Hades—The Invisible Side of Mathematical Thinking

Walther Paravicini, Jörn Schnieder and Ingrid Scharlau

Abstract The chapter presents a philosophical approach to teaching and learning mathematics in which five philosophical methods are translated into methods of reading and writing mathematical texts. The philosophical approaches are Hermeneutics, Analytical philosophy, Dialectics, Experience based (phenomenological) and Speculative philosophy. We use the acronym HADES for the combination of these approaches. For each of them we present reading and writing material which can be used for teaching peer tutors and by them in their interaction with students.

Keywords Reading mathematics · Writing mathematics · Philosophical approaches · Teaching and learning mathematics

Introduction

Mathematics is a language of written text. Doing mathematics thus involves the close reading of mathematical texts. Mathematical language often consists of an intricate, strictly regulated interplay of prosaic and formal language. Importantly, however, the meaning of mathematical texts is not restricted to their deductive or logical consistency. It also results from additional historical aspects of theory and research and aspects of discourse which often go unnoticed—such as what is left out of a text or how a question is framed. Thus, we view mathematics—as all disciplines—as practice. Under the perspective of their purposes and means, a

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conclusive understanding of mathematical contents, theorems and even theories presupposes comprehensive knowledge and appraisal of the interests which govern research—as well as those interests which have been supplanted.

One very important task for student tutors is to encourage first-year students to independently learn mathematical thinking. Among the obstacles for learning are mathematical texts, especially the economical style of presenting mathematical content in introductory textbooks, which abandons the context of discovery in favour of the context of justification, as well as the very common focus on a correct solution. In the present chapter, we develop a systematic approach for dealing with and understanding mathematical texts. This educational approach combines specific mathematics-related writing, reading, and understanding-related tasks with meta-reflection. In the short run, the methods help tutors to richly understand mathematical thinking; in the long run, they help to question and analyse how mathematical content is typically and appropriately presented, that is to engage in critical self-examination. We expect that the approach we have developed (termed HADES—explained below) will firstly support tutors to make content intelligible both by means of enhancing their reading and writing skills and by increasing their understanding how mathematics is done. They will secondly be able to integrate (parts of) the methods into their own teaching.

The basic idea is to develop philosophical methods of thinking and working into reflective reading and writing strategies for mathematical texts. We assume that comprehensive understanding of mathematical texts can result from the interplay of five philosophical or "elementary" methods of thinking (Martens, 2003). These methods are based on the idea that rational scholarship can be explained by and construed of elementary thinking tools which can be learned and taught. These are,

- H: hermeneutic methods of understanding and interpreting texts and images
- A: analytical methods of precise definitions
- D: dialectical methods for analysing (apparent) contradictions and structured dialogs
- E: experience-based or phenomenological methods of exact descriptions
- S: speculative methods of (thought) experiments.

These methods can be considered the didactic essence of five important schools of philosophy (Nida-Rümelin, Spiegel, & Tiedemann, 2015) and were developed in the context of a didactic of philosophy with the goal of fostering comprehensive text understanding. They were derived by simplifying classical philosophical positions to their essential features for educational purposes (Schnieder, 2013) and combined with appropriate tools for reflective and exploratory reading and writing.

In the present chapter, we describe these five elementary methods. We provide examples of how the methods can be turned into mathematical reading and writing tasks. Our main focus is on peer tutors and our aim is to provide them with tools for rich and reflective communication about mathematical content or, more precisely, mathematical discourse.

Hermeneutics

As a theory and practice of understanding texts, hermeneutics provides methods to incorporate what has been said by others, especially what has been passed on in writing, into one's own theoretical thinking. Hermeneutics thus relates a text systematically to an individual horizon of understanding. Besides objective understanding, which is constituted by historical reasons and effects, this approach accentuates subjective understanding: gaining access to meaning by starting from prior knowledge in order to subsequently relate "objective and subjective interpretations" to each other. Thus, hermeneutics addresses problems that arise because the language of the arguments in a text is not or is only partly one's own language. At its core is the hermeneutic circle (Gadamer, 2010), a process of prior understanding (Vorentwurf), text understanding (Textverstehen), and the fusion of horizons (Horizontverschmelzung). Readers inevitably move in a hermeneutic circle because they read texts with expectations and understand individual statements in a general context and, conversely, the general context from single statements.

The hermeneutic circle allows focused work with mathematical texts, structured reading, slow reading, and self-clarification. Within this circle, students establish connections to their prior knowledge. The main operation is translating text into one's own language. Specifically, the hermeneutic circle requests readers to check the suitability of the concepts used in translation by making them explicit and showing how a term is a suitable translation or why its meaning cannot (yet) be confirmed with the help of the text. Naming these differences and similarities of text and translation allows the reader to put forward new interpretations and test hypotheses which might reduce the differences.

The hermeneutic circle has two dimensions, *depth* combining the historical development of a mathematical proof with its critical reconstruction, and *breadth* containing aspects of prior knowledge and assumptions which can be used for developing hypotheses. Following Martens (2003), interesting aspects besides the logical structure of a text could be its cultural or historical context—particularly in combination with its scientific claims—, its structure, ruptures, gaps, and fringes and, last, but not least, its effect on the reader.

Rhetorically speaking, the primary goal of the hermeneutics approach in HADES is to engage in reflecting and making systematic and rich connections to personal thinking and prior knowledge. This goal can be pursued by a *writing circle method* of which we give an example: Tutors are presented with different historical texts which are the starting point of a central mathematical idea, e.g., Barrow's (1976, Lectio X, Prop. 11) preliminary version of the fundamental theorem of calculus together with the corresponding picture (Jahnke, 2009, p. 87). Firstly, the tutors translate the texts according to their actual, unaided mathematical understanding, similar to the situation of a student. Secondly, they work through the hermeneutic circle and note their thoughts as inner monologue. For *prior understanding*, they collect all their impressions of the text, including subjective impressions. In a preliminary translation they then hypothesize what the text is about, including an analysis of the 'prejudices' governing their current understanding. In the *text interpretation* stage, they test how far their hypothesis summarizes its content and whether it deviates from the latter. They validate their impressions with the help of the text. Finally, in the *fusion of horizons*, they summarize their thoughts by enumerating similarities between the topics of the text and the translation. They then start the circle a second time, developing a deeper prior understanding. They compare their inner monologue with the translation from the first step which was made without help. Finally, they discuss the advantages and disadvantages of the hermeneutic circle and how it can be presented in their classes. The intensive step-by-step analysis of the text puts the recipients into the role of reflecting producers who, in a written inner monologue, describe their thoughts while translating, making them explicit and, in turn, comprehensible. The writers emphasize translation and conceptual issues, compare meanings, evaluate translations and make the evaluation comprehensible. This resembles methods from teaching writing, e.g. the exploratory methods developed by Bean (2011). Although the task itself already encourages reflection, a necessary final step is *meta-reflection*: describing and evaluating experiences and discussing transfer of methods.

Analytics

Analytic philosophy developed in close interplay with modern fundamental research in mathematics (Frege, 1953). Its methods therefore well match the analysis of mathematical concepts and theory. Analytic philosophy focuses on methodological aspects of scientific thought and especially on establishing clear rules of how to define concepts, how to argue and how to criticize. Consequently, epistemological, aesthetic, ethic, metaphysical problems are all put down to questions of language, concept, or argumentation analysis (Russell, 1948).

More precisely, analytic philosophy discusses norms and procedures for establishing concepts and relations such as explicit and implicit, predicative and impredicative definitions, abstraction and ideation. It furthermore develops methods for analysing arguments, for instance by uncovering hidden implications, and testing internal consistency. One important claim underlying this approach is that these procedures serve to clarify central elements of everyday and scientific language without taking any stand content-wise.

It goes without saying that mathematical texts and in particular mathematical proofs follow strict rules. Notwithstanding they are only partly deductive. Actually *doing* mathematics, solving a specific problem, cannot be reduced to applying the rules of formal logic. Consequently, formal logic cannot capture the complexity and creativity of mathematical research with its sudden insights and the personal involvement of researchers.

What is more: Mathematical proofs consist of an intricate interplay of prose and formal language in which the true claims are neither easy to find nor easy to validate—especially for beginners. For instance, arguments in proofs may be highly condensed: premises, background knowledge or warrants may remain implicit and tacit. This is where analytic procedures come into play: Their point is to uncover and highlight the formal structure behind the verbal form of mathematical texts.

Turned into a didactic approach, this means to introduce and practise the (classical) concept of argument and how to complete an argument by making its premises and warrants explicit. Teachable and learnable tools can be developed which aim at precise and detailed analysis and evaluation of arguments by deductive reconstruction (Toulmin, 1958).

To instruct students to pay close attention to the deductive structure of a mathematical argument when reading it line by line has been shown to be a simple and effective way to enhance student comprehension, at least to a certain degree (Hodds, Alcock, & Inglis, 2014). With the following example task we want to sensitise tutors to how complex the analytic procedure is that expert mathematicians seem to carry out automatically when reading mathematical argumentations.

The tutors are given an elementary, but non-trivial, textbook proof showing, for example, the existence of Feuerbach's nine-point circle or some number theoretic fact. They now extract, individually and by reading it line by line, every single argument that is given in the course of the proof, where an argument consists, quite classically, of a premise, a warrant and a conclusion. In some steps of the proof, a premise or a warrant might be implicit, and the tutors are asked to fill in these gaps. The individual three-step arguments are written down, each in a separate box, and the boxes are arranged into a tree-like diagram that reflects the global deductive structure of the argumentation. The tutors are also asked to add arguments that they feel are missing to the proof. What each tutor ends up with should be a diagram that represents the argumentation to a degree of completeness satisfying to her (see Fig. 1 for an example—with the actual arguments removed from the boxes). Note that similar diagrams are used for research purposes in the argumentation analysis literature (e.g. Knipping & Reid, 2014; Krummheuer, 2003).

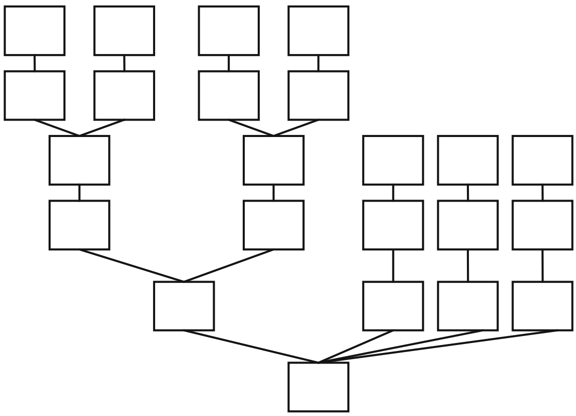

Fig. 1 The global deductive structure of a proof of the existence of the nine-point circle

Now the tutors are asked to compare their diagrams among each other and to reflect on how the ideals of completeness and exactness are met in academic mathematical writing—and on what (other) didactical tools might be suitable for helping novice readers to understand the argumentative structure of mathematical texts.

One central advantage of this approach is that it can provide students and tutors with a (low-threshold) search strategy that helps them find gaps in argumentations. It thus allows for the fact that mathematical proofs are composed as written texts which—at least on the surface—only gradually differ from other scientific or everyday texts although they are geared to the ideal of deductive conclusiveness. Note also that all arguments which can be correctly reconstructed with this method can also be reconstructed as deductively valid. In principle, thus, the method is independent of the teacher's demands and thus supports student autonomy.

Dialectics

Dialectics does not have a good standing in academia. It is often associated with pointless talk, formalistic debate, hair-splitting, and even manipulation. It might therefore surprise to see methods for reading and writing derived from such an approach. However, an essential goal of dialectics is and always was to develop methods which help to reason with specific addressees and in an open and undogmatic way (Martens, 2003; Rohbeck, 2008).

Mutual dialogs with concrete persons and within historical contexts play a prominent role in mathematics. Doing mathematical research does not happen in a silent inner dialog or purely rational thinking. Quite the reverse, argumentation that cuts off experiences, moods, or spontaneous ideas and that views itself as context-independent is only the final result of a highly communicative process. This process is both dependent on and oriented towards comprehensibility and communication. Talking with a concrete person is therefore expected to aid conceptual and argumentative clarity and thus problem solving and idea and knowledge generation.

Thus, mathematical arguments do not result from well-aimed manipulation. Yet, this notion can help to test the logical validity of an argument by systematically searching all possible criticisms which are then put into words and invalidated. Strictly speaking, a mathematical proof is understood only if all potential objections have been disproved. Thus, it is beneficial to be provided with methods for scrutinizing proofs from all perspectives in claims and replies.

This can be achieved by a reduced form of dialogical logic as developed by the Erlangen school of logical constructivism (Lorenzen, 1969): Mathematical proving and arguing is schematised as a dialog. In this dialog, a proponent and an opponent *stage a dialog* in order to attack and defend a thesis. This staging uses few predefined rules which suffice to formulate logically relevant objections (Lorenzen & Lorenz, 1978).

A further advanced writing method is a *dialog with the author* in which students take one part and the tutor the other. This approach dedicates special attention to how authors and their (assumed) readers are present in all kinds of texts and negotiate content. In order to foster this, attention is first drawn to the linguistic means by which this presence is realised: There is always crosstalk between authors and readers (or metadiscourse) in texts, even though readers (or writers) may not be aware of it. Linguistic signs of metadiscourse are rare or invisible in mathematical texts, but not absent. An important approach to this crosstalk is to *identify instances of metadiscourse* such as stance and engagement in a text (Hyland, 2010) and re-write it with a different amount of both (possibly more). So-called interactive means guide readers by transitions, for instance by frame markers, references, or code glosses, that is, words which help readers grasp meanings of material. Interactional means involve the reader in the text, for instance by hedges and boosters which withhold or emphasise the writer's commitment to a statement, attitude or engagement markers. The reading/writing task consists of (1) learning the different linguistic means, (2) identifying signs of interactive and interactional means in a mathematical text, including their absence where they might be helpful, and (3) rewriting the text with more (or even less) of those means. A variant of (3) would be to create a dialog of the author with another person, the core of the dialectic approach. It should follow the text structure, but complement it by including thoughts of the persons which can be deduced from stance and engagement markers and are turned into questions and the author's possible answers.

Experience/Phenomenology

Phenomenology as founded by Edmund Husserl (1970) is one of the most influential trends of current philosophy. Phenomenological methods uncover, in a complex analysis, how things present themselves. The phenomenological approach neither defines nor puts forwards theoretical propositions, but focuses on experience: What something is results from how different aspects form its actual experience. The question "What is X?" thus turns into the question "How does something present itself as X (to me, us)?" (Waldenfels, 1992).

Phenomenological methods foster deep understanding of mathematical concepts and ideas by illuminating the interplay between definition/concept and example—an interplay which is very important for learning. They provide processes by which one can approach the ideas or essence of mathematical concepts, that is the motivation behind their invention and study. Phenomenological writing methods should therefore encourage variation and support reduction to the essential. Such methods can be adapted from creative writing. For instance, concepts—imagine something like *continuity*—could be described with several of the following instructions:

> Consider an example for continuous functions of your own choice and its graphic representation. Then imagine you are standing on a ladder and look at the graph from high above. What becomes almost invisible in this perspective, what presents itself clearly? Or imagine yourself lying on the floor with your nose almost in the dust and looking at the graph with a magnifying glass. What becomes difficult to grasp? What is very present?

The students write down answers to the questions of at least a few sentences, capitalizing on the figurative aspect of the task, but without bothering much about what they suspect or imagine is correct (variation). The exercise ends with a first attempt at distilling the core of the concept by discussing which of the aspects visible in each perspective are essential and which are not (reduction).

This task provides a structured procedure to patiently and richly describe—for the time being without comparison without reference to formal definitions or statements. The example is described unhurriedly until ideas are gained which can be developed in a more theoretical manner. For instance, all examples of continuous functions found might be differentiable. Is that coincidence? Is there a counterexample?

The observations are then compared to the formal definition or scientific facts. Thereto, the descriptions are formalised and compared to the exact definition: Can the descriptions be identified as necessary or sufficient conditions for continuity? Can the claims be proved and illustrated with suitable examples and counterexamples? The opposite direction is possible as well: A concept could be illustrated by examples which are described precisely and in much detail, in what is visible as well as invisible. Again, the description is formalised and compared to the formal definition.

After the comparison, generation of examples continues (variation). Are there other examples—possibly more extreme ones? An important aspect of this procedure is that it can be executed a number of times. That means that it is not necessary that the first examples are good or exciting—better or richer examples will turn up in the process. A point of this method is that it conveys understanding as a process of sharpening of a concept. Thereby it allows to independently explore and understand mathematical content.

The task can be transferred to concepts that do not or only barely lend themselves to visualization. Instead of the very imaginative perspectives of the ladder scenario, other perspectives should be chosen, for instance logical, symbolical, pattern-related perspectives as well as metaphorical, spatio-temporal, social, communicative and even emotional models of presentation of a concept. In order to attenuate the possible strangeness of the task, a set of specified formulations might be used. Again, meta-reflection constitutes the last step, here with a focus on the question how perspective enables aspects of a concept to become visible.

Speculation

Speculative philosophy provides methods which support finding or devising new and relevant approaches to solve difficult problems—within philosophy itself, everyday life and not least science (Bloch, 1963; Peirce, 1997). Being able to have good ideas is an essential part of doing mathematics and an important starting point for developing mathematical ideas.

There are various methods of speculation in philosophy. As an example for the didactics of mathematics, we introduce an adapted version of freewriting. Freewriting as advocated, for example, by Elbow (1973) is a standard method in the teaching of academic writing. At its core is fast, uninterrupted and uncensored writing. To this end, all thoughts are written down without reflecting on them, evaluating them or revising them, even if they are only fragments. Writing uninterruptedly is meant to prevent reflection stopping writing flow.

Speculation draws on a combination of favourable attitudes, methods, sensitivity and art rather than a strictly methodical, step-wise procedure. Thinking speculatively, especially in science, often means to detach oneself from familiar perspectives, see through apparent necessities and constraints, but without giving way to arbitrariness. Newness as such has not value; it requires a purpose which arises from the particular context of research.

The method presented below meets these claims in its first two steps: Pre-set material helps the student tutor because it can be combined into sentences in a playful, non-committal way and thus with little risk. At the same time, they ensure that the sentences refer to a structured field and thus meet the need that new ideas are interesting insofar as they relate to current research. The method postpones the goal of refuting or finally proving and thus provides more space for thinking in everyday language or in a preformal or semiformal way: It provokes tentative deliberation of pros and cons, speculation and experimentation. Thinking is driven by questioning, raising objections, as an experiment holding a view which is not one's own and putting the results down on paper. The way towards an exact proof is staged as a slow approach. It is not the formalism alone which is the measure for correct work.

In the first step, writers are provided with concepts and (few) standard formulations (a variant of focused freewriting). Such a list might contain elementary concepts relating to functions—*injective*, *surjective*, *bijective*, *composition*, *range*, *domain*, *identity function*, *inverse function*, *group*, *unique*, *linear*, *homomorphic*, *continuous*, *iterated*, *and*, *or*, *not*, *if then, for all* and *there are*. The writers then produce as many linguistically and mathematically appropriate, complete sentences as possible, but they do not bother about the correctness of the sentences. This task has a given time, e.g. five or ten minutes.

In the second step, the claims are examined. In a second round of freewriting, first impressions, presumptions and thoughts about these sentences are put down as a stream of thought. This can be supported by a list of hedges such as “maybe”, “possibly”, “it is possible that”, “one could argue/imagine/point out” etc. Providing the list and encouraging to hedge one's own thoughts makes speculation easier for the students.

Two aspects alternate in this kind of writing: Firstly, ideas are generated by making use of intuition and imagination and thus gaining insights or raising risky presumptions. This is the core of the first step. The second step focuses on testing the insights and guesses with respect to their relevance and logical or argumentative conclusiveness. These steps can be repeated several times, for instance if no appropriate material has been gained in the first step or the second step makes it necessary to focus more closely on a certain area.

An optional (but fruitful and important) third step is meta-reflection. This can be done in pairs who think about attitudes and strategies which help them to think speculatively.

Scientific progress and creativity cannot be enforced; they cannot be put down into general instructions or trained. Planned creativity is a contradiction in terms. On the other hand, mathematical creativity and speculative thinking are more than a simply irrational, incomprehensible event which somehow leads to sudden insights. And even if creative insights cannot be produced, the ground for creativity can be prepared, for instance by helpful attitudes, methods and strategies, and fostered by internal maxims. Conversely, anxiety or little willingness to take chances in thinking can block creativity. How to develop the courage to take risks and the stamina to take novel and incalculable ways towards a solution, how to support composure facing the uncertainty of not yet having the solution and not being certain how to develop it, are all productive questions for the last step.

Outlook

We understand HADES as a descriptive as well as a normative system: We suggest that the five modes of thinking sketched above are central to how mathematical experts read and write texts, but also that the modes are usually chosen and applied unconsciously. We further suggest that tutors can use the methods we have elaborated above to improve how they read and write mathematical texts and, perhaps more importantly, to productively reflect on how they teach how to do mathematics. It might even prove to provide a reasonable scheme to plan their teaching.

In the future, we plan to improve the HADES tasks sketched above (and develop further tasks) by analysing how student tutors interact with them and how tutors incorporate them in their teaching. We suppose that specific aspects of (proof) comprehension can be improved in students by training them in one or the other of the five thinking modes—and that this compares favourably to a general training like the SET (Hodds, Alcock, & Inglis, 2014). We also consider the categories suggested by HADES to be a useful tool to analyse how mathematical experts read texts and solve problems.

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