhile, a large variety of standardized assessments to measure quantity and number knowledge of young children exists. For example, in German-speaking countries, the *Osnabrücker Test zur Zahlbegriffsentwicklung* (OTZ) (Van Luit, van de Rijt, & Hasemann, 2001, German version of the Utrecht Early Numeracy Test), the *Neuropsychologische Testbatterie für Zahlenverarbeitung und Rechnen bei Kindern* (ZAREKI-K) (Von Aster, Bzufka, Horn, Weinhold Zulauf, & Schweiter, 2009), and the TEDI-MATH (Kaufmann et al., 2009) are popular to assess children's quantity and number knowledge.

Although there are many tests addressing children's competence in the content area of *quantity*, there are hardly any instruments addressing mathematical competence as a broad construct covering different content areas and different cognitive components. In the literature, we found a few examples of such assessment instruments which cover content areas beyond quantity: the item pool developed within the *Early Numeracy Research Project* (ENRP) (DEET, 2001), the *Research-Based Early Math Assessment* (REMA) (Clements, Sarama, & Liu, 2008; Weiland et al., 2012), and the *Kieler Kindergartentest* (KiKi) (Grüßing et al., 2013).

The ENRP is a one-to-one interview including items addressing the content areas number, measurement, and space. The interview is primarily developed as a diagnostic tool for primary school teachers in grade 1 and includes special parts for children before entering school. It can also be used as a research instrument. Because of its diagnostic focus, the ENRP interview also provides qualitative information about children's knowledge, for example, the use of strategies for solving tasks.

The REMA focuses on number, geometry, measurement, patterns, and data analysis (classification) (Clements et al., 2008). As the authors used the instrument to evaluate a mathematics curriculum, they applied item response theory to estimate children's competence on a unidimensional scale (ebd.). The authors reported that the Rasch model is an appropriate statistical tool to analyze the data collected by the REMA item pool and to describe children's mathematical competence. Moreover, even a short version of the REMA was able to measure children's mathematical competence with acceptable reliability and validity (Weiland et al., 2012).

The KiKi is also a one-to-one interview available in three different interlinked versions for children of age 4 (easy version), 5 (medium version), and 6 (difficult version) (more information below). The whole KiKi item pool (items from all test versions together) can be analyzed by the unidimensional Rasch model with acceptable item fit values (Grüßing et al., 2013, 76). Moreover, Jordan et al. (2015) analyzed data of 4-year-old children collected with the easy version of the KiKi with a confirmatory factor analysis (CFA). They reported that it is possible to distinguish empirically three mathematical competence dimensions for this age cohort: *quantity (Q)*, *space and shape (S & S)*, and *change and relationship (C & R)*. This result is represented in Fig. 8.1. Each ellipse represents a factor of preschool children's mathematical competence. The curves and numbers label significant correlations between factors. As all three factors are assumed to be parts of mathematical competence, the correlations are expected to be high.

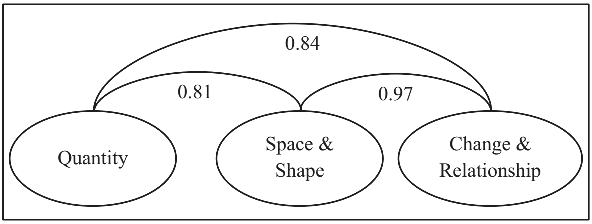

Fig. 8.1 Structure of 4-year-old children's mathematical competence (Jordan et al., 2015)

Research Question

As described in the previous section, the assessment of early mathematical competence is an important research field which in the last years has mainly focused on the mathematical content area of *quantity*. There is no doubt that *quantity* is a very important content area, especially for connecting preschool and primary school learning. However, during everyday activities, preschool children also gain experiences in other mathematical content areas. These content areas beyond *quantity* are also addressed in modern preschool curricula, and there is empirical evidence that preschool children acquire corresponding competences (e.g., Clements & Sarama, 2007). Available assessments allow a holistic view on children's mathematical competence as a broad construct and its development during preschool years. Moreover, for some content areas, it is possible to assess diagnostic information about children's mathematical competence. There is still a lack of evidence whether children's competences in different mathematical content areas are empirically distinguishable, i.e., whether competences in the different content areas develop in parallel or whether they are to some extent independent. Results concerning this question are relevant for the decision whether there is the necessity to design specific learning environments for each content area. Moreover, we still do not know which role mathematical competence beyond quantity plays for the development of preschool children after the transition to primary school. The presented study is therefore guided by the following research questions:

1. Is it possible to replicate a multidimensional structure of mathematical competence based on content areas as identified by Jordan et al. (2015)?

Like Jordan et al. (2015) we used the KiKi but collected data from 6-year-old children instead of 4-year-old children. Moreover, we included items for the content areas *units and measuring* and *data and chance* which were not part of the analysis of Jordan et al. (2015). Accordingly, our second research question is:

2. Does the inclusion of the content areas units and measuring as well as data and chance allow a more detailed model to describe children's mathematical competence?

Method

Instruments

Our study is based on data collected with the medium version of the KiKi. This version includes 31 items from five mathematical domains (*quantity*, *space and shape*, *change and relationship*, *units and measuring*, *data and chance*) (Grüßing et al., 2013). As described above, the KiKi covers mathematical competence in a broad way and provides reliable and valid data. Moreover, there was a pragmatic reason to use KiKi data for our study: the KiKi was chosen as mathematics test in the project KOMPASS (*Kompetenzen alltagsintegriert schützen und stärken*) by the University of Rostock (Jungmann et al., 2012). KOMPASS was an evaluation study of in-service preschool teacher trainings addressing children's mathematics, language, and social-emotional development. The authors supported the KOMPASS project team by a training of research assistants for the test administration and by analyzing the KiKi data.

Some items of the KiKi encompass different kinds of materials to illustrate or to process the item. Moreover, all items include precise action and voice instructions for the interviewers so that a maximum standardization is given (see example items below). The items were administered by trained interviewers in a one-to-one interview in the kindergartens. The interviews took about 30 minutes which was an acceptable time for the children. Furthermore, a puppet was used which could function as an ice breaker. This decision was made because most German preschool children are not familiar with test situations, especially when the test is carried out by an external interviewer and not by the preschool teacher. Moreover, in some items, the puppet served as a third person, for example, to offer a counting mistake for the children to correct.

The items require a verbal answer or an answer by an action (see Figs. 8.2 and 8.3). A standardized sheet was used to document the answers during the test situations. Children's responses were coded dichotomously (right or wrong answer) or in a partial credit model so that partially correct answers could be respected. In case a child did not give a response, the interviewer chose the code "no response." As the interviews were not video- or audiotaped, the answers of the children were directly coded in the documentation sheet during the interview. For example, for the item in Fig. 8.2 the interviewers had to code a child's response dichotomously ("showing 3 triangles" versus "showing less than 3 triangles and/or a shape which is not a triangle"). To ensure that all children had the same conditions, the test developers trained the interviewers by using videotaped examples from earlier interviews. Moreover, the KiKi manual included instructions for the interviewers, for example, in which case it is allowed to repeat the question or if an answer is right or wrong.

In the KiKi, the content area of *quantity* is covered by 11 items. The items address counting activities on different levels, the cardinality of numbers, and ordering quantities. The content area of *space and shape* is represented by five items, focusing on spatial thinking and perspective taking as well as dealing with a coordinate and identifying two-dimensional shapes in a picture. Figure 8.2 presents an

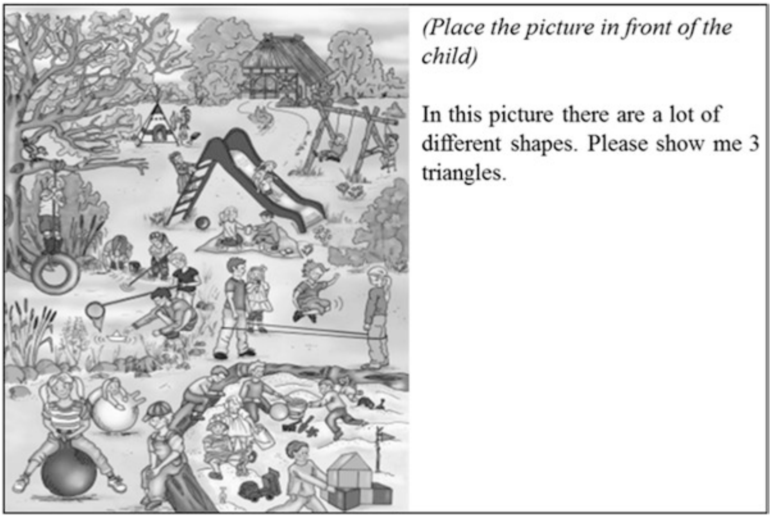

Fig. 8.2 Item playground (*space and shape*)

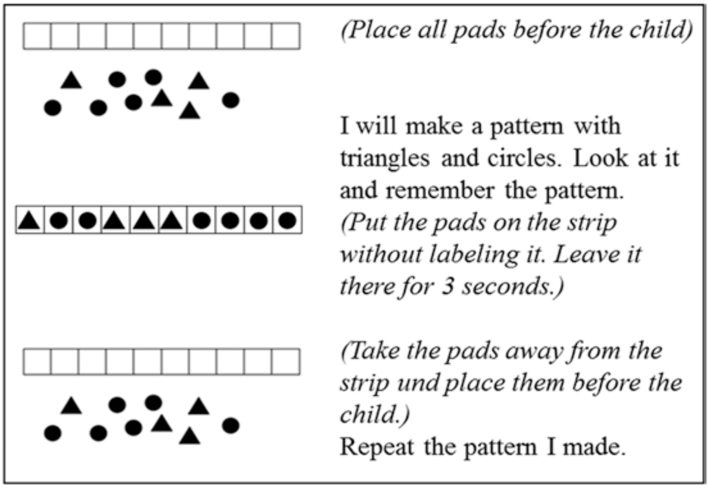

Fig. 8.3 Item notice patterns (*change and relationship*)

example for an item (with interviewer instruction) from the domain of *space and shape*, where the children have to identify triangles. The content area of *change and relationship* is also represented by five items focusing patterns (e.g., copy a pattern shown for a short moment) and easy numerical relationships (e.g., six pieces of cheese are given; how many slices of bread could be glazed when you need two pieces for each slice).

Figure 8.3 gives an example for dealing with patterns. This is also an example for an item which is coded dichotomously ("repeated the pattern completely correct" versus "repeated the pattern wrong or not completely"). The presented pattern is based on the repeating rule and a pattern of growth (Benz et al., 2015, 295). Repeating rule means that the presented items recur in a specific way (triangle, circle, triangle, etc.). Pattern of growth indicates that it must be figured out that the number of the used items is growing for each repetition (one triangle, two circles, three triangles, etc.). The five items covering *data and chance* ask children to interpret and use data sets (to compare amounts of animals represented by dots), to create this kind of data representation, and to deal with chance (e.g., draw a specific candy from glass bowls with two kinds of candies in different ratios and decide where the probability is highest). *Units and measuring* is represented by five items which ask the children to compare different lengths or to measure with unstandardized instruments (pieces of chocolate to measure a chocolate bar).

Sample and Analysis

The study is based on a sample of $N = 335$ children at the end of their last preschool year. As the data collection took place in the course of the KOMPASS project, the participants visited preschools in Northern Germany, especially in the city of Rostock and surroundings. The mean age was 5.7 years (SD = 0.62) and 53.1% of the sample were boys.

To answer the research questions, the data was analyzed with a confirmatory factor analysis (CFA) using the software Mplus 5.2 (Muthén & Muthén, 2007). The CFA allowed analyzing different latent factors (based on theoretical assumptions) in one model. The usual model fit indices CFI, TLI, and RMSEA were used to compare the fit of the different models computed by the CFAs. According to Schermelleh-Engel, Moosbrugger, and Müller (2003), CFI and TLI should be greater than 0.97 and the RMSEA should be smaller than 0.05 for a good fit. In some cases, the model fit gave no definite answer which model is to be preferred because the fit indices took identical or similar values. In these cases, we computed additionally a χ^2-difference test to compare the model fit by using the DIFFTEST option implemented in the Mplus software (Muthén & Muthén, 2007).

In the first step, we tried to replicate the result Jordan et al. (2015) reported for 4-year-old children by only using data of the KiKi items on *quantity*, *space and shape*, and *chance and relationship*. Hence, we computed three CFAs to answer the first research question: firstly, a three-dimensional model (separating *quantity*, *space and shape*, and *chance and relationship*); secondly, a two-dimensional model (*quantity* vs. non-*quantity*) because Jordan et al. (2015) found a high correlation between the two factors *space and shape* and *chance and relationship*; and finally, a one-dimensional model as a default model. In the second step, we extended the three models and included the data for *units and measuring* as well as for *data and chance* as additional factors to answer the second research question.

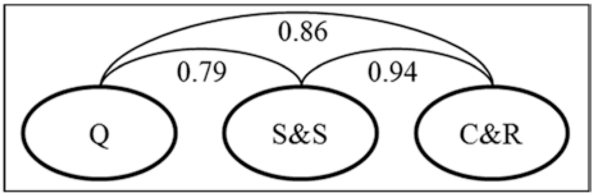

Fig. 8.4 Three-dimensional model of 6-year-olds' mathematical competence

Table 8.1 Model fit indices of the CFA models including the mathematical domains *quantity* (QU), *space and shape* (S&S), and *chance and relationship* (C&R)

Dimensions	CFI	TLI	RMSEA	*N*	$\chi^2/df/p$
3 (QU vs. S&S vs. C&R)	0.988	0.992	0.026	335	84.545/69/0.11
2 (QU vs. S&S+C&R)	0.988	0.992	0.025	335	84.804/70/0.11
1 (QU+S&S+C&R)	0.977	0.985	0.035	335	98.678/70/0.01

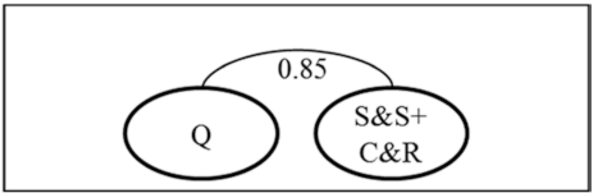

Fig. 8.5 Two-dimensional model of 6-year-olds' mathematical competence

Results

Figure 8.4 presents the result of the first CFA, replicating the structural analysis of Jordan et al. (2015) based on the data of items from the content areas *quantity (Q)*, *space and shape (S&S)*, and *change and relationship (C&R)*. Like in Fig. 8.1, the ellipses represent the factors of preschool children's mathematical competence. The curves and numbers label significant correlations between factors. Table 8.1 additionally presents the model fit indices of the estimated models. Row 1 indicates that the model fit for this three-dimensional model is good.

However, like Jordan et al. (2015) we identified a high correlation between the two latent factors *space and shape* and *change and relationship* (see Fig. 8.4). Therefore, we also estimated a two-dimensional model where we combined *space and shape* with *change and relationship* to one factor. This model is presented in Fig. 8.5. Table 8.1 indicates good model fit indices for this model which is very similar as for the three-dimensional model. As described in the methods section, we furthermore estimated a one-dimensional model as a default model. As Table 8.1 indicates, this model reaches a worse model fit compared to the other two models.

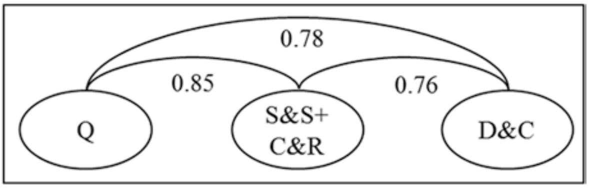

Fig. 8.6 Model of mathematical competence including data and chance (D & C)

We conducted a χ^2-difference test comparing the more restrictive two-dimensional model with the less restrictive three-dimensional model to decide which model represents the data best. The test indicates that the two-dimensional model fits better to the data ($\chi^2 = 1.226/df = 2/p = .54$).

In the second step, we investigated whether the content areas *data and chance* and *units and measuring* could be identified as competence dimensions for children at the end of preschool time. We addressed this question by analyzing a CFA with the two-dimensional model identified in Table 8.1 plus latent factors for *data and chance* as well as *units and measuring*. It turned out that only the CFA with one additional latent factor for *data and chance* converges. The model shows a good fit ($\chi^2(101) = 138.239(101)$, $p = .01;$ RMSEA = 0.033; CFI = 0.973; TLI = 0.983).

Figure 8.6 presents this model of 6-year-olds mathematical competence. All correlations are significant. They indicate strong relationships between the three factors but also provide evidence that they are distinct factors.

Discussion

Early mathematical competence is an important field of research in mathematics education and psychology. In the past, most studies concerned with mathematical competence in the preschool age focused on the mathematical content area of *quantity*, e.g., the development of the number concept (Krajewski & Schneider, 2009). Young children also encounter situations covering other mathematical content areas so that research started to consider content areas like *space and shape* or *data and chance* (Benz et al., 2015; Clements & Sarama, 2007). During the last decade, some assessments to measure mathematical competence have been developed which also cover mathematical content areas beyond quantity. However, up to today, we still do not have much research examining the structure of mathematical competence of preschool children. In a study with 4-year-old children in Germany at the beginning of preschool, Jordan et al. (2015) administered the *Kieler Kindergartentest* (KiKi) (Grüßing et al., 2013) and found evidence for a three-dimensional competence structure covering the content areas *quantity*, *space and shape*, and *change and relationship*. In the present study, we had the aim to replicate this three-dimensional structure for children at the end of preschool (aged 6). Moreover, in the second step, we analyzed whether the mathematical content areas *data and chance* and *units and measuring* represent additional dimensions of early mathematical competence.

Similar to the study with the 4-year-old children, our results indicate that the two- and three-dimensional models have a comparable model fit. In contrast to the study with 4-year-olds, the χ^2-difference test indicates that the two-dimensional model fits better to the data. The better fit of the two-dimensional model might be explained by the methodical reason that *quantity* was measured by 11 items whereas *space and shape* as well as *change and relationship* each were measured by only five items. Irrespective of the question whether mathematical competence has a two- or three-dimensional structure, our findings confirm the result that mathematical competence of young children is not represented by only one competence dimension *quantity*. This finding implicates for further educational research, such as curricula evaluation as well as assessment development, that it is worthwhile to consider different mathematical competence dimensions. This might provide more sophisticated results, for example, about the effectiveness of curricula or children's mathematical competence development.

In the second step of our analysis, we also included the mathematical content areas *data and chance* as well as *units and measuring*. A three-dimensional model covering the content areas *quantity*, *space/shape/change/relationship*, and *data and chance* showed a good fit to the empirical data. As expected, the correlations between the three latent factors were substantial and significant but supported the result that the factors are separate dimensions. Hence, *data and chance* seems to be an aspect of mathematical competence of preschool children which should not simply be ignored. As a limitation, it must be mentioned that the content area *data and chance* was only measured by five items. This could be considered as strength of these items but from a content validity perspective, five items are not sufficient to mirror a complex mathematical content area.

The models, including the latent factor *units and measuring*, did not converge when processed in Mplus. A possible reason might be the small number of only five items which were too diverse to mirror the content area *units and measuring*. From a cognitive perspective, some items might be too narrow to the content area *quantity* and others too narrow to the content area *space and shape*. This is not surprising because *units and measuring* can be considered as "a bridge" between these two domains, since it combines aspects of *quantity* (count how often a unit is given) with aspects of *space and shape* (make a length or content area "countable") (Clements & Sarama, 2007).

Our study has some limitations. As we already mentioned before, there is a limited number of items for some mathematical content areas. We plan further studies which include the whole item pool of the KiKi (i.e., all three versions of the KiKi) so that we have a larger basis for each content area. Furthermore, the convenience sample of this study is not representative since it is limited to only one region in Northern Germany.

Overall, our study contributes to the research on the structure of preschool children's mathematical competence. The results indicate that mathematical competence in this age group can be considered as multidimensional, not only from a theoretical but also from an empirical perspective. This extends the results of Clements et al. (2008) as well as Weiland et al. (2012), who developed an instrument to assess preschool children's mathematical competence in a broad way. Moreover, our findings are in line with the results of Maier and Benz (2014), who showed differentiated competences of young children in the domain of space and shape. However, the descriptive results of our study do not provide causal evidence that preschooler's mathematical competence is already a multidimensional construct. Therefore, further research investigating, for example, the development of the competence structure as well as the predictive function of the different dimensions of early mathematical competences for children's mathematical development in school is needed.

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Chapter 9
TellMEE – Telling Mathematics in Elementary Education

Rebecca Klose and Christof Schreiber

Abstract The recently launched project 'TellMEE' (Telling Mathematics in Elementary Education), at the University of Giessen (Germany), focuses on the individual concepts of preschoolers. Their explanations pertaining to arithmetic and geometric content are of particular interest. Hereby the focus will be on 2-D shapes. In order to investigate how preschoolers verbalise their individual ideas and concepts, they undergo an interactive process with the aim of creating an audio recording. The procedure will be illustrated through the presentation of an empirical example. The analysis of interaction is used to analyse the preschoolers' utterances.

Keywords Concepts · Concept definitions · Concept image · Verbal explanation · Shapes · Audio recording

Developing Mathematics Concepts in Preschool

Mathematics lessons in school are assigned a central role in constructing mathematical concepts. The goal is for students to develop sustainable ideas with reference to mathematical content. Nevertheless, it should be noted that, 'many mathematics concepts, at least in their intuitive beginnings, develop before school' (NCTM, 2000, p. 73). Already before coming to school, children gather various types of mathematical experiences from daily life and playful settings. Concept formation takes place in preschool and at primary school age particularly, by using objects actively together with language (Franke & Reinhold, 2016). Hereby, the development of basic mathematics competences is of great importance for future learning in school (Koch, Schulz & Jungmann, 2015). The didactic community agrees on mathematical education playing an important role in preschool (see Kortenkamp, Brandt, Benz, Krummheuer, Ladel & Vogel, 2014). Early childhood mathematics

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education should align itself with the fundamental ideas of the subject. The kind of mathematical learning opportunities that children in kindergarten should be confronted with is still subject to discussion in Germany (Schuler, 2013). Based on the German national education standards for primary schools, Steinweg (2008) distinguishes four content-related competence areas for kindergarten, namely, ‘Number and Structure’, ‘Space and Shape’, ‘Data and Probabilities’ and ‘Dimensions and Time’, which correspond to the process-related competences ‘Communicating and Arguing’, ‘Justifying and Testing’, ‘Being Creative and Solving Problems’ and ‘Sorting and Using Patterns’ (ibid., p. 147).

Young Children’s Concepts of 2-D Shapes

In this paper, we want to place our focus on the content-related competence ‘Space and Shape’, namely, on young children’s individual concepts of 2-D shapes. Research studies have shown that young children indeed form ideas and individual concepts about 2-D shapes before entering school (e.g. Clements, 2001; Eichler, 2007; Maier & Benz, 2014). Thus, by the time they start formal schooling, they have at least some ideas about common shapes such as circles, squares, triangles and rectangles. Clements, Swaminathan, Hannibal and Sarama (1999) investigated criteria young children use to distinguish geometric shapes, which are common in the social-cultural environment. They collected data primarily through clinical interviews. Hereby, their focus was on the children’s responses while they were performing shape-selection tasks (pencil-and-paper tasks). The authors found out that the children identified circles with a high degree of accuracy. The six-year-olds performed better than the younger children, who also chose the ellipse and curved shapes. If the children described the circles at all, they mostly did it by using the word ‘round’. The children’s accuracy in identifying squares was only slightly less. Due to this, only a minority of the children’s selection reasons referred to the properties of a square. However, the children were less accurate in recognising triangles and rectangles. Slightly more than half of the rectangles were identified correctly. The children appeared to accept ‘long’ quadrilaterals, which had at least one pair of parallel sides, to be rectangles. The fact that young children showed a better identification accuracy for circles and squares was explained by the aspect of symmetry:

> Those figures that are more symmetric and have fewer possible imagistic prototypes (circles and squares) are more amenable to the development of imagistic prototypes and thus show a straightforward improvement of identification accuracy. Rectangles and triangles have more possible prototypes. (Clements et al., 1999, p. 207)

The young children’s verbalisations were limited (ibid.). If children gave verbal responses, they mostly referred to visual aspects or to some properties. With their results, the authors suggested reconsidering van Hielian research and their levels of geometric thinking. They assumed an earlier prerecognitive, syncretic level than originally described.

Our project attempts to develop an approach, which focuses especially on the verbal explanations of preschoolers. By doing so, we hope to get further insights to the preschooler's individual concepts.

Concepts: Concept Image and Concept Definition

When we look at the meaning of a concept, we draw upon cognitive linguistics. In cognitive linguistics, a concept refers to an 'idea of how something is in our experiential world' (Pörings & Schmitz, 2003, p. 15; translation by the authors). A concept can refer to a single mental unit (entity) or to a whole set of entities. Furthermore, a concept is structured if it refers to a whole set. The concept of a 'quadrilateral', for example, consists of a specific set of mental units (e.g. square, rhombus, kite, rectangle) that systematically exclude other entities (e.g. triangle or circle). A large number of individual semantic elements or entities are thus compiled in a structured way and categorised. Mathematics lessons in school are assigned a central role in constructing structured, mathematical concepts. When it comes to the investigation of concept formation in the mathematics classroom, Tall and Vinner's theory of concept image and concept definition (1981) is often drawn upon. Although their theory focus is on older students, certain aspects are still relevant for this paper. With regard to a mathematical concept, the two components of concept image and concept definition are key. Concept image includes 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes' (Tall & Vinner, 1981, p. 2). It builds on various experiences over time and is flexible, such that it changes when interacting with a new stimulus. When facing mathematical vocabulary (e.g. 'body'), a collection of ideas may arise. However, evoked images might not necessarily pertain to a formal, mathematical kind of concept but rather to an individual one. Concept definition is further characterised by 'a form of words used to specify that concept' (Tall & Vinner, 1981, p. 2). Depending on the context, concept definitions may be rather personal or formal, self-constructed or linguistically defined or rather adopted. In each case, people use these forms of words to explain their own evoked concept image (Tall & Vinner, 1981). Thus, the concept image can be accessed through concept definitions.

However, apart from verbalising the individual concepts, they can also be expressed through nonlinguistic actions (Pörings & Schmitz, 2003). When people want to convey meaning due to the evoked concept image, it might also be expressed through verbalised actions. Thus, it is also possible that concept definitions and actions can occur and interact at the same time (see Fig. 9.1).

The concept image is not accessible, as indicated by the continuous line surrounding it. However, it is possible to access a concept image through a concept definition and actions. The dashed lines surrounding both of them indicate this. In our project, the focus lies more on the linguistic side, in other words, the concept definitions.

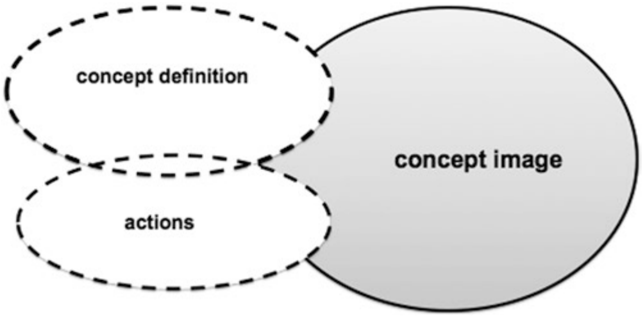

Fig. 9.1 Accessing a concept image by concept definition and actions

TELLMEE Project (Fig. 9.2)

The individual concepts of kindergarten children are investigated by producing audio recordings. The idea was inspired by a previous project ('PriMaPodcast'), in which school students produced audio podcasts on mathematical topics. The production process of mathematical audio podcasts served as a special communication tool for promoting reflection and deepening the understanding of mathematical content (Klose & Schreiber, 2014). Developed by Schreiber (2013a), this method has been used in various practical teaching and research-oriented learning arrangements in schools and teacher education. The production of audio podcasts has also been used as a research method within the scope of a thesis that studies the individual concepts of primary students, who are taught bilingually (German and English) (see Klose, 2015). The special feature that the use of digital media brings forth is the focus on a certain form of representation. In the 'PriMaPodcast' project, the students were challenged to produce verbal explanations pertaining to mathematical topics as precise as possible without the use of written and graphical elements. The multi-staged production process of audio podcasts has been modified and aligned to the objective of the 'TellMEE' project as well as the age and developmental state of the preschoolers.

The project 'TellMEE' aims to gain insight into the ideas and individual concepts of kindergarten children. The way children are *Telling Mathematics* to others is of particular interest. In order to investigate how preschoolers verbalise their ideas and individual concepts, we want to examine the following research questions:

1. How do the preschoolers express their ideas and individual concepts?
2. To what extent do they use mathematical vocabulary?

To obtain information with regard to the abovementioned questions, the preschoolers undergo a four-staged process in teams of two (see Fig. 9.3):

Fig. 9.2 TellMEE logo

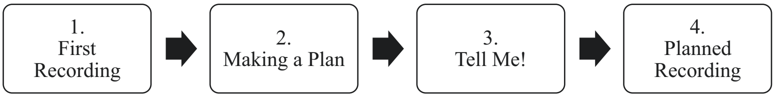

Fig. 9.3 Production stages

Production Process

1. First Recording (FR): A question to a mathematical topic is read aloud to the preschoolers. A voice recorder is used to record their response. By talking aloud and for others, the children verbalise their individual ways of thinking. The children express their individual concept image by making use of certain words and language structures (concept definition).
2. Making a Plan (MaP): To record the final audio file, the children need to make a plan first. Therefore, they are given paper and pens as well as some material (templates and objects). They are free to decide on how to realise their planning as well as the content and structure to be used. At this stage, it is interesting to observe, how the children proceed in describing mathematical content and what arrangements they settle on.
3. Tell Me! (TM): Up to this stage, the team has worked autonomously. Before the children produce the recordings, they present their plan to an instructor. At this point, the instructor may pose questions and ask for more precise explanations to get further insights into the children's individual concepts and ways of thinking. At the same time, the children's ideas can be addressed and discussed.
4. Planned Recording (PR): Based on the previous planning (stage 2) and the instructor's feedback (stage 3), the preschoolers produce an audio file using a voice recorder.

So far, the project 'TellMEE' focuses especially on the individual preschool concepts of numbers and geometric shapes. In this paper, an example taken from the field of geometry, namely, that of 2-D shapes, will be described in greater detail. Moreover, a closer look will be taken at the interaction analysis before the example is presented and analysed.

Analysis of Interaction

The interaction analysis is based on the ethnomethodological conversation analysis, developed by Bauersfeld, Krummheuer and Voigt at the IDM Bielefeld. It deals with processes of interaction that take place in school (Bauersfeld, Krummheuer & Voigt 1988). This form of analysis is based on symbolic interactionism:

> The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing. Their actions operate to define the thing for the person. Thus, symbolic interactionism sees meaning as social products, as creations that are formed in and through the defining activities of people as they interact. (Blumer, 1969, p. 4-5)

The meaning of a thing is thereby negotiated through interaction. This negotiation occurs during processes of social interaction from which understanding and cooperation emerge on a semantic level.

To enable this negotiation of meaning, the participants' interpretations of a situation must accommodate that of the other. The definitions of the situation are not necessarily identical, but they must harmonise sufficiently to continue the development of the interaction. Therefore, the participants' products are not seen as a shared meaning but rather as a 'taken-as-shared-meaning' (Krummheuer & Fetzer, 2005, p. 25). Such an 'interim product' of the interaction is generated by the process of meaning negotiation. It signals a thematic openness towards the continuing progress of the interaction (see Naujok, Brandt and Krummheuer, 2004). Through the reciprocal interpretation attempts, there is an ongoing process of 'clarification' during the attribution of meaning by the participants (see also Schreiber, 2013b).

By using the interaction analysis, the way in which individuals create and negotiate taken-as-shared-meaning is reconstructed (Krummheuer & Naujok, 1999; Krummheuer, 2000). The aim is to reconstruct any operations in the situation that are meaningful for the participants and to construct as many interpretations of these actions as possible. These initial interpretations are then reinforced or rejected in order to ensure the most convincing interpretation of the episode.

In the empirical example, the production stages were recorded as screen videos and transcribed. These scenes were then interpreted in detail using the interaction analysis. There are five steps of analysis (Krummheuer 2010; Krummheuer & Naujok 1999):

1. Outline of the Interaction Units

First, excerpts or interaction scenes are selected (Krummheuer, 2010) according to certain criteria. Depending on the research interest, subject-specific/didactic, interaction-theoretical or linguistic categorisation criteria can be determined. In the empirical example, we consider the four production stages as relevant interaction units. The focus lies particularly on the verbalised individual concepts and ideas of the participants as well as their use of mathematical linguistic means.

2. General Description

By means of a general description, the anticipated semantic content of the selected scenes will be explained.

3. Detailed Analysis of Individual Utterances

Thereafter, the individual utterances will be closely examined and analysed in groups. A joint interpretation through group work provides multi-perceptivity in the sense that it enables many different views on the same situation. By doing so, many interpretations of the utterances and actions can be collected. Yet, it is important to keep the utterances in chronological order as well as maintain openness towards the interpretations. Plausible interpretations can only be justified and linked backwards, as they can only be based on utterances made previously. Another aspect is the fact that alternative interpretations need to be justified during the course of the interaction analysis.

4. Turn-by-Turn Analysis

The turn-by-turn analysis in groups compares, applies and even restricts the forth-brought interpretations to the actual course of conversation. A conversation analysis therefore occurs step by step, turn by turn. During the comparison, differences in opinions or views can be discussed or corrected. Then, the agreed on aspects are considered to be shared knowledge.

5. Summarised Interpretations

The agreed upon knowledge is presented as a coherent interpretation. The goal is to justify the diversity of interpretations. Such coherent interpretations can often be found summarised in publications, instead of the detailed process description.

Our empirical example will present the transcripts and some interim products as well as the summarised interpretations. The children's explanation processes will be described in greater detail with reference to the individual production stages. Squared brackets will be used to indicate single utterances. For example, the 7th utterance in the transcript 'Making a Plan' will be denoted as: 'I see a triangle' [MaP7].

Empirical Example

The following example shows the production process of two boys, who are soon to enrol in school. Child 1 (Ch1) is five years old and already able to read. Child 2 (Ch2) is six years old. He can read fluently and speaks Russian, English and German. The instructor's (I) question is 'What shapes do you know? Describe all, which you know.'

In addition to the original German transcripts of the production process, the English translations are presented as well. The first column depicts the utterance number and the second column shows the speaker. In the third column, the English translation is presented, and in the last column, the original German text serves as comparison. A transcription legend can be found at the end of this paper. Paralinguistic details are given in brackets and italics (*like this*). The various transcripts that follow are in conjunction with the four stages 'First Recording' (FR), 'Making a Plan' (MaP), 'Tell Me!' (TM) and 'Planned Recording' (PR), respectively. We have tried to translate the statements of the children and the instructor meaningfully, yet would like to point out that some terms do not always have a direct English equivalent. For example, in German the colloquial terms for the polygons 'Viereck' (engl. quadrilateral), 'Fünfeck' (engl. pentagon) and 'Sechseck' (engl. hexagon) have no English equivalent translation. If translated literally, they mean 'four-corner', 'five-corner' and 'six-corner', respectively. Therefore, we could not realise a translation that may be content-wise close enough to the German words, but which is taken into account in the summarised interpretations. Moreover, the German term 'Formen', which is stated in the task, refers to shapes. While in colloquial language this can refer to 2-D and 3-D objects, German mathematics textbooks usually make reference to 2-D shapes.

First Recording

The recording begins the moment the instructor starts to read out the question, 'What shapes do you know? Describe all which you know.'

FR		English (*translation by the authors*)	German (*original*)
1	I	what **shapes** . do you know . describe all . which you know . .	welche **Formen** . kennst du . beschreibe alle . die du kennst . .
2	Ch1	(*raises hand*)%	(*meldet sich*)%
3	I	just say it	einfach sagen
4	Ch1	rectangle/	Rechteck/
5	Ch2	triangle and quadrilateral	Dreieck und Viereck
6	Ch1	circle/	Kreis/
7	Ch2	hexag o n . . pentag o n (*5 s*)	Hexag o n . . Pentag o n (*5 s*)
8	Ch1	hexagon/	Sechseck/
9	Ch2	**I** already said that	das hab **ich** schon gesagt
10	Ch1	really/	echt/
11	Ch2	yes (*11 s*)	ja (*11 s*)
12	I	okay . can you think of anything else/	okay . fällt euch noch was ein/
13	Ch1	I can't	mir nicht
14	Ch2	no	nein

After the task was read out aloud, child 1 raises his hand. This indicates that the child experiences this setting as a teaching-learning situation. The instructor quickly points out that he does not have to raise his hand, but simply talk away. Child 1 is the first to name the shape 'rectangle' [FR4]. Mentioning this 2-D shape first is rather unexpected. Child 2 adds two (typical in German mathematics) shapes:

‘triangle and quadrilateral’ [FR5]. However, which triangle and quadrilateral child 2 exactly refers to remains unclear. Child 1 takes over again and adds ‘circle’ [FR6]. Surprisingly, child 2 responds by literally mentioning the terms, ‘hexagon’ [FR7] and ‘pentagon’ [FR7]. Using the Greek words is rather unusual for German-speaking children, as they have simpler terms like ‘Sechseck’ (six-corner) and ‘Fünfeck’ (five-corner). It can be observed that child 1 also makes reference to the hexagon by using the more common German term ‘Sechseck’ [FR8], where child 2 then points out that he has already named it prior [FR9]. Child 1 was unaware of this [FR10]. At this point, it becomes clear that child 2 has an individual concept for the hexagon and two corresponding words for it. Perhaps, knowing these specific mathematical terms can be attributed to child 2’s multilinguistic background. Moreover, it can be observed that at least for child 2, it is important that no answer is mentioned twice. Therefore, it suggests that the children mean something else instead of the rectangle when they mentioned the quadrilateral (Viereck) prior. After a long pause, the instructor asks if the children could think of more shapes. Both children decline.

In this first stage, it becomes apparent that both children associate the ambiguous term ‘Form’ (shape) with 2-D shapes. This could not be assumed previously. The children address the task by listing terms without describing them any further.

Making a Plan

After the ‘First Recording’, the children were asked to explain the topic to others. For this, they had to work out a plan together. Various geometric shapes (circles, squares, rectangles, triangles) were provided on a template in different sizes and positions (see Fig. 9.4) to support them. This template was created for the pilot study on the basis of German schoolbooks for the first school year. The children were given pens and two sheets of paper. They were free on how to make their plan.

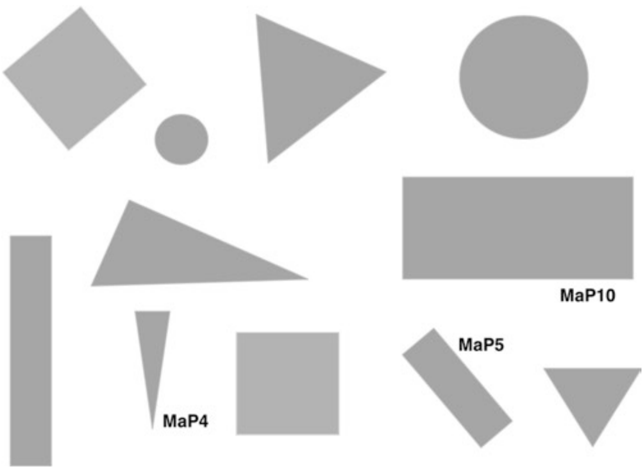

Fig. 9.4 A3 template for the children. Annotations to the template were added later

MaP		English (*translation by the authors*)	German (*original*)
1	I	(*points on the template*)% you can also (*points on the template*) make the plan on this%	(*deutet auf die Vorlage*)% ihr könnt euch auch (*deutet auf die Vorlage*) hier den Plan machen%
2	Ch2 Ch1	< (*looks at the template*) wow wow% < (*looks at the template*)%	< (*blickt auf die Vorlage*) wow wow% < (*blickt auf die Vorlage*)%
3	I	and the question/ what shapes do you know . describe all . which you know . .	und die Frage/ welche Formen kennst du . beschreibe alle . die du kennst . .
4	Ch2	I see a tip (*points with a finger on a triangle*)%	ich sehe eine Spitze (*deutet mit dem Finger auf ein Dreieck*)%
5	Ch1	I see a rectangle (*points with a finger on a rectangle*)%	ich seh ein Rechteck (*deutet mit dem Finger auf ein Rechteck*)%
6	Ch2	I see a **long** *rectangle* (*moves his left hand to the front and his right hand to the back*)% (*4 s*) and a large tip (*3 s*)	ich seh ein **langes** Rechteck (*fährt mit der linken Hand nach vorne und mit der rechten Hand nach hinten*)% (*4 s*) un eine große Spitze (*3 s*)
7	Ch1	I see a triangle	ich seh ein Dreieck
8 9	Ch2 I	< I see a circle < you can also point on it so that you all know which one is being referred to	< ich seh ein Kreis < ihr könnt auch drauf zeigen damit ihr gegenseitig wisst welches ihr meint
10	Ch2 Ch1	< a **broad** (*moves both arms to the side*) rectangle% (*3 s*) (*points with a finger on a rectangle*)% (*3 s*) I'll draw it (*both children begin to copy the figures onto their respective blank sheet of paper*)% < (*takes a colour pencil*)%	< ein **dickes** (*breitet beide Arme zur Seite aus*) Rechteck% (*3 s*) (*deutet mit dem Finger auf ein Rechteck*)% (*3 s*) mal ich mal auf (*beide Kinder beginnen, die Figuren auf eigenen Blankopapieren abzuzeichnen*)% < (*nimmt sich einen Stift*)%

Thereafter, each child copied the shapes of the template free-handedly onto a sheet of paper (see Figs. 9.5 and 9.6). After a while, child 2 requested a second sheet of paper and drew a large triangle (see Fig. 9.7) on it.

MaP		English (*translation by the authors*)	German (*original*)
11	Ch2	boah . that is a long one . and I have already drawn all shapes	boah . das is ja ein langes . un ich hab schon alle Formen aufgemalt
12	I	that you know of/	die du kennst/
13	Ch2	**these** (*points to the template with a pen*) that are on here%	die **hier** (*deutet mit dem Stift auf die Vorlage*) drauf sind%
14	I	but you are also supposed to draw those that you know . . do you know any others/	aber du sollst auch die aufmalen die du kennst . . kennst du noch andere/
15	Ch2	nope	nee
16	I	but just now you mentioned others	aber du hast doch eben noch andere gesagt

(continued)

MaP		English (*translation by the authors*)	German (*original*)
17	Ch2	a rhombus . . . but how does a rhombus go/ (*7 s, Ch2 draws something*)% (*looks at his drawing, see* Fig. 9.7) actually it goes like this like a kite (*9 s*) besides these I do not know any other shapes (*5 s*) I do not know any others . I have already copied those that I know of (*turns second paper over and draws something*)% finished . I have no others . I cannot think of any others anymore	eine Raute . . . aber wie geht eine Raute/ (*7 s, Ch2 zeichnet etwas auf*)% (*blickt auf seine Zeichnung, siehe* Fig. 9.7) eigentlich geht so eine Raute wie ein Drachen (*9 s*) sonst kenne ich keine Formen mehr (*5 s*) ich kenne keine Formen mehr . ich habe schon alle abgeschrieben die ich kenne (*dreht sein zweites Blatt um und zeichnet etwas auf*)% fertig . ich hab sonst keine . mir fallen keine mehr ein mehr
18	I	have you considered how you want to describe them/ . . to someone who doesn't know them . how they look/	hast du dir schon überlegt wie du die beschreiben möchtest/ . . jemandem der die nicht kennt . wie die aussehen/
19	Ch2	in any case there is no rhombus here	hier steht jedenfalls keine Raute drauf
20	Ch1	yes but you are supposed to take all those you know . . even if they are not on it	ja aber du sollst ja auch alle nehmen die du kennst . . wenn die da nicht drauf ist

In the second production stage, the focus lies initially on the template with the shapes. The instructor notifies the children that they can use the template for their own plan [MaP1]. Thereupon, the children view the template. As child 2 looks at the template, he appears to be impressed [MaP2]. After the instructor has read out the

Fig. 9.5 Shapes drawn by child 1

Fig. 9.6 Shapes drawn by child 2

Fig. 9.7 Triangle drawn by child 2

task again, child 2 points onto the acute-angled triangle and calls it ‘a tip’ [MaP4]. Hereby, he describes what he sees. At this point, he either means to describe the shape as ‘a tip’ or he means to refer to the sharp angles of the shape, particularly the property of it being highly acute-angled. Both scenarios are possible. Child 1 picks up the phrase ‘I see’ and identifies ‘a rectangle’ [MaP5]. Child 2 then describes two other forms as ‘a long rectangle’ and ‘a large tip’ [MaP6]. His hand movements to the rectangle, point to the fact that he means the rectangle in the lower left corner. He distinguishes this rectangle from the rectangle previously identified by child 1 through the use of the adjective ‘long’. At this point, it is not clear to which shape he refers to with ‘large tip’, even though one can assume that it may also refer to a triangle. It does not seem to bother child 1 that child 2 identifies two shapes (presumably also a second triangle) as ‘tip’. At least he does not disagree or question it. Nevertheless, child 1 now brings in ‘a triangle’ [MaP7]. It appears that he is referring to another triangle, presumably an equilateral or isosceles triangle. Child 1’s utterance seems to follow the phrasing of child 2. He begins by saying what he sees and forgoes the use of gestures. This procedure reminds of the German version of the children’s game ‘I spy with my little eye’. They take turns in describing something and introduce their utterances with the phrase ‘Ich sehe…’ (engl. I see…).

Child 2 moves on to the next shape and names ‘a circle’ [MaP8]. As the template shows two circles, it again is not clear which circle child 2 is referring to. Even though the children seem to be satisfied with their approach, the instructor reminds them that they can point to the shapes their utterances correspond to. By doing so, the instructor in particular, wants to know the shapes the children refer to. Child 2 introduces another rectangle by distinguishing it from other rectangles through gestures and verbally referring to it as ‘a broad rectangle’ [MaP10]. Child 2 uses the adjective ‘broad’ to indicate the special position and size of this rectangle. Thereupon, he expresses that he would like to draw it. Child 1 and child 2 each begin to draw the shapes on paper (see Figs. 9.5 and 9.6). Hence, the children refer to the given shapes (rectangles, triangles, circle, squares) presented on the template in a graphical way.

After child 2 informs the instructor that he has ‘already drawn all shapes’ [MaP11], the instructor asks further and points out that he is to refer to all shapes known to child 2 [MaP14] and those he mentioned previously [MaP16]. Perhaps he wants to focus again on the previously mentioned shapes, such as ‘hexagon’, ‘pentagon’ and ‘quadrilateral’, as well as the term ‘tip’. Yet instead, child 2 adds the mathematical vocabulary ‘rhombus’ [MaP17] and the word ‘kite’ [MaP17]. When child 2 ponders about how to draw a rhombus, he makes the reference to a kite. Thereafter, he continuously emphasises that he does not know any other shapes. The instructor asks child 2 if he already knows how he will describe his shapes to others [MaP18]. By doing this, the instructor tries to return the focus to the second part of the task. Child 2 does not respond to the question, instead he still seems to be in thought with the rhombus. Even though his rhombus has taken an odd shape (see Fig. 9.8) – the drawing reminds of a Santa Claus hat – he does appear to know that this shape is not to be found on the template [MaP20]. Child 1 has understood the task as having to describe all shapes that they know of [MaP20].

Fig. 9.8 Child 2’s drawing called ‘rhombus’ [MaP17]

In this stage, the children proceed as follows: They mostly refer to the template and try to name some of the shapes. By pointing at three shapes, a deeper insight into their understanding of the terminology is given. In our example, child 2 uses adjectives to differentiate the shapes and characterises them further ('long rectangle' [MaP6], 'large tip' [MaP6], 'broad rectangle' [MaP10]). Child 2 initiates both the descriptive phrasing of 'I see' as well as the drawing of shapes. Child 2 introduces two new shapes, which are not found on the template and places them in association: a rhombus and a kite. The children do not touch on the before-mentioned polygons 'quadrilateral' [FR5], 'hexagon' [FR7] and 'pentagon' [FR7] anymore.

Tell Me!

Upon inquiry, the children did not need any additional time to prepare. In the next stage, the children were supposed to present their ideas to the instructor. Before presenting their ideas, they were required to introduce their topic once again.

TM		English (*translation by the authors*)	German (*original*)
1	Ch1	(*holds his hands before his face*) oh I cannot think of anything%	(*hält sich die Hände vors Gesicht*) oh mir fällts nicht ein%
2	I	shall I say it again/	soll ichs nochmal sagen/
3	Ch1	(*nods*)%	(*nickt*)%
4	I	the shapes . yes/ what shapes you know . and that you describe all the ones you know . you have already mentioned so many . perhaps you can start with one . . with which one would you like to start/	die Formen . ja/ welche Formen du kennst . und dass du alle beschreibst die du kennst . ihr habt ja eben schon ganz viele gesagt . vielleicht könnt ihr mal mit einem anfangen . . mit welchem wollt ihr anfangen/
5	Ch2	with the rectangle	mit dem Rechteck
6	I	describe how it looks like	beschreibt mal wie das aussieht
7	Ch2	it has four corners and it is small	das hat vier Ecken und das ist k lein
8	Ch1	the triangle it has **three** corners (*traces the sides of the triangle with the right index finger on his drawing that lays in front of him*)%	das Dreieck/ das hat **drei** Ecken (*fährt mit dem rechten Zeigefinger die Seiten eines Dreiecks seiner vor sich liegenden Zeichung nach*)%
9	Ch2	the circle- has no corners- and it is round (*12 s*) the quadrilateral has four sides and is squarish . . .	der Kreis- hat keine Ecken- und er ist rund (*12 s)* das Viereck hat vier Seiten und ist quadratisch . . .
10	I	so which quadrilaterals are squarish since you put it that way	welche Vierecke sind denn quadratisch weil du das so sagst
11	Ch2	these (*points to a square on the template*)% these (*points to a second square on the template*)%	diese (*deutet auf der Vorlage auf ein Quadrat*)% diese (*deutet auf der Vorlage auf ein zweites Quadrat*)%
12	I	I see . and what is with those (*points to three rectangles consecutively*)% is it also a quadrilateral/	aha . und was ist mit denen (*deutet nacheinander auf drei Rechtecke*)% ist das auch ein Viereck/
13	Ch1	nope	nee

(continued)

TM		English (*translation by the authors*)	German (*original*)
14	I	but has four corners	aber hat doch vier Ecken
15	Ch1 Ch2	< yes but it is not a quadrilateral < yes	< ja aber es ist kein Viereck < ja
16	Ch2	this (*points to a rectangle on the template*)% is a rectangle	das (*deutet auf der Vorlage auf ein Rechteck*)% ist ein Rechteck

Initially, child 1 does not know how to begin [TM1]. Hence, the instructor explains the task again [TM4]. When asked with which shapes they would like to start, child 2 names the 'rectangle' [TM5]. Since the boys have been quick to talk about the rectangle in previous stages, it appears to be a familiar or fascinating shape to them. The instructor asks him to describe the rectangle [TM6]. Child 2 now mentions the four corners of a rectangle and describes the rectangle as being small [TM7]. The latter may indicate to his first rectangle reference on the template (see Fig. 9.4, MaP5). The sides of a rectangle are not discussed further. Then child 1 begins to describe 'the triangle' [TM8]. His phrasing follows child 2's phrase as already seen in stage 2. His utterance is accompanied by gestures, tracing the sides of the drawn shape that lays before him. It is possible that he may identify the sides as corners. He may even implicitly gesture and indicate another property, namely, the three sides of a triangle. Yet, the actual intention remains hidden. Thereafter, child 2 describes the circle in a complete sentence, naming specific properties, such as 'no corners' and 'it is round' [TM9]. By doing so, he considers the shape as well as the circumference. Since child 1 is not taking his turn to describe another shape, child 2 goes on to mention 'the quadrilateral' [TM9]. For child 2, a quadrilateral 'has four sides and is squarish' [TM9]. Unlike the description of the rectangle, child 2 only implicitly focuses through the property 'squarish' on the corners of this shape. However, the sides are the explicit focus. Perhaps he refers to the square without naming it as such. At this point, the instructor asks specifically for the quadrilaterals that he considers to be 'squarish' [TM10]. Using the template, it becomes apparent that when he says quadrilaterals, he actually means squares [TM11]. The instructor uses the opportunity to ask him, if the depicted rectangles are also considered quadrilaterals [TM12]. Child 1 rejects this [TM13]. It appears as if for him, the word 'rectangle' is the designated mathematical term for this particular shape. The instructor points out that a rectangle also has four corners (and therefore also is a quadrilateral) [TM14]. At this point, the instructor indicates the connection between the word 'quadrilateral' and the property of having four corners.[1] Both boys agree on this [TM15]. Child 2 emphasises, however, that a rectangle is still not a 'quadrilateral' (in terms of a square); thus, it should not be called 'quadrilateral' [TM15]. To support his argument, he points on a rectangle on the template and names it 'rectangle' [TM16].

Unlike before, the children are more precise in their descriptions by differentiating between shapes according to their properties (e.g. 'corners' [TM7], 'sides'

[1] In German, the corners play a significant role, as the literal translation of the German term 'Viereck' is 'four corners'.

[TM9], ‘round’ [TM9] and ‘squarish’ [TM9]) and using the definite article. Their utterances are hereby supported through the template, their own drawings and gestures. The gestures in combination with the utterances provide deeper insights into the children’s individual concepts.

In the following, the instructor attempted to clarify that rectangles and squares are both quadrilaterals, highlighting – from a German point of view – the most obvious property of quadrilaterals: having four corners.

Planned Recording

After the practical phase the children recorded the final version. A voice recorder was used to record their response. They did not follow any script, instead, child 1 used the template, while child 2 used his own drawings to support their individual explanations.

PR		English (*translation by the authors*)	German (*original*)
1	Ch1	the circle has no corners but it is **round** (*while Ch1 speaks he traces the circle on his template twice using his right index finger*)	der Kreis hat keine Ecken sondern er ist **rund** (*während Ch1 redet umfährt er den Kreis auf seiner Vorlage zweimal mit dem rechten Zeigefinger*)
2	Ch2	the quadrilateral . has . four corners and is squarish	das Viereck . hat . vier Ecken und ist quadratisch
3	Ch1	the triangle/ . has three corners . (*traces the sides of a triangle on his template using his right index finger*)%	das Dreieck/ . hat drei Ecken . (*fährt mit dem rechten Zeigefinger die Seiten eines Dreiecks auf seiner Vorlage nach*)%
4	Ch2	the rectangle . has four corners (*17 s*)	der Rechteck . hat vier Ecken (*17* s)

In stage 4, ‘Planned Recording’, most statements from earlier on (see stage 3 ‘Tell Me!’) were taken up again. This took place without prior planning or agreeing to a particular structural sequence. This time, child 1 begins with describing a circle [PR1]. The ‘but’ emphasises the property of the shape being ‘round’. At this point, he uses the template and gestures to describe the circle and its properties. Child 2 describes the quadrilateral by not explicitly mentioning the four sides, instead he emphasises more on the four corners [PR2]. This may be due to the instructor having highlighted this property previously. He addresses the sides through the property ‘squarish’ [PR2], which implicitly describes the position and length of the sides. Child 2 sticks to the term ‘quadrilateral’. As already seen in stage 3, child 1 again paraphrases the triangle by stating the property of having ‘three corners’ [PR3] and only refers to the three sides using gestures. This time, ‘the rectangle’ is described last by child 2. He only discusses the ‘four corners’ but not the size of the rectangle [PR4].

Conclusion

Carrying out these individual stages has proven to be quite useful for providing a deeper insight into the ideas and individual concepts of preschoolers. Therefore, our four-staged production process of audio recordings is an appropriate tool. The tool is adjusted to the children's age and communication skills. On the whole, all four stages are important to investigate the children's individual concepts. The 'First Recording' has shown to be very informative in demonstrating what the children were individually capable of expressing with regard to a particular topic. By addressing the topic verbally, the children expressed their individual ways of thinking and activated their knowledge. In the second stage 'Making a Plan', it was observed how the children developed a plan through the supporting material, in order to explain the topic to others. Hereby, the children worked autonomously to a large extent. The plans corresponded to the individual capabilities of preschoolers, who were not alphabetised yet. Child 1's utterances were oriented towards child 2's approach and phrasing. Based on his own drawing, he began describing his shapes. Child 2 attempted to describe another shape, 'the rhombus' and created his own drawing. The stage 'Tell Me!' has been particularly insightful, as the preschoolers addressed the properties of the shapes in more detail. At this stage, the content, intended for the final recording, was rehearsed and practised. In order to get deeper insights into the children's individual concepts, the instructor posed further questions and could refer to the template. The 'Planned Recording' constitutes of the final stage and the project's defined goal of the given task. By explaining the topic to others, the children's descriptions increasingly gained precision.

Pertaining to our first research question, so far, we have observed that throughout the stages, the preschoolers are able to express their individual understanding and ideas of 2-D shapes. These were evidently growing in precision throughout the stages of the empirical example. To address the second research question, the transcripts of the empirical example show that the children did use mathematical vocabulary. Also, their utterances expanded from labelling the shape as a whole to referring more and more to the properties of the 2-D shapes.

Transcription Legend:

. .. …	Pause: . 1 s .. 2 s … 3 s
(*4 s*)	Pause with given duration from 4 seconds onwards
(*Text written in italics*)	Describes actions, gestures, body movements and, for example, whispering or incomprehensible expressions
%	Described actions, gestures, body movements, etc. end here
bold	Emphasis
b l o c k e d	Stretched pronunciation
/	Pitch inclination
-	Pitch stays constant
\	Pitch declination
#	One utterance is followed immediately by another
<	Two participants are talking both at the same time, for example: Ch 1 < yes but it is not a quadrilateral Ch 2 < yes

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Part III
Design and Evaluation of Mathematical Learning Settings

Chapter 10
Designing Playful Inquiry-Based Mathematical Learning Activities for Kindergarten

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Abstract This study addresses issues related to the process of designing mathematical activities for kindergarten based on the constructs of playful learning and inquiry. The mathematical activities were designed for 5-year-old children in a Norwegian kindergarten setting. In order to address this design process, we focus at issues of mediating design principles and ideas to kindergarten teachers, who are the ones to orchestrate and implement the mathematical activities with children. The study is situated within a design research methodology in which observations, written and personal communication have been used as sources of data. Our analysis shows that the kindergarten teacher closely followed the written instructions given, that she only occasionally orchestrated an inquiry approach to the learning of mathematics, and a limited implementation of playfulness in the activities. These results were due to issues with too detailed written instructions, a possible experienced expectation to carry out all parts of the activities, a possible experienced power relationship between researchers and the kindergarten teacher as well as limited experience in orchestrating mathematical activities and lack of time to adopt playfulness and inquiry as a way of being.

Keywords 5-year-old children · Design research · Guided play · Inquiry · Kindergarten teacher · Mathematics · Orchestration · Playful learning · Power relationship · Written instruction material

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Introduction

This article reports from an early phase of an intervention programme in Norwegian kindergartens, called "The Agder project".[1] This research and developmental project focuses at nurturing 5-year-old children's development within four competence areas, social skills, self-regulation, literacy and mathematics, adopting a playful learning approach. The study reported here focuses at taking a playful learning and inquiry approach to the orchestration and learning of mathematics in the kindergarten area. The aim of this article is to communicate insights into a design process in which researchers were to initially design mathematical learning activities and where one kindergarten teacher (KT) was to adopt and carry out these learning activities designed by us as researchers.

One core idea within the Agder project in general, and this study in particular, is that high-quality early childhood programmes may influence children's early learning in school and also later influence the children's success in school and working life. European intervention programmes and teaching materials in early childhood education in mathematics have proven to have positive effect on children's learning of mathematics (Stehler, Vogt, Wolf, Hauser, & Rechsteiner, 2013). Similar positive effects have been proposed from the Building Blocks programme in the USA (Clements & Sarama, 2011). In particular, some researchers claim that an emphasis on playfulness in early years' mathematics is of particular importance when it comes to long-lasting effects of intervention programmes compared to learning settings which are described as highly instructional (Marcon, 2002; Singer, Golinkoff, & Hirsh-Pasek, 2009).

In order to motivate our study of the design process with respect to developing mathematical learning activities for kindergarten, it is important to align with previous researches of the design process. There are research considering how to design the content and form of professional development programmes for KTs, such as the study of Tirosh, Tsamir, Levenson and Tabach (2011). They suggest that professional programmes should consist of a combination of relevant mathematical content for the KTs to learn and knowledge of children's development, in particular related to mathematics with concrete ideas. They quote Schwan Smith (2001, in Tirosh et al. (2011)) and propose four main areas in professional development programmes: (1) focus on student's learning as a goal, (2) grounded in mathematics, (3) designed to support the teacher's day-to-day practice and (4) appropriate to the participants' contexts.

The curriculum materials such as the Building Blocks material often come with detailed instruction materials and a learning module in written and/or digital form for its users, the KTs, which are supposed to use the materials with children. These curriculum materials usually also come in the form of a booklet for users without following a training programme in using it. The study reported in this paper concerns

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cases where KTs receive curriculum materials in a written format. We have not come across research discussing how to design such curriculum materials in order to support KTs' implementation of them with children. However, there exists a lot of research considering school mathematics teachers' use of curriculum materials, in particular use of textbooks and teachers' guide. Evidence of the strong impact of such curriculum materials on teachers' orchestration of teaching has been reported (e.g. Davis & Krajcik, 2005).

Concerning curriculum materials, Ahl, Gunnarsdóttir, Koljonen and Pálsdóttir (2016) distinguish between traditional teacher guides and educative teacher guides. The first approach typically offers activities ready for classroom practices, where teachers often end up using page by page. The second approach, the educative teacher guides, does "not only provide resources for instruction, but also support teaching as a design process rather than depicting instruction as prefabricated procedures" (p. 192). Teachers in the study by Ahl et al. apparently liked both kinds of teachers' guides, and research elsewhere has shown that in general inexperienced teachers use textbooks and their progression more directly page by page than the experienced ones (Grave & Pepin, 2016). However, Ahl et al. report previous research which have found evidence that curriculum materials emphasising educational features such as "key goals, relevant content, appropriate strategies, and available concrete materials" (p. 192) have a more positive effect on teachers' development of their teaching (Davis & Krajcik, 2005) compared with traditional teacher guides.

The paragraph above considered teacher guides for teachers in schools where the pupils usually have their own textbooks and the teacher guide is supposed to help the teacher in supporting pupils' use of the textbook. The curriculum material we have developed for kindergarten level is not meant for direct use by the children, but as resources for shared adult-lead activity with the children. Thus, it shares the form of a teacher guide but where the activities for the children are included in the guide. The reason for developing a manual of mathematical learning activities somewhat similar to teacher guide was based in developmental part of the Agder project as a whole, in which activities were to be designed within all the four areas of competence followed by KTs' implementation of these activities.

In order to study the subtleties of the process of designing mathematical activities for kindergarten, we focus at issues of mediating design principles and ideas to kindergarten teachers. The following research question has been formulated: *What issues with researcher-designed mathematical activities for 5-year-old children are discerned when a kindergarten teacher orchestrates these activities?*

We mainly use the term orchestration in accordance with Kennewell (2001): "The teacher's role is to orchestrate the supporting features – the visual cues, the prompts, the questions, the instructions, the demonstrations, the collaborations, the tools, the information sources available, and so forth…" (p. 106). However, we want to emphasise that the KT also has to think through and plan the carrying out of the mathematical activities. Thus, the KT's role is both to implement and orchestrate the mathematical activities.

Playful Learning and Inquiry

In this study we adopt a sociocultural perspective on learning (Rogoff, 1990; Vygotsky, 1978, 1986). Thus, we view learning as an individual process of appropriation, mediated by interaction and active participation with others, in which the child takes over "what someone else produces during joint activity for one's own subsequent productive activity" (Moschkovich, 2004, p. 51). It thus follows that a child, in a kindergarten setting, constantly appropriates tools and actions. The basic and fundamental activity of children is play, and as Vygotsky (1978, p. 96) claims: "The influence of play on a child's development is enormous". In the Agder project and in our study, we use the term "playful learning" to emphasise the importance of play and its close relation to learning. The concept of playful learning combines play and learning and takes into consideration that for a child play and learning are one and the same thing (Hirsh-Pasek, Golinkoff, Berk, & Singer, 2009). It has been documented that children of all ages experiment with mathematical concepts through play. "Playful learning, and not drill-and-practice, engages and motivates children in ways that enhance developmental outcomes and lifelong learning" (Hirsh-Pasek et al., 2009, p. 4). According to Weisberg, Kittredge, Hirsh-Pasek, Golinkoff and Klahr (2015), the concept of playful learning captures two types of play, both free play (child-initiated and child-directed play) and guided play (adult-initiated and child-directed play). In both types of play, children are active and lead most of the play. In free play children play without interference from adults. In guided play the KT organises the environment and guides the play with respect to aims for the children's play. It is nevertheless important that the adult makes room for children's self-directed exploration. Fisher, Hirsh-Pasek, Golinkoff, Singer and Berk (2011) argue that children can learn from both free play and guided play. This perspective also addresses the overall theme of this book, early mathematics learning within the poles of construction and instruction. Free play may be associated with the pole of construction while guided play may be associated with the pole of instruction. However, we argue that guided play also may be associated with the pole of construction, as child-initiated self-directed explorations are encompassed and emphasised in guided play. The child thus, when participating in the orchestrated mathematical learning activities, gets opportunities to construct or, in our sociocultural parlance, appropriate mathematical tools and actions.

According to Fisher, Hirsh-Pasek, Newcombe and Golinkoff (2013), "*Free play* generally refers to self-directed activities that are fun, engaging, voluntary, and [sic] flexible [sic] have no extrinsic goals, and often contain an element of make-believe... *Guided play* is a discovery-learning approach intermediate between didactic instruction and free play" (p. 1872, emphasis in original). Within guided play the teacher's role is to orchestrate the activity so that the children's interest, curiosity, engagement, (mathematical) sense-making and ultimately learning are nurtured. It is this balance between freedom and structure that makes guided play such an effective teaching tool. Weisberg et al. (2015) argue that children learn a huge amount through free play and that "free play is a wonderful realm for children to explore their social and self-regulatory skills" (p. 9). However, these authors argue that "guided play is most effective for achieving specific learning goals in areas such as… number sense" (pp. 8–9). We agree with this stance, even though mathematics in the kindergarten context is different from mathematics as an academic and scientific discipline. According to Weisberg et al. (2015, p. 13), "guided play is a powerful tool for enhancing young children's learning". Our study is thus focusing on the guided-play type of playful learning. Similar to Weisberg et al. (2015), who highlight the balance between freedom and structure in playful learning, van Oers (2014) argues that playful activities should contain some elements of instruction. He argues that the variation between instructions and children's self-directed exploration is important, but that children's play should be the starting point. "The nature of the actions embedded in play can vary with respect to their degree of freedom allowed, as long as the activity as a whole remains a playful activity" (van Oers, 2014, p. 121). Thus, the playful activity has to be founded in rules acknowledged between the players, the activity has to be engaging and the activity has to emphasise the player's possibilities to deliberately play in his/her own way.

The concept of inquiry that we take into account is based on a sociocultural perspective on learning and development. It cannot be regarded as a method for solving mathematical problems, nor can it be regarded as a personal exploration of mathematical ideas. According to Jaworski (2005), inquiry is "a way of being in practice" (p. 103), and it is about "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (Wells, 1999, p. 121).

Based on the theory of Wells (1999), we see inquiry as a way children work together and together with the KT to seek answers to meaningful questions. The aim of the activities was to encourage the KT to guide the children into an "inquiry-mode" by his/her way of being – by asking questions, by being curious and by presenting the content in an exciting way. We argue that the natural curiosity amongst children to a great extent coincides with adopting inquiry as an approach to learning mathematics. Moreover, an inquiry approach runs in parallel with the practice of Norwegian kindergartens, where adults and children, to a great extent, interact, play and communicate with each other. This view upon inquiry also illuminates the overall theme of this book. Inquiry as an approach to learning mathematics stresses the importance of children making the mathematics their own (construction) while at the same time stresses the importance of the KT's nurturing of this own-making (instruction).

Furthermore, a characteristic of adopting inquiry as an approach to orchestrate mathematical activities in the kindergarten context is the use of questions. In Carlsen, Erfjord and Hundeland (2010), six different kinds of questions are identified in a similar context to the ones studied here. These authors analysed a KT's orchestration of a measuring activity with a pair of scales. During this half-hour session, more than 150 questions were addressed to the six participating children. The six identified categories of questions were suggesting actions, asking for argument, problem-solving invitation, rephrasing, concluding and open. This last category was labelled open not necessarily because these questions had multiple solutions or answers. They were open in the sense that the content of those questions were linked to the children's opinions, e.g. "Do you think this one weighs the most?", "How can we decide which one of them are the heaviest?" and "What has happened now?" (Carlsen et al., 2010, p. 2571). For the purpose here, it is the first three categories that are of most relevance: suggesting actions, asking for argument, and problem-solving invitation.

Design Research

In our study we have designed mathematical activities for KTs to implement. Our study falls under the critical research paradigm (Carr & Kemmis, 1986), because our intention is to adopt a critical stance towards current practice in Norwegian kindergartens and attempt to make changes to the practice by designing mathematical activities. Furthermore, our methodology may be argued to be situated within design research, since design research "is directed primarily at understanding learning and teaching processes when the researcher is active as an educator" (Kelly, 2003, p. 3). In our case, we as researchers were active in designing the mathematical activities for the KT to implement. Three or all of us were present, observed, videotaped and made notes when a KT selected for the study made the implementation of the activities. Furthermore, we all discussed each session and reflected on how the activities were implemented, whether they were orchestrated in our intended and anticipated way, and to what extent each orchestration could inform revisions of future activities.

According to Hjalmarson and Lesh (2008), design research is a perspective on research which focuses on "simultaneous and parallel knowledge development and product development" (p. 521). That means that both the process of developing knowledge and the process of developing some product are intertwined. Our design process is thus in line with this argument. We have focused on both the process of developing knowledge, i.e. developing our knowledge of how to design, how to adapt and how to format mathematical activities for 5-year-olds in Norwegian kindergartens, and on developing the written product which we will give the KTs.

In order for the KTs to nurture children's development of competence within the area of mathematics, we as researchers were to design a portfolio of mathematical learning activities for the KTs to orchestrate. This study thus focuses at conducting research into the first cycle of developing those activities. The aim is to achieve insights into the process of designing such activities.

We agree with Smit and van Eerde (2011) that design research is a suitable approach for possibly developing mathematics teachers' expertise. Furthermore, we agree that the core features of design research are predicting and reflecting. In our design of the mathematical activities, we built on existing research literature on children's mathematical development (e.g. Bishop, 1988; Clements & Sarama, 2007; Fischer, 1992; Fuson, 1988; Gelman & Gallistel, 1978). Based on what may be expected from 5-year-olds, we have predicted the children's mathematical competence as well as predicted challenges for the children to reach within their zone of proximal development (Vygotsky, 1978). Based on the KT's orchestration and implementation of the mathematical activities, we discussed and reflected on these experiences.

To be even more specific, we label our study as a design experiment, in accordance with Cobb, Confrey, diSessa, Lehrer and Schauble (2003): "Prototypically, design experiments entail both "engineering" particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (p. 9). These two characteristics of design research are further elaborated when Cobb et al. claim that "Design experiments are conducted to develop theories, not merely to empirically tune "what works". However, they also claim that "Design experiments are pragmatic as well as theoretical in orientation in that the study of function – both of the design and of the resulting ecology of learning – is at the heart of the methodology" (p. 9).

These authors argue that "[d]esign experiments ideally result in greater understanding of a learning ecology – a complex, interacting system involving multiple elements of different types and levels – by designing its elements and by anticipating how these elements function together to support learning" (Cobb et al., 2003, p. 9). We view the kindergarten setting, with the KT, the children, the activities with tasks and questions and the various tools used, as an interacting system, that is, metaphorically speaking, as an ecology. A further characteristic of this ecology is the discourse and mode of participation that we encourage the KT to promote, playful learning and inquiry. Thus, the design experiment has a theoretical foundation (cf. Cobb, 2000).

Moreover, our design experiment is pragmatic, as we want to find out "what works" as regards mathematical activities with 5-year-olds in a kindergarten setting. With respect to Cobb et al.'s (2003) list of various experiments, our study share most commonalities with what they call a classroom experiment. However, we are also concerned in our analysis about how the design works (cf. Hjalmarson & Lesh, 2008), under what circumstances it works as well as why the design works.

Principles for our Design of the Mathematical Activities

The national curriculum for kindergarten in Norway (Ministry of Education and Research, 2011) highlights three main areas of mathematics, number, spaces and shapes, while the curriculum for school (Ministry of Education and Research, 2006), for grades 1–2, highlights four main areas of mathematics, number, geometry, measuring and statistics. From a mathematical point of view, we decided to integrate mathematics from the four main areas in the school curriculum into the activities. This was done in order to support more fluent transition between kindergarten and school. We were also inspired by a detailed study of the Building Blocks material (Clements & Sarama, 2009). The Building Blocks consist of a comprehensive collection of mathematical activities sorted in three main areas of mathematics:

number and quantitative thinking, geometry and spatial thinking, and geometric measurement. In particular, the Building Blocks materials helped us in the process of developing activities within the different areas of mathematics. From a didactical point of view, we built on previous experience and insights from research (e.g. Carlsen et al., 2010; Erfjord, Hundeland & Carlsen, 2012) and designed the mathematical activities for KTs based on two main principles: playful learning (Hirsh-Pasek, Golinkoff, Berk, & Singer, 2009) and inquiry approach to the teaching and learning of mathematics (Jaworski, 2005; Wells, 1999).

The first principle related to the implementation of the mathematical activities was to encourage the KT to promote playful learning. Thus, in the written instructions we tried to encourage the KT to vary between structure and freedom throughout the activity, for example: The children were given high degree of freedom through comments like “let the children make their own stories”. On the contrary, low degree of freedom was communicated through comments like “how many bears are currently on the bus?”. We tried to give suggestions in the activities. However, the degree of freedom is something the KT has to reflectively consider in the moment.

The second principle emphasised in the activities was related to KTs’ use of an inquiry-based philosophy in their orchestrations. Adopting an inquiry approach to the teaching and learning of mathematics may be exemplified through the use of questions that make children wonder and investigate mathematical concepts and ideas. An inquiry approach may also be exemplified through nurturing children’s own imagination and creativity in investigating mathematical concepts and ideas.

The Mathematical Activities

The context of play served as a starting point in our process of designing the activities. We therefore designed activities within the contexts of a bus, a farm, shoes and paper airplanes. Play is a concrete and practical activity, not abstract. In the activities we thus contextualised (abstract) mathematics with respect to (concrete) situations and objects the children were familiar with. We further developed the four activities with respect to emphasising inquiries in mathematics, and sent them in written form to the KT in beforehand of the intervention. Additionally, we emphasised in our design of the activities that the activities should be orchestrated focusing at children’s participation and oral contributions. This was due to the Norwegian kindergarten tradition in general and due to the content and aim of the framework plan (cf. Ministry of Education and Research, 2011) in particular. To focus on written symbols, etc. is alien to both the Norwegian kindergartens and to the KTs. We further emphasised to include children’s use of concrete materials in the activities, e.g. paper airplanes, plastic bears, ordinary shoes, etc. This was deliberately done in order to facilitate the children’s process of mathematising within the activities. There was also strong emphasis that the activities ought to be guided by the KT (cf. guided play) and that the children were supposed to engage and participate in the activities in a collective way in order to nurture their mathematical learning process (cf. a sociocultural perspective on learning and development). Collective reasoning supports the individual child's process of making the activity-incorporated mathematics concepts her/his own (cf. Moschkovich, 2004).

KTs' use of these activities is intended to make a pedagogical shift in the KTs' actions. In a traditional didactic triangle in a kindergarten setting, mathematics is more incidentally offered to the children. However, by a modified didactic triangle, the aim of the activities is directly linked to the learning of mathematics (Erfjord et al., 2012). In that way the KT got the opportunity to gain insight into the activities and prepare details for her orchestration. Below, we present key elements from the four activities and relate them to the earlier outlined design principles.

The Bear Bus

In this activity, the materials for the children were one sheet of paper with a picture of a bus and 40 plastic bears to each of the children.

In this activity the KT is meant to start by introducing the plastic bears, about 20, and for example, ask the children: "How many bears are there in total?", "How many bears of each colour?" and "Can we sort the bears relative to their sizes?". Then, the bus is supposed to be introduced, by drawing on children's stories from their own experience travelling with buses. Later the KT introduces "bear stories", as, for example: "In one kindergarten, seven bears enter the bus. On the next bus stop, three new bears enter the bus. How many bears are there now on the bus?" Since each child has its own picture of the bus (Fig. 10.1), they can each simulate the situation and count 7 + 3 bears. The KT gives each child an original problem different from the other children. In the end, the KT may offer subtraction problems, as, for example, 11 bears – 5 bears =? or 12 bears – 4 bears =?, in an oral way, as a story, and not as a written problem. The written instruction sheet suggest that the

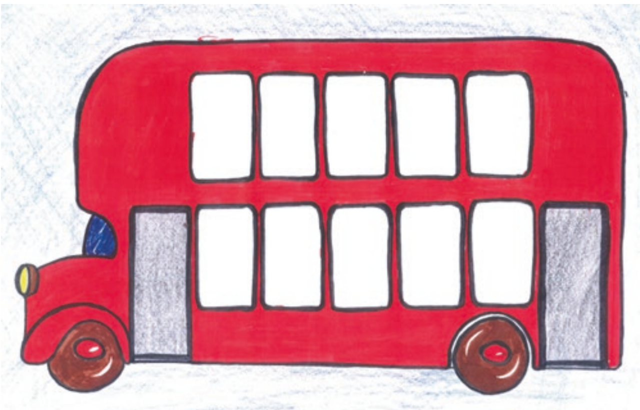

Fig. 10.1 A picture of a bus distributed on paper to each child

children collaborate and that they all, children and adult, discuss the different problems.

From the perspective of play, the activity gives opportunities on different levels. First, the picture of the bus and the bears is a starting point for free play (ref Hirsh-Pasek et al., 2009). The equipment can, for example, be offered to the children beforehand for free use, independent of this learning session. Second, the KT may carefully direct the bus activity from free play to guided play (ref 2009, se over) by using the initial instructions to give the children counting challenges and to inspire them to make "bear stories". Gradually, the activity will be more influenced by the KT, but still dominated by play (guided play).

In order to promote the children into an inquiry mode related to mathematics, the KT can support the children with something that starts their wondering and motivation to think mathematically. A classification process is one way. The KT may challenge the children to find ways to classify the bears and compare their classifications with the other children. Such processes potentially raise questions and a need for answers, thus classified as an inquiry process. Later the KT can offer particular mathematical challenges based on stories that are developed in the group.

The Farm

In this activity an A3-sheet of a farm was prepared as associated material. The activity starts when the KT distributes an amount of play animals on the table surrounded by children. She may ask the children to sort and count the different types of animals. The KT writes the numbers of animals with numerals on a sheet of paper and discusses them with the children (Fig. 10.2).

Each child chooses three different animals, for example, cows, sheeps and hens. They are then encouraged to count how many animals they have now.

After a while, the KT introduces the farm (Fig. 10.2) in the format A3. Each child is encouraged to make stories about the farm. The KT is recommended to assist the children with questions, as, for example: "How many horses are eating grass?" or "Two of the cows walked away, how many animals are left?" or "How many legs do your animals have all together, the cows, the sheeps and the hens?". The children and KT can try to figure these questions out together. In the end of the activity, the KT can then invite the children to suggest other things that may be counted, by studying the picture of the farm. Amongst possible countable objects, we can observe fence posts, flowers, ducks and stair steps. Patterns may also be studied, like the patterns of stones between the buildings and how the fences are built. Different geometrical shapes may be observed, as rectangular windows, triangular walls, pyramid-shaped timber piles and traffic cones.

Fig. 10.2 A picture of a farm similar to the one distributed to the children on paper

The farm context may easily be associated with play. The children should have the opportunity to play freely with the animals and the farm picture before the more structured session begins. After a while the play may be more structured when the KT guides and directs the focus on mathematical objects (counting, number, pattern, geometrical shapes).

Both the children's own stories from the farm and their chase of things to count in the bus picture may be perceived as play. The KT plays an important role in regulating to what extent she guides the process.

The suggested questions for the KT above focus on "how many", but not only as counting challenges. As soon as the questions focus on stories, as in "Two of the cows walked away, how many animals are left?", the children have to reflect and make actions in order to address the challenge. In this question, they may simulate the situation by putting all the cows on a line, then removing two cows and in the end counting cows left. In order to come up with such a plan, they have to be taking an inquiry approach in order to find a strategy for solving the problem. The inquiry approach will be visible when the KT together with the children chases different objects in the farm picture to count. There will also be inquiry opportunities in the study of patterns. For example, they can try to find out what sort of pattern is observed amongst the white and grey stones? The KT may inspire the children to suggest other sort of patterns which can be investigated by the help of manipulatives that may represent the stones.

Shoes

The activity starts as different pairs of shoes are put into two bags in such a way that shoes belonging together are placed in different bags. First only one bag is used. The KT picks up a long adult shoe and a short child shoe and discusses with the children about concepts such as long, short, tall and low related to the shoes. The KT may challenge the children by asking how we may figure out what shoe is the longest or the tallest. The next step is to ask the children, one after another, to pick a shoe with a particular property without looking into the bag, as, for example: "Find the shortest shoe" or "Find the longest shoe". Each child compares his/her shoe with the shoes of the other children. When all children have picked one shoe, the KT may raise a discussion about how they can sort the shoes? That can be done in many ways, for example, using height, length or shoe number as sorting criteria.

This activity uses everyday objects such as shoes that the children know well. The children will probably find it amusing to experience using other's shoes, maybe two different shoes. In the start of the activity, the play concerns investigating the shoes; try to walk with them, in a regular way or with "mice steps". Later, the KT gets the opportunity to guide the play in direction of measurement. For example, there may be a need for discussion of what is meant by "longest" and "highest" and how to measure these variables. Then the children are involved in an inquiry process. That will also happen when the children are involved in sorting the shoes. There will be a need for reflection about how they want to sort the shoes. Additionally, the children were supposed to measure the length of the room by using different shoes. Thus, the challenge was to compare the number of shoes and size of shoes (Fig. 10.3).

Fig. 10.3 Children measuring length with shoes

Paper Airplanes

The activity starts as every child gets one paper airplane each. First, there is probably a need to practise in throwing the paper airplanes. After a while the KT is supposed to measure the length. The children may be organised in a line and one after another throws his/her airplane. The landing place should be marked. The KT should at this stage focus on concepts such as "longest", "second longest", etc. and get help from the children to sort out the results.

The KT can vary the measurement of lengths by using sticks and rope (non-standard units of length) as well as a measuring tape with metres and centimetres (standardised units of length). The KT may discuss issues as accuracy with the children and write down the numbers involved, for example, the number of metres each child throws his/her airplane. The play may start by engaging the children in making the airplanes. This might be difficult for them, but the amount of play may be large. The paper folding will even give some implicit experience of symmetry. They will probably also value the need for accuracy during this process. When the children start to throw airplanes, the play factor will probably be associated with perceived competition between them. Who is throwing the longest? If the children stand inside a circle and throw the airplanes in different directions, a question will emerge: How can we decide who threw the longest? The KT may now inspire the children to inquiry into different ways of measuring length using non-standard and standard units of length in order to come up with solutions to the problem stated (Fig. 10.4).

Fig. 10.4 Children who throw airplanes

Analysis

This section reports from analyses of a KT's implementation of the four mathematical activities outlined above, *The Bear Bus, The Farm, Shoes* and *Paper Airplanes*. Our interest was to develop insight into how the KT interpreted and used the written instruction material and associated practical material such as the bear bus sheet, plastic bears and a drawn picture of a farm. The data considered in this section are the written instruction material we developed for the four activities, observation of the KT's orchestration of them and conversations ahead of and after the teaching took part. Data was collected with the help of video recordings, and all four researchers made individual field notes from the observations. We documented whatever came to our minds and reflected on how the activities functioned as learning activities.

We invited one local, public kindergarten to implement the activities. In our design of the activities, we emphasised that the activities were supposed to be orchestrated involving a group of 5–6 children of approximately 5 years of age. Previous to the implementation, we (four researchers) visited the kindergarten and met one voluntary KT and her leader. The KT was an experienced, pedagogically educated kindergarten teacher with a 3-year university kindergarten education. We presented briefly the four activities. We emphasised that our motive was to learn and that we were grateful that the KT would help us in implementing the activities. We pointed out that it was not our intention that the KT strictly had to follow the written instructions and that we would welcome feedback based on her experience. She was asked to read through the first two activities, *The Bear Bus* and *The Farm,* and plan her use of the activities. We told her that we wanted her to consider how to use the activities in *her* way with the children. This was followed up by a conversation with her and us in the kindergarten a few days later, before her use of them with children. In this meeting we gave her the associated materials we had prepared, talked about the activities and asked her if there were particular things she found unclear. The first two activities took part on 2 consecutive days. After ending the first of these, *The Bear Bus*, there was time for a brief talk. We also talked with her in between the activities and ahead of her use of the latter two activities, *Shoes* and *Paper Airplanes.* We consciously did not want to add much to what was expressed in the written form except from some brief sharing of experiences, clarifying the content of the written instruction material if the KT was unsure about things and discussion of the practical materials developed. Our experience overall was that the KT reported that she found the material easy to grasp and she had no specific need for clarification. She had made herself checklists in order to remember all aspects of the activities.

The approach to data analysis was to compare the content of the written material of the four activities and how we observed the KT's orchestration of them. We were particularly interested in investigating to what degree inquiry as an approach to the learning of mathematics came through in her orchestration of the materials. We were also interested in possible changes from one activity to the next activities where the role of the brief conversation we had with KT was considered. Analyses of all four activities show that the KT (1) followed the instruction material to a large extent, i.e. by using the associated materials, the suggested practical arrangements and comments; by literally using the mathematical concepts suggested; and by using the suggested questions; (2) only occasionally orchestrated children's mathematical inquiry; and (3) implemented playful learning to a limited degree.

1. Followed the Instruction Material to a Large Extent

The first characteristic is that we experienced that the KT used the associated materials we had developed and followed the practical arrangement and comments we had suggested in the written instruction material. In the first two activities, we observed several similarities between what we had written and what was orchestrated.

Using Associated Material

From the written instruction	From the transcript of the observation	
Now the A3 sheet of paper with the bus is introduced. What experiences do the children have with buses? What can a bus be used for? Introduces a bus story about some bears that are taking the bus from the kindergarten to some place	279 KT: 280 children: 281 KT: 282 children: 283 KT: 297 KT:	(she gives each child the sheet of paper with a picture of a bus) (the children immediately start to put bears on the windows of the bus) But, [] so that you get some space next to your bus. But, ehm, all the bears are not going to enter the bus yet. So they have to be placed on the outside. (the children remove the bears from the bus) What can a bus be used for? (the KT introduces the bus story) This bus is going from the kindergarten, but where is it going to?

Literally Using the Mathematical Concepts Suggested

The second characteristic is associated with the mathematical content of the activities. Throughout all the activities we observed that the KT followed what we had suggested. Below is an example from the second activity, *The Farm*, where we suggested a particular number of animals to be used at a certain point. As evident from the excerpt, the KT also chooses these numbers.

From the written instruction	From the transcript	
Let every child choose three different animals for their farm	149	Like this. And then you may choose, you may choose three animals, three different animals (shows three fingers)
Then they are going to count four animals of each type. All children are given their own container for their animals	177	Pick one of your animals, and then you show me. Yes, the animal you show me, you are going to have four of those in your container.

Using the Suggested Questions

The use of questions is important in order to facilitate an inquiry approach to the learning of mathematics. We therefore analytically searched for occasions were the KT indeed used various questions. We found that a third characteristic was the KT's use of the suggested questions. In *The Bear Bus* we altogether counted 213 questions from the KT during the 70 minutes the activity lasted. Of these 213 questions, 67 questions were of the form "How many…", 25 questions were of the form "Can you count…/shall we count…", 15 questions were of the form "Who has most of …./ Who has less of…", while the 9 remaining questions concerned her writing of the numerals "How do we write this numeral…/ "Shall I write the numeral…". All these types of questions were suggested by us in the written instruction, and the KT often repeated each of the questions to each of the 6 children participating in the activity.

In the activity *Paper Airplanes*, we also observed that the KT used questions with the children as we suggested in the written instruction.

From the written instruction	From the transcript	
Let the children throw their airplane once more. What airplane came the longest? What airplane came the shortest?	33 KT:	But you, who came the longest?
	34 tom:	Per
	35 KT:	Per came the longest. Who came the second longest? Or should I rather ask: Who came the shortest?

From these three characteristics, which were typical for the KT's orchestration of the activities, we have strong evidence that the KT followed the instruction material and guidelines to a large extent.

2. Children's Inquiry Only Occasionally Orchestrated

The overall analysis of the four activities showed that there is little evidence of situations where the KT implemented an inquiry approach to the learning of mathematics. Rather, we found examples of "lost opportunities" for adopting an inquiry approach for investigating the mathematical issues at stake.

We experienced situations where children responded differently than the KT apparently seemed to expect. In a counting situation within *The Bear Bus*, where the KT asked the children how many bears she had on her bus, the child responded with "four" assumingly because she subitized the number four. However, the KT's response was to ask her to count, not to explain how she came to the result of 4.

Later on, in the same activity, each child is working on counting seven bears that are supposed to be on the bus. The KT is very busy checking that every child gets exactly seven bears on the bus. Suddenly a child prompts:

357	Gro:	Look it's three! (she is pointing at the free seats (empty spaces) in the bus).
358	KT:	Yes you can see that it is three free seats in the bus.
359	Mia:	It is four on my bus
360		(then the kindergarten teacher rushes further, to the next child and checks if she got seven bears on her bus)

From this excerpt we see that the opportunity to inquire into the mathematics involved in adding and subtracting the various numbers of bears is not followed up by the KT. Moreover, in this excerpt the initiative to inquire into the mathematics comes from one child. It is thus evident that the KT loses this opportunity.

However, we also found incidents where the KT was able to adopt an inquiry approach for investigating the mathematical issues at stake. In the activity called *Shoes*, the children and the KT discuss the name of various shoes, and why one particular shoe is called high heeled shoe. The following question was asked: "But is this a high shoe or a low shoe?" (Utterance 172). This question generated a discussion amongst the children and the KT as regards the mathematical meaning of the terms "high" and "low". We argue that this question initiated the use of an inquiry approach to engage with mathematical concepts.

3. Playful Learning Was Implemented to a Limited Degree

In our thorough analysis of the orchestration of the four activities, we found evidence of a rather limited emphasis on the principle of playful learning. Above we have, in accordance with the research literature, defined playfulness in mathematical activities as characterised by engagement and opportunities to play through both self-directed explorations by the children and goal-directed explorations suggested by the KT.

As mentioned in our descriptions of the four mathematical activities, all of them were designed based on playfulness as a principle and guided play as the suggested approach for the KT to adopt. The contexts of a bus, a farm, shoes and paper airplanes were assumed to create playful points of departure with respect to plausibly awake the children's interest, curiosity and engagement.

To exemplify the limited focus on playfulness, we have chosen an episode from the initial phase of the KT's orchestration of the activity called *The Farm*. After some practical arrangements, the following conversation took place:

10	KT	Today we're going to work with animals
11	Eva	Animals?
12	KT	Yes, animals.
13	Mia	Are we going to use all those? (looks at the amount of animals in front of the KT)
14	KT	(distributes the animals at the table, so that all children are able to see) Here there are several animals, and we are going to find out how many animals there are. Shall we count them? Shall I move them with my finger while you are counting?

We notice that the KT accurately expresses what they are supposed to do in this activity. She says that they are going to do something with the animals that she has in front of her (Utterance 10). No open, initial questions are asked with respect to the children's imagination of what they think could be done with the animals. However, it seems that at least Mia is curious about the animals when she asks whether they are going to use all the animals (Utterance 13). Mia's question is not explicitly dealt with. The KT rather continues with her agenda and says that they are going to find out how many animals there are all together (Utterance 14). From this short excerpt from the initial phase of the activity, we recognise a limited degree of playfulness with respect to nurture the children's curiosity and interest.

The children together with the KT counted the animals and got, after several minutes, 66 animals all together. The activity continued with the following:

149	KT	Then you may choose three animals, that is three different animals (shows three fingers)

When all the children had got three different animals, after several additional minutes, the KT gave the following task:

177	KT	Choose one of your animals and show me! (all the children do this). Yes, and the animal that you now show, you are going to have four of.

The activity continues with the KT asking the children to find four of the secondly chosen animal and four of the thirdly chosen animal. After about 25 minutes from the start of the activity, after a quite long period of counting various animals with each individual child, the KT says:

254	KT	That's it. Now everybody has four of each animal. Then you are going to count how many animals you have all together.

From the episode above we see that the children are more or less occupied with counting animals for an extensive time period. We interpret this episode as showing a limited degree of playful learning on behalf of the children due to the severe emphasis on counting the animals. In the written instruction, we said that the children were going to have some animals each, but in this case we argue this counting to take too much time and effort. The picture of the farm (cf. Fig. 10.2) was not shown yet, even after 25 min into the activity. We see opportunities for playful learning to occur when the animals are linked to the farm environment, but this does not happen due to too strong emphasis on the counting.

However, the nurturing of playful learning was also taking place. For example, in the activity called *the Bear Bus*, the following conversation occurred, when the KT showed the picture of the bear bus (cf. Fig. 10.1):

89	KT	(shows the picture of the bus)
90	Child	A bus! (several children simultaneously)
91	KT	A bus. Yes, a bus. But this is not <u>any</u> bus. It's a bear bus!
92	Child	Oooh! (all of them smiles)
93	KT	(laughs a bit) Isn't it cool? A bear bus. Afterwards you are going to get your own bus and you are going to get 15 bears

From this short excerpt we argue that the children's interest and curiosity are made explicit. Both in utterance 90 and 92, we interpret the children's simultaneously expressed words to show eagerness and engagement. In particular, the expression "oooh" in utterance 92 is a usual and strong utterance used by kindergarten children to express excitement and curiosity. In this excerpt we thus argue the KT to be successful in nurturing the children's playful learning process.

In the written instructions for *The Bear Bus* activity, we proposed the suggestion: "let the children make their own stories". This was deliberately done in order for the children to possibly participate in a playful learning process. After approximately 45 min of counting bears with each individual child and questions asked by the KT about each of them having seven bears on their bus, the following conversation took place:

391	KT	Now you are going to decide yourself how many bears are entering the bus. Let's see, we start with Ida. Where is your bus going?
392	Ida	Let me think… to the beach
393	KT	Okay. But how many bears are going to catch your bus?
394	Ida	My… (she counts the bus' windows and the driver's seat with her finger)
395	KT	How many bears are going to the beach?
396	Ida	Eleven!

In this short excerpt we see that Ida is nurtured in her learning process through the KT's questions. The KT asks Ida both where the bus is going, to make Ida imagine a bus ride on her own, and also how many bears are catching Ida's bus. We argue this excerpt to show playful learning through guided play. Moreover, we argue that the excerpt show a fruitful way of combining imagination (the bus ride with a bear bus) with mathematical learning goals (on Ida's own choice to estimate how many bears are catching the bus). We also see that Ida responds with the number eleven, which shows her mathematical reasoning with respect to the picture of the bus (cf. Fig. 10.1). There are ten windows for passengers and one window for the bus driver, 10 + 1 = 11 bears all together.

Discussion

The analysis above identified three issues that are discerned when a kindergarten teacher orchestrates the researcher-designed mathematical activities. The first issue regards the KT's close following of the written instructions. The second issue concerns that the inquiry approach that was intended to pervade the activities was not adopted. The third issue emerges with respect to a limited implementation of playfulness in the activities.

With respect to the first issue, that the KT was closely following of the written instructions, we suggest the following possible reasons: (1) the written instructions were too detailed making it difficult to personalise them for the KT, (2) the KT thought she were expected to carry out all parts of the activities and (3) the KT experienced a power relationship due to the researchers' role as developers of the activities and observers of her orchestration of them.

The activities were thoroughly described in the written instructions, which contained a precise and chronological description of how to carry out the activity, including suggestions for questions that may be asked. We observed that the KT strictly followed the instructions. She used a checklist in order to remember all the instructions. This observation is in accordance with the argument of Davis and Krajcik (2005) that curriculum materials have deep impact on teaching. The KT during the first two activities ran out of time without reaching the end of the activities. It seems as if the KT perceived the written instruction material as what Ahl et al. (2016) call a "traditional teacher guide" which they characterise as "page-by-page" implementation of learning activities. Our intention was rather the opposite, to design an "educative teacher guide" that included both specific learning activities and the design principles behind the activities.

Moreover, a trustful and personal relationship between the KT and the researchers was not yet developed. The researchers had designed the activities, chosen the kindergarten where the activities were to be tested out and observed the KT's orchestration of them. Thus, the KT was left with the task to carry out the researchers' ideas. The fact that the KT had prepared a checklist, in order to help her not missing out any aspects in the instructions, indicates that she executed a mission without having a personal ownership of the activities. It seems reasonable to claim that the design process was not in line with what Wells (1999) defines as "collaborative inquiry", where researchers and practitioners work together and benefit from each other's competence. This probably shows that the relationship between the researchers and the KT was unbalanced from the start of the process. Thus, it is reasonable to argue that the KT possibly experienced a power relationship between her and the researchers. In the third and fourth activities, we observed that the KT made the activities more her own and thus orchestrated the activities more independently and free from the written instructions. This development may have been a result of the ongoing process between her and the researchers, during the days we spent together.

The second and third issues, which concern the lost opportunities for adopting an inquiry approach to the learning of mathematics and that playful learning was implemented to a limited degree, we suggest follow from the first issue and consequently from the reasons behind.

Because the KT was strictly following the written instructions and did not use the implicit opportunities for playful learning and inquiry, we argue that her strict focus on following the instructions as the overall reason for the lack of inquiry and play. The KT interpreted the written instructions as a manual and missed out opportunities for inquiry and play. It seems reasonable to assume that the KT's concern about thoroughly following the instructions limited her possibilities for nurturing children's inquiry. From the data we observed several opportunities to include inquiry in the learning process, and to follow children's play initiations, but the KT did not utilise these opportunities. Instead her focus seemed to be the continuation of the activity.

An additional reason which may explain her lack of adopting an inquiry approach and her choice to follow the instructions is that mathematics as a subject area in kindergarten which is still rather new in Norway. This may explain why she hesitated to leave the "safe" written instructions. It may also explain why time was spent differently than the researchers had intended. Both in *The Bear Bus* and in *The Farm*, the KT spends much more time on the counting of the exact number of animals and bears instead of getting more rapidly to situations where she could awake the children's interest and curiosity for additional mathematical problems. In these two activities the KT followed the instructions to such a detail that she wanted each of the children to have exactly 12 animals each and exactly 15 bears each. From our perspective, those numbers of animals and bears were not the main mathematical objectives behind the activities. The KT also interpreted our suggested questions very literally and used them over and over. Even if we had given suggestions for possible questions, our intentions were not that these were the only possible questions. Instead we expected the KT to bring her own questions into the activities. To conclude, we believe the lack of inquiry and playfulness happened due to the detailed instructions, which she thought she was expected to follow in detail. This was due to a possible lack of experience in orchestrating mathematical activities, and an experienced power relationship in being observed by researchers in mathematics education. Thus, it is reasonable to argue that it is mostly us as researchers to blame that playful learning and inquiry was implemented to a limited degree.

Emphasis on playfulness in early years' mathematics is important in order for the children to develop their mathematical competence (cf. Marcon, 2002; Singer et al., 2009). Furthermore, in order for the children to be engaged in meaningful mathematical learning processes, there is a need for guided play (cf. van Oers, 2014;

Weisberg et al., 2015). Despite the researcher-designed activities' shortcomings and unclearness, we see from the excerpts analysed above evidence of a KT who seeks to take a discovery-learning approach (cf. Fisher et al., 2013) and to balance between freedom and structure in her implementation and orchestration of the mathematical activities. She also occasionally manages to adopt an inquiry approach to the learning of mathematics. Thus, the children participating in this study still got opportunities to come further in their mathematical learning process.

Concluding Remarks

We had specific intentions for the KT's orchestration, a playful and inquiry-based approach to the mathematics which was not clearly communicated by us. In order for a KT to develop inquiry as a way of being, oral and written instructions are not sufficient. Questions were designed with the aim to promote children's inquiry. The questions were in line with categories suggested by Carlsen et al. (2010), i.e. questions that invited the children to be included in a problem-solving process. Jaworski (2005) describes inquiry "both as a tool for developing practice, and as a way of being in practice" (p. 103). Our analysis suggests that it is unlikely to believe that KTs will develop such an inquiry mode, as described by Jaworski (2005), automatically due to mathematical learning activities offered by researchers with suggestions for questions which may promote inquiry. To strengthen possibilities for developing inquiry as a way of being, a co-learning agreement between the researchers and the KT is a useful methodology in designing a professional development programme (cf. Schwan Smith, 2001; Tirosh et al., 2011; Wagner, 1997).

Additionally, we agree with Tirosh et al. (2011) that KTs need to work and explore mathematics themselves in order to be able to foster children's inquiry into mathematics. Erfjord et al. (2012) argue that in some Norwegian kindergartens a shift has taken place when it comes to the overarching didactic triangle governing the kindergarten practice. The shift has occurred from a situation where pedagogical activities have been orchestrated in which mathematics only plays a minor and ad hoc part, till a situation where mathematics is the core of the mathematical pedagogical activities. From our data we cannot conclude whether the KT would associate herself with the kind of practice characterising the former didactic triangle or the latter didactic triangle. However, the fact that the activities were externally designed is, at least from the outset, a critical issue encountered in our study with respect to possibilities for making the activities her own. Our analysis which shows that the KT followed the instructions carefully and did not extensively take advantage of inquiry opportunities to engage with mathematics, further indicates that she had limited experience in orchestrating mathematical pedagogical activities.

In this study, we have particularly paid attention to two concepts, *guided play* and *inquiry*, that have informed our design of the mathematical activities. We argue

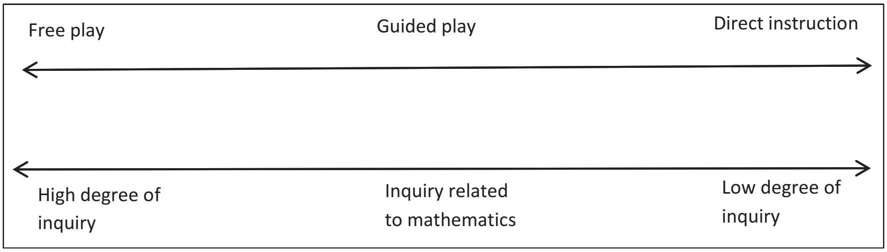

Fig. 10.5 Characterisation of inquiry and play

that the lack of both inquiry and playfulness stem from the same reasons. Thus, we propose that there exists a reciprocal relationship between the two concepts. Below in Fig. 10.5 we suggest an illustration of this reciprocal relationship of the concepts:

We argue that if the KT gives children high degree of freedom in play, it makes sense to suggest that the children's opportunity to wonder, investigate and pose their own questions will increase. However, there is no guarantee that the inquiry process will be about mathematics. If the KT wants to link the play to particular mathematical learning goals, the play has to be guided (cf. van Oers, 2014; Weisberg et al., 2015). Direct instructions in play may result in that the KT gives particular instruction of actions and closed questions linked to particular mathematical topics. During this process, the degree of the children's wondering and own investigation will be low.

With respect to the overall theme of this book, Fig. 10.5 signifies that in order for the child to construct mathematical concepts and ideas, (s)he has to participate in guided play in which the KT orchestrates an inquiry approach to the learning of mathematics. Guided play also incorporates a dimension of instruction as the KT needs to afford mathematics learning amongst the children by asking questions, leading the activity, introducing concrete materials to be used and so forth. The child-initiated self-directed explorations, afforded by the "instructor", give opportunities to construct, or appropriate, mathematical tools and actions.

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Chapter 11
Talking About Measuring in the Kindergarten: Linguistic Means in Small Group Interactions

Birgit Brandt and Sarah Keuch

Abstract This paper deals with the support of language learning in settings planned for mathematical learning by kindergarten teachers in small group interactions. Using qualitative and linguistic analysis tools, we will reconstruct patterns of language use, the correction strategies of kindergarten teachers, and their language sensible organization related to measuring and magnitude.

Keywords Academic language · Correction strategies · Interaction analysis · Kindergarten · Language deviation · Language learning · Language sensible organization · Language usage/language use · Linguistic analysis · Measurement/ measure/measuring · Qualitative content analysis · Small group interaction

The preoccupation with early education has become increasingly popular within the last few decades in many countries, in research and politics alike. This is also the case for all 16 German federal states in Germany. The education system is under the responsibility of each federal state, but for school education, there exist standards of education (Bildungsstandards; e.g., for mathematics education in primary school: Kultusministerkonferenz (2005)), which are obligatory for the individual curriculum of every German federal state. For the elementary level, however, there exists only a joint framework, which is not obligatory in the same way:

> Elementary education focuses on the conveyance of fundamental competences and the development and strengthening of personal resources. Those competences and resources are to motivate and prepare the child to act on and overcome future tasks in life and learning. Furthermore, they help the child to take part in social life and lifelong learning. (Kultusministerkonferenz & Jugendministerkonferenz, 2004, p. 3; translation by the authors)

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For this purpose, the guidelines emphasize the support of communicating basic skills. Thus, supporting language education has to be a main principle of daycare centers for children up to starting school (s. Kultusministerkonferenz & Jugendministerkonferenz, 2004, p. 9). Early mathematics education is mentioned only in one sentence and restricted to experiences with numbers and geometrical shapes. Nevertheless, almost all German early education curricula include mathematics as an area of education and mention more mathematical topics, like measuring, problem-solving, and logical thinking, or statistic representations. We collected the data for our project in the federal state Hessen, where the curriculum of early education includes the years 0 up to 10. In this curriculum, early mathematics education is linked to language learning in a differentiated manner:

> In the first years, the basis for later mathematical thinking is built up, as the child gains experiences with regularities, patterns, forms, sizes, weight, time and space. (…) Mathematical learning has a close connection to other subjects, like music, rhythm and sports and specifically to language development. On the one hand, language serves as a basis for mathematical thinking. Mathematical problem solving on the other hand is developed and improved through communication. (Hessisches Ministerium für Soziales und Integration & Hessisches Kultusministerium, 2007, p. 75; translation by the authors)

On this administrative basis, our project addresses aspects that foster language skills in mathematical situations in kindergarten. In particular, this paper deals with the (academic) language in mathematic experimental situations in daycare center concerning the content measuring and magnitude. Measuring can be regarded as a mathematic basic activity that is culturally-historically important for the development of mathematics as an academic discipline and therefore also for the development of mathematical thinking in children (Bishop, 1988). Measuring connects different mathematical as well as real-life topics (Skoumpourdi, 2015), and it is included in most curricula for young children in Germany.

Hereafter we first introduce some aspects of current research concerning subject-oriented language education and hence develop our research questions. Then we briefly present our research design. The core theme consists of the presentation of the qualitative and linguistic analysis tools, which we will introduce by means of generic examples. Based on some general observations concerning correction strategies, we will focus on significant differences between the three kindergarten teachers in our concluding remarks.

Supporting Language Acquisition in Subject-Related Contexts

The importance of language for cognitive (subject-specific) learning processes is undeniable and well established with regard to research in early mathematic education. Academic language proficiency is seen as an important factor for successful education and schooling. There are still unsatisfied needs for Germany to appropriately support children with disadvantageous starting conditions (e.g., migration, socioeconomic background, developmental speech disorder), in order to give them an equal chance to participate in education processes (Gogolin & Lange, 2010; Prediger, Renk, Büchter, Gursoy, & Benholz, 2013).

Early education in kindergarten, which puts emphasis on supporting language education, could provide a remedy. As mentioned above, the guidelines for early education in Germany focus primarily on general language competences and less on language education that is linked to subject-related learning processes. Prediger (2015) however requires academic language education processes to start as early as possible, to design it age-appropriately and to orientate it by specific contents. In this sense, the Hessian education plan (Hessisches Ministerium für Soziales und Integration & Hessisches Kultusministerium, 2007) emphasizes the important role of language for subject- and mathematic-related learning processes. Nevertheless, it merely asks for dealing with certain terms (numerals, names for geometrical shapes, and basic terms for spatial or temporal information like yesterday, today, tomorrow) but leaves out other aspects of academic language, like specific grammatical structures. Concerning our focused section, that is, measurement, it asks for phrases that express comparisons, like bigger–smaller, thicker–thinner, and so on. In the area of early education, Germany is especially lacking approaches considering language education that integrate subject-related learning processes and not only punctually training single academic language terms. Hence, with our analysis of language usage, we focus on precisely this gap.

Rudd, Satterwhite, and Lambert (2010) describe how mathematical learning and language learning can be combined in (natural) kindergarten situations. They introduce the concept of math-mediated language (MML). This means that mathematical learning is embedded in dialogues, which include mathematical as well as linguistic knowledge (Rudd et al., 2010). They give concrete examples for different mathematical topics, e.g., how to foster complex counting strategies by modeling them in concrete situations or by requesting them from children by corresponding questions.

Even though the concept of MML emphasizes the mathematical learning in kindergarten, it points to the need that kindergarten teachers have to consider both: the mathematical context and linguistic effort of the dialogues – and they have to address this connection in their planning as well as in spontaneous situations. Thus, MML deals with the integration of language education and topic learning in everyday activities for kindergartners. MML requires a certain amount of language awareness. For preservice early childhood educators, Moseley (2005) found out that their perceptions of MML is restricted to technical terms and basic mathematical terminology. In our qualitative-empirical project, we are interested in the language awareness of kindergarten teachers in everyday situations.

Kindergartners are – independent from the individual language background – language learners. Therefore, comparing to MML we turn around the priorities of mathematical learning and language education in our investigation. This means that we put our focus on the support of language learning in settings planned for mathematical learning. This idea corresponds to the underlying idea of supporting language development in the subject (Leisen, 2013; Prediger, 2013; Prediger & Wessel, 2013) like it is discussed in the schooling context. Often, these concepts trace back to the immersion model for bilingual education for children with migration background in school context (e.g., Cohen & Swain, 1976).

Vollmer and Thürmann (2013) took a first step toward determining and systematizing the academic language needs in subjects (Fig. 11.1). They have developed a model for the description of academic language requirements and competence

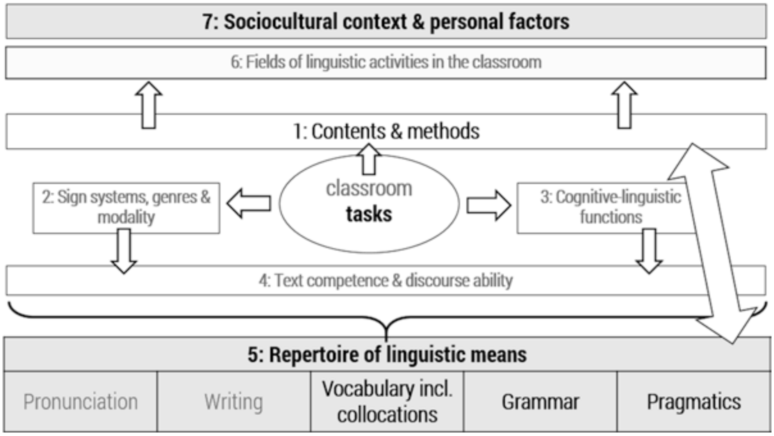

Fig. 11.1 Model for the description of academic language requirements and competence expectancies in the classroom (cf. Vollmer & Thürmann, 2013)

expectancies in the classroom that serves as the foundation of our (linguistic) analysis. While this model – based on activities in the classroom – consists of seven dimensions, we mainly concentrate on three of them that we find most relevant in the kindergarten context (grayed in Fig. 11.1). In the future, one might have to think about how to adapt this model to meet the requirements of kindergarten, namely, of a non-alphabetized environment.

The focus of our study lies on dimension five, the repertoire of linguistic means. We are especially interested in the vocabulary including collocations and grammar. Since we are dealing with interpersonal communication, we also consider pragmatic aspects in our analysis. The concrete use of vocabulary and grammar always depends on the contents and methods applied in the particular situation (dimension one). In our case, this is measuring. Of course, you cannot analyze natural communication without considering dimension seven, the sociocultural context and personal factors, although it is not the focus of this article. Furthermore, such analyses highly depend on the actual language(s) used. For different languages, the manifestation of each dimension can vary widely. In our case, we mainly concentrate on the German language.

In particular, our aim is to reconstruct the empirical language in use, to detect aspects of language support, and to show the connection to specific meanings and concepts that are negotiated in certain situations. In this paper, we will concentrate on grammatical aspects of the empirical language and the corresponding questions:

I. Which essential patterns of language use for *measuring* can be identified in mathematical-orientated situations in the kindergarten?
II. Which kind of correction strategies can be traced back to the preschool teacher in terms of a language sensible organization?

Research Design

The data basis for our analysis consists of mathematical situations designed by kindergarten teachers and taken from the project erStMaL (early Steps in Mathematical Learning) (Acar Bayraktar, Hümmer, Huth, & Münz, 2011); within this longitudinal project, there are 19 situations in total dealing with measuring and magnitude. So far, we have only looked at a contrastive partial corpus of three situations, in order to develop and test our analysis tools (see Table 11.1 for basic information on all three situations). In the sense of a comparative analysis, those three situations differ in terms of their children's age and language background but agree in terms of the kind of magnitude they are dealing with, which is length.

Table 11.1 Basic information on the focused situations

(A) Playing measuring				
23 min	Measuring devices: leveling board (for children), wool			
Doris: pre-school teacher, trained in mathematics				
Nikola	Female	4;2	Bilingual[a]	Greek/German
Orania	Female	3;10	L1	–
Regina	Female	4;4	L1	–
Uwe	Male	3;11	L1	–
(B) Talking about measuring				
15 min	Measuring devices: rulers, carpenter's rule, wool			
Sabine: pre-school teacher, trained in mathematics				
Mona	Female	5;5	L1	–
Omara	Male	4;11	L2	L1: Tamil
Oslana	Female	5;3	L2	L1: Croatian
Sadira	Female	5;11	L2	L1: Urdu
Theresa[b]	Female	Unknown	L2	L1: Unknown
(C) Things for measuring				
45 min	Measuring devices: rulers, carpenter's rule, measuring tape, building blocks, chalk			
Berna: pre-school teacher, L2 (L1 unknown)				
Bella	Female	6;0	L1	–
Can	Male	6;0	Bilingual	German/Turkish
Denis	Male	6;0	L1	–
Friedel	Male	6;2	L1	–

[a]L1 means the child learned and uses German as a first language; L2 means the child learned another language than German as a first language and now learns and speaks German as a second language; bilingual means the child learned German and another language as first languages and now uses both languages at home

[b]Theresa was not a child of the original erStMaL sample, but she participated in this situation. Thus, we do not have her basic metadata

Analyzing Methods

The model described above by Vollmer and Thürmann (2013) works as the foundation of our analysis. We correlate our research questions to dimension five and work on each of them with specific analyzing methods that we will explain in the following. Although we concentrate our analyses on linguistic moments, it is not possible to look at spoken language without analyzing the negotiation process between the speakers. Hence, we put a content-related analysis of the negotiation processes in front of the language-related analyses. For this purpose, we traced back the interaction analysis (Krummheuer, 2007), which we introduce subsequently with concrete data.

Sentence Structure Analysis

For research question I, we look at how the preschool teachers and children use the verb *to measure* syntactically. More precisely, we analyze the grammatical valence of the verb in use. The term *valence* derives from chemistry, where it describes the ability of an atom to link with other atoms. Linguistics takes this model to explain the fact that in valence theory, a verb asks for a certain number and kind of sentence constituents (like subject, different objects, or adverbial phrases) in order to form a correct sentence (Herbst & Götz-Votteler, 2008).

Granzow-Emden (2013) uses this analogy to illustrate vividly the verb as the core atom linking with other atoms – the complements – in order to form a molecule or a sentence. The following example shall explain what linguistic valence means when put into practice. When using the verb "to give," you need to know *what* is given, *who* gives, and *who* receives, in order to form a sentence that native speakers find acceptable in the standard variety. Therefore, neither "She gave the book." nor "She gave her." sounds acceptable in conceptually written English without any further context. The right number and kind of complements vary from language to language and sometimes even within one language, especially when a verb has several meanings. Thus, speakers need an elaborate feeling for that certain language. In conceptually oral contexts like the situations in our study, however, it can be accepted to use fewer complements or to use deictic expressions as complements. In our study, we want to look at the verb *to measure*, its grammatical valence and how it is used within the situations. In German, someone (subject, sub) measures something (accusative object, acc) with something (adverbial phrase, adv.). In order to produce a correct German sentence, you need at least the subject and the accusative object to go with the verb (Fig. 11.2).

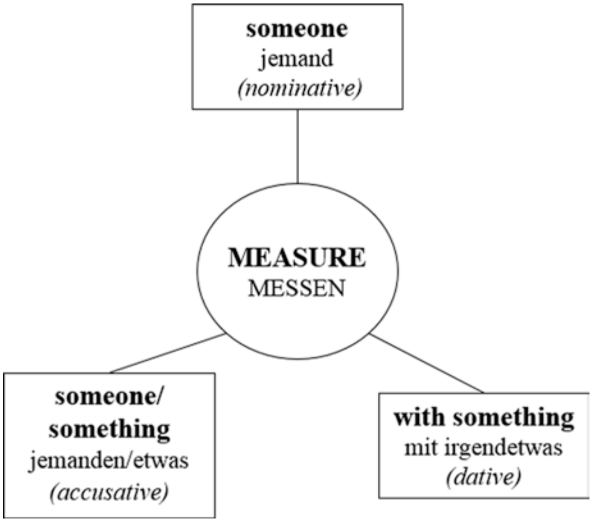

Fig. 11.2 Atom model for linguistic valence (cf. Granzow-Emden, 2013)

Analysis of Correction Strategies

For research question II, we look for correction strategies following utterances that contain deviations concerning the vocabulary, the grammatical structure, or pragmatic aspects. If the preschool teacher utters a corrective phrase, we divide the utterances into different categories. Based on the qualitative content analysis (Mayring, 2000), we generated different theoretical categories:

0. Deviation without situational response
1. Direct/explicit
   (a) "No"/"wrong" (possibly with further clarification).
   (b) The kindergarten teacher utters the correct word or sentence.
   (c) The kindergarten teacher asks another child to correct the (perceived) mistake/deviation.[1]
   (d) Metalinguistic commentary to the utterance:
      (i) With explicit reference to the erroneous structure
      (ii) Without explicit reference to the erroneous structure
2. Indirect/implicit
   (a) The kindergarten teacher takes up the utterance through a correct paraphrase.
   (b) The kindergarten teacher takes up the language structure and uses it for expressing their own ideas.

[1] Depending on the reaction, the kindergarten teacher (sub-) consciously marks the utterance as mistake or a somewhat less severe deviation.

Analysis and Interpretation of Empirical Data

Here we firstly present our concrete situations with an interaction analysis and secondly apply our linguistic analyses. The qualitative language analyses are as well always interpretations of the situations. The summaries of the analyses at the same time give answers to our research questions, as described in the analyzing methods.

Interaction Analyses of the Three Situations

The interaction analysis is a turn-by-turn-organized analysis, which is based on conversation analysis. Here, we only present the summary of this analysis for each situation and the final comparison of the interaction processes. The names we have chosen for the situations (A, playing measuring; B, talking about measuring; C, things for measuring), however, are based on the overall impressions we got in the course of this analysis.

Situation A: Playing measuring Doris, the preschool teacher, provides several crayons and wool hanks as well as cardboard paper, pairs of scissors, and glue sticks. In this situation, an interaction script unfolds along these materials, which head toward her targeted goal: sticking woolen strings as long as the before measured body length on the cardboard. In order to execute this design idea, the children are allowed to choose one color among the balls of wool. In a next step, they are supposed to find the matching crayon. In each case, Doris initiates and supervises these actions. In this opening situation, Doris repetitively asks for names of colors. It becomes clear though that this color matching only serves organizational purposes:

7	Doris	magste de die nehm Nathalie / nimmst du den blauen Stift / damit ma eure Farben um unterscheiden kann wie groß ihr seid -guck ma an der Farbe deshalb mach ma so \\. [do you want to take these Nathalie / you take the blue crayon / so that you can distinguish your colors how big you are – look at the color that's why do it so \\]

Transcript 01 Sequence A.1

After choosing colors, Doris measures the children's sizes one after another at a measuring stick that belongs to the kindergarten. She follows a consistent pattern in which she introduces the children by giving constant organizing commentaries. The children are supposed to put away the wool, stand at the measuring stick in a way Doris can make a check mark in the matching color. She uses a book as an auxiliary mean. She reads out the determined sizes and writes them down next to the child's name in the matching color:

13	Doris	soo wer möchte zuerst sich messen / der Uwe \ komm ma her \ gib mir ma den Stift / so \ an die Wand stelln / Füße ganz ganz an die Wand ihr werd das schon öfters machn \ so das graade wird / nehmn wir ein \ huh / geht ihr [who wants to measure first / the Uwe \ come here \ give me the crayon / so \ stand at the wall / feet to the wall you will do that more often \ so that it'll be straight / we take one \ huh / you go]
16	Doris	bisschen zur Seite bitte / und jetzt mach ma mit dem roten Stift ein [a little to the side please / and now make with the red crayon a]
	Nena	geelb\ [yellow \]
17	Doris	Strich (unverständlich) okee \ guck ma hier Uwe / geh ma zur Seite / guck soo groß bist du \ da das wär hier drei Meter / ein Meter neun \ [line (incomprehensible) okay \ look here Uwe / go to the side / look you are soo big \ there that would be here three meter / one meter nine \]

Transcript 02 Sequence A.2

After identifying and writing down all children's sizes, Doris and the children generate woolen strings with the according length. Here again, Doris follows strict patterns: The children take the wool and hold the beginning of the string to the colored marker next to the measuring stick. Doris cuts off the string at the lower end of the measuring stick. Because of the colored mark, the children are able to identify their own mark and size without being able to read. It becomes obvious that during the measuring phase, the children have to follow Doris's narrow instructions, while she performs all the meaningful activities. The children serve as measured objects or assistants.

In the following phase, which consists of designing the cardboard, the children have more room for expressing their own idea. Some children also measure up their heads' circumference with a woolen string of the matching color and glue them to the cardboard.

142	Doris	du kannst auch rund kleben \ wie Du willst \ [you can also glue round \ as you like]
143[2]	Orania	ich mag noch Augen malen \ [I want to color eyes \]
	Doris	ja dann machs \ [yes then do it \]

[2] Because we use GAT transcription with autograph score style, the change of speaker does not require a new line.

Transcript 03 Sequence A.3

The measured length, for which the woolen string serves as a representative, becomes a design element, which can be transformed following aesthetic moments.

Situation B: Talking about measurement The whole situation lasts 34 min; the kindergarten teacher Sabine encourages measurement activities for the magnitude length (14 min) and for the magnitude volume (20 min). Here, we only look at the sequence that deals with length, for which Sabine prepared several rulers, a folding meter stick, wool, and a pair of scissors. At the beginning, these things are hidden, and Sabine opens the situation with the following impulse:

3	Sabine	so / so ihr Süßen heeeute / wollen wir mal über Messen sprechen \ kennt jemand dieses Wort messen / [so / so you sweethearts todaaay / we want to talk about measuring \ does someone know this word measure /]
4	Sadira	messen \ (messen) [measure \ measure]
5	Sabine	(was kann ma \) kennst du des Wort / kennst du das Wort Messen / [what can you \ do you know this word / do you know the word measure /]
6	Mona	mhm \ [mhm]
7	Sabine	ja \ (was is n des \) [yes \ what is it \]
8	Sadira	(ich auch \) [me too \]
9	Mona	immer wenn man krank ist \ dann muss man das da so in Arm machen oder in Popo dann muss man messen \ [every time when you are sick \ then you have to put it there in arm or in bum and then you must measure \]

Transcript 04 Sequence B.1

The children use this opening in order to understand the word linguistically and to reproduce it (Sadira) but also to link to their everyday experience (Mona). Sabine positively evaluated Mona’s everyday experience but seems not to have prepared adequate material. Therefore, the group collects further ideas concerning the word *measure*.

13	Sabine	du hast noch ne Idee \ [You’ve got an idea \]
14	Sadira	kann man die Menschen messen \

		[can you measure people \]
15	Sabine	die Menschen kann man messen \ genau \ und was misst man der bei den Menschen \ [You can measure people \ right \ and what do you measure when you measure people \]
16	Sadira	wie man groß ist \ [how you are big \]

Transcript 05 Sequence B.2

Sabine takes up the impulse and initiates at first a direct size comparison, where the children stand "back-to-back." The children carry out several comparisons in pairs, always following the same pattern. The children who do not actively take part in the direct comparison can participate with simple linguistic means, for example, by saying who is bigger or smaller. The children can decide if and with whom they want to compare themselves. They also include Sabine as an object of comparison. Omara does not actively participate as an object of comparison.

After this phase of direct comparison, Sabine focusses upon the prepared measuring devices. Sabine asks for names and explicitly acknowledges neologisms ("Messer") and deviations in the pronunciation ("Linal"), even though she finally mentions the technical terms *Zollstock* (carpenter's rule) and *Lineal* (ruler) (<87;119>). The group finally concludes that the ruler is "too small" to measure people and they choose the carpenter's rule for the following measuring activity.

Sabine measures Omara first, who before did not participate at the comparison activity. She names Omara's height in meter and centimeter (1 meter and 10 centimeter) and shows the children the (red) number in centimeter (110) at the carpenter's rule. Furthermore, they cooperatively produce a woolen string of the according length. Mona helps to produce the woolen string for Omara by holding it to the zero point. Omara cuts the wool at the indicated red number:

190	Sabine	Mona \ hilfst du uns \ halt doch mal bitte hiier fest (.) okay / und jetzt bis eein Meter zehn \ eein Meeter zehn \ guck mal / das ist hier \ und jetzt darfst du hier mal abschneiden \ [Mona \ can you help us \ please hold heere (.) okay / and now until one oone meter ten \ oone meter ten \ look / that's here \ and now you may cut here \]
191	Mona	ja \ [yes \]
192	Omara	ta s noch Wollee \ [Tere s still wool \]
193	Oslana	das rosa \ [that pink \]
194	Sadira	ich nehme rosaa \

		[I take pink \]
195	Sabine	soo \ bei der roten Zahl \ da darfst du abschneiden \ jawoll / super \ und daas darfst du nachher \ super (..) ja gut gemacht Mona (.) mit runter nehmen \ und dann kannst du jedem zeigen so \ lang \ bin ich \
		[soo \ at the red number \ there you may cut \ yes / super \ and you can later \ super (..) yes well done Mona (.) take that downstairs \ and then you can show everybody so \ tall \ am I \]

Transcript 06 Sequence B.3

In a similar way, the children produce all woolen strings, one for every child. Sometimes, Sabine relates the children's height in meter and centimeter (1 meter and 19 centimeters) to the denotation only in centimeter (119 centimeters). In this phase, Sadira decides that she does not want to be measured but is nonetheless used as a helping hand.

Sabine repeatedly points out that the children can use their woolen string later on in order to compare their length with other children that were not part of this situation <195>. Measuring with the woolen string as a mediator would make an indirect comparison in contrast to the direct comparison "back-to-back" activity conducted before. Therefore, children could optically perceive the comparison result themselves. However, Sabine frames this comparison possibility only hypothetically.

Situation C: Things for measuring Berna, the kindergarten teacher, has prepared a box with different materials (rulers, measuring tapes in different lengths, a carpenter's rule, and blocks), as well as chalk, crayons, and paper. At first, she presents the box, which is covered with a blanket. The children one after another grab into the box and try to name the things they touch and feel in it. Afterward, they take out different things and name them with their technical terms. Berna puts special emphasis on the right naming, which becomes obvious in the following sequence that happens, a while later in the situation:

298	Berna	wie heißt des / nein
		[what do you call it / no]
	Deny	Maß \ Meß \ Metermaß \
		[Measure \ measure \ tape measure \]
	Friedel	Lineaal \
		[ruleer \]
299	Berna	Lineal dankeschön Deny \ merk es dir / ein Linnneaal \ (.) super \ *(seufzt)*
		[ruler thank you Deny \ remember it / a rullleeer \ (.) super \ (*sights*)]
	Friedel	(unverständlich)
		[(unintelligible)]
	Can	Lineaal
		[ruleer]

Transcript 07 Sequence C.1

Before Berna measures the children's lengths with different measuring devices, she particularizes the meaning of the measuring scale on the measuring devices. She implicitly explains the meaning of those dashes and numbers and the relationship between centimeters and meters. She describes 1 centimeter as one "small box," and she asks the children one after another to show 1 centimeter or sections of 3–7 centimeters at the different measuring devices they grabbed out of the box beforehand. Berna chooses a 1-meter-long measuring tape as a representative for 1 meter and comments with reference to the number at the end of the measuring tape: HUNdert Zentimeter \ also / das heißt / in EINEM solchen Meter sind hundert Kästchen drin \ ... ganz viel oder / [One HUNdred centimeter \ so / that means / in ONE such meter there are one hundred. little boxes \ … a lot or /] <85–86>.

Subsequently, the group uses different measuring devices in order to measure the children's body length. Berna asks the children to write down the size and the measuring device on prepared sheets of paper. This notation process takes a lot of time since the children have not yet gained sufficient writing competences, which results in difficulties writing their names as well as numbers.

As a fixed component of all measuring procedures, the child that is to be measured lays down on the floor and the other participants mark the body's length with chalk lines. In the beginning, the group clarifies that you can measure the distance between the two chalk lines on the ground because they work as a representative for the body length. Accordingly, you have to locate the measuring device. Can is the first person to be measured. Deny chooses a measuring tape of 1 meter.

113	Berna	so wie machst du jetzt überleg al \ [so how do you do now think about it\]
	Deny	muss den wieder da hinlegen \ und dann [you have to put down this/him[3] there \ and then]
114	Deny	kann man genau wieder messen (.) wie lang er war \ [you can measure exactly here (.) how long he was \]
	Berna	muss Can sich wieder [again Can has to]
115	Berna	hinlegen / wir ham doch des jetzt festgehalten \ so lang / [lay down / we have recorded this now \ so long /]
	Deny	ja man muss es \ [yes it has to \]
116	Berna	(zeigt mit dem Zeigefinger gleichzeitig beide Markierungslinien) [*(points to both marks at the same time with her index finger)*]
	Can	man muss es in [you have to put]
117	Can	die Mitte legen \ [it in the middle \]
	Deny	genau da kann man immer wissen wie lang er ist \ [exactly there you can always know how long he is \]

[3] The German pronoun *den* can relate to things (this) with masculine genera or male people (him).

Transcript 08 Sequence C.2

For each measuring device, Berna predetermines the measuring procedure and asks the children to do specific assisting jobs:

(a) In order to measure Can's body length with the measuring tape that is too short, the group measures the entire length in two sections and relocates it, respectively. Berna formulates this addition process explicitly:

180	Berna	also \ ein Meter sind hundert Zentimeter \ plUs noch die fünfunzwanzig dazu \ dann kann ich euch jetzt verraten \ dass der Can ein Meter und fünfundzwanzig Zentimeter lang ist \ [so \ one meter are hundred centimeter \ plUs the other twenty five to it \ then I can reveal \ that the Can is one meter and twenty five centimeter long \]

Transcript 09 Sequence C.3

(b) The group uses a cuboid-shaped (10 centimeters edge length) as a measuring unit, in order to determine Bella's body length. They repeatedly measure Bella's body length with one building block until they completely stride through the section. A ruler serves as an auxiliary means to avoid overlapping units or gaps. The children count aloud; in a second attempt, the group finds out that the building block fits into Bella 13 times.
(c) The group measures Friedel's size with the help of a ruler (30-centimeter long) that they use like the building block as a measuring unit. Again, the children count out aloud. In a first attempt, they identify four as a measured value. In a second attempt, it is three, which the group then records as a measurement result.[4]
(d) The children determine Deny's body length with a carpenter's rule. Friedel lays out the carpenter's rule with the zero point at one marking line, and Deny reads out aloud the number at the second marking line: one one seven. The children then write down this series of numbers.

While Berna emphasizes the correct labeling of measuring devices, she is less precise with the determination and labeling of measurement results. The concrete body lengths of the children are less important. Moreover, the measuring processes with building blocks and rulers put stress on the meaning of the measuring unit, which also becomes obvious in the denomination of centimeters as "little boxes" between the lines.

[4]Although the first attempt seems to be the right one, since you would assume Friedel's body length rather to be about 120 cm than 90 cm. Berna, however, does not use this as a correction strategy. The counting process receives un-scrutinized validity.

Comparison of the Interaction Processes

Crayons and matching colors shape the introduction to the first scene (situation A). Doris puts most emphasis on the tinkering activity, while measuring the children and reproducing their height as wool strings seems to be of secondary interest. It seems to serve only as the fundament for the later activities. When Doris measures the children's height with the measuring stick, she does so without actively including them. The children act as spectators and only peripherally participate to the concept of measuring. On the linguistic level, Doris' speech also stays very notional. The difference between a rather vague language immersion in situation B concerning the word field of *measure* and a concrete activity immersion with measuring devices (situation C) becomes already obvious in the two opening scenes. Berna presents several measuring devices with their according technical terms. By asking the children what you do with those things, she establishes the word *measure*. Sabine, on the contrary, starts with an explicit – but theoretical – subject orientation by stating, "Today, we want to talk about measuring," although the children might not know the word and therefore this orientation might stay vague for them.

In all three situations, the body lengths of the children serve as measured quantities, but with different idea. In situation A, the lengths are the basis for the tinkering activity, whereas they serve as examples for different measuring procedures in situation C. In situation B, the concrete body length seems to be an important feature of each child, which is recorded as a woolen string.

Sentence Structure Analysis: *Measure*

The basis for our following analyses is the valence model presented above for the verb "measure" that is normatively oriented toward standard written language. In spoken language, deictic expressions can replace those constituents by or are missing completely, as shown in the following examples from our corpus[5]:

Doris (A):	soo wir (*sub*) messen mal	[soo we (*sub*) measure now]
Sabine (B):	wir backen nicht wir *(sub)* messen	[we don't bake, we (*sub*) measure]
Berna (C):	wir *(sub)* messen jetzt mit dem Zollstock *(adv)*	[we (*sub*) measure now with the carpenter's rule (*adv*)]

[5] The letters behind the speakers refer to the label of the situation.

Friedel (C): mit dem Zollstock *(adv)* kann
man *(sub)* das *(acc)* messen

[with the carpenter's rule (*adv*)
can you (*sub*) measure that (*acc*)]

In the sentence structure analysis we coded the valence of every use of *measure* in the different forms concerning numerus and time. Table 11.2 shows how often the preschool teachers use the verb *to measure* with the according complements and if and how the children take up those structures.

Subject learners as well as language learners can learn a lot from the correct application of those complements. The more complements are used with the verb *measure* in a sentence, the more concrete the process and concept is described and the easier it gets to deduce the conceptual meaning from the context. Furthermore, for German using complements with a verb offers a chance to practice often-neglected case endings. In German, case endings occur with the accusative and the genitives complements but are hard to learn and seldom discussed in school. When kindergarten teachers leave out complements, they are missing out important learning opportunities; on the other hand, concentrating on single complements and cases gives them the opportunity to focus on certain aspects without overwhelming the children.

When complements are missing completely, the verb's meaning gets very abstract; Doris and Sabine use *measure* as a generic term for the whole situation. Doris utters *measure* only five times during the whole situation, which is the lowest number in our corpus and compared to the length of the situation as well. In correlation with the situation's length, Sabine uses the verb *measure* far more often than Berna does. Sabine utilizes *measure* most often with a subject complement only, leaving the meaning of the utterance very vague. Berna on the other hand accompanies concrete measurements with her speech and uses mainly subject and accusative complements, placing the focus on the measured object.

Correction Strategies

In order to analyze correction strategies, it is necessary to look at utterances that – in the according situation – do not meet the syntactical or lexical norm or appear as less adequate. Since we are dealing with spoken language, which often includes aspects

Table 11.2 The use of sentence structure in the actual situations

	Situation 1		Situation 2		Situation 3	
Complements	Doris	Children	Sabine	Children	Berna	Children
Without/sub	1	0	19	3	8	4
sub + acc	2	0	14	4	13	2
sub + adv	0	0	0	0	2	2
sub + acc + adv	1	0	3	0	6	2
Total	4	0	36	7	29	10

of dialectal variation and language change phenomena, it is not always trivial or even possible to decide whether one utterance is correct or not. In German, for example, there are nouns with locally varying genders (cf. *der Joghurt*: male or *das Joghurt*: neuter, both possible in standard German; and in eastern parts of Austria *die Joghurt*: female[6]).

In principle, language deviations can be divided into lexical and syntactical ones. While the first deals with semantically inappropriate utterances, neologisms, and wrong pronunciation, the second is associated with wrong conjugation or flexion and word order. Here as well one has to look at the context in order to decide if something is a language deviation and to which category it belongs, as shown in the following examples from our corpus. The first example is from situation B:

82	Sabine	(hält Lineal hoch an beiden Seiten mit Zeigefinger) was ist das \ [(*holds ruler between her two index fingers*) what's that \]
83	Mona	mmhh \
84	Omara	messen \ [measure \]
85	Sabine	das ist zum Messen und weiß jemand wie das heißt / [that's for measuring and does somebody know what you call it /]
86	Oslana	Mii
87	Sadira	Messer \ [knife (measure-er) \]
88	Theresa	Messer \
89	Sabine	mhh guta guter Name \ [mhh good good name \]

Transcript 10 Sequence B.4

The word *messer* itself is a usual concrete noun in German meaning *knife*. In this context, however, the children use *messer* (knife) as a name for *ruler*, resulting more or less in a neologism. Although it seems to be a completely wrong word at first sight, there probably lies a very interesting word building process behind it. Sadira uses the stem of the verb *messen* with the ending "-er" that works similar in English with the verb *play* and *play-er* although the word *messer* (knife) used by the children already has another meaning. Here Sadira tries to actively extend her vocabulary and seems to implicitly know principles for German word formation. Unfortunately, the correct word is not *messer* but *lineal.*

The second example derives from transcript 05: Sequence B.2. Sadira says: "Can you measure people," which is a correct question on the syntactical level with the modal verb *can* in the first position. The intonation – her voice is noticeably going down, not up, at the end of the phrase – reveals that her utterance is not meant as a question but a

[6] Duden, 2013; 26. Ed., Dudenverlag, Berlin.

statement. The context as well (Sabine asks “what else can you measure /”) points to the conclusion that Sadira wants to give an answer and not ask a question.

As a third example, we found a real grammatical mistake in situation C:

Berna wir haben ja auch das Zollstock \ wo ist das jetzt hin /
[we also have the *(n)* carpenter‘s rule *(m)* \ where is that *(n)* now /]

Transcript 11 Sequence C.4

In contrast to *Joghurt*, *Zollstock* is a masculine noun only.[7] You use the definite article *der* and the pronouns *der* or *er*. Using *Zollstock* with the neuter article or pronoun can neither be associated with a dialectal variation nor a language chance phenomena or any other register.

For our analysis of correction strategies, we try to identify erroneous statements. This decision is closely bound to the interaction analysis. Then we look at the following utterances. We want to know how the participants deal with putative deviations. The preschool teachers’ reactions are of special interest to us since they offer valuable moments for supporting (or at least influencing) the child’s language and conceptual development.

In a first step in order to classify a correction strategy, we decide if one of the following utterances can be interpreted as a correction. If this is not the case, we label the detected erroneous statement as category 0: deviation without situational responding. This only means there is no detectable reaction to the linguistic deviation in this situation; the preschool teacher might still pick up the content of identified erroneous statements in the negotiation process.

Doris (A), for example, never corrects any grammatical mistakes or deviations from standard German.

Uwe will aauch machn
[wanna do tooo] (toddler speak)
Doris dann hol ma deine Wolle
[then get your wool]

(…)

Doris und dann schneid ich unten ab \
[and then I cut it below \]
Regina wer gewinn hat kriegt was ja /
[who win has gets something yes /]
Doris meine Schere \
[my scissors \]

[7] Duden, 2013; 26. Aufl., Dudenverlag, Berlin.

Transcript 12 Sequence A.4

When Uwe talks in a way you would expect from a 2-year-old, she reacts to the content, but not to the language structure. Some minutes later, she is so absorbed in the tinkering activity that she seems not to recognize Regina's incorrect utterances.

Hereafter, we will elaborate on our category system by giving examples and results of first observations concerning the distribution and frequency of certain categories as well as differences in the corrective behavior related to the type of mistake or deviation that precedes it.

We will start with the category 1 (direct/explicit). The following sequence is a perfect example for the combination of categories 1.a. and 1.b: Berna asks for the distance between two marks on the ruler (indicating 1 centimeter):

166	Friedel	Meter [meter]
167	Berna	ne, ein Zentimeter . aber du warst schon richtig nah dran, Friedel . sehr gut [no, one centimeter. But you were very close Friedel very good]

Transcript 13 Sequence C.5

After negating Friedel's answer (1.a), Berna gives the right answer (1.b). Nevertheless, she praises Friedel's attempt. In Transcript 07: Sequence C.1, however, Berna answers with a clear "no," and then Denis gives the right answer, so here we have correction strategies 1.a. and 1.c.

We could not find any metalinguistic comment with an explicit reference to the erroneous structure (1.d.i), but we were able to detect a metalinguistic comment without explicit reference to the erroneous structure (1.d.ii) in Transcript 10: Sequence B.4, when Sabine replies "good name" to two children calling the ruler a *messer* (knife). We categorize Sabine's reaction as 1.d.ii), because she comments on the name in a very positive way" good name," which indicates a differentiated language awareness but does not explicitly say what is wrong with the word (resp. with the use of this specific word building).

In some situations, we detect indirect or implicit comment on the erroneous structure (category 2). When, for example, Sadira says, "Can you measure the people" (Transcript 5: Sequence B.2), Sabine takes up the utterance through a correct paraphrase. Her utterance is a repetition of Sadira's proposition with the correct word order, which can be filed under correction strategy 2.a. The following excerpt serves as an example for correction strategy 2.b:

Oslana (B) Aber ich **gemesst**

[But I measured]

Sabine (B) Du wirst jetzt gleich **gemessen**[8]

[You will now be measured]

Transcript 14 Sequence B.5

Oslana uses the past participle but chooses the wrong form. Because of the missing auxiliary, Oslana's assertion is vague. Sabine indirectly corrects Sadira's utterance by taking up the wrong language structure (participle) and using it to express her own ideas (correction strategy 2.b).

Apparently, correction strategies vary in terms of the mistake or deviation they accompany. Thereby, the discrimination of lexical and syntactical deviations seems to be crucial for correction strategies in kindergarten situations.

The following transcript serves as an example for analyzing the correction strategies following lexical deviations:

32	Berna	oder wie heißt das sonst noch / das ist eine . ein (…) / [or what else do you call it / that's a.a (…) /]
33	Friedel	Maßbrett [measure plank]
34	Berna	ein Lineal ist das [a ruler is this]

Transcript 15 Sequence C.6

Friedel describes the ruler as a plank (*Brett*) for measuring (*Maß*) and uses a common way to build composite words in German. Berna's reaction could be classified as 1.b, because she mentions the technical term *lineal* (ruler). However, she does not react to the thoroughly linguistically plausible word construction (see also Transcript 07: Sequence C.1). In a comparable situation, Sabine reacts with a direct correction (without explicitly negating the answer) to the neologism *messer* (see Transcript 10: Sequence B.4) as well and finally mentions the correct term *lineal* but identifies the children's *messer* as a good name ("guter name"). She seems to recognize the children's language awareness and evaluates it positively (1.d.ii, followed by 1.b.).

[8] There are two ways of building the past participle in German. Depending on the kind of verb (so-called strong or weak verbs), you have "ge-stem-t" or "ge-stem-en." *Messen* belongs to the latter class.

On the syntactical level, on the contrary, we only find indirect correction strategies, which mostly go along with a positive evaluation of the content, as illustrated in Transcript 14: Sequence B.5. This reaction could be classified as 2.b, because Sabine takes up the participle, which Oslana does not correctly construct in her utterance, and forms a new sentence with the correct form. In doing so, she faces her in a positive way (cf. Transcript 14: Sequence B5) (Brandt & Tatsis, 2009; Goffman, 1972; Leisen, 2013; Tatsis & Koleza, 2006).

To sum it up, we were able to detect patterns in the use of correction strategies based on the type of deviation on which the kindergarten teachers comment. While we discovered direct corrections mainly on the lexical level, we found less direct corrections concerning syntactical deviations. If at all, syntactical deviations, in contrast, are followed by indirect corrections, which are usually connected to a positive evaluation. Generally, kindergarten teachers seem to put special emphasis on technical terms and only limited focus on complex language structures when using language in mathematical situations, similar to Moseley's results (2005).

Discussion and Conclusion: Language Sensible Organization

On the personal level, the choice of correction strategy reveals aspects of language sensible acting in the situations. We want to conclude our analyses by summarizing our observations with regard to differences in the preschool teachers' language sensible organization.

Doris' (A) reactions to detected erroneous statements – addressing the youngest children – can be classified as *without situational responding*. This might be due to the age of the children. Furthermore, Doris herself speaks with a strong dialect and some grammatical inaccuracies, mainly elliptical constructions, which may be related to her lack of corrections. Even so, concerning the toddler-like speech of some of the children, she misses out on important opportunities to support the individual (and maybe the whole group's) language and conceptual development.

Sabine (B), on the contrary, corrects very often, mainly through indirect corrections by giving corrective feedback. Her corrections on the lexical level can only be associated with category 1.b or 1.c. Thereby, she praises neologisms and language awareness. As mentioned above, with her utterance, she comments on the underlying, creative but still legit German word building process of adding an "-er" suffix to the stem of a verb ("mess-"). On the syntactical level, in situation B, you mostly find units from category 2.a and 2.b. Being the only kindergarten teacher who uses indirect corrections, she might see herself as a language role model.

Berna (C) only offers corrections on the lexical level and puts special emphasis on the correct use of technical terms, which becomes especially obvious by looking at her correction strategies. She is the only preschool teacher who uses correction strategy 1.a (often followed by 1.b or 1.c), but only in connection with technical terms concerning measurement devices. While she chooses strategy 1.b in the case of mistakes by the use of units, Berna reacts with direct, increasingly negative and even face-threatening corrections concerning technical terms (Brandt & Tatsis, 2009; Goffman, 1972; Rowland, 2000). You have to take into consideration that Berna's mother tongue is not German and that she makes grammatical mistakes herself. This might be because she has to put more attention on her own language (and the mathematical content), which leaves less working memory to focus on the children's language on the syntactical level.

With a last example taken from situation C, we want to point out how missing corrections on the linguistic level impede subject learning. We chose an example that is not correctly used on the semantical level that has to do with the concept of measuring. When you say you measure *with something*, you refer to the measuring tool – the tool with which you identify the quantified value of the special characteristic of the object. When Can says "I measure it with the chalk," he uses this construction to express his desire to draw a line from one limiting line to the other, as it was done before in order to measure length. Can might use this expression because he does not know how to express himself otherwise, or he tries to justify the attractive action of drawing on the floor which is normally forbidden, by calling it *measure*. The group measures "with a building block" in this situation, hence using an everyday life object as unit, which could confuse language learners as well as subject learner linguistically and semantically alike. Since Berna does not react to this incorrect utterance, she misses an important chance to clarify the concept of measuring, not only for Can but also probably for the whole group.

Finally, the children use the situation very differently, some of them without any active participation to the word field *measure*, including associated adjectives in different forms. However, in two of the situations, we were able to find children who use the situation as a discovering field, in terms of behavior patterns as well as linguistically. Interestingly, these are children with another first language besides German (Sadira (B) and Can (C)). From a mathematics education point of view, it would now be desirable for the preschool teachers to make the whole group profit from the children's proactive behavior. Despite some weaknesses in her language usage, Sabine partially seems to be able to do so.

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Chapter 12
Early Maths Via App Use: Some Insights in the EfEKt Project

Laura Birklein and Anna Susanne Steinweg

Abstract Nowadays, computer, tablets, and mobile phones are part of everyday life. This leads to an integration of ICT into schools and into curricula. Especially mobile devices are offering new possibilities for kindergarten education. The digital learning environment MaiKe has been developed to foster mathematical competencies in kindergarten. The research study EfEKt evaluates the effects of MaiKe use in different settings. First results of the pilot study are illustrated by insights into two case studies outlining the effects of the digital features offered by the app MaiKe on habits and competencies performed.

Keywords Attitudes towards ICT · Casxee studies · Development of competencies · Different settings · Digital learning environment · Early maths approaches · Early maths topics · EfEKt (Effects of an early maths app use on the development of mathematical competencies) · Evaluation · ICT · Intervention · MaiKe (Mathematics in Kindergarten)

Introduction

Family homes, including those with young children, are usually equipped with different media devices (e.g. Feierabend, Plankenhorn, & Rathgeb, 2013). It is not surprising that an increasing number of specific offers of applications for the younger ones are available online on the software market or at app stores. Most of the apps are alleged to have been developed with a focus on playing and learning, often in connection with each other.

In the context of education and learning in kindergarten, the use of digital media, like tablet apps, is still being controversially discussed in Germany. In addition, the huge number of apps offered on common websites is vast and confusing. Google Play Store, for instance, shows more than 250 results for the search terms 'mathe

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vorschule' (maths preschool[1]). However, a critical view at these applications has to be taken, because quantity must not be confused with quality here (e.g. Krauthausen, 2012). Many of the designs are far from satisfactory from a mathematical and educational perspective (Steinweg, 2016). In order to give a digital playground a special educational value, it is important to consider the respective empirical research findings. Krauthausen (2012) points out that there are very few findings in regard to evaluating mathematical learning apps and programmes.

The EfEKt project presented here wants to evaluate effects on children's mathematical competencies of using one particular maths app (MaiKe) in different settings in kindergarten. Currently, the pilot study has been completed. In this paper, design and main questions of the research study are described, and an exemplary insight in the pilot results is given by example of two case studies in a qualitative approach.

Theoretical Framework

The EfEKt study is embedded in two different broader theoretical research fields. On the one hand, it focusses on learning via ICT. On the other hand, early mathematical learning in kindergarten and its special approaches and topics frame the project.

One important branch of the theoretical background of the EfEKt project is the question how ICT in general and tablet apps in particular may support and provide learning opportunities. For this purpose, research on fundamental attitudes towards learning with ICT and research studies in the field of ICT use are being considered and the position of our own study outlined.

Hereafter, the second relevant branch in the theoretical framework of the project, which lays in research on approaches and content-related considerations about early mathematics education, will be the focus. Consequently, the position taken in our own study is set out.

Fundamental Attitudes Towards Learning with ICT

Süss et al. (2013) distinguishes three basic attitudes concerning learning with ICT: cultural pessimism, media euphoria, and critical optimism.

The perspective of cultural pessimism has been dominating common public attitudes and discussions time and again throughout the last decades and centuries. Buzzwords like 'trash movies' or 'reading mania' illustrate the negative connotation, which is central to this attitude. Nowadays, representatives of cultural

[1] In this paper preschool indicates no special institution but the last year in kindergarten before the children start school.

pessimism focus especially on so-called screen media, in particular on computer and video games. It is assumed that ICT use endangers the psychosocial development of the adolescent or that it at least cannot contribute anything positive to it. Spitzer (2011), for instance, blames screen media to be the reason for obesity and attention deficits, to cause a drop in performance at school, and to lead to more violence in the real world. In this perspective often no distinction is made between certain kinds of screen media or the specific contents offered by them.

The position on the opposite side of the spectrum is called media euphoria. It has become especially famous in the twenty-first century in regard to screen media, i.e., computer or internet. This perspective focusses solely on the positive effects of new media. For instance, growth of children's competencies is attributed to media use. Potential risks are seldom mentioned or discussed. Beck and Wade (2006), for example, applaud the advantages of the 'gamer generation' especially towards social learning and mutual support behaviour: 'Gamers are surprisingly good at teamwork. They love working together and helping each other' (p. XV).

The attitude called 'critical optimism' is a position located between both extremes. So-called secondary experiences via ICT use are regarded as a valuable supplement of primary experiences in the real world, but not as compensation. Moreover, researchers in this approach differentiate between different kinds of media and their specific pros and cons. They also consider the content provided by ICT in evaluative studies. Krauthausen and Lorenz (2008) point out that ICT can be useful in teaching-learning situations. Using digital learning environments, for example, may be a valuable complement in mathematic classroom lessons. However, the digital tool cannot completely replace the interaction with professional teachers and the organization of learning situations through them. Furthermore, Neuß (2013) emphasises that learning support via ICT use depends on quite a few variables. Successful support correlates at least with the quality of the software, the pedagogical involvement, and the individual, additional support by educators. The approach of the EfEKt project presented here is based on the principles of critical optimism.

Some Research Studies on Learning with ICT

Herzig (2014) identifies a research desideratum from the perspective of media education. Researchers often get carried away in theoretical discourses on whether the use of tablets in class has more value than working with traditional concepts. Research studies should focus on evolving teaching and learning scenarios and analyse evidence-based output instead. Empirical research studies show that ICT environments do not always have immediate effects on learning outcomes, but new possibilities and potentials may be opened up, if the use is based on a sound educational and professional concept (e.g. Kerres, 2003).

Different key issues regarding the use of ICT in early mathematics education are addressed in a great amount of current research studies. Some references exemplify the wide range of foci: Pilner Blair (2013) evaluates different kinds of feedback in a freely available iPad app for preschoolers. Lembrér and Meaney (2016) as well as Ladel and Kortenkamp (2013) focus on opportunities offered by interactive tables. Zaranis, Kalogiannakis, and Papadakis (2013) designed applications for kindergarten classrooms based on the concept of realistic mathematics education (RME). The activities are evaluated in relation to 'their integrity and educational use compared to the traditional method of teaching' (Zaranis et al., 2013, p. 6). Krauthausen (2012) addresses the issue of mathematical content offered by digital media. He urges experts and researchers to take a clear position in regard to topics and contents used in the digital media while evaluating programmes and software.

The EfEKt project works in line with Herzig (2014) who asks for evaluation studies with regard to potential effects on the development of the mathematical skills of the children. The app at the heart of the project is purposefully designed, and content provided in the app is carefully chosen in regard to the latest research findings in early mathematics focussed on below.

Early Mathematics Education: Approaches

Different approaches are possible to encourage children to think mathematically and to support their competencies. One approach can be identified as programmed learning. In this approach, especially conceived training programmes mostly aim to support specific mathematical competencies. They often focus on practicing skills separately. Because of their highly fixed structure, they usually provide exactly the same learning programme for all participating children (e.g. Gasteiger, 2010). Famous examples of this kind and frequently used training programmes in Germany are 'Entdeckungen im Entenland' [discoveries in duck land] by Preiß (2007), 'Komm mit ins Zahlenland' [Come to number land] by Friedrich et al. (2011), or 'Mengen, zählen, Zahlen' [Quantities, counting, numbers] by Krajewski, Nieding, and Schneider (2007).

In contrast, another approach to support early mathematics assumes potential in everyday situations. Mathematical learning opportunities are provided through everyday life and play situations (e.g. Gasteiger, 2014). In such situations, purposeful impulses and suggestions to communicate about the objects and situations can support mathematical learning. It is challenging for educators to recognise such situations and to use them productively (e.g. Benz, 2016; Ginsburg, Lee, & Boyd, 2008). Communication processes contribute to independent problem-solving competencies and support the development of transferable knowledge. Furthermore, it is supportive and important to offer mathematically rich activities on purpose (e.g. Ginsburg & Ertle, 2008). A large number of research projects underpin the potential of playful learning environments for the development of mathematical competencies, especially in preschool (e.g. Benz, Peter-Koop, & Grüßing, 2014).

Despite the positive effects of different forms of play situations described in the research findings, continually systematic training programmes including strictly prescribed educator-child interactions are being developed still (Gasteiger, 2010).

In German kindergarten practice, unfortunately, these trainings are popular and widely used. One reason for this popularity lies in the missing mathematical education of kindergarten educators (e.g. Steinweg, 2016) and their therefore naive attempt to do the right thing. According to Benz et al. (2014), concepts, which focus on motivational processes and exploit daily learning opportunities, have greater potential for learning success than solely cognitive-oriented and regimented offers.

The app MaiKe, used in the EfEKT project, is understood as an enrichment of play and learning situations in kindergarten. Although the app is a digital learning environment, it is not meant to contribute a programmed approach but offers virtual opportunities to meet mathematical challenges and tasks in play situations.

Early Mathematics Education: Topics and Contents

A useful and early mathematical support should be based on key competencies of mathematics, because first basic skills and ideas found the basis for further development in school mathematics. Key competencies of school mathematics are one possible orientation while choosing specific topics for early mathematics to support solid evolvement in school. Consequently, the German standards for mathematics education in primary school (KMK, 2004) are suitable as one starting point to consider kindergarten topics and contents in order to take connectivity into account (e.g. Benz et al., 2014).

German maths standards list topics in the fields of

Numbers and operations
Space and shape
Patterns and structure
Quantities and measurement
Data and probabilities

Of course, there is no worldwide agreed maths topic catalogue neither for kindergarten nor primary school. Nevertheless, most of the proposals overlap. Even if headings differ, in the end they describe nearly the same contents and topics (e.g. Lorenz, 2012). In literature, a wide range of topic lists for early mathematics education can be found, which are vastly corresponding to the topic lists for primary education (Brownell, Chen, & Ginet, 2014; Montague-Smith, 2002; NAEYC & NCTM, 2010; Sarama & Clements, 2009; Steinweg, 2008; Wittmann & Müller, 2009).

Special attention should be paid to predictive competencies, which have empirically proven impact on performance in the first and the second grade of primary school. Dornheim (2008) identifies the following competencies to be predictive: counting; simultaneous perception (subitising); flexible counting; part-whole relations, e.g. first additions; one-to-one relation; seriation; and certain knowledge about digits. If starting early with supporting the competencies, elements of spatial awareness like redrawing, bilateral symmetry, or pattern are predictive as well.

The design of the app MaiKe, which will be evaluated within this research study, is oriented on fundamental mathematical ideas and essential competencies for early mathematics (Steinweg, 2016; Steinweg & Weth, 2014). Especially predicative competencies, such as those outlined above, which influence performance in the first grades of primary school, are considered (Dornheim, 2008; NAEYC & NCTM, 2010; Steinweg, 2008; Wittmann & Müller, 2009).

The app MaiKe intends to enrich the mathematical learning environment in kindergarten alongside nondigital play materials. The app itself can be provided for free play. All app tasks can be used as suggestions to play the offered ideas with real-world materials, too. Moreover, the app provides a basis for rich mathematical conversations and impulses, if kindergarten educators or parents play together with the children. Benz et al. (2014) note that children benefit from a targeted use of selected board games and mathematical educational games. This is in particular the case, if they are supported by intensive verbal and content-related communication by an adult person.

The EfEKT Project

The EfEKt project (Effekte durch den Einsatz einer App zur mathematischen Frühförderung auf die Entwicklung mathematischer Kompetenzen) evaluates effects of implementing the early maths app MaiKe (http://sw-software.net/; Steinweg, 2016) in different settings in kindergarten.

The MaiKe app design tries to allow both keeping a playful character and encountering fundamental ideas of mathematics (e.g. Steinweg, 2007) or 'big ideas of mathematics' (NAEYC & NCTM, 2010; Sarama & Clements, 2009). The mathematical topics are framed and embedded in an enriched digital learning environment, to initiate specifically and systematically young children's mathematical thinking and learning processes.

Van Oers (2014) notes that 'it is clear that both creative construction and sensitive instruction are necessary elements for a developmentally productive organization of play and the development of mathematical thinking' (p. 121). The EfEKt project aims for empirical evidence of in what sense the balance of construction and instruction is effective in the special case of playing an app. For this purpose, the implementation of the maths app MaiKe will be evaluated in different settings in kindergarten.

Research Questions

The main EfEKt project research questions, derived from the theoretical reflections considering the current state of research outlined above, are as follows:

1. Will there be effects on the development of mathematical competencies of children who use the app MaiKe compared to a control group?
2. Will there be effects due to the setting in which the app MaiKe is offered on the development of mathematical competencies of children?
3. Which thinking and learning processes can be identified while using the app MaiKe in regard to certain competencies?
4. Do the digital features of the app MaiKe evoke any particular behaviour?

The first two questions lead to quantitative research methods and hypotheses to be tested. The third and fourth questions require a qualitative approach.

Methodology and Design

The EfEKt project aims to evaluate if playing the app MaiKe has any effects on the mathematical competencies of children (question 1). Hence, effects on the learning process and competencies have to be tested by quantitative methods prior and past the intervention phase. For the study, a pre- and post-test design with experimental and control group is used (e.g. Bortz & Döring, 2009). Children in the control group receive no special support apart from the usual kindergarten's daily activities (Fig. 12.1).

As a basis for the pre- and post-test, a school entry test is adapted and extended. As a typical school entry test, it focusses on fundamental mathematical competencies (see above). The realization of the test is adapted to preschool conditions. The test consists of two parts. The first part, a paper-pencil test, is carried out with two children simultaneously. The instruction is given verbally. The second part is a personal interview with each child individually, in order to test verbal counting skills, etc. The test also includes working with manipulatives and material to investigate

<table>
<tr><th colspan="2">experimental group</th><th>control group</th></tr>
<tr><td colspan="3">pretest</td></tr>
<tr><td>setting A</td><td>setting B</td><td>no intervention</td></tr>
<tr><td colspan="3">posttest</td></tr>
</table>

***MaiKe* available for free use**	***MaiKe* with personal attendance**
Organisation by the kindergarten	Regular play sessions
Observation sheets	Guided interview (video recording)
Log files	Log files

Fig. 12.1 Design of the EfEKt project

various predictive competencies. For instance, printed images of fingers or structured black dots, e.g. like on a dice, are used to determine the ability to perceive quantities simultaneously (conceptual subitising), first arithmetic operations are posed by image stories or with the aid of little cubes or counters, which can be covered under a hand, children are asked to place number cards (from zero to ten) in the correct order, and so forth.

In order to discover whether different settings have an effect on the development of mathematical competencies, the implementation of the app MaiKe takes place in two different settings (A and B) supplemented by a control group (question 2). In setting A MaiKe is made available in the kindergartens for free play. In contrast, regular play sessions organised and participated by the researcher take place in setting B. While playing together and interacting with the children, a deeper insight into their thinking processes and their reactions and behaviour during the play sessions is possible.

Additionally to the pre- and post-tests, log files automatically written by the app provide insight into the use of the app within the intervention phase. Each participant has got their own account for using the app. In this way, it is possible to back up the game progress and to create individual log files. The files document start and end time of each game, the percentages of correct swiping actions and trial-and-error-attempts, and the duration of the time played. The log file data is only available for the researcher and not for children, parents, or kindergarten educators.

In setting A the free use of the app MaiKe excludes the researcher from direct interaction and observation. Hence, observation sheets about each participating child are regularly filled in by the educators. The documentation includes notes about the playing behaviour of the child. The educators note, for example, how self-reliant the children play, if and when adult assistance is needed, or if and when children talk about the app contents with educators or other children.

The researcher-child interaction in setting B is recorded by video and documented in transcripts. The interaction is structured by the individual progress playing the app as well as especially designed interview questions (guided interview). Specific topics are chosen, and standardised questions are theory-based and designed in order to enable comparison between individual reactions and answers of the participating children given in the interviews (question 3). To illustrate this idea, one example (concerning the cardinality of numbers) is given in the following paragraphs.

One of the MaiKe app games shows fields of ten with different amounts of black dots on the left. These are presented in a structured representation regarding the power of five (Krauthausen, 1995). This means the standard representation fills in the first row of five dots before colouring dots in the second row ($6 = 5 + 1$, etc.). The dot fields in the movable area have to be matched to an egg carton on the right. In this specific game, the eggs are filled in, unstructured, in the carton (Fig. 12.2). Structured and unstructured representations of the same amount of dots or eggs therefore have to be compared. Although the representation in the egg carton is labelled as unstructured in this context, it should be mentioned that the children have the option to use and see individual structures (e.g. Benz et al. 2014).

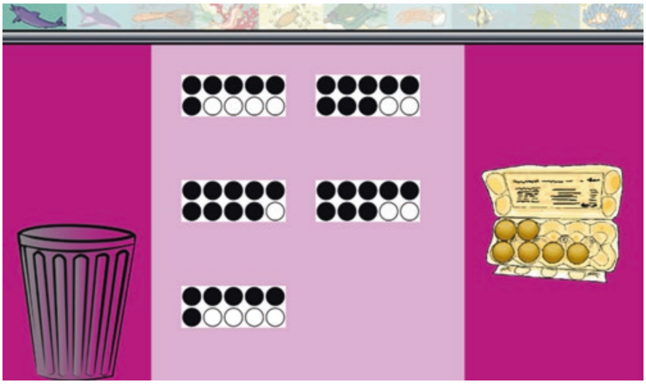


Fig. 12.2 Exemplary app game concerning the cardinality of numbers

The example in Fig. 12.2 shows two different possibilities to represent a quantity of six, which have to be matched. Whether and how a structure is perceived in one of these representations may vary considerably between the children. This perception may also change over time due to increased experience or due to the task's demands and size of the quantity presented.

The corresponding interview questions focus on special learning opportunities offered in each specific game. While playing the game described above without commenting or thinking aloud, it is not clear, if a perceived structure is actually used to determine the quantity. Consequently, each child is asked to answer the same following question: 'Where can you see faster how many there are?' The interviewer is correspondingly pointing at the black dots and the egg cartoons while posing the question.

Depending on the child's answer, additional questions allow a deeper insight into his or her thoughts, e.g. 'Why can you see it better here?' Thus, it is possible to get some indication, whether or not the individual child perceives the structure of the representations at all and if he or she uses the structure to determine the quantities.

Similar games throughout the whole app allow the pursuing of possible developments, in this case concerning the identification of quantities.

The digital environment provides features impossible in real-world environments. These features may evoke special behaviours (question 4) to be found in a qualitative analysis of case studies. Exemplary results of two case studies, which illustrate the scope of this research question, are presented below in this paper.

Pilot Study

The pilot study has already been carried out. As a matter of course, a pilot study aims for testing the used methods (test, interview questions, structure, and chosen topics). The hope is that the results provide information for improvements concerning design and methodology for the main study. The pilot study uses case studies without a control group in a qualitative approach.

The intervention phase of the pilot was scheduled from October to December, which allows a focus on participating children almost a year before they commence primary school. The participants attended two different kindergartens. Four children participated in setting A. For these children the tablets with the MaiKe app have been available for free play throughout the intervention period. Four children of another kindergarten participated in setting B. Weekly play sessions with the researcher and interviews took place. The intervention phase was framed by the pre- and post-test.

The analysis of interviews indicates that particular digital features of the play environment, like special forms of presentation, have special effects on the children's behaviour (research question 4, see above). Hence, besides the general and concurrent aim of pilot studies (testing the design), two particularities evolved in the analysis of the cases concerning

- Differences in competencies performed in the test vs. the digital environment
- Particular habits exploiting digital features

These two remarkable points are a clear focus in each in the cases of Sarah and Karin.

Some Results in the Cases of Sarah and Karin

In this paper, insight in the pilot study is exemplarily given by the cases of Sarah (4 years 10 months) and Karin (6 years 1 month). Nine months after the intervention, they will be starting school. Both participated in the setting B group. Thus, they were taking part in regular play sessions with the researcher.

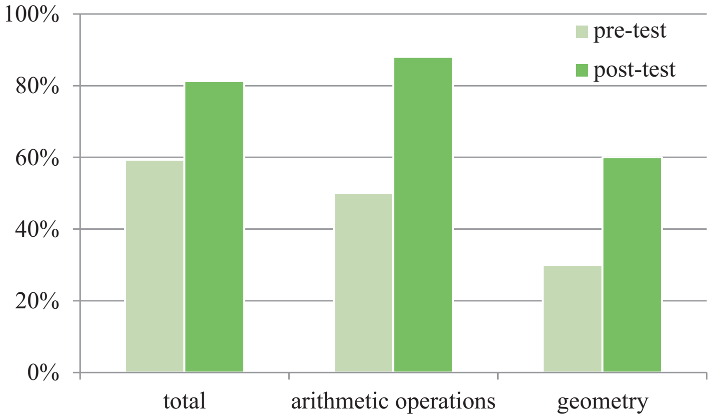
In Sarah's case, the differences in competencies performed in the paper-pencil environment versus the competencies shown in playing the app are striking. Two topic areas are picked out for a more detailed examination regarding the development of her mathematical competencies and her learning process and progress.

The case of Karin illustrates peculiar experiences and habits while playing the app and answering the interview questions. Her handling utilising special opportunities offered by the app is outlined and illustrated by exemplary transcripts.

Sarah's Case

Before considering individual sections and exemplary games in detail, the results of Sarah's pre- and post-test can provide a first impression about the development of her competencies (Table 12.1).

Table 12.1 Sarah's results prior and post the intervention



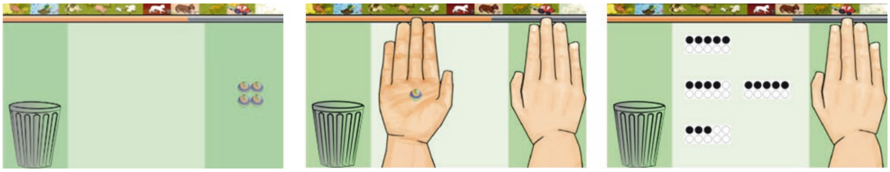


Fig. 12.3 Covered addition (4 + 1)

Her competencies in the pretest are already fairly high. She solves 59% of the tasks in total. After the intervention the post-test results indicate an increase by 22 percentages points. Sarah is now able to solve 81% of the tasks correctly.

The development in the overall outcome indicates some learning progress. A more detailed analysis concerning content fields shows that the development of Sarah's competencies is especially pronounced and remarkable in two topic areas (Table 12.1).

The topic areas arithmetic operations and geometry examined in the test correspond to certain games in the app MaiKe. These games focus on the part-whole concept, addition, and subtraction in the field of arithmetic operations. In geometry, they require competencies in symmetry, composing shapes, and spatial orientation. For some deeper insight into Sarah's reactions and possible explanations of her progress, first a closer look at *arithmetic operation* is taken.

The addition and subtraction games are designed as so-called covered operations, which have been invented by Spiegel (1992). First, one addend is shown (as a certain amount of marbles) and then covered by a hand (Fig. 12.3). The other hand adds zero to three further marbles. After all the marbles are hidden under the hand, fields of ten with various numbers of black dots are shown. The field of ten, for which the number of black dots corresponds to the amount of marbles under the hand, matches the hand.

Sarah participated in the interviews in setting B. By analysing sections from these interviews, it is possible to trace some of her thoughts and learning processes during her gameplay. In this paper, attention will be drawn to the two topic areas, in which her competencies significantly changed from pre- to post-test (Table 12.1). The following transcript (translated by the authors) describes Sarah's statements and actions during her first encounter with the game covered addition. App actions seen on the screen are indicated by A (app).

1	A	*Four marbles are shown.*
2	S	Four.
3	A	*The four marbles are covered by a hand, and a second hand adds one marble.*
4	S	And one to it.
5	I	Mhm.
6	A	*Fields of ten are shown.*
7	S	(*5 s pause)* Huh? (*Assigns the field of ten with five black dots. Swipes non-fitting fields to the recycling bin.)*
8	I	Very good.
9	A	*Two marbles are shown.*
10	S	Two.
11	A	*The two marbles are covered by a hand, and a second hand adds three marbles.*
12	S	*(Tries to assign the field of ten with four black dots, five times. Questioning gesture.)* Four?
13	I	You can watch it once again.
14	S	(*Clicks on the digital image of the hand on the screen.)*
15	A	*The animation (two marbles, two covered, three added) is repeated.*
16	S	(*Shows two fingers under the table.)*
17	I	You can go ahead and do it with your hand.
18	S	*(Raises two fingers at once. Adds then three times one on the same hand until five fingers are raised. Assigns the field of ten with five black dots.)*

While the animation is running, Sarah comments on the action (#2 and #4). Thereby she names the amount of the four marbles without hesitation, which indicates that she likely did not count them. Her first correct assignment suggests that she is mastering the task (#7). The next task (2 + 3) challenges Sarah. She tries to assign the wrong field of ten several times (#12). The interviewer encourages Sarah to watch the animation once again (#13). After watching the animation a second time, Sarah uses her fingers hidden under the table (#16). The interviewer invites her to openly show her finger operations (#17). Sarah uses two fingers as starting position and counts on by raising another three fingers one after the other, until she reaches five fingers (#18).

Fig. 12.4 Exemplary app games concerning geometry

Finger use can be observed in this game only two times out of six tasks (3 + 2 and 2 + 3). If one addend is 1, the sum can be determined by identifying the successor. Likewise, counting strategies are leading to the desired result quickly. Possibly, the relation between predecessor and successor (ordinal number aspect) may be anchored as factual knowledge. The tasks, where Sarah uses her fingers, need a mediator (fingers) to determine the sum by counting-on strategies. However, if the sum is being worked out, no counting processes can be observed, while the correct field of ten is assigned promptly. The immediate assignment indicates that Sarah is able to match the five dots in the field of ten with five as a finger number (#18).

Finishing a game is rewarded by a progressively completing illustration as in a jigsaw puzzle in the starting screens. Because of the relatively high trial-and-error rate in this first round, the reward picture of this particular game is not fully coloured. Sarah therefore starts a second round. This time, she uses her fingers implemental while solving three tasks (3 + 2, 3 + 1, 2 + 3). For solving the task 4 + 1, she decided to watch the animation a second time via tapping on the image of the hand.

In the EfEKt project, playing the MaiKe app with or without interaction with the researcher is accompanied by recording background data in log files. These files offer quantitative data which can be compared to the qualitative data and validate or indicate similar findings. Overall, Sarah's solution rate in the covered addition game increases from 30% in the first round to 75% in the second round. This becomes visible in the log files documented by the app. In the first round, Sarah needs 5:17 min to solve the six addition tasks. In the second round the time is nearly halved (2:39 min).

During the game session a week later, Sarah is playing the game on covered subtraction. In this first subtraction game, the amount of the marbles covered by the hand first (minuend) is not more than four. Then, the second hand takes zero to four marbles away (subtrahend). Sarah solves these eight tasks in 3:34 min with a solution rate of 85%. She is not using her fingers at any time.

As mentioned above, the second content area in which Sarah's progress is remarkable is *geometry*. In the topic field geometry, games on symmetry, composing shapes, and spatial orientation (Fig. 12.4) are presented.

The analysis of Sarah's test answers and her behaviour and success playing the game reveal one striking point in her case: Although Sarah scores low on spatial orientation and geometry tasks in the pretest, the log file data from the corresponding games in this specific topic show high solution rates between 84% and 100% often even combined with fast processing. Sarah solves especially the symmetry games with an above average speed. The time required for the other games can be classified as average.

During the post-test, Sarah shows higher competencies in geometry (increase by 30 percentage points now to 60%). She solves the tasks focussed on symmetry correctly for the first time. She still has difficulties to assign building blocks to a given construction in the post-test, although she masters the matching app game very well with a solution rate of 84%.

Karin's Case

As mentioned above, Karin also participated in setting B and, therefore, in the interviews during the pilot study. She is already 6 years old at the beginning of the study and scored quite well in the pretest (86.6%). Although, her scores in the post-test increase to 94%. The particularity in Karin's case is her handling of the digital features of MaiKe.

The following interview (translated by the authors) portrays her reaction to a part-whole-relation game in the app MaiKe. In this game a certain amount of black dots in a row (whole) and another amount of red dots (part) in the second row below is given. The parts are framed, and the whole amount is shadowed in grey. The part pieces appear as if they have been cut (straight line on one side of the frame) to underline the fragment characteristic. Potential matching parts are presented in the moveable area (Fig. 12.5).

1 A *First row with five black dots (whole) and second row two red dots (part).*
2 K Funny, how should this work? No clue. (*Swipes a four dot part to the gap.*) Like this?
3 I Aha, good.
4| K (*Swipes the second four dot part towards the bin.*) Recycling bin? Ah. (*Lifts her finger.*)
5| A *The first four dot part is faded out, and the gap is empty again.*
6 K Huh? Where is that again?
7 I That matched. Now you can have a look, if another one matches.
8 K That there? *(Swipes the second four dot part to the gap.*) Yes, fits.
(*Tries to swipe a three dot part several times to the gap*) Does this match?
9 A *The three dot part bounces back.*
10 K No. (*Tries to swipe a different three dot part to the gap.*)
11 A *The three dot part bounces back.*
12 K Anything has to match still. (*Tries again to swipe the three dot part to the gap.*) Then, into the bin. (*Swipes both three dot parts to the recycling bin.*)

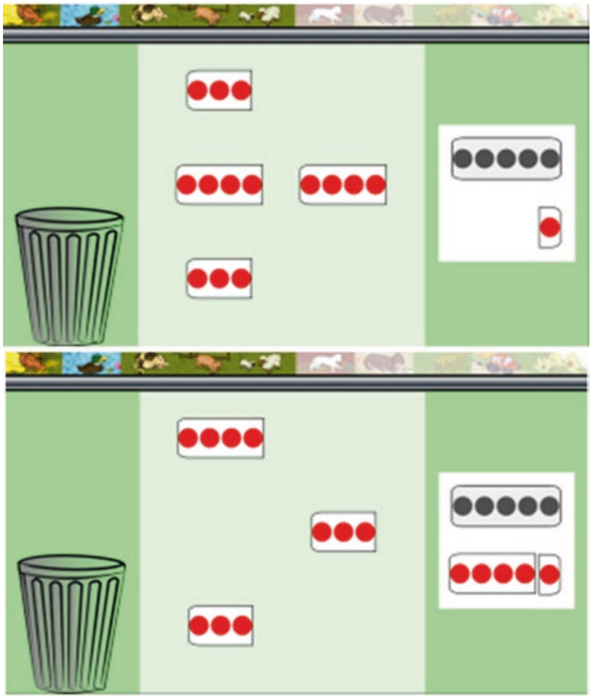


Fig. 12.5 Part-part-whole app game

In Karin's first encounter with this game, she expresses her lack of knowledge (#2) but finds a matching part nevertheless. It is unclear, whether this first match is found accidentally, because she tries to swipe the second fitting part to the recycling bin first (#4). She feels uncertain doing so, which is indicated by her question, her interjection, and the fact that she does not finish her assignment but lifts her finger again (#4). She, therefore, takes advantage of the digital learning environment which allows tentative and uncertain trials without any consequences, if the swiping process is not finished.

In synch, the animation fades out the first correct match, to offer the gap again for possible other correct objects (#5). This function is irritating Karin (#6) and is being explained by the interviewer (#7). Now it looks like Karin has understood the game, and she assigns the second fitting part (#8). But again, she is unsure and tries to assign a non-matching three dot part several times (#8–#11), before she gives up and decided to swipe the last two parts to the bin (#12). To summarise her first solving strategy, she finds correct answers by trial and error.

While playing the second task, the interviewer gives impulses by posing questions. Karin solves the other four following tasks in this first round independently and without any problems. The following interview transcript shows one of these four tasks:

13 A *First row with three black dots (whole) and second row with two red dot (part).*
14 K *(Swipes a one dot part next to the two dot part.)* One more can fit, I keep this. *(Swipes the other two and three dot parts into the recycling bin. Swipes the second one dot part next to the two dot part.)* Exactly!

This short transcript shows furthermore that Karin is able to change between the different possibilities of swiping. First, she swipes one of the two correct parts to the right side. She spots the second correct part immediately but decides to 'keep this' and to swipe the non-fitting parts into the recycling bin before she matches the remaining one to the right side, too (#14). This scene indicates that she is handling the opportunities of swiping flexibly and confident.

Karin plays this game a second time. The following transcribed part shows that she is now using even more extensive opportunities, the digital playground offers.

15 A *First row with five black dots (whole) and second row with one red dot (part).*
16 K Five. *(4 s pause)*
Need to be actually four more, or what? *(Swipes a four dot part next to the one dot part but she does not lift the finger to finally place it there.)*
Are they? 1, 2, 3, 4, 5. Yes*! (She lifts the finger to fix the part.)*
17 I Yes, this fits. Good.
18 K *(Swipes the non-fitting parts into the recycling bin and identifies another four dot part.)*
And that may fit, too. (*Assigns the part.) Five, again. (And lifts her finger immediately.)*

At the beginning of this scene, Karin presumes that the four dots are fitting next to the one dot part, but she is not entirely convinced. This becomes evident because she swipes the four dot part next to the one dot part without releasing it finally (#16). The app makes it possible to move the dot parts, which are placed in the middle, anywhere on the screen. Only when releasing through lifting the finger or the pen, the dot part is either fitting or bouncing back to the original position in the middle of the screen. Holding the parts next to each other, without releasing her finger, gives Karin the opportunity to check her presumption by counting the dots of both parts (#16). Reaching five by counting, her presumption is confirmed. Therefore, she lifts her finger now (#16). After swiping the non-fitting parts into the recycling bin, she identifies the second four dot part. She considers this one a suitable solution, too. She assigns the four dot part next to the one dot part. This time, she recognises the amount of the five immediately and lifts her finger (#18). At this, she does not need to count anymore to assure herself.

The log files of this game show a solution rate of 84% for Karin's first round. Her few mistakes mainly result from her trial-and-error strategy in solving the first task (see above). In the second round, she exploits the possibility of the digital learning environment, e.g. by swiping the movable dot part next to the given ones in order to check her intuitive attempt. She is verifying her presumption by using a counting strategy to gain confidence. Afterwards she does not use any tentative or counting strategy. The solution rate of 100% in this second round of playing validates the process data identified in the transcripts. The period of time, Karin needs to solve all tasks in this game dropped by one third; first she needs 3 min and in the second go 2 min. This quantitative data can be interpreted as another indicator of her growing mastery.

Discussion

The case of Sarah indicates some substantial difference between the abilities and competencies shown in the test versus the digital play environment. One possible explanation for the diverging results might be the forms of presentation, which have to be interpreted differently by the children in the paper-pencil test and in the digital playground MaiKe. In the first case, immobile printed images (building blocks, mirror images, etc.) expect mental rotation and reconstruction. By contrast, the app provides animated representations and allows concrete (though virtual) actions. For example, building blocks can virtually be moved, or line patterns in a matrix are actually drawn by the app, so that the children can imitate this action directly.

The app offers opportunities a paper-pencil learning environment does not. The meaning of movability of the elements and the explicit option to tentatively match objects is a special value of the digital learning environment. To have a whack at solving the task is not considered makeshift, but a possibility and customary way of behaviour in gaming situations. The case of Karin reveals and exemplifies these opportunities in the part-whole game.

Fixed and, therefore, immobile printed images in a paper-pencil learning environment do not allow movable features exploited by Karin. The same task would require more mental activity and mental moving of the representations. Furthermore, the children do not have the chance to check their presumptions in a real-world environment without any consequences, like Karin.

Of course, it is conceivable to create a task like this with real-world materials like cube strings or something like that. In principle, all of the app tasks can be used as suggestions to pose analogue tasks with real-world materials. Working with manipulatives and real-world materials usually requires support by an adult. In the part-whole relation example, the cubes have to be placed in suitable rows. If this silent impulse is not sufficient, only a question posed by the educator or parent can initiate the problem-solving. To find a solution, the child then has the option to change the position of the cubes presented or add other ones. In contrast to the digital environment, the attempts to find missing parts mainly depend on counting processes because cubes and other manipulatives are usually presented in units. The app allows and asks for more sophisticated approaches like simultaneous perception.

The materials sometimes enable the children to check their solutions self-reliantly. If the task demands to compare parts and the whole, the parts can be placed beneath the given whole and be compared by length. Though, this elaborate idea does not come to every child. Thus, often support by educator feedback is needed or demanded by the children. The automatic feedback of the app can, therefore, be regarded as another special feature of the digital playground. At least a distinction between correct and incorrect solutions is made by the app directly. Children do not take offence at the digital feedback and are not intimidated at all. On the contrary, they try another solution or a wrong solution again and again, as long as it takes them to figure out the mathematical relation and the correct interpretation of the task. One may object that this feature encourages children to stick to trial-and-error strategies. Our findings – like in Karin's case – dissent this fear. Karin needs the trial-and-error phase to become confident in the task's demands and overcome it quite fast in the second attempt. Of course, for a more detailed response and individual support during the learning process, adult assistance is useful while playing the app as well.

In summary, the pilot study indicates that the interviews, like in Sarah's and Karin's case, allow interesting insights into thoughts and learning processes in a qualitative way. The findings can be compared and related to quantitative data gained by the test results and the individual log files, especially concerning solution rate and duration of time playing a certain game, as exemplarily shown above.

The participating children benefit from playing the app and their mathematical competencies increased. The pilot study implies that the competencies performed in the app are even higher than the ones shown in tests in some cases. Of course, the results are only first indications and tentative. Deeper insight into the effects and evidence for dependencies of results will only be made possible by the main study.

Furthermore, the pilot study expertise leads to important implications for the main study design regarding:

Sample size
Age of the children
Tasks used in the tests
Interview impulses and focus

The experiences gained during the pilot study allow a sound estimation regarding the possible amount of participants manageable in the main study. The sample, of course, will be extended. Sixty-six children from six different kindergartens will participate. The design of settings A and B will be maintained. The main study will be complemented by a control group, which enables reliable interpretation of effects as well as attribution to the intervention.

Additionally, the chosen sample will be widened with respect to the age of the children participating in the main study. The pilot study indicated that several of the preschool children master the pretest – which is a school entry test – already quite successfully. Likewise, the app tasks are worked on with ease and very well by some. Hence, younger children (aged 4–5 years), who's school start is going to be 1.5 years ahead, will be included in the main sample.

The preschool children, who constitute half of the sample, participate for half a year, before the post-test is taking place shortly before they start school. The younger children participate 1.5 years in the intervention. During this period, one intermediate test is taking place half a year before the children's school entry. This intermediate test makes it possible to compare the results with the results of the preschool group. Both groups consist of children of the same age and utmost equal experiences in the same kindergarten but different experiences concerning the app. Finally, the post-test is carried out shortly before their school beginning, too.

The questions and tasks used in the pre- and post-test are improved and complemented by some further tasks for the main study in order to receive additional information on specific competencies. Moreover, the guideline for the interviews in setting B is elaborated and refined. In particular the guidelines and impulses are focussed on two specific topics rather than broaching the issue of every possible topic sketchily. Because of the longer intervention period over 1.5 years, regular meetings with the educators in the setting A kindergartens are scheduled. This allows for maintaining an overview of the progress playing the MaiKe app, e.g. by checking the log files. Furthermore, it enables the researchers to present an intermediate result to the educators to keep them informed about the current game status of their children.

The adaptations have been worked on and the design of the main study has been completed. The pretests were actually already carried out in spring 2016. At the moment, the intervention phase in the two different settings is currently running.

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Part IV
Mathematical Learning in Family Settings

Chapter 13
How Can a Father Be Supportive for the Mathematics Learning Process of a Child? – The Relationship Between Scaffolding and the Interactional Niche in the Development of Mathematical Learning in the Familial Context

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Abstract This chapter focuses on a father-child interaction during block play, which shapes the child's mathematical experiences and mathematics learning process. With the aim of analyzing and discussing such interaction process in detail, the negotiation of taken-as-shared meanings during block play is observed. For this, the concept of the interactional niche in the development of mathematical thinking is used. This concept sheds light on questions of how a father, as one of the main parts of family systems, uses some scaffolding functions and how such interaction process enables a child to learn mathematics in a play situation. The result demonstrates that the play with father takes place as a social act for the child, and the interaction process with father provides the child an effective mathematics learning process, and an interactional niche in the familial context emerges. It can be concluded that familial systems have crucial effects on the scaffolding process.

Keywords Family interactional niche · Father · Scaffolding · Block play · Early childhood · Familial context · Geometry · Interaction · Scaffold learning · Family systems · Leeway of participation · Negotiation of taken-as-shared meanings

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Introduction

NCTM reports that early experiences in mathematics have major importance on children's learning in the first 6 years of life, and young children in every setting experience mathematics through familial practices (NCTM, 2013, p. 1). In this regard, the activities, toys, materials, and social events introduced to children in their home environments shape their thought processes and performances in mathematics. So indeed, Connecticut State Board of Education suggests that family supports children's thinking and play in the emergence of their skills and abilities in each developmental domain (2007, p. viii). Thereby the familial environment gives children various opportunities to experience mathematical activities, which are potentially significant for learning mathematics. Furthermore mathematical thinking and learning come to be a "jigsaw" (Pound, 2006, p. 23) in which the child can make connections between things that are known and new information and experiences.

Research results reveal that early learning within play activities and with the participation of a family member turn out to be more productive and fruitful for the child than playing without an adult (Acar Bayraktar, 2014a, 2014b, 2016; Acar Bayraktar & Krummheuer, 2011). Regardless if the family member has adequate knowledge about mathematical issues, the interaction leads the child to learn something about mathematics (Acar Bayraktar, 2014a, 2016). In addition, emotional motivations of family members can suffice to provide different mathematics learning situations for the child (Acar Bayraktar, 2014a). Furthermore, while the network of family members links closely with everyday lives of children, playing with different family members is likely to provide various learning opportunities about mathematical ideas (Acar Bayraktar, 2014a, 2014b, 2016).

Similarly, Pound (2008) points out that children profit from discussing mathematical ideas with adults. Parents are the first adults in children's lives and the first and the most continuous provider of services and care for their children. They inform their children about any issue, from birth to death, while they also satisfy the emotional, physical, and motivational needs of their children. In this regard parents have a crucial role in the development of children in mathematics as well as in any other realities of life. Collins, Madsen, and Susman-Stillman (2002) point out that the education level of parents has an influence on the communication and parents' styles of interaction with their children. Their education level enters into their communication styles with their children, the children's social environment, and "daily informal and formal activities, which promote or discourage children's peer relationships" (Parke, 2004, p. 371). Parents with lower levels of education have less frequent interactions with their children in middle childhood, and when these children start school, the frequency of their interactions become less than half (Collins et al., 2002, p.79). In contrast, high parental guidance, often with advice-giving strategies, efforts to keep children from being influenced by peers, and talking to them about the future consequences of their behavior, lead children to low levels of antisocial behavior and higher levels of academic achievement (ibid.).

At the age of 5, children enter a wider social world and begin to "determine their own experiences including their contacts with particular others" (Collins et al., 2002, p. 73). Both fathers and mothers increase their attention to their children's school achievement and homework during middle childhood (Collins et al., 2002, p. 80). Furthermore they each make the uses of mathematics apparent so that children can benefit from them and learn complex mathematical meanings and understandings. The questions then arise on which roles fathers and mothers take in the mathematical development of their children and how and in which ways they separately provide and make possible such mathematical learning situations. While we already know something about children's engagements in mathematical situations with their mothers (e.g., Brandt & Tiedemann, 2010; Fisher, Hirsh-Pasek, Golinkoff, & Gryfe, 2008; Miller, Kelly, & Zhou, 2005; Tiedemann, 2013; Vandermaas-Peeler, 2008), we know only a little of children's engagements in mathematical situations with their fathers (e.g., Acar Bayraktar, 2014a, 2014b; Hawighorst, 2005). In many research cultures, father's role is issued less often than mothers' role (Parke, 2002, p. 62). Regarding this, this paper responds to this research need and focuses on the question of how fathers support the learning of early-year mathematics of their children.

According to traditional models of society, fathers are "financial providers" (Tamis-Lemonda, 2004, p. 220), and thus in western industrialized nations, they spend less time in direct one-to-one interaction with their children than mothers (Bornstein & Sawyer, 2008). Therefore usually they take less responsibility than mothers for child caring. According to family systems theory,[1] while mothers mostly attend to "the child's calm and comfort," fathers foster children's "openness to the world" (Tamis-LeMonda, 2004, p. 220). Fathers tend to encourage risk taking while simultaneously protecting their young from danger. During play activities with their fathers, children experience standing up for their own beliefs, while their fathers encourage them to face up to unfamiliar occurrences and their own mistakes, hence justifying themselves and taking risks in new sets of circumstances (ibid.). Such occasions lead children's social competences and functions to develop; they open children up to the outside world. Besides these, fathers encourage their children to complete tasks in the shortest amount of time, which is the primary goal in problem solving (Laakso, 1995, p. 447; see also Acar Bayraktar, 2014a, 2014b). Laakso points out that in the parent-child conversation, children experience more communicative breakdowns with their fathers than with their mothers, and thus there occur different communication styles between mother-child and father-child dyads (Laakso, 1995, p. 446). Moreover fathers ask questions more frequently than mothers, offer their children more information, use more elaborative labels, and come up with more imperative and short utterances in the interaction process with their

[1] Family system theory lays emphasis on the internal and external factors of a family and regards the family as a social system (for more, see Bornstein & Sawyer, 2008). This approach considers "the interdependence among the roles and functions of all family members" (Parke, 2004, p. 366) and helps me "to understand fully the nature of family relationships" and how family members deal with each other and these relationships affect the child's development.

children (Mullis & Mullis, 1986 see also Acar Bayraktar, 2014a, 2014b). Furthermore fathers give more responsibility to their children in completing their given tasks, while they pose more questions and vary the instructions given to their children more flexibly. They tend to make more requests for information, give more exact and elaborative descriptions in play situations, and use a greater proportion of verbalizations describing form, shape, and direction relations than mothers in course of interacting with their children (Laakso, 1995; see also Acar Bayraktar, 2014a, 2014b). Thereby they evoke the "activation function" during play interactions with their children, which involves an exploratory system whereby children experience novel issues in physical and social environments (Tamis-Lemonda, 2004, p. 222; see also Acar Bayraktar, 2014a, 2014b).

On the basis of theoretical aspects above, I observe a father-child dyad in game playing and try to answer some sub-questions:

1. How can a father make the uses of mathematics apparent so that his child can benefit from them and learn complex mathematical meanings and understandings?
2. How does a father in turn "scaffold" his child toward higher levels of mathematical development (Wood, Bruner, & Ross, 1976)?
3. How do education level of a parent and the role of father affect interaction and scaffolding process in the mathematical context?

I pursue these questions in an empirical and qualitatively laid out work, which is in line with the interactionistic research paradigm (Cobb & Bauersfeld, 1995). Thus, in line with abovementioned approaches to mathematical learning, I focus on the emergence of mutual understanding and coordination in discourses between a child and a father. This research has important implications for the fields of mathematics education research. Because the role of fathers in mathematics learning of their children is mostly overlooked or neglected in everyday practices of mathematics education research, this study can bring about any further questions and research themes in this field and maybe also in early childhood education research.

Specific Issues of the Theoretical Approach

The Theoretical Concept of NMT-Family

One of the central research purposes of this work is to examine the relationship between the participation of children and a family member in play situations and to find out how they interact with each other and how individual content-related learning occurs. In this regard, the concept of "interactional niche in the development of mathematical thinking in the familial context" (NMT-Family) (Acar Bayraktar & Krummheuer, 2011; see also Acar Bayraktar, 2014a, 2014b, 2016) is used.

The interactional niche in the development of mathematical thinking (NMT) is developed by Krummheuer (2014) and particularly based on "symbolic

Table 13.1 The structure of NMT-Family (Acar Bayraktar, 2016)

NMT-Family	Component: content	Component: cooperation	Component: pedagogy and education
Aspect: allocation	Mathematical issues, mathematical play	Play as a familial arrangements for cooperation	Developmental theories of mathematics education and proposals of activeness for parents on this theoretical basis
Aspect: situation	Interactive negotiation of the rules of play and the content	Leeway of participation	Folk theories of mathematics education, everyday routines in mathematics education
Aspect: child's contribution	Individual actions	Individual participation profile	Competence theories

interactionism (Blumer, 1969), the cultural historical approach of Vygotksky and Leont'ev, (see Bruner, 1996; Ernest, 2010; Wertsch & Tulviste, 1992) and the phenomenological sociology of Schütz (Schütz & Luckmann, 1979) and its expansion into ethnomethodology (Garfinkel, 1972)" (Krummheuer, 2014, p. 73). It comprises of "the aspect of the interactive local production" of mathematical developmental processes in "the micro-environment of the child" (Krummheuer & Schütte, 2016, p. 173) and answers the question, "How can the situationally emerging form of participation of a child in a social encounter be conceptualized as a moment in the child's development in mathematical thinking?" (Krummheuer, 2014, p. 72).

NMT-Family is the concept of an "interactional niche in the development of mathematical thinking in the familial context" (NMT-Family) (Acar Bayraktar & Krummheuer, 2011; see also Acar Bayraktar, 2014a, 2014b, 2016) and constructed as a sub-concept of NMT. Similar to the concept of NMT, it consists of the aspects of allocation, situation, and the child's contribution. This structure of NMT-Family (Acar Bayraktar, 2014a, 2014b, 2016) is shown in Table 13.1.

The aspect of *allocation* refers to the provided learning offerings of a group or a society, which specifically highlight cultural representations. The aspect of *situation* consists of the emerging performance occurring within the process of negotiating meaning. The aspect of the *child's contribution* involves the situational and individual contributions of the child in focus.

Scaffolding

Bruner (1983) highlights that parents elicit interactive play settings, which promotes child development to sophisticated levels. Furthermore he assumes scaffolding as one of geneses as parents' initiative for supporting children's learning. Thereby parents reflect on the child's perspective voluntarily and obviously, which enables the child an increasing or decreasing autonomy during play. According to Boekaerts, scaffolding refers to a metaphor which "captures the idea of an adaptable and temporary support that helps an individual during the initial period of gaining expertise" (1997, p. 171). Similarly, Brandt and Tiedemann (2010) define scaffolding as a kind of support, of which "key function is to arrange a situation, which allows the child to participate as a competent community member" (p. 430).

The term "scaffolding" extensively appeared in the work of Wood et al. (1976) about the role of tutoring in problem solving. They define scaffolding as an "adult controlling those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon and complete only those elements that are within his range of competence" (Wood et al., 1976, p. 90). In the work of Wood and his colleagues, the adult person is referred to as an "expert," who "tutors" children during 3D structure building, and the "novice" or "tutee" is referred to as a person who is less adult or less expert and thus gets help from an "expert" (Wood et al., 1976; see also Bruner, 1996; Hammond & Müller, 2012; Nader-Grosbois, Normandeau, Ricard-Cossette, & Quintal, 2008). Their work aimed to examine "some of the major implications of [the] interactive, instructional relationship between the developing child and his elders for the study of skill acquisition and problem solving" (1976, p. 89). Wood and his colleagues define the usual type of tutoring as an "actual pattern of instruction," "in which one member *knows the answer* and the other does not, rather like a *practical* [situation] in which only the instructor *knows how*." (ibid). Thereby the tutor enables children to learn a subject through his or her instructions in the interaction process. For this, (s)he realizes six scaffolding functions called "recruitment, reduction in degrees of freedom, direction maintenance, marking critical features, frustration control, demonstration" (Wood et al., 1976, pp. 98). This process is called scaffolding, which is an "interactive system of exchange that tutors operate with an implicit theory of the learner's acts" (ibid, p. 99).

Bibok and his colleagues referring to Wood et al. define scaffolding as a process that an adult person "simultaneously aims to regulate both children's motivation (recruitment, frustration control) and cognition (reduction in degree of freedom, marking critical features, demonstration)" (Bibok, Carpendale, & Müller, 2009, p. 18). In addition to this, Anghileri (2006) points out scaffolding is not a teaching process but rather *flexible and dynamic* practice that an adult person is *responsive to individuals*, while they are learning independently and autonomously. In my study the focal medium is families; thus I think of scaffolding not as a teaching method but rather as a support that can also focus on the development of the child in a familial context (cf. Van de Pol, Volman, & Beishuizen, 2010). For me such kind of scaffolding differs from teacher scaffolding, which particularly aims at schooling or realizing school culture. Similarly Hammond and Müller (2012) consider parental scaffolding as "unique among potential forms of parental influence on children" at attempting to improve a child's problem solving (2012, p. 280). Tiedemann (2013) also perceives scaffolding as a support that adult and child realize and co-construct together in the situation of negotiation of meaning. In this regard, I perceive scaffolding as a kind of methodology of family members that they "intuitively and informally" realize scaffolding functions in order to support their children during play. For further aspects of scaffolding discussed in the literature, see also, e.g., Bakker, Smit, and Wegerif (2015), Belland, Walker, Olsen, and Leary (2015), and Van de Pol et al. (2010).

Block Play and the Baden Family

In this section, I present the empirical instrument that is embedded as a sub-project in the project of "early Steps in Mathematics Learning-Family Study" (erStMaL-FaSt) (for more, see Acar Bayraktar, 2014a, 2016). The example mathematical game selected from erStMaL-FaSt is the block play "Building 02." In the following sections, first, the game "Building 02" is analyzed. Subsequently, an empirical material is brought in, which comprises of a video recording and its transcription of the Baden family while playing game "Building 02."

A Block Play: "Building 02"

The game "Building 02" is based upon the game "Make 'n' Break" (Lawson & Lawson, 2008) and refers to a block play. It is constructed according to the specific design patterns of erStMaL-FaSt (Vogel, 2014), which means play situations focus on "one mathematical task or problem, which is presented in a playful or exploratory context according to the age of the child and represents the starting point of a common process of dispute" (Vogel, 2014, p. 225). One particular mathematical domain is addressed, and compatible materials, arrangement of space, and mathematical task are chosen. In a brief description, a specific design pattern contains (1) a definition of the play situation, (2) an application field, (3) an intended mathematical domain, (4) a mathematical context, (5) materials and playroom, and (6) an instruction manual (Acar Bayraktar, 2014a). Regarding all these facts (1) "Building 02" can be defined as a block play, which refers to the sum of all actions of building of three-dimensional versions of different geometrical shapes depicted on different playing cards with wooden blocks. (2) The application field is a familial context for the children ranging in age from 4 years upward. (3) The intended mathematical domain of this play situation is geometry, which includes two-dimensional (2D) and three-dimensional (3D) spaces. (5) Materials comprise of playing cards and wooden blocks.

The playing cards are scaled representations in four different levels of difficulty (Fig. 13.1). This means that the size of three-dimensional version of a geometrical shape does not match precisely the size of its two-dimensional version depicted on a card (Fig. 13.2).

(6) The instruction manual explains the rules: The cards are placed on the table face down. Players play five rounds in total by turns of each player in the game. In

Fig. 13.1 The wooden blocks and the game cards in different levels of "Building 02"

Fig. 13.2 The difference between sizes of the wooden block and the image on a playing card

Fig. 13.3 Recording position and the chosen card

each round, one player chooses a card from the deck and builds the figure depicted as a 2D representation. The aim of play is to build the figure shown on the chosen card. To check the compatibility between the built figure and the figure seen on the card, the other players give feedback. If it is correct, then the player is awarded the number of points shown on the card.

The Block Play of the Baden Family

Baden family is a German family who lives in a major German city. Conrad is the focus child who is aged 7 years and 1 month old. He has a younger sister, who is about 1 year old. His parents have higher education. His mother works as an architect, and his father is an engineer. While the parents are at work, a nanny looks after both children.

In the extract from the video recording to be discussed, Conrad is playing with his father. I first describe the beginning moves observed in this episode and then highlight and analyze key points of Conrad's turn at building this 3D object from its 2D image.

The extract comes from the first round of the play. The play begins with Conrad's turn. Conrad picks up the card from the deck. In other words, this is the first round of play, and this is the first card Conrad picked. The chosen card is shown in Fig. 13.3 and has the difficulty level 4. This means it is one of the hardest cards in the deck. The image on the card that Conrad chose technically comprises eight blocks. To be specific, eight blocks are set on top of each other, the frontal view of this structure is drawn as a picture, and then transitions between each block are made fluid. Thereby an image is produced which refers to a rectangle.

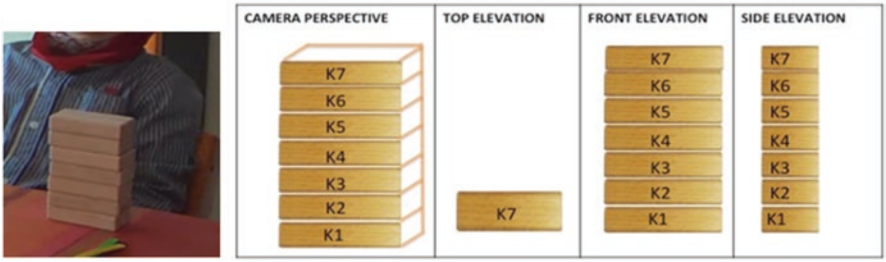
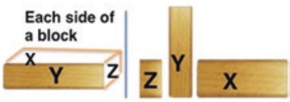


Fig. 13.4 The first block tower that Conrad built

Conrad looks at the chosen card very closely and scans the image on the card for about 15 s while moving his right point finger repeatedly from left to right along the length of the image on the card and keeping on moving his lips soundlessly. Then he says that he needs seven blocks and takes seven wooden blocks from the box. Thereupon he leaves the chosen card on the table and starts to build a block tower. He puts seven blocks (K1, K2, K3, K4, K5, K6, K7) on their X sides[2] horizontally on top of each other. As can be seen in Fig. 13.3, the card is located to Conrad's right, in front of the father, and the child is building the block tower in front of himself, to the left, but within reach, of the father. The block tower that Conrad built is shown in Fig. 13.4.

When the built block tower and the image on the chosen are compared, the front elevation of the built tower does match the image on the chosen card (see Figs. 13.3 and 13.4). Regarding the standard developmental phases of geometrical and educational issues (KMK, 2004; NCTM, 2000), it seems that Conrad is "parts of shapes identifier," "congruence determiner," and "3D shape composer" by building an identical block tower to the image on the chosen card (Clements & Sarama, 2014). By virtue of his visualization, he may be able to represent blocks at the detailed level of shapes to identify shapes in terms of their components. Moreover, he gives the impression of being very capable of coordinating both structures topologically and realizing that the built block tower and the image on the card ostensibly are the same frontal elevation. Furthermore he shows sufficient spatial abilities by composing shapes with anticipation, producing arches, corners, and crosses systematically. In this sense, he gives the impression of determining the congruence by comparing all attributes and all spatial relationships. Ultimately Conrad seems to achieve a vertical block tower identical to the image on the chosen card, although transitions between the various blocks in the image on the chosen card are fluid, and it is purposely complicated to predict how many blocks are needed and how they should be set to achieve an identical tower to the image on the card.

[2] Each side of wooden blocks 

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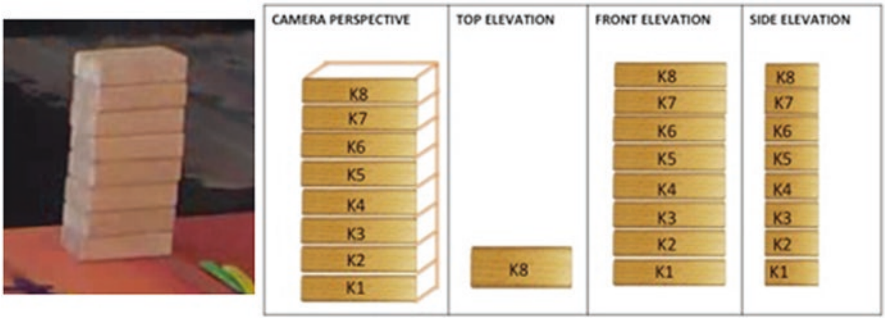


Fig. 13.5 The second block tower

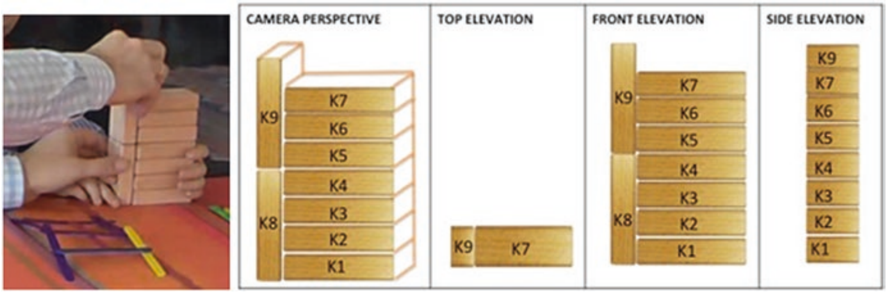


Fig. 13.6 The third block tower

Conrad then folds his arms, smiles, and looks at his father. After about 10 s, his father shakes his head from left to right and proceeds as described in the transcription[3] provided:

Transcript

1.	F:		*Takes a block (K8) from the box and sets it on its X side horizontally on K7* (see Fig. 13.5)
2.			Do you know, why?
3.	C:		No

[3] Rules of transcription

Column 1	Column 2	Column 3	Column 4
Serially numbered lines	Abbreviations for the names of the interacting people.(F, father; C, Conrad)	<: Indicates where people are talking or acting at the same time	Verbal (vocal) actions: regular fontNonverbal actions: italic font

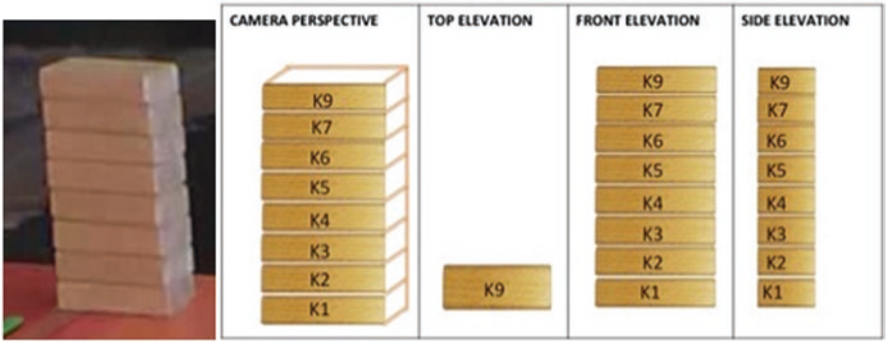


Fig. 13.7 The final block tower

4.	F:		*Takes K8 and sets it on its Z side vertically adjacent to the block tower*
5.		<	*Conrad had built and takes K9 from the box and puts it on top of K8* (see Fig. 13.6)
6.	C:	<	*Keeps both hands around and close to bottom of the tower he built* (see Fig. 13.6)
7.	F:		*Takes K9 away and places it on its X side horizontally on the chosen card on the table*
8.			Look (*shows the image on the card with his left index finger*)
9.	C:		*Looks at the card*
10.	F:		There are two quadrates. (*Shows the image on the card with K9*)
11.	C:		Yes
12.	F:		(*Holds K9 with his right hand.*) Thus another one comes upon it
13.		<	*Takes K8 away*
14.	C:		*Grasp K9 from his father's hand and puts it horizontally*
15.		<	*on its X side upon K7* (see Fig. 13.7)

Interaction Analysis

The father takes one more block (K8) from the box and sets it on its X side horizontally on top of K7 < 1>. Thereby he reconstructs the built block tower. But when the image on the card and the rebuilt tower are compared, the front elevation of the tower still matches the image on the chosen card. Perhaps the number of blocks "matters" for the father, and the block tower should exist just one more block. Conrad's father is an engineer, and, broadly speaking, in his job the mathematical exactness has crucial importance. Maybe, thus, he can predict how many blocks are exactly needed to achieve such an image as a block tower and tries to let Conrad experience such a block-building activity, in which built block tower matches exactly and successfully the image on the card. Maybe therefore he sets one more block upon K7. In this regard the father seems to realize one of *scaffolding* function, namely, either *demonstration* or *marking critical features*. He appears to *demonstrate* either how many blocks actually should be set more or how they set. Thereby he might also show how an ideal block tower can be built. In the sense of *demonstration*, the father seems to perform a perfection of building an ideal block tower and idealization of the act, which involves completion or even explication of the building action. By means of *marking critical features*, he seems to provide information either about Conrad's act, that he should have set one more block upon K7, or about the built block tower, that it was to comprise eight blocks. In both possibilities, he remarks on the critical feature of the built block tower in action that he sets "one more" block "upon" the built tower or "adds 8th block on 7th one" in the built block tower in order to achieve an ideal structure, which is completely identical to the image on the chosen card. Therewith he demonstrates this feature. By *marking critical features*, he accentuates geometrical and numerical features of the built block tower. In this regard his action seems to be made up of both geometrical and arithmetical approaches, which touch on folk psychology and folk pedagogy (Bruner, 1996).

Thereupon he asks whether Conrad knows why <2>. Most probably he asks Conrad whether he knows the reason why the father set one more block upon the block tower or why the block tower should exist eight blocks. Maybe he tries to keep Conrad partly in the field and to let Conrad think about the reason for setting one more block upon the block tower or why he reconstructs the block tower that Conrad built. In this sense the father gives the impression of realizing a scaffolding function called as *direction maintenance*. The father's reactions <1–2 > bring to the mind an aspect of family systems theory that fathers ask questions more than mothers, offer their children more information, use more elaborative labels, and come up with more imperative and short utterances in the interaction process with their children (see Mullis & Mullis, 1986). Furthermore, during play activities with their fathers, the fathers encourage their children to face up to unfamiliar occurrences and their own mistakes, hence justifying themselves and taking risks in new sets of circumstances (see Tamis-LeMonda, 2004). In this regard, Conrad's father might try either to demonstrate correct solution or to give definite instruction to Conrad about the way of building a right block tower and the reason for his action at the line <1>. Maybe he tries to let Conrad understand his point of view, and by posing such question <2>, he tries to encourage Conrad to think exactly about block tower and the image on the chosen card. Moreover he might try to encourage Conrad to face up to unfamiliar occurrence and his own mistake, hence justifying himself in this set of circumstance. In this regard he also seems to be an *activator*, who gives the impression of trying to activate Conrad's spatial knowledge by means of scaffolding. Moreover the family systems theory reinforces this idea that fathers evoke the "activation function" during play interactions with their children, which involves an exploratory system in which children experience novel issues in physical and social environments (see Tamis-LeMonda, 2004). By posing such question to Conrad, the father might try to offer Conrad such a situation that he can exchange his own ideas and so they can strive to reach an agreement with each other. Thereby the father seems to maintain the negotiation with Conrad using exploratory talk, and the interaction process gives the impression of rendering an expanded leeway of participation to Conrad.

Conrad replies him by saying no <3>. His reaction gives the sign of either not knowing why his father set one more block on the block tower or not understanding what his father really tries to do or show. So indeed, when the image on the card and the built tower are compared, the front elevation of the built tower still matches the image on the chosen card, and, thus, from Conrad's part, it seems to remain actually unclear why the father set one more block in the built tower.

The father takes K8 and K9 and sets them successively on their Z sides vertically adjacent to the block tower (see Fig. 13.6) <4–5>. Thereby he again rebuilds the block tower and somehow seems to highlight spatial relationships of 3D objects. When the image on the chosen card and the rebuilt tower are compared, the top-front-side elevations of the built tower do not match the image on the chosen card. Thus, it is unclear whether he tries to build a new block tower or to justify his argument or to show the reason for setting one more block in the tower that he did previously <1>. Wooden blocks are half unit blocks, sized 8 by 4 by 2 cm. The length of each unit is twice the width, which is twice the thickness. In this regard the length of each block is fourfold with the thickness. This means to reach the length of one block, one should set four blocks on top of each other. In this regard the father appears to show or emphasize the height or the length of the built block tower, which Conrad built. Maybe therefore he uses two blocks (K8, K9) in order to show or check in detail the extent of the block tower. In this sense, he might try to find a way to justify his argument at the line <1 > by setting both blocks on top of each other adjacent to the block tower. Regarding family systems theories, he seems to offer Conrad more information, vary the instruction given to his child and thus use more elaborative labels, give more exact and elaborative descriptions, and try to show direction relations in course of interacting with his child (Laakso, 1995; Mullis & Mullis, 1986).

At the same time, Conrad is keeping his both hands around the built block tower (see Fig. 13.6) <6>. Thereby he gives the impression of struggling to avoid hazard of the tower falling. Furthermore his reaction reinforces the idea that he is very capable of coordinating the 3D-structure topologically that he can predict the vertical built tower can fall down. In this regard the negotiation process between Conrad and his father seem collaborative that they build a block tower together collectively. Therefore, from a participatory point of view, they ascribe the role of collaborative game partner to each other. In this regard Conrad and his father seem to engage in the interaction process critically but collectively and constructively.

Thereafter the father takes K9 away and sets it on its X side horizontally on the chosen card which lays on the table and shows the image on the card with his left index finger while saying "look" <7–8 > (see Fig. 13.8). Most probably he tries to justify his argument either at the line <1 > or at the lines <4–5 > by showing the image on the chosen card. Bearing in mind the idea of family systems theory that fathers vary the instructions given to their children more flexibly and tend to make more requests for information, give more exact and elaborative descriptions in play situations, and show direction relations in course of interacting with their children (Laakso, 1995; Mullis & Mullis, 1986), he seems to vary the instruction about the built tower and to give his descriptions more precisely. Maybe thus his utterance is imperative and directive that Conrad should look at the card on the table so that he can “see” or “get” his point of view. Furthermore, by saying “look” to Conrad, he gives the impression of calling Conrad’s attention to the image on the card. Regarding family systems theory, it does not seem to be surprising that he again comes up with an imperative and short utterance in the interaction process with his child as in the line <2 > (see Mullis & Mullis, 1986). By saying “look,” by means of *scaffolding*, he seems to emphasize to Conrad that he should focus on the image on the chosen card and try keeping Conrad partly in the field. In this sense he gives the impression of realizing a type of scaffolding function, namely, *direction maintenance*. He might try to ensure that Conrad can exactly observe and explore the reason for setting one more block upon the block tower, which Conrad built. Thereby the father uses instant *directivity*, and the block-building activity of Conrad can be directed toward achieving particular outcomes that contribute to completion of the building of the matching tower. Hence looking from a participatory perspective, the father seems to be an *expert*, while he is reserving the role of *novice* for Conrad.

While Conrad is looking at the image on the card <9>, the father says that “there are two quadrates” by still showing the image on the card with his left index finger and keeping the block K9 on the card <10> (see Fig. 13.8). By looking at the image on the card, Conrad seems either to pay attention to his father’s argument or to see the reason why one more block should come in the built tower. Thereby he gives the impression that he orients his father’s utterances and actions by his reactions in the situation of negotiation of meaning.

By saying “there are two quadrates” <10> while still showing the image on the card, the father most likely emphasizes that the image on the card comprises of two quadrates. The term “quadrate” refers to the term “square,” and two squares in equal measure put together make a new shape, a rectangle. So indeed, when the image on the chosen card is reviewed carefully, it is obvious that it is a rectangle and comprises two squares in equal measure (see Fig. 13.9). Considering the technical information about the structure of the chosen card, one should also emphasize that in each square fit exactly four blocks (see Fig. 13.9).

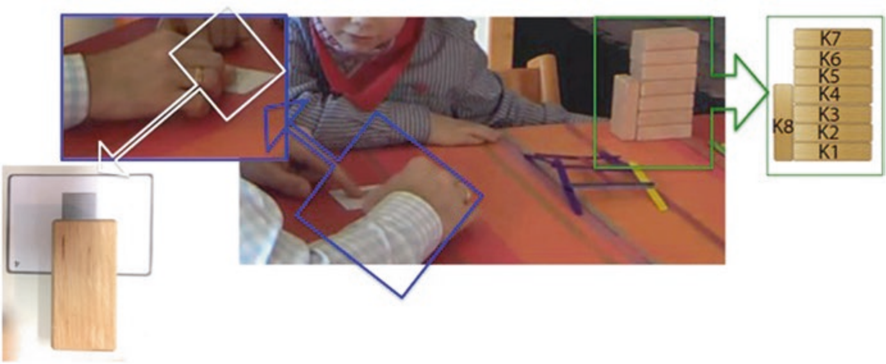
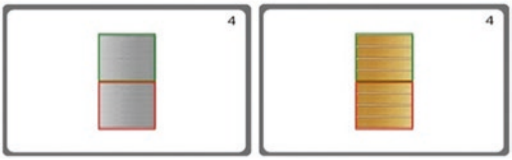


Fig. 13.8 The father shows the card with the help of the block K9

Fig. 13.9 The review of the image on the chosen card



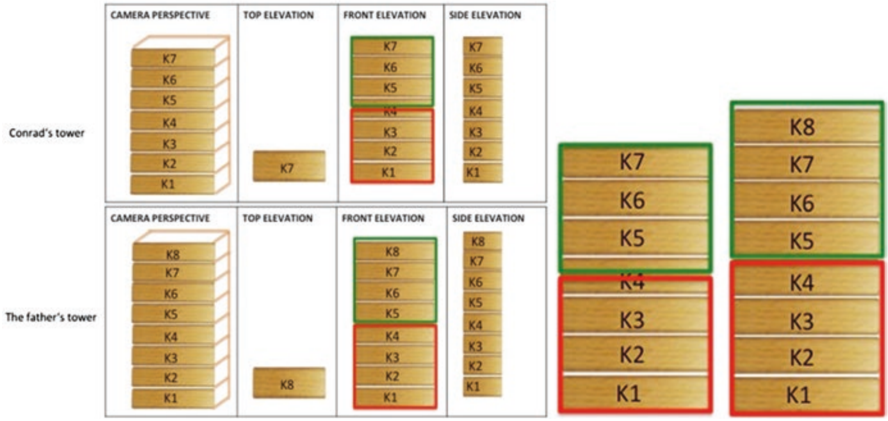


Fig. 13.10 Comparison from the frontal elevation between the first and the second built towers

Regarding this, when the first tower that Conrad built and the second tower that the father built (Fig. 13.10) are compared from the frontal elevation, their differences (see Fig. 13.10) can be scrutinized clearly that they both represent rectangles, but the rectangle of the father's tower can be divided into two squares in equal measure (outlined with red and green) easily, whereas the rectangle of Conrad's tower can be separated into another two rectangles in equal measure (outlined with red and green) only.

Furthermore, considering the idea that the image on the chosen card comprises two squares, one should also take into account the idea that 3D structures can only be divided into groups by computing the amount of the blocks. In this regard father's 3D tower can be divided into two equal groups, of which front elevations refer to squares and consist of four blocks (outlined with red and green), whereas Conrad's cannot (see Fig. 13.11).

As mentioned before <1>, Conrad's father is an engineer and presumably attaches great importance to the mathematical exactness in the game. The father's reaction <10> reinforces this idea and the interpretation in line <1 > that the number of blocks in the built tower "matters" for the father in order to let build a tower, which matches exactly and successfully the image on the card (see <1>). In respect of previous arguments of Conrad's father at lines <1, 4–5, 7–8>, he might still try to ensure the perfection of the built tower, of which frontal elevation is completely identical to the image on the card. Considering Figs. 13.6, 13.9, and 13.11, by setting K8 and K9 next to the built block tower (see <4–5>), the father might try to

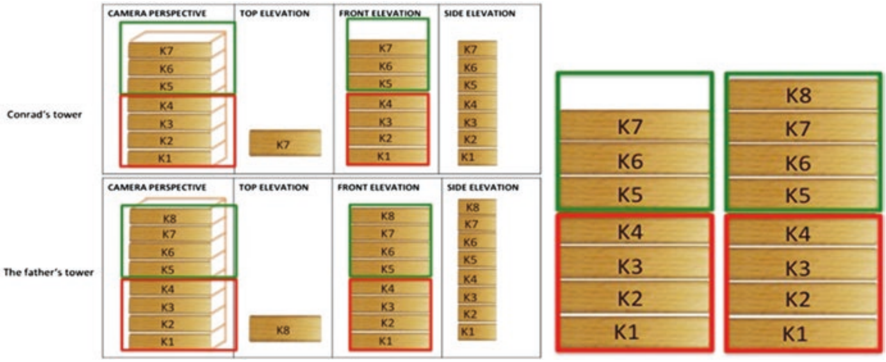


Fig. 13.11 Division of the first and the second built towers into two groups and their comparison from the frontal elevation

show that two quadrates (outlined with red and green) have the same length as these wooden blocks, and the length of each block is fourfold with the thickness (see Fig. 13.11). According to him, by setting four blocks on top of each other, one can reach the length of one block. Maybe therefore he tries to describe the distinctive nature of the image by diving in pieces. By setting eight blocks on top of each other <1>, one can reach two quadrates and thereby achieve the length of two blocks on top of each other that exactly matches the length of tower in the image on the chosen card. In this regard the father might try to suggest the ideal block tower should be built in the way of reaching two quadrates.

In any event the father seems to come up with the geometrical and numerical arguments that the image on the card consists of "two quadrates." Thereby he might emphasize that seven blocks cannot be equally divided into two quadrates (see Figs. 13.10 and 13.11). Maybe therefore he tries to call Conrad's attention to the point that the built tower should be made with eight blocks in order to get two quadrates perfectly. In this sense he gives the impression of coming up with the geometrical and numerical arguments together that the tower should comprise of two quadrates. "Two" represents the amount of the quadrates, and "two quadrates" represents one rectangle, namely, the image on the chosen card. From a developmental perspective, his reaction might activate both Conrad's geometrical and numerical skills in that he can consider his father's both geometrical and numerical arguments and produce 3D block tower properly with the image on the chosen card. Furthermore the geometrical argument of the father seems to enable Conrad to explore *composing and decomposing spatial fields unit by unit in both 2D and 3D spaces* to investigate and predict the results of combining, subdividing, and changing shapes, to understand the variety of ways in which geometric shapes and objects can be measured, and to explore and apply the concepts of congruence.

In addition, the father seems to realize a *scaffolding* function *marking critical features*. He appears to provide different information about the extent of the image on the chosen card. Thereby he gives the impression of accentuating a certain feature of the image on the chosen card, which is relevant. Furthermore his marking provides Conrad information about the way of building an exact block tower, which is totally identical to the image on a card. In this regard the father seems to be an *expert*, while he is ascribing the role of *novice* to Conrad**.** Furthermore, regarding some aspects of family system theory (see Mullis & Mullis, 1986; Tamis-LeMonda, 2004), one can say that Conrad's father gives exact and elaborative descriptions of the card and the block tower and uses a greater proportion of verbalizations describing form, shape, and direction relations in the course of interacting with his child. Moreover, he seems to encourage his child to face up to unfamiliar occurrence and his own mistake and hence enable Conrad in justifying himself.

Conrad gives an affirmative response <11>. Most probably he gets either the point of his father's view, or what he means, or what actually should be done to accomplish an ideal tower. Maybe the father's elaborated elucidation enabled Conrad to judge and justify his idea about the way of building the block tower. Maybe therefore he affirms his father and says "yes" in order to emphasize that he agreed with the necessity of setting 8th block in the block tower. Conrad's reaction gives the impression of accepting his father's argument and ascribes the role of *expert* to his father, while he takes the role of *novice*. Concordantly he seems to assign his father the role of *activator* who activates Conrad's knowledge about geometrical and spatial issues that he can judge the properties of the ideal block tower and the rightness of his father's assertion and make a decision – that the father is right. In this regard Conrad gives the impression of activating his spatial abilities through which he can recognize and operate geometric shapes and structures in the environment and specify their location (see KMK, 2004; NCTM, 2000).

The father holds K9 with his right hand and states "thus another one comes upon it" <12>. His utterance looks like a description of his action in line <1> that "another one block" should be physically added to the top surface of the block tower; in other words "another one block" should be set "onto" the block tower. His reaction reinforces the idea in line <1> that the number of blocks in the built tower "matters" for the father in order to achieve exact and successful match of the built tower and the image (see <1>). Furthermore this reaction of the father reinforces the idea at line <11> that the image on the card comprises of two quadrates, but seven blocks cannot be equally divided into two quadrates, and thus the built block tower should be made up with eight blocks in order to get these two quadrates and build a block tower that matches the image perfectly (see Figs. 13.9, 13.10, and 13.11). In this regard the utterance of "another one" might be also interpreted as a kind of elucidation of "one more block." In any event he obviously comes up with the numerical argument that "another one" block should be set upon the built block tower. By emphasizing that one block should come "upon" it, the father uses the vertical directionality term "upon" as "onto." In this way he does not only call attention to the point that another one block should come on top of the block tower but also verbalizes and namely expresses this action and, respectively, his action in line <1> vocally. In this regard the father seems to again highlight spatial relationships of 3D objects. Thus he gives the impression of coming up with the geometrical argument while continuing to provide numerical information. Therefore he seems to maintain fulfilling the scaffolding function *marking critical features* so that he interprets spatial relationships of the built block tower and the image on the chosen card. In this regard he seems to act as an *expert*, while he is reserving the role of *novice* to Conrad.

Conrad grasps K9 from his father's hand and puts it horizontally on its X side upon K7 (see Fig. 13.7), while the father is taking away K8 from the side of the block tower (see Fig. 13.6) <13–15>. By taking K8 away <13>, the father might try to help and leave Conrad a kind of block tower as the first block tower that Conrad already built with seven blocks (see Fig. 13.4). Thereby he seems to provide Conrad with an opportunity that he can go on his building action by setting "another one (block) upon" the first block tower built by Conrad (see line <12>). In this regard, considering family systems theories (Laakso, 1995; Tamis-LeMonda, 2004), the father seems to give more responsibility to Conrad in completing his given tasks – here it is building a block tower – while letting Conrad set the 8th block on to the block tower <15>. From another point of view, in the light of family systems theory, the father might try to complete the game in the shortest amount of time and might have not wish to waste time still with keeping on negotiating about the block tower. So indeed he does not enter into any further discussion about the built block tower and just takes the block K8 away.

At the same time, Conrad takes K9 from his father's hand and sets it on the top of built corpus <14–15>. Thereby Conrad gives the impression of understanding and performing the point of view of his father and what actually should be done to accomplish an ideal tower. Maybe therefore he sets "another one" block "onto" the built block tower (see line <12>). In this sense he appears to come to an agreement on the necessity of setting 8th block in the block tower. Thereby a *working consensus* between Conrad and his father seems to emerge about setting "another one" on the 7th block in order to reach ideal block tower (see lines <1, 12>), which perfectly matches the image on the card. Furthermore Conrad's reaction shows that the father's activities and responses work on Conrad that he executes the building activity in the same way as his father. By means of *scaffolding*, the father's *demonstration* in line <1 > seems to turn out well that Conrad got the idealized form of building an ideal block tower and imitated it back in a more appropriate form. Thus his reaction can be interpreted as a completion of a solution already partially executed. Additionally, the father's reactions (Tamis-LeMonda, 2004) evoke the "activation function" for Conrad so that he involves himself in such novel experiences in the block-building activity, through which an exploratory negotiation process can emerge. In that respect, Conrad's reaction gives the impression of accepting his father's argument and ascribes the role of *expert* to his father, while he takes the role of *novice*. Furthermore, from the developmental point of view, he acts as "units of shape composer" (Clements & Sarama, 2014, p. 182) that he seems to able to make adult-like structures with blocks from pictured models *unit by unit* perfectly and systematically.

Considering lines <1–15>, the negotiation of meaning between Conrad and his father is a collective argumentation process in that they engage collaboratively and communicatively in the block-building activity. They offer justifications and alternative hypotheses, while they are overcoming challenges. They perform collective argumentation in that they offer hypotheses, which can be made publicly accountable, and try to reach an agreement with each other. Conrad first offers his justification and hypothesis about building the block tower and then builds the first block tower. After that the father comes up with alternative hypotheses about the way of building an ideal block tower. Subsequently they reach an agreement with each other so that the father is taking one block away while Conrad is setting 8th block on the 7th block and building the last version of the block tower (see Fig. 13.7), and they achieve a perfectly built block tower, which is completely identical to the image on the chosen card. In this way Conrad succeeds in his turn. Ultimately Conrad's play turn in the first round ends.

The Relationship Between Scaffolding and NMT-Family

In the chosen sequence, from an allocative perspective, the father is the official game partner of Conrad, but he – situationally – sets about the scaffolding process. They realize a *collective argumentation* process in which the father uses and adopts *intuitively and informally* some scaffolding functions in the negotiation process with Conrad. Through his father's scaffolding and his referential verbal and nonverbal acts, Conrad explores and performs whole spatial consequences in the block-building activity. The negotiation process between Conrad and his father is accomplished in an exploratory way in that they are collaborating, reaching agreement with each other, and understanding each other's points of view. In this sense, the learning process for Conrad can emerge through his participation, in which he experiences to build an ideal and perfect matching block tower. Therefore his father takes the role of *activator*, who evokes Conrad's "activation function" so that Conrad exploratory experiences novel issues and the father's perfection and idealization about building an ideal block tower that enable Conrad a learning situation. In this sense the father takes on the role of an *expert*, while he is ascribing the role *novice* for Conrad. Within this context then, I argue, there can emerge a developmental niche for Conrad. According to the whole analysis, the three aspects of an interactional developmental niche in Conrad's familial context can be structured as follows:

The Aspect of Allocation

Content In the chosen scene, Conrad and his father are confronted with a spatial play situation. For more see the section "A Block Play: Building 02" in this paper.

Cooperation In the play situation Conrad and his father are game partners. Conrad's father is the adult person and his *official* conversation partner, who allocates the right to take the next play turn.

Pedagogy and Education Block building provides a view of children's initial abilities to compose 3D objects. In the chosen game, four goals are pursued: spatial structuring, operating on shapes and figures, static balancing between blocks, and identifying the faces of 3D shapes with 2D shapes. These competencies reflect an initial development of thinking at the level of relating parts and wholes.

The Aspect of Situation

Content The chosen play situation enables Conrad and his father to negotiate interactively about building a block tower, which perfectly and ideally matches the image on the card. A *dyadic* interaction process between Conrad and his father emerges as the father comes up with geometrical and numerical approaches to the building block tower. During block-building activity, Conrad and his father put forward their justifications, alternative hypotheses, and agreements. Moreover they share relevant information, strive to reach an agreement, and dedicate themselves to pursuit of the best solution. Thus they engage in the interaction process critically but constructively and collectively. In this respect the negotiation process between father and son emerges as an exploratory one. The father's perfection and his *geometrical* and *numerical arguments* enable Conrad to explore and build an ideal tower. Thus Conrad is exposed to examine spatial relations in great detail and experience of composing and decomposing spatial structures perfectly. In the course of the negotiation process, a working consensus occurs between Conrad and his father about the need of setting one more block upon the block tower.

Cooperation In this dyadic interaction process, Conrad and his father are collaborative game partners. They perform block-building activities *collaboratively* and mostly negotiate in an exploratory way so that Conrad actively experiences how to compose and decompose 2D and 3D shapes *unit by unit* and comprehensively. Thus the negotiation process generates for Conrad such a leeway of participation that he acts as *activated* to complete and achieve the ideal built block tower that matches the image on the card perfectly. In this regard the father acts an *activator*, who evokes Conrad's "activation function" so that he exploratory experiences novel issues and the father's perfection and idealization.

Pedagogy and Education In the chosen play situation, the father strikes a balance between playing with Conrad and at the same time realizing a scaffolding process. Regarding the six scaffolding functions, he exposes Conrad to three of them, namely, "demonstration, direction maintenance, and marking critical features," whereas he does not draw on the other scaffolding functions called as "recruitment, frustration control, reduction in degree of freedom" (see Wood et al., 1976):

- Demonstration: The father models the idealized form of building a perfect matching block tower <1>. This means that he performs an idealization of the act and completes the Conrad's *solution* <1> in order to reach perfect matching tower. Thus, the father provides Conrad with a position in which they become able to "imitate" it back in a more appropriate form. So indeed the father's *demonstration* in line <1 > works on well that Conrad got the idealized form of building an ideal block tower and imitated it back in a more appropriate form in lines <14–15 >.
- Direction maintenance: The father tries to ensure that Conrad can exactly think about, observe, and explore the reason for setting one more block upon the first built block tower <2, 8>. Thereby the block-building activity of Conrad can be directed toward achieving particular outcomes that contribute to completion of building the perfect matching tower. Hence the father uses instant *directivity* and tries to keep Conrad in pursuit of a particular objective so that Conrad can be kept in the field, can directly maintain the building activity, and hereby become involved only in building an ideal block tower, which matches the image on the chosen card perfectly.
- Marking critical features: The father obviously emphasizes the geometrical and numerical features and different aspects of the building activity that are important or relevant for its completion <1–2, 4–5, 10, 12>. By approaching block-building activity from geometrical and numerical perspectives, the father accentuates certain features of the building of block tower and the image on the chosen card. In this regard, his markings let Conrad review spatial relationships of the built block tower and the image on the chosen card in great detail. Thereby they also provide Conrad information about the way of building an exact block tower, which is totally identical to the image on a card.

In this sense the father fulfills three scaffolding functions. Bearing in mind the idea of family systems theory (Laakso, 1995; Mullis & Mullis, 1986; Tamis-LeMonda, 2004), Conrad's father varies instructions about the built block tower, offers Conrad more information, shows direction relations between block tower and the image, and gives more exact and elaborative descriptions of the card and the block tower <1–2, 4–5, 10, 7–8, 12>. He also ensures the mathematical exactness in the block-building activity too. Moreover, he encourages his child to face up to unfamiliar occurrence and his own mistake and hence enables Conrad in justifying himself <8>. Additionally, the father gives more responsibility to Conrad in completing block-building activity and thus encourages his son to face up to unfamiliar occurrences and their own mistakes, hence justifying themselves and taking risks in new sets of circumstances (see Tamis-LeMonda, 2004).

The Aspect of Child's Contribution

Content Conrad builds a vertical block tower identical to the image on the chosen card, although transitions between the various blocks in the image on the chosen card are fluid and it is purposely complicated to predict how many blocks are needed and how they should be set to achieve an identical tower to the image on the card. In this regard, Conrad acts as "parts of shapes identifier," "congruence determiner," and "3D shape composer" by building and matching block tower to the image on the chosen card (Clements & Sarama, 2014, pp. 164–175).

Cooperation Conrad collaborates with his father in the course of whole block-building activities in the play situation. In both turns Conrad apparently cares for his father's elaborative descriptions, demonstrations, verbal stimulations, and instructions. By accepting the geometrical and numerical arguments of his father, imitating the idealized form of building an ideal block tower shown by his father <1>, Conrad takes the role of *novice* while ascribing the roles of *expert* to the father. Furthermore he ensures himself such a leeway in which he participates in the play situation actively so that he effectively explores and experiences spatial features of building ideal block tower.

Pedagogy and Education Conrad has learning opportunities for building the perfect matching tower by exploring different spatial features and relations in great detail. Through his father's perfection and activation in the negotiation process, he can learn to compare, compose, and decompose 2D and 3D structures unit by unit comprehensively. The *collective* argumentation process with his father enables him to reconstruct *geometrical and numerical meanings*. Thereby Conrad accomplishes the perfect matching block tower. He represents 3D transformations, regulates their relations, links them with each other, and comes to conclusion in a short amount of time. He explores and examines directly the stability of the building towers and builds a sturdy tower. Through the father's usage of three scaffolding functions (demonstration, marking critical features, and direction maintenance), the father directs and maintains elaborations whereby Conrad's development is facilitated. Furthermore, by means of family systems theory (Tamis-LeMonda, 2004), Conrad is encouraged to face up to unfamiliar occurrence and to judge and justify his idea about the way of building the block tower. Thereby he gets the idealized form of building an ideal block tower and imitated it back in a more appropriate form so that he realizes a completion of a solution already partially executed. In this regard he acts as "units of units shape composer" (see Clements & Sarama, 2014) that he seems to become able to make adult-like structures with blocks from pictured models *unit by unit* perfectly and systematically, whereas at the beginning of his turn as a *3D shape composer*, he didn't. On a metacognitive level (Bruner, 1996), by providing explicit directions on how to build the ideal and perfect block tower, the father emphasizes crucial actions, guides at key points, and indicates alternatives as he leads Conrad to "internalisation of schemes, concepts and reasoning that are the

Table 13.2 The NMT-Family Baden

NMT-Family	Component: *content*	Component: *cooperation*	Component: *pedagogy and education*
Aspect: *allocation*	Geometry, spatial structuring, operating on shapes and figures	Playing with father	Development of spatial skills and transformational abilities in spatial thinking and learning
Aspect: *situation*	Negotiation between father and Conrad, geometrical and numerical arguments of Conrad's father; working consensus	Leeway of participation for Conrad; Activator (Father)-activated (Conrad)	The father's *idealization and perfection* of building block tower perfectly *Three Scaffolding functions* by father familial systems
Aspect: *child's contribution*	Operating on shapes and figures; "parts of shapes identifier"; "congruence determiner"; "3D shape composer"	Expert (Father)-novice (Conrad)	Building the perfect matching tower composing and decomposing 2D and 3D structures *unit by unit*; "units of units shape composer"

subject of intra-psychic regulations" (Boekaerts, 1997; Nader-Grosbois et al., 2008). Moreover through reaching, grasping, balancing, stacking, and moving blocks, Conrad gets an opportunity to learn hand-eye coordination. The negotiation process with his father thus inherently enables Conrad's temporal and representational cognitive developments (Bibok et al., 2009).

Regarding all these facts, interactional niche in the development of Conrad's geometrical thinking and learning occurs. Due to these three components, the interactional developmental niche in the Baden family is structured as follows (Table 13.2).

Conclusion

Mathematical play situations conducted in the familial context seem to be a possible contribution to the child's mathematical development. Conrad experiences mathematical learning opportunities during block play with his father. By profession as an engineer, the father has a higher education level. These facts seem to affect the quality of arguments about block-building activities, while Conrad and his father negotiate about mathematical meanings between each other. The father's perfections, directiveness, and usage of three scaffolding functions enable Conrad to become activated while acting as a novice. Furthermore the realizations of family functions offer Conrad's father such situation that he makes the uses of mathematics apparent so that he can evoke Conrad's activation functions during play. By virtue of the father's perfections of building an ideal and perfect block tower and realizations of some scaffolding functions, Conrad explores and reviews different spatial features in great detail. Thereby the father provides to Conrad a learning situation from spatial and numerical perspectives in terms of his folk psychology and pedagogy. Thus he has direct influence on Conrad's geometrical and numerical developments through which Conrad learns complex mathematical meanings affectively. In this manner, his father renders mindfulness of spatial features directly for Conrad. Therefore I argue that an interactional niche in the mathematics learning in the familial context emerges for Conrad.

Regarding the chosen example, it can be concluded that the usage of some scaffolding functions and realization of some family systems functions offer a child different opportunities to be exposed to different mathematical features and relations through which a mathematics learning situation can occur. It seems that the flux of this interaction process between child and family member underscores the developmental importance of coordination and dynamic match, i.e., reciprocity, mutuality, and synchrony of family member's and child's behaviors. Maybe therefore not all scaffolding functions have to be fulfilled while realizing some family system functions in order to achieve a learning situation for a child. The factors of the roles taken can change dynamically and mutually so that individuals can facilitate different types of learning and the way of negotiating can take place in different characters. But one factor stays stable that such mathematical play situations lead them to achieve different kinds of scaffolding processes in which one do not have to fulfill all scaffolding functions in order to offer a child a learning situation.

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Chapter 14
Instruction and Construction of Mathematics at Home: An Exploratory Study

Ann Anderson and Jim Anderson

Abstract In this paper, we draw on data from a longitudinal study of mathematics engagement during adult-child joint activity at home. Drawing from across case and within case analysis of 44 videotaped activities, gathered in 6 middle class homes over 2 years, this paper provides evidence of either Instruction *or* Construction being predominate in each activity. Examination of one Instruction (i.e., Play-Doh pizza) and one Construction activity (i.e., family photos) suggests each activity shared common characteristics, but there were also distinct characteristics pertaining to concepts, control, and interactions. We discuss two considerations (i.e., blending across experiences and valuing big ideas) to stimulate further reflection and research.

Keywords Parent-child interactions · Preschool children's mathematics · Early childhood mathematics · Mathematical construction and instruction · Mathematics at home

As we debate the ways in which, and the extent to which, we *should* educate children in mathematics prior to primary school, young children's experiences with mathematics in their home environments can and should inform our discussions. On the one hand, for many young children, experiences with parents and extended family members run concurrent to, and interact with, their experiences in other early childhood settings from birth to 5 years (e.g., Bronfenbrenner, 1979, 2005). For other children who do not have access to, or whose families elect not to involve them in, more formal early childhood programs, these at-home experiences

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constitute the prior experiences and knowledge they bring when they enter primary classrooms (at 4–6 years). No matter which pathway families follow, the ways in which parents and children engage with mathematics at home can and should provide valuable insights for early childhood educators and early years' teachers when children enter school. For as Tiedemann (2013) argues, "Everyone who wants to teach children mathematics has to know about their earlier ways of learning mathematics ... "(p. 2218).

Theoretical Framework

The claim that "parents are the child's first teacher" is ubiquitous in the popular press and much of the early childhood and parenting literature in North America and other jurisdictions. The image of *teaching* that this phrase invokes seems more akin to that of an early childhood educator than a school teacher. That is, parents are seen as providing opportunities, resources, and support for their children's learning in an exploratory- and inquiry- or curiosity-driven manner and not the more formal or direct instruction that is often associated with school. Thus, we argue, both parents and early childhood educators appear to be positioned as caregivers, who teach in an almost surreptitious manner. And where mathematics is concerned, just as "… many early childhood teachers are reluctant to embrace an active role in the teaching of mathematical concepts…" (DeVries, Thomas, & Warren, 2010, p. 719), many parents tend to believe they do not "do mathematics" with their child prior to school. Considering the similarities between early childhood educators and parents then, it seems reasonable that child-parent interactions in the home and child-educator interactions in the early childhood classroom may also share commonalities, such that insights gained from researching one would likely benefit the other.

Research into Adult Mediation of Young Children's Mathematics

What then do we already know about the ways in which parents mediate (i.e., teach) mathematics to children in the early years? Walkerdine's (1988) foundational study of audiotaped conversations between 36 mothers and preschoolers (mainly daughters) in the home introduced the idea that many at-home tasks are mapped onto a dichotomy of *instrumental* and *pedagogical* activities:

> Instrumental referred to tasks in which the main focus and goal was a practical accomplishment and in which numbers were an incidental feature …In the pedagogic tasks … numbers were the explicit focus …predominantly the teaching and practice of counting. … (Walkerdine, 1988, p. 81).

In 2003, Aubrey, Bottle, and Godfrey, drawing on data from a 3-year study of 9 children's early mathematics in homes across a range of social classes, reported on two cases chosen to represent the maximum variation in parental mediation styles. In one case, the daughter and mother experienced mathematics through everyday routines and play activities, while the other mother and daughter engaged in mathematics largely through games and puzzles with an explicit pedagogical focus. Thus, Aubrey et al.'s (2003) findings are in accord with Walkerdine's typification. In our own research (Anderson & Anderson, 2014) with 6 middle class, mainly Euro-Canadian families, we expanded Walkerdine's dichotomy to a 5-point scale in which *math as a major portion*, *math as an equal focus*, and *math as a minor portion* of a task were added to the polar constructs of *math as the goal* (pedagogical) and *math as incidental* (instrumental). Our analysis suggested that a continuum of parental styles, and not a dichotomy, more aptly captured parent-child engagement in mathematics at home. In addition to these studies of "naturally" occurring events, researchers have also investigated parent-child interactions observed during researcher-designed tasks, approximating at-home games and activities. For instance, Tiedemann and Brandt (2010) reported on two cases where children and adults played a card game, which the researchers provided, to ascertain distinctions between structured learning and game playing, which they characterized elsewhere (Brandt & Tiedemann, 2010) as different forms of guided participation between the poles of enculturation and acculturation. In 2013, based on a study of 10 German mother-child dyads, who read books and played games that the researcher provided during home visits, Tiedemann illustrated three support roles associated with different families' MASS (Mathematics Acquisition Support System), namely, *participation* where the focus is on playing the game smoothly, *improvement* where the interactions are about improving the child's mathematics and not just playing the game, and, *exploration* where the child and/or parent explore math as they speculate or imagine possibilities beyond the game. In contrast, Acar Bayraktar (2013) drew on Krummheuer's (2012) concept of "interactional niche in the development of mathematical thinking (NMT)" to analyze familial math learning occasions according to a 2 (allocation and situation) X 3 (content, cooperation and pedagogy) matrix. Using a short excerpt from one 6-year-old child's interactions with her father while playing the "assigned" tower building game with her and her mother who was off camera, Acar Bayraktar demonstrated how the developmental niche emerged, with the father restricting the child's leeway while the mother encouraged it. Similarly, Solmaz (2015) examined one 5-year-old Turkish immigrant child's interactions with both parents, while they played a block tower game and reaffirmed the presence of NMT-Family, showing how both parents offered opportunities for their son to actively participate in mathematics. Thus, although the researchers employed different frames to analyze family strategies and orientations, these studies demonstrate that parents do/can engage their young children in activities that support the development of early mathematical knowledge and thinking. And, the diversity in the ways in which parents/families mediate that knowledge (e.g., Vygotsky, 1968) seems to point to a range of intentionality to "teach" math.

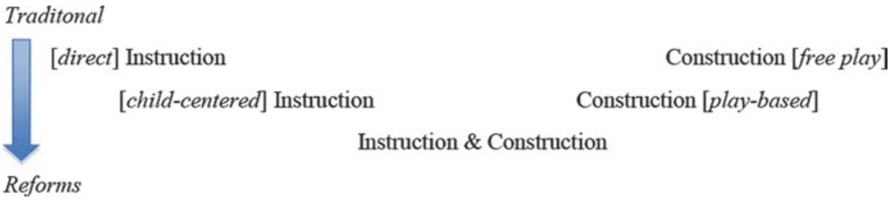

Fig. 14.1 Instruction versus Construction within early childhood mathematics education debates

Of note here is the parallel between the research findings on parent mediation and the ongoing debates in Canada (and elsewhere) regarding early childhood mathematics education (ECME) within preschool (and Kindergarten) classrooms. For as Sherman-LeVos (2010, p.2) stated:

> Views differ with respect to what ECME should consist of and how it should be infused into preschoolers' lives, with a continuum that represents the amount of intervention or instruction proposed. On one end of the continuum is a very direct, didactic, and teacher centered approach to ECME, while the other end of the spectrum represents a play based, child centered, non didactic approach to ECME.

Similarly, in Germany and other European countries, "a mathematics education perspective on early mathematics learning in the strain between Instruction and Construction" (Benz et al., 2014, p. 3) has guided many researchers' investigations into adult mediation of children's mathematics in early years settings (e.g., Kortenkamp et al., 2014; Meaney, Helenius, Johansson, Lange, & Wernberg, 2016). For instance, Meaney and colleagues have modified Walkerdine's categorizations and Bishop's mathematical activities over several iterations giving rise to their framework of *didaktic space* for analyzing preschool teachers' and children's interactions, related to mathematical learning. (e.g., Helenius et al., 2016). On the other hand, Krummheur and colleagues developed "the interactional niche in the development of mathematics thinking (NMT)" through their ongoing longitudinal research of learning in everyday situations of mathematics classes (Krummheuer, 2014). Not surprising, much of this research "oscillates between concerns for children's construction as active human agents and the need to be instructed in socially valued mathematics knowledge" (Helenius, Johansson, Lange, Meaney, & Wernberg, 2016, p. 15) and points to the multifaceted experiences of young children's mathematical experiences prior to school, rather than prescriptions for practice.

As indicated, recent calls for increased attention to children's mathematics learning in early years settings appear to be cast as a choice between free play (early childhood care) versus formal instruction (school-like mathematics) (see Fig. 14.1). When positioned in this way (i.e., at the poles), the term *instruction* connotes direct instruction, whereby the teacher transmits mathematical knowledge to the student; on the other hand, *construction* connotes a child, alone or with minimal adult input or guidance, constructing knowledge from, and through interactions with, the environment (e.g., Piaget, 1958). However, reforms in mathematics education and early childhood suggest a shifting away from such polarity in more recent years, wherein play-based pedagogies of early childhood education and child-centered approaches to mathematics learning in primary classrooms (Clements & Sarama, 2009; Copley, 2004) are fairly compatible. In such a climate, *instruction* connotes teacher as guide and *construction* connotes a child building knowledge with peers and significant others in socially and culturally mediated ways. Finally, as Presmeg's (2014) "dance of instruction *with* construction" (p. 2) suggests, perceptions of instruction and construction continue to shift toward a more blended metaphor, whereby they are intimately linked (e.g., two sides of a coin). Indeed, Presmeg (2014) argued, "effective instruction can facilitate students' making constructions that lie within the canons of mathematically accepted knowledge, and yet there is room for creativity and enjoyment." (p. 3). When positioned in this blended way, *instruction* connotes teacher as facilitator working in concert with the autonomous learner and *construction* connotes a child who is agentive and generative in her/his sense making. Interestingly, from one perspective, we might see these shifts as chronological, with the traditional "at the poles" dichotomy being in the past and the reform-based, interconnected interpretation being the present. However, from another perspective, we might see the shifts as a trajectory, along which pedagogy (e.g., individual teacher's practice) develops. As such, moving toward a blended metaphor within (school) settings, in which instruction tends to be privileged, at the very least requires increased attention to children's sense making (construction). For early years settings, in which construction has been privileged, moving toward a blending of instruction with construction suggests teachers' appropriate and sensitive interventions (instruction) during child-centered activity should receive increased attention. In the home setting, where pedagogy is somewhat figurative, the implications and value, of learning *between* the poles of instruction and construction, seem less clear. How then might exploring the tension between instruction and construction as conceived within classrooms inform or be informed by research in the home?

Our Research into Parent Mediation of Preschoolers' Mathematics

Our work with young children and their families is informed by sociocultural theory (Vygotsky, 1968; Wertsch, 1998) and the perspective that learning is initially social as significant others guide children in learning the cognitive tools such as language and mathematics that are important in their cultural/social context. That is, children first use the tools *inter-psychologically* supported by parents and significant others; then, as support is withdrawn gradually, children (or novices) use the tools *intra-psychologically*, or independently without the support of others. However, we are also mindful of the work of cultural psychologists such as Barbara Rogoff (2003) who remind us that expectations of and for children, and how their learning is mediated, differ significantly across cultural groups. We also draw on Bronfenbrenner's (1979, 2005) ecological theory of human development, which postulates that children's development and learning are influenced by overlapping systems or spheres. For example, in thinking about children's mathematical development and learning in their homes and communities, we need to be mindful that ideologies and policies in the larger society influence what occurs in the context of the family.

We have been researching parent mediation of preschool mathematics learning for two decades. Beginning with a longitudinal case study of our daughter's mathematics learning (Anderson & Anderson, 1995), we reported on the role mathematics played in her construction of meaning as we shared storybooks with her. Over the years, we have documented a range of parent-child engagement with mathematics in storybook reading (e.g., Anderson, Anderson, & Shapiro, 2004, 2005). In an earlier study (Anderson, 1997), Ann investigated parent-child mathematical interactions across activities involving four sets of materials (blocks, story book, drawing materials, worksheets), and we have also looked at parent-child interactions when playing a board game (Moffatt, Anderson, Anderson, & Shapiro, 2009). As indicated earlier, in Anderson and Anderson (2014), we reported on a continuum of mathematical engagement found across six middle class families that we proposed moved beyond the *instrumental* and *pedagogical* dichotomy (Walkerdine, 1988). In this chapter, we reexamine these 6 families' activities with respect to *instruction* and *construction* to explore how such a framing (orientation) characterizes the ways in which the children were being "educated" in mathematics during their preprimary or preschool years.

Methods

Six well-educated, middle class mothers of preschoolers (aged 2.5 years at the outset) agreed to participate in a 2-year study of parent-child mathematics engagement in their homes. Every 6–8 weeks, the mothers chose a joint activity to have videotaped, in which they "normally" supported their preschooler's mathematics. For the duration of the study, the same research assistant videotaped three mother-daughter dyads and one mother-son dyad in their homes at their convenience, while one mother videotaped her daughter interacting with the child's father and the other mother videotaped her preschool daughter with several family members (i.e., daughter-older sister; daughter-grandmother; daughter-a family friend, or daughter-mother dyads). Each session lasted at least 15 min, although the number of sessions (usually one every 4–6 weeks) as well as their duration varied for each family (See Table 14.1).

We transcribed each videotaped joint activity in its entirety and observational notes from multiple viewings of each activity augmented the transcripts. In Anderson and Anderson (2014), we reported that the 44 activities documented could be classified as either play or everyday activities and the mathematics verbalized in each activity was characterized along a 5-point scale (i.e., (a) math is core; (b) math is prominent; (c) math and other focus equal; (d) math is minor; (e) math is incidental).

Table 14.1 Summary of participants in relation to preschooler

Family name	Preschooler gender	Sibling gender/age	Dyad in video	Video by (sessions #)
Adam	Girl	Boy/younger	Mother/daughter; mother/daughter/brother	Research Assistant (7)
Liu	Girl		Father/daughter	Mother (4)
Penn	Boy	Girl/older	Mother/son; mother/son/sister	RA (5)
Star	Girl	Infant[a]	Mother/daughter	RA (5)
Beet	Girl	Infant[a]	Mother/daughter	RA (6)
Pimm	Girl	Girl/older	Mother/daughter; sister/D; grandmother/D; other adult/D	Mother (6)

[a]Sibling born during the 2nd year of the study

In the current analysis, upon rereading each transcript multiple times, *Instruction* or *Construction* was assigned as a best-fit descriptor for the "tenor" of the parent-child interactions evident in each joint activity. We propose that coding in this holistic, qualitative manner assists in identifying a pedagogical approach which was sustained throughout the activity, rather than for a short portion thereof. Keeping in mind that construction occurs regardless of the type or amount of instruction and that joint adult-child activity by design is a context in which adult mediation (i.e., instruction) of some sort is present, we sorted the 44 activities according to more or less *Instruction* (I), whereby those activities where there was no (or minimal) instruction were labeled *Construction (C)*. In Table 14.2, we report our current categorizations (I or C) juxtaposed with our previous 5-point scale (a to e) for comparison purposes. In addition to identifying any trends in the activities, across and within families, we selected two activities (i.e., pizza and photos), one from each family (i.e., Adam and Pimm, respectively) who appeared to lie *at* the poles of Instruction and Construction, for further analysis. In Appendix A, we provide a vignette, a transcript excerpt (space limitations precluded full transcripts), and a short description of the tenor. For the current analysis, the full transcripts and videos of these two joint activities were again reread and reviewed, and emergent themes were noted.

Results

Considering that the two independent analyses of this data were separated by several years, it was interesting, although perhaps not surprising, to find that most activities now categorized as Instruction had previously been coded as having math as core or a major portion (*a* and *b*) and those activities categorized as Construction were previously coded as math being minor or incidental (*d* and *e*) (see Table 14.2). There were four exceptions, however, where Instruction mapped onto two activities with minimal mathematics content (*d*) and Construction mapped onto two with mathematics more prominent (*a, b*) (See Table 14.2). For example, playing

Table 14.2 Summary of each family's activities by type, math goal, and instruction level

Family	ADAM	LIU	PENN	STAR	BEET	PIMM
Play-based activities (15 I: 12 C; 2 I and C)						
Puzzle	Number-a-I	Number-a-I	Jigsaw-c-C and I		Jigsaw-e-C	Jigsaw-c-I
Play	Store-a-I	Trains-b-I		Stickers-a-I	Tea party-d-C	[a]Pegboard-d-I
Board game	Snakes and ladders-a-I		Bingo-a-I	Hungry hippos-b-I	Checkers-d-C	
Toys		Pop-up-b-I	Cars-b-I and C	Trains-e-C		Dolls-e-C
Play-Doh	**Pizza-a-I**			Face-d-C	Food-d-C	
Physical	[a]Hopscotch a-C				Sprinkler-e-C	Follow the leader-e-C
Matching	Cards-a-I	Rods-a-I				Images-e-C
Games		Pasta-a-I				Dreydel-e-C
Everyday activities (6 I: 9 C)						
Story time	Math-a-I			Felt-e-C	[a]Objects-d-I	Sounds-e-C
Family time		b-C		Lunch-d-C	Baking-a-I	**Photos-e-C**
School	Problems-a-I	Computer-a-I				Yearbook-d-C
Songs	1,2,buckle… -a-I	ABCs-e-C		Row, row-e-C		
Misc						Penny-e-C
	8 I: 1 C	6 I: 2 C	3 I: 2 C	2 I: 5 C	2 I: 5 C	2 I: 8 C

NB: Math goal: (a) math is the core and goal of the activity; (b) math occupies a major portion of the activity but was not the original goal, necessarily; (c) math occupies an equal part of the event, wherein other aims and content are present; (d) math occupies a minor portion of the activity but seems apparent to the participants; (e) math is incidental for the most part and may/may not be apparent to participants. Instruction (I), "parent" appears to be "teaching" mathematics more directly to child; Construction (C), "parent" appears to be "teaching" mathematics less directly to child

[a]Exceptions in mapping I and C onto 5-point scale (a to e)

hopscotch (coded a-C) involved math (number and counting) as a goal (a), but entailed the ADAM child hopping onto a numbered square while she (and her mother) "counted" aloud (Construction). Similarly, identifying objects in a storybook (coded d-I) involved math (number names, shape) in a minor way (d), but entailed the BEET mother continuously asking questions and explaining about objects (animals, colors, shapes) that she and her child identified (Instruction). It would seem then that Instruction was somehow related to increased verbalization of mathematics. Is it that during Instruction the adult tends to talk (tell) more and thus the naming of mathematics terms increases? Could it be that in activities where mathematics is a goal or a major focus, it inevitably leads to conversations that sound more like Instruction (Q-A-E)? In contrast, is it possible that for parents oriented toward Construction, the mathematics we identified as incidental or minor was more intentional? Without further research, what the relationship is and what it may mean for children's mathematical learning remain unclear.

While there was a balance between Instruction and Construction (i.e., 21 Is; 21 Cs; 2 C and Is) across all 44 activities, when we look across the activities, within individual families, there was a tendency for most activities to cluster around one or the other poles (e.g., ADAM and LIU, I > C; STAR, BEET, and PIMM C > I). However, the PENN family seemed to demonstrate a more blended approach (i.e., 2 of 3 activities coded I and C). Thus, it would seem that each of the six families tended to support mathematical engagement somewhat consistently, adopting either an Instruction or Construction stance for most of the joint adult-child activities. In a study where parents were asked to identify the mathematics-related activities they normally do, it is interesting that three of the mothers shared activities which we designated Construction. What remains unclear, without further research, is whether these parents knowingly showcased these orientations or if, with 4–6 weeks between videotaping, it was more happenstance. Also more research is needed to determine whether a larger sample of daily at-home experiences within each family would point toward a balance of Instruction and Construction over time.

Instruction and Construction in Context

As seen in Table 14.2, two families ADAM and PIMM appeared to be polar opposites in terms of the ways in which they engaged their preschooler with mathematics. While both families shared play-based and everyday activities, eight of the activities the ADAM family shared were categorized as Instruction, while eight of the activities the PIMM family shared were designated Construction. That is, even though the specific activities differed considerably, the tenor of the adult-child interactions was consistent across all eight of them. Since we imposed the designation Instruction or Construction holistically, a reasonable next step was to look more closely at an illustrative example of each. Two particular activities (Play-Doh pizza, family photos), one from each home, caught our attention. While the Play-Doh pizza activity (Adam mother and daughter) and the family time photo activity (Pimm grandmother and daughter) (Appendix A) were comparable in duration, they varied otherwise (e.g., play vs everyday, hands-on vs pictorial materials, mother vs grandmother, shape vs measurement, instruction vs construction). Unlike games and storybooks, both activities (i.e., children playing with clay; children viewing photos) have received little attention in the research literature in children's early mathematics learning and yet are familiar experiences for children in daycares and their homes. Also, on separate occasions over the years, we have presented short excerpts of these activities at workshops and conferences, and we have noted how audience members' (e.g., practicing teachers, mathematics educators researchers, graduate students) remarks usually align with the Instruction and Construction designation, increasing our confidence in the coding. However, some caveats are warranted. First we remind the reader that the one Instruction activity and one Construction activity that we highlight, while illustrative examples, are not necessarily representative of other activities engaged in by the two families nor by the other four families. As we point out elsewhere, the findings are not generalizable and serve to raise questions and provoke further inquiry about the intriguing dance metaphor. Second, it is important to keep in mind that the type of adult support (i.e., Instruction or Construction) was only *one* of multiple factors (e.g., types of materials, types of mathematics, social and cultural expectations, and so on) implicated in the themes we report. Thus, any attempts to read causality are misplaced.

We begin by describing aspects of both activities which appear anomalous to what we might normally associate with Instruction and Construction. For example, Play-Doh pizza is a child-centered activity, where the child interacts with hands-on material. It is play-based since the child initiates the making of pizza with the modeling clay and the pizza was being served to imaginary visitors, who are attending a birthday party. The mother raises the idea of cutting the pizza for a specific number of guests and invokes the need for same size slices and engages her daughter in cutting the pizza, all the while using a question-answer-evaluation (Q-A-E) discourse pattern, often associated with more didactic or formal teaching. In contrast, the materials used in family time photos are pictorial and adult-centric in that the child's access to the related social-conventional knowledge (who, when, where) is through her grandmother. While both the child and grandmother query each other about what they see, with minimal evaluation of the answers, the grandmother also elaborates on most of the child's statements and inevitably tells the child unsolicited information for many of the photos. Other than pointing occasionally to a particular photo or person/thing in a photo, there is no manipulation of materials. Throughout this everyday activity, the conversation is relaxed and playful, with both participants laughing and teasing one another. Thus, while this at-home Instruction activity with pizza then seems at a surface level to have aspects of child-centered pedagogy (Fig. 14.1) with some direct teaching overtones, this at-home Construction activity does not appear to entail features we usually associate with child-centered pedagogy such as manipulating materials (Fig. 14.1). Also, contrary to what we might expect, neither Instruction nor Construction within these contexts provided the children more leeway (or control) over pacing or the amount of time they could spend at the task. In light of the contradictory nature of, and the differences between, these particular instances of Instruction and Construction, we found ourselves searching for themes of commonality that might be hidden in such polarized contexts.

In both Instruction and Construction activity, the adults introduce the 3-year-old child to mathematics *concepts* at, and *beyond*, what might normally be considered *age-appropriate*. The ADAM mother explicitly discusses and model's dissection of circles and solving fair-share word problems, as well as naming fractions during the pizza activity. The PIMM grandmother discusses and illustrates time (dates, ages, time periods), size (height), scale, and the relationship between size and time (growth) during the Photos activity. Indeed, in the Province of British Columbia, Canada, where we work, the curriculum guide for K-9 mathematics (British Columbia's New Curriculum: Mathematics) introduces many of these ideas in postprimary (grades 4 and beyond). Thus this mother and grandmother appear to support these children's mathematical learning beyond what they would achieve independently (Vygotsky's zone of proximal development) and beyond what some educators and curriculum developers would expect for this age group. They did so employing different pedagogical orientations or stances (i.e., Instruction in the first case and Construction in the second) calling into question what we believe are assumptions held by some researchers and educators that such challenging mathematical concepts or constructs necessitate (direct) instruction.

In both Instruction and Construction contexts, the child and adult *co-construct* the activity as it unfolds. That is, neither activity appears to be preplanned or scripted. In the Instruction activity, the ADAM mother introduces the "pretend visitors sharing pizza" scenario after her daughter reveals her plans to make a pizza with Play-Doh. Her daughter participates fully by making, and cutting, the Play-Doh pizza and counting the slices as well as expanding on the story elements of the context, intermittently adding details for characters such as ages and names and that they are attending a birthday party. While the ADAM mother occasionally acknowledges her daughter's storyline, she maintains a focus on generating problems (e.g., "so how many pieces for 6 visitors?") for her daughter to answer. During the Photos activity (i.e., Construction), both adult and child seem immersed in the task of viewing photos to remember (imagine) family, with the PIMM grandmother and the child engaged in identifying persons (things) in the photos. The PIMM grandmother expands on the story elements of the context (e.g., "it was a very big doggy that she got for (inaudible)") as well as providing background knowledge (e.g., ages, places), while the PIMM child asks about the photos (e.g., "who are these two?") and inquires about (e.g., why?) and confirms her grandmother's storyline (e.g., "oh, that is so big"). Thus, whether through an Instruction or a Construction orientation, these adults address bigger ideas of mathematics in contextually relevant ways. Interestingly, within both contexts – and not just in Construction as we might intuitively surmise – the mathematics (e.g., dissecting a circle and change over time) remains embedded in the broader storyline (i.e., cutting pizza slices and reminiscing about the past).

In both Instruction and Construction, the *adults respond to* child initiations, through *redirection* and *elaboration.* For instance, in the Pizza activity, when the ADAM child suggests making parallel vertical cuts, the ADAM mother redirects her daughter to cut along two perpendicular lines (diameters). On another occasion, when the child mentions, "... 6 people," the ADAM mother immediately responds with a corresponding problem (i.e., how many slices for 6 visitors?) and elaborates on the need for radial cuts and explains how to make them (i.e., to get 6 slices of the same size). That is, during Instruction the mother (both verbally and with gestures) corrects or refines the child's actions (i.e., how the child manipulates the Play-Doh) and in so doing elaborates on how to model the mathematical solutions to the problems posed. As such, many of the redirections in the Play-Doh context could be seen as forms of elaboration. In the photo activity, the PIMM grandmother answers the child's questions (e.g., "who is this") with the same detail as the grandmother's self-initiated descriptions. In addition, the child's comments (e.g., "so short") are elaborated on or explained by her grandmother (e.g., "short? well, I was only 8 years old"). On occasion, the PIMM grandmother redirects the child's attention (e.g., do you want to see photos? Do you know who this is?). Thus, whether their orientation is Instruction or Construction, these adults build (on) from child-initiated comments or questions to scaffold learning.

Discussion

Due to the exploratory nature of this study, we caution the reader that the findings may not be generalizable and that we do not see them necessarily as defining features of Instruction and Construction in the home. Recognizing that the sample of activities from each home was small compared to the plethora of joint adult-child activity these preschoolers experienced in their homes, we acknowledge that these preliminary results suggest the need for further research. We see them more as serving more to generate and to provoke discussion. To that end, we now reflect on what we have learned from the current study regarding the "dance of Instruction with Construction" within at-home settings.

A "Dance of Instruction with Construction" Metaphor

Findings from this study show that parents can and do engage preschool children in at-home joint activity we might characterize as Instruction and as Construction. However, for these families whose activity and interactions we analyzed, the "dance of Instruction with Construction" occurred across a set of activities over time. Within specific activities, families enacted either a Construction or Instruction orientation or stance, and their proclivity toward that stance was fairly consistent across the set of activities. In other words, although all families at times engaged in Construction and Instruction, the dance was imbalanced as each tended to favor one or the other posture. Thus, while one interpretation of the "dance" metaphor might suggest adults should or can avail of "teachable moments" during children's play and everyday activity blending Instruction and Construction as necessary within each activity, this study indicates that except for the PENN family, this tended not to occur. This study lends support to an interpretation of the "dance" metaphor, where adults should and can engage children in a variety of play and everyday activity such that some are instruction oriented and others are construction oriented. However, again, such an interpretation of a "dance" metaphor, which blends Instruction and Construction across activities, seems to imply a balance between the two orientations, which was not the case for these families. Thus, how might a metaphor for blending Instruction with Construction across multiple activities portray a more asymmetrical "dance" and would there be advantages in doing so? Or put differently, are young children disadvantaged if a Construction or Instruction mode is favored by, and thus is more prominent within, their family? We do not see this call for a more contextual or nuanced interpretation of the metaphor of the dance as a discredit of it, but we think it should give us pause. If we are to acknowledge parents' (and ECE caregivers') funds of knowledge, which may or may not include mathematics competence or mathematical knowledge for teaching, we must value (and validate) their intimate knowledge of their child and their everyday contexts and daily mathematical practices. Based on the current study, further research into the nuances and features of adult-child joint activity which appears to support blending of Construction and Instruction within experiences (e.g., PENN family) as well as families whose activities might be discordant with our current views is warranted. In particular, future research in which more activities are collected more often to more closely approximate the plethora of daily activity in home than was the case in the current study would be of value. Finally, research which examines adult-child joint activity (conversations) (e.g., parent-child; ECE teacher-child; docent-child) in multiple early years settings (e.g., homes, day cares, museums) is needed to broaden our understandings of the expanse of adult-mediated mathematical experiences prior to (outside) school.

Instruction and Construction: Mathematics Content or Big Ideas

To reiterate, in the current study, we reexamined only 2 of the 44 activities to inform a sense of what an Instruction (i.e., Pizza) and Construction (i.e., Photos) activity might look like for each family (i.e., ADAM and PIMM, respectively). As evidenced in the current study, *big ideas* of mathematics (e.g., linear relationships) were present in both the Instruction (pizza)- and Construction (Photos)-oriented activity. Interestingly, within both of these activities, the big ideas remained implicit (i.e., seldom elaborated on verbally) and yet are repeatedly attended to throughout each (e.g., the ADAM mother continually talks about cuts and fairness, the PIMM grandmother continually speaks of size and age). Indeed, the sources of both big ideas appear embedded in the social conventions of each context (i.e., sharing a pizza fairly and growth and aging of family members). However, much remains unknown in the current study. What sense did the ADAM child make of the underlying properties of a circle that impact the area of the slices? Did she notice the differences between her proposed cuts and those of her mother? What frames of reference did the PIMM child use to make sense of scale and her grandmother's changing size in the photos? Did she notice how the height of the child (her grandmother) in the photo might resemble her own? What sense did she make of the varied scales of time (e.g., age in years, past-present continuum)? While constructivist theory suggests each child is potentially "making sense" to different degrees of these underlying relations and connections, it appears that during the Instruction-oriented activity, the child's attention may have been drawn away from the big ideas and onto more explicit math conventions (i.e., counting to solve how many pizza slices mask the relation between the increasing number of visitors and the decreasing size of each equal share). Does Instruction inevitably reduce big ideas to school-like math and Construction permit big ideas to remain more prominent in conversations and interactions with young children? Is it possible that parents/caregivers may maintain a focus on big ideas through Construction, because the activity is enacted by talking about the world which is inherently mathematical (Hunting, 2010) whereas Instruction activates a parent's/caregiver's traditional, school-like view of the mathematics they know? Of course, the design of this study does not allow us to conjecture about the answers to these questions. Therefore, further in-depth research into the type of mathematics children experience during adult-child joint activity at home, in light of the orientation that the adult takes, is needed to further our understanding of the value of Instruction and Construction, in preprimary settings.

In closing then, we believe the questions raised by this study are important ones that researchers and educators in early childhood mathematics education need to ponder as we continue our work with young children and their families.

Appendix A

Pizza Activity

The Adam mother and her preschooler are seated at adjacent sides of the child's table. Once the mother assists her daughter in getting a "lump" of Play-Doh onto the table, the child begins to roll a wooden dowel onto it. When the Adam mother asks what the child plans to make, she answers pizza. At times, the mother helps flatten the dough to make it easier for the roller and converses with her daughter as the child concentrates on rolling out the pizza. Initially the mother asks what shape she is trying to make ("what's the shape of a Pizza?") and the daughter responds ("triangle") in such a way that the mother qualifies it ("when it is sliced") before turning their attention to cutting the pizza to share with friends who visit. As the episode unfolds, the child is encouraged to cut the circular Play-Doh pizza into equal-sized slices for various numbers of imagined visitors. The mother's questions ensue about the number of pieces and on occasion about whether the sizes are fair ("same size for each visitor") while the child makes the cuts and counts the slices (Fig. A.1).

18. Mother: **How is the pizza? OK you know what? Let's pretend we are going to have four people over for supper, four people are coming. We want to cut this pizza so everybody gets the same. So the first thing we need to do is cut it in half.**
19. Daughter: **Like this? (gestures cutting middle of circle)**
20. M: **Yeah right down the middle so there is two pieces the same size.**
21. D: **Cut it that way?**
22. M: **Yeah, good.**
23. D: **And that way? (gestures to cut parallel and to right of previous cut)**
24. M: **So you cut it in half first, good girl. And there is how many pieces?**
25. D: **Two.**
26. M: **Two. But we need more than two don't we? How many people are coming?**

Daughter holds up four fingers.

27. M: **Four people are coming for supper so can you cut it in half this way? Or this way? Yeah**.
28. D: **I am going to go this way** (inaudible) (vertical cut gestured)
29. M: **OK but let's do this first so you already cut it in half this way so let's cut it in half this way. Yeah. Let's see what happens. That's a girl. So we cut it in half that way. Now how many pieces do we have? One, two---**
30. D: **One, two, three, four.**
31. M: **Four.**
32. D: (inaudible) **six people.**
33. M: **Six people coming? OK How can we divide it for six people?**
34. D: **You go like that. Go like that and then you go like that.**
35. M: **Should I build it back together or do you want to keep it like this?**
36. D: Build it back.

Fig. A.1 Excerpt from Adam mother-daughter Play-Doh activity transcript

NOTE: As the mother repeatedly asked math (number) related questions, explicitly directed the child's actions for cutting equal pieces, redirected the child's attention to the problems being posed, and maintained a focus on a mathematical goal, the tenor of the session was deemed Instruction.

Photos Activity

The Pimm grandmother and her preschool granddaughter sit side by side on the edge of the child's bed, with large sheets of paper on the floor or bed near them. Multiple photos of various family members, captured in groups or alone, with or without other props, have been printed on these sheets. The grandmother holds one sheet at a time, at a slight angle while resting the bottom edge on her lap, so as to be readily visible to her and the child. The child is able to point to, and readily touch, most of the photos. The Pimm mother is behind the camera and videotaping the conversation between the grandmother and her preschooler. She comments on one occasion only, although the child is seen looking in the direction of the camera (her mother) on a few occasions typically when the grandmother speaks of the mother's (Eema) presence in the photos. The grandmother shares varied information about selected photos and those captured, asks the child questions, answers the child's questions, and at times expands on the child's statements (Fig. A.2). In the latter third of the session, the older sibling joins her grandmother and sister, and all three continue to view and talk about the family photos together.

14. Grandmother **That is your Ohma, when she was teensy weensy.**
15. Daughter: (laughs)
16. G: … you know her. … **and that was your Eema when she was teensy weensy**
17. D: **Hey, what is that doggy?**
18. G: **That was a very big doggy that she got for (inaudible).**
19. D: Why?
20. G: **It was such a big doggy that you could sit on it.**
21. D: **Yeah, and get a ride, get a ride**.
22. G: Yes, that is (inaudible) **when she was about 17 or 18.**
23. D: **Oh that is so big.**

:

29. D: **Who are these two?**
30. G: **That is my mother after the war with me and that is my father.**
31. D: **You are so so short.**
32 G: **Short? I was short, I was eight years old.**
33. D: **The dress is so nice.**
34. G: Yes, that was **my first new dress after the war.**
35. D: Oh

Fig. A.2 Excerpt from Pimm grandmother-daughter Photos activity transcript

NOTE: As the grandmother repeatedly identifies the persons in the pictures, providing details about relationships to each other and the child, sharing short stories of related events, interweaving math-related terms on occasion when quantity, age or size were invoked to describe the photo, the tenor of the session was deemed Construction

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Hedwig Gasteiger and Christiane Benz

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The Fig. 4.1 in this chapter has been inadvertently published in German, which has been updated in English.

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