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Paul Ernest *Editor*

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A Plea for a Critical Transformative Philosophy of Mathematics Education

Luis Radford

A disciplinary educational research field cannot, I think, avoid tackling general questions about the educational aims it pursues, as well as more specific questions concerning the teaching and learning of its contents, the nature of these contents and its methodology and theoretical foundations. Mathematics education is not an exception. These questions and their possible answers define a specific area of inquiry that has been termed the *philosophy of mathematics education*. Ernest (1991a, b, 2009), whose work has been influential in shaping this area of inquiry, suggests that the philosophy of mathematics education revolves around two axes. On the one hand, the philosophy of mathematics deals with the philosophical aspects of *research* in mathematics education. On the other, it deals with the *aims* of mathematics education (Ernest, this volume). Taking both axes together, the philosophy of mathematics education tackles questions such as our understanding of, and the meaning we attribute to, mathematics and its nature. It also includes questions about the purposes of teaching and learning mathematics, the meaning of learning and teaching mathematics and the relationship between mathematics and society.

The answers that we can offer to the previous questions go beyond mathematics itself. In order to tackle those questions, we need, indeed, to go beyond mathematics and step into new territory. We need to immerse ourselves in a series of theoretical domains like history, politics, ontology, metaphysics, aesthetics, epistemology, anthropology, ethics and critical philosophy (Ernest, this volume).

Consider for instance, the question about the relationship between mathematics and society. Since ancient times, what we call today “schooling” has been related to societal needs. The education of the scribes in Mesopotamia is a case in point. Mesopotamian scribes were instrumental in the organization and administration of the City (Høyrup, 2007). The mathematics that they learned and practised was influential in the measuring and distribution of lands, the collection of taxes, the calculation of the amount of food to be distributed to the soldiers, etc. One of the

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three oldest known problems goes back to *ca*. third millennium BC. It was found in 1975 by an Italian Archeological Mission while excavating the site of the Royal archives of the city of Ebla. The problem, contained in text TM.75.G.1392, is about the amount of cereal that is required to be distributed among a large number of individuals. In Fribergs' (1986) reconstruction, the problem reads as follows:

> Given that you have to count with 1 gu-bar for 33 persons, how much do you count with for 260,000 persons? (Friberg, 1986, p. 19).

The mathematics that the Babylonian produced and that scribes learned in school (what they called the "House of Tablets") was not a disinterested endeavour. It was related to the way the Babylonian administrative and political body sought to respond to societal needs.[1] It does not mean, however, that all Babylonian mathematics was about solving *practical problems*. This is the case, for instance, of many geometric problems at the basis of what has been called "Babylonian algebra." The first problem of a tablet known as AO 8862 that goes back to *ca*. 1750 BC reads:

1. Length, width. Length and width I have made hold:
2. A surface have I built.
3. I turned around (it). As much as length over width
4. Went beyond,
5. To inside the surface I have appended:
6. 3` 3. I turned back. Length and width
7. I have accumulated: 27. Length, width, and surface w[h]at? (Høyrup, 2002, p. 164)[2]

Without being "applied problems," many geometric problems like this, formulated as a kind of riddle, evoke the sensible actions of walking around a field while measuring parts of the field and operating on those measures. More than being simply inspired by surveying practices, problems such as the above, and Babylonian mathematics at large, convey ideas, values, interests and needs of the society from which they emerged (Nemet-Nejat, 1993). The same could be said of the mathematics of other historical periods. For example, the mathematics produced by the masters of Abacus in the Renaissance responded to problems that arose with the emergence of Western capitalism (see Swetz, 1989).

A general formulation of what these examples offer is that the mathematics that is produced and imagined in a particular historical and cultural context is related to the ideas, values, interests and needs of mathematics' cultural-historical context. In other terms, mathematics always *refracts* ideas, values, interests and needs of the

[1]Kramer (1949) and Lucas (1979) present a portrait of the scribal education. A more recent account can be found in Robson and Stedall's 2009 book. For an overview of the Sumerian administration structure, see Diakonoff (1974).

[2]Rephrased, the text talks about a (rectangular) surface built out of a length L and a width W to which the difference $(L - W)$ is added. The result is (in the Babylonian sexagesimal system) 3`3. The text also tells us that $L + W = 27$ (see Høyrup, 2010, p. 25).

society from which it emerges. It is in this sense that mathematics can be said to always be *ideological* (that is, not as something that conveys a false portrait of culture's reality, but as something that *embodies* the ideals and tensions of its own sociocultural context). It is in this sense that mathematics in general is not, as I claimed above when discussing Babylonian mathematics, a disinterested endeavour. It has never been so—not even in Plato's Academy, where mathematics, as opposed to the sensuous and kinesthetic Babylonian mathematics, was conceived as unrelated to practical matters. To conceive of mathematics as unrelated to practical matters is already the result of an ideological posture.

Plato's ideas about philosophy in general and mathematics in particular arose and evolved during the turmoil of the Peloponnesian War between Athens and Sparta and the post-war oligarchic Athenian regime established by Sparta. Just before the war Athens experienced a population growth. Athenians of the time saw the rise of commerce, and the emergence of new social classes, leading to a social restructuring where the old values of the aristocratic elite were shaken. The concept of the "good," related to manliness and good birth, which had been progressively elaborated since Homer's time, was challenged by the new context shaped by the arrival of "[r]ootless foreigners in their origins; skeptical, nominalistic, subjectivistic, and relativistic thinkers . . . [who] had no axioms, no epistemic certainties, no fixed axes of value, no ancestral pieties" (Levi, 1974, p. 61). "We can be certain," states Beavers (n.d.) in his biography of Plato, that the Peloponnesian War, "the establishment of a government by Sparta (after the chaos of Athens' final defeat in 404), and the events that followed, dramatically affected the direction of [Plato's] thinking."

Plato grew up in an aristocratic family. His "father's lineage went back to the first kings of Athens" (Levi, 1974, p. 58). Because of his aristocratic ancestry, he was destined to become a member of Athens' ruling class. His path, however, was interrupted by the Peloponnesian War and the subsequent course of events, which led to the decline of the Athenian empire and the Athenian aristocracy's loss of political power. Greatly affected by the execution of Socrates, Plato turned to other endeavours and travelled for several years, seeking comfort in philosophy. It is in this historical and political context that Plato fought for the restoration of the Greek world ruled by a "cultured elite" (Levi, p. 58) and that, during his return to Athens in *ca*. 387, he founded the Academia—"to instruct a new generation to become the legislators and the aristocratic statesmen of a future world" (p. 60). It is against this historical-political backdrop and the aristocratic outlook that opposed epistemological relativism and despised social and political change, practical labour, commerce and all mundane activities, that Plato came to formulate his philosophy of permanent Forms and the ensuing idea of truth as something immutable, perfect and timeless. Truth was conceived of as something that was accessible not through practical labour with artefacts but through "λόγος" (*logos*), the reasoned discursive activity of cultured citizens whose aim was to rise to higher levels of knowledge.

In Plato's view, mathematics was not about calculations or using mechanical instruments (Radford, 2003, 2008). Plutarch reports how Plato got offended when he learned that Eudoxus and Architas were resorting to mechanical instruments in their geometric inquiries. Plutarch says:

> But Plato took offense and contended with them that they were destroying and corrupting the good of geometry, so that it was slipping away from incorporeal and intelligible things towards perceptible ones and beyond this was using bodies requiring much wearisome manufacture. (Plutarch, Lives: Marcellus, xiv; quoted by Knorr, 1986, p. 3)

In Plato's conception, the forms of mathematics (the mathematical objects) have delimiting boundaries that make it possible to clearly distinguish one form from another (e.g. a triangle can be distinguished from a square with certainty; by contrast the boundaries separating courage from cowardice are not necessarily clear). In addition to this boundary feature of its objects, in mathematics, through *reason*, "we gain access to [a] purely intelligible, formal stable entity" (Roochnik, 1994, p. 559). This is why, in Plato's view, mathematics offers a paradigmatic model of clear and authoritative knowledge, where one can "shift one's sights, away from the sensible towards the noetic" (p. 559), and that mathematics becomes invested with moral value: "The study of mathematics is good for turning [away from the sensible world] the souls of the future philosopher-kings" (p. 560).

It is in this discursive society, torn by the distinction between appearance (doxa) and truth, with its scorn of the material and the sensible, that speech and its social use took on an epistemological dimension that remained unthinkable to the Babylonians, the Mayas, the Inuits, the Azande, the Maori, etc.

As an expression of its society, mathematics appears as the refraction of the manner in which knowledge is ideologically expressed and power is exerted. However, the manners in which mathematics in general and the mathematics that we teach at school embody such an ideological refraction need to be spelled out in detail. It is here that I find the promises of a philosophy of mathematics education most welcome. In my view, a philosophy of mathematics education should not appear merely as another field of inquiry, but as an urgent endeavour. For if there has always been a relationship between mathematics and society, this relationship has taken a very particular turn during the period in which we are living. Our historical period can sadly be characterized as the unprecedented historical age of the most radical assault on schools and educational systems at large by the economic forces of society. No school system before has ever been engulfed in such a virulent manner by one of society's components. The school of today appears, indeed, as an appendix to political economy, defined by global capitalism. And it is against this background that curricular contents are determined and that expectations about students and teachers are set.

Referring to public education in the USA, Lavallee notes that

> "public" schools have not only had their educational practices and curriculum taken over by edu-businesses, but schools' hidden curricula have also been likewise infiltrated by capitalism . . . Like a colonial occupying force, the for-profit publishers, test makers, test-prep profiteers, tutoring companies, curriculum designers, and so on are determining what our children learn and how their futures (economic, ideologic, etc.) will be shaped – not the community and parents, not the teachers, and least of all not the students themselves

> (who should actually have the greatest say). One cannot deem an occupied territory a "public" space. (Lavallee, 2014, pp. 6–7)

The assault on education that Lavallee talks about is happening farther north too. A central document that defines the goals of education in Ontario is *Achieving Excellence: A Renewed Vision of Education*. In this ministerial document, which is the reference *par excellence* in our province and frames all the initiatives of our Ministry of Education, achievement is explained as "raising expectations for valuable, higher-order skills like critical thinking, communication, innovation, creativity, collaboration and entrepreneurship" (Ontario Ministry of Education, 2014, p. 3). Then, candidly, the document continues: "These are the attributes that employers have already told us they seek out among graduates" (p. 3). The term entrepreneur/entrepreneurship appears 10 times in this document of 19 pages—a very worrisome frequency! In the opening lines of the document, we are told that we have one of the best educational systems in the world. What is the evidence? It comes from "respected international organizations such as the Organisation for Economic Co-operation and Development (OECD), McKinsey & Company, and the National Center on Education and the Economy in the United States." They "have all applauded Ontario, our programs and our results" (Ontario Ministry of Education, 2014, p. 2). We are on the right track. We are developing the taskforce that capitalism requires to keep the machinery going on—the same machinery that produces as many commodities as inequalities.

A philosophy of mathematics education should, I think, denounce the current political trend that defines human existence in mere economic terms and that reduces education to the development of actions (competencies) that are necessary to maintain, expand and refine the current capitalist forms of production.

In my view, a philosophy of mathematics education is the space to investigate and to denounce what Ferreira de Oliveira calls "the ideology of the market;" that is, the "transformation of things, inanimate or alive, in passive elements of commercialization" (Ferreira de Oliveira, in Freire, 2016, p. 113). The ideology of the market, with its emphasis on competitiveness, reduces the human to a means; it reduces the student to human capital: an atom that is trained to jump later in the inclement machinery of supply and demand to produce, consume and reproduce. It perverts the basis of true human relations, leading to a model of alienated society that schools repeat again and again. Within this context,

> Nature, water, the air, the earth, the world, the planet, the universe, the human beings, and all other beings, their minds, their organs, their feelings, their sexuality, their beauty, their workforce, their knowledge, their existence, their homes and their lives, are considered as merchandise. (Ferreira de Oliveira, in Freire, 2016, p. 113)

The articles in this volume ask different questions and try to answer them through different perspectives. Some chapters move around philosophical matters about language, pedagogy and conceptions of mathematics. Other chapters interrogate our often taken for granted assumptions about teaching and learning, about the nature of mathematics, and the role that mathematics plays in society and in the shaping of teachers and students.

Platonism is featured in several papers. In his contribution to this volume, Skovsmose reminds us of the influential role played by Platonism in referential theories of meaning. Platonist referential theory of meaning, Skovsmose notes,

> provides the basis for logicism and for many attempts to construct mathematics on a foundation of logic. It also provided a basis for the whole New Math movement, establishing set theory not only as the logical but also as an educational foundation of mathematics. (Skovsmose, this volume)

Otte (this volume) distinguishes various forms of Platonism. In the discussion, he refers to the distinction between the object and its representations and the role of representations in our knowing of the object. There is an often-quoted passage in *The Republic* where Plato deals with this problem:

> And you will also be aware that they [the geometers] summon up the assistance of visible forms, and refer their discussion to them, although they're not thinking about these, but about the things these are images of. So their reasoning has in view the square itself, and the diagonal itself, not the diagonal they have drawn. And the same with other examples. (Plato, 2000, 510d, p. 217)

Otte (this volume) mentions an acquaintance of his for whom ideal objects (mathematical, musical, etc.) appear in "the classical sense of a universe of eternal ideas," a conception that has been largely considered as "a foundational conception of pure mathematics." Otte writes:

> Once we had a colleague at our mathematics department at the University of Bonn, who would not listen to music, but would read it from the partiture [score]. He did not visit music performances because he thought music becomes distorted by playing it.

In this Platonic view about ideal objects, a human intervention would ruin the purity of the object. Kant held a similar, although not exactly equal, position: since all knowledge starts with our senses, or as Kant puts it, in our capacity to be affected by the representation of the objects (Kant 1781/2003, p. 93; A51/75), what we come to know of the ideal object is not the object itself but what results from the mediation of our senses (Radford, Arzarello, Edwards, & Sabena, 2017). As a result, we cannot know the object *itself*, but only its *appearance*. Consider the drops of rain that you feel when it suddenly starts raining and you hurry to find some shelter. These drops are appearances, objects of the phenomenological experience you are undergoing, not drops of rain as ideal objects. What we get to know is precisely *that*: the drops of rain that we feel over our body, not the transcendental object. Kant says:

> We then realise that not only are the drops of rain mere appearances, but that even their round shape, nay even the space in which they fall, are nothing in themselves, but merely modifications or fundamental forms of our sensible intuition, and that the transcendental object remains unknown to us. (Kant 1781/2003, p. 85; A46/63)

This is in a nutshell the argument behind Kant's epistemological relativism. In the case of Otte's acquaintance, the problem is not the impossibility of the human accessibility to the ideal object, but the fact that its representation (here the musical performance of the orchestra) seems to end up representing something else—a distorted version of the musical work.

Without a doubt, Platonism has had a privileged seat at the table of the mathematicians. The mathematicians' ontological position that attributes to the ideal objects an existence independent of human labour certainly has consequences in the manner in which research is conducted. It is not the same to assume that you create something as to assume that you are discovering it. The French 2010 Fields medallist Cédric Villani put this question as follows:

> Of course, philosophical thinking can influence the way in which research is done in mathematics, in the sense that if one is persuaded that there is something intrinsic to discover, one will not look in the same way as if one is persuaded that it is a human movement of construction. We will not have the same reflexes, not the same tension. (Villani, in Cartier, Dhombres, Heinzmann, & Villani, 2012, p. 60)

And as many mathematicians (Bernays, 1935), Villani recognizes himself as one of those that adopt a pre-existing harmony that is already waiting to be discovered:

> I am one of those who believe that there is a pre-existing harmony and that, on a given problem, will seek the nugget, persuaded that it exists. I am one of those who seek the miracle, not of those who will create it or seek something very clever in their own resources. (Villani, in Cartier et al., 2012, p. 60)

A number of papers in this volume deal with another range of questions identified by Ernest in his overview of the philosophy of mathematics education. These questions have more to do with the relationship between mathematics and society. Andrade-Molina, Valero and Ravn's contribution examines the role of mathematics education in producing children of a certain kind: rational and logical children. Their inquiry features Euclidean Geometry as a model of inquiry that, historically speaking, grew up entangled with a worldview that provides explanations about the natural world. Mathematics loses here its innocence. Rather than being beyond the vicissitudes of cultures, mathematics, as well as its teaching and learning, unavoidably refract a conceptual view of the human world that is political through and through. As Walshaw notes in her chapter, "Objectively derived and propositionally formulated, it [i.e., school mathematics, although this is even truer of mathematics itself—LR] is constructed from the experiences of a privileged group of people." What is specific to our contemporary world that incessantly produces and reproduces inequalities through its own economic machinery, is the fact that, theoretically, it aspires to erase the same inequalities that it produces through its own individualist conception of democracy. Hunted by its own contradictions, global capitalist societies (and those that without being such are affected by them) imagine that the solution to the riddle of inequality is to be found in the achievement of an impossible equity and the dream that the mathematics that has been constructed from the privileged groups are "paradigmatic for all" (Walshaw). A critical philosophical attitude helps us understand that the uncertain solutions offered by neoliberal political benevolent and naïve discourses that tackle the question of diversity through the conflation of equity with equality are doomed to fail. Such a conflation is

> based on the understanding that full opportunities to learn within the classroom and respectful exchanges of ideas about mathematics between a teacher and her students' outcomes, yield a comprehensive picture of equitable mathematical access for students, irrespective of any social determinations. (Walshaw; this volume)

Underpinned by a utilitarian logic, this conflation of equity with equality assumes that it is possible to erase the social, cultural and historical pillars of human existence through an equalitarian repartition of positions and possibilities in the social web of a competitive market.

The ideological substrate of mathematics and its teaching and learning—one of its features being the one discussed by Walshaw—is a topic that appears in various chapters (see, e.g. Schürmann's contribution to this volume). One of the questions that surfaces in this regard is the one concerning the conditions for the emergence of genuine critical thinking (e.g. Barwell, this volume). Another question revolves around the possibility to move beyond the oppressing and alienating framework circumscribing most of the current practice of mathematics teaching and learning. Seeking some alternatives, Walshaw (this volume) turns to Foucault's idea of *governmentality*. Through this concept, she sees a possibility for us to come up with "an interpretation of individual experiences in which domination and resistance are no longer conceived of as ontologically different but as opposing effects of the same power relations."

The previous brief overview of some of the problems that haunt mathematics education and mathematics education research makes clear, it seems to me, the need for an urgent space of critical reflection that can be filled by a philosophy of mathematics education. Ernest formulates a possible role for such a philosophy as the endeavour directed "to analyse, question, challenge, and critique the claims of mathematics education practice, policy and research" (Ernest, this volume). Yet, I would like to contend that we must go one step further and *act*, *take action*, so that our analyses, questions and critiques come to make, through concerted movements, a *transformation* of mathematics education as it is practised today. This is why a philosophy of mathematics education today appears to me as the space in whose interior an encompassing struggle against the reduction of education in general, and mathematics education in particular, to a technical consumerist view can be organized and deployed. It is in this sense that a philosophy of mathematics education appears as a land of hope—the hope to understand, criticize and *transform* the aims of mathematics education and its concrete practice. This is why I would like to submit that what we need is a *critical and transformative* philosophy of mathematics education.

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Part I
Introduction to the Field

The Philosophy of Mathematics Education: An Overview

Paul Ernest

Abstract This chapter offers an overview of the philosophy of mathematics education. This sub-field is characterised in both narrow and broad terms, concerning the aims of mathematics education and all philosophical aspects of research in mathematics education, respectively. The sub-field is also explored in terms of its questions and practices, which can be called a bottom-up perspective, as well as in terms of the applications of branches of philosophy to mathematics education, which might be called a top-down perspective. From the bottom-up one can characterize the area in terms of questions, and I have asked: What are the aims and purposes of teaching and learning mathematics? What is mathematics? How does mathematics relate to society? What is learning mathematics? What is mathematics teaching? What is the status of mathematics education as knowledge field? In characterizing the sub-field from a 'top down' perspective I look briefly at the contributions of ontology and metaphysics, aesthetics, epistemology and learning theory, social philosophy, ethics, and the research methodology of mathematics education. This reveals both how rich and deep the contributions of philosophy are to the theoretical foundations of our field of study. But even these different approaches leave many questions unanswered. For example: what are the responsibilities of mathematics and what is the responsibility of our own subfield, the philosophy of mathematics education? I conclude that the role of the philosophy of mathematics education is to analyse, question, challenge, and critique the claims of mathematics education practice, policy and research.

Keywords Philosophy of mathematics education · Mathematics education
Aims · Research methodology · Philosophy · Epistemology

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Introduction: What Is the Philosophy of Mathematics Education?

In the past quarter century the philosophy of mathematics education has emerged as a loosely defined area of research, primarily concerned with the philosophical aspects of mathematics education. This chapter aims to briefly map out some of its terrain, and attempt a synoptic vision of the breadth and depth of this area. This will prepare the ground for the more detailed and specific enquiries in the chapters that follow. This task is made all the more urgent as the question of what constitutes the philosophy of mathematics education is not without ambiguity and multiple answers. For example, is *the* philosophy of mathematics education a particular, specific approach to mathematics teaching and learning or to mathematics education research? The definite article might be taken to imply a claim for the definitiveness of the account, namely, that a unique philosophy, rationale or direction is proposed. This is not what is intended here, and 'the' is meant to indicate a definite area of enquiry, a specific domain, not a single fixed ideological perspective. Thus, *the* philosophy of mathematics education is not a dominant interpretation but rather a particular, if not completely defined, area of study and investigation, a sub-specialism within mathematics education.

Understood in its simplest sense mathematics education is about the practice or activity of teaching mathematics. The philosophy of some activity or domain is its aim, rationale or underlying purpose. So the simplest sense of 'philosophy of mathematics education' concerns the aim or rationale behind the practice of teaching mathematics. This issue is a vitally important one, central to the philosophy of mathematics education, as well as to mathematics education as a whole. The purpose of teaching mathematics also implicates the aim of learning mathematics, because learning is inseparable from teaching, although they can be conceived of separately. In practice an active teacher presupposes one or more learners, and only in pathological situations can one have teaching without learning. However, the converse does not hold, for informal learning is often self-directed and takes place without explicit teaching.

Further to this, the aims, goals, and rationales for teaching and learning mathematics do not exist in a vacuum. They belong to people, both to individuals and to social groups (Ernest, 1991). Since the teaching of mathematics is a widespread and highly organised social activity, its aims and goals need to be related to social groups, social interests and to society in general, acknowledging all the while that there are multiple and divergent aims and goals among different groups and people. Aims are expressions of values and so the educational and social values of society, or some sections of it, are implicated in this enquiry. Further, the aims discussed here are for the teaching of *mathematics*, so the aims and values implicated centrally concern mathematics and its role and purposes in education and society.

Thus a first consideration of the meaning of the philosophy of mathematics education raises the issues of the teaching and learning of mathematics, their underlying aims and rationales, the roles of the teacher, learner, and of mathematics in society, as well as the underlying values of the relevant social groups. To a great extent this mirrors the issues arising from applying the curriculum theorist Schwab's (1961) *four commonplaces of teaching* to mathematics. His commonplaces, the basics of the curriculum, are the subject (mathematics), the learner, the teacher, and the milieu of teaching, including the relationship of teaching and learning, and its aims, to society in general.

Philosophy is a substantial discipline and so a broader interpretation of what the philosophy of mathematics education might be brings further issues to the fore from a philosophical perspective. We can apply philosophy to mathematics education as well as exploring the specific implications of applying the philosophy of mathematics or philosophy of education (Brown, 1995). Philosophy is about systematic analysis and the critical examination of fundamental problems. It involves the exercise of the mind and intellect, including thinking, analysis, enquiry, reasoning and its results: judgements, conclusions, beliefs and knowledge. There are many ways in which such processes as well as the substantive theories, concepts and results of past enquiry in philosophy can be applied to and within mathematics education. So the philosophy of mathematics education should also be understood to include the application of philosophical concepts and methods, such as a critical attitude to claims as well as detailed conceptual analyses of the concepts, theories, methodologies or results of mathematics education research, and mathematics itself (Ernest, 1998; Skovsmose, 1994).

But introducing philosophy in its broader sense raises questions. Why does philosophy matter? Why does theory in general matter? How do they enhance mathematics education research? These questions need addressing at some point in justification of the furtherance of the philosophy of mathematics education, and I offer some provisional answers here. First of all, these philosophical domains and approaches help to structure research and inquiries in an intelligent and well-grounded way, offering a secure basis for knowledge. They provide an overall structure for accommodating the results of cutting edge research within the hard-won body of accepted knowledge. Second, secure philosophical foundations ensure the 'hygiene' of research providing sound warrants and justification for results and claims. But thirdly, they enable people to see beyond the widely accepted but unexamined presuppositions and official stories about the world, and about society, economics, education, mathematics, teaching and learning. They provide thinking tools for questioning the status quo, for seeing that 'what is' is not 'what has to be'; to see that the boundaries between the possible and impossible are not always where the 'received wisdom' says they are. It enables commonly accepted notions to be probed, questioned and many implicit assumptions, ideological distortions or unintended prejudices to be revealed and challenged. Fourthly, and most importantly, they enable us to imagine alternatives. Just as literature can allow us to stand in other people's shoes and see the world through their eyes and imaginations, so too philosophy and theory can give persons new 'pairs of glasses' through which to see the world and its institutional practices anew, including the practices of teaching and learning mathematics, as well as those of research in mathematics education. Thus philosophy enables us to 'question the unquestionable', including the problematising of some of the sacred Shibboleths of mathematics education, as I exemplify below.

Mathematics education is a field straddling research and public policy concerning education, and it is not without heated public controversies concerning the practices of teaching mathematics. Thus a central role for the philosophy of mathematics education is to examine such controversies dispassionately, to analyse the concepts and identify the underlying values in order to illuminate, clarify and if possible resolve the conflicts. In the sequel a number of such outstanding controversies are identified as areas where the philosophy of mathematics education has been or can be fruitfully applied.

At the very least, this analysis suggests that the philosophy of mathematics education must attend not only to the aims and purposes of the teaching and learning of mathematics and the philosophy of mathematics and its implications for educational practice. It suggests that we must look more widely for philosophical and theoretical tools for understanding all aspects of the teaching and learning of mathematics and its milieu. At the very least we need to look to the philosophy of Schwab's (1961) other commonplaces of teaching: the learner, the teacher, and the milieu or society. So we also have as areas of interest the philosophy of learning (learning mathematics in particular), the philosophy of teaching (mathematics) and the philosophy of the milieu or society (in the first instance with respect to mathematics and mathematics education) as further elements to examine. We should also consider the discipline of mathematics education as a knowledge field in itself, to establish the epistemological status of the reasoning, analyses and answers given, even when they are claimed to be no more than provisional.

Looking at each of these four commonplaces in turn, plus mathematics education as a knowledge field itself, a number of questions can be posed as issues for the philosophy of mathematics education to address. In addition, each of the topic areas designated has associated controversies needing attention and clarification, which I sketch below.

Question 1: What Is Mathematics? (The Basic Question of the Philosophy of Mathematics)

What is mathematics, and how can its unique characteristics be accommodated within a philosophy? Can mathematics be accounted for both as a body of knowledge and a social domain of enquiry? Does this lead to tensions? What philosophies of mathematics have been developed? What features of mathematics do they pick out as significant? What is their significance for and impact on the teaching and learning of mathematics? What is the significance of the emergent movements on the social construction of mathematics and on the philosophy of mathematical practice for education, if any? What is the rationale for picking out certain limited elements of mathematics for schooling? Given that most school mathematics content is centuries old is there a place for recently developed mathematical concepts and theories in education? How can and should mathematics be conceptualised and transformed for educational purposes? What educational and social values and goals are involved? Is mathematics itself value-laden or value-free? How do mathematicians work and create new mathematical knowledge? What are the methods, values and aesthetics of mathematicians? Do students need to be exposed to such working methods, or are they just for a tiny specialist grouping, if for anyone at all in education? How does the history of mathematics relate to the philosophy of mathematics, and indeed to current day mathematics itself? Is mathematics changing as new methods and information and communication technologies emerge? Is there a place for the history and philosophy of mathematics within its teaching in schools? Or should mathematics simply be taught as a set of instruments and tools for solving abstract problems? What is the difference between research mathematics and school mathematics? In school mathematics, what balance between calculation, proving and modelling is appropriate for students at different levels? What should be the role of the traditional topics of algebra, geometry, and arithmetic in the contemporary curriculum, and why? How important should probability, statistics, and computing be in school mathematics and why?

Controversies 1 There is a 'hot' controversy in the philosophy of mathematics, between traditionalist mathematicians and philosophers versus fallibilist and social constructivist philosophers of mathematics. The former claim that mathematics is certain, cumulative and untouched by social interests. The latter argue that mathematics is fundamentally social, with cultural limitations to its claims of certainty, universality and absoluteness, and is as much subject to conceptual revolutions as is science. This controversy is part of what has been termed the 'Science Wars' between realists and constructivists. It is an age old conflict going back to the Ancient Greek arguments between the dogmatists and sceptics (Lakatos, 1962), and is unlikely to be resolved in the immediate future.

Another related controversy arises over the question of whether mathematics is value-laden or value-free. For example, is mathematics ethically neutral or does it bear ethical responsibility for its role and uses in society and education? (Ernest, 2016a). If mathematics is ethics and value-laden where do its responsibilities lie and where do they end? How should this be reflected in education? This controversy is a heated one, because to many mathematicians and others simply asking the question of whether pure mathematics is value-laden or has any ethical dimensions or responsibilities appears to be an oxymoron, that is self-contradictory and hence absurd.

Question 2. How Does Mathematics Relate to Society? (The Philosophy of the Milieu)

How do mathematics and mathematics education relate to society? What are the aims of mathematics education, i.e., the aims of teaching mathematics? Are these aims valid? Whose aims are they? For whom? Based on what values? Who gains and who loses in the process? Should the goals of learning mathematics be the same for everybody, or should different groups of students have different goals? How are or should these grouping and associated goals be chosen? By whom and with what criteria? How do social, cultural and historical contexts relate to mathematics, the aims of teaching, and the practices of teaching and learning mathematics? What values underpin different sets of aims? How does mathematics contribute to the overall goods and goals of society and education? What is the role of the teaching and learning of mathematics in promoting or hindering social justice conceived in terms of gender, race, class, (dis)ability and critical citizenship? What might feminist and/or anti-racist mathematics education mean and are they possible? What might be their implications for the teaching and learning of mathematics? How is mathematics viewed by the public and perceived in different sectors of society? What impact does this have on education? What is the relationship between mathematics and society? What functions does it perform? Which of these functions are intended and visible? Which functions are unintended or invisible? To what extent do mathematical metaphors, such overall quantification, the 'measurement' of qualities, the profit and loss balance sheet, and the spreadsheet permeate social thinking? What is their social and philosophical significance? To whom is mathematics accountable? Is mathematics an untrammelled good for society? In addition to the benefits mathematics brings to society through its applications in science, technology, business, and so on, are there costs and collateral damage brought about by its universal prioritisation in education? Can the teaching and learning of mathematics be reformed or supplemented to compensate for or ameliorate any such damage caused? (See Ernest, 2018, this volume).

Controversies 2 There have been heated controversies over the aims of teaching mathematics for a long time. This was manifested in the conflict between groups with different aims, values, epistemologies in UK National Curriculum of the 1980s and 1990s. The UK conflict has been analysed as involving five different competing groups. These are

1. Industrial Trainers, with authoritarian, back-to-basics aims;
2. Technological Pragmatists, with industry-centred aims;
3. The Old Humanists, with pure mathematics-centred aims;
4. The Progressive Educators, with learner-centred aims; and
5. The Public Educators, with social justice-centred aims (Ernest, 1991).

In the UK case, groups 1, 2 & 3 dominated the formulation of the National Curriculum in the 1980s, with lip service paid to group 4 aims and with group 5 aims completely ignored. In subsequent developments even this lip service to group 4 aims has been expunged. To what extent has this analysis been borne out? Is it still relevant today?

Conflict over the aims of teaching mathematics has also been manifested more widely. For example, in the USA a conflict over the aims of teaching mathematics together with its pedagogy, termed the Math Wars, erupted between traditional mathematicians and mathematics education specialists in the 1990s and 2000s (Ernest, 2014). Are such conflicts inevitable, or are there compromises and resolutions possible? To what extent can the findings of research projects and international comparisons be used to evaluate the efficacy of the particular implementations of the aims?

Question 3: What is Learning and Learning Mathematics, in Particular? (The Philosophy of Learning)

What assumptions, possibly implicit, underpin views of learning mathematics? Are these assumptions valid? Which epistemologies and learning theories are assumed? What are the philosophical presuppositions of traditional reception, information processing, radical constructivist, social constructivist, enactivist, sociocultural and other theories of learning mathematics? How can the social context of learning be accommodated in what are often individualistically-oriented and traditionally cognitive learning theories? Does the adoption of different learning theories help or hinder the learning of mathematics? Do such theories have any differential impact on classroom practice, and if so what? Are there discernible differences between classroom practices based on the underlying theories of learning adopted? What elements of learning mathematics are most valuable? How can they be and should they be assessed? What feedback loops do different forms of assessment create, impacting on the processes of teaching and learning of mathematics? How strong is the analogy between the assessment of the learning of mathematics and the warranting of mathematical knowledge? What is the role of the learner? What powers of the learner are, could or should be developed by learning mathematics? How does the identity of the learner change and develop through learning mathematics? Does learning mathematics impact on the whole person for good or for ill? To what degree do such beneficial/deleterious outcomes occur, under what learning conditions and how do these relate to the cultural context? Does learning mathematics impact differentially on students according to social and individual differences and identities, and if so how? How is the future mathematician and the future citizen formed through learning mathematics? How important are affective dimensions including emotions, attitudes, beliefs and values in learning mathematics? What is mathematical ability and how can it be fostered? Is the learning of mathematics accessible to and for all? How do cultural artefacts and technologies, including information and communication technologies, support, shape and foster the learning of mathematics? To what extent should student experiences of learning mathematics mirror or model the practices of research mathematicians? Is the learning of mathematics hierarchical, progressive or cumulative, as traditional theories tell us, and if so, to what extent? To what extent mathematical learning transferrable to new and different situations?

Controversies 3 There have been heated controversies between proponents of different theories of the learning of mathematics for many years. Early in the post-World War 2 Era, these were between behaviourism and cognitive psychology. From the 1980s on these have primarily been between traditional cognitivist mathematics learning theories and radical constructivism, especially over epistemological issues such as the claim that all knowledge is constructed by the learner. As Ernst von Glasersfeld, perhaps the most prominent advocate of radical constructivism said "To introduce epistemological considerations into a discussion of education has always been dynamite" (1983, p. 41). More recently there has been further controversy between constructivism versus socio-cultural theories based on the issue of whether knowledge and learning are primarily individual or social. This latter controversy remains alive and unresolved, although scholars like Sfard (2008) have tried to bridge the gaps with novel theories like her 'commognition', that combines individual cognition with communicative dimensions. How successful is this and other learning theories? Are there measureable differences between the outcomes of different learning theories, as Boaler (2002) claims, in a contribution to the Math Wars?

Question 4. What is Teaching and Teaching Mathematics, in Particular? (Pedagogical Philosophy for Mathematics)

What theories and epistemologies underlie the teaching of mathematics? Are there any adequately articulated theories of teaching mathematics? Is the French 'Didactique' such a theory? Or the related Anthropological Theory of the Didactic? What assumptions, possibly implicit, do mathematics teaching approaches rest on? Are these assumptions valid? Do different philosophies of mathematics underpin different teaching approaches? Are there unique outcomes that stem from particular philosophies of mathematics applied in teaching? Or are additional assumptions, such as values, in addition to the assumption or adoption of particular philosophies or epistemologies, needed to make any discernible differences in teaching practices or learning outcomes? What means are adopted to achieve the aims of mathematics education? Are the ends and means consistent? Can we uncover and explore different ideologies of education and mathematics education and their impact on teaching mathematics? What methods, resources and techniques are, have been, and might be, used in the teaching of mathematics? Which of these have been helpful and under what circumstances and conditions? What theories underpin the use of different information and communication technologies in teaching mathematics? What sets of values do these technologies bring with them, both intended and unintended? Is there a philosophy of technology that enables us to understand the mediating roles of tools between humans and the physical and cultural worlds? What is it to know mathematics in a way that fulfils the aims of teaching mathematics? How can the teaching and learning of mathematics be evaluated and assessed? What is the role of the teacher? What range of roles is possible in the intermediary relation of the teacher between mathematics and the learner? What are the ethical, social and epistemological boundaries for the actions of the teacher? What mathematical knowledge, skills and processes does the teacher utilise or need? Are these specifiable or are they context dependent? What is the range of mathematics-related beliefs, attitudes and personal philosophies held by teachers? How do these attitudes, beliefs and personal philosophies impact on mathematics teaching practices? What disparities between espoused beliefs and enacted beliefs can be observed? How should mathematics teachers be educated? What is the difference between educating, training and developing mathematics teachers? What is (or should be) the role of research in mathematics education and teaching and the education of mathematics teachers? Is mathematical pedagogy primarily a design science concerned with optimizing the efficiency of teaching techniques? Or does mathematical pedagogy necessarily rest on its philosophical and epistemological foundations?

Controversies 4 Competing pedagogies for mathematics teaching have long been the site of contestation and recently the battle ground for the 'Math Wars'. Progressive educationists promoting child-centred teaching methods, including discovery learning, problem solving and investigational approaches, e.g., Boaler (2002), have been pitted against proponents of traditional teacher-centred instructional approaches, e.g., Stephan (2014). But is there any evidence that either of these positions is more effective in practice, and in which contexts? Klein (2007) argues that the alignment of the progressive versus traditionalist pedagogies dichotomy with left, liberal versus right wing, conservative politics, respectively, as most certainly happened in the United States of America only served to entrench opposition and took the focus away from the goals and outcomes of school mathematics.

Additional related controversies concern the preferred methods for teaching number and arithmetic. One question is whether arithmetic should be introduced to young children based on children's own spontaneous approaches and mental calculation methods, as opposed to teaching the standard written algorithms for arithmetic (the 'four rules'). Another question is whether electronic calculators should be introduced alongside or prior to the teaching of standard written algorithms, as opposed to afterwards, and indeed whether there should be less emphasis altogether on these algorithms? Strong opinions have been voiced on both sides of these controversies.

Lastly, there are controversies over the role of contexts in the teaching of mathematics. Are applied problems the best way to introduce, teach or reinforce mathematical concepts and methods? Can mathematics be effectively taught via ethnomathematical practices? Or should pure mathematics be taught before applying it, so that concepts and methods are mastered in their most powerful and abstract forms? All of these issues have generated a great deal of heat and conflict, especially in the media, but also in professional mathematics and mathematics education circles.

Question 5: What is the (Philosophical) Status of Mathematics Education as Knowledge Field?

What is the basis of mathematics education as a field of knowledge? Is mathematics education a discipline, a field of enquiry, an interdisciplinary area, a domain of extra-disciplinary applications, or something else? Is it a branch of applied mathematics or a special part of educational theory? Is it a science, social science, art or humanity, or none or all of these? (Kilpatrick, 2008). What is its relationship with other disciplines such as philosophy, mathematics, sociology, psychology, linguistics, anthropology, etc.? How do we come to know in mathematics education? What is the basis for knowledge claims in research in mathematics education? What research methods and methodologies are employed and what is their philosophical basis and status? How does the mathematics education research community judge knowledge claims? What standards are applied? How do these relate to the standards used in research more widely in education, or the social sciences, humanities, arts, mathematics, the physical sciences and applied sciences such as medicine, engineering and technology? Is mathematics education a design science, as some researchers have claimed, or does it address more fundamental research questions? What is the role and function of the researcher in mathematics education? Should we (such researchers) focus on technical aspects of improving the teaching and learning of mathematics, or are we also public intellectuals whose responsibilities include critiquing mathematics and society? What is the status of theories in mathematics education? Do we appropriate theories and concepts from other disciplines or 'grow our own'? Which is better? What impact on mathematics education have modern developments in philosophy had, including phenomenology, critical theory, post-structuralism, post-modernism, Hermeneutics, semiotics, linguistic philosophy, inferentialism, etc.? What is the impact of research in mathematics education on other disciplines? What do adjacent STEM (science, technology, engineering and mathematics) education subjects have in common, and how do they differ? (The same again for STEAM: science, technology, engineering, arts and mathematics.) Can they be unified or do they need to be taught separately? Can the philosophy of mathematics education have any impact on the practices of teaching and learning of mathematics, on research in mathematics education, or on other disciplines? What is the status of the philosophy of mathematics education itself? Is it a legitimate sub-field of mathematics education, or a loosely grouped family of cognate inquiries and methods? Does calling the philosophy of mathematics education a sub-field of mathematics education have legitimacy? How central is mathematics to research in mathematics education? Does educational research that does not draw on deep knowledge of mathematics have any right to proclaim itself as research in mathematics education? Does mathematics education have an adequate and suitable philosophy of technology in order to accommodate the deep issues raised by information and communication technologies? Are information and communication technologies merely a set of tools or do they have a deeper impact on the very nature of mathematics education itself, or on human knowing?

Controversies 5 Should mathematics education, as a university discipline, be accommodated within education departments or mathematics departments? Different countries answer this in different ways, and are not always fully consistent within themselves. This question matters, for location in a mathematics department within a scientific faculty often brings significantly better resourcing than housing in an education department, within a social science faculty. However, in some traditional mathematics departments mathematics specialists are looked down upon as not being ‘real mathematicians’, whereas in many education departments mathematics educationists are on a par with their education colleagues.

Another controversy in mathematics education research that has a bearing on funding issues concerns research methods and methodologies. The conflicts between proponents of the scientific research paradigm and those of the interpretative research paradigm have been termed the ‘paradigm wars’ (Gage, 1989). To this day research in the scientific paradigm using randomized controlled trial methodologies (that is a quantitative, experimental versus control group design) is regarded as the ‘gold standard’ of educational and social science research. This receives the lion’s share of resourcing and prestige, despite widespread critique of its applicability in educational research (Sullivan, 2011; Thomas, 2016). Is this legitimate? What is the status and validity of the critical research paradigm? (Ernest 1994; Habermas, 1972). Is this a threat to the other two research paradigms or does it complement them?

Overall, these five sets of questions encompass much of what it is important for the philosophy of mathematics education to consider and explore. As revealed by the various areas of overlap the sets of questions are not wholly discrete. Many of the questions are not essentially philosophical, in that they can also be addressed and explored in ways that foreground other disciplinary perspectives, such as sociology and psychology. Indeed the questions, when suitably refined, could form the bases for hundreds of PhDs in mainstream mathematics education. However, when such questions are approached philosophically, they become part of the business of the philosophy of mathematics education. Also, a move to exclude any of these questions right from the outset as not legitimate for research in mathematics education, without considering them, risks promoting a particular ideology or indeed a slanted philosophy of mathematics education. In addition, perhaps more so than philosophy, sociology or psychology, mathematics education is a multi- or inter-disciplinary area of study, so that it is perhaps the most appropriate and accessible area where the questions and sub-questions listed above can be explored together, from a philosophical perspective.

As the controversies associated with each of the five questions demonstrate, there are issues in mathematics education, often with a philosophical dimension, that remain hotly contested. The role of the philosophy of mathematics education is not primarily to adjudicate in these contests, but to analyse and disambiguate the concepts, questions and arguments involved to bring clarity to the debates, rather than to endorse one or other side of a dichotomy. In fact, in several of the cases cited a deeper conceptual analysis reveals that neither side is correct in its claims, but often a compromise, a dialectical synthesis or a third position is best supported by the available evidence.

An example of such a controversy is that between progressive and traditional pedagogies. Let me start by saying that progressive child-centred pedagogies conducted well are very likely more successful than traditional teacher-centred pedagogies done badly, both in terms of student learning outcomes and student satisfaction. But likewise, traditional teacher-centred pedagogies conducted well very likely have better outcomes than progressive child-centred pedagogies done badly. The key factor is not one of technique, or even ideology, but of effective application of teacher knowledge and teaching skill.

Concerning this controversy, a very interesting piece of research is reported in Askew, Brown, Rhodes, Wiliam, and Johnson (1997). This is based on a project that correlated primary student achievement gains in numeracy with their teacher's beliefs or philosophies of mathematics education. In the study three different belief set clusters emerged from the data as important in understanding the approaches teachers took towards the teaching of numeracy. These were termed the transmission, discovery and connectionist belief sets. Transmission beliefs were based around the primacy of teaching and a view of mathematics as a collection of separate routines and procedures. This corresponds to a greater or lesser extent with a traditional teacher-centred pedagogy. Discovery beliefs clustered around the primacy of learning and a view of mathematics as being discovered by pupils. This corresponds to a progressive child-centred pedagogy (constructivism and discovery learning). Connectionist beliefs were based around both valuing pupils' methods but also teaching strategies with an emphasis on establishing connections within mathematics. Although this is partly child-centred it is also traditionally mathematics-centred and thus does not fall neatly one side or the other of the progressive-traditional divide. Over the two terms of the study neither the classes of teachers with strong discovery or transmission orientations made the greatest achievement gains. Instead the teachers with a strong connectionist orientation were more likely to have classes that made greater gains. Thus, in this study, the discovery/reception learning or progressive/traditional teaching dichotomies were not helpful in identifying the most successful pedagogical styles. Rather it was a different mathematical and pedagogical orientation beyond the traditional dichotomies that was the most successful.

Although this is only a single study and it has been reported here in an extremely simplified manner, it nevertheless represents a significant initial finding with respect to this particular controversy. The relationship between teacher beliefs and learning outcomes, as in this study, has long been a focus in philosophy of mathematics education theory and research (Ernest, 1989). By conceptualizing research in terms of the concepts and relationships foregrounded in the philosophy of mathematics education, as in this case, there is the potential to clarify controversies in mathematics education and to take the heat out of them. This success is based on putting empirical findings and successful classroom practices (while acknowledging that this phrase raises a host of problematic philosophical issues) before ideological commitments.

Applying Philosophy to Mathematics Education

The questions listed above interrogate and problematise the practices of teaching and learning mathematics and related issues from a low- or non-theoretical perspective. Starting from questions in this way represents a 'bottom up' introduction to the scope and nature of the philosophy of mathematics education, Simply speaking, this is putting practice before theory. In contrast, a 'top down' approach can use the branches of philosophy to provide conceptual frameworks for analysing philosophical concerns in research in mathematics education. In what follows, research and theories in mathematics education are analysed according to the branches of philosophy they draw upon, including metaphysics and ontology, epistemology, social and political philosophy, ethics, methodology, and aesthetics. Ontology and metaphysics have as yet been little applied in mathematics education research (Ernest, 2012). Work drawing on aesthetics is still in its infancy (Ernest, 2015a, 2016a; Sinclair, 2008). However extensive uses of epistemology and learning theory, social and political philosophy, ethics and methodology can be found in mathematics education research and literature.

Ontology and Metaphysics

Ontology is that part of metaphysics that studies the nature and conditions of existence and being in itself. Although as yet little applied in mathematics education research ontology raises two immediate problem areas including first the nature of mathematical objects and second the nature of human being (Ernest, 2012).[1] Platonism, which concerns the first of these issues, has been a dominant philosophy of mathematics for over two thousand years. It is the view that mathematical objects exist independently of the physical world in some ideal realm. However, there have been longstanding disputes in this area, historically between Platonists and realists versus conceptualists and nominalists. Although sociologists and social constructivists have challenged Platonism it is only recently that mainstream philosophy has

[1] I use the term 'nature' here and elsewhere without presupposing essentialism in being.

countenanced the idea that there is a fully existent social reality (Searle 1995) and that mathematical objects are part of this social reality rather than some other reality (Cole, 2013; Hersh, 1997). Such thinking will doubtless also have consequences for the philosophy of technology and the status of the virtual realities brought into being by information and communication technologies, as well as for the philosophy of mathematics. All I will signal here is that this is a controversial but burgeoning area of inquiry of potentially great significance for our field. For it is largely through the teaching and learning of mathematics that learners meet, develop relationships with, and come to believe in the reality of mathematical objects and the certainty of mathematical knowledge (Ernest, 2015b).

The nature of human being is another deep question that has implications for the teaching and learning of mathematics. What is the deep nature, the non-essential but enduring character of learners, teachers and persons as beings in general, which is subsequently presupposed by teaching, learning and research in mathematics education? For all areas of humanistic knowledge depend on and presuppose implicit or explicit models of what it is to be a human being, with all the agency, capabilities and rights this entails. Of course such concerns have immediate ethical consequences, but what do we add to mathematics education research by focussing on and clarifying these deep ontological issues? What new researchable projects are suggested and brought within our reach? As yet these are open questions, but the past decade or two has seen a growth in research on learner and teacher identities which begins to touch on this area (Grootenboer, Smith, & Lowrie, 2006; Lerman, 2005).

Aesthetics

Relevant work drawing on aesthetics is still in its infancy but is growing (Ernest, 2015a, 2016a; Inglis & Aberdein, 2016; Sinclair, 2008). Aesthetics has been associated with mathematics since the time of Plato, but what does the theoretical focus on aesthetics in research in mathematics education add beyond letting learners experience some of the beauty of mathematics? It is a commonly remarked that there are beautiful proofs, equations and theories in mathematics. But why are there such divergences of opinion between those who exalt the sublime beauty of mathematics and those who fail to see any beauty at all in mathematics? Are these differences of opinion intrinsic or are they down to the unique learning trajectories of some individuals? What can a focus on beauty in mathematics and its teaching and learning add to research and classroom teaching? Since the experience of beauty is usually associated with interest, admiration and other positive attitudes, can these be harnessed to improve learning experiences and overall engagement with mathematics?

Epistemology

Epistemology concerns theories of knowledge and can be taken to include both the nature of mathematical knowledge, including most centrally its means of verification, and knowing, including the processes of coming to know and learning. Thus some of the questions posed above: 'What is mathematics or mathematical knowledge?' and 'What is learning mathematics?' fall under this heading. There is a literature exploring the relationships between epistemologies of mathematics and mathematics education (Ernest, 1994, 1998, 1999; Sierpinska & Lerman, 1997). This literature provides frameworks for examining some of the main epistemological questions concerning truth, meaning and certainty, and the different ways they can be interpreted for our field. It surveys a range of epistemologies and epistemological issues including the contexts of justification and discovery, foundational and non-foundational perspectives on mathematics; critical, genetic, social and cultural epistemologies, and epistemologies of meaning. A recent stimulus for research in mathematics education has been the application of inferentialism, which rejects representational theories of meaning. Instead it interprets the meaning of a word or concept as the holistic web of reasonings that connects it to other words and concepts (Brandom, 2000; Derry, 2017).

Looking within mathematics education a number of epistemological controversies can be mapped out including the subjective-objective character of mathematical knowledge; the role in cognition of social and cultural context; the transfer of knowledge and the transfer of learning from one social context to another; relations between language and knowledge; and tensions between the major tenets of constructivism, socio-cultural views, interactionism and French Didactique, from an epistemological perspective. Relationships between epistemology and a theory of instruction, especially in regard to didactic principles, can also considered, thus addressing the question 'What is teaching in mathematics?', since teaching is the deliberate attempt to direct and foster learning.

Work by sociologists on epistemology and the sociology of knowledge, including that of Bloor (1991) and Bernstein (1999) have impacted on our field through foregrounding sociological theories of knowledge. Even more radical impacts stem from the post-structuralism of Foucault (1980) and others, and the post-modernism of Lyotard (1984) and Derrida (1978). However, the impact of their theories cannot be confined solely to epistemology since they question and critique the traditional divisions of philosophy and knowledge. These theories and accounts serve to destabilize traditional conceptions of the fixity of knowledge and the definiteness of concepts. Consequently, there is a growing body of literature and theory that applies the insights of these recent social theories, if I can term them that, to mathematics education research (Hossain, Mendick, & Adler, 2013; Llewellyn, 2010).

Learning Theory

Although the natural home of learning theory is in the domain of psychology, much has been made of the epistemological assumptions and implications of learning theories within mathematics education research. Many tyro researchers in our field cut their philosophical teeth on the controversy over radical constructivism. The heated public debates at Psychology of Mathematics Education (PME) Conference no. 7 in Montreal in 1987 between Ernst von Glasersfeld, Jeremy Kilpatrick, David Wheeler and others foregrounded these issues for the international mathematics education research community. Striking and important philosophical differences can be found between the leading learning theories in our field. Although the controversy has calmed down since those first heady days it remains understood that there are major differences in the philosophical presuppositions of information-processing, constructivist, social constructivist, enactivist, and sociocultural theories of learning mathematics. These are primarily epistemological differences, although proponents and critics of the various theories also bring ontological, ethical, social and methodological analyses and reasoning into their arguments.

Social and Political Philosophy

Social and Political Philosophy is harder to pin down than some of the other branches of philosophy since the emergence of sociology, for the latter has contested and colonised some of its terrain. But there is a long and honourable tradition of political and social philosophy going back to Plato's Republic. In it Plato suggests how a society might best be organised on philosophical lines. In addition Plato also enunciates what might be termed the first philosophy of mathematics education. He argues that the learning of mathematics not only prepares philosophers to be future rulers, and provides important practical knowledge for builders, traders and soldiers, but more importantly also introduces its students to truth, the art of reasoning, and also to the key ideas of ethics. Such knowledge, he argues, is necessary at all levels in society, especially the top. As is well known, his academy, possibly the first university in history, and certainly one of the longest enduring, required that all who enter be versed in geometry.

The political philosopher *par excellence* of modern times is Karl Marx. His social and political analysis is primarily based on a critique of the economic structure of society and the role of capital. However, there is a strong ethical dimension to his work because his critique focuses on the exploitation of one social class by another and his outrage at this is palpable. Several schools of philosophy have built on Marx's insights including the Frankfurt School of critical theory, post-structuralist philosophy including Foucault (1980), Pierre Bourdieu's social theory (e.g., Bourdieu & Passeron, 1977), and few modern continental philosophers have been untouched by his ideas, from Hegel's successors to the present. All of these are extensively used in mathematics education research. All of the movements mentioned are continental (primarily French and German) but the most widely cited non-continental social thinker in mathematics education research, Basil Bernstein, does not base his work on Marx. All of these named scholars or movements, whether primarily social or philosophical, have been used to make important philosophical contributions within mathematics education research.

Some of the other contributions of social and political theorising in mathematics education research have been critiques of individualistic conceptions of learning, persons and knowledge and some use of the social construct of ‘identity’ as a unit of analysis in researching and teaching mathematics (Lerman, 2005).

Ethics

Ethics enters into mathematics education research in a number of ways including a concern with values, with social justice and equity approaches, and through the ethics of research methodology. Several authors have argued that despite its traditional value-free absolutist image mathematics is value laden (Ernest, 2016a). Others draw on Freire’s (1972) emancipatory philosophy, again based on Marx, to argue that learning mathematics can be a revolutionary activity and should be emancipatory and empowering through fostering a critical citizenry. Prominent in taking these ideas forward, although not necessarily drawing on Freire, are the movements of critical mathematics education (Skovsmose, 1994) and ethnomathematics (D’Ambrosio, 2007). Because of the prominent role of ethics in these movements I mention them here, but their powerful social critiques could just as easily have been included under the heading of social and political philosophy, especially since critical mathematics education explicitly draws on the Frankfurt school.

Another dominant strand of ethics-driven research in mathematics education concerns social justice and its deficiencies in the education of special groups such as females, ethnic minorities, students with disabilities, special needs students, second language learners, students of lower socio-economic status, and so on. These righteous concerns have spawned a vast literature over the past forty-plus years with many thousands of publications as well as dedicated conferences and research groups. Once again much of such research could also be labelled social and political but that which has an overt philosophical dimension often predominantly focuses on the ethics of exclusion or disadvantage, so it fits here.

Methodology

Lastly, an area of mathematics education research in which philosophical issues are influential and overtly utilised is that of research methodology. Serious research in our field, whether in the form of smaller projects such as doctoral investigations, or larger funded research projects, is expected to address the philosophical issues in research methodology. Beyond techniques and methods, research methodologies are expected to have a sound basis with explicit awareness and treatment of the ontological and epistemological assumptions underpinning the study, not to mention its ethics. Non-empirical research, being conceptual or philosophical, is even more required to be on top of its philosophical assumptions. Mathematics education is an interdisciplinary field of study straddling the sciences, social sciences, humanities and perhaps even the arts, so it is not surprising that a wide range of research methodologies and paradigms are employed in research. Indeed this diversity of research paradigms, approaches and methodologies is one of the great strengths of our field. Nevertheless, philosophical justification is needed for the appropriateness of whatever research approach is chosen and employed, as well as for the reliability, validity and trustworthiness of the knowledge produced.

Conceptual Analysis

In addition to the contributions of the substantive branches of philosophy to mathematics education, there are also benefits to be gained from applying philosophical styles of thinking in our research. For example, many of the constructs we utilise need careful conceptual analysis and critique. I have in mind, for example, such widely used ideas as understanding, development, progress, progressivism, mathematical ability, nature/natural, values, objectivity/subjectivity, identity, working like a mathematician, learning, discovery learning, problem solving (including pure, applied, 'real' and 'authentic' problems), teaching, assessment, mathematics, knowledge, sex/gender, special needs in mathematics, multiculturalism/antiracism, ethnomathematics, context, both social and task-related, and so on.

Deconstructing some of these ideas and terms might seem 'old hat', but even an apparently everyday idea like understanding, for example, contains hidden assumptions and pitfalls. First of all, it is based on the metaphor of something under our standing, providing a foundation for our position. In what way does this capture its meaning? Synonyms like 'grasping', 'getting a handle on' or 'seeing' it are all based on familiarity through a sensory encounter with meaning, and on being able to control or possess it ('getting it'). Thus these metaphors presuppose a static 'banking' model, interpreting understanding as the acquisition, ownership or possession of knowledge (Sfard, 1998). But secondly, there is an ideological assumption that understanding a concept or skill is better, deeper and more valuable than simply being able to use or perform it successfully. Skemp (1976) distinguished 'relational understanding' from 'instrumental understanding', and posited the superiority of the former. However, his co-originator of the distinction, Mellin-Olsen (1981), used it to distinguish the modes of thinking of academic students from that of apprentices, thus bringing in a social context and even a social class dimension to the distinction, and imposing less of an implicit valuation hierarchy. If we want to assert the superiority of 'relational understanding' over 'instrumental understanding' it needs to be done on the basis of a reasoned argument, and not taken for granted as obvious. Skemp's own argument was based on the psychology of schemata, based on Piagetian theory, but this has been challenged by a number of alternate theories of learning including socio-cultural theory and social constructivism, drawing on Vygotsky's (1986). According to Vygotsky knowledge is not something that the learner possesses but is a competence inferred from the learner's manifested ability to complete a task, either unaided, or, with the help of a more capable other, in what is termed the learner's zone of proximal development. Given current challenges to the underlying theories of learning, the assumption that relational understanding is superior to effective performance stands in need of justification.

Some scholars have challenged the unquestioned pre-eminence of relational understanding. Hossain et al. (2013) question the accepted good of the related notion of 'understanding mathematics in-depth' because, as they show, its role in the identity work of some student-teachers is troubling to them.[2] for example, one student teacher with the pseudonym Lola experiences a conflict between the imposed *good* of relational understanding, when studying in England, and her own success within the norms of instrumental understanding that she internalized in her Nigerian upbringing (Ernest, 2016b). Others have challenged the uncritical promotion of understanding within the mathematics education community because of its incoherence. Llewellyn (2010) questions 'understanding' partly because of slippage in the use of the term so that it encompasses both its relational and instrumental forms. However, her deeper critique is that in use it carries with a whole host of problematic assumptions about who can own 'understanding' in terms of ability, gender, race, class.

> Understanding is produced as hierarchical, particularly in relation to gender, social class and ability. It belongs to the privileged few, the 'naturally' able, which are often boys (another unhelpful and unnecessary classification). To suggest that girls have a 'quest for understanding' is over simplistic and gendered and in the first instance we should unpack how each version of understanding is constructed. … Finally I suggest that student teachers do not produce understanding as cognitive; the child is not an automaton who performs as the government text prescribes. Pupils and understanding are tied up with notions such as gender, confidence and emotions. (Llewellyn, 2010, pp. 355–356)

What a brief look at one example shows is that a widely presupposed good in the discourse of mathematics education, the concept of understanding, is a worthwhile target of philosophical analysis and critique. Although such analysis does not mean that we have to abandon the concept, it does mean that we need to be aware of the penumbra of meanings revealed and aporias unleashed through its deconstruction. We need to use the term with caution and precision, clarifying or sidestepping its troubling connotations and implications. Thus the philosophy of mathematics

[2]In later work Skemp (1982) refers to instrumental understanding as 'surface' and relational understanding as 'deep' understanding, thus prefiguring the depth metaphor in the more recently coined term 'understanding mathematics in-depth'.

education, as well as offering valuable overarching and synoptic views and explanations of our field, also serves as an under-labourer.[3] It can clear the conceptual landscape of unnoticed obstacles and perform the hygienic function of targeting, inoculating and neutralizing potentially toxic ideas circulating, like viruses, in our discourse.

Conclusion

In this introductory chapter I have offered a perspective on the nature and problematique of the philosophy of mathematics education. I have characterized the fields in both narrow and broad terms, and from bottom-up and top-down perspectives. From the bottom-up perspective one can characterize the area in terms of questions such as: What are the aims and purposes of teaching and learning mathematics? What is mathematics? How does mathematics relate to society? What is learning mathematics? What is mathematics teaching? What is the status of mathematics education as knowledge field? Using a 'top down' perspective the field can be characterised based on the branches of philosophy involved. Looking briefly into the contributions of ontology and metaphysics, aesthetics, epistemology and learning theory, social philosophy, ethics, and the research methodology of mathematics education reveals both how rich and deep the contributions of philosophy are to the theoretical foundations of our field of study.

This synoptic account ends up asking many more questions than it provides answers. Some of unanswered questions about the field include, for example: what are the overall responsibilities of mathematics education as an overall field of study and practice, and what is the responsibility of our own subfield, the philosophy of mathematics education? What are the responsibilities of mathematics education researchers? Does this depend on our philosophical stances, whether we see ourselves as critical public intellectuals or as functional academics probing deeper into narrow specialisms? (Ernest, 2016b). Do we see ourselves as needing to go beyond understanding the complex issues that arise from researching the teaching and learning of mathematics? Is it our business to also transform the practices of mathematics education, as Luis Radford suggests in the Preface (this volume)? After all, Marx argued that "philosophers have only interpreted the world in various ways; the point, however, is to *change* it." Marx and Engels (1969, pp. 13–15). However, unless you understand the interests and social forces at work in any social practices, you are limited in the efficacy of your actions. Interpreting Marx's thesis, West (1991, p. 67) argues that

[3] "[I]t is ambition enough to be employed as an under-labourer in clearing the ground a little, and removing some of the rubbish that lies in the way to knowledge" (Locke, 1975, p. 10).

> The task at hand then becomes a *theoretic* one, namely, providing a concrete social analysis which shows how these needs, interests, and powers shape and hold particular human conventions and in which ways these conventions can be transformed.

Thus philosophy (theory) and practice need to advance together in a praxis of transformative mathematics education. Radford is right that the philosophy of mathematics education needs to go beyond mere understanding. It needs not only to articulate competing sets of values, but to also commit to a set of values for human flourishing through mathematics. Beyond this, we need to engage with transformative practices to better the teaching and learning of mathematics, mathematical practices, society and the world we live in.

Philosophy emerged from the dialectics of the ancient Greeks where commonplace beliefs and unanalysed concepts were interrogated and scrutinised; where the role of the rulers was questioned and challenged through speaking truth to power. Thus the role of the philosophy of mathematics education is to analyse, question, challenge, and critique the claims of mathematics education practice, policy and research. Our job is to unearth hidden assumptions and presuppositions, and by making them overt and visible, to enable researchers and practitioners to boldly go beyond their own self-imposed limits, beyond the unquestioned conceptual boundaries installed by the discourse of our field, to work towards realizing their own ideals, visions and dreams in their classrooms, societies and the world.

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Part II
The Nature of Mathematics

The Who and What of the Philosophy of Mathematical Practices

Jean Paul Van Bendegem

Abstract In the first part an outline is presented of the emergent new field of the study and the philosophy of mathematical practices, including (the philosophy of) mathematics education. In the second part the focus is on particular themes within this field that correspond more or less to my personal contributions over a thirty-year period. As the title of this contribution indicates the relations and connections between the study of mathematical practices and 'mainstream' philosophy of mathematics need not be antagonistic but rather are to be seen as mutually beneficial.

Keywords Mathematical practices · Lakatos · Kitcher · What-if mathematics
Sociology of mathematics · Beauty · Explanation · Style · Analogy
Foundational studies

Introduction

This contribution will serve simultaneously a number of different purposes but the two most important ones are, on the one hand, a sketch of what the philosophy of mathematical practices is all about and, on the other hand, my specific involvement within this new field. It is thus a summary not only of the state of affairs within the domain or, if you like, what the domain looks like and who is involved but also of my work of the past thirty years. This approach to the topic implies that the originality of this chapter will be found in the bringing together of quite diverse elements that, as far as I know, have not been brought together as such. It also implies that I will rely on previously published materials. The first part of this chapter covering the domain as a whole is based on the first part of Van Bendegem (2014), itself an elaboration of the rough outline presented in Giardino, Moktefi, Mols and Van Bendegem (2012). When I write 'based on' I mean that I have

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included new material and I have made several annotations thus slowly moving towards a more complete history. The second part brings together a number of my papers that have dealt with one or another aspect of the study of mathematical practices. Alternatively this text can thus be seen as an imaginary book proposal for a fuller and more comprehensive study of mathematical practices as I see it.

A Short and Incomplete Historical Outline

Lakatos as the Starting Point

As should be repeated over and over again, any historical outline is always partially a reconstruction. Therefore we always seem to desire a historical starting or turning point, that will be inevitably of a sufficiently symbolic nature. Or, if you like, if the philosophy of mathematics underwent a similar change as the one that 'shook' the philosophy of science (at least, according to some), then it seems fair to take Lakatos's (1976) seminal *Proofs and Refutations*, a work that was in part inspired by Pólya's (1945) *How to Solve It*, which discusses heuristics and problem-solving techniques in an educational setting.[1] Its focus on mathematical practice was clear by the mere fact that it boldly presented a 'logic' of mathematical discovery. What did the book offer? Nothing less but the *history* of a mathematical statement and its proofs and it should be noted that the plural is justified. That fact alone should amaze everybody for is not mathematics supposed to be timeless, eternal even, unchangeable for sure and is mathematical proof not supposed to be absolutely certain, undoubtable, secure? This implies that, if a proof is found for a statement A, then A has been proven and that is that. To which is usually added: and a wrong proof is not a proof, it is not even wrong.[2] Lakatos disagrees and takes as an example the statement that for polyhedra (in three-dimensional Euclidean space), V(ertices) − E(dges) + F(aces) = 2. Take, e.g., a cube. There are 8 vertices, 12 edges and 6 faces and indeed $8 - 12 + 6 = 2$. Euler himself had found an ingenious proof or so he believed. As soon as the proof was around, counterexamples appeared,

[1]This, in itself, is a quite interesting phenomenon. One of the main sources of inspiration for Lakatos came from an educational setting and not solely from a reflection on academic mathematics. I will come back to this connection as it presents a separate difficulty in the philosophy of mathematical practice. And then I do not even mention the other source that was just as important but just as often forgotten, due to the difficulty, I assume, of giving it its proper place, namely the influence of Georg Wilhelm Friedrich Hegel. See Larvor (1988) for a thorough discussion of this relation.

[2]I have to confess that I am sensitive to this argument. If one chooses an ontological viewpoint where proofs exists in some sense independent from any epistemic subject then a wrong proof cannot be 'out there'. 'Out there' only genuine proofs exist. It then becomes a matter of belief: "I thought this was a proof but apparently I have made a mistake". But, of course, if the ontological viewpoint is abandoned then only we remain to decide whether something passes the test of being a proof or not. And tests can fail, that we all know.

making it necessary to (sometimes seriously) modify the proof and, in that sense, the proof has a history. Even more importantly is that Lakatos found patterns in this game of proof and refutation, explaining thereby the title of the book. In this book, written in a dialogue form, stressing the dynamics of the search for a 'final' proof, a totally different picture was drawn of what mathematics is and what it is about.

An important warning should be made at this point. As I have remarked at several occasions, selecting a symbolic starting point is not without its dangers. The most prominent one is the exclusion of important contributions not related to that particular starting point. One such exclusion—and this would require a thorough study of its own—is the French historico-philosophical approach to mathematics, including such thinkers as Brunschvicg (1912), and his pupils Lautman (1977) and Cavaillès (1962). In its turn this would invite us to consider the work of François Le Lionnais and Raymond Queneau and, inevitably the *Ouvroir de Littérature Potentielle*, see Van Kerkhove and Van Bendegem (2014) for a more detailed presentation of this latter development. Another exclusion, equally important, is the Dutch group of researchers, known as the *Significs*, see Pietarinen (2009) for a historical overview (with a focus on the philosophy of language). Although L. E. J. Brouwer must be the most famous member of this group, it would be unwise to identify him with it. Of equal importance but with a much stronger focus on the social aspects of (the development of) mathematics is Gerrit Mannoury. I just mention here Mannoury (1917, 1924) that are quite astonishing given the period wherein they were written.[3] That brings me to another matter apart from the exclusion problem, namely the delicate matter of precursors. In fact, apart from Mannoury, there is the intriguing paper Kneebone (1957)[4] that rightly deserves the title.

Kitcher as the Next Step

The next step that drew attention was set in 1983, when Philip Kitcher proposed in his book *The Nature of Mathematical Knowledge* (Kitcher, 1983) a more or less formal model of how mathematics as an activity can be described, clearly inspired by the developments in the philosophy of science, where attempts to develop formal models have always been present. It is sufficient to think of the logico-mathematical work of Joseph Sneed, formalizing a Kuhnian outlook, as a prime example, see Sneed (1971). The model itself might appear at first sight as being rather crude. The model consists of a five-tuple $\langle L, M, Q, R, S\rangle$, containing

- a *language* L,
- a set of accepted *statements* S,

[3]Unfortunately both texts are written in Dutch (as are many other papers by the Significs). It partially explains why they seem to remain largely unnoticed in present-day discussions, a rather unfortunate observation to make.

[4]Thanks to Brendan Larvor for having brought this paper to my (and others') attention.

- a set of accepted *reasonings R*,
- a set of important *questions Q*, and
- a set of philosophical or *metamathematical views M*.

Elementary as it may seem, the important thing is that it includes elements such as questions and metamathematical views that are usually excluded in a formal(ist) presentation of a mathematical theory[5] and that comes closer to mathematics as it is usually practiced. That being said, it is not correct to see Kitcher's approach as a continuation of what Lakatos had initiated. For one thing, Kitcher discusses philosophical problems that clearly belong to the traditional philosophy of mathematics (and I will return to that topic as well in the second part), such as questions of realism, of the existence of mathematical objects, of our capability (or lack of it) to get to know these objects, and so forth. In short, in Kitcher's approach we get a mixture of the more radical Lakatosian view and more traditional philosophy of mathematics, using a Kuhnian framework to accommodate both.

The Kitcherian outlook proved to be more successful, as several authors embraced this trend, as is shown in subsequent volumes that made connections between philosophy and history of mathematics, as did Tymoczko's (1986) *New Directions in the Philosophy of Mathematic*s and, later and more explicitly, Aspray and Kitcher's (1988) *History and Philosophy of Modern Mathematics*. It must be explicitly stated here that historians of mathematics have always paid attention, in varying degrees to be sure, to practices and have produced significant studies with often implicit, but also sometimes explicit, claims about their philosophical relevance, see, e.g., the recent volumes Ferreiros and Gray (2006) and Gray (2008) and, especially, the recent Ferreiros (2015) where history and philosophy clearly interact.

A First Tension Is Introduced to Stay

This initial tension between Lakatos and Kitcher has never left (up to now) the study of mathematics through its practices. The nature of that tension is anything but new. We have known it in the philosophy of science in the form of the context of discovery versus context of justification divide, where preferably the latter is seen as independent of the former. Its mathematical counterpart is the supposed independence between the search of a proof and all the processes it involves and the 'finished' proof, whose justification consists mainly in checking its formal correctness or validity. Obviously, Lakatos' claim is exactly the opposite: to make sense of how mathematics develops, an understanding of these discovery processes

[5]I am of course aware of the fact that there is a highly sophisticated domain labeled metamathematics but it is fair to claim that it is 'mathematized' to such an extent that the classification of the *American Mathematical Society* considers it to be a branch of mathematics proper. Hence the importance of the term 'views': Kitcher refers to metamathematical views and not necessarily and/or exclusively *theories*.

is essential. Kitcher is prepared not to defend the *total* independence view but is willing to allow such elements from the context of discovery that are needed to understand the final results, in most cases, the proofs. All this means that, right from the start, two approaches were being initiated and developed. Has the situation changed must since then? To be honest, not that much.

In the introduction to *The Philosophy of Mathematical Practice* Mancosu (2008, p. 3), identifies two main traditions within this new research 'paradigm', namely the study of mathematical practices. The first is, what he calls, the 'maverick' tradition which remains close(r) to the Lakatosian approach,[6] while the second one settles itself within the modern analytical tradition, thus remaining closer to Kitcher than to Lakatos, and focuses, among other things, on the naturalizing programme that started with Willard V. O. Quine, and where, e.g., Maddy (1997, 2007) has played and still plays an important role. That being said, there is not a particularly impressive unity in this analytical approach, as the following quote, taken from the entry on naturalizing epistemology in the *Stanford Encyclopedia of Philosophy* makes immediately clear:

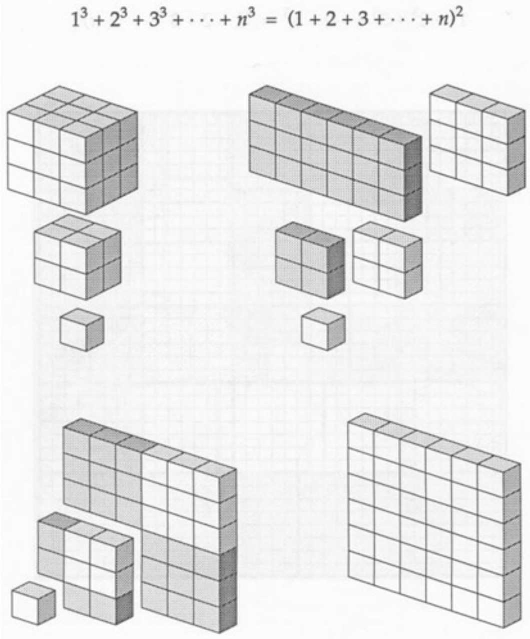
> Naturalized epistemology is best seen as a cluster of views according to which epistemology is closely connected to natural science. Some advocates of naturalized epistemology emphasize methodological issues, arguing that epistemologists must make use of results from the sciences that study human reasoning in pursuing epistemological questions. The most extreme view along these lines recommends replacing traditional epistemology with the psychological study of how we reason. A more modest view recommends that philosophers make use of results from sciences studying cognition to resolve epistemological issues. A rather different form of naturalized epistemology is about the content of paradigmatically epistemological statements. Advocates of this kind of naturalized epistemology propose accounts of these statements entirely in terms of scientifically respectable objects and properties. (Feldman, 2012, introduction)

Similar oppositions, as the ones sketched in this quote, exists in the analytical tradition in the philosophy of mathematical practice. Essentially it boils down to the question whether the philosopher intervenes or not. Or, if you like, we are dealing here with an extremely old tension, namely do we prefer a more descriptive outlook versus a more normative outlook. In Van Bendegem (2014) I gave the example of computer proofs—famous examples being of course the four-color theorem[7] and Kepler's conjecture.[8] Another fine example are so-called proofs by looking.

[6] The author of this chapter is supposed to belong to this tradition and I do most certainly not object!

[7] This is the statement that says that, given a planar map, divided up in regions, this map can be colored using four colours only, satisfying the condition that neighboring (i.e., sharing a border) countries get different colors.

[8] This statement is about sphere packing. In classical three-dimensional space one can think of spheres that have to be arranged in such a way that the space between the spheres is minimized. The regular packing, familiar from supermarkets where spheres are oranges, turns out to be the most efficient.



Above is one of my favourites, reproduced from Nelsen (1993, p. 86), the 'proof' is attributed to Alan L. Fry. Does this constitute a proof or not? Or is it 'merely' a source of inspiration to formulate a 'decent' proof? And who do we ask these questions? The mathematicians and/or the philosophers (or some third party, not yet identified[9])? And what do we do if the mathematicians either say yes or no? Do we accept it as a final verdict or do we explain that they have it wrong as they are obviously ignoring the finer philosophical details? This is in fact a rather serious matter as it seems to affect a number of such discussions that are at the heart of the study of mathematical practice, such as what proofs mathematicians consider to be beautiful, or what their explanatory power can be, or whether revolutions (in the Kuhnian sense) do or do not occur in mathematics (see Gillies, 1992).

So, at the end of this section, we have already identified three major approaches in the philosophy and study of mathematical practices:

- The Lakatosian approach, also referred to as the 'maverick' tradition,
- The descriptive analytical naturalizing approach,
- The normative analytical naturalizing approach.

This, however, is absolutely not the end of the story.

[9]The pupils in the classroom seem an excellent candidate for such a third party.

Enter the Sociologists (and a Second Tension), The Educationalists and the Ethnomathematicians

I have already stressed the importance of the historians of mathematics but in this section I want to pay some attention to some more recent developments. Partially inspired by Lakatos as well as Kuhn's (1962) *The Structure of Scientific Revolutions*, some researchers developed a sociology of mathematics, where one of the major focuses was mathematical practice as a group or community phenomenon. Two works should be mentioned in this line: Bloor's (1976) *Knowledge and Social Imagery* and Restivo's (1985) *The Social Relations of Physics, Mysticism, and Mathematics*, but see also Restivo (1992). In contrast with history and philosophy of mathematics, the sociological approach did not merge easily with the above mentioned traditions, although some 'brave' attempts in this direction should be noticed, e.g., Restivo, Van Bendegem and Fischer (1993), Rosental (2008) and Löwe and Müller (2010). This unease finds its roots in another crucial tension, namely that between the internal and external development of mathematics.[10]

One outcome of such discussions has been to draw the attention to other, so far neglected, areas where mathematics is involved, prominently mathematics education and ethnomathematics. To avoid all misunderstandings, I do not mean, of course, that these domains did not exist before the practice turn in the philosophy of mathematics, witness on the one hand the pioneering work[11] of the already mentioned Pólya and, on the other hand, the equally pioneering work of Ubiratan D'Ambrosio.[12] Quite the opposite but they were not perceived as being *relevant* to the philosophy of mathematics. In the standard view, mathematics educators are interested in how pupils can learn to grasp the concept of a mathematical proof or the certainty involved in a geometric construction or develop the ability to translate a verbal problem into a mathematical problem. That, of course, has little or nothing to do with questions such as what are reliable foundations for the whole of mathematics. In that very same standard view, ethnomathematics is a branch of anthropology and, as such, not relevant to a study of a high-level abstract mathematical problem in western mathematics.

[10]In Van Bendegem (2014) I gave the example of the use in Ancient Egypt of a rope with twelve knots—$3 + 4 + 5 = 12$ and these numbers are the sides of a right-angled triangle as $3^2 + 4^2 = 5^2$—to construct a right angle to divide the land after the flooding of the Nile. The obvious yet all too often unasked question is: why a right angle? The answer is simple: it is easier to calculate the surface of a rectangular piece of land in order to calculate the taxes. So it is an economical motive that prompted the mathematical development.

[11]Today one of the core figures in the field is Bishop (1988). For a 'local' contribution see François and Van Bendegem (2007).

[12]Although the first time the concept is mentioned in a paper is 1985, see (D'Ambrosio, 1985), informally the term was circulating much earlier and even in the 1985 paper it is clear that the Lakatos approach did not play a role (although Thomas Kuhn is mentioned). See also by the same author D'Ambrosio (1990, 2007).

From the practice point of view, however, the links are evident. Firstly, practices are 'carried' by people and people have to be educated, that forges the link with education, and, secondly, practices are socially embedded and thus culturally situated, that forges the link with ethnomathematics. Or, if you like, educations concerns the *diachronic* dimension of how mathematical knowledge is situated in time, whereas ethnomathematics concerns the *synchronic* dimension of how mathematical knowledge is situated in space. Finally, it must be added that mathematics education and ethnomathematics have an extensive, non-empty intersection.

And still the picture is not complete.

Brain and Cognition Complete the Picture

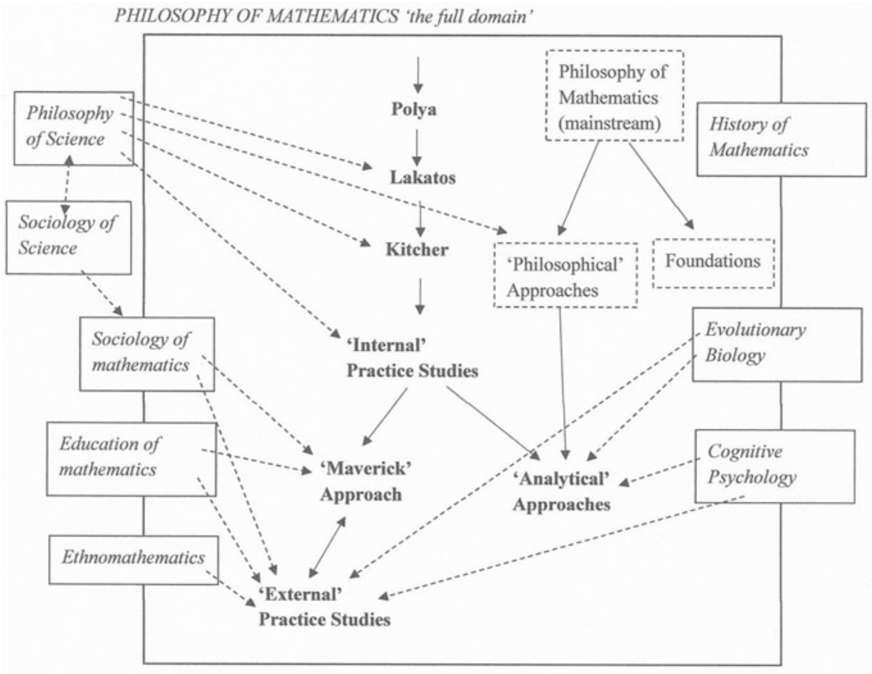
Other sciences play a part as well and I just mention the two most important ones, namely (evolutionary) biology and (cognitive) psychology. In the first field, the challenge is to determine how much mathematical knowledge is biologically, perhaps genetically, encoded in the human body and how it affects our mathematical abilities. The work of, e.g., Stanislas Dehaene (see Dehaene & Brannon, 2011 for an excellent overview), is the best illustration of this type of work. In the second field, one of the main topics is the study of human thinking and, quite trivially, since mathematical thinking is a perhaps highly particular and extraordinary form of thinking, it should and does attract their attention. What immediately comes to mind is the well-known study of Lakoff and Núñez (2000). And, to further increase the complexity, these studies can range from brain activity studies of mathematical tasks human subjects have to perform, to interviews with mathematicians on the way they (think they) solve mathematical problems.

We did already identify three major approaches in the philosophy of mathematical practice whereto five approaches need to be added:

- The Lakatosian approach, also referred to as the 'maverick' tradition,
- The descriptive analytical naturalizing approach,
- The normative analytical naturalizing approach,
- The sociology of mathematics approach,
- The mathematics educationalist approach,
- The ethnomathematical approach,
- The evolutionary biology of mathematics,
- The cognitive psychology of mathematics.

A complex picture indeed! The figure[13] below tries to catch some of this complexity without any pretence at completeness.

[13]A few remarks on the diagram: full arrows sketch dependencies and dotted arrows influences. I did not use any arrows for the history of mathematics as it is present everywhere. Partial overlap



A Methodological Note

One of the major consequences of the complexity of this situation as sketched above, is that, as a philosopher-mathematician myself, I find that I have to work in different 'registers' when reading different texts in the field (or that I, at least, consider to be relevant). There is a world of difference reading papers (a) on representations of numbers in the brain, including all the nicely coloured pictures of brain scans, or (b) on a cognitive-analytical approach to the use of diagrams in mathematical reasoning, where the thought processes are described in detail, or, (c) on a sociological analysis of the mathematical concepts of space and time, where anthropological methodologies are of crucial importance to be aware of possible cultural biases, or, finally, (d) a formal approach to what explanatory strength of proofs could be.

with the large rectangle means that there is an interest in mathematics within that domain. Not indicated are overlaps between the smaller rectangles and only some of the dependencies are indicated.

Let there be no mistake: one might think that perhaps we are dealing here with a division of labour of a vast field to explore, but such a division suggests that all the parts can be put together again to form a minimally coherent whole. And that is (at present) definitely not the case. There are, as I have indicated, fundamental oppositions and tensions at play. Therefore, it follows, I believe, that the two major tasks for the future are, first, to develop a greater coherence in the field and, two, to keep the conversation going with the other philosophers of mathematics (that so far have not yet been mentioned). These are difficult tasks, no doubt, but without any such attempts, in the former case, decoherence can prove to be fatal for the survival of this new and emerging domain and, in the latter case, one should not forget that the mainstream in the philosophy of mathematical practice is itself not mainstream at all in the larger field of the philosophy of mathematics where, e.g., foundational studies still form a major part. Setting up the dialogue can, quite frankly, again be a matter of survival.[14]

Topics in the Philosophy of Mathematical Practices and What They Can Tell Us

What follows is not meant as an exhaustive listing of topics that are relevant for philosophers (and all the other researchers involved of course) in the domain of the study of mathematical practices. Although the list will not be a short one, it is closely related to my own work, spread over thirty years,[15] that, as I had not fully realized, did tend to focus on the practices *internal* to mathematics. I am using the past tense as this is less the case today and I will come back to this feature in the conclusion of this chapter. I have grouped a number of themes under four headings but obviously these themes are not independent from one another and hence

[14]A historical note: in 2002 a conference was organized in Brussels, Belgium, where the organizers, Jean Paul Van Bendegem and Bart Van Kerkhove, tried to realize their ambition in bringing together representatives of some of the disciplines mentioned. This conference has led to the book or proceedings, titled *Perspectives on Mathematical Practices*, with the overambitious subtitle *Bringing Together Philosophy of Mathematics, Sociology of Mathematics, and Mathematics Education* (Van Kerkhove & Van Bendegem, 2007). There was a follow-up conference in 2007, whereof the proceedings were published in two volumes: see Van Bendegem, De Vuyst and van Kerkhove (2010) and Van Kerkhove (2009), with our ambitions slightly moderated, but another important outcome from all these developments, overall, has been the confirmation of the rather heterogeneous character of this field known as the ‘philosophy of mathematical practice’, as can be seen in the recently founded *Association for the Philosophy of Mathematical Practice* (*APMP*) (see the website of the association at http://institucional.us.es/apmp/), wherein both of Mancosu’s traditions are clearly present and at times happily interacting.

[15]As a matter of fact, it is *precisely* thirty years since my first paper on mathematical practices was published in 1987, see Van Bendegem (1987). In that paper I presented a view of the history of Fermat’s last theorem from the perspective of evolutionary epistemology that goes together perfectly fine with a close attention to practices and the practitioners.

cross-references will abound. At the same time I will stress their philosophical importance in order to show that the study of mathematical practices really adds to the discussions in the broader field of the philosophy of mathematics.

The View from Above: From Bird to Frog

In "The creative growth of mathematics", (Van Bendegem, 2004), I formulated the ambitious plan for a description of (nearly) *all* mathematical activities. An essential feature of the model was its three-layered structure, including a macro-, meso- and micro-level. The object was to make a distinction between the global development or, if one likes, the 'great' periods and their transitions,[16] and the daily life of a non-specified mathematician who is working on finding a solution to a particular detailed and highly specialized mathematical problem. At the macro-level, discussions about revolutions[17] make sense (independently from the outcomes) but not at the micro-level where the timescale is such that revolutions, if any, tend to disappear in the background. There was however a need for an intermediate level that I identified with the formulation of specific research programs, ranging from Hilbert's famous list of 23 problems that would determine the research agenda for the 20th century (and to an amazing extent, it actually did), over the Erlanger Program, initiated by Felix Klein to get geometry unified once again, to category theory as a new foundational theory for mathematics. Where the macro-level determines to a certain extent the methods one can use, the meso-level puts forward the 'interesting' problems wherein one should invest one's time (with or without the supervisor's approval). Both the macro- and meso-level already fix a great number of aspects of the micro-level. What kind of proof methods one can use, how rigorously a proof has to be formulated, what kind of heuristics are considered fruitful, what consequences might open new research avenues, …, these are all the ingredients that a trained mathematician has available (often in the form of skills that partially operate on an unconscious level) to tackle a mathematical problem.

One of the mean reasons why I still defend this three-layered model is that it immediately explains some interesting features of mathematical practices:

The Dieudonné Challenge I will come back to this problem in the conclusion of this chapter but the challenge is to show how mathematics can be influenced by society in such a way that is has an impact on how mathematics is done. There is, of course, the straightforward connection with applied mathematics—and that in itself is an intriguing and highly complicated story—but here the matter is whether society also has an impact on what we now label as 'pure' mathematics. Or as expressed in quite strong terms by Dieudonné himself:

[16]A fine example of such a description is to be found in Koetsier (1991).
[17]See, e.g., François and Van Bendegem (2010).

> To the person who will explain to me why the social setting of the small German courts of the 18th century wherein Gauss lived forced him inevitably to occupy himself with the construction of a 17-sided regular polygon, well, to him I will give a chocolate medal. (Dieudonné 1982, p. 23).[18]

Although it is easily imaginable that for many the question itself seems rather ludicrous, nevertheless I believe that 'decent' answers can be provided (as I hope to show in the conclusion).

The False Isolation of the Individual Mathematician Related to the previous item is the discussion about the usefulness of what mathematicians do and the products they generate (mainly but not exclusively proofs). After all, what could be more 'pointless' than a mathematician at the blackboard writing down formulas that perhaps a handful of colleagues can understand, desperately searching for a proof, wasting his or her lifetime in pursuit of a form of ethereal reward purely for its own sake.[19] Though it may seem so at the micro-level, the meso- and macro-level let us see the rich environment that is the mathematician's *niche*, that is not be confounded with the actual room the mathematician happens to be in but rather with the journals and papers from other mathematicians, lying around on his or her desk.

The Remaining Unexpectedness Very often our 'quest' to understand nature is seen as a two-person game involving the researcher and Mother Nature. Equally often she is represented as a hard player, quite reluctant to divulge her secrets. The same applies, I believe, to the Queen of the Sciences. She is equally playful and unruly meaning that she likes to surprise us. Does it not seem downright 'silly' (for want of a better word) that the expression $\left(\left(1/10^5\right)\left(\sum_n e^{-n^2/10^\wedge 10}\right)\right)^2$, where n goes from $-\infty$ to ∞, is equal to π up to the first *42 billion* digits and only then do things go wrong. In short, it is a warning that theories about mathematical practices that are too 'neat' must have serious gaps.

There is another important consequence that I could have added to this short list but I prefer to treat it separately in the next section in this chapter as I have written a number of papers to address this issue.

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[18]Our translation of: "Celui qui m'expliquera pourquoi le milieu social des petites cours allemandes du XVIII^e siècle où vivait Gauss devait inévitablement le conduire à s'occuper de la construction du polygone régulier à 17 côtes, eh bien, je lui donnerai une médaille en chocolat.".

[19]A very fine recent example is the rather romantic film *The man who knew infinity*, telling the dramatic story of the Indian mathematician Ramanujan and his collaboration with, in first instance Godfrey Hardy, himself the author of the famous *A Mathematician's Apology* (Hardy, 1940).

Could It Have Been Otherwise?

It is an old discussion to say the least but still quite alive: does the development of mathematics involve any form of necessity? Is there an internal 'mechanism' (for want of a better term) that pushes mathematics forward to some specific final state, close by or at an infinite distance? In short, do we have to view mathematics as satisfying the basic conditions of linearity and accumulation? At each future stage mathematical knowledge is added to the existing stock and no knowledge already obtained gets lost (up to a form of translation[20]).

Obviously, to deny this implicit necessity is a 'tough call'. There are basically two strategies to be employed. Either philosophical arguments can be formulated against this position and this has of course been done many times over. Or why not simply demonstrate that it could have been otherwise by providing examples? To be honest, I have always favoured this second option. It does seem a legitimate question to ask: if mathematics could have developed differently from what we know today, what are we supposed to think of? Over the years I have come up with three examples that I will briefly discuss without the formal details that can be found in the papers mentioned:

Vague Mathematics The starting point of this investigation, see (Van Bendegem, 2000a) was the question what would have happened if mathematicians would not have had this 'obsession' with the elimination of all vague concepts? If today a mathematician uses a universal quantifier "For all x, so-and-so" then it is really meant that "so-and-so" is valid for all x, without any exception. This had led, I believe, to this curious form of mathematical theorems that we know today: "For all x, if conditions C_1, C_2, …, C_n are satisfied, then so-and-so". However, in many cases the conditions cover nearly all the x and the remaining x's are not really 'typical'. At the same time we do have good formal analyses of vagueness. So why not use these models to formulate a vague form of mathematics? Such a reformulation of elementary number theory allowed me to *prove* that theorems such as "Small natural numbers have few prime factors" actually hold. Although I did not yet investigate this, it seems to me that the implications for mathematics education are to be taken seriously if only for the fact that mathematics will resemble ordinary language that embraces vagueness, thus reducing the gap that exists now.

'Proofless' Mathematics The challenge that I put myself in (Van Bendegem, 2005) is whether it is possible for a community not necessarily of mathematicians

[20]Think, e.g., of complex numbers. The translation from the expression a + i.b with the infamous imaginary unit present to couples of reals (a, b) did not make the imaginary unit disappear. It now reappears as (0, 1) with the interesting property that (0, 1) * (0, 1) = (−1, 0) (with * representing multiplication) or, translated back, says that $i^2 = -1$.

to arrive at all arithmetical truths without using any notion of proof. To be a bit more precise, the set of arithmetical truths are all statements expressible in first-order predicate logic with the additional functions of addition and multiplication and equality as the unique predicate. The inspiration for this model came from (two-dimensional) cellular automata where the cells contain arithmetical statements and the community performs a random walk, only obeying a local rule that only involves neighbouring cells.[21] If this is indeed to be considered an alternative, it shows that the 'proof game' is a very particular game indeed!

Complex Numbers We all know the rough details of the genesis of complex numbers starting with Cardano's famous square root of −15. Now, what is really necessary to deal with such an impossible entity, generating a whole set of problems? In (Van Bendegem, 2008a) I have presented an alternative history whereby a square root of a negative number had an unproblematic meaning right from the start. In a nutshell the idea was to consider a geometry with a reference place where surfaces can be added to and where holes could be created. If a square has a surface A then, since filling up a hole of the same size, restores the reference place, the surface A* of the missing square must be such that A + A* = 0, hence A* = −A. And thus the side of a missing square is the square root of A*. The number is not so much imaginary as quite specifically the side of a square that is missing. A mathematically interesting and philosophically intriguing detail is that the standard addition and multiplication of complex numbers can be recovered in those terms.

A warning must be made however at this point: concepts such as 'an alternative development' or 'a contingent development' are not as easy to tackle as they might seem. To mention just one difficulty: an alternative development can be viewed as a 'what-if' story, that need not be a problem. But how far can one deviate from what is given? It is easy to imagine that the complex number story is more plausible than the proofless mathematics story but what are the grounds for this plausibility? The interesting other side of the contingent-necessity coin is inevitably(!) that the notion of 'necessity' is not all that clear either.

The Rich Nature of 'Real' Proofs

Once we leave behind the ideal notion of a mathematical proof—I mean thereby the familiar phrasing that 'a proof is a list of formulas such that the initial formulas are the premises and all subsequent formulas can be justified according to the logical rules given and were the last statement is the conclusion'—we have no other option

[21]To be a bit more specific, if, say, 3 + 5 = 8 has been accepted as a correct addition then the neighbours are also accepted: 2 + 5 = 7, 4 + 5 = 9, 3 + 4 = 7 and 3 + 6 = 9. In two dimensions it has been shown that a random walk will eventually encounter every cell so all possible additions are covered in the (very) long run.

than to look at the 'real' proofs as they appear in journals and in handbooks, as they are presented at conferences and in classrooms, and as they are pieced together in the construction phase.[22] A good starting point is to treat these proofs as texts and wonder what features such texts need in order to 'work' as a proof. That turns out to be a very rewarding investigation. I list here some of these features:

Beauty Mathematicians often indicate that a proof is beautiful or ugly. It is not an easy matter to pin down what exactly is meant and what it is that is considered to be beautiful. Can a formula be beautiful or a proof or a concept or a construction method (say, in geometry) or a theory? Is there a general idea of beauty or do we need to consider separate cases? This is a still on-going study so no general results are to be reported at present but the philosophical relevance seems clear enough as it seems clear that aesthetic considerations guide mathematicians in their search, e.g., for proofs. See Van Bendegem and Desmet (2016) for more details.

Rhetoric and Style Connected to the previous feature is the specific way a proof is presented. After all, a 'good' proof should convince an 'audience', hence the style of presentation becomes an important element and, by extension, all the rhetorical aspects become relevant. The pioneering work of Reviel Netz must be mentioned here, see Netz (1999, 2009). To give just one perhaps obvious example: proofs presented in too much detail are 'bad' as the overview is lost (and do note that such overly detailed proofs actually approximate the ideal case). See also Van Bendegem (2008b).

Explanation Although I did not contribute myself directly to this theme, it must be mentioned because of its similarity with beauty: mathematicians use this concept to guide their activities. Explanatory proofs are interesting proofs because they make it possible to answer why-questions that go beyond the 'mere' establishment of the correctness of the proof. Answers to why-questions also provide insights in the structure and coherence of the mathematical theory that serves as a background for the question to make sense.

Analogy and Metaphor Mathematics in the creative phase faces quite similar problems as scientists do when they produce new 'discoveries' and are in need of words and images to explain what and why they are doing what they are doing. The standard 'way out' is to talk in old terms about new things, i.e., to use metaphors and analogies. And, as it happens, mathematics is full of them. To mention the most famous example of all: how to talk about and reason with infinity except in metaphorical terms? See Van Bendegem (2000b).

An important source of inspiration for the study and analysis of 'real' proofs are the domains of semiotics and argumentation theory. I already mentioned the *Significs Movement* in the Netherlands starting at the end of the 19th, up to the first quarter of the 20th century, but there is also a present-day interest, ranging from the

[22]For obvious reasons I resist using the term 'discovery'.

papers of Ernest (2006) on semiotics, and Andrew Aberdein on argumentation theory (see Pease & Aberdein, 2011).

A Second Look at Foundational Studies

As I mentioned before the well-established foundational studies need not be thought of in terms of opposition but rather in terms of mutual inspiration and collaboration. If one is pushed to give a distinguishing feature of foundational studies then surely it must be that it is to a very large extent formalized, both in terms of the tools used as of the modelling techniques to explore philosophical questions. These have the power, I believe, to perform a 'bridge function' between the two domains. Roughly speaking one can look for 'local' and 'global' bridges:

Local Bridges Examples are, e.g., (the already briefly mentioned) diagrammatic reasoning, and (formal models in) argumentation theory. Let me say a few words about these two topics. At the present moment diagrammatic reasoning has been approached from different angles. On the one hand there are studies, closer to mathematical practice, such as, e.g., Carter (2010), and, on the other hand, there are logical studies to understand the correctness of diagrammatic reasoning, such as, e.g., (Shin, 1995). Recently the work of Mumma (2010), brings the two together, so the first bridge has been built. In argumentation theory there have always been, quite early, attempts to formalize arguments in some sense. One can think of the pioneering work by Hintikka (1985), and Barth and Krabbe (1982). In recent times, a new impetus has been given to the field from studies in Artificial Intelligence, namely, the study of so-called argumentation networks and problem-solving networks, see (Dung, 1995) for a pioneering study. And in even more recent times, groundbreaking work has been done in Pease, Lawrence, Budzynska, Corneli and Reed (2017). What these studies clearly show is that the incorporation of insights from these diverse domains in the study of mathematical practices is a quite natural move to make, as these formal models will allow us to represent and study the interactions between arguments and counterarguments in a group setting, thus making it possible to build local bridges.

Global Bridges The more daring task, of course, is to construct 'global bridges'. Here too several options are available. One can think of models where mathematical 'agents' exchange information with one another, where they argue together, where they practice a division of labour, where they convince one another of the relevance of a bit of mathematics, and so on. The outlines of such models exist at the present moment, see, e.g., Van Benthem (2011), as one of the major contributors, although rarely specified for mathematics. But all indications point in the same direction: it can be done. And it should be done, for it will allow the practitioners of the study of mathematical practice to tap into these rich sources. A more modest attempt is to 'complexify' existing models such as in Kitcher's case. In Van Kerkhove and Van Bendegem (2004) we presented such an extension, leading to a seven-tuple instead of the original five-tuple, including additional elements such as mathematical communities, research programs, proof methods, concepts, argumentative methods and proof strategies.

Another way to express these bridges is in terms of idealized mathematicians. Classical mathematics presupposes the omniscient God-like mathematician,[23] the intuitionist Brouwer's creative subject, constructivist mathematics the human agent with potentially infinite capacities, the strict finitist the human agent with strictly finite capacities, and, if the social element is introduced we move closer and closer to the real-life mathematician. We need not ignore the idealizations, far from it, but they play a different role once the bridge is crossed.

Conclusion

As I wrote at the beginning of this chapter, the outline of domain and topics within the study and philosophy of mathematical practices was not meant to be exhaustive but rather a state of affairs of what the author of this contribution has done over the past thirty years. If I would be forced to formulate in a single phrase what these studies have revealed to me, the answer would be: *heterogeneity first, homogeneity after*.[24] Note that I did not use 'second' instead of 'after'. The idea rather is that whoever observes mathematical activities at the micro-level must be struck by its heterogeneous nature: one cannot confuse an algebraist with a topologist, a set theorist with a statistician, an academic research mathematician with a classroom teacher, a philosopher of mathematics with a mathematician, a mathematician with the so-called 'lay' person, … At the same time, manifold are the books that present us the whole of mathematics in one fell swoop, suggesting that there is a high degree of homogeneity. It is my conviction that its prime function is to enable us to deal with the heterogeneity. It is thus not 'secondary' but a consequence and thereby is supposed to play a specific role, namely to reduce the complexity, comparable to many foundational stories whose main function is to explain the whole in as short a story as possible. For mathematics such a story could start with "In the beginning there was nothing but the empty set".

There are still two things that need to be dealt with. I promised to present a solution to the Dieudonné challenge. Here it is. So how did life at the small German courts of the 18th century force Gauss to occupy himself with the construction of a 17-sided regular polygon? Of course, if the answer is to show a direct connection, the task is outright silly. But, if we are allowed to have a broader outlook, then we could think of the following.[25] What were the political-social circumstances that

[23]One is reminded of the famous words of Bishop (1967, p. 2): "If God has mathematics of his own that needs to be done, let him do it himself.".

[24]In Van Bendegem (2016) I have developed this idea in more detail.

[25]A strong inspiration for this answer is to be found in Restivo (2011, p. 47): "The sociological way is first to look to both "external" and "internal" contexts, networks, and organizations.

gave rise to a system such as the small German courts of the 18th century? This question is answerable and it will also shed light on the status of the sciences, including mathematics, during that process. We will understand why mathematicians were given the opportunity to continue their work and we will probably also understand why particular concepts were deemed more important than others, e.g., because of theological connections (I am thinking here about infinity). This in its turn invites us to see how this relative isolation of the mathematicians at these courts, led to an internal dynamic wherein certain topics were favoured over others, where the research community of 18th century mathematicians decided what problems were interesting and what problems not, to a certain extent co-determined by societal elements. Now we can focus on the internal, more local development and understand the importance of the study of polygons and, if we focus on the most detailed micro-level, the importance of the 17-sided regular polygon. Then we see Gauss drawing such a polygon and understand the social act that he is performing.

The very last thing I should mention is that, together with two of my fellow researchers, Karen François and Kathleen Coessens, I have participated in a research network, *Philosophy and History of Science of Pedagogy*, initiated by Paul Smeyers, Marc Depaepe and myself, for more than fifteen years now on the philosophy of pedagogy and education. Different themes were explored in yearly conferences and a book series, *Educational Research*, edited by Smeyers and Depaepe, resulted out of it. Our contributions[26] focused on mathematics, mathematics education and the arts. Although this collective work is highly relevant for the philosophy of mathematics education (and hence to this volume), nevertheless a survey of our joint work should involve the three of us and hence I will end here this outline of an emerging research discipline that has the power to let us see mathematics as we have never seen it before because we were looking at the sky and did not notice what has happening at our feet.

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Dieudonné's error is to imagine that only "external" milieux hold social influences. Second, the sociological task is to unpack the social histories and social worlds embodied in objects such as theorems. Mathematical objects must be treated as things that are produced by, manufactured by, social beings through social means in social settings. There is no reason why an object such as a theorem should be treated any differently than a sculpture, a teapot, or a skyscraper.".

[26]To avoid a truly overburdening of the list of references, I refer the reader to my website where all our contributions are listed: http://jeanpaulvanbendegem.be/home/papers/mathematical-practice/.

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The Philosophy of Mathematical Education Between Platonism and the Computer

Michael Otte

Abstract Mathematics causes some specific epistemological and educational difficulties. On the one hand, it must accept the merely representational character of all knowledge. On the other hand, mathematical knowledge claims a special truth status, compared to other knowledge. The difference between a symbol and a thing is that symbols have meaning and that meaning is always something personal and thus subjectively biased. To counteract this fact, logic and mathematics have tried to formalize sense or meaning, reducing it to formal syntax, to sets of rules and recipes. Hence result specific educational difficulties. For example, difficulties to stimulate creativity and encourage insight.

Keywords Platonism · Instrumentalism · Semiotics · Complementary

Introduction

The basic question in philosophy is: how should I live? To some mathematics seems attractive as part of an answer, because of the calm and clear order of its universes. To others mathematics appears more like a confusing labyrinth and out of any question or perspective. As everybody has had some experience of mathematics at school as part of general education the various philosophies of mathematics do play an important role. Mathematical education is not just a bundle of methods. If we take our task seriously—trying to offer some reflections about mathematical education and philosophy—we should, at least hypothetically, accept that mathematics, although being ruled by formal sense or algorithm in its methods might help to acquire real knowledge and we should also assume that formal mathematics might possibly be helpful to strengthen the creative and emphatic human subject.

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Ideas represent, first, possibilities of personal growth. A person who believes in the reality of ideas usually connects them to a world view that is somehow teleological and in which the progress to the general better plays an essential role. Such a world view seems to be of Platonic origin. It is not by accident that this kind of Platonism returned during the Industrial Revolution in Europe as part of Romantic natural philosophy and pure mathematics, challenging the utilitarism of 18th century Enlightenment.

Paul Ernest distinguishes three philosophies of mathematics

> because of their observed occurrence in the teaching of mathematics, as well as in the philosophy of mathematics and science.
>
> First there is the Platonist view of mathematics as a static but a well-established and unified body of certain knowledge. Mathematics is discovered, not created.
>
> Secondly, there is the instrumentalist view that mathematics is an accumulation of facts, rules and skills to be used in the pursuance of some external end. Thus, mathematics is a set of unrelated but utilitarian rules and facts.
>
> Thirdly, there is the problem-solving view of mathematics as a dynamic, continually expanding field of human creation and invention, a cultural product. Mathematics is a process of enquiry and coming to know, not a finished product, for its results remain open to revision. (Ernest, 1989)

Let us reflect on these "philosophies".

Platonism 1

Mathematics is a science, not an art, or is not just an art and, like every science it craves for the truth. Any science refers to truth, at least as a regulative principle and as a goal of its aspirations. It is thereby trying to establish itself as a genuine science, in contrast to mere fiction or fairy tale. To this end, it must distinguish between the sense or meaning and reference of its representations, that is, between signs and objects. If theories were identical with their languages, there would be no false theories and no truth.

Mathematics is no empirical science like physics or biology, but is a formal science and it must therefore refer to objects of a different kind, ideal objects. Pure mathematics is established by the idea "that intelligible means identical or as the modern logicians say, the highest principle of thought is the principle of tautology" (Mouy, 1971, p. 50). Only the well determined could be known, as Plato insisted. *The red is not the green*, but things might be red and green, or white and blue, like the Bavarian banner. Snow is white, but it brings all shades of blue, violet, grey and yellow to the eye of the meticulous observer.

Linguistic philosophers like Wittgenstein claim, in contrast, that if there were no language there would be no logic and this would mean that there were no necessity, since all necessity is linguistic necessity. Therefore they say "*nothing is simultaneously red and green*" (Wittgenstein, Tractatus, 6.3752). His friend Frank Ramsey pointed out to him, however, that the impossibility of a particle being in two places at the same time expresses a feature of the world, rather than of language. Nevertheless, Wittgenstein maintained, the singular mind that he was, that, "The world is the totality of facts, not of things. The world is determined by the facts. The facts in logical space are the world".

Wittgenstein considered himself a student of Frege but after many years of trying both gave up on each other's theories. Frege was a kind of Platonist conceiving of a realm of objective content of thought. People talk in this context sometimes of "semantic realism" or of a "third world", beyond the world of physical objects or physical states, and the world of states of consciousness. Karl Popper explains:

> What I call 'the third world' has much in common with Plato's theory of Forms or Ideas, though my theory differs radically … from Plato's. It has more in common still with Bolzano's theory of a universe of propositions in themselves and of truths in themselves, though it differs from Bolzano's also. My third world resembles most closely the universe of Frege's objective contents of thought. (Popper, 1973, p. 106)

Mathematicians as a rule do not believe that their discipline is just a linguistic enterprise, as was said and therefore Platonism in the classical sense of a universe of eternal ideas comes natural as a foundational conception of pure mathematics. And universal ideas require universal human spirits and souls. As Rorty says critically and with a derogatory undertone, "The distinction of the mental and the physical is parasitic on the universal-particular distinction rather than conversely" (Rorty, 1979, p. 31). And the question is not "do mathematical objects, like numbers exist", or "what is the difference between empirical objects, like tables and chairs and ideal objects, like numbers and conics, although such question bother teachers and students in school most frequently. The question is about mind and nature (brain), between humans and robots. Looking into the history of philosophy we find all sorts of answers to this question, from Plato to modern philosophers like Gilbert Ryle (1900–1976), who is principally known for his critique of Cartesian dualism.

Once we had a college at our mathematics department at the University of Bonn, who would not listen to music, but would read it from the partiture. He did not visit music performances because he thought music becomes distorted by playing it. In fact, no two performances by different artists transmit the same impression. However, to the majority of humans real performances are essential to gain access to the quality of music. And the same is true for the majority of students with respect to their having access to the purity and power of ideas.

With respect to education it is essential therefore that knowledge and theory must appear as embodied and animated by the enthusiasm of a teacher, rather than being presented as mere information. We had once called teachers "exemplary intellectuals" (Otte, 1994).

We had meant the teacher would influence his students not primarily through pedagogical methods and techniques, but by what he himself is. Not the explicit instructions and the individual teacher's words are decisive, but especially important seems the spirit and the credibility that he radiates in his activities. The teacher acts efficiently primarily by the nature of his own intellectual life. The student needs to experience the personalized embodiment of knowledge and of the thought, which are expressed in it. Theories remain to him mere dry paper if they are not animated by the thought and attitude of a human being.

In Plato's dialogue *Protagoras* (314a–b) Socrates has one last thing to tell Hippocrates before they go off to inquire of the famous Protagoras. He says,

> if you are a knowledgeable consumer, you can buy teachings safely from Protagoras or anyone else. But if you're not, please don't risk what is most dear to you on a roll of the dice, for there is a far greater risk in buying teachings than in buying food. … You cannot carry away teachings away in a separate container. … you take the teaching away in your soul and of you go either helped or injured.

And the essential fact, responsible for both, mathematics not being simply and straightforward algorithmic knowledge and humans not being just calculators or Sophists—"masters of rhetoric",—or fixers of something arbitrary, refers to the necessity of idealization and generalization. So, we must create new concepts and ideas, or ideal objects. To generalize means just this, introduce new ideal objects. In this way, by claiming that knowledge is more than information Platonism seems at least partially and conditionally be justified.

Platonism 2

Knowledge is for Plato <u>direct</u> perception of an ideal object. As Socrates says to Theaetetus: "But how utterly foolish, when we are asking what is knowledge, that the reply should only be, right opinion with knowledge of difference or of anything! And so, Theaetetus, knowledge is neither sensation nor true opinion, nor yet definition and explanation accompanying and added to true opinion" (Theaetetus, 210b)

The world of common objects becomes intelligible, because it shares its essence with the universe of ideal objects or ideas. One should "loudly exclaim that you do not know how else each thing can come to be, except by sharing in the particular reality in which it shares, and … you do not know of any other cause of becoming *two* except by sharing *Twoness*" (Phaedo, 101c).

Abstract notions like *Twoness* are established by a kind of axiom of extensionality for concepts. Mathematically on might express this like:

F = G, if and only if for every object x: F(x) = G(x).

This axiom is a counterpart of Leibniz' fundamental *Principle of the Identity of Indiscernibles*, which says that no two things have exactly the same properties, such that:

X = y, if and only if for every function F: F(x) = F(y).

Gödel has explained this axiom of extensionality thus:

> *Two* is the notion under which fall all pairs and nothing else. There is certainly more than one notion in the constructivistic sense satisfying this condition, but there might be a common form or nature of all pairs. (Gödel, 1944, p. 138)

Gödel was a Platonist and he believed in the existence of ideas like *Twoness* He writes “It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases, it is impossible to interpret the propositions one wants to assert about these entities as propositions about data” (Gödel, 1944, p. 137).

Later in life even a Platonist like Gödel did no more believe “that generally sameness of range is sufficient to exclude the distinctness of two concepts”. Such a statement would be “an assertion of a very unlikely possibility of the structure of the world which includes concepts” he said (Wang, 1996, p. 275).

General Ideas and Social Knowledge

For Hume and the British empiricists perceptions retained the function which ideas had possessed for Descartes or Leibniz. “The really important difference between Hume and Descartes is that Hume had come to perceive the acute difficulty of vindicating, underwriting, guaranteeing the world attained by rational exploration” (Gellner, 1992, p. 20). But this does not concern us at the moment, because empiricism is not much *en vogue* as a means of justifying mathematical assertions.

What does concern us, however, is the process of discovery and generalization. And this process of generalization poses some rather challenging epistemological as well as socio-historical questions. The representation of universal ideas has complex social and cognitive premises and obstacles. “It must have required many ages to discover,” says Bertrand Russell, for example, “that a brace of pheasants and a couple of days were both instances of the number two.”

And the mathematician and historian Salomon Bochner said with respect to the creation of pure mathematics at the turn of the 18th century:

> No matter how much Lagrange may assert and insist that a function is for him an abstract object, in his thought patterns it somehow is residually a mechanical orbit or perhaps a physical function of state, whereas in Cauchy orbits and forces and pressures are always functions, as they are for us today. (Bochner, 1974, p. 837)

And it is even more interesting to notice that freeing the number concept of definite reference came along with the advent of symbolic algebra and this means it came with a separation between meaning and objective reference or between communication and mathematical application. Hence Wittgenstein’s credo that the only necessity be linguistic necessity.

Therefore, sociologists like Emile Durkheim have considered general concepts or universal ideas as socially established entities:

> If concepts were only general ideas, they would not enrich knowledge a great deal,…. But if before all else they are collective representations, they add to that which we can learn by

> our own personal experience all that wisdom and science which the group has accumulated in the course of centuries. Thinking by concepts is not merely seeing reality on its most general side, but it is projecting a light upon the sensation, which illuminates it, penetrates it and transforms it. … Each civilization has its organized system of concepts, which characterizes it. Before this scheme of ideas, the individual is in the same situation as the *nous* of Plato before the world of Ideas. (Durkheim, 1912, pp. 407–410)

Every society considered in its socio-historical context has its own notions of mathematical concepts and its own way of communicating or transmitting mathematical knowledge. In fact, the controversy over the respective cognitive or educational advantages or disadvantages of geometry vs. algebra (including arithmetic) runs through the entire modern history. In his work, *Scholae Mathematicae* of 1569 Petrus Ramus (1515–1572) bitterly criticizes Euclid for the methodological and pedagogical uselessness of the geometric representation and he tries arithmetizing geometry in a way, which did later-on influence Descartes (Otte, 1984).

However, as the individual mind is not directly connected to social communication, theories of meaning and theories of truth have to be distinguished. Peirce *Pragmatic Maxim* presents the resulting problem or dilemma. The original 1878 statement of the *Maxim* runs as follows:

"Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object" (Peirce, 1966, p. 124).

Peirce comments on this about 25 years later, in 1902, by a contribution to Baldwin's "Dictionary of Philosophy and Psychology". The Pragmatic Maxim, there he says,

> might easily be misapplied, … The doctrine appears to assume that the end of man is action —If it be admitted, on the contrary, that action wants an end, and that end must be something of a general description, then the spirit of the maxim itself, which is that we must look to the upshot of our concepts in order rightly to apprehend them, would direct us towards something different from practical facts, namely, to general ideas, as the true interpreters of our thought. (Peirce, CP 5.3)

Peirce wants by this comment to justify the creation of such entities as the imaginary numbers or the "doctrine of incommensurables" which represent a contradiction between the definite determinations of the everyday world, on the one hand, and the world of theory and creative development, on the other hand. Sense or meaning cannot be constituted exclusively either from psychological content or from the real-world correlates of our representations, nor from the semantics of communication.

Knowledge as a Historical Process

Another difficulty with Platonism stems from the fact that our knowledge is always incomplete and evolves through history. The distinguished mathematician Stan Ulam once said: "The mathematical method as it is in use today would possibly appear familiar to the Greeks as well. The objects, however, to which the mathematical is applied today, have changed enormously" (Ulam, 1969).

Kant had already emphasized that mathematics seems characterized by its methods rather than its objects:

> Those thinkers who aim at distinguishing philosophy from mathematics by asserting that the former has to do with quality merely and the latter with quantity haven mistaken the effect for the cause. The reason why mathematical cognition can relate only to quantity is to be found in its form alone. For it is the conception of quantity only that is capable of being constructed that is presented a priori in intuition. (Kant, *Critique of Pure Reason*, B 742)

This conception of mathematics as a science of quantities dominated from Descartes unification of arithmetic and geometry until Cantor and the advent of modern set theory. Set theory was meant to establish the ontological furniture of modern pure mathematics and finally even of mathematical education in school. The essential common characteristics of these ideas about the subject matter consist in the fact that sets are established exclusively in terms of relations of equality and difference.

Until the Renaissance, all knowledge seemed the result of an interpretation of the *Great Book of the Universe*. Then at a certain period in history it happened that words and things parted ways and direct interpretation of our sense impressions seemed to become utterly unreliable. It is a merit of Michel Foucault to have brought these facts into the centre of our attention. At the beginning of the 17th century he says "writing has ceased to be the prose of the world, resemblances and signs have dissolved their former alliance, similitudes have become deceptive. And just as interpretation in the sixteenth century … was essentially a knowledge based upon similitude, so the ordering of things by means of signs constitutes all empirical forms of knowledge as knowledge based upon identity and difference" (Foucault, 1973, pp. 47–51 and pp. 56–57).

Putting *A* and, say, *Two* on the same conceptual level requires to see *Two* as a concept that no longer intends a definite number of specific things. *Two* means "in Vieta … the general *concept* of Twoness in general. … It no longer means or intends a determinate number of things, but the general number-character of this one number" (Klein, 1985, p. 25).

The Complementarity of Syntax and Semantics

We have mentioned above, referring to Plato vs. Wittgenstein the differences between an extensional versus intensional view of mathematics or logic. We believe that the complementarity of the intensions and extensions of symbolic representations, that is, the complementarity of sense and reference of symbols is of fundamental importance to understand mathematical knowledge or discourse (Otte, 2003).

The interesting thing is that algebraic sense is syntactical and operative. The sense or meaning of an algebraic equation consists in its being transformed and eventually solved according to formal rules. Viete's algebraic notation and Descartes algebraization of geometry, did not only provide mathematics and the exact sciences with a new symbolic form, mechanizing and formalizing thinking and rendering it more precise, but also enabled people to experience the yet unknown, by reifying it symbolically. The famous unknown "x" of symbolic algebra permits to make the yet unknown an object of mathematical activity, elaborating its relationships with the known numbers or quantities by a series of diagrams.

In the diagrams of Euclidean geometry reference prevails, in the algebraic diagrams sense prevails and sense or meaning and reference become relatively independent from each other. This makes it possible to provide varying interpretations of such diagrams. If one wanted—in school, for example, to verify the commutative law of addition or of multiplication of arithmetic one might either use a model, a box of beer bottles, counting them in various ways or one might establish the desired outcome by formal operations. The number of paths through a rectangular lattice with sides m and n is $\binom{m+n}{m}$. By cutting the lattice along different axes and counting the paths according to where they cross the cut quite a number of rather difficult formal identities could be derived (Greene & Knuth, 1981, p. 7ff).

In 1975 Sabetai Unguru published an article in which he emphatically criticized the geometric algebra interpretation of Book II of Euclid's *Elements*. That article ignited a fierce controversy with some well-known mathematicians, such as B. v. d. Waerden, A. Weil or H. Freudenthal about the question of whether the term "geometric algebra", coined by the Danish historian Hieronymus Georg Zeuthen (1839–1920) is appropriate for the content of the II. Book of Euclid's *Elements*. But nobody came to discuss the matter in terms of the relation of sense and reference in the diagrams and in terms of the historical changes which this relationship or complementarity underwent in the course of history (Otte, 2003).

The essence of mathematics lies in the complementary roles of meaning and reference and in their independence against each other.

Some elementary examples:

> *How do you justify the commutative law of the arithmetic multiplication in school?* You do it either syntactically, starting from the axiomatic description of the arithmetic operations and the principle of complete induction, or referring to a model, for example trying to count the number of beer bottles in a box and realizing that this can be done in different ways.

> *How to get to introduce a non-commutative product, and thus create a new algebraic structure.* Again, either syntactically—for instance multiplying two matrices and observing that the outcome depends on the order—or again using a model, like the representation of the area of a parallelogram in terms of the anti-commutative vector-product of its sides. Grassmann was the first to make this bold step and he used it to establish an analogy between Newton's law of mass attraction and electromagnetic attraction. (Grassmann in fact corrected Ampere who remained an empiricist and did not see the possibility and importance of the anti-commutative vector-product, see: Lenhard & Otte, 2010, p. 312ff)

Descartes has been first person to realize that the essence of mathematics lies in the combination of calculation, on the one hand and of perception and geometric generality, on the other hand. Descartes had in 1619 already tried to design a program and a method by which the problems of continuous and discrete magnitude could be treated analogically. He programmatically outlines the new ideas in an important letter to Beeckman of March 26, 1619 (Adam & Milhaud, 1936, pp. 5–11). Descartes formulated a program for his future investigations in this letter. The essential idea was to try and establish a productive analogy between arithmetic and geometry. Descartes writes:

> In arithmetic, for instance, some questions can be solved by rational numbers, some by surd numbers only and other can be imagined but not solved. For continuous quantity I hope to prove that similarly certain problems can be solved by using only straight or circular lines, that other problems require other curves for their solution, but still curves which arise from one single motion and which therefore can be traced by the new compasses… and finally that other problems can only be solved by curved lines generated by separate motions…. (Shea, 1991, p. 44)

The Double Nature of Mathematics

Mathematics seems to be that area of intellectual activity, where the difference between concepts and definitions and consequently the difference between seeing or intuiting something, on the one hand, and calculating or formally deriving it, on the other hand, gapes apart most strongly and widely. Philosophy is reasoning by means of concepts, says Kant (1787, B 742) and Deleuze echoes him, "philosophy is the discipline that involves creating concepts" (Deleuze & Guattari, 1994), while mathematics and science frame formal definitions and engage in logical deductions.

The strength as well as the curse of mathematics lies in its exactness. The striving for exactness, precision and explicitness renders the representations of things rather complicated such that these representations often become confused and useless. If the map is so detailed and precisely adapted to the landscape that it becomes a true image of it, it is useless.

Concepts, theories, or works of art should only provide orientation or guidance to stimulate reflection and cognitive activity. If, however, the teacher wants to construct a Nuremberg funnel to guarantee the transmission of knowledge to his students, then exact language and meticulous steering is important. And if someone wants to build a machine or program a computer then too precision and exactness are required. So, mathematics education might satisfy itself and attempt to teach students how to use a coherent system of mathematical reasoning or calculation.

But, theories function on the basis of laws of nature and of theoretical concepts, i.e. on the basis of meanings. Algorithms work with data. Just one example:

Photography was created after physics and chemistry had discovered the relevant fundamental laws of nature. Today we have *flatcams* which are as flat as a credit card. And the camera lenses were replaced by algorithms, which compose an image from many tiny partial pictures, single pixels (internet site of Richard Baraniuk, Victor E. Cameron Professor of Electrical and Computer Engineering at Rice University/USA).

All the disputes and controversies on the question of computer thinking and the shock that computers are now better able to master even the game of Go after they had already surpassed humans in the game of chess for some time now, as well as Dreyfuss's and Searle's comments on the importance of intuition in human thought, are all pointless. They are wiped out by Turing's remark: give me an exact definition of human thinking, and we will build a machine that surpasses this thinking. The true contrast between the formal and algorithmic thinking of computers and human thinking lies in the historicity of man. Humans have a history which must be honoured.

Mathematics as Problem Solving and as a Universal Language

The British philosopher Gilbert Ryle introduced about 1946 a distinction between *knowing-how* and *knowing-that* and he maintained, correctly, we believe, that: "knowledge-how cannot be defined in terms of knowing-that and further knowing-how is a concept logically prior to the concept of knowing-that" (Ryle, 2009, p. 225).

For example, since remote and ancient times quite an amount of mathematical knowledge had been accumulated between Babylonia, Greece and Egypt. Then, about 300 BCE, a man, *Euclid of Alexandria* set out to try and organize this knowledge into a set of textbooks, called Euclid's *Elements.* These were serving as the main textbooks for teaching mathematics (especially geometry) from the time of its publication until the 19th century.

Some 25 years ago Joan Richards reported on philosophical and educational problems with respect to the pursuit of Geometry in Victorian England (Richards, 1988). In Britain Euclidean Geometry did not become less relevant or uninteresting, because the advancements of pure mathematics towards analytical or projective geometry. Geometry's place in the picture of knowledge was originally bound up with its special truth status due to the fact that the objects of geometrical reasoning were clear and distinct, while in analysis and algebra it was often unclear what the symbols meant. The excitement about the status of geometry has perhaps been greater in England and the USA with their empiricist traditions in philosophy than on the continent.

In Euclid's *Elements* Hobbes believed to have discovered a demonstrative science, which could for the first time explain the true foundations of political science and justice (Skinner, 1996). And Hobbes clearly favoured geometry over the new algebraic language:

> Symbols are poor though necessary, scaffolds of demonstrations. … Symbols … do not make the reader understand it sooner than if it were written in words. For the conception of the lines and figures (without which a man learns nothing) must proceed from words either spoken or thought upon. (Thomas Hobbes, Works vol. 7, p. 248, 329)

Hume later adopted similar tones. In algebra, he says, the ideas are always clear and determinate, as well as in geometry. "But the operations are more complex and lengthy so that the conclusion comes at last very wide from of the premises". In the 19th century the cultural preferences looked similar. William Whewell, Britain's most eminent philosopher of science, said in 1838: "It is desirable, not so much to define good arguments, as to feel their force".

Whewell was Master of Trinity College and as a staunch defender of the Cambridge mathematical curriculum he wrote: "The peculiar character of mathematical truth is that it is necessarily and inevitably true; one of the most important lessons which we learn from our mathematical studies is a knowledge that there are such truths and a familiarity with their form and character (quoted from Richards, 1986, p. 300).

Joan Richards comments on this by emphasizing that "the curriculum at Cambridge was designed to introduce upper-class men to a body of truths transcending particular time or place, expressions of 'humanity in its general and permanent character' (Whewell). Within this scheme geometry was one of the permanent studies shared by the nineteenth century British and ancient Greeks" (Richards, 1986, p. 300).

The new developments that grew out of research into the mathematical nature of space like the formal axiomatic approach of Grassmann, Peano or Hilbert called the paradigmatic status of Euclidean geometry into question and caused a lot of excitement with examples of a wholly new approach to mathematical theory and methodology. The new abstract axiomatic or invariant theoretical approach of Hilbert, Cayley, and others, pleased the "modernizers" in the educational camp.

Joan Richards writes, with respect to these modernizers in Victorian England: Although their primary interests were mathematical, these men were active members of the larger intellectual community of post-Darwinian England. … Much of what concerned men like Huxley and Tyndall, or their more mathematical friends like Hirst and Clifford was that science in England needed to be more adequately supported" (Richards, 1988, p. 132). And in this context, "projective geometry was presented as the new naturalistic study of space… In this way, it could be pursued as an integral part of the progressive scientific vision being propounded by a new generation of post-Darwinian scientific publicists" (Richards, 1988, p. 137).

Euclid's geometry was a theory of figures, not a theory of space, while the new geometries were exactly this, such that their methods became more varied while the subject matter became more abstract—think of the ideal and infinitary elements of projective geometry which are harder to conceive outside the theoretical enterprise.

Ultimately all discussions about the nature of geometry are discussions about its foundations, whether they take place at the forefront of mathematical research or on the apparently mundane level of elementary teaching. This is nowhere clearer than in England in the 1870s and 1880s. During this period, virtually all of England's mathematical enthusiasts were directly involved in considering the optimal form of elementary geometry texts (Richards, 1988, p. 163).

On the continent the problems were not so different. In 1905 the geologist Lapparent gave a series of lectures at the *Catholic Institute* in Paris examining the new science and its implications for religion:

> Before 1800 every age had believed in the existence of absolute truth; men differed only in the choices of sources: Aristotle the fathers of the Church, the Bible, philosophy and science all had their day as arbiters of objective eternal truths. In the eighteenth century, human reason alone was upheld and this because of what it had produced in mathematics and in the mathematical domains of science. The hope of mathematical truths had been especially comforting because these held out hope of more to come. Alas the hope was blasted. The end of the dominance of Euclidean geometry was the end of the dominance of all such absolute standards. (Paul, 1979, p. 112)

But let us get back to Euclid's *Elements* themselves!

People start with problems of construction, land surveying, commercial exchange, etc. And then the solutions to these problems are put into an order of their own. As was said, the first who apparently made a more profound effort in these matters, was Euclid. In this way, a first textbook of scientific mathematics emerged.

The geometrical objects and problems became now objects of a theory. And this theoretical context provided them with a new quality. Just as a painted merchant or mayor or a painted house becomes an object of art or architecture, all things become transformed into symbols as elements of a new context. This also happens with the original geometrical problems of measurement and construction. Euclid's elements have not become so famous, because of the geometrical problems themselves, but because of their composition as parts of a theoretical structure. In painting, too, one has understood that the work of art is a space *sui generis* with its own principles. Precisely the same happens to the theoretical connections of classical geometry. This structure is not based on a logical-deductive connection, but it arises from the activity of solving geometric problems themselves.

Take the proof of Theorem 44 of Book I.

- To prove it one refers back to Theorem 42, Theorem 29 and Theorem 15

and in addition to Axiom 8 and Postulate 5.

- and to prove Theorem 42 we refer to Theorems 10, 31 and 41.
- and to prove Theorem 41 we refer to Theorems 34 and 37, etc., etc.

Or open Book III: Theorem 22 is reduced to Theorem 21 and this is reduced to Theorem 20 etc. …

There are no general rules or logical syllogisms involved. The student cannot solve a somewhat complicated problem like that of Theorem 44, if she does not know at least some of the conditions represented by Euclid. One has to closely follow the progress of the text. In addition on should also remind oneself that the essential postulates of Euclid's *Elements* are assumptions about possible constructions. We read, for example: "Let the following be postulated: "To draw a straight line from any point to any point", or: "a circle can be drawn with any point as its centre and with an arbitrary radius", …, etc. etc. Under these conditions, a proof in the elements is nothing different than the demonstration that if certain operations or constructions are licensed, something can be constructed. One might think of Euclid's first theorem which requires: "On a given straight line to construct an equilateral triangle".

It is always a matter of solving certain construction problems, which in turn require the solutions of other problems. This character of Euclid's *Elements* is reflected already in the commentaries of Proclus:

> Now some of the ancients, however, such as the followers of Speusippus, insisted on calling all propositions 'theorems', considering 'theorems' to be more appropriate designations than 'problems' for the objects of the theoretical sciences since these sciences deal with eternal things. There is no coming to be among eternals, and hence a problem has no place here, proposing as it does to bring into being or to make something not previously existing - such as to construct an equilateral triangle …. Thus it is better, according to them, to say that all these objects exist and that we look on our construction of them not as making, but as understanding them …. Others, on the contrary, such as the mathematicians of the school of Meneachmus, thought it correct to say that all inquiries are problems. (Morrow, 1970, pp. 63–64)

There is another remarkable feature of Euclid's text to be mentioned. It is addressed to the active individual and it is not a typical textbook to be used in larger school classes.

L. E. J. Brouwer (1881–1966) was a famous Dutch mathematician, a philosopher and an extremely individualistic personality. As a philosopher of mathematics, he became known for his intuitionism and his rejection of the logical law of excluded middle in mathematical reasoning. Brouwer's intuitionistic constructivism has a somewhat very Euclidean touch, assuming a fixed set of capacities on the subject's side and concluding that this implies a fixed set of trustworthy mathematical operations.

Andrey Kolmogorov, one of the greatest mathematicians of the 20th century, interpreted Brouwer's intuitionistic logic in terms of problems and solutions. To assert a formula is to claim to know a solution to the problem represented by that formula. For instance, "*P implies Q*" is the problem of reducing *Q* to *P*; to solve it, requires a method to solve problem *Q,* given a solution to problem *P*. Kolmogorov writes:

> In addition to the theoretical logic which systematizes the proof schemes of the theoretical truths, one can also systematize the solutions of problems, e.g. of geometric construction problems. In analogy to the principle of syllogism, the following principle holds here: if we can reduce the solution of *b* to the solution of *a* and the solution of *c* to the solution of *b*, then we can also reduce the solution of *c* to the solution of *a* …. Thus, in addition to the theoretical logic, a new calculus of problems is obtained. One does not need any special epistemological or intuitionist assumptions. The following remarkable fact applies: According to its form, this task calculus coincides with Brouwer's intuitionist logic, as formalized by Mr. Heyting …. It is shown that this formalized intuitionistic logic should be replaced by the calculus of problems, since its objects are in reality not theoretical statements, but rather problems. (Kolmogorov, 1932, p. 58, our translation)

Up to about 1800 mathematics consisted mainly in problem solving and the search for more powerful methods and new applications ruled the scene. Since then mathematics became most valued as a language (Effros, 1998, p. 131). The view of mathematics as language has especially been prominent in educational contexts and to the attempts to teach students how to use a coherent system of mathematics.

The pedagogical principles underlying mathematics instruction, says Effros “are quite similar to those used in language instruction” (Effros, 1998, p. 135). No protests please! In educational contexts problems play an auxiliary role only. We do not, says Effros, “include algebra in the high school curriculum in order to enable students to solve ‘word problems’” (op. cit.).

The core of the conception of mathematics as language consists in the axiomatic method and in deductive reasoning. This method came about as soon as one realized that one was free to create all kinds of algebraic systems in which the variable stood for completely new objects. The works of Evariste Galois (1811–1832), Hermann Grassmann (1809–1878), A. DeMorgan (1806–1878), George Boole (1815–1864) or Benjamin Peirce (1809–1880) in the United States, to name just a few, provide a clear idea of this new algebraic spirit.

There is Another Side to Plato

How can mathematics be applied? Plato reflected about this problem too. He begins with a comparison: Hunting, he writes in the dialogue *Euthydemus*,

> does not extend any further than pursuing and capturing: whenever the hunters catch what they are pursuing they are incapable of using it, they and the fishermen hand their prey over to the cooks. And again geometers and astronomers and calculators … since they themselves have no idea of how to use their prey, but only how to hunt it, hand over the task of using their discoveries to the dialecticians. (290c)

And the dialecticians argue not from truths, but from more or less arbitrary hypotheses. Thus, mathematics can and must be applied by hypothetical-deductive reasoning. Ironically this fact has bestowed mathematics with some extraordinary power and efficiency in exploring the real world, because the new can only be represented, or rather indicated, but cannot be interpreted or described. If all thinking would be conceptual, it would be a mere recognition of the implicitly already known, in the manner of Plato’s dialogue *Meno* and all knowledge would become analytical.

In this dialogue the protagonist asks how it is at all possible, to determine something completely unknown. Meno asks Socrates: “How will you aim to search for something you do not know at all? If you should meet it, how you will know that this is the thing you did not know” (80d). Socrates refers to the immortality of the soul, in which all ideas are already engraved, such that we can know about things, because we already carry the associated idea within ourselves. We recognize things, simply because all knowledge is based on recollection and recognition, he says.

If we happen to hit on something completely unknown, however, the only thing one can do is to start with some more or less arbitrary hypothesis and develop its consequences. There is a difference between mathematics, on the one side and philosophy, on the other side. Mathematics is no hermeneutic science, based on interpretations of the continuously varying meanings of words or concepts. The mathematician, confronted with a certain problem, presents a definition and a hypothesis and proceeds to explore the consequences, by formal hypothetic-deductive reasoning.

In Plato's *Meno*, Socrates employs a geometrical example to demonstrate that the hypothetic-deductive method of mathematics helps even philosophy and complements philosophical conscience, because it takes the unknown as an object of activity and arrives at a solution even without being fully aware of the eternal ideas beneath. The discourse in the *Meno* dialogue shows, for example, that if it is possible to construct a quadrilateral, then it is also possible to construct a quadrilateral of twice the size of the original one.

Plato cites still another geometrical problem (see Meno 87a) showing that the method of hypothesis might be useful, even if it does not lead to a solution. Socrates says: "It seems we have no choice but to investigate a particular quality of a thing whose general nature we do not know. … Please consent to investigate whether virtue is teachable or not by means of a hypothesis. I have in mind a method geometers often employ to get on with their investigations".

Episteme Versus Techne

The importance of philosophy to mathematics education results not least from the fact that the concept of "explanation" is central to our educational practices and aims. Mathematics could not fruitfully be organized and pursued at school as a primarily professional topic or as a mere language without content. Mathematical education has, like other subjects, also to contribute to a common search for clarity on fundamental issues and to the formation of the person, such that philosophical formation and historical sensibility seem as important as logical exactness or mathematical literacy.

When trying to educate the younger generation within to-day's technological "knowledge society", it seems worthwhile to remember, that knowledge should when considered from an educational point of view, fulfils two major roles in human society: a technical one and a philosophical one. General education is to be based on proven scientific knowledge not the least because "it seems that science came into being with the requirement of […] coherence and that one of the functions it performs permanently in human culture consists in unifying […] practical skills and cosmological beliefs, the *episteme* and the *techne* […] despite all changes that science might have undergone, this is its permanent and specific function which differentiates it from other products of human intellectual activity" (Amsterdamski, 1975, pp. 43–44).

One might doubt, however, that Amsterdamski's vision can become true, simply because there is a distinction between *knowing-how* and *knowing-that* and knowers are not doers, as a rule. Theories are not theories of their own application, one the hand. And on the other hand, technical discovery is not brought about by the mere implementation of a scientific discovery, but is, when it occurs autonomously inventive. Even Plato himself seemed undecided about this relationship between men's technical skills and his human wisdom. In his dialogue, *Protagoras*, there is a discussion between Socrates and the sophist Protagoras and others about whether *virtue* were teachable. Protagoras argued in favour of the necessity of its teaching, because of the forgetfulness of *Epimetheus*, who was *Prometheus'* brother.

The brothers were entrusted with distributing all the necessary traits among the newly created living beings, animals as well as humans. Prometheus agreed to let Epimetheus do the job satisfying himself with supervision after the job had been done. However, while Prometheus is characterized as ingenious and clever, Epimetheus is depicted as foolish. Epimetheus distributed everything to humans and animals that was necessary to survive in hostile Nature, but when it came to the human virtues and men's social characteristics, he found that, lacking *foresight,* he had given out everything, such that there was nothing left.

This affected the whole nature of humankind having become beings in whom thought follows production. Humans thus came to represent Nature in the sense of pragmatism, positivism and materialism, assuming that thought comes later than thoughtless bodies and their thoughtless motions Philosophers like Husserl (1970) or Stiegler (1998) became worried with the reduction of the idea of science to mere factual science and technology, especially so, because technological sets the pace of development, rather than human reflection. They claimed that mathematized science has "decapitated philosophy" because "merely fact-minded sciences make merely fact-minded people" (Husserl, 1970, pp. 6–9). Paolo Rossi, on the other side, accuses philosophers like Husserl, Horkheimer or Heidegger to try and resume the process of the Inquisition against Galilei (Rossi, 1992).

Mathematics: Theories and Algorithms

When we observe the animals in the wilderness and perceive how clever they are and what extraordinary physical strengths and skills they have, we might start to wonder what had made humans still superior or more successful with respect to the challenges of evolution. Philosophers in trying to answer such questions have become convinced of the importance of tools of all kind.

> In back of the development of tools and machines lies the attempt to modify the environment in such a way as to fortify and sustain the human organism: the effort is either to extend the powers of the otherwise unarmed organism, or to manufacture outside the body a set of conditions favourable toward maintaining its equilibrium and ensuring its survival. (Mumford, 1934, p. 10)

Scientists, like Mumford see the difference between tool and machine in the independence of the efficiency of the machine from the experience and individual skill of a worker. This principle created the whole modern science of technology, logic, and computer-science. Such was the conclusion of the Vienna Circle logical empiricist too. Rudolf Carnap, for example, writes in his lectures on logic:

Logic is regarded as a tool "needed for the task of getting and systematizing knowledge. As a hammer helps man do better and more efficiently what he did before with his unaided hand so a logical tool helps a man do better and more efficiently what he did before with his unaided brain, that is, by means of instinctive habits, rather than through deliberate acts guided by explicit rules" (Carnap, 1947, p. VIII).

And in his book *The Advent of the Algorithm* Berlinski claims that two ideas have determined our history:

> The calculus and the algorithm. The calculus and the rich body of mathematical analysis to which it gave rise made modern science possible; but it was the algorithm that made the modern world possible. … The algorithm gave us–is still giving us–the computer (or, more precisely, the computer program) and this is what made the modern world possible. Without the algorithm, there would have been no computer, no Internet, no virtual reality, no e-mail, or any other technological advance that we rely on every day.

The *New Math* wanted to bring theory and algorithm into harmony, but failed. The question of mathematical objects is traditionally regarded as the central problem of our human self-understanding, in which all the questions about the relationship between mind and world, man and history, etc. etc. bundle up and it has therefore to be clear and open to direct simple understanding. The educational movement of the *New Math* had started from such an assumption.

The proof methods of mathematics in contrast have become ever more complicated and ever more distanced from human understanding. And the computer plays an important role in this distancing process. Davis and Hersh comment on the computer proof of the four-color theorem by Haken and Appel:

> When I heard that the four-color theorem had been proved, my first reaction was, Wonderful! How did they do it? I expected some brilliant new insight, a proof which had in its kernel an idea whose beauty would transform my day. When I received the answer, …. I felt disheartened. (Davis & Hersh, 1981, p. 384)

Vladimir Alexandrovich Voevodsky, Field medallist for the year 2002 tells a different story with the same result however:

> In October 1998, Carlos Simpson submitted to the arXiv preprint server a paper called "Homotopy Types of Strict 3-groupoids." It claimed to provide an argument that implied that the main result of a paper I had published in cooperation with Kapranov contained a mistake. But I was sure that we were right until the fall of 2013 (!!). I can see two factors that contributed to this outrageous situation: Simpson claimed to have constructed a counterexample, but he was not able to show where the mistake was in our paper. Because of this, it was not clear whether we made a mistake somewhere in our paper or he made a mistake somewhere in his counterexample. Mathematical research currently relies on a complex system of mutual trust based on reputations. … I think it was at this moment that I largely stopped doing what is called "curiosity-driven research" and started to think

seriously about the future. I didn't have the tools to explore the areas where curiosity was leading me and the areas that I considered to be of value and of interest and of beauty. So, I started to look into what I could do to create such tools. And it soon became clear that the only long-term solution was somehow to make it possible for me to use computers to verify my abstract, logical, and mathematical constructions. When I first started to explore the possibility, computer proof verification was almost a forbidden subject among mathematicians. … Today, only a few years later, computer verification of proofs and of mathematical reasoning in general looks completely practical to many people. (Oral presentation Bielefeld University, July 2016)

Conclusion

Today's academic community is led to observe a deep abyss between mathematicians and historians of mathematics. The same is true, albeit to a lesser degree with respect to the relationships between mathematics and philosophy, especially linguistic or analytic philosophy. So, mathematical education might experience difficult times, looking for orientation. Mathematics could not fruitfully be organized and pursued at school as a primarily professional topic. But, seeking shelter in the calm bays of psychology is not really an alternative. The discussion above was meant to indicate some further possibilities to reflect about the philosophy of mathematics education.

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A Dialogical Conception of Explanation in Mathematical Proofs

Catarina Dutilh Novaes

Abstract In this chapter, I argue that the issue of explanatoriness in mathematical proofs can be fruitfully addressed within the dialogical conceptualization of proofs that I have been developing in recent years. The key idea is to emphasize the observation that a proof is a piece of discourse aimed at an intended audience, with the intent to produce explanatory persuasion. This approach explains both why explanatory proofs are to be preferred over non- or less explanatory ones, and why explanatoriness is an audience-relative property of a proof. This account is also able to clarify a number of features of mathematical practice.

Keywords Explanation · Mathematical proofs · Dialogues

Introduction

In 1959, Stephen Smale proved that all immersions of the *n*-sphere into Euclidean space are regularly homotopic; this implies that the sphere can be everted (turned inside out in a three dimensional space with possible self-intersections but without creating any crease). The result was surprising, and moreover Smale's proof (an indirect proof) did not reveal details of the process of eversion (Jackson, 2002). As such, while the proof itself was considered to be correct, it failed to explain the very process whose existence it proved. It was only when Bernard Morin created clay models that exhibited a homotopy carrying out the eversion in 1967 that the process became fully understood (Dutilh Novaes, 2013). (Later on, different eversions were described.) Until then, Smale's proof was viewed as so puzzling that it was often (and is still now) referred to as *Smale's paradox*.

Smale's proof can be viewed as a quintessential example of a proof that, although correct and accepted as such (thus establishing the truth of the conclusion)

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fails to be *explanatory*. It is precisely its lack of explanatoriness that makes it seem paradoxical, and which led Morin and other mathematicians to continue to work on the problem, despite the fact that the possibility of eversion itself had already been proved.

The distinction between non-explanatory and explanatory proofs is often formulated in terms of the difference between proofs that merely establish *that* the conclusion is the case (that is, if the premises are the case) and proofs establishing *why* the conclusion is the case; the former merely demonstrate, while the latter *explain* (Colyvan, 2012, p. 76). Explanatory proofs are those that 'seem to reveal the heart of the matter' (Davis & Hersh, 1982; quoted in Resnik & Kushner, 1987). Thus described, explanatoriness is (presumably) a desirable property in a proof, and all things being equal, mathematicians would prefer explanatory over non-explanatory proofs. But what makes it so that some proofs are explanatory and some are not?[1]

The issue of the explanatoriness of mathematical proofs has received significant attention in recent decades, both in philosophy (Mancosu & Pincock, 2012) and in mathematics education (Hanna, Jahnke, & Pulte, 2010).[2] The topic itself has an old and distinguished pedigree: it was discussed by ancient authors such as Aristotle and Proclus, as well as by Renaissance and early modern authors (Mancosu, 2011: Sect. 5). In recent decades, the debate was reignited among philosophers by the work of Steiner in the late 1970s (Steiner, 1978), and in mathematics education (as well as in some philosophical circles) by Lakatos' emphasis on the epistemic, explanatory (as opposed to merely justificatory) role of mathematical proofs in *Proofs and Refutations* (Lakatos, 1976).

One prominent strand within philosophy of mathematics takes as its starting point discussions on explanation in the (empirical) sciences. According to a widespread, broadly Aristotelian account, a scientific theory is truly explanatory iff it accurately tracks the causal processes underlying the phenomena that it seeks to explain. Transposed to mathematics, the question then becomes: are mathematical proofs explanatory in the same way as scientific theories are, i.e. insofar as they track causal processes? It is not obvious how such a causal story for mathematical proofs might be told. Does it make sense to say that some mathematical truths can *cause* some other mathematical truths? For this to be the case, one would presumably have to accept not only the independent existence of mathematical entities, but also the idea that they can causally influence each other—which would entail a very inflated and somewhat implausible conception of mathematical ontology.

Most authors involved in these debates seem to think that a full-blown causal story is unlikely to be satisfactory in the case of mathematical proofs. Different accounts of the explanatoriness of mathematical proofs that are not fully causal/

[1]See Baracco (2017) for a critique of this idea, which is described as 'someism'. Instead, Baracco defends the view that *all* proofs are explanatory, at least in some sense.

[2]It is important to bear in mind that the whole discussion here pertains to so-called 'informal' deductive proofs (such as proofs presented in mathematical journals or textbooks), not to proofs within specific formal systems.

metaphysical have been proposed (Lange, 2016), but a number of them (Colyvan, 2010; Steiner, 1978) still seek to establish purportedly objective properties of proofs (or of the relevant mathematical entities/structures) as the ground for explanatoriness (or lack thereof), thus attempting to establish a non-causal but still objective version of explanation for mathematics.[3]

Within debates on scientific explanation, there are those who lean towards metaphysical, causal conceptions of explanation (Salmon, 1984), and those who adopt a pragmatic perspective. The 'explanation as answer to why-questions' approach developed by van Fraassen (1977, 1980) is the most prominent example of the latter. A similar divide can be found among accounts of the explanatoriness of mathematical proofs, and some of those endorsing a pragmatic, contextual approach are Heinzmann (2006) and Paseau (2010).[4] But even those who adopt a pragmatic perspective on explanation tend to formulate 'disembodied' accounts of mathematical proofs as freestanding entities floating in the air, as it were: neither produced not consumed by anyone.[5]

As may be expected, the literature on the explanatoriness of proofs within mathematics education is much less prone to agent-less approaches. Mathematics educators are after all ultimately interested in the very concrete, multi-agent phenomenon of teacher-student interaction, and in fairly tangible epistemic questions pertaining to understanding mathematical proofs and mathematical concepts.[6] A number of them (Alibert & Thomas, 1991; Balacheff, 1991; Ernest, 1994) emphasize the role of proofs as devices for interpersonal persuasion and explanation. After all, what is an explanation if not a discourse produced by someone with the intent to explain something to someone else? (Or to oneself, in the limit case.)

My main claim in this chapter is that there is much to be gained for a *philosophical* account of the explanatoriness (or lack thereof) of mathematical proofs by adopting a multi-agent perspective. More specifically, I here apply the dialogical account of mathematical proof that I've developed elsewhere (Dutilh Novaes, 2016)

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[3]An interesting recent development is Gijsbers' quasi-interventionist theory of mathematical explanation, which draws on Woodward's interventionist theory of causation (Gijsbers, 2017). However, the resulting conception of explanatoriness is according to the author himself "subjective in the sense that it does not depend on the objective structure of mathematics itself" (p. 47).

[4]A third approach might be described as 'nihilist': explanation is simply not a useful concept with respect to mathematical proofs (Resnik & Kushner, 1987; Zelcer, 2013).

[5]It is revealing that in van Fraassen's account of scientific explanation based on the idea of why-questions, explanation "is a three-term relation, between theory, fact, and context" (van Fraassen, 1980, p. 153). What is conspicuously missing in van Fraassen's account from the present perspective are the *agents* consuming and producing these explanations, i.e. the agents asking the why-questions and the agents answering them. (More on van Fraassen shortly.)

[6]Recently, in the literature on scientific explanation, epistemic approaches focusing on *understanding* have also gained some traction (Grimm, Baumberger, & Ammon, 2016; de Regt 2009; Khalifa, 2012). To my knowledge, this trend has not yet reached discussions on mathematical explanation (with the exception of Dufour (2013a, b), who discusses explanation and understanding with respect to mathematical proofs), though the question of understanding with respect to proofs has been discussed by some philosophers (Avigad, 2008).

to the issue of mathematical explanation.[7] This will result in a resolutely 'embodied', pragmatic and independently motivated dialogical account of the explanatoriness of mathematical proofs.

The dialogical perspective provides a very natural rationale for why we want mathematical proofs to be explanatory: because they are pieces of discourse offered by the proof producer to an intended audience, with the intent to produce explanatory persuasion.[8] The dialogical perspective also explains a number of other phenomena related to the explanatoriness of proofs: context-dependency, granularity, the search for new proofs of already proved theorems. At the same time, it allows for the formulation of a notion of explanatoriness *simpliciter* in the limit case, in terms of the rhetorical concept of a *universal audience*.

I proceed in the following way: Sect. "Explanatoriness in Mathematics" presents a brief overview of some of the philosophical debates on mathematical explanation; Sect. "A Dialogical Conception of Mathematical Proof" presents my dialogical conception of mathematical proof; Sect. "Mathematical Proofs as Explanatory Fictive Dialogues" unpacks what the dialogical perspective has to say about the explanatoriness of mathematical proofs.

Explanatoriness in Mathematics

A presupposition running through philosophical discussions on mathematical explanation and proofs is that there really is a meaningful, useful distinction between proofs that are explanatory and proofs that are not ((Baracco, 2017; Resnik & Kushner, 1987; Zelcer, 2013) are the exceptions). However, it is a legitimate question whether mathematicians themselves do or do not view the categories of 'explanatory' and 'non-explanatory' as relevant for their work. Colyvan (2012, p. 79) recognizes that judgments of explanatoriness are typically left out of the published work of mathematicians; he also notes that "it is difficult to find a great deal of agreement in the mathematical literature on which proofs are explanatory and which are not".[9] Nevertheless, he maintains that these categories are in fact relevant for the practices of mathematicians, and thus that explanatoriness is an

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[7]Naturally, I am not the first one to emphasize the dialogical nature of mathematical proofs; this idea is at the heart of Lakatos' *Proofs and Refutations*, and has been developed in detail by Ernest (1994).

[8]For other dialectical, argumentative approaches to mathematical proofs, see Dufour (2013a, b) and Aberdein (2013). I will comment on these approaches later on.

[9]Whether mathematicians converge in their attributions of explanatoriness to proofs is essentially an open question. Much of the philosophical literature seems to presuppose that they do, but this is of course ultimately an empirical question. The philosophical significance of consensus or lack thereof on the explanatoriness of proofs, as well as preliminary empirical results, will be discussed in more detail in the final section of this chapter.

important topic for the philosopher of mathematics.[10] Hafner and Mancosu (2005) reach a similar conclusion on the relevance of explanation for mathematical practice on the basis of a number of case studies.

But assuming that explanatoriness is an interesting property of (some?) mathematical proofs, a number of questions immediately arise. For example, is the distinction between explanatory and non-explanatory proofs a sharp distinction, or is it one allowing for degrees, such that explanatoriness becomes a comparative notion? In other words, perhaps a proof is not explanatory as such, but rather more explanatory than some proofs, as well as less explanatory than others.[11] Moreover, is explanatoriness an objective, absolute property of the proofs themselves, or is it a property that is variously attributed to proofs first and foremost based on contextual, variable elements? Are judgments of explanatoriness agent-relative?

As mentioned above, Steiner's seminal work in the late 1970s has been and continues to be very influential in debates on explanatoriness. He introduced the notion of a *characterizing property* as what distinguishes explanatory from non-explanatory proofs: "an explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property" (Steiner, 1978, p. 143). This notion of dependence seeks to capture a non-causal but realist analogue of the notion of causation in scientific explanation more generally. As described by Mancosu and Pincock (2012, p. 16), "Steiner argues that these dependence relations require both that the entity or structure be uniquely picked out by some characterizing property, and that the explanatory proof be part of a family of proofs where this property is varied." Importantly, the notion of a characterizing property is attributed to mathematical entities or structures, and is thus conceived of in realistic, objective, non-contextual terms. However, there is still an irreducibly epistemic component in how the characterizing property is (or is not) suitably captured in the proof.

As pointed out by Mancosu (2011), Steiner's is a local account, i.e. explanatoriness is a local property of a given proof. In contrast, another influential approach, namely Kitcher's (1989), is aptly described as global/holistic in that the explanatoriness of proofs is viewed in the broader context of (mathematical) knowledge as a whole. For Kitcher, the key notion is that of *unification*: "an explanatory proof in pure mathematics is one that is part of a small collection of argument patterns that allows the derivation of the mathematical claims that we accept." (Mancosu & Pincock, 2012, p. 15). So while Kitcher's notion of explanatoriness is not located at the level of mathematical entities or structures, but rather at the level of argument patterns, it is presented in non-contextual, absolute terms, i.e. with respect to the

[10]Some philosophers may wish to maintain that whether mathematicians themselves view explanation as an important concept is irrelevant to determine the *philosophical* significance of the topic. But to disregard mathematical practice in this way seems to me to be a highly problematic move.

[11]Pincock (2015) adopts the comparative perspective, thus asking the question 'what makes a proof *more* explanatory than other proofs?' rather than 'what makes a proof explanatory *simpliciter*?' (More on this point shortly.)

totality of our mathematical knowledge. In particular, thus described, attributions of explanatoriness do not seem to be agent-relative in any way.[12]

A natural reaction to such approaches would be to emphasize the inevitable contextual, pragmatic elements involved for a mathematical proof to be considered as more or less explanatory—in other words, the human factor. One motivation to adopt this perspective is the observation that explanation may be fruitfully viewed as a triadic concept, involving the *producer* of the explanation, the explanation itself (the proof), and the *receiver* of the explanation. The key idea would be that an explanation is typically addressed at a potential audience; one explains something *to someone else* (or to oneself, in the limit case) (Walton, 2004).

Though perhaps promising at first sight, turning to a pragmatic account of scientific explanation such as van Fraassen's does not seem to offer the right starting point for an analysis of mathematical explanation in this vein. As argued by Sandborg (1998, p. 604), "explanations do answer why-questions and explanatory evaluations are context dependent, but […] the why-question approach nonetheless misses crucial aspects of certain explanatory evaluations." Indeed, as noted above, van Fraassen's account leaves out the agents producing and consuming explanations, despite its focus on why-questions. And yet, it can be argued that disregarding the perspective of the agents involved makes it virtually impossible to have a meaningful discussion on proofs being explanatory or not. For instance, Paseau (2010) presents an insightful overview of various proofs of the compactness theorem, and concludes:

> Finally and perhaps most obviously, our discussion highlights the context dependence of mathematical arguments' explanatoriness. Explanatoriness seems to depend on what areas of mathematics one is familiar with. […] Whether an argument is elegant or explanatory therefore turns on what assumptions it employs and whether *the audience to which it is targeted* is familiar with those assumptions. (Paseau, 2010, pp. 96–97, emphasis added)

In other words, perhaps the most relevant contextual element determining whether a proof will be explanatory is its suitability for its target audience, in terms of prior knowledge and familiarity with the relevant conceptual tools. For readers coming from the field of mathematics education, this may seem like an almost trivial observation; but among philosophers, it remains controversial.

In what follows, I argue that an independently motivated dialogical account of mathematical proofs based on the insight that proof (just as explanation) is essentially a discourse aimed at a putative audience leads to a compelling analysis of mathematical explanation, one that accounts for a number of features of mathematical practice in relation to proofs. But before moving on, let me quickly

[12]"Like Hempel, Kitcher remains half-pragmatic and says nothing about dialectical or interactive features in the use of explanation and argument. His very formulation is symptomatic: "scientific explanation advances our understanding". Whose understanding? Yours, mine?" (Dufour, 2013b, p. 11).

comment on some recent work on mathematical proofs from the perspective of argumentation theory, since this approach comes quite close to some of the ideas that I will develop in the next sections.

Dufour (2013a, b) provides a useful overview bringing together different debates which thus far have developed largely independently of each other: the philosophical debate on scientific explanation as well as on the explanatoriness of proofs involving Steiner, Kitcher, Mancosu etc.; and discussions on explanation within argumentation theory. One question he addresses is whether a mathematical proof is an argument, and whether it is/can be an explanation (a question further taken up by Aberdein's (2013) commentary on Dufour). He discusses some of the arguments that have been given to the effect that mathematical proofs are best not conceived of as arguments, but finds them unpersuasive. He thus joins the growing pool of authors who view mathematical proofs as specific kinds of arguments (see Aberdein & Dove, 2009, 2013, and my own dialogical approach to proofs). If this is so, and if some mathematical proofs are indeed explanatory, then at least some arguments (proofs) must also count as explanations.[13]

A Dialogical Conception of Mathematical Proof

Within the (broadly conceived) practice-based approach adopted here (as well as in most of my work in the philosophy of mathematics), *functional* questions become particularly important. With respect to proof, the relevant questions become: what is the function (or what are the functions) of a proof? Why do mathematicians bother producing proofs at all? What effects do they seek to obtain? Mathematicians and philosophers of mathematics tend to overlook these important questions, but they have been addressed by a number of authors (Auslander, 2008; Dawson, 2006; Hersh, 1993; Rav, 1999).

One promising vantage point to address these questions is what could be described as a *genealogical approach* (Dutilh Novaes, 2015). When it comes to functionalist questions, it makes sense to inquire into what the first practitioners of a given practice thought they were doing, and why they were doing it when the practice first emerged. (But note the likelihood of *shifts of function* along the way.) In this case, the historical emergence of deductive proof in ancient Greek mathematics is a particularly relevant data point, and on this topic the most authoritative study remains (Netz, 1999). Netz emphasizes the importance of persuasion, orality, and dialogue for the emergence of classical, 'Euclidean' mathematics in ancient Greece:

[13]Some authors take the two notions of argument and explanation to be disjoint, but Walton, Dufour, and Aberdein have convincingly disputed this idea.

> Greek mathematics reflects the importance of *persuasion*. It reflects the role of orality, in the use of formulae, in the structure of proofs… But this orality is regimented into a written form, where vocabulary is limited, presentations follow a relatively rigid pattern… It is at once oral and written… (Netz, 1999, pp. 297–298)

Netz's interpretation relies on earlier work by Lloyd (1996), who argues that the social, cultural and political context in ancient Greece, and in particular the role of practices of *debating*, was fundamental for the emergence of the technique of mathematical deductive proofs (see also Jahnke, 2010). So from this perspective, it seems that one of the main functions of deductive proofs (then as well as now) is to produce *persuasion*, in particular what one could call *explanatory persuasion*: to show not only *that* something is the case, but also *why* it is the case.[14] As well put by Dawson (2006, p. 270):

> [W]e shall take a proof to be an *informal* argument whose purpose is to convince those who endeavor to follow it that a certain mathematical statement is true (and, ideally, to explain *why* it is true).

What I add to Dawson's description is an explicit multi-agent, dialogical perspective, which is only implicit in his description. On this conception, a deductive proof corresponds to a dialogue between the person wishing to establish the conclusion (given the presumed truth of the premises), and an interlocutor who will not be easily convinced and who will bring up objections, counterexamples, and requests for further clarification and precision. A good proof is one that convinces a fair but 'tough' opponent; as allegedly noted by mathematician Mark Kac, "the beauty of a mathematical proof is that it convinces even a stubborn proponent" (Fisher, 1989, p. 50). Now, if this is right, then mathematical proof is an inherently dialogical, multi-agent notion, given that it is essentially a piece of discourse aimed at a putative audience, typically composed of 'stubborn' interlocutors.

To be sure, there are different ways in which the claim that mathematical proofs are essentially dialogical can be understood. For example, the fact that a proof is only recognized as such by the mathematical community once it has been sufficiently scrutinized by trustworthy experts can also be viewed as a dialogical phenomenon, perhaps in a loose sense (the 'dialogue' between the mathematician who

[14]For Hersh (1993), proof is also about convincing and explaining, but on his account these two aspects come apart. According to him, convincing is aimed at one's mathematical peers, while explaining is relevant in particular in the context of teaching. Similarly, in argumentation theory, persuasion and understanding are often thought to be orthogonal phenomena: persuasion would be related to a mild form of coercion via argumentation—one cannot but assent to the conclusion—whereas understanding via explanation presupposes a certain amount of cognitive freedom. (See Wright, 1990 on the two phenomena.) On my story however, persuasion and explanation go hand in hand in mathematical proofs; a proof will be more persuasive precisely if it is viewed as (more) explanatory. Smale's proof of the eversion of the sphere, for example, was viewed as paradoxical precisely for its lack of explanatoriness.

formulates a proof and the mathematical community who scrutinizes it).[15] But in what follows I present a more precise rational reconstruction of the (quite specialized) dialogues that would correspond to deductive proofs.

On this conception, proofs are semi-adversarial dialogues of a special kind involving two participants: Prover and Skeptic.[16] Prima facie, the participants have opposite goals, and this is why the adversarial component remains prominent: Prover wants to establish the truth of the conclusion, and Skeptic will not be easily convinced. The dialogue starts with Prover asking Skeptic to grant certain premises. Prover then puts forward further statements, which purportedly follow from what has been granted. (Prover may also ask Skeptic to grant additional auxiliary premises along the way.)

It may seem that most of the work is done by Prover, but Skeptic has an important role to play, namely to ensure that the proof is persuasive, perspicuous, and correct.[17] Skeptic's moves are: granting premises so as to get the proof going; offering a counterexample when an inferential move by Prover is not really necessarily truth-preserving (or a global counterexample to the whole proof);[18] asking for clarification—why-questions—when a particular inferential step by Prover is not sufficiently compelling and clear. These three moves correspond neatly to what are arguably the three main features of a mathematical proof: it starts off with certain premises; it proceeds through necessarily truth-preserving inferential steps; these steps should be individually evident and compelling.

From this point of view, a mathematical proof is characterized by a complex interplay between adversariality and cooperation: the participants have opposite goals and 'compete' with one another at a lower level, but they are also engaging in a common project to investigate the truth or falsity of a given conclusion (given the presumed truth of the premises) in a way that is not only persuasive but also

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[15]Take for example the ongoing saga of Mochizuki's purported proof of the ABC conjecture, which is for now still impenetrable for the mathematical community at large, and so it remains in a limbo. See http://www.nature.com/news/the-biggest-mystery-in-mathematics-shinichi-mochizuki-and-the-impenetrable-proof-1.18509. See also (Auslander, 2008) for proof as certification.

[16]This terminology comes from the computer science literature on proofs. The earliest occurrence that I am aware of is in Sørensen and Urzyczyn (2006) who speak of *prover-skeptic games*. One may think of the interplay between proofs and refutations as described in Lakatos' (1976) seminal work as an illustration of this general idea: Prover aims at proofs, Skeptic aims at refutations. The 'semi' qualification pertains to the equally strong cooperative component in a proof, to be discussed shortly. See also Pease, Lawrence, Budzynska, Corneli, and Reed (2017) for a formalized version of Lakatosian games of proofs and refutations, again very much in the spirit of the Prover-Skeptic dialogues here described.

[17]Moreover, again on a Lakatosian picture, refutations and counterexamples brought up by Skeptic may play the fundamental role of refining the conjectures and their proofs (see quote in footnote below).

[18]Lakatos (1976) distinguishes between global and local counter examples.

(ideally) explanatory.[19] If both participants perform to the best of their abilities, then the common goal of producing mathematical knowledge will be optimally achieved.[20] The cooperative component also stands out from the observations that Prover seeks to assist Skeptic in truly understanding why the conclusion follows from the premises, and that (by asking appropriate why-questions) Skeptic may help Prover to formulate an illuminating proof.[21]

At this point, the reader may be wondering: this is all very well, but obviously deductive proofs are not really dialogues, as they are typically presented in writing rather than produced orally.[22] If at all, there is only one 'voice' that we hear, that of Prover. So at best, they must be viewed as monologues. But a mathematical proof is arguably an instance of what linguist E. Pascual describes as a *fictive interaction*, that is, a piece of discourse that is apparently monological but in fact reproduces a multi-agent communicative scenario.

> Specifically, we present the premise that there is a conversational basis for language, which serves to partly structure cognition, discourse, and grammar. Stemming from this tenet, we discuss the notion of fictive interaction or 'FI', namely the use of the template of face-to-face interaction as a cognitive domain that partially models: (i) thought (e.g. talking to oneself); (ii) the conceptualization of experience (e.g. 'A long walk is the answer to headache'); (iii) *discourse organization (e.g. monologues structured as dialogues)*; and (iv) the language system and its use (e.g. rhetorical questions). (Pascual & Oakley, 2017, p. 348, emphasis added)

Given the observation that a mathematical proof is a (specific, regimented and specialized) form of discourse, and the idea that interactive, conversational structures permeate much of what appears to be monological discourse at first sight, it is in fact not so surprising that a mathematical proof too would have a conversational (dialogical) basis. My main contribution here is to spell out in detail the kind of conversation that a mathematical proof re-enacts: semi-adversarial dialogues following fairly strict rules, involving the fictive participants Prover and Skeptic.

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[19]As well put by one of the characters in *Proofs and Refutations*: "Then not only do refutations act as fermenting agents for proof-analysis, but proof-analysis may act as a fermenting agent for refutations! What an unholy alliance between seeming enemies!" (Lakatos, 1976, p. 48). See also recent discussions on 'adversarial collaboration' in the social sciences (Tetlock & Mitchell, 2009).

[20]Compare to what happens in a court of law in adversarial justice systems: defence and prosecution are defending different viewpoints, and thus in some sense competing with one another, but the ultimate common goal is to achieve justice. The presupposition is that justice will be best served if all parties perform to the best of their abilities.

[21]The cooperative component becomes immediately apparent if one considers that a one-line 'proof' from premises to conclusion—say, from the axioms of number theory straight to Fermat's Last Theorem—will be necessarily truth-preserving, and yet will not count as an adequate proof. Of course, Skeptic may also make misuse of why-questions and refuse to be convinced even when a particular inferential step is as clear as it can get (such as the tortoise in L. Carroll's famous story of Achilles and the Tortoise).

[22]Though of course they can also be presented orally, for example in the context of teaching, but even then writing also typically occurs.

Indeed, Skeptic may have been 'silenced', but he is still alive and well insofar as the deductive method has *internalized* the role of Skeptic by making it constitutive of the deductive method as such. Recall that the job of Skeptic is to look for counterexamples and to make sure the argumentation is perspicuous.[23] This in turn corresponds to the minimal requirements that each inferential step in a proof must be necessarily truth preserving (and so immune to counterexamples), and that each step of a proof must be evident and persuasive.

Further empirical support for the claim that a mathematical proof is best conceived as a fictive dialogue is the work by Hodds and colleagues on self-explanation training to increase students' proof comprehension (Hodds, Alcock, & Inglis, 2014). The training consists in teaching students to ask themselves questions about proofs that are very similar to the questions that Skeptic asks: Do I understand the ideas used in this inferential step? Do I understand the general idea of the proof? Does the information provided in the proof contradict my beliefs on the topic thus far? (Which may prompt a search for counterexamples.) Students are even instructed to provide answers to these questions out loud, as if engaging in a real, oral dialogue, thus enacting the fictive interaction of the written form. Their results indicate that this approach substantially improves proof comprehension, thus suggesting that 'going back' to a dialogical version of the proof has a significant cognitive impact. (Recall Netz's description of mathematical proofs above as "at once oral and written".)

Mathematical Proofs as Explanatory Fictive Dialogues

That explanatoriness is a desirable property in a proof follows straightforwardly from the dialogical conceptualization just presented, in particular with respect to the cooperative, didactic component of such dialogues: Prover does not want simply to force Skeptic to grant the conclusion, but also to truly enlighten Skeptic as to *why* the theorem holds. In this section I spell a number of upshots of this dialogical conception for the question of the explanatoriness of mathematical proofs.

The main feature of the dialogical account of explanatoriness in mathematical proofs defended here is that explanatoriness is 'in the eyes of the beholder': it is not a property of a proof as such, but rather it emerges from the *relation* between a proof and a given audience. In other words, explanatoriness is an audience-relative property of a proof; it can be explanatory for a given audience, familiar with certain portions of mathematics (say, topologists), but not for a different audience

[23]The work of L. Andersen interviewing mathematicians on their refereeing practices shows that, when refereeing a paper, mathematicians behave very much like the fictive Skeptic (Andersen, 2017). This suggests that, in the broader social context of mathematical practices, Skeptic does remain active insofar as this role is played by members of the community who scrutinize proofs (in particular, but not exclusively, in their capacity of referees).

(say, mathematical logicians). Paseau (2010) makes this point very convincingly with respect to the different proofs of the compactness theorem that he discusses.

The audience-relativity of explanatoriness pertains not only to different fields within mathematics, but also to different levels of expertise. This is made apparent by phenomenon of different *levels of granularity* for mathematical proofs. It is well known that the level of detail with which the different steps in a proof are spelled out will vary according to the intended audience: for example, in professional journals, proofs are more often than not no more than proof *sketches*, which do not spell out in detail all inferential steps.[24] The presupposition is that the intended audience, namely professional mathematicians working on similar topics, would be able to reconstruct the details of the proof from its gist, should they feel the need to do so (e.g. if they somehow doubt the results). In contrast, in the context of textbooks or in classroom situations, proofs tend to be presented in much more detail, precisely because the intended audience is not expected to have the level of expertise required to reconstruct the proof from a proof-sketch. What is more, the intended audience is in the process of *learning* the game of formulating and understanding mathematical proofs, and so proofs where each step is clearly spelled out are required. Furthermore, different areas within mathematics tend to have different standards of rigor for proofs, again in function of the intended audiences.

What the phenomenon of different levels of granularity suggests when it comes to the explanatoriness of proofs is that, for a proof to be explanatory for its intended audience, the right level of granularity must be adopted.[25] If a proof is to be explanatory in the sense of making "something that is initially puzzling less puzzling; an explanation reduces mystery" (Colyvan, 2012, p. 76), mystery reduction is inherently tied to the agent *to whom* something should become less puzzling.

From this perspective, it is not surprising that mathematicians should prefer certain proofs over others, in particular those that achieve 'mystery reduction'. Now, if the primary or perhaps sole function of a mathematical proof were to establish the truth of a certain mathematical conjecture, then the phenomenon of mathematicians preferring certain proofs over others would not occur. Indeed, if establishing truth were the only function of a proof, then a mathematician would be equally satisfied with two correct proofs establishing the same theorem. But naturally, this is not what happens (as also noted by Bueno, 2009, p. 52), and in fact mathematicians often work on re-proving theorems, i.e. on finding more satisfying proofs for a given theorem that has already been proved (Dawson, 2006). What makes certain proofs more satisfying than others is most likely a multi-dimensional affair (Inglis & Aberdein, 2015), but features such as simplicity, elegance,

[24]"[T]he mature mathematician understands the entire proof from a brief outline." (Lakatos, 1976, p. 51).

[25]The reader versed in argumentation theory may notice some similarities with the 'New Rhetoric' framework introduced by Perelman and Olbrechts-Tyteca: "since argumentation aims at securing the adherence of those to whom it is addressed, it is, in its entirety, relative to the audience to be influenced." (Perelman & Olbrechts-Tyteca, 1969, p. 19) (More on connections with the New Rhetoric shortly.)

generality, depth, clarity, are some of the features typically associated with satisfying proofs.

Mathematicians often use aesthetic vocabulary to indicate the proofs they like (beautiful, elegant) or dislike (ugly) (Inglis & Aberdein, 2015; Montaño, 2014). The exact nature of the properties that these aesthetic terms are tracking in mathematical proofs is a matter of contention (Inglis & Aberdein, 2015), and it has been argued that these apparently aesthetic appraisals are in fact tracking epistemic properties (Todd, 2008). In particular, Rota (1997, p. 182) singles out *enlightenment* as the main property that these apparently aesthetic judgments track, which he describes in the following terms:

> We acknowledge a theorem's beauty when we see how the theorem "fits" in its place,[26] how it sheds light around itself, like a *Lichtung*, a clearing in the woods. We say that a proof is beautiful when such a proof finally gives away the secret of the theorem, when it leads us to perceive the actual, not the logical inevitability of the statement that is being proved.

It is immediately apparent that the property that Rota describes here is very much what is usually understood as the explanatoriness of proofs in the literature we've been discussing so far. From these observations, we may conclude that the phenomenon of proof predilection among mathematicians (as tracked by use of aesthetic vocabulary, among others) is by and large (though not uniquely) related to explanatoriness.

If this is correct, then explanatoriness is best seen as a matter of degrees, i.e. as a comparative notion (as suggested in Pincock, 2015): a proof may be *more or less* explanatory (for a given audience) than another proof (of the same or of another theorem). For example, *reductio ad absurdum* proofs are typically viewed as less explanatory than direct proofs, and similarly proofs by cases and proofs by mathematical induction are generally thought to be less explanatory (Lange, 2014). But this does not mean that such proofs lack explanatoriness completely. On occasion, a *reductio* proof may even be more explanatory and convincing than a direct proof of the same theorem, for example if the direct proof is inordinately long whereas the *reductio* proof is short, perspicuous, and elegant. In fact, Colyvan (2012, p. 83) convincingly argues that discussing explanatoriness or lack thereof in terms of structural families of proofs (*reductio*, by induction, by cases etc.) may not be the most promising approach to take; one should look at the details of individual proofs to make pronouncements on (comparative) explanatoriness.

Another prediction of the approach to the explanatoriness of proofs defended here is that mathematicians will typically *not* converge in their judgments of explanatoriness (as well as of associated features such as beauty and elegance). Given the epistemic diversity observed among mathematicians (different kinds of training, familiarity with different sub-areas of mathematics) despite institutional and methodological commonalities, a proof that is viewed as explanatory by some mathematicians will most likely be viewed as less (or not) explanatory by others.

[26]See also Raman-Sundström (2012) on fit.

This is an empirical prediction that has been investigated, albeit indirectly, in recent years by Inglis and Aberdein. In (Inglis & Aberdein, 2015), the collected data suggest that, in mathematicians' appraisals of proofs, the explanatoriness of a proof correlates with (non-)intricacy, utility, and precision. In (Inglis & Aberdein, 2106), they in turn show that mathematicians quite substantially disagree on their appraisals of proofs, in particular with respect to intricacy, utility, and precision, which, given their previous results, is indirect evidence for the claim that mathematicians disagree on their attributions of explanatoriness to proofs. But more work is needed at this point to investigate the empirical claim that mathematicians disagree on their judgments of explanatoriness.

Does this mean that there are no prospects for a non-relative, objective notion of explanatory mathematical proofs within the present account? Not so, thanks to the notion of an *internalized ideal Skeptic*. The internalized Skeptic is what could be described as the *universal, arbitrary* Skeptic: a mathematical proof may aim to be convincing and explanatory for the widest possible range of Skeptics (understood as those having the necessary mathematical credentials), and thus presuppose as little as possible that is specific to particular Skeptics (e.g. background knowledge, styles of argumentation).[27] So we may say that an explanatory proof in an absolute sense is a proof that would be considered as explanatory by an arbitrary, universal Skeptic.

But of course, the arbitrary Skeptic is an idealization, and in particular if the goal is to remain close to actual mathematical practices, it is not immediately obvious that it is a *useful* idealization. (As argued, actual mathematical proofs invariably display features of context-sensitivity such as different levels of granularity.) But this means that those seeking a non-relative, objective notion of explanatoriness need not reject the dialogical approach proposed here altogether. From this perspective, the non-relative notion emerges from maximally broadening the range of contexts/putative audiences considered, and is thus a limit case of the relative notion.

Smale's proof of the eversion of the sphere, which we started this chapter with, may well count as a proof unanimously viewed as non-explanatory. Perhaps there are also examples of proofs unanimously viewed as highly explanatory (though in the literature on the explanatoriness of proofs, one is hard-pressed to find examples). However, it seems that the wide majority of proofs will inhabit a grey zone; they will be considered as explanatory by some but not by others, which is not surprising given the presence of significant cognitive and epistemic diversity among mathematicians.[28]

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[27]See the distinction between the concepts of a particular vs. a universal audience in the New Rhetoric framework of Perelman and Olbrechts-Tyteca (1969). See Dufour (2013a, b) for an application of the New Rhetoric framework to mathematical argumentation. See also Malink (2015) on the emergence of formal logic in the *Prior Analytics* as related to the concern of making premises fully explicit, which can be understood in terms of addressing a universal audience.

[28]There may well be wide agreement on what counts as a correct, valid proof (Azzouni, 2007), but even this idea has been contested recently (Weber, Inglis, & Mejia-Ramos, 2014).

Conclusion

In this chapter, I have argued that the issue of explanatoriness in mathematical proofs can be fruitfully addressed within the dialogical conceptualization of proofs that I have been developing in recent years. The key idea is to emphasize the observation that a proof is a piece of discourse aimed at an intended audience, with the intent to produce explanatory persuasion. This approach explains both why explanatory proofs are to be preferred over non-explanatory ones (or less explanatory ones), and why explanatoriness is an audience-relative property of a proof. Nevertheless, this pragmatic, 'embodied', multi-agent approach still leaves room for the possibility of an absolute sense of explanatoriness by means of the concept of a universal audience. I have argued that this account is able to clarify a number of features of mathematical practice, such as proof predilection, different levels of proof granularity, and non-convergence of judgments of explanatoriness. Notice though that the arguments presented here are not intended as definitive rebuttals of non-dialogical, objective accounts of explanatoriness, but rather as offering a different, potentially fruitful approach to these issues.

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The Amalgam of Faith and Reason: Euclid’s *Elements* and the Scientific Thinker

Melissa Andrade-Molina, Paola Valero and Ole Ravn

Abstract Problematizing the truths of mathematics education is one of the roles of the philosophy of mathematics education. That mathematics education is a matter of reason and science—not of faith and religion—and that mathematics is timeless, universal and immutable, objective knowledge that is independent from people’s work and sense-making are two strong taken-for-granted statements that navigate in common understandings of mathematics education. Using a Foucault-Deleuze inspired analytical strategy, we examine the contention that mathematics education for the making of the rational and logical child intertwines with what was ought to be the ‘scientific thinker’ to Christianity. We focus on how Euclidean geometry, taken as a proper method of inquiry amalgamated with the Christian worldview to provide explanations about the natural world. The effect of power is the making of the Modern scientific thinker.

Keywords Platonism in mathematics education · Faith · Sacralisation of mathematics · Scientification of education

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Introduction

Part of doing philosophy of mathematics education, as Ernest et al. (2016) express, is about a systematic analysis and a critical examination of problems that are fundamental to mathematics education. It enables us:

> [T]o see beyond the official stories about the world, about society, economics, education, mathematics, teaching and learning. It provides thinking tools for questioning the *status quo*, for seeing 'what it is' is not 'what has to be'; to see that the boundaries between the possible and impossible are not always where we are told they are. (Op. cit., p. 4)

Such phrasing resonates with our positioning on the cultural politics of mathematics education. We are interested in exploring the practices of mathematics education by making evident how mathematics and its inclusion in the school curriculum made part of the technologies of power/knowledge for the making of Modern subjectivity (e.g., Andrade-Molina & Valero, 2017; Valero & Knijnik, 2016). Blurring the division between fields of study such as history, cultural studies, educational sciences, and mathematics education research itself, we broaden the possibilities to understand schooling as a social institution in concrete historical configurations, and the desire to strive for the fabrication of rational, enlightened subjects. Our analytical strategy, drawing from the toolboxes of Foucaultian and Deleuzian studies, invite to historicize the present in a rhizomatic search for how rationalities about mathematics and school mathematics have been constituted and have found a solid place in the current narratives of the undeniable necessity of mathematics for citizenship, society, economics now and in the future. From this perspective, the philosophy of mathematics education is concerned with moving beyond the official stories about what are the objects and subjects of mathematics education, and challenge the *status quo* of what has come to be accepted as taken-for-granted truths about school mathematics.

Elsewhere (Andrade-Molina & Ravn, 2016) we have discussed the value granted to Euclidean geometry for the shaping of scientific thinking since the structure of the *Elements* was taken, by scientific research, as a proper method of inquiry. This is important in understanding how in the desire for making the rational subject through education, the scientific and mathematical rationalities have been intertwined through history. Now, we take a step further in problematizing how modern narratives about the fabrication of the 'reasonable citizen' (Andrade-Molina, 2018) through the learning of school mathematics bring science, mathematics and religion together. Our contention is that current naturalized truths about the role of mathematics education for the making of the rational and logical child are intertwined with what was ought to be the 'scientific thinker' to Christianity. We problematize two truths that navigate in the way mathematics in the school curriculum and educational practices are currently conceived of. First, school mathematics forms the rational mind, and such formation is distinctly separated from faith and religion. In other words, mathematics education is conceived a secular project of rationality, advancing the Enlightenment for the expansion of human reason over obedience to the rule of faith. Second, it is the idea that mathematics is timeless, universal and immutable, objective knowledge that is independent from people's work and sense-making. In other words, particular notions of mathematics are embedded in the common practices of school mathematics despite efforts to introduce pedagogies rooted on the socio-cultural-political theories of knowledge and learning (e.g., Planas & Valero, 2016; Radford, 2008). We problematize these truths as interconnected statements tracing the historical justifications for the inclusion of Euclidean geometry as a topic of teaching and learning.

The dominant narratives of mathematics education position mathematics as an objective knowledge completely independent from faith, ideologies, culture, society or politics. However, the teaching of mathematics, in particular of Euclidean geometry, shaped scholastics and the expansion of Christianism on a quest for certainty and for a closer understanding of God. The view of mathematics as the language used by God in Christianity had a great impact in the forming of Western, Modern education. On the one hand, the Modern languages of education as cultural expressions emerged in an amalgamation of Christian notions of morality belonging to the different confessional orders in Europe in the 18th century and political ideals (Tröhler, 2011). Education as a tool of State governing through the fabrication of notions of the "moral man" also articulated the growing desire for scientific and mathematical knowledge (Valero & García, 2014). On the other hand, the quest for God in natural philosophy also promoted the advance of mathematics (Kvasz, 2004). Here we will explore the entanglement between modern discourses of mathematics education and the discourses on faith from Christianity.

Setting the Scene

Before beginning to unpack the discourses on faith, we want to set the scene. All taken events occur in a time and place where there is no such thing as 'scientists', yet. What we currently identify as science had until the 18th century occurred in the realm of "natural philosophy" (Beltrán, 2009) or "natural history" (Foucault, 1971). And 'the Philosopher' *par excellence* of that time was Aristotle, given that his work was taken as the "eminent representation of science" (Beltrán, 2009, p. 284, our translation). His ideas "transformed the way the West thought about the world and its operations" (Grant, 2004, p. 14). For many years, the *Elements* of Euclid were considered as a particular expression of Aristotelian logic. Scholars made efforts to establish 'one-to-one correspondence' between Aristotelian logic—axiomatic—and Euclid's postulates (Gómez-Lobo, 1977), sometimes assuming that Euclid had put Aristotle's 'dictum' into practice while configuring the *Elements* (Mueller, 1969).

According to Descartes, what Euclid accomplished should be understood as a model for inquiry in all areas (Toulmin, 2001). His insistence on the "Euclidean model of knowledge planted some seeds in natural science between 1600 and 1650" (Op. cit., p. 43). Since then and until the 19th century, the *Elements* were "taken as the paradigm for establishing truth and certainty. Newton used [its form] in his Principia, and Spinoza in his Ethics" (Ernest, 1991, p. 4). The geometry on Euclid's books became a logical system:

> [The *Elements*] is one of the great achievements of the human mind. It makes geometry into a deductive science and the geometrical phenomena as the logical conclusions of a system of axioms and postulates. The content is not restricted to geometry as we now understand the term". (Chern, 1990, p. 679)

This is precisely what other philosophers saw in Euclid's books: Content not only restricted to geometry. The *Elements* became a guide to produce science through a particular way of describing reality. And its status as an example of Aristotelian logic led it to expand throughout most fields of inquiry. That Euclid's axiomatic was the model to produce secure knowledge became a part of Western culture and appeared repetitively in many cultural expressions. For instance in Botticelli's fresco St. Augustine in his study from 1480.

> Botticelli portrayed Augustine as a scholar-saint wearing clerical robes with an open treatise on geometry and a weight-driven clock nearby. He looks heavenwards, seeking the order that the Christian God (like Plato's demiurge) imposed on creation by dividing light from darkness. (Gamwell, 2015, p. 476)

The Amalgamation of Faith and Science

In the Middle Ages, Aristotelian logic was perceived as "the indispensable instrument for demonstrating theoretical knowledge" (Grant, 2004, p. 10). The use of principles instead of axioms led Aristotle's work to be considered as an axiomatization of science (Geréby, 2013). On the one hand, as aforementioned, natural philosophers relied on the work of 'the Philosopher' for their inquiries. Philosophy was viewed as a "wish to gain a rational understanding of the world in which we live, and the fundamental processes at work in nature, society and our own way of thinking" (Grant & Woods, 2002, p. 25). So, to understand the Universe is to comprehend the nature of things "by observation and reflection, to discover the causal principles, the forces, the powers and potentialities of the things that govern their behaviour" (Tiles, 2003, p. 351). On the other hand, there was a conflict among Christians on the symbolic or literal reading of the Bible (Midgley, 2005). Within this conflict, 'science' was taken as dangerous given that God forbid Adam and Eve to eat from the Tree of Knowledge or as archangel Raphael told Adam to be 'lowly wise' when he began questioning the nature of the universe (Wolpert, 2013). With the flourishing of natural philosophy, conservative theologians "were alarmed […]. They were concerned that Aristotle's natural philosophy was circumscribing God's absolute power to do anything" (Grant, 2004, p. 12). According to Murray and Rea (2016, p. 2), "early Christian thinkers such as Tertullian were of the view that any intrusion of secular philosophical reason into theological reflection was out of order". Natural philosophy sought for a kind of certainty that faith could not grasp.

The Christian Church took its wishes of expansion to spread the evangelic message as an agreement between faith and reason (Beltrán, 2009). And the Church used Aristotle's language to articulate its documents without an interest on "establishing the truth […] but to only capitalize some possibilities of the Greek cosmogony in conditions to make more explicit the sense of the proper mysteries of religion" (Op. cit., 2009, p. 283, our translation). The need to reconcile faith and 'science' emerged in connection to this desire of expansion:

> [S]ome of the greatest Christian theologians clearly had defended the position that the concrete contents of religious truth should be based on reason alone. […]A rationalist position was particularly tempting in times in which there was a clear awareness that a legitimate interpretation of "revelation"—of scripture and tradition—was itself a work of reason. If the last meaning of scripture was allegorical, tropological, and anagogical, then this meaning had to be based on rational arguments, which alone could have the power to transcend literal meaning. (Hösle, 2013, pp. 2–3)

Thomas Aquinas balanced a Catholic discourse of faith and science rooted in Aristotelian philosophy; he believed that God's existence was rationally demonstrable (Hösle, 2013). How else can one approach God, who is unreachable by the senses, if not by the use of logic and reason? In this need of amalgamation, "Aquinas' *Summa Contra Gentiles* is a good example of how dialectical investigations have been carried out in philosophy and theology" (Bovell, 2010, p. 70). The claims made by either theology or philosophy under the Thomistic model, were not believed to conflict anymore (Murray & Rae, 2016), since "some truths can be known only through faith, some can be known only through reason, and some can be known through either faith or reason" (Garcia, 2003, p. 623). In his *Summa,* analogue to philosophy, "theology consists of (theological) principles, and (theological) theses derived from these principles" (Geréby, 2013, p. 175):

> The genius of Aquinas articulated itself in the fact that he transformed the insecurity of Christianity that resulted from the discovery of the Aristotelian corpus, […], into a positive development and, despite many hostilities that culminated in suspecting him of heresy, conceived a great synthesis of Christianity and Aristotelianism that satisfied both the religious need and the need for knowledge of the empirical reality. (Hösle, 2013, p. 151)

Prior to Aquinas, Augustine established a connection between faith and natural philosophy where philosophy complemented theology "but only when these philosophical reflections were firmly grounded in a prior intellectual commitment to the underlying truth of the Christian faith" (Murray & Rea, 2016, p. 2). His work had a significant impact in Christianity (Finocchiaro, 1980), "in the process that would eventually lead to the rationalization of medieval theology" (Grant, 2004, p. 39). Augustine's theory of illumination is embedded in Malebranche and Descartes' works (Spade, 1994). Grant (2004) argues that Augustine had a Platonic interpretation of the 'valid rules of logic', which made him believed that

> [L]ogic was a valuable tool that would enable them to infer the correct conclusions from the initial [Scriptural or doctrinal] premises […] With this attitude toward logic and reason, Augustine was not reluctant to use analytic tools – especially Aristotle's categories – in his analysis of doctrinal truths, as he did in one of his greatest works, the fifteen books of *On the Trinity*. (p. 39)

The amalgamation of faith and science, through Aristotelian logic, fulfilled the Christian need to base ‘revelations’ on reason and logic. This allowed adding certainty to the allegorical interpretation of the Bible. From this merging, thinkers of the world of faith made contributions to the world of empirical philosophies, for example, Aquinas’ and Augustine’s contributions to astronomy (see Campion, 2014). And despite being regarded as two separated, even opposite, fields of knowledge in modernity, both are not “games that can be played independently of each other. Both are about truth, and reality. Their divided claims cannot stand with the assumption of there being one single reality” (Geréby, 2013, p. 177).

When Scholastics Met Euclid

A religious search for knowledge and certainty has not to be reduced only to a simplistic discussion about God, or to what we currently consider theology as “the science of God” (Hösle, 2013). Within the Church’s structure, Bishops, the teachers of Christianity, had a preeminent authority among scholastics to provide instruction. Scholastics were not to rely only on their faith and their beliefs. And so, ‘students’ of the religious orders were educated under the oeuvre of classical pagan philosophers of Ancient Greece, for example, Plato’s theories of the soul: “Platonism held that the soul could exist apart from the body after death. This would obviously be appealing to Christians, who believed in an afterlife” (Spade, 2016, par. 9). Scholastics were encouraged to study ‘sciences’: geometry, astronomy, Aristotelian logic, Platonic tradition, among others (Clavius, 2002). They were also encouraged to translate and reproduce the most prominent books of the time. For example Boethius’ translations from Greek into Latin of Aristotle’s and Plato’s most dominant works, including Boethius’ commentaries to ‘illuminate’ their philosophies (Marenbon, 2009). They also translated the *Element*s of Euclid, since they recognized in these books more than just geometrical knowledge of Ancient Greece. They used them to study proof, common notions, and axiomatics; and through such study the reasonable and scientific thinker needed for Christianity could be shaped:

> [The student] offer new proofs of some of the propositions of Euclid, thought out by himself; in these places, let praise be given to those who best solve the problem proposed, or who commit the fewest false syllogisms, which occur not rarely, in the invention of the new proofs. For it would happen thus, that they would become not a little eager for these studies, when they see such honor given to them, and at the same time would understand the eminence of these same studies, and they would make greater progress in these things through this exercise. (Clavius, 2002, p. 467)

The structure of the *Elements* became a very powerful model for achieving certainty through axiomatics. The *Elements* began intertwine with the productions of faith. Scholastic made “efforts to build theological systems from scriptural texts as plane geometry is built from the postulates of Euclid” (Pals, 2011, p. 919). Nicholas of Amiens, a French theologian from the 12th century, wrote

Ars Catholicae Fidei [Art of catholic faith] based on Euclid's books. Amiens provided a sequence of arguments to set the rules of the Catholic faith by assembling "definitions [descriptions], postulates [petitiones], and common notions, or axioms [conceptions]" (Grant, 2004, p. 67). Aquinas' book *Summa Contra Gentiles* became the best example of the amalgamation of faith and science, theology and philosophy. According to Bovell (2010), Aquinas' work is rarely related with Euclid's books, but his *Summa Contra Gentiles* 'mimics' the configuration of the *Elements*. "These seem the syntactical equivalents to Euclidean proofs of demonstration and play analogously the same role in Aquinas as proofs of demonstration do in Euclid" (Op. cit., pp. 70–71). As his predecessors, Herbert de Cherbury, an English philosopher of the 17th century, borrowed the term 'common notions' from Euclid as the foundation of reasoning in his book *De veritate* (Serjeantson, 2001). De Cherbury contends that "the being of God is indicated by the structure of reality" (Pailin, 1983, p. 198) and, so, his existence can be determined by reason and by observation of the natural world (de Cherbury, 1633).

To Christianity the *Elem*ents were special; they encapsulated "a form of reasoning, and a handmaiden of natural theology" (Cohen, 2007, p. 164). It had common notions, rigorous mathematical proofs, and a deductive system.

> [C]ommon notions are the ultimate and indisputable principles by which understanding ought to be governed and are God's way of ensuring that every person has what is essential 'for this life and for life eternal'. They are not, however, principles which everyone is always aware of. They emerge to consciousness only when the mind has been aroused by appropriate experiences. What is common to all people is the basic structure of understanding through which any individual, suitably provoked, may come to recognize them and perceive their certainty. (Pailin, 1983, p. 198)

In this regard, it is not rare that the *Elements* were used in missions to expand Christianity. The Italian Jesuit Matteo Ricci recognized in Euclid's books "something different, namely, propositions presented in order and so definitely proven that even the most obstinate could not deny them" (Gallagher 1942, p. 471). As an example, the Elements were brought to China as a mean to introduce Aristotelian logic (Yuan, 2012). Matteo Ricci and the Chinese mathematical scholar Xu Guangqi translated the firsts books into Chinese. The Jesuits selected these books to deal with the main differences between Western and Chinese culture. According to Yuan (2012), Chinese logic is pragmatic, inscribed in a world of concrete objects that is always flowing, always changing. On the contrary, Aristotelian logic is a hierarchical order system of abstract concepts. According to Gallagher (1942), Euclidean geometry pleased Chinese as much as any other form of knowledge.

> Though no Chinese understood Aristotelian logic at the beginning, *Jihe yuanben* [Euclid's book I–VI], as a mathematical text and source of logical training, became more and more popular in China over the last four hundred years […] By contrast, studying Aristotelian logic itself is still considered as difficult work. (Yuan, 2012, pp. 83–84)

Sacred Mathematics and the Path to God

To Christianity, the *Elements* were not only an instrument to teach scientific reasoning through Aristotelian logic for the shaping of the scientific thinker. Neoplatonism helped giving a different status to geometry, and mathematics was conceived as a form of thinking for approaching Immutable Truths (Grant, 2004). The Greek Proclus based his philosophy on Plato's idealist thoughts, the Neoplatonic, in which, "beings exist in a cave of impaired perception, a profane realm of limited, imperfect things: matter, decay, ever-changing shapes" (Cohen, 2007, p. 19). To Proclus, human existence occurs in the realm of muddled existence. The divine, above humans, is a 'sphere' of purity and eternal Truth. And "[m]athematics plays a special role in this divided universe—it ascends from the world of impermanence to this higher, heavenly plane. (Cohen, 2007, p. 19). The notion of infinity was considered as negative in Ancient Greece, however it was taken as a path to God for medieval scholars: "Theology made the notion of infinity positive, luminous, and unequivocal [infinity] was interpreted as the consequence of human finitude and imperfection" (Kvasz, 2004, p. 114). For example Leibniz's soul-like monads metaphysical system, in which "God is needed to ensure that the components of the universe interact as harmoniously as possible" (Francks & Macdonald, 2003, p. 665). Or Descartes attributing God a fundamental role in the conservation of momentum.

> Descartes is one of the earliest philosophers who sees in the conservation laws of physics an expression of God's immutability and even if he adduces as an example the conservation of momentum, he still regards momentum as a scalar magnitude, not as a vector. Therefore, he can believe that the *res cogitans* may influence the mere direction of the *spiritus animales* without altering the quantity of momentum. (Hösle, 2013, p. 99)

Natural philosophers "saw all regular phenomena as marks of God's Rational Order" (Toulmin, 2001, p. 51), It is in this sense that Christianity sacralized mathematics as the path to access divinity. The realm of the primary causes, the cause without cause, is reserved to God. But the realm of the second causes, the natural world derived from the first cause, can be understood, studied and known through science and reason. Conservative theologians saw in the natural order of things a clear proof of the hands of God regarding the creation of the Universe and of humanity. Even today, primary causes are questions that science cannot solve for Christianity. Pope John Paul II claimed, at a conference about cosmology held in the Vatican in 1981, that "there is needed that human knowledge that rises above physics and astrophysics and which is called metaphysics; there is needed above all the knowledge that comes from God's revelation" (John Paul II, 1981, par. 5). And he told the participants they could "study the evolution of the universe after the big bang, but [physicists] should not inquire into the big bang itself because that was the moment of creation and therefore the work of God" (Hawking, 2003, p. 67).

Reading the *Elements* as the vehicle to reach an understanding of the "first cause", enables to describe the connection that mathematics establishes between man and God:

> John Dee inherited this occult quest and was convinced that mathematics was the special language that would transport its conjurer to that higher plane of divine truth. Dee's introduction to Euclid's Elements encapsulated the purpose and efficacy of mathematics in a manner that resonated with the mathematical idealism of the early Victorian age […] Dee divided all things in the universe into three categories: the natural, the supernatural, and the mathematical. Natural things are perceivable, changeable, and divisible. Supernatural things are invisible, immutable, and indivisible. Mathematical concepts occupy a critical middle position between the natural and the supernatural, thus mediating between these realms. (Cohen, 2007, pp. 21–22)

The *Elements* were not only to teach scholastics how to reason and to be logical, since it was believed that the "knowledge of God cannot be achieved by means of science, it was thought to be beyond the reach of reason" (Hösle, 2013, p. 3). Geometry was thought to be the architecture of divinity. Through the understanding of natural things scholastics could approach the purity and eternal realm of God's structure of the universe. The medieval expression from 13th century, the great architect of the universe or God the geometer, portrayed God with a compass in his right hand in an act of creating the universe, which entails that he first created geometry as his own language for structuring the cosmos and mankind. And since then this idea has navigated widely in a variety of expressions of Western culture, from Christian medieval expression of God as the geometer of the Universe, to Mandelbrot's fractal geometry of nature theory (Mandelbrot, 1983) to the molecular composition of the DNA.

In the book *The fractal geometry of nature*, it is set a discussion about how "nature has played a joke on the mathematicians. The 19th century mathematicians may have been lacking in imagination, but Nature was not" (Mandelbrot, 1983, p.3), in such discussion Mandelbrot agues that "imagination tires before Nature" (Op. cit., p.4). The manifesto Mandelbrot poses is the one of mathematics as a dual constitution, one from Nature—that needs to be study—and the other from human invention—mathematicians discovering what is already made in Nature. Here Mandelbrot uses the medieval icon of *God the geometer* with the inscription "here God creates circles, waves, and fractals" (Op. cit., p. c1). Such interpretation gives to fractals an origin from the divine. The latter is a modern example of the sacred character given to geometry "The [DNA] helix, which is a special type from the group of regular spirals, results from sets of fixed geometric proportions" (Lawlor, 2002, p. 4). "Fixed" geometric proportions not necessarily mean that DNA is a creation of God, but gives to geometry a Platonic character, a supernatural thing, according to Dee's division. As Lawlor (2002) continues these fixed geometric proportions "can be understood to exist a priori, without any material counterpart, as abstract, geometric relationships. [The helix] existence is determined by an invisible, immaterial world of pure form and geometry" (p. 4).

In medieval schools, mathematics was included as part of the quadrivium (astronomy, geometry, arithmetic, and music) and it was taught by scholastics. The aim was to "yield knowledge concordant with both human reason and the Christian faith" (Garcia, 2003, p. 620). The Jesuit mathematician Clavius expressed that "one cannot understand various natural phenomena without mathematics" (Smolarski, 2002, p. 258). A need to teach geometry and mathematics emerged: Mathematics to achieve Truth in the divine.

> [B]ecause of [mathematics] participation in both the perfect and imperfect spheres of existence, mathematics provides a mental pathway for ascending out of the material realm and attaining an ideal comprehension of the universe. (Cohen, 2007, p. 20)

The Discourses on Faith

Aristotelian logic and Neoplatonism gave to Christianity a solid foundation on which to ground their beliefs on the existence of God. Both gave to their philosophy the certainty needed in the Middle Ages. The *Elements*, taken as the perfect example of Aristotle's understanding of science, helped in shaping the 'scientific thinker'. *Geometry* was taken as a deductive science with logical conclusions (Chern, 1990). And so, Euclid's books were taken to be the core of any science, to Christianity. The learning of common notions, propositions, axioms, and proof enable scholastics to engage in scientific (philosophical) discussions with the books produced by Christianity. For example, de Cherbury belief in God "is not derived from history, but from the teachings of the Common Notions" (Pailin, 1983, p. 200). The discourses on faith about science became intertwined with what we know today as science.

Medieval schools emerged as a previous step towards the formation of the university. The latter, "was a wholly new institution that not only transformed the curriculum but also the faculty and its relationship to state and church" (Grant, 2004, p. 29). And although it seems that Christian beliefs, in contemporaneity, have been distinctly separated from school curricula, it seems fair to conclude that there is no religious beliefs been reproduced in schools nowadays. Buchardt (2016, p. 1) argues that, while "it is common sense in the educational field that religion before modernity has played a central role in education, opinions differ when turning to a perspective of the present" She argues that the apparent secularization of education through its increased scientification has created the idea that contemporary schooling is about science—even in subjects such as "Religion". However, a close analysis would reveal how religious notions guiding education reconfigured into new, secularized and scientified forms of school subjects (Buchardt, Tröhler, & Valero, 2016). In the case of school mathematics, the discourses on faith that have historically made part of the amalgamation of religion and science through mathematics are the connecting thread that binds faith with reason. Such fine thread remains though unexposed in current understandings of school, school mathematics, mathematics and science, although some work has pointed in that direction (e.g., Peñaloza & Valero, 2016; Restivo, 2008; Valero & García, 2014).

As we showed, the teaching of mathematics shaped scholastics and supported the expansion of Christianism on a quest for certainty and for a closer understanding of God. The historical amalgamation of faith and reason through the articulation of theology and science in Christianity positioned Euclidean geometry and its axiomatics as a privileged element in education for the making of a desired knowledgeable, scientific and faithful self. In schools, mathematics is still mainly thought of as a sacred, timeless, universal, objective knowledge, and an immutable truth in the sense of Christianity. It is not the path to access the purity of God, but to access the purity of the Platonic world of ideas. School mathematics seems not to be mutable, as the history of mathematics and mathematics education reveal. Despite this, it keeps on being conceived as fixed. Some kind of essence seems to escape the possibility of transformation. Not all students nowadays are meant to be mathematicians, nonetheless all are expected to recognize in mathematics the key to access knowledge… just as medieval scholastics did.

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Part III
Critical Mathematics Education

Students' Foregrounds and Politics of Meaning in Mathematics Education

Ole Skovsmose

Abstract Beginning with the presentation of two fictitious South African children, Nthabiseng and Pieter, I outline the idea of a politics of meaning in mathematics education. Such a politics refers to the social, economic, cultural and religious conditions for experiencing meaning. It refers as well to the layers of visions, assumptions and preconceptions that might construct something as being meaningful. In order to articulate the socio-political formation of meaning, I present a foreground-interpretation of meaning. The basic idea is to relate meanings and foregrounds, acknowledging that foregrounds are formed by a range of factors, as well as by the person's experiences of such factors. Politics of meaning becomes analysed further with reference to foregrounds being polarised, destroyed, pointed and multiplied.

Keywords Meaning in mathematics education · Politics of meaning Students' foregrounds · Foreground-interpretation of meaning · Polarised foregrounds · Destroyed foregrounds · Pointed foregrounds multiplied foregrounds

Nthabiseng and Pieter are two fictitious South African children, who appear in the opening pages of the World Bank's *Equity and Development: World Development Report 2006*.

> Consider two South African children born on the same day in 2000. Nthabiseng is black, born in a poor family in a rural area in the Eastern Cape Province, about 700 kilometres from Cape Town. Her mother had no formal schooling. Pieter is white, born in a wealthy family in Cape Town. His mother completed a college education at the nearby prestigious Stellenbosch University.
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> On the day of their birth, Nthabiseng and Pieter could hardly be held responsible for their family circumstances: their race, their parents' income and education, their urban or rural location, or indeed their sex. Yet statistics suggests that those predetermined background variables will make a major difference for the lives they lead. Nthabiseng has 7.2 percent

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> chance of dying in the first year of her life, more than twice Pieter's 3 percent. Pieter can look forward to 68 years of life, Nthabiseng to 50. Pieter can expect to complete 12 years of formal schooling, Nthabiseng less than 1 year. Nthabiseng is likely to be considerably poorer than Pieter throughout her life. Growing up, she is less likely to have access to clean water and sanitations, or to good schools. So the opportunities these two children face to reach their full human potentials are vastly different from the outset, through no fault of their own. (Word Bank, 2006, p. 1)

In her talk in 2008 at the symposium *on Mathematics Education, Democracy and Development*, at the University of Kwazulu-Natal in Durban, Renuka Vital referred to these two children, and since then I have referred to them several times.[1]

Now let us imagine these two children sitting in a mathematics classroom. Since we are taking part in a thought experiment, we can really imagine many things—that they are sitting in the *same* classroom, listening to the same mathematics teacher for one. We imagine that there is a nice welcoming atmosphere in the classroom, where the teacher does her best to engage the children in shared activities. The teacher has presented some problems to do with traveling on holiday. Catalogues from travel agencies have been circulated, the students have been divided into groups, and Nthabiseng and Pieter are in the same group. Each group have been invited to choose the destination for their holiday and to decide on a budget for the trip, including possible expenses for extra equipment and new clothes. The teacher has followed many recommendations presented in the mathematics education literature about the relevance of contextualising and of making students active in their learning process. The teacher has done her best for bringing meaning into mathematics education.

One could assume that Nthabiseng and Pieter might experience more or less the same, as they are listening to the same explanations and presentation for the same contextualisation. However, as outlined in World Bank's report, Nthabiseng and Pieter face radically different prospects in life. Such different prospects concern the length of life, economic conditions, educational possibilities, and many other things. My point is that such prospects also form their experiences of meaning which brings me toward the notion of the politics of meaning. *By a politics of meaning I refer to the social, economic, cultural and religious conditions for experiencing meaning as well as to the layers of visions, assumptions, presumptions and preconceptions that might establish something as being meaningful or meaningless.* (When I go on to talk about meaning, I always have in mind experiences of meaning as well as of meaninglessness.)

To a large extent, interpretations of meaning in mathematics education have provided a de-politicisation of the issue. This resonates with many philosophical theories of meaning, which as well mark a de-politicisation of the topic. Contrasting such diverse interpretations, I will provide a *foreground interpretation of meaning*, which helps to reveal the complex socio-political formattings of meaning in mathematics education.

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[1] See, for instance, Skovsmose (2011).

Meaning as a De-politicised Educational Issue

It is broadly assumed that mathematics education should provide meaning to the notions and techniques that become presented for the students. One finds, however, no general agreement about how to establish such meaning or how to interpret the notion of meaning. Here I restrict myself to present two principal positions.[2]

One position states that meaning of concepts and notions have to be expressed in terms of their references. During the 1960's, a referential theory of meaning accompanied the implementation of the so-called New Math. It was assumed that the meaning of mathematical notions would be clarified accurately, if one provides an adequate landscape of references, first of all having to do with sets. Thus the meaning of, say, the notion of function had to be established through a carefully elaborated route that introduced the notions of set, ordered couple, and set of ordered couples, before reaching the very notion of function.

During time, different attempts have been made to establish meaning through a careful decomposition of mathematical concepts. This approach remained, also after the New Math movement had faded away. The general assumption is that meaning of a complex concept can be adequately grasped in terms of the meaning of its constituting elements.[3]

Contesting the referential theory of meaning, one meets the claim that meaning is first of all established through the use of mathematical notions and techniques. The general idea is that students would come to experience meaning when they become familiar with applications of mathematics. The idea has guided the contextualisations referring to examples like, buying something, traveling somewhere, cooking something, etc. A further elaboration of the use-oriented theory of meaning is found in the many proposals for giving mathematical modelling as principal role in mathematics education.[4] Thus it has been assumed that when engaging students in the very modelling process, the relevance of mathematics becomes directly experienced "from within".

Both the referential and use-oriented theory try to bring meaning into mathematics education by relating mathematical notions and techniques to phenomena and situations that is familiar to the students. Thus *familiarity* has been highlighted as the principal carrier of meaning.[5]

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[2]For broader presentations of perspectives on meaning in mathematics education, see, for instance, Howson (2005), Kaiser (2008), Kilpatrick, Hoyles, Skovsmose, and Valero (2005), Otte (2005), Thompson (2013), and Vollstedt (2011).

[3]See for instance, Biehler (2005), Dreyfus and Hillel (1998), Hillel and Dreyfus (2005). Discussions about the use of computers in mathematics education have emphasised the importance of providing mathematical notions with illustrative references. See, for instance, Hoyles (2005); and Laborde (2005).

[4]For critical discussions of mathematical modelling, see, for instance, Jablonka (2007, 2010).

[5]This idea can be substantiated with references to, for instance, Ausubel (1963, 1968).

Neither the referential nor the use-oriented theory of meaning would easily reveal the profound differences in Nthabiseng and Pieter's experience of being involved in the same group work, as these two theories do not recognise the socio-political formation of these experiences.

Meaning as a De-politicised Philosophical Issue

That the discussion of meaning appears de-politicised is not particularly related to mathematics education. It is as well a characteristic of classic philosophical theories of meaning, and let me briefly summarise two of them.

Platonism represents a most elaborated suggestion for a referential theory of meaning. The meaning of a notion has to be identified as the entity to which the notion refers, and the world of proper references makes up the world of ideas. Since antiquity, Platonism with respect to mathematics has found many different expressions.

Gottlob Frege, assuming a version of Platonism, elaborated a referential theory through which he wanted to demonstrate that mathematics rests on a logical basis.[6] This idea brought Frege to claim that the world of ideas that constitutes all possible references for mathematical concepts is the world of sets. This referential theory provides the basis for logicism and for many attempts to construct mathematics on a foundation of logic. It also provided a basis for the whole New Math movement, establishing set theory not only as the logical but also as an educational foundation of mathematics. Frege's referential theory of meaning turns the discussion of meaning into an analytical enterprise.

In *Tractatus*, Ludwig Wittgenstein (1922) elaborated a referential theory of meaning in terms of a picture theory. He pointed out that the whole purpose of science is to picture reality, and further that this picturing has to be formulated in the language of mathematics. This idea formed the outlook of logical positivism, which included the formulation of a specific theory of meaning. According to this, a proposition has a meaning if and only if one can point out a possible procedure for its verification.[7] With departure in this principle, meaning became related to conditions for providing empirical justifications. This way the discussion of meaning became located as a strictly de-politicised and analytical issue.

In *Philosophical Investigations*, Wittgenstein (1958) abandoned any referential theory of meaning by suggesting that the meaning of a concept should be discussed in terms of its possible uses. By this suggestion, Wittgenstein provided a shift in the analytic philosophy of meaning. According to Platonism, to Frege and to the principle of verification, the social has no role to play in identifying the meaning of concepts, while Wittgenstein's suggestion established the social as crucial for

[6]See Frege (1960) as well as Dummett (1981, 1991).

[7]See, for instance, Carnap (1932) and Hempel (1959).

interpreting meaning. Important, however, is to recognise that Wittgenstein's opening for the social does not make space for any politics of meaning. His use-oriented interpretation was developed within an analytic approach. This also applies to other versions of a use-oriented theory of meaning, as John Austin's analysis of how to do things with words and John Searle's speech act theory.[8]

Referential and use-oriented theories of meaning, both with respect to mathematics education and to philosophy, shape an overall de-politicisation of discussions of meaning. In order to provide a re-politicisation of this discussion, it becomes necessary to provide new interpretations of meaning, and in the following, I will suggest a foreground-interpretation of meaning.[9]

A Foreground-Interpretation of Meaning

As a first clarification of the notion of foreground, I want to relate it to the notion of intentionality, which has deep philosophic roots.

Intentionality was addressed in Antiquity as well as in scholastic philosophy. The notion also played a crucial role in Franz Brentano's characteristic of the human mind, which anticipated Edmund Husserl's formulation of a phenomenological outlook. To Brentano, intentionality is the defining feature of human consciousness: this is always directed towards something. Consciousness is an always floating stream. Its directedness is an a priori phenomenon; it takes place before any particular psychological phenomenon does occur. In this sense, intentionality is a pure and universal human category, which appears before any form of social structuring. Also in Husserl's formulation of phenomenology, intentionality works as a basic category for interpreting human experiences.[10]

I am also interested in the notion of intentionality, and I consider intentionality to be fundamental for interpreting human actions. I see actions as directed towards something. However, I completely disagree with both Brentano and Husserl with respect to the stipulated purity of intentionalities. Contrary to them, I see intentionalities as being socio-political structured. The directions of one's intentionalities are formed by all kind of socio-political factors: cultural, religious and economic. They might be directed by presumptions, misconceptions and any kind of ideologies. There is no end to the range of factors that may form our intentionalities and purposes for actions. In this sense, I consider intentionalities as being profoundly structured.

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[8]See Austin (1962, 1970), and Searle (1983).

[9]In Skovsmose (2016a), I present what I referred to as an intentionality-interpretation of meaning. Besides some terminological difference, we have to do with a foreground-interpretation of meaning.

[10]See Brentano (1977, 1995) and Husserl (1970, 1998, 2001). See also Albertazzi (2006) and Smith (1994).

When we pay particular attention to the directedness of the socio-political structured intentionalities, we reach the notion of foreground.[11] An action can be directed towards possibilities, and the landscape of possibilities constitutes a foreground. We should, however, not only refer to possibilities as being positive phenomena. A foreground is also structured by absence of possibilities. Something might appear achievable, while other things appear to be out of reach. In this sense foregrounds are structured in terms of possibilities as well as of impossibilities. They are structured by hopes and aspirations as well as by fears and aversions.

Considering the World Bank's presentation of Nthabiseng and Pieter, we see profound differences between their foregrounds. As mentioned, these differences are formed by parameters expressing life expectancy, opportunities for education, likely economic conditions, and so on. They establish a profound structuring of possibilities and impossibilities. A foreground, however, is not a simple expression of such statistical parameters. It is as well structured through layers of sexism, racism and of presumptions in general. It is structured through dominant discourses. Furthermore, a foreground reflects the person's expectations, hopes, fears and frustrations. This observation does not only apply to Nthabiseng and Pieter, but to everybody. This is the basic idea of claiming a socio-political formation for foregrounds.

One can talk about foregrounds in the plural, and this plurality might be considered an intrinsic feature of foregrounds. Thus it does not make sense to talk about *the* foreground of a person as if it were a well-defined entity. The person makes interpretations, changes interpretations, comes to grasp new possibilities, or to recognise new obstructions.

As part of the learning process, students engage in activities or stay passive. The students might do their homework or try to escape this task. They may pay attention to what the teacher is explaining or may lose interests for it all. The students' learning or not-learning includes a range of actions based on how their intentionalities become directed, redirected or blocked. When I consider the meaning experienced by students in learning mathematics, I always consider the meaning that could be related to the actions that constitute their learning process.

A basic idea of a foreground-interpretation of meaning is that students' experiences of meaning first of all emerge when they recognise that their learning actions can be directed towards features of their foregrounds.

Let me illustrate by a though experiment. Imagine that we, mathematics educators, arrive to the village where Nthabiseng is living. Our purpose is to develop a meaningful mathematics education. We make careful investigations of what is taking place in the local community. We find out about traditions; we find out about the trading going on; we find out about the handcraft being done; we find out about the cooking; and so on. By the way, we have noticed that so many chickens are around. We recognise that much mathematic can be related to the selling of chickens: the costs of the transport to the market and their value at the market.

[11]See Skovsmose (2014a).

And how to divide the profit? Mathematics can as well be related to the cooking of chickens and to recipes in general. We mathematics educators might become rather pleased with ourselves identifying all these possibilities for ensuring meaning to the mathematical activities in which we are going to engage the children.

However, a foreground interpretation brings our attention into a different direction than this interpretation focusing on familiarity. What about a pilots' mathematics? Maybe that would provide another kind of meaning to Nthabiseng and her friends from the village. A pilot's mathematics could take departure in that fine line of smoke up there high in the sky. How did it appear? How high up might it be? In what directed does it point? To which city might the airplane making this line be heading? How to use a compass? What could a map tell us? If we consider the dream of travelling—the dream of flying—one may be open to the possibility that pilot's mathematics would bring meaning to the learning of mathematics, also in Nthabiseng's village.[12]

In the following, I am going to discuss different socio-political formattings of foregrounds, and indicate how they structure experiences of meaning. I will talk about polarised foregrounds, destroyed foregrounds, pointed foreground, and multiplied foregrounds.

Polarised Foregrounds

In his study *Interplay of Citizenship, Education and Mathematics: Formation of Foregrounds of Pakistani Immigrants in Denmark*, Sikunder Ali Baber (2007) makes profound observations with respect to students' foregrounds.[13] These observations are based on in-depth interviews with students and families from Pakistan living in Denmark. For short I refer to these students as Pakistan students even though they might be born in Denmark and speaking Danish without any sign of foreign accent. I refer to students with longer roots in Denmark as Danish students.

The interviews addressed the students' experiences in school, their experiences related to mathematics, and in particular how they saw their future opportunities in

[12]While the pilots' mathematics represents a thought experiment, real examples document the same phenomenon. Skovsmose and Penteado (2015) and Skovsmose (2016b) tell about a school in the interior of Brazil where a project having to do with surfing was completed in the mathematics classroom. The school was situated in a poor neighbourhood where maybe none of students would ever have seen the ocean. If we think of meaning as established through familiarity, the project appears devoid of meaning. However, if we consider meaning as having to do with students' foregrounds, their hopes and aspirations, the situation appears different. In fact, the surfing project was concluded with much dedication from the students.

[13]See also Baber (2006, 2012).

Denmark. In some cases the students and their families were interviewed together, and it became clarified how it had turned into a family concern that the students did well in school.

One of Baber's important observations can be referred to as the *polarisation of foregrounds*. The students from Pakistan described their future opportunities in Denmark as beings principally different from Danish students' opportunities. They found that polarised opportunities opened in front of them. If one performs *far above* average in school, one will have the same opportunities in Denmark as any Danish students. If, however, one's performances turn out to be a bit above average or average, not to mention if they are below average, one is left with no other possibilities in Denmark that turning into a helper in "father's kiosk". This expression could be understood literally, but also as a metaphor for any unqualified job. According to the Pakistan students, this polarised prospect is radically different from the one spreading out in front of Danish students. An average Danish student well enjoy a range opportunities for the future. The focus of Baber's study was not to try to verify to what extend the Pakistan students in fact were correct in claiming such polarisation; instead it was to identify the nature of this *experienced* difference.

Baber's observation can be related to another observation made by one of the teachers who checked the Danish national tests in mathematics after the ninth school year. He told me about a phenomenon that he had observed, and which he found strange. He had noticed that when the test reached the more difficult questions, the students reacted in two different ways. The majority of students let the white paper stay white. Apparently, they could not provide an answer, and silently recognised that there were no more for them to do. The other reaction, less common, was a visible aggression when the too difficult questions emerged. The questions became crossed out; the paper became crumbled and flattened out again; it got stokes and dots and spots. What surprised him was that this last aggressive reaction mostly came from immigrant students. Considering Baber's observation, the reaction appears less surprising. Sitting there in the middle of the test, some of the immigrant students experience that attractive features of their foregrounds are blown to pieces and substituted by "father's kiosk".

Through Baber's interviews, it became revealed that the students' experiences of meaning had little to do with relating mathematical notions to the students' background and to what might be familiar to them. Rather, the experiences of meaning related to experiences of possibilities for the future. In particular, activities that demonstrate a clear function in ensuring a better mark was experienced as meaningful.

The polarisation of foregrounds is a socio-political phenomenon, which might concern any group of people that have been displaced. It might concern any group of people that have been stigmatised through racist or any presumptuous discourses. Today the hostilities towards immigrants get more and more force and, as a consequence, the polarisation of foregrounds makes part of political reality that forms the experience of meaning for many students.

Barber's research also reveals the profound way immigrants respond to stereotypical labelling such as "Paki's". Aggressive Danish political discourses establish sharp eyes on immigrants. If a single person commits anything wrong, the entire Muslim Pakistani immigrant community become mapped. That brings pressure on Pakistani immigrants to redefine themselves and to engage in processes justifying themselves and their actions. This creates huge pressure on the formations of their foregrounds, which turns limited and polarised. This polarisation creates the need of spaces of belongings such as religious groups where Pakistani immigrants might feel comfortable—maybe not because of their religious choice but because negative political discourse pushes them to such groups. In this way experiences of belongings, of possibilities and of impossibilities becomes socio-political formed.

Destroyed Foregrounds

In his study *Who does not Dream about Becoming a Football Player? Working with Projects to Redevelop Foregrounds* [*Quem não sonhou em ser um jogador de futebol? Trabalho com projetos para reelaborar foregrounds*], Denival Biotto Filho (2015) describes how he has been working together with children living in an orphanage. They might have lost contact with their parents; their parents might have been involved in crimes and gone to jail; they might have suffered an extreme poverty and disappeared in search for at least some opportunities.

As an initial characteristic, we can refer to the foregrounds of these children as being destroyed. Certainly, it does not make sense to think of this in terms of self-destruction. We have to do with prospects in life that have suffered heavily due to the context in which the children are situated. We have to do with a destruction that only can be accounted for in socio-political categories. Still it is such foregrounds that foster experiences of meaning as well as of meaninglessness.

In what sense could one work with these children's foregrounds and try to redevelop them, as suggested by Biotto Filho? One can hardly imagine changing principal socio-political parameters through specific educational initiatives. What one could hope for, however, is to make some changes for some students in some situations. Thus foregrounds also include students' experiences of opportunities, and education can provide students with new experiences.

Biotto Filho organised a project among the children. The topic of the project was football, suggested by the children. Sure there have been many projects about football before, however the children were eager about this project. As part of the project, the children learned about what it could mean to be a professional player. We all know about the fashionable life of well-known players, as it becomes portrayed by the media. Less known is the reality of regular professionals. Such a professional visited the orphanage, and it became clarified that more than half of the professional football players in Brazil earn less that the minimum salary.

The football project was concluded with a presentation made by the children for invited teachers, adults and some researchers from the university. The presentation took people with storm. Here appeared the children with self-confidence and enthusiasm. They demonstrated profound knowledge about the topic, and they could present it with clarity and self-confidence. Afterwards one of the teachers cried: she had never imagined that these children could perform like this.

The football projects might have been the first case where they experienced success in an educational setting: they came to master a topic, and they could present it for a bigger audience. For a child with only negative experiences in school, the prospects in life becomes devastating reduced as anything presupposing further education tends to fade away from their foregrounds. However, a first experience of success in learning might establish a new element in otherwise destroyed foregrounds. Experiences of success might open for new experiences of meaning. We have to do with the inverse phenomenon of the immigrant students sitting at the national test experiencing that attractive features of their foregrounds becomes blown into pieces.

Destroyed foregrounds are a common phenomenon. One can think of the many refugees that are drifting around in Europe. They have left behind their belongings, except for what they have been able to carry along with them. (Had they at a certain time got to Denmark, some of their jewelleries might have become confiscated due to a most contemptible proposal made by the Danish government.) Displacement causes socio-political dumping and destroyed foregrounds. Certainly the apartheid regime in South Africa belongs to the past, but many structures and discourses makes a continuing dumping operate, as for instance revealed in the statistics forming Nthabiseng's foregrounds. When we have to do with students, who for one reason or another have suffered socio-political dumping, one has to be very attentive to what they might experience as being meaningful or devoid of meaning. Destroyed foregrounds as well as attempts to provide foregrounds with new features are socio-political issues.

Pointed Foregrounds

Mellin-Olsen (1981) describes *instrumentalism* as a rationale for learning mathematics not related to the content of the learning, but to the benefits that can be achieved through the learning. Thus one can try to master some mathematical techniques, not simply in order to understand mathematics better, but in order to be able to pass a coming test. Such phenomena I will refer to in terms of pointed foregrounds.

In the USA, the Algebra Project was organised by Bob Moses.[14] The aim of the project was to improve the quality of mathematics education in poor communities and to provide better access to further education for black students. Mathematics exercises a powerful gatekeeping function, and Moses wanted to ensure that black

[14]See Moses and Cobb (2001).

students were not obstructed in their career opportunities by low scores in mathematics. In order to overcome such obstructions, it became crucial to engaging back students in the existing curriculum. This forms the logic of the gatekeeping, and the aim of the Algebra Project was to help black students mastering this logic.

Black students participating in the project might experience meaning in what they were doing. However, we will get only a superficial understanding if we address the meaning associated to the Algebra Project in terms of, say, familiarity. Rather we have to pay attention to the opportunities that might emerge in front of the students. The meaning has to do with the students' hopes, priorities and imaginations; it has as well to do with overcoming fears and aversions. In fact one can easily imagine that Nthabiseng would experience much meaning by participating in the Algebra Project.

A foreground turns pointed when it comes to include a specific element that dominates the meaning-making process. This element could refer to particular aims set for the future as, for instance, coming to dominate a certain topic like algebra. One can also think of the foregrounds of the Pakistan immigrant students in Denmark as turning pointed as a strategy for coping with the polarisation of their experienced opportunities. Pointed foregrounds forms experiences of meaning, and certainly we have to do with a socio-political phenomenon.

Multiplied Foregrounds

In the literature of critical mathematics education and mathematics education for social justices one finds very many examples of how to address important social issues.[15] Mathematics can be used in reading and writing the world as suggested by Erik Gutstein. In terms of these metaphors coined by Paulo Freire: reading the world by mathematics provides tools for interpreting a range of social phenomena, and writing the world by mathematics helps identifying relevant political actions. The topics that have been presented includes for instance: pollution, violence, social exclusion, displacement, election, distributions of income, minimum wages, tax payment, sexism and racism.[16]

The introduction of such issues in mathematics education represents concerns for putting in practice a mathematics education for social justice. And certainly also concerns for establishing a meaningful mathematics education. Thus it seems broadly assumed that socio-political significant examples become experienced as meaningful by the students. Without any hesitation, I find that such examples are relevant. However, we cannot assume that political issues and topics concerning

[15] I do not try to differentiate between critical mathematics education and mathematics education for social justice, see, for instance, Skovsmose (2011).

[16] See, for instance, Gutstein (2006, 2009, 2012), Bartell (2012), Peterson (2012), Frankenstein (1983, 1989, 2012) and Skovsmose (2014b).

social justice or injustice *automatically* become experienced as meaningful by the students. Nor in this case, we can associate meaning as a property of the issues that become addressed.

When students perceive something as being meaningful, we have to do with a constructed perception. So when projects about pollution, violence, social exclusion, etc. may be experienced as meaningful by the students, it has much to do with the complex interactive processes that relate such issues to their foregrounds.

This aspect of a politics of meaning can be illuminated by a study provided by João Luiz Muzinatti (in progress). He is teaching mathematics for students coming form an upper-middle class neighbourhood in São Paulo. His concern is to address how mathematics-based arguments can help pointing out preconceptions broadly assumed among the students with whom he is working. This could, for instance be preconceptions with respect to poor people: they are lazy; they do not want to work and earn their own money; they cost society a lot of money. Muzinatti tries to provide a reading of the world by means of mathematics that challenges such general socio-political assumptions. To me this is an important example of an acted out politics of meaning. In content, it is similar to the projects proposed by, for instance, Eric Gutstein and Marilyn Frankenstein, but in terms of context it is different. While Gutstein and Frankenstein work with groups of students who suffer suppression, Muzinatti addresses groups benefitting from economic inequalities. In both cases, however, we have to do with a politics of meaning.

Through his work, Muzinatti opens new features within the students' horizons. In a most direct way, he tries to add new elements of their foregrounds in terms of possible preoccupations and new concerns about social justice. This phenomenon I refer to as a multiplication of foregrounds. Thus I do not claim that one set of foregrounds becomes substituted by another. Rather, we have to do with a multiplication, meaning that students come to operate with different horizons, which opens for different layers of experiences of meaning.

Politics of Meaning as Research

Foregrounds can be shaped in different ways. They can be *polarised*, meaning that the students see sharply contradicting prospects within their horizon. Students' foregrounds can be *destroyed* if they, according to statistics, are left with only a few opportunities for transgressing the miserable situation in which they are located. Poverty can destroy foregrounds, displacement as well. Foreground can turn *pointed*, when students see only one option to head for. Foregrounds can also be *multiplied* when the classroom practice tends to locate additional features within the horizon of the students.

The basic idea of a politics of meaning is that experiences of meaning are socio-political structured, and I ague that this is the case due to the fact that foregrounds are socio-political structured. Nthabiseng and Pieter experiences of meaning, even when sitting in the same classroom, can hardly be accounted for by referring to, say, degrees of familiarity. It appears obvious that we need to consider the socio-political formation of their opportunities in life as well as of the obstructions they are facing.

The socio-political formation of foregrounds concerns polarisation, destruction, pointedness and multiplicity. There is no "naturalness" to be expected with respect to experiences of meaning. Any such formation is a challenge for research in mathematics education, and I will refer to this research area as the *politics of meaning in mathematics education*.

During time research in mathematics education has got much inspiration from philosophy, including the philosophy of mathematics. As pointed out, discussions of meaning in mathematics education has drawn on such inspiration. However, this source of inspiration is also imposing certain limitations. In particular, I find that analytic philosophy has tended to obstruct the formulation of a politics of meaning in mathematics education. In fact it becomes important to consider to what extend a foreground-interpretation of meaning, not only makes space for a politics of meaning in mathematics education, but as well for a more general politics of meaning in philosophy.

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The Struggle Is Pedagogical: Learning to Teach Critical Mathematics

Eric "Rico" Gutstein

Abstract Paulo Freire's and Amílcar Cabral's orientation of social struggles as pedagogical suggests that when teachers are involved in community issues and struggles, they develop some of the necessary political-pedagogical knowledge to teach critical mathematics (or any subject). One does not learn to swim in a library, wrote Freire, and one will not grasp nuances of social movements and community resistance by watching alone—though active engagement is a process and has multiple entry points. In this chapter, I examine one teacher's participation in a community struggle rooted in Black self-determination, and how her learning supports her own political development and radical pedagogies.

Keywords Critical mathematics · Equity · Freire · Mathematics for social justice

The students in my "math for social justice" class at the Social Justice High School in Chicago (or *Sojo*) were grappling with the idea that when one pays for a mortgage on a house, one cannot always afford the house! Our class was trying to determine whether a family earning the median income in the neighborhood (around $32,000 a year) could afford a home mortgage of $150,000, with an interest rate of 6% a year on a 30-year loan. According to the US Department of Housing and Urban Development, paying more than 30% of one's income for housing is considered a "hardship". We worked through the mathematics to learn that such a family could pay $808 a month without hardship. But given the specific terms on the mortgage, they could not pay off a $150,000 loan. In fact, after paying $808/month for 30 years (about $291,000), they would still owe about $92,000. This prompted me to tell students, "understand how capitalism works, how banking works…" and to write the following equation on the board:

$$150{,}000 - 291{,}000 = 92{,}000.$$

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Throughout this 13-week unit, students deepened their grasp of neighborhood *displacement* through using mathematics to model and understand their social reality—mortgages (including subprime ones), various predatory lending practices, immigration and deindustrialization trends in their community, and how global financial processes impacted their immediate lives in racialized and unjust ways (Balasubramanian, 2012; Buenrostro, 2016; Gutstein, 2016).

The students were in their last year of high school (17 and 18 years old), in a "neighborhood public school", which means, in an urban US context, that any young person in the area could attend, with no selection criteria (i.e., no *tracking* or *streaming*). The school itself was created out of a political struggle for a new high school, a 19-hunger strike in 2001 by *Latinx*[1] residents of Chicago's Lawndale neighborhood. Sojo opened in 2005, along with three other small high schools (each with about 400 students, of whom 70% were Latinx and 30% African American) on a new campus. I was part of the design team that started Sojo and worked closely with the school, teachers, and students for its first five years of its existence through about 2010, including teaching the math for social justice class in 2008–09.

I use this short vignette to introduce my critical mathematics teaching in urban US settings, and to frame my argument for how sociopolitical struggles and movements are pedagogical sites for learning to teach *radical* pedagogies (in Freire's sense, 1970/1998, 1998). We are in an historical period that challenges us to action in ways that we probably cannot fully understand. It is a time when, as Yeats (1920) pointedly wrote, "Things fall apart; the centre cannot hold". On our immediate horizon are unprecedented and unknowable climate catastrophe; war, famine, and the world's worst refugee crisis since 1945 in the Mideast and Africa; extreme polarization and the rise of neo-fascism in Europe; a "soft" coup in Venezuela (a la 1973 Chile) engineered with US and OAS support; and a transactional, sexist and racist billionaire in the White House, perhaps the most powerful person on the planet—and certainly one of the most dangerous with his finger on nuclear weapons. The list goes on from there. Teaching in critical ways is not optional in the present juncture. We have a responsibility to our future and our planet, to life and all species. What we do in the classroom matters, for today and tomorrow, and the myriad possibilities for resistance and transformation are inextricably and dialectically related to the intensity of the crises we face.

[1] I use *Latinx* as a gender-neutral and meant-to-be inclusive word for *Latina* and *Latino*. This refers to people who are from (or their families are from) Latin America.

Teaching Critical Mathematics

My experience teaching critical mathematics or "mathematics for social justice" (as people in the US often call it) stretches back to 1997 when I was teaching 7th grade mathematics ("middle-school" students age 12–13) at "Rivera," a neighborhood school in a working-class, low-income Mexican immigrant community in Chicago (Gutstein, 2003, 2006). I refer to this work as *reading and writing the world with mathematics*, building upon Freire (1994), Frankenstein (1983, 1990), Skovsmose (1994), and others. While I acknowledge the multiple meanings of these terms, for me, they all mean the same: to use and learn mathematics to study social reality, as a way to deepen learners' understanding of the roots of injustice and to prepare them to change the world, as they see fit, in both the present and future.

I arrive today, in 2017, with the sense that I am a competent and confident critical mathematics teacher, after 20 years of trial and error, learning through "failure" and "success", studying my practice and the literature, and working and learning with and from others (students, teachers, parents, researchers). As a university professor, I have developed and taught a good deal of critical mathematics curricula—both alone and in collaboration (Gutstein, 2003, 2006, 2007, 2008, 2012a, 2012b, 2013, 2016). I taught my own middle school mathematics class for four years at Rivera (between 1997 and 2003), and one year of high school (at Sojo). Over the years, I have worked with teachers and co-taught many critical mathematics classes (beyond my own), and I have spent many hours in the classroom working with students, though I have never been a full time, K–12 teacher.

I have also worked outside of critical mathematics. In my 10 years at Rivera in various capacities (see Gutstein, 2006, p. 18), I worked with and supported teachers in adopting various curricula aligned with the US-based NCTM Standards, primarily *Mathematics in Context* (MiC) (NCSRME, 1997–98). MiC was originally developed at the Freudenthal Institute in the Netherlands and was an enactment of the *realistic mathematics education* philosophy (Freudenthal, 1983). At Rivera, during my four years of teaching, I taught MiC 75–80% of the time and spent roughly 15–20% of the time on social justice projects (what I called, "real-world projects"; see Gutstein, 2013) and perhaps 5% on other miscellaneous mathematics.

In five years at Sojo, I worked in teachers' classrooms where we co-developed and co-taught social justice mathematics projects that we interjected into the regular curriculum (the *Interactive Mathematics Program*, Fendel, Resek, Alper, & Fraser, 1998). This was similar to how I worked at Rivera, in that in both schools there was a strong, conceptually based mathematics curriculum into which I interspersed critical mathematics units and projects.

At Sojo, I began working closely with the entering class of 98 ninth-graders (14 and 15 year olds) and followed them through their next three years until graduation. In the school's third year, the mathematics faculty (including myself) decided to have a fourth year mathematics option that would be entirely social justice math. This was in contrast to what we were doing at Sojo or what I had done at Rivera, with "stuck-in" critical mathematics. Our plan was that I would teach it, as an experienced social justice mathematics instructor, and the other three math teachers would rotate in and co-teach. Unfortunately, all four of us taught simultaneously, and we were unable to actualize the co-teaching plan. So I taught the 2008–09 class by myself (see Gutstein, 2016) and explain more below.

One more point, to which I return below, is relevant to this brief introduction. My trajectory as a mathematics teacher educator is relatively unusual. I grew up in a densely populated working-class neighborhood of colour in New York City and came of age in the 1960s and 1970s. I became involved in social movements when young and did not seriously start college until I was 32. When I did go, it was with the conscious intention to continue doing political work, but differently from what I had been doing in organizations "outside the system". I eventually became a mathematics teacher educator in 1994 in Chicago, where I have been since. I continue to be active in political struggle and am substantively involved in the city's active education justice movement.

In summary, over the past 20 years, I have slowly and painstakingly learned to teach critical mathematics in urban neighborhood schools in Chicago. My students and I both claim that they have learned to read and write the world with mathematics, entailing also their capacity to read the (mathematical) word—that is, learn mathematics. The primary manner in which I have learned how to do this has been through the actual work of developing and teaching critical mathematics curriculum. I have studied my own practice and have tried to "sneak up on the theory imbedded in … [my] practice" (Freire, 1994, p. 126), using the extant literature to help me make sense of my work.

Teaching Others to Teach Critical Mathematics

But I have been challenged in supporting others to teach critical mathematics. This provoked me to try to understand why. My starting point is to unpack what I do as a critical mathematics teacher and understand the various competencies upon which I draw (including knowledges, dispositions, orientations, and ideologies). In so doing, I also bring in the story of another radical teacher, Monique, and reflect on our journeys, both of which bear upon the relationship of political engagement to learning to teach in critical ways.

In hindsight, certain things made sense to me. There is good deal of research on what teachers can draw upon to foster learning: content knowledge, pedagogical content knowledge, curricular knowledge, knowledge of students' thinking in general and specific to the domain (e.g., mathematics), and more (Hill, Ball, & Schilling, 2008; Shulman, 1986). The literature deepens our understanding of teaching, learning, assessment, etc., and can help teachers and researchers in our own learning. However, the knowledges needed to develop and teach critical mathematics are not the same as what one needs to teach a curriculum like MiC (challenging in its own right). A teacher who supports students in learning conceptually based mathematics with understanding has important pedagogical knowledge and experience to teach critical mathematics—in fact, necessary, in my view. But it is not sufficient and does not address many issues. For example, how does one shift contexts from ostensibly apolitical ones to those that are explicitly political (assuming one's mathematics teaching is contextualized at all)? How does a teacher know what contexts to choose? How to teach about them in ways that bring community wisdom into the experience? How to support young people to "convert merely rebellious attitudes into revolutionary ones in the process of the radical transformation of society"? (Freire, 1998, p. 74).

These are complicated questions. Some researchers have done work with teachers in schools and with teacher-education students at the university level to explicitly politicize them (Bartolomé, 2004; Gutiérrez, 2015). This important work supports teachers to disrupt systems of power and push back against an oppressive status quo. However, it may be insufficient in the present global context, because political knowledge does not develop primarily within the classroom. Freire and Macedo (1987) wrote that:

> …a radical and critical education has to focus on what is taking place today inside various social movements and labor unions. Feminist movements, peace movements, and other such movements that express resistance generate in their practice a pedagogy of resistance. (p. 61)

What does it mean, that social movements generate a "pedagogy of resistance"? What does it imply for the learners? Freire was suggesting that participants in these struggles learn, and their learning is about how to resist. Of course, how and what people learn will vary and is, in part, contingent upon the movements, organizations, forms of struggle, and individuals themselves, and also dependent on the structural possibilities and limitations (Novelli, 2004). But for Freire, "educators' comprehension of what is taking place in these social movements and spheres of public action is most vital for critical pedagogy" (p. 62).

Freire repeatedly revisited this idea. He wrote consistently of the need for educators to have political clarity, by which he meant "a level of comprehension of the complex whole of relations among objects" (p. 131), and he had a particular stance on how political clarity emerged. In the excerpt below, he was discussing the relationship between education and liberation in a pre-revolutionary situation. He wrote this in 1968, in Chile, in the dynamic period just before the 1970 election of Salvador Allende, the first socialist to be elected president in any country, and only five years before the US-backed coup overthrew Allende's government and ushered in Pinochet's years of dictatorship. His point was that people, through the process of writing themselves into history (engaging in political struggle in their own behalf), would learn how to make the new world they were building:

> The revolutionary process is dynamic, and it is in this continuing dynamics, in the praxis of the people with the revolutionary leaders, that the people and the leaders will learn both dialogue and the use of power. (This is as obvious as affirming that a person learns to swim in the water, not in a library). (Freire, 1970/1998, p. 118)

Of course Freire, as a radical intellectual, valued libraries and theory highly, although he always framed academic study and theoretical reflection in dialectical relation with practice, the unity of which he referred to as *praxis*. But he understood that we do not learn to swim in a library through his experience as a Brazilian and student of the history of African people who fought against slavery in his own country. He wrote that the *quilombos*, like other maroon settlements in the Americas, were places of safety and creativity where enslaved peoples "took their existences and history in hand" and found their freedom through "a reinvention of life" (Freire, 1994, p. 107). These spaces "constituted an exemplary moment in that learning process of rebellion" (p. 107). It was not that people knew, beforehand, how to resist and rebel; rather, it was *through* the process of resisting and rebelling that people, as subjects of history, learned how to do so.

This perspective that one does not learn to swim—or engage in political struggle—in a library was a consistent experiential theme of Freire's. His work at the World Council of Churches in Geneva in the 1970's allowed him to participate in national liberation movements in Africa in various ways and profoundly influenced him (Freire, 1978, 1994; Freire & Faundez, 1989). Shortly after gaining its independence from Portugal in 1974, the revolutionary government in Guinea-Bissau led by the PAIGC (African Party for the Independence of Guinea and Cape Verde) invited Freire to help in its literacy campaign, an effort tied closely to national reconstruction after 5 centuries of devastating colonialism. The PAIGC, too, thought that one learns to swim in the river. Amílcar Cabral, PAIGC's leader, was known, even by his enemies, as a serious political theorist, revolutionary humanist, and exceptional leader (Chabal, 2002). He wrote in 1969, when the independence movement had already liberated perhaps two thirds of the country, "We must consider that we were learning how to wage struggle in step as we were advancing (on the path)" (Cabral, 1979, p. 46).

From the Community to the Classroom

Freire, and Cabral's, understanding of learning revolution while making revolution has implications for developing and teaching critical (mathematics) curriculum in an urban, US public school, far from the global South contexts in which they worked and propagated their ideas. The fundamental question is: What political competencies does a critical (mathematics) teacher need to do this work and how might she develop them? To address this question more, I delve farther into my own and then Monique's evolution as radical teachers and focus on aspects which tie the classroom to social movements.

When I taught at Rivera, I selected all the real-world projects my students pursued. They were based on my reading of the world, not theirs. It is true that students mostly accepted the projects and engaged in them, and were generally more involved in them than in other math. In this sense, the projects resonated with who my students were as people, touched upon their lives, and built upon the sense of justice that they carried into the classroom; this was based upon their and their families' and communities' life experiences (Gutstein, 2006).

However, it always troubled me that it was not my students who decided what specific projects we would do—even if they accepted them. I wanted students—not teacher—to determine the starting point of the critical mathematics in which they engaged. My understanding was based on Freire's (1970/1998) notion of the departure point of a liberatory education to be the *generative themes* of the learners, by which he meant the dialectical relationship of the objective reality facing people (e.g., displacement from one's community) and the ways in which people understood and interacted with and upon that reality. For Freire, the starting point for radical pedagogies "must be the present, existential, concrete situation, reflecting the aspirations of the people" (p. 76). He argued, "only by starting from this situation… can they begin to move" (p. 66). While one might argue that a teacher could potentially grasp the themes facing a community and then derive a liberatory education program from that, it struck me as not the same as if and when students themselves named their reality and their own starting points. But at Rivera, for a variety of reasons, I did not make that happen (see Gutstein, 2006).

When I moved to Sojo, the situation changed (Gutstein, 2012a, 2016), and the "math for social justice" class I taught in 2008–09 had a different beginning. Students had to take a final year of mathematics at Sojo (more than what the school district required of them), and 21 chose the class, among other mathematics offerings, some months before the actual class began (and, more importantly, from a curriculum creation standpoint, before the summer break). After they selected the class, I met with them twice as a group, and with many students individually. I told them that it was their responsibility to articulate various generative themes for us to study, and that it was then mine to create the mathematics curriculum through which they would investigate the issues and their questions—and in the process, learn college-preparatory mathematics. Students proposed some units, and I also put forward some ideas, but students themselves determined what we would study and the unit order and pacing (Gutstein, 2016).

The idea of having students present their ideas about what they should study, and their reasons, is not novel. But it presupposes a particular orientation toward students' knowledge, lives, and experiences. From a Freirean perspective, it is about uncovering and locating that "present, existential, concrete situation" from which to begin the process of creating curriculum that would simultaneously support students' learning of the content (mathematical knowledge in this case) and their deeper understanding of their reality from a more critical and multi-faceted perspective. It entails both curriculum development and teaching, from a radical stance. Gandin (2002) explained what this looked like in the *Citizen School* project in Porto Alegre, Brasil, which was an enactment of a Freirean educational process:

> The starting point for the construction of curricular knowledge is the culture(s) of the communities themselves, not only in terms of content, but in terms of perspective as well. The whole educational process is aimed at inverting previous priorities and instead serving the historically oppressed and excluded groups. The starting point for this new process of knowledge construction is the idea of Thematic Complexes. This organization of the

> curriculum is a way of having the whole school working on a central generative theme, from which the disciplines and areas of knowledge, in an interdisciplinary effort, will structure the focus of their content. (p. 140)

This is no simple process. In both Porto Alegre and São Paulo (where Freire was Secretary of Education from 1989 to 1991), the educators in charge of schools tried to support teachers in learning the generative themes of the communities as a starting point to creating curriculum. In the Citizen School project, this was a complicated 10-step process (Gandin, 2002), while in São Paulo, it was quite challenging for many reasons including that some teachers were not from the community and looked down upon its knowledge and experience (O'Cadiz, Wong, & Torres, 1998). At one school, researchers found that teachers with significant social class distances from the community thought that students lived a middle-class life. In

> … conducting the Study of Reality [investigation of the life of the community and its generative themes], teachers were shocked to find that these assumptions completely disproved. They found that many of their students lived in the nearby *favela* [informal settlement] and many others lived two or three families to a one-family dwelling unit in a poorly maintained *cortiço* (tenement house). (op. cit. p. 217)

So how might teachers uncover, understand, and appreciate the generative themes of their students' lives as a step toward designing and teaching critical curriculum? Freire's and Cabral's orientation of social struggles as pedagogical space suggests that teachers have to be involved, at some level, in community issues, or at least more deeply aware of them. As an example, I describe some of Monique's political development, which partly derives from her participation in a dramatic community struggle.

Briefly, Chicago has been an epicenter of both neoliberal urban education policies (e.g., closing of public schools while opening market-based "charter schools" in their place) as well as the concomitant destruction of public housing and the massive displacement of low-income communities of colour, especially African American families (Lipman, 2011). Since 2002, the Chicago public school district (CPS) has closed roughly 160 public schools and opened about an equal number of privatized charter and similar "contract" schools (Pauline Lipman, personal communication, April 1, 2017). People in neighborhoods with almost zero public schools refer to their community as a "school desert"—this in a school system of around 650 schools and 380,000 students, CPS, 2017). The Bronzeville neighborhood is arguably the most storied and prominent Chicago Black community, but in 2012, the CPS Board of Education (appointed by the mayor, rather than elected as are over 99% of US school boards) voted to close, over a 3-year period, the last public high school in Bronzeville, Walter H. Dyett High School (Dyett).

The Bronzeville community has vigorously resisted the school closings in their neighborhood. In fact, it is often stated that while Chicago may have been a test-bed for neoliberal urban education privatization, it is also a center of resistance, led by a strong coalition of community-based organizations in neighborhoods of colour and a relatively powerful and radical teachers union with the explicit goal of becoming a "social movement union" (Compton & Weiner, 2007; Gutstein & Lipman, 2013). The effort to save Dyett HS was a key focus of a long-time Bronzeville group, the Kenwood Oakland Community Organization (KOCO). KOCO led the fight to prevent the school from being closed which including increasingly militant actions by community members and solidarity activists, including sit-ins, various protests, civil disobedience (including a group "chain-in" of 11 people immediately outside of the mayor's office resulting in arrests), and more. Finally, in a decisive act of resistance, 12 people, mainly Bronzeville parents, underwent a 34-day hunger strike in the summer of 2015 that ultimately won a major victory (see https://vimeo.com/179167070). CPS reopened Dyett HS after a one-year hiatus, and the struggle continues by the Bronzeville community for educational self-determination (see http://teachersforjustice.org).

Monique was a hunger striker. She is an African American CPS teacher who lived just 2 blocks from the school and was the mother of a preschooler at the time. She was an active union member and leader of the Black teachers caucus, but the hunger strike was definitely a higher level of commitment for her. Through it, she directly engaged with the desires and emotions of people in her neighborhood, especially parents and elders who have fought for decades against racism and marginalization on many fronts (jobs, housing, schools, health facilities) and worked for a vision of long-term, viable and sustainable schools that are grounded in the lives and experiences of students—schools, like Dyett, which would hopefully prepare their children for global leadership and environmentally just futures (Dyett Proposal, Chicago Public Schools, 2015).

Throughout the 34 days, the hunger strikers spent their daytime hours in the public park in which Dyett sits, on a main street. Over a thousand people from around Chicago and across the US, even from other countries, stopped by to show solidarity, bring water and ice, do art projects with the hunger strikers' (and neighborhood) children, and just share in the experience in limited, but important, ways that expressed support, often across race, class, geographic, and ethnic lines.

This experience assumed its pedagogical role for Monique. She was engaged with young people and organizers from Chicago and elsewhere, and this influenced her profoundly in multiple ways. It put her in direct contact with how people in her community expressed their generative themes. She said this about the hunger strike:

> From community resistance and this hunger strike, we pushed CPS and the mayor to commit to reopening Dyett as a public, open-enrollment neighborhood school. We have been able to shine a light, both nationally and internationally, on the plight of working-class Black and Brown [i.e., Latinx] communities, and the separate and unequal education system that exists in these neglected communities. We have rebuffed the narrative that privatizers have feasted on for way too long—this narrative that Black and Brown parents do not care about their students' education. The history of Dyett has helped to illuminate the fact that our schools are not failing, but they have been failed, and disinvested in purposely, in order to justify privatization. So we will continue to fight until Dyett Global Leadership and Green Technology is a reality.

Though I am obviously not suggesting that every teacher engage in a 5-week hunger strike to fight for a just education, Monique's involvement in the struggle supported the development of her political knowledge—concretely, her deeper grasp of the generative themes of her own community. It is from this basis that she will be able to create curriculum and educational experiences for her future students. It is not knowledge that came from reading about the hunger strike, but was gained through the ongoing, daily, back-and-forth struggle with the city and Board of Education. In her own estimation, there could be no substitute for her actual participation in terms of her learning and overall political development. This struggle was pedagogical, and, as Freire (1970/1998, p. 118) said, "The revolutionary process is dynamic, and it is in this continuing dynamics…that the people and the leaders will learn both dialogue and the use of power".

It is important to appreciate that social movements (like genuine mathematics problems) have multiple points of entry. Monique's life experiences are such that being a hunger striker for her is a natural extension of who she already is. But an important theme, often stated by the hunger strikers, was that rather than honor them individually, we should honor the larger effort of which the hunger strike was part. There was a broad net around the hunger strikers of people who helped out in so many ways, from (daily) medical care, logistical support, constant replenishment of water and ice, provision of nutritional supplements, transportation to and from sleeping quarters to events or press conferences, cleanup and setup, media contact, public relations, childcare, legal support, cultural and educational programming, fundraising, social media, video and photo documentation, security, spiritual uplift, phone calls to the mayor's office, attendance at rallies and other public events, and more. The hunger strike could not have happened—and would not have won what it did—without many dozens of people supporting it in myriad ways. One does not have to starve oneself for 34 days to enter the movement.

In short, if one wants to develop and teach critical mathematics curriculum based upon the generative themes of a community, then by engaging in its justice struggles, one will develop some of the political experience and knowledge to more deeply understand the neighborhood. With that more profound grasp, one will be better able to teach in ways that genuinely connect to students' lives and the deeply felt issues of their communities. Walking the picket line at the people's side, physically or metaphorically, is a swim in the river of political struggle. So a pre-requisite, or perhaps co-requisite, of developing and teaching critical mathematics pedagogy is to become involved in political struggles of the community, to learn its generative themes. Freire (Freire & Macedo, 1987) put it this way: "Educators who do not have political clarity can, at best, help students read the word, but they are incapable of helping them read the world. A literacy campaign that enables students to read the world requires political clarity". After all, if we want our students to read and write the world with mathematics, we ourselves have to (learn to) read and write the world with mathematics—and without.

In summary, learning to teach critical mathematics needs to include a process of active engagement in the sociopolitical struggles of the communities in which one teaches—through this, one deepens one's understanding of generative themes and (further) develops necessary political relationships and solidarities with the people (adults, elders, youth). This is, for me, the meaning of the phrase "the struggle is pedagogical", and is how I understand my and Monique's political evolution and radical teaching. Involvement in the "struggle" is part of our education, part of what we need to know to contribute to changing the world. Furthermore, as professionals (teachers, teacher educators, researchers), we have resources, knowledge, and skills that communities need and can (and do) put to use in the battles for their own liberation. Given our present historical moment, in the words of the US Black Liberation Movement of the 1960s and 1970s, it is for us to "seize the time" in this period of crisis and to further contribute to this momentous opportunity of transforming the world.

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Some Thoughts on a Mathematics Education for Environmental Sustainability

Richard Barwell

Abstract Planet Earth is beset by myriad environmental problems attributable to the unsustainable nature of human activities. These problems include climate change, species loss, pervasive pollution, and ecosystem degradation, all of which are examples of 'post-normal situations': they generate urgent, complex problems involving a high degree of risk and uncertainty. Mathematics is essential to our understanding of post-normal situations, to describe them, to make predictions about how they will progress, and to communicate about them. For citizens to participate in democratic debate about how to respond to these challenges and develop more sustainable ways of living, they need to engage with this mathematics at some level, as well as understand its role in making possible contemporary industrialised ways of life. I argue that *critical mathematics education* offers a perspective with which to conceptualise how mathematics teaching and learning might educate future citizens to participate in post-normal science, drawing in particular on the idea of the 'formatting power' of mathematics and the importance of reflective knowing.

Keywords Environmental sustainability · Risk · Uncertainty · Critical mathematics education · Reflective knowing · Post-normal science

The Planetary Context

The United Nations Sustainable Development Goals were approved at the headquarters of the United Nations in New York in 2015. The declaration sets 17 goals for the world with the aim of achieving them by 2030. The preamble to the declaration summarises the overall purpose of the goals:

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> This Agenda [for Sustainable Development] is a plan of action for people, planet and prosperity. It also seeks to strengthen universal peace in larger freedom. We recognise that eradicating poverty in all its forms and dimensions, including extreme poverty, is the greatest global challenge and an indispensable requirement for sustainable development. All countries and all stakeholders, acting in collaborative partnership, will implement this plan. We are resolved to free the human race from the tyranny of poverty and want and to heal and secure our planet. We are determined to take the bold and transformative steps which are urgently needed to shift the world onto a sustainable and resilient path. As we embark on this collective journey, we pledge that no one will be left behind. (United Nations, 2015, preamble)

A notable feature of the 17 integrated Sustainable Development Goals (SDGs) is the emphasis on environmental sustainability. Several goals are explicitly related to different aspects of environmental sustainability, including goals on clean energy, clean water, sustainable cities, responsible consumption and production, climate action, and the protection of terrestrial and marine ecosystems. Other goals, such as those on eliminating poverty, improving education, reducing inequality and creating sustainable economic growth, are also related to environmental sustainability. There are important links, for example, between environmental degradation and poverty, between economic growth and climate change, and between education and peace. The reason for this emphasis on environmental sustainability is made clear in article 14 of the declaration:

> Natural resource depletion and adverse impacts of environmental degradation, including desertification, drought, land degradation, freshwater scarcity and loss of biodiversity, add to and exacerbate the list of challenges which humanity faces. Climate change is one of the greatest challenges of our time and its adverse impacts undermine the ability of all countries to achieve sustainable development. Increases in global temperature, sea level rise, ocean acidification and other climate change impacts are seriously affecting coastal areas and low-lying coastal countries, including many least developed countries and small island developing States. The survival of many societies, and of the biological support systems of the planet, is at risk. (United Nations, 2015, article 14)

Our planet is not in good health. Nevertheless, while this portrait of the planet is depressing, the next article begins by stating that we live in “a time of immense opportunity” (United Nations, 2015, article 14). The SDGs are, after all, an exercise in hope rooted in a belief in progress.

How can mathematics education, both as a practice and as a field of research, respond to the appalling state of the planetary ecosystem? What, if anything, can mathematics education contribute to sustainable development? In this chapter (which revises and updates Barwell, 2013), I consider what role mathematics education can play in understanding and responding to current environmental issues. I argue that critical mathematics education offers a theoretical perspective with which to conceptualise how the teaching and learning of mathematics can engage with environmental issues. To do so, I draw on two sets of ideas: the notion of ‘post-normal science’, which highlights the uncertainty that science can entail in response to major environmental challenges mentioned in article 13 (Funtowicz & Ravetz, 1993, 1994); and ideas from critical mathematics education (e.g., Skovsmose, 1994). Much of my argument is illustrated by more specific reference to climate change as the most significant of environmental challenges facing humanity.

Environmental Sustainability as Post-normal Science

The prevalence of mathematics in the environmental sciences suggests an important role for mathematics education in promoting a better understanding of the kinds of challenges mentioned in the SDGs. How can citizens engage with debates about the construction of a new power plant, or the installation of wind turbines, or the allocation of fishing quotas or the choice to buy locally, without some appreciation of mathematics? The mathematics involved might include statistical information about the current state of different aspects of the planetary ecosystem, predictions relating to the impact of specific actions (e.g., on local habitats) or more general impacts (e.g., on future climate), and communications originating from different actors (Coles, Barwell, Cotton, Winter, & Brown, 2013). Citizens need to be able to critically engage with this kind of mathematical information in order to participate in debate about future policy, evaluate the positions put forward by different actors (e.g., politicians, activists, business leaders), and reflect on and adjust their own behaviours. In this section, I examine this idea in more depth, drawing on the idea of post-normal science.

One of the major characteristics of current environmental challenges is the high level of risk involved. For Beck (1992), such risks are a defining feature of post-industrial society:

> The gain in power from techno-economic ‘progress’ is being increasingly overshadowed by the production of risks. In an early stage, these can be legitimated as ‘latent side effects’. As they become globalized, and subject to public criticism and scientific investigation, they come, so to speak, out of the closet and achieve a central importance in social and political debates. (p. 13)

Beck’s point is that while technology has in many ways enhanced life for individuals over the past century or so, it also brings new and important risks. And indeed, public debates about major environmental issues are largely oriented to problems of risk, as well as the costs of reducing such risks. In relation to climate change, for example, the Stern Review (2006), commissioned by the UK government, evaluated the economic risks of climate change and concluded that pre-emptive action now would make more sense, economically, than reacting to the increasingly costly effects of climate change in the future: “The costs of stabilising the climate are significant but manageable; delay would be dangerous and much more costly” (p. vii). This conclusion compares the risks of making immediate costly structural changes to the global economy with the risks associated with allowing climate change to continue on its current course. The important thing to notice in this analysis is that both options (a simplistic binary choice) involve high levels of risk. It is also worth noting the highly mathematical nature of Stern's analysis. Beck's (1992) argument is that political debate, such as the debates relating to climate change, is now all about risk:

> At the center lie the risks and consequences of modernization, which are revealed as irreversible threats to the life of animals, plants, and human beings. Unlike the factory-related hazards of the nineteenth and the first half of the twentieth centuries, these can no longer be limited to certain localities and groups, but rather exhibit a tendency to globalization which spans production and reproduction as much as national borders, and in this sense brings into being supra-national and non-class-specific global hazards. (p. vi)

The risks inherent in modernity, based on technological development, have become globalised, adding complexity and uncertainty to any analysis or proposed responses.

Political responses to environmental problems often involve a turn to science as both a source of solutions, and to legitimise political positions. Beck's (1992) risk society, however, is characterised by *uncertainty*, with scientists offering one perspective in competition with several others. Funtowicz and Ravetz (1993) refer to this kind of situation as *post-normal science*:

> To characterize an issue involving risk and the environment, in what we call 'post-normal science', we can think of it as one where facts are uncertain, values in dispute, stakes high and decisions urgent. In such a case, the term 'problem', with its connotations of an exercise where a defined methodology is likely to lead to a clear solution, is less appropriate. We would be misled if we retained the image of a process where true scientific facts simply determine the correct policy conclusions. (p. 744)

Funtowicz and Ravetz argue that 'traditional' science, conducted through controlled experiments and standardised procedures, is not particularly effective in the broader context of global environmental problems and the related public debates. Current environmental challenges are too complex and too embedded in the structure of our societies and ways of life for a laboratory-developed quick fix to be effective, not least because of the likely unintended consequences of any simplistic 'solution'. First, it is not possible to fully observe and describe the complex ecosystem of which we are a part. There is a high degree of uncertainty in our understanding of phenomena like climate change (which does not, by the way, undermine the scientific consensus, but rather epistemologically recasts it) (Hauge & Barwell, 2017). Second, despite this uncertainty, the situation is urgent and action is needed. Inaction is highly likely to be catastrophic, as Stern, for example, argued. Any significant measures designed to tackle climate change will, however, have unpredictable effects, some of which might also be catastrophic, at least for human beings. Finally, even where there is consensus that human activity is a major driver of climate change, there is disagreement about whether it matters, or, if it does matter, about what measures should be taken. The uncertainty of climate science occurs as much at the level of policy (what to do) as it does in the pursuit of scientific understanding (Hauge & Barwell, 2017).

Funtowicz and Ravetz (1993, 1994) highlight several features of post-normal science, of which I shall mention three. First, post-normal science must deal with the contradictions that arise in any system:

> we can consider contradictions as being of several sorts. One is of complementarity, where the opposed elements are kept in dynamic balance. Another is of destructive conflict, where the struggle results in the collapse of the system in which they coexist. Finally there is creative tension, in which the resolution is achieved by the qualitative transformation of the system. (Funtowicz & Ravetz, 1994, p. 573)

In the context of environmental issues like climate change, for example, a contradiction arises in our individual desire for material comfort (and its associated consumer lifestyle) and the impossibility that all human beings can achieve this same level of material comfort (Funtowicz & Ravetz, 1994, p. 580). Second, post-normal science features a high degree of irreducible uncertainty. Data, for example, may be inadequate or unobtainable (e.g., high quality data for the Earth's climate in the past). And the complexity of the systems involved make observation, measurement and intervention difficult. Funtowicz and Ravetz (1993) include mathematical modelling of complex systems as an example of methods that are "untestable" and hence include a degree of uncertainty. The uncertainty inherent in climate models is well understood by mathematicians (Nychka, Restrepo, & Tebaldi, 2009). Third, Funtowicz and Ravetz (1993) highlight the shift from a traditional view of science in which facts are seen as distinct from values. This perspective led to a model of the relationship between science and policy in which science provided the facts and politicians or bureaucrats or individual citizens decided what to do with these facts. In post-normal science, values and facts cannot be separated, in part due to the problem of uncertainty (Funtowicz & Ravetz, 1993, p. 751). Climate models, for example, include uncertainty and any possible action to deal with climate change will have uncertain effects to a greater or lesser extent. Deciding which information to use, which voices to hear and which methods to try depends as much on values as it does on scientific facts.

Based on their analysis, Funtowicz and Ravetz (1993) argue that post-normal science must involve engagement with a much wider community of participants. Scientists can no longer simply seek and present facts: the increasing role of contradiction, uncertainty and values mean that dialogue between scientists and wider society is needed:

> When problems lack neat solutions, when environmental and ethical aspects of the issues are prominent, when the phenomena themselves are ambiguous, and when all research techniques are open to methodological criticism, then the debates on quality are not enhanced by the exclusion of all but the specialist researchers and official experts. The extension of the peer community is then not merely an ethical or political act; it can positively enrich the processes of scientific investigation. Knowledge of local conditions may determine which data are strong and relevant, and can also help to define the policy problems. Such local, personal knowledge does not come naturally to the subject-specialism experts whose training and employment predispose them to adopt abstract, generalized conceptions of genuineness of problems and relevance of information. Those whose lives and livelihood depend on the solution of the problems will have a keen awareness of how the general principles are realized in their 'back yards'. They will also

> have 'extended facts', including anecdotes, informal surveys, and official information published by unofficial means. It may be argued that they lack theoretical knowledge and are biased by self-interest; but it can equally well be argued that the experts lack practical knowledge and have their own unselfconscious forms of bias. (pp. 752–753)

Funtowicz and Ravetz's arguments for a 'wider peer community' suggest an important role for education in contributing to the wider understanding of environmental issues. It also underlines, however, that public understanding and individual and collective action draw on a variety of social and institutional practices.

What, then, does this all mean for mathematicians and mathematics educators? First, mathematics plays a central role in describing, predicting and communicating the nature and future course of environmental impacts (Barwell, 2013). Mathematics is, therefore, central to environmental sustainability research as post-normal science. Second, the development of mathematics has made possible the industrial-technological economic system that has led to the environmental problems listed in Article 14, both directly (e.g., through industrial pollution) and indirectly (e.g., through population growth). And third, participation in the wider peer community of science proposed by Funtowicz and Ravetz requires mathematically literate citizens; in particular, future citizens need an education that includes a critical understanding of the role of mathematics in understanding and creating current environmental problems. As D'Ambrosio (2010) says, we have a responsibility, as mathematics educators, "to question the role of mathematics and mathematics education in arriving at the present global predicaments of mankind" (p. 51). In the next section, I discuss mathematics education research that can help us to think about what that role might be.

A Critical Mathematics Education Perspective on Environmental Sustainability

There is little research in mathematics education that addresses topics of environmental sustainability. The question then arises of what research could contribute to developing a mathematics education for environmental sustainability. In this section, I argue for the potential of critical mathematics education as one useful foundation for such a project. First, however, I briefly examine and (partially) reject two alternative bodies of work: statistical literacy and research on modelling.

For Shaugnessy (2007), statistical literacy concerns the skills necessary for the consumption and production of statistical information (p. 961):

> There is a clear, overarching importance for students and adults alike to be able to critically read and evaluate information in tables, graphs, and media reports. And to adopt a healthy questioning attitude towards what is presented by sellers and buyers, by scientists and by the government, by politicians and by the news media. (p. 964)

Much of the research on students' statistical literacy emphasises the importance of learning statistics in context (e.g., Watson & Moritz, 1999; Watson, 2001).

An engagement with context supports students to go beyond mechanical calculation to consider the meaning of data within realistic situations, and to ask critical questions about the data, its presentation, and the situation. Various terms have been proposed to capture this approach, including "reading beyond the data" (Curcio, 1987), or "reading behind the data" (Shaugnessy, 2007). Research on statistical literacy is valuable in highlighting the nature of statistical reasoning and its importance in a data rich society. It does not, however, and with some exceptions, go beyond a sense of students or citizens as consumers of statistical information. Funtowicz and Ravetz's (1993) analysis suggests that with issues of environmental sustainability, citizens will need to take an increasingly active role.

Much of the research on the teaching and learning of mathematical modelling (see Blum, Galbraith, Henn, & Niss, 2007; Lesh, Galbraith, Haines, & Hurford, 2010) shares a similar set of assumptions, including a distinction between the 'real world' and a mathematical world. The purpose of mathematical modelling is to solve problems in the real world through the development of mathematical models. The modelling process involves an iterative cycle of moving between the real world and the mathematical world to develop and refine both the real world problem and a suitable mathematical model with which to tackle the problem (see Niss, Blum & Galbraith, 2007, p. 9). This process involves working from data to develop and validate a model, and working from models to produce data. Much research on mathematical modelling is concerned with students' or teachers' 'modelling competency':

> the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics, and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model etc. (Niss, Blum, & Galbraith, 2007, p. 12)

The idea of modelling competency entails a degree of critical thinking, particularly in relation to, for example, the focus on assumptions or the attention to scope. Nevertheless, there is much less attention to deeper critical issues, such as the role of mathematical modelling in a modern technological society.

Given the central role of mathematical modelling in understanding the ecosystem, such as, most notably, the use of complex climate models, there is a good case for strengthening education about mathematical modelling. At the same time, the mathematics involved in such models is generally too advanced to include in, say, high school mathematics curricula. Ignorance about the nature of mathematical models can, however, lead to them being dismissed as little more than fiction (see Hauge & Barwell, 2017). Mathematics education needs to find a way to fill this gap, and for that, I turn to critical mathematics education, as developed, in particular, by Skovsmose (1994).

A Critical Mathematics Education Approach

Skovsmose's approach is valuable for two reasons. First, it recognises the role of mathematics in *creating* our world, which he refers to as 'formatting' (1994, p. 43). Second, Skovsmose argues that mathematics teaching and learning can include a critical analysis of the role that mathematics plays. A critical mathematics education can offer students some insight into how mathematics is part of their lives and the consequences it can have—including consequences for the Earth's ecosystem.

In order to explain how mathematics formats or constitutes social reality (although not exclusively), Skovsmose (1994) proposes a distinction between two forms of abstraction. *Thinking abstraction* refers to:

> the mode of thought used to facilitate reasoning, and this type of abstraction is exemplified by mathematical concepts and mathematical modelling. Here we are concerned with explicit mathematics. (p. 51)

In effect, this definition describes the commonly understood meaning of the word 'abstraction'. Much of 'applied' mathematics involves working with thinking abstractions to solve problems, often based on some form of mathematical model. The environmental sciences now rely heavily on mathematical modelling, so that, in Skovsmose's terms, environmental problems can be understood as forms of abstraction. Climate projections, for example, are based on complex climate models, so that climate change becomes a kind of abstraction. This idea is illustrated by a climatologist's summary of how climate models work:

> A climate model starts with a set of equations governing the dynamics of the climate system and translates those equations into a model grid that represents the Earth. Each of the subcomponents (ocean, atmosphere, land surface, cryosphere) interacts and exchanges heat, moisture, and momentum. The resulting system is then driven by specified radiative forcings, including energy from the sun and emissions of human produced greenhouse gases. (Weaver, 2008, p. 183)

This account is similar to the prevailing perspective on mathematical modelling in mathematics education discussed in the previous section. The equations and the links between them are not the climate of the Earth; they are abstractions. The 'resulting system' is a system of equations, designed to represent the climate but, inevitably, a great simplification.

In contrast to thinking abstractions, Skovsmose proposes *realised abstractions*:

> Thinking abstractions are (though perhaps rather imprecise) 'images' of reality, but we also may witness the reverse phenomenon that real structures can be 'images' of thinking abstractions, and these we call realised abstractions. They are taken for granted and become reifications of modes of thought. (p. 52)

Examples discussed by Skovsmose include money, taxation (p. 52) and airline booking systems (2001). In each case, a mathematical system or model has become 'real': it has become part of the structure of social life and, as such, influences what we experience and how we act. In most circumstances, we pay little attention to the mathematics embedded in the abstraction.

Climate change, to continue the example, can be thought of as a realised abstraction. Governments around the world are formulating policies and laws in response to the threat of climate change. These measures include attempts to regulate greenhouse gas emissions or changes in energy policy, as well as mitigation measures, such as the construction of new flood defences. Such responses are largely in response to the predictions of mathematical models but have increasingly real effects. It is impossible for an individual to experience climate change on a planetary scale. Our response to climate change is, therefore, changing the structure of our society based on climate models. In this sense, climate change is becoming a realised abstraction. The same idea applies to other environmental challenges, such as air pollution, deforestation, or species loss. For the majority of the population, these major environmental challenges are largely abstract. Air pollution is difficult to perceive and has imperceptible but significant long-term impacts. Deforestation and species loss are, for most citizens, happening far away, to someone else.

For Skovsmose (1994), the process through which thinking abstractions become realised abstractions depends largely on information technology and hence on mathematics. Moreover, the language of mathematics is transferred from one of description in the form of models, to a way of organising human behaviour. A model of airline ticket sales, for example, is ostensibly a description of various factors that influence the sale of tickets—time of day, time of year, route, etc. Skovsmose's argument is that this description becomes prescriptive. In effect, it is a simplified, mathematised, model of human behaviour, but once established through technology, our behaviour increasingly must conform to the model. We are caught in a mathematical reality:

> Mathematics intervenes in reality by creating a 'second nature' around us, by giving not only descriptions of phenomena, but also by giving models for changed behaviour. We not only 'see' according to mathematics, we also 'do' according to mathematics. (p. 55)

This account of how mathematics formats reality underlines the role of mathematics in modern society. It is information technology based on mathematical models and algorithms that makes possible, for example, the creation of global supply chains through which raw materials are shipped around the world and made into products that are shipped again across the world before being distributed to arrive on the shelves of supermarkets. One by-product of this kind of activity is environmental degradation, due to, among other things, pollution from the transportation, greenhouse gas emissions, and the waste arising from a consumer culture of disposable products. The same kind of globalised system could be described for every other sector of modern life—agriculture, mass media, medicine, warfare, etc. And it is these technological systems, in which mathematics plays such a key role, that are the origin of the globalised risks highlighted by Beck (1992).

Mathematics also formats how we interact with the ecosystem (of which we are a part). Through the mathematised, model-based perspective prevalent in much environmental research, the planetary ecosystem is constructed in particular ways: as, for example, measurable, predictable, technical and controllable by humans. This construction of the ecosystem does not include the stories of our ancestors about how the weather has changed or the anguish of people whose way of life has been disrupted by deforestation or floods or the disappearance of their fish. Current approaches to teaching mathematical modelling do not generally focus on the social or environmental effects of mathematical models or incorporate human values or social relations. These kinds of considerations are exactly the kinds of "keen awareness of how the general principles are realized in their 'back yards'" that Funtowicz and Ravetz (1993, p. 753) argue must be taken into account in the conduct of post-normal science.

Reflective Knowing, Post-normal Science and Environmental Sustainability

Post-normal science is a way of responding to problems with particular features: high levels of uncertainty, urgency, high stakes and the interrelation of facts and values. Such problems feature contradictions and our collective response needs to involve an extended peer community. Normal science with its standardised procedures, ways of defining problems and separation of facts from values is insufficient. Indeed, some have argued that global environmental challenges are largely insoluble: the ecosystem has already changed. Some authors have, therefore, argued that such challenges like climate change must be understood as opportunities for collective dialogue and action in order to change our society (Hulme, 2009; Jamieson, 2014). Critical mathematics education provides a perspective with which to prepare students to participate in such a dialogue.

Drawing on his examination of the formatting power of mathematics, Skovsmose distinguishes three kinds of knowing. *Mathematical knowing* "refers to the competencies we normally describe as mathematical skills" (p. 100) which constitute the kind of formal mathematics that is embedded in technology. In the case of climate change, mathematical knowing includes the statistical methods involved in describing climate change and the mathematical methods used in predicting climate change: differential equations, non-linear systems and so on. It also includes mathematical aspects of the communication of climate change, such as the formal properties of graphs and charts. This kind of mathematical knowing is well represented in mathematics curricula and is the focus of much research on mathematical literacy. This mathematical knowing is important for citizens to be able to understand information about environmental challenges.

Technological knowing "refers to the ability to apply mathematics and formal methods in pursuing technological aims" (pp. 100–101). The emphasis here is on operating and applying mathematical tools and methods, not for the sake of advancing mathematics but in order to perform some external task. The difference between mathematical and technological knowing is largely analytical and depends on the goal involved. Technological knowing can be used without an equivalent level of mathematical knowing. For example, airline sales personnel sell tickets by operating their company's booking system. They do not need to understand the underlying models. Even when the mathematics is understood, in the context of technological knowing it is not necessary to pay attention to it. This is part of the power of mathematics.

In the case of climate change, technological knowing is relevant in two ways. First, climate scientists use technological knowing to conduct much of their work. Most climate scientists are not mathematicians. They use mathematical models and other techniques embedded in software and measuring instruments to understand the climate or some part of it. Second, we all rely on technological knowing to live in our technological society. Operating a mobile phone is a form of technological knowing that does not need to connect to the complex mathematical knowing (and other kinds of knowing, for that matter) that makes the phone work. By not connecting the two, however, the role of mathematics is hidden. Our technological society, however, is responsible for changing the ecosystem of the planet in risky, difficult to predict ways.

Aspects of technological knowing are also well represented in mathematics curricula. Teaching mathematics through mathematical modelling, or approaches to statistics education based on exploratory data analysis, educate students how to use mathematics to solve basic problems. Again, technological knowing is valuable in understanding aspects of environmental challenges, particularly in terms of how scientists use mathematics to describe these challenges and make predictions.

Reflective knowing "has to do with the evaluation and general discussion of what is identified as a technological aim and the social and ethical consequences of pursuing that aim with selected tools" (p. 101). Reflective knowing, then, is a meta-level of knowing, which goes beyond the narrower, formal knowing of mathematics or the operational knowing of technology. It is necessary to distinguish reflective knowing from technological or mathematical knowing because the latter are insufficient in themselves for an awareness of their own social or ethical consequences. It is through reflective knowing that questions about the role of mathematics in climate change can be asked. In mathematics classrooms, all three forms of knowing can be present and are interlinked. Little of the research relating to different aspects of mathematical literacy seriously engages with reflective knowing in mathematics.

Funtowicz and Ravetz argue that for complex environmental problems, a post-normal science approach is needed, involving dialogue and engagement with a wider peer community. Reflective knowing can be thought of as a small-scale analogy of post-normal science. To do the latter, the former is needed. It is reflective knowing that makes possible an awareness of the way mathematics formats society and, as such, is a key part of critical mathematics education. Teaching mathematics from a critical mathematics education perspective therefore entails teaching for reflective knowing (as well as mathematical and technological knowing).

Mathematics Education and Environmental Sustainability: Some Concluding Suggestions

In my concern that mathematics education needs to engage with environmental sustainability, I have drawn together a number of different ideas. In particular, I have highlighted how environmental problems are best understood as post-normal science and I have argued that critical mathematics education is a theoretical approach that can form the basis for engagement with such issues in mathematics education, not least because of the many points of connection between critical mathematics education and post-normal science. Reflective knowing, for example, is an important concept in critical mathematics education that connects well with the idea of the extended peer community proposed in post-normal science.

It is important to reiterate that there has been little sustained mathematics education research with an environmental focus. The primary purpose of this chapter, therefore, has been to propose a theoretical starting point with which to frame practice and research that addresses environmental sustainability through mathematics teaching. Nevertheless, since an earlier version of this writing, focused on climate change (Barwell, 2013), some initial work has been completed. To conclude the chapter, then, I will briefly summarise some of this work.

Research published so far that draws on and develops the approach I have set out in this chapter for the most part remains exploratory, with most attention devoted to the issue of uncertainty. Uncertainty is an important aspect of post-normal science, being one of its defining features. In Hauge and Barwell (2017), Kjellrun Hiis Hauge and I expand on the theorisation of uncertainty in post-normal science, in which the following distinctions are proposed:

- Technical uncertainty: inexactness arising in particular methods and techniques, which can be dealt with through standard methods, such as statistical measures of error or the use of probability.
- Methodological uncertainty: unreliability arising from the choice of methods, data, etc., where, for example, judgment and academic traditions play a role. For instance, connections between system components are basically known, but cannot be accurately quantified.
- Epistemic uncertainty: the "border with ignorance", arising from lack of knowledge, information or suitable methods, or the lack of awareness of some features of the situation. This sort of uncertainty is, in particular, related to complex issues with conflicting stakes. (Based on Funtowicz & Ravetz, 1993, pp. 743–744, cited in Hauge & Barwell, 2015)

We point out that these types of uncertainty relate to Skvosmose's (1994) three types of knowing in mathematics. Technical uncertainty involves mathematical knowing, such as the statistical methods used to control the uncertainty. Methodological uncertainty is related to technical knowing, since it concerns an understanding of the application of mathematical and scientific methods. Epistemic uncertainty is related to reflective knowing, since it involves broader meta-level understanding of the use of mathematical and scientific methods and their impact in the world. This work reinforces the complementarity of critical mathematics education and post-normal science in relation to environmental sustainability.

These ideas are supported by two empirical analyses. In the first, Hauge and Herheim (2015) report on the treatment of uncertainty in a secondary school mathematics project on traffic safety. The class investigated a stretch of road in their community in which there had been a number of accidents, some fatal. The students conducted a traffic survey, inspected the stretch of road and conducted statistical analyses. Their work was reported in the local newspaper. Hauge and Herheim used the typology of uncertainty and the three types of knowing to analyse their data, which consisted of classroom observations, video recordings and an interview with the teacher. Their findings showed that the students dealt primarily with technical and methodological uncertainty, and mostly engaged in mathematical and technical knowing to deal with these forms of uncertainty. There was less evidence of epistemic uncertainty coming into play, and less evidence of reflective knowing. Nevertheless, the authors argue that the types of uncertainty and the types of knowing that came up during the project would make it possible to open up questions of epistemic uncertainty in the context of reflective knowing. They point out that the students did begin to question local planning decisions and critique some aspects of the teacher's mathematical input, implying that the students engaged with the topic authentically.

In a second analysis, Hauge and I analysed the treatment of uncertainty in the communication of climate change projections (Hauge & Barwell, 2015). We traced multiple uses of one graphic, which showed multiple possible global temperature projections, derived from different emissions scenarios, for the period until 2300. The graphic appears in the most recent technical report of the Inter-governmental Panel on Climate Change, in the accompanying non-technical summary, in a newspaper report about the IPCC findings, in a newspaper opinion column, and in a university mathematics education class discussion. We were able to show how the treatment of uncertainty is transformed as the graphic moves into different communicational domains. For example, mathematical uncertainty was highly visible in the scientific texts, but largely disappears in the newspaper texts, while in the opinion article, uncertainty is discussed in terms of errors, rather than as an intrinsic aspect of scientific and mathematical processes. This finding suggests how a mathematics education for environmental sustainability, based on a critical mathematics education perspective, could address the nature of uncertainty in mathematics and in scientific work, so that future citizens have a better understanding of the information they consume on environmental topics.

There remains much work to be done, in particular to develop a clear sense of what a mathematics education for environmental sustainability can look like. The research on statistical literacy and mathematical modelling suggests that students need to work with real world data, represented numerically or in graphical form. From a critical mathematics education perspective, however, the danger is that such an approach will only address mathematical and technological knowing: environmental data as a 'fun' way to learn statistical methods. Critical mathematical education emphasises the development of reflective knowing, in addition to mathematical and technological knowing. Such an approach entails an examination of the formatting role of mathematics. In the context of environmental sustainability, this formatting role has two aspects: the role of mathematics in creating the technological industrialised society that has led to the many environmental challenges we face; and the role of mathematics in constructing our understanding of these challenges, including aspects of uncertainty. Such an approach needs to get 'behind the data' to consider the associated political issues, where questions of risk and epistemic uncertainty come into play. Such an approach would allow students to engage with the contradictions inherent in post-normal science, such as the apparent contradiction between maintaining a standard of living and taking action on climate change. To take this kind of approach further, students need to develop and engage with a wider peer community, through, for example, communication with climate scientists, politicians or community representatives (see Coles et al., 2013, for some ideas on what aspects of this approach could look like in mathematics classrooms).

Our planet is not in great shape, but, as the Sustainable Development Goals imply, there is much hope that the huge environmental challenges we face can be addressed. Mathematics education, as perhaps the most widely and intensively taught school subject in the world can play an important role in preparing tomorrow's citizens to engage in the debates and participate in the extended peer communities through which change will take place. In this chapter, I have set out some ideas for how critical mathematics education, combined with ideas from post-normal science, can map out its role in this process. At the same time, we should be realistic: mathematics education is not going to save the world, but it can help children understand what is going on. As Lovelock (2009) points out, humans tend to overestimate their power to act: "it is hubris to think that we know how to save the Earth: our planet looks after itself. All that we can do is try to save ourselves" (p. 9).

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Epistemological Questions About School Mathematics

Margaret Walshaw

Abstract In proposing that mathematics pedagogy limits and regulates epistemological legitimacy, the chapter raises thorny questions about the assumption that mathematical knowledge is co-constructed and shared through respectful exchanges between a teacher and her students. In the proposal, questions of differential power and privilege figure centrally. The approach that conceives of mathematics as constructed, situated within institutions, historical moments as well as social, cultural and discursive spaces signals dilemmas in relation to the question of who qualifies as knowing mathematics. Once school mathematics is conceptualized as a political and ethical project—a conceptualization that simultaneously disavows the possibility of utopian transformation—questions of epistemic responsibility come to the fore. The chapter's response is to offer a praxis that attends to failures and refusals. In the slippage from taken-for-granted truths about underachievement amongst specific groups of students, Foucault's ethics of the self asks: How are learners constituted as moral subjects of their own actions? What his work makes possible is a politics which embraces a recognition of the multiple and contradictory aspects of both our individual and collective beings.

Keywords School mathematics · Epistemology · Responsibility

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Introduction

Mathematics, it is agreed, should be accessible to all. Contemporary democratic societies subscribe to the view that, for individual and society progress, *all* students should have right of access to mathematical knowledge and that teachers should develop students' capacity to learn. Such modern 'enlightened' visionary meta-narratives, relying on a commitment to the ideals of critical reason, individual freedom, and benevolent change, sustain the idea of progress through pure and untainted knowledge. Much earlier, an articulation of these ideas had received a very clear expression in Déscartes' belief that a more objective, rationally ordered and controllable society would evolve from mathematics. In his view mathematics was the hope for the world, the only reliable means for realising an 'enlightened' vision.

Over the next few centuries this view came to be the view of others, underwriting all forms of social and intellectual life. It became naturalized, equated to thinking and hence fundamental to western democratic society. Indeed democratic social life, in both its structural and processual terms, and in all its various forms today, reveals an a priori commitment to mathematics. At this historical juncture the impressive achievements of mathematics make such respect and authority understandable. Concepts drawn directly from mathematical thought are central to the way we map out, frame, create, model and articulate, for example, the techno-scientific theorised world of material process, and the business world of money circulation and its instrumentality (Rotman, 1993). Moreover, these concepts fold into and are constitutive of the very abstractions we form, providing metaphors, idealisations and perceptions for us to understand the many realities we inhabit.

However, contemporary debates about mathematics education have drawn our attention to the problems confronted by teachers, schools, and district boards, with respect to student disaffection with and disengagement from mathematics. These discussions, providing an arresting reminder of the trend of systemic learner underachievement, have been a key rallying point for administrators and policy makers. Policy's most recent engagement with the issue of students' knowledge of mathematics has arisen, in part, through reflections on the changing nature of our mathematics classrooms, which increasingly cater to diverse groups of learners. The response to diversity often pivots around the conflation of equity with equality, in which unequal approaches, unequal access, and unequal opportunities are deemed to fully explain why specific groups of students do not succeed with mathematics. Equality, in these explanations, is privileged over any other advocacy, based on the understanding that full opportunities to learn within the classroom and respectful exchanges of ideas about mathematics between a teacher and her students outcomes, yield a comprehensive picture of equitable mathematical access for students, irrespective of any social determinations.

However, in trying to paint an idealized picture of mathematics for all, issues as fundamental as the nature of mathematics cannot fail to intervene. This chapter's key interest, then, is in that nature, and within that, the epistemological question concerning who can know mathematics. It examines conditions of possibility of constructing and using mathematical knowledge in a discussion that moves away from more typical philosophical preoccupations that are centred on with what ideal students ought to do, think and know within mathematics classrooms. It is not written as a philosophy chapter nor, particularly, a chapter in philosophy since it does not pretend to offer a rigorous argument in accordance with the exacting standards created within philosophical institutions. Rather, in troubling what Pierre Bourdieu has named as the "ritualized institution of philosophy," it might be read as an exploration into the options open to learners whose mathematical knowledge of the world does not necessarily conform to that judged by authorities as legitimate.

In proposing that mathematics pedagogy limits and regulates epistemological legitimacy, the chapter raises thorny questions about the assumption that mathematical knowledge is co-constructed and shared through respectful exchanges between a teacher and her students. In the proposal, questions of differential power and privilege figure centrally. The approach that conceives of mathematics as constructed, situated within institutions, historical moments as well as social, cultural and discursive spaces signals dilemmas in relation to the question of who qualifies as knowing mathematics. Curricula that are implemented to grant pride of place to the co-construction of knowledge within the classroom tend to cover over the complexity of the pedagogical relation in which students occupy positions of minimal epistemic authority. Since epistemological questions like these invoke ethical deliberations over what might be done, I offer a Foucauldian way forward.

Traditions of 'Knowing Mathematically'

Initial modernist justifications for including mathematics in the formal school curriculum revolved around the idea that in developing mathematical reasoning in all people, society would be provided with a more secure rational foundation for attaining knowledge and making progressive change. Thus school mathematics became an objective historical force and became central to Western political thought, politics and social organizing. Indeed, the understanding of mathematics as a prestige school subject and as instrumental to social and economic progress tends to legitimize the work of research and educators across the world. It has typically been viewed as the means to Truth and Knowledge.

Mathematics education has taken great strides over recent years to move "toward a broader understanding of what it means to know mathematically" (Maheux & Roth, 2011, p. 41). Those strides have resulted in a shift from cognitive psychology towards a strong reliance on socio-cultural theory (Jablonka, Wagner, & Walshaw, 2013). For example, in many reforms documents, to know mathematically, in a way that counters the effects of social or material disadvantage, involves teaching that is focused on building on student interests in a collaborative way; co-participating as a learner in a community of learners; engaging in dialogue between and with the students; and developing relationships between the teacher and student in less traditional ways. In these various formulations, knowing mathematics is embedded within a cultural and social context; it is a collaborative process rather than a function of the individual or the social setting; it is an apprenticeship that occurs through guided participation in social activities with a community that supports and extends understanding; and/or it is mediated by language and other cultural tools. Crucially, then, in these formulations, knowing mathematics is more about interactions between the teacher and students than it is about transmitting and consuming knowledge. To that end, knowing mathematics is characterized by an enhanced, integrated relationship between teachers' intentions and actions, on the one hand, and learners' disposition towards mathematics learning and development, on the other.

In an attempt to sharpen the modalities that shape equitable practice, Roth and Radford (2011) have proposed a cultural-historical perspective in which learning is created in the spaces and activities that the classroom community share within a web of economic, social, and cultural differences. The proposal comes hard on the heels of a renewed respect for 'the other' within mathematics education (e.g., Adler & Sfard, 2017; Ernest, Greer, & Sriraman, 2009; Gutstein, 2006). If knowing mathematics is relational and is able to tell us about the nature of an equitable mathematical experience, then it needs to be expressed as something dynamic rather than as a static process or as a property of people as formulated in the conventional take-up of socio-cultural work. In underscoring the relational nature of Vygotsky's (1978) ontology, Roth and Radford (2011) merge the cognitive, the social, the historical and the affective together. They do not stop at an analysis of the social context nor at an analysis of the individual. Their reading emphasizes that culture and history are embodied drivers of thinking and being.

Cultural-historical approaches amplify the contingent, and in doing so, embrace the post-human in preference to the fixed and unitary subject. Such approaches frame identity differently and in doing so, inspire a re-shaping of the equity imaginary. Radford's (2012) reading of emancipation troubles the clarity of the natural order, lifting it out of the narrower interpretation expressed within sociocultural theory and moves us towards appraising the enlightenment modern project as "a chimeric and unfulfillable dream" (p. 101). His view of the modern-day social justice project is aligned with the view of Foucault and other thinkers (e.g., Bingham & Biesta, 2010). For him, rethinking mathematical knowing away from the modality of self-sufficiency invites a very divergent mobilization of the emancipation project and the role the project plays in everyday classroom practice for individual learners.

For Radford, at the heart of the issue is the relationship between freedom and truth and, with it, the relationship between the individual and the social. By way of example, Radford discusses the student-centred approach, advocated in many official curriculum documents as the pivot of mainstream education policy and practice. In these documents the teacher is positioned within a learning community as a guide on the side, facilitating interchanges, providing structured and purposeful activities and connecting mathematics to everyday contexts, all the while mindful of the cultural and mathematical knowledge the students bring to their learning. It is the teacher who is tasked with orchestrating thoughtful discussion around meanings and understandings. The difficulty arises when the ideas generated by students in the classroom are reconciled by the teacher with the conventional mathematical ideas as outlined in curriculum documents. As Radford notes, students' ideas are invited yet they can never be autonomous since they are "unavoidably engulfed in discourses and epistemes (i.e., systems of thinking) that are not [the students'] own" (p. 104).

Foucault (1972), in his conceptual interrogations, has shown how bodies of knowledge, like school mathematics, tend to homogenize epistemological agency on the basis that such knowledge is shaped round the notions of rationality and pure objectivity. Indeed, mathematics is the site of ideal, controlled and objective knowing *par excellence*, embodying an immediate presumption of truth. Knowers, for their part, are detached, neutral and separate from mathematical objects. In his work, Foucault has shown, however, that knowledge, like school mathematics, is premised on a certain set of claims to truth, and thus is caught up in 'regimes of truth'. What comes to count as school mathematics does not pre-exist certain normalizing and regulating practices. Objectively derived and propositionally formulated, it is constructed from the experiences of a privileged group of people and presented as paradigmatic for all.

Thus, irrespective of definitive articulations of school mathematics, knowledge about school mathematics is an *effect* of particular rules of formation. While these rules are often unknown to the actors involved, they circumscribe the possibility of thought concerning what precisely school mathematics is. They also set boundaries on what is taken as mathematical truth. The unmasking of school mathematics as intimately tied to the social organization of power then becomes crucial to our understanding of the emancipatory project focused on the question: 'who can know mathematics?' As Radford has asked: "How can the modern subject be the locus of meaning, feeling and intentionality if it has to talk, feel and intend through thoughts and words that are not its own?" (p. 106).

Radford's question draws attention to an issue that many of us have pondered upon from time to time, yet have not been able to address from within the traditional horizon of philosophical endeavours. Despite pronouncements that school mathematics is intersubjectively negotiated in classrooms by teachers and students who are, themselves, produced as epistemic subjects, the structures of school mathematics, nevertheless, operate in a way that masks the links between power and knowledge. When mathematical knowledge is invited, shared and then suppressed in the classroom context within which cognitive resources and positions of expertise and authority are unevenly balanced, cognitive agency is severely undermined. When we can no longer propose learners as knowing, willing and judging subjects, able to act in an autonomous fashion, the possibility of emancipation is ruled out a priori. Instead, we become aware of the contingency of our very being. Trapped within Nietzsche's dilemma, we are confronted with a difficult situation: that if no Truth exists then there can be no possibility of an ethics and a politics possessed of genuine emancipatory values. When old orthodoxies are unsettled, like this, we are presented with a new set of challenges for mathematics education.

An Ethics of the Self

Foucault's final work provides us with the possibility of finding a new impetus for "the undefined work of freedom" (Foucault, 1984a, p. 46). In "The Use of Pleasure" (1984c) and "The Care of the Self" (1984b), Foucault offers a theory which encompasses a more complex and differentiated analysis of relations; one that is able to acknowledge the potential of creativity and agency within social constraints. In proposing a modern ethics of the self, infused with emancipatory potential, Foucault aimed at promoting "new forms of subjectivity through the refusal of [a] kind of individuality which has been imposed on us for several centuries" (Foucault, in Dreyfus & Rabinow, 1982, p. 217). This work offers a set of vocabularies for articulating practice which is at odds with the language offered within, for example, the Cartesian master narrative of progressive change.

Foucault's argument is that a progressive politics might best be served not through adherence to externally imposed moral obligations, but rather upon an ethic of who we are to be, and what, therefore, it is possible for us to become. As he explains, what matters is how the individual turns herself into a subject. The proposed critical ontology of the self is generated from the idea of *critique as its enabling condition* and it is this idea which forms the basis of his modern ethics of the self. In the classroom and in everyday life, it is through *technologies of the self* that the learner conceives of herself as a thinking, speaking and acting mathematical learner and where she fashions her mathematical identity. These particular practices and techniques influence the ways in which her subjectivity is constituted and the ways in which her mathematical experiences and identity will be shaped. Her thoughts and actions are also governed by them.

It is through the notion of *governmentality* that Foucault is able to offer an interpretation of individual experiences in which domination and resistance are no longer conceived of as ontologically different but as opposing effects of the same power relations. Institutions and social groups do not own power; rather, power exists in relationships that are discursively produced. Since discourse is both an instrument and an effect of power, discourse can be the medium through which an opposing strategy is prevented as well as the medium through which an oppositional strategy might be put in place. What derives from the notion of governmentality is the idea that individuals are *active agents with the capacity to fashion their own existences*. That is, whilst governmentality targets the individual as the means with which to maintain social control, at the same time it provides the individual with the very techniques with which to resist this government of individualisation. The category of the 'self' suggests an understanding of the learner as an active and never-completed process of enculturation across a vast number of subject positions, some to a much greater degree than others, *but over all of which the learner may exert some degree of autonomy*.

An ethics of the self requires a distinction between socially imposed 'ethics' (which determine which acts are permitted or forbidden and which acts are ascribed positive or negative valence in a constellation of possible behaviours), and internally constructed 'morals'. In a similar vein to Roth and Radford (in Roth, Radford, & LaCroix, 2012) who argue for activities as a means to praxis and transformation, Foucault argues that in order to understand how individuals constitute themselves as moral subjects of their own actions, we need to examine the practices that "constitute, define, organize, instrumentalize the strategies which individuals in their liberty can have in regard to each other" (Foucault, 1988, p. 19). Such an examination would reveal the different ways in which one's self is formed as an ethical subject.

In the examination the concept of autonomy is reworked to one that is linked to a questioning of what appears as natural and inevitable about one's identity. In its redefinition autonomy assumes an analytic function through which to explore the ways in which learners act in the mathematics classroom and the ways in which they give meaning to their experiences, activities and identities. That is, the examination looks closely at technologies (or practices) of the self—those rules of conduct the individual learner sets herself intentionally and voluntarily in order to create herself in daily school life as a 'work of art'. In Foucauldian understanding, learners, like teachers, curriculum planners, researchers, and so forth, despite their stable appearance, are all merely productions of practices through which they are subjected. The identities these actors have of themselves are made in and through the activities, desires, interests, and investments of others. Hence, the truth about oneself is not something given, not something in our nature, but something we have to discover for ourselves.

There are significant points of convergence between the notion of the subject (mathematics learner) as conceived by Roth and Radford (in Roth, Radford, & LaCroix, 2012) and that put forward of the subject by Foucault (1970). All promote a historically variable account of subject-constitution. For all of them, the subject is never fully constituted. When they talk about identity, they talk about identity as fluid in nature, forever in process. Theorizing identity in this way allows them to differentiate their work from the Cartesian effort to conjure a foundational status for the subject that is claimed to pertain to a rational 'man'. It allows them to escape the cause-effect logic which is at work in the construction of an essential or fixed identity. It also allows us to move beyond that logic in descriptions of the pedagogical relation involving teachers and underachievers.

From that reading we can map out new coordinates for the identity of a mathematical learner. In Foucauldian understanding, the idea is that the subject is, on the one hand, an agent, and, on the other hand, has a connotation of being subjected to. The subject is internally contradictory since it has both the status of position of agency and the status of being acted upon. That is to say, the subject is embodied with a double valence: it is an ensemble, and never in one place only. A reading of the subject, like this, invested as it is in dynamism, requires the shift in language that Roth and Radford argue for in order to convey the point that the verb 'to be' is "always in transformation" (p. 9). Thus a learner's aesthetic self-fashioning is not oriented towards the recovery of an essential inner mathematical identity but towards an exploration of the scope for potential and ways of existing in the world. Blake, Smeyers, Smith, and Standish (1998) elaborate:

> It is because she has no essence that the subject enjoys…a freedom of fragmentation: a freedom that arises in the constellation of differences that constitute a lineage of loose alliances, relations of resistance and mastery, and configurations of fluid interests. The freedom of fragmentation remains real in response to the constant transformation of problems. It puts in question the firmest of principles and established practices. The result is an ethic of responsibility for the truths one speaks, for the political strategies which these inform, and for those ways of relating to ourselves that make us either conformists or dissidents. (p. 62)

The construction of an identity as a mathematical learner is an ongoing process, ever-changing as the student works at reconciling her sense of self as a learner with the normative activity established within her mathematics classroom. Choosing one's mathematical identity is a process that is not wholly conscious but nevertheless accessible to consciousness. It involves the interpretation of a classroom reality which is weighted with existing sanctions and prescriptions. The important point is that through an ethical self-analysis the learner examines everyday classroom practice, problematizing and unravelling the tangled complexities of the pedagogical relations, all the while maintaining an awareness of the possibility of resistance. Thus, on the one hand, a mathematical identity is the locus of cultural interpretations, that is, it is always already caught up and defined within a classroom context. On the other hand, it is also the site at which the individual learner is required to receive and actively interpret that set of interpretations.

In Foucault's words, these interpretations guide the self-fashionings by which they "seek to transform themselves, to change themselves in their singular being, and to make their life into an oeuvre that carries certain aesthetic values and meets certain stylistic criteria" (Foucault, 1984b, pp. 10–11). The self is always open to reinscription. However, as Bibby (2010) points out, the self has a tendency to "move uncomfortably between the individual and the social or cultural without resolving, or satisfactorily exploring, the tensions inherent in this tussle' (p. 37). The student is complicit, subsumed by more global structures but the reality is that the student *is never reduced to them* (Walshaw, 1999). Transforming the self is a 'work of art'. As Foucault (1984a) says: "Modern man…is not the man who goes off to discover himself…; he is the man who tries to invent himself. Thus modernity…compels him to face the task of producing himself" (p. 42). Through Foucault's concept of ethical self-formation, mathematics classroom learners constitute themselves continually as a work of art, and as the subject of a critical practice of freedom, ever mindful of what is not able to be surrendered: that is, ever mindful of the limits of pedagogical practice.

Moving Forward

Classroom life can never be fully dictated by the western philosophical tradition. At stake are the rhetorical spaces of mathematics and the implementation of curricula by teachers in which a student's complicity in structures of power and privilege is obscured. Foucault's theorizing provides a point of departure for thinking about subordination and complicity in knowing mathematics. His conceptual tools for a project of freedom take us beyond the 'limits' of human knowledge, and, in particular, the limits that obtain when mathematics is mapped out so that certain understandings are privileged and others are not because those that are not are deemed incompatible with the established discourse. His work also allows us to move beyond the point when credibility of mathematical knowledge is presumed, or otherwise, merely in relation to the speaker's hierarchical position within the pedagogical context. His work gives new direction to the undefined work of freedom to be achieved by working at the limits that have been imposed on us. But the strength of Foucault's work, as opposed to the display of its individual gems, is that it creates a politically constructive moment by offering the tools for invoking ethical deliberation. Specifically, it embraces the potential for creativity and agency within social constraints. This is important, because, as Foucault sees it, what we might become stands as the political, ethical, social, and philosophical problem of today.

In undercutting the foundations upon which the notions of empowerment and emancipation lie, Foucault's work sets itself up against grand narratives of social progress and the united calls to social justice as a macro-political struggle. Based in practice, the ethical analysis he proposes is able to address the realities of inequitable experiences. For him, our freedom is not located in our so-called transcendental nature; rather it is found in our capacities to contest and change those practices that constitute who we are. Based on the principle of self-critique, the proposal is for an ethics of the self that is experimental, endless, and relinquishes any hope of attaining a complete and definitive knowledge of what may constitute our historical limits.

The proposal, it is argued, has far-reaching implications for equity projects in schools focused on lifting the mathematics achievement of specific groups of students. The political possibilities centre on the individual learner as both the site for a range of possible forms of subjectivity, and subjected at any particular moment of thought or speech, to the regime of meaning of a particular discourse but which enables the learner to act. The learner is neither the origin of the particular classroom arrangement in which she finds herself; but neither is she a passive member of it, or of the wider education system. The learner has the capacity to modify the relations and arrangements in which she finds herself. If the learner shuns the responsibility of authentic self-creation she comes to be entirely fabricated by others. She cannot simply, of course, wilfully fashion an entirely new self but she can, as Waugh (1992) has argued, use aesthetic strategies to reformulate available resources. Learners do not have a hidden essential mathematical identity waiting to be discovered. Learners are, rather, an artefact, an aggregation of available forms from which they must choose to shape into a coherent identity.

From the perspective of interventionary work in the field of mathematics education, a whole new space for critical reflection on the scope and limits of freedom becomes available. What becomes possible is a response to the question relating to how the growth of mathematical capabilities might be disconnected from the intensification of mathematical understandings that are 'handed down'. On offer is the possibility of the learner moving beyond her current self, and what she might be doing or thinking. The project involves a consideration of how the learner comes to an understanding of the elasticity of her current individual freedom by constantly exploring the limits of her subjectivity. It is to concede that the learner, in general, and the disaffected and disengaged learner, in particular, does this, in the first instance, by questioning the boundaries of the taken-for-granted understandings of learners in school mathematics and revealing how these established forms of identity are necessarily contingent and historically specific. At this point the possibility of transgressing the so-called limits is made available and this, too, is the point where the potential for new forms of subjective experience and new forms of mathematical identity are established.

Once school mathematics is conceptualized as a political and ethical project—a conceptualization that simultaneously disavows the possibility of utopian transformation—questions of epistemic responsibility come to the fore. The chapter's response is to offer a praxis that attends to failures and refusals. In the slippage from taken-for-granted truths about underachievement amongst specific groups of students, Foucault's ethics of the self asks: How are learners constituted as moral subjects of their own actions? What his work makes possible is a politics which embraces a recognition of the multiple and contradictory aspects of both our individual and collective beings.

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The Concept of Culture in Critical Mathematics Education

Brendan Larvor and Karen François

Abstract A well-known critique in the research literature of critical mathematics education suggests that framing educational questions in cultural terms can encourage ethnic-cultural essentialism, obscure conflicts within cultures and promote an ethnographic or anthropological stance towards learners. Nevertheless, we believe that some of the obstacles to learning mathematics are cultural. 'Stereotype threat', for example, has a basis in culture. Consequently, the aims of critical mathematics education cannot be seriously pursued without including a cultural approach in educational research. We argue that an adequate conception of culture is available and should include normative/descriptive and material/ideal dyads as dialectical moments.

Keywords Culture · Ethnomathematics · Critical mathematics education · Gender

Ethnomathematics and Its Discontents

The concept of culture is rarely invoked explicitly in the research literature on critical mathematics education. The reason for this is not hard to find. Ole Skovsmose, the founding father of critical mathematics education, was deeply affected by the apartheid system in South Africa (Skovsmose first arrived in South Africa in 1993 and later started to collaborate with the South African researcher Renuka Vithal). Consequently, he is very careful in his use of the notion of culture. A so-called

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'appreciation' of pupils' cultural background (such as Zulu culture) could be associated with people "out there" and "down there" (Skovsmose, 2014: 64). Cultural approaches run the risk of essentialism and worse. For Skovsmose, the fact that pupils have cultural backgrounds is not a "simple truth" (Skovsmose, 2014: 64).

Meanwhile, in the literature on mathematics education, 'culture' has become associated with the ethnomathematics movement, because the ethnoi of ethnomathematics are "identifiable cultural groups".[1] Thus, 'culture' (at least when used as a count-noun) is part of the founding definition of ethnomathematics. This movement was the object of a withering critique from the critical mathematics education research group by Vithal and Skovsmose (1997). The principal point of this critique was that the ethnomathematical approach is uncritical with respect to the cultures that pupils belong to, the discourses around culture in general and the role of mathematics in formatting the world as we experience it. Regarding the cultural backgrounds of pupils, Skovsmose and Vithal argue that these might not be entirely benign or free from internal conflict. Indeed, it would be a very unusual human culture that had no internal conflicts, egregious power relations or structural injustices. Turning to the broader discourse, talk about 'culture' can easily fall into a reductive ethnic essentialism and even outright racism. They make this point dramatically by shifting the context of the discussion from Brazil (where ethnomathematics emerged) to South Africa. They argue that, in that context, the proposal that pupils from different ethno-cultural backgrounds should have different kinds of mathematical education to reflect their cultural differences sounds rather like the rationale that used to be given for the educational arrangements under apartheid.

Skovsmose and Vithal do not suggest that the ethnomathematics movement is somehow guilty of implicit racism or secretly supportive of apartheid. On the contrary, they affirm that they share the progressive goals of the ethnomathematics movement. Rather, their point is that the ethnomathematics research programme lacks the theoretical resources to distance itself from apartheid education. In his programmatic works on ethnomathematics, D'Ambrosio took a broad view of his core notion, thus, "we will call *ethnomathematics* the mathematics which is practised among identifiable cultural groups, such as national-tribal societies, labor groups, children of a certain age bracket, professional classes, and so on" (1985: 45). The *ethnoi* of his ethnomathematics are not solely racial or national-tribal groups. But they do include national-tribal groups, and this makes it difficult to explain in technical, scientific, value-neutral terms why the proponents of apartheid education should not be part of the ethnomathematics movement. The reason they cannot join is that the apartheid regime and the ethnomathematics movement have quite different political goals, but this reference to political goals cannot be part of the technical core of a scientific research programme.

In general, the *ethnoi* of D'Ambrosio's ethnomathematics are groups to which learners are assumed to belong simply and unproblematically. Part of Skovsmose and Vithal's argument is that membership of such groups is often involuntary and

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[1] D'Ambrosio (1985, p. 45).

may be established, achieved or imposed through the exercise of power-relations. Discovering mathematics in the practices of a group to which one is proud and glad to belong is one thing. It is another to have one's mathematics education tangled up with a group-membership that is uneasy or coerced, or which one is unwilling to discuss in class. Above all, ethnomathematics addresses the learner as a member of a cultural group, rather than as an individual. Of course, it does not follow that teachers who bring an ethnomathematical approach into their pedagogy must be guilty of treating their pupils as instances of types. This is unlikely to happen. The point rather is that ethnomathematics as pedagogical theory has no resources to explain why they should not treat pupils solely as instances of their ethnic types. For example, Presmeg (1998), working in the USA, identifies four principles for introducing ethno-cultural material into school mathematics classes. The first two are:

1. Each student is considered as having a unique sociocultural history; each student has ethnicity.
2. This ethnicity is a mathematical resource; mathematics may be developed from associated cultural practices.[2]

The first principle is a clear expression of the conflict between treating learners as individuals and treating them as members of cultural groups. Each student does indeed have a unique sociocultural history—that is part of what makes each student a unique individual, not reducible to membership of any group. It is also the case that each student has ethnicity, that is, belongs to some cultural group(s). But the second of these truths cannot possibly be an explication of the first. This is because membership of such groups is not a simple, unproblematic relation that we might symbolise thus: $x \in A$. Rather, belonging to a group is a task—one has to work out what it means for oneself to be female, or Irish, or heterosexual, or black, or the child of immigrants or market traders, or whatever it might be. Each person might carry out these tasks differently. Each of us might come to a different accommodation with membership of whichever groups we happen to find ourselves in. In other words, it's not quite right to say that "each student has ethnicity", as each student has a blood group and a shoe size. Rather, each student constructs (or struggles with) his or her ethnicities. Ethnicity is not a fixed property but a dynamic process of giving meaning to the different constituents of identity. This too is part of our individuality.

This is not Presmeg's confusion alone—it arises from the combination of an ethnomathematical approach and a proper concern for individual difference. The second principle states starkly that the learners' ethnicities are a resource for the teacher (who may belong to some other ethnicity altogether) to use in the teaching of the school curriculum. Learners are likely to notice when their cultures are being instrumentalised. They may resent it, and they could not be blamed if they did. Here

[2]Presmeg (1998, p. 321).

again, we cite Presmeg only because she expresses the logic of ethnomathematical pedagogy with unusual clarity, and thereby displays its aporia.

Ethnomathematics suffers from some other logical tensions that will have a bearing on the present argument. It pretends to treat all mathematical practices equally, but it a) seeks to valorise some practices and denigrate some forms of education and b) imposes the concept of mathematics on practices found in cultures where this concept is not present—specifically, a concept of mathematics that is abstract and modern (dating from the early twentieth century). Until relatively recently, European mathematicians understood mathematics to be the science of number, magnitude and geometrical space. This was Kant's (1724–1804) view; his philosophical account of mathematical knowledge founded it on arithmetic and geometry (each in turn grounded in a suitable pure intuition).[3] When his near contemporary Euler (1707–1783) received the Königsberg bridges problem, he wrote in reply, "…this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle."[4] We now recognise the Königsberg bridges problem as belonging to mathematics, specifically, to graph theory. In order to see the mathematics in basket-weaving, string forms, kolam-drawing and city tour-route design, it is necessary to move beyond the antique conceptions of Euler and Kant, and to deploy an up-to-date notion of mathematics such as "the science of patterns" or "the science of detachable relational insights".[5] Such broad definitions reflect the rise in the twentieth century of mathematics concerned with such abstracta as sets, graphs, topologies, morphisms and categories. (For the argument that twentieth-century academic mathematics marks a deep shift from its precursors, see Lautman, 2006 or Gray, 2008.) In other words, ethnomathematics breaks the first rule of anthropology in spectacular fashion. It analyses 'ethnic' practices using a concept imposed from without, a concept, moreover, which is only intelligible to the relatively small number of people with a university-level education in contemporary pure mathematics.

Aside from this spectacular violation of anthropological method, the appeal to this modern, abstract notion of mathematics presents a practical pedagogical difficulty. One way of using ethnomathematics in the classroom is to offer pupils activities that can function as a bridge from the mathematical practices they already engage into the school mathematics curriculum and thence to practical mathematics for their future lives. This may be plausible if the pupils' mathematical practices are relatively close to the curriculum—for example, if they are already using money or

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[3]*Critique of Pure Reason.* The details do not matter for the present purpose—all we need is that for Kant, if you have arithmetic and geometry, you have mathematics.

[4]Leonhard Euler, letter of April 1736, quoted in Hopkins and Wilson (2004, p. 201). He went on to complicate this point by wondering whether the bridge-problem might be the sort of thing that Leibniz had gestured at with his talk of 'geometry of position'. Nevertheless, the example stands up well enough for the present purpose.

[5]Thomas (1996) and Steen (1990) respectively, both quoted in Presmeg (1998) p. 328.

weights and measures. However, ethnomathematics tends to elide mathematical practices with practices susceptible of mathematical analysis. As Presmeg reports, "Millroy's… carpenters, Saxe's… candy sellers, Lave's… grocery shoppers—none of these considered themselves to be engaged in doing mathematics" (Presmeg, 1998: 328). She goes on to make the point vivid with a study that finds a dihedral group of order eight in the Warlpiri kinship system. We might persuade people that measuring and counting wood, candy, groceries and money are examples of the same activity as school mathematics. We are unlikely to persuade the Warlpiri that in forming marriages, they're doing algebra. Similarly, there is in print a mathematical analysis of tying a neck-tie (which activity is, from a mathematical point of view, equivalent to a short walk on a triangular lattice).[6] No-one supposes that tying a neck-tie is a mathematical practice, even though it consists of iterations of elementary operations. Certainly, the authors did not take themselves to be writing ethnomathematics. Activities such as tie-tying and basket-weaving only count as mathematics because modern mathematicians can recognise abstract structures in them, but this is unlikely to help learners. If the path from ethnomathematical practices to book-keeping, engineering and statistical analysis has to go through group theory or graph theory, it is unlikely to offer much improvement on existing routes. Pupils might do better starting school mathematics from a blank slate after all.

In their analysis, Vithal and Skovsmose (1997) drew on some of Pierre Bourdieu's ideas on the reproduction of the social structure and the mediating role of the school. They argue that identifications of students aligned with cultural categories (class, race, gender, …) establish expectations regarding their academic success. These expectations may suppress or repress the student's academic performances, both predicted and recorded (since the former influences the latter). This role of the school system in reproducing social structure is rarely discussed (Bourdieu, 1996). Vithal and Skovsmose (1997) introduce their notion of foreground, which they understand as 'the set of opportunities that the learner's social context makes accessible to the learner to perceive as his or her possibilities for the future' (Vithal & Skovsmose, 1997: 147). Skovsmose emphasises the political and cultural situation as an important aspect of the foreground since they condition the opportunities for the learner (Skovsmose, 2005). The notion of foreground also makes the political nature of the learning process explicit. From this context we can understand that Skovsmose gives more attention to the notions of politics and sociology—rather than the concept of culture—to analyse educational processes in contemporary societies (Alrø, Ravn, & Valero, 2010).

Skovsmose's work deeply influenced research in critical mathematics education and increased the focus on the micro-level of mathematics education. The notion of culture within the research community of critical mathematics education almost disappeared, as may be seen in the festschrift for Ole Skovsmose (Alrø, Ole, & Paola, 2010). The preface indicates that Skovsmose's work mostly contributed to

[6] Fink and Mao (2001).

mathematics (education) in relation "to society, educational thinking that emphasises the politics and sociology of educational processes in mathematics, and pedagogical thinking that connects the philosophical and educational reflections with issues of relevance for the micro-processes of teaching and learning of mathematics" (Alrø et al., 2010: vii).

Note that Skovsmose and Vithal direct their critique principally at ethnomathematics as a pedagogical proposal (p. 135). They do not oppose the project of investigating ethnic practices that fall under the broadest possible conception of mathematics, broadening the history of mathematics or examining mathematics in non-academic contexts. The debate has evolved since 1997—see Pais (2010) for a summary of developments after Skovsmose and Vithal's critique, Skovsmose (2015, 2011: 84–87) for a recent statement of his reservations about ethnomathematics as pedagogy, and Knijnik (2012) for an attempt to make good the shortcomings of ethnomathematical theory. None of the points in this section is original—they all occur in the cited literature. The purpose of this first section is just to explain why people in the critical mathematics education research group rarely deploy 'culture' as a technical term, even if they sometimes use the word.

Culture, Again

One of the claims of this paper is that the concept of culture is important for understanding mathematical practices and mathematics education. Having sketched some of the problems arising from the word's association with ethnomathematics, we are in a position to ask whether it is possible to recover a concept of culture that can escape these difficulties. To that end, this section will develop a reflection on the notion of culture in general. It draws on arguments elaborated in Larvor (2016a).

The word 'culture' is semantically rich. Its many meanings divide into two broad groups: normative/educative and descriptive/scientific. On one side, there are the normative senses in which culture is a good thing, valuable in its own right and for the people who have it, both individually and collectively. Thus, a person may be *cultured*. Such use of the term 'culture' requires a high level of confidence in the associated evaluations. One must firmly believe that one knows what is valuable in order to deploy the word in this way, and be confident of culture's good effects. Culture, thus understood, stands in contrast to nature and ennobles the youth fortunate to be educated in it. Culture in this sense need not be 'high culture', that is, great works of high art, but it does require distinctions of quality and some sense of sustained tradition. Even the most recently emerged genres of art and music have agreed masterpieces and landmarks. Any aspiring rappers now must know their Public Enemy from their Run DMC. Mathematics can constitute culture in this sense—one reason for keeping it in the curriculum is that it is part of the common cultural treasury of humanity. Paul Ernest, speaking at a conference in London on mathematical cultures, spoke of introducing learners to the 'poetry of mathematics'.[7] We may wonder—without cynicism—whether it would be possible to justify the teaching of mathematics on the same grounds as the teaching of poetry.[8] Notice that in this paragraph, 'culture' functions as a mass-noun.

This family of normative/educative senses of 'culture' forms a dyad with the family of descriptive/scientific senses. We say 'dyad' because every distinction establishes a relation, and traces of the normative/educative sense are detectable throughout the descriptive and scientific literature on culture in general and mathematical cultures in particular. In any case, we should expect to see a normativity implicit in any research directed at the question 'How can we teach mathematic *better*', even if that research makes explicit use of descriptive or scientific concepts of culture. Indeed, Skovsmose (without mentioning culture) explicitly refers to this interplay between the normative and descriptive aspects of critical mathematics education.[9] We therefore think that the normative/educative and the descriptive/scientific senses of 'culture' stand in dialectical relation rather than absolute separation.

On the descriptive, scientific or anthropological senses of the term 'culture', a good place to start is the magisterial literature review in Kroeber and Kluckhohn (1952) *Culture: A critical review of concepts and definitions.* This report starts with the first recorded scholarly uses of the term and tracks its evolution and differentiation up to their time of writing. This is of some interest, given the discussion so far, because Kroeber and Kluckhohn explore both sides of the descriptive/normative distinction, and their historical survey includes those German thinkers of the nineteenth century who saw this distinction as the ground of a relation.

At the end of their historical journey, Kroeber and Kluckhohn land on this definition of culture, which they take to be an approximation of the view of "most social scientists":

> Culture consists of patterns, explicit and implicit, of and for behavior acquired and transmitted by symbols, constituting the distinctive achievements of human groups, including their embodiments in artifacts; the essential core of culture consists of traditional (i.e. historically derived and selected) ideas and especially their attached values; culture systems may, on the one hand, be considered as products of action, and on the other as conditioning elements of further action.[10]

In this definition, we see a tension that turns up more widely in the descriptive senses of culture. The first part of this definition presents culture as patterns of

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[7] https://sites.google.com/site/mathematicalcultures/.

[8] There is, for example, more than a trace of Mathew Arnold in Lockhart's (2009). Ernest (2000) makes a distinction between capability and appreciation that emphasises the value of mathematics as cultural achievement.

[9] See the interview in Alrø, Ravn and Valero (2010, pp. 1–9), in which he relates his thinking to philosophers associated with critical theory such as Habermas, Adorno and Foucault. Especially: "A critical activity cannot *only* represent uncertainty. It also represents concerns, and the formulation of concerns immediately brings us to the formulation of visions, aspirations, and hopes" (p. 7).

[10] Kroeber and Kluckhohn (1952 p. 181).

behaviour, embodied in and transmitted by symbols and artefacts. From a positivist or behaviourist point of view, that sounds scientifically respectable. Symbols, artefacts and behaviours are all empirically detectable (setting aside the question of how a purely empirical consciousness could recognise them *as* symbols, artefacts and behaviours). However, they go on to say that the core of (a) culture is "ideas and especially their attached values". Ideas and values are not so easy to detect empirically, except perhaps indirectly through their effects. This mention of values is not an afterthought; on the contrary, for Kroeber and Kluckhohn, values are central to anthropology: "Values provide the only basis for the fully intelligible comprehension of culture, because the actual organisation of all cultures is primarily in terms of their values."[11] It is also worth noting their use of the term 'achievement'. This is a success-word, and therefore invokes a normative sense of culture, though presumably the criteria of success are culture-specific.

Social science has not stood still since 1952, but the core of Kroeber & Kluckhohn's account is recognisable in more recent definitions of culture. Writing in 1989, Banks, Banks, & McGee claim that,

> Most social scientists today view culture as consisting primarily of the symbolic, ideational, and intangible aspects of human societies. The essence of a culture is not its artifacts, tools, or other tangible cultural elements but how the members of the group interpret, use, and perceive them. It is the values, symbols, interpretations, and perspectives that distinguish one people from another in modernized societies; it is not material objects and other tangible aspects of human societies. People within a culture usually interpret the meaning of symbols, artifacts, and behaviors in the same or in similar ways.[12]

We should expect to find this taut duality in any study of culture because the two ends need each other. The behaviours and artefacts do not classify themselves or explain themselves; the ideas and values must body forth in words, deeds and things if they are to have any presence at all. We can find this tension in the philosophy of mathematical practice. Mathematics is obviously concerned with ideas, but studying mathematical practices directs attention to artefacts (blackboards, notations, diagrams, models and computers) and behaviours (gesticulating, writing, sketching, gathering mathematicians in groups of various sizes, etc.).

Kroeber and Kluckhohn's emphasis on values as "the only basis for the fully intelligible comprehension of culture" conforms to some popular usages, especially when culture is contested. 'Cultural issues' in politics are precisely those that are not driven primarily by prudence or efficiency, and they tend to form blocs—support for rote-learning of times-tables tends (in the UK at least) to coincide with social conservatism and a yearning for deference, while enthusiasm for whole-book teaching of reading sits with environmentalism and new-age spirituality. These blocs do not make sense in practical terms—there is no logical reason why a person's view about teaching basic literacy or numeracy should predict their foreign

[11] Kroeber and Kluckhohn (1952, p. 340). Note that 'culture' occurs in this sentence both as a mass-noun and as a count-noun.

[12] Banks and McGee (1989). Note the reference to 'modernized societies'.

policy stances or their openness to religious innovation. Rather, these apparently technical judgments (such as, how best to teach multiplication or the extent of human influence on the global climate) come to seem expressive of some general values and attitudes towards authority, society and the self. That is what expressions like 'culture wars' seek to capture. Similarly, there is a whole industry dedicated to achieving cultural changes in organisations, as managements have come to believe that changing policies and practices alone do not achieve their desired ends. In December 2015, there was a scandal because the UK Financial Conduct Authority dropped its report into banking culture. The suspicion fuelling the sense of scandal was that political pressure was brought to bear precisely because a review that addressed the culture of banking might achieve changes that piecemeal adjustments to rules and practices cannot.

This rather abstract discussion then gives us two thoughts:

1. There is a dialectical relationship between the material aspects of culture (the artefacts and practices) and the ideas and values, such that culture is unintelligible without reference to values.
2. There is a relation between descriptive and normative approaches to culture such that the normative aspect can never be eliminated. To understand a culture is to understand its values, but this always involves some dialogue with the researcher's own values (even if these are solely norms of scientific practice).

The Interest of Mathematics Education in Culture

Our emphasis on the centrality of values for the intelligibility of culture has consequences for both approaches to mathematics education under discussion here. For ethnomathematics, the *ethnoi* are defined culturally, which on our view means, through their values. Thus teachers who instrumentalise ethnic practices for mathematics teaching bring value-laden practices into the classroom without knowing what those values are—nor how they will stand in relation to the values of the individual students, the classroom, the school and the curriculum.[13] So, our claim that cultures must be understood in terms of their values clarifies and deepens the existing critique of ethnomathematical pedagogy. On the other hand, this same claim of ours supports D'Ambrosio's insistence that all groups are socioculturally determined, including Western academic mathematicians. Bishop (2008, 2016) identified and analysed values connected to Western academic mathematics (see also Ernest, 2016). If Kroeber and Kluckhohn are right that the core of a culture is its values, then Bishop's work shows us the core of the culture of Western mathematics. Our development of the notion of culture articulates more clearly the

[13]For analyses of national differences in school and classroom cultures, see essays by Andrews and Gosztonyi in Larvor (2016b).

ethnomathematicians' claim that the Western mathematics curriculum also brings value-laden practices into the classroom. It remains to be seen what effect this might have on the grander aspirations of the ethnomathematics movement, which are to build a civilization that rejects the colonial politics of imposing one curriculum (the Western one) worldwide (Pinxten & François, 2011). They hope to restore cultural dignity, reinforce cultural self-respect and empower currently excluded people and societies (D'Ambrosio, 2016). Nevertheless, articulating these aims in terms of values (which can overlap and inform each other) rather than discrete cultures might offer more practical possibilities than does the simple narrative of oppression and resistance.

Our development of the notion of culture may have another consequence for the ethnomathematics programme. Our suggestion that descriptive and normative aspects of culture cannot be wholly separated entails that wholly value-neutral anthropology is impossible. This may offer researchers in ethnomathematics a defence against the charge that their work is too interested in pedagogy and politics to count as science. If we are right, then all ethnology is an encounter between value-imbued research practices and value-imbued 'ethnic' practices. Then the question is not, how can this encounter be rendered value-free, but rather, how can it be negotiated?

Turning to critical mathematics education, recall Kroeber and Kluckhohn's claim that culture systems are "conditioning elements of further action". In other words, pupils' background and foreground (in Skovsmose's sense) are determined in part by culture. Culture tells you what is possible and what is worth doing, and it tells you where you fit in. Skovsmose and Vithal developed the notion of foreground precisely to criticize and to overcome the stereotyping effects of pupils' backgrounds. Stereotyping affects the recorded and expected performances of pupils and therefore it affects their futures. The notion of foreground is developed to make learning obstacles visible as political issues based on the students' backgrounds rather than as individual phenomena. The notion of foreground frames the teaching and the learning of mathematics within a social and political context of 'social justice education' in order to overcome cultural determination (Burton, 2010; François, 2016). Although the notion of foreground and the programme of social justice are part of culture, critical mathematics researchers emphasise more the social and political aspect due to the association of culture with determinism, as we discussed in the first section. We argue that this is a missed opportunity, because an appropriate notion of culture can forge analytic connections between ethnomathematics, critical mathematics education and work on values in mathematics by Bishop and Ernest.

One example—omnipresent in the social justice debate—is the gendering of mathematics, particularly the widespread conviction that it is not for girls. This shows itself to some degree in measurable effects such as stereotype threat.[14] This is the reduction in cognitive test performance caused by reminding subjects that they

[14]See Doyle and Voyer (2016).

are members of groups that are stereotypically less good at tasks of the type tested. For example, the reminder might take the form of requiring subjects to write their genders on the front of their answer-books. There is a measurable effect in the case of mathematics, though this varies with age and with the degree of commitment to mathematics. One might think that the solution is to avoid such reminders, both in the testing procedure (use of candidate numbers rather than names, for example) and in the test content. However, experiments on stereotype threat do not record all the effects of sexist culture. They merely record the effect of a reminder of group membership on the day of the test. Girls going into a mathematics exam already know that they are girls, even before they write ‘F’ in a box on the front of their answer-book or fill in their gendered names. The experiments on stereotype threat do not capture the effects of this prior knowledge on their learning. Nor do they measure the effects of gendered stereotypes on girls’ choices—we do not know how many girls choose not to study mathematics as a result of their being persuaded that it is not for them. What the stereotype threat experiments do suggest is that these effects may be significant.

In general, sexism, racism, class-prejudice, etc. are cultural (that is, value-laden) structures that may find expression in all sorts of practices—this is the material-ideal dyad in action. If we think about practices in isolation without considering that they are expressive of and ramifying of clusters of values, we won’t get them right or act effectively. Consider how ineffectual appeals to reason are in discussions of gun ownership in the US or fox hunting in the UK—precisely because such appeals fail to recognise the role of these issues as expressive of values relating to identity and belonging. This is not to deny that changing practices can change culture; recall that we claimed that the material-ideal dyad must be part of any viable conception of culture. Changing rules, policies and practices can make a difference, but the case made here is that real change requires us to understand such changes at the material end of the dyad as exemplary and expressive of a shift at the ideal end, in the corresponding system of values.

To conclude: the aims of the critical mathematics education movement require attention to culture, but this requires a conception of culture that avoids the shortcomings of ethnomathematics, and is especially apt to avoid the risks of stereotyping and essentialism. While speaking of ‘culture’, we must recall that pupils belong to multiple cultures: school culture, peer culture, home culture, and so on (this may be what Presmeg’s first principle was trying to capture). These all interact in unpredictable ways, and pupils all have their own ways of relating to the cultural groups that they find themselves in. Recall our observation that ‘culture’ sometimes functions as a mass-noun rather than as a count-noun. The cultural life of a pupil may be more like the turbulent flow of liquids than a stable relation to a static object. Since culture is an unavoidable topic given the aims of the critical mathematics education movement, we need a notion of culture that captures this dynamism. We believe that the two-dimensional framework offered here (the two dimensions being material/ideal and normative/descriptive) may be a useful contribution to this endeavour.

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The Ethics of Mathematics: Is Mathematics Harmful?

Paul Ernest

Abstract In this chapter I challenge the idea that mathematics is an unqualified force for good. Instead I show the harm that learning mathematics can inadvertently cause unless it is taught and applied carefully. I acknowledge that mathematics is a widespread force for good but make the novel case that there is significant collateral damage caused by learning mathematics. I describe three ways in which mathematics causes collateral damage. First, the nature of pure of mathematics itself leads to styles of thinking that can be damaging when applied beyond mathematics to social and human issues. Second the applications of mathematics in society can be deleterious to our humanity unless very carefully monitored and checked. Third, the personal impact of learning mathematics on learners' thinking and life chances can be negative for a minority of less successful students, as well as potentially harmful for successful students. I end with a recommendation for the inclusion of the philosophy and ethics of mathematics alongside its teaching all stages from school to university, to attempt to reduce or obviate the harm caused; the collateral damage of learning mathematics.

Keywords Critical mathematics education · Ethics · Collateral damage
Harm · Instrumentalism · Philosophy of mathematics

Introduction

Mathematics is a rich and powerful subject, with broad and varied footprints across education, science, culture and indeed throughout all of human history. Both the academic world and society in the large accord mathematics a high status both as an art and as the queen of the sciences (Bell, 1952). Mathematics has a uniquely privileged status in education as the only subject that is taught universally and to all ages in schools. Hidden behind this elevated status is the assumption that mathematics is an unqualified force for good. Is this ethical assumption correct?

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Does nothing but good flow from mathematics? In this chapter I argue that mathematics does harm as well as good. My claim is that mathematics in school has unintended outcomes in leaving some students feeling inhibited, belittled or rejected by mathematics. In sorting and labelling learners and citizens in modern society, mathematics reduces the life chances of those labelled as failures or rejects. In addition, even for those successful in mathematics, the discipline serves as a training that shapes thinking in an ethics-free and amoral way. Thus mathematics supports instrumentalism and ethics-free governance. This is exploited in warfare, heartless corporate activity, the misuse of humans and the environment, and in many acts that treat persons as objects rather than moral beings, who are entitled to respect and dignity. I conclude by suggesting solutions. To avoid or remedy the negative effects in schooling we need to attend more closely to the causes of success and failure, and become fully aware of how these have far-reaching impacts on learners. Further, in order to forestall the harmful effects of mathematics in society we need to teach the social responsibility of mathematics through including philosophy and especially the ethics of mathematics alongside mathematics itself. All students of mathematics and fully fledged mathematicians should be able to view the uses and applications of mathematics critically, seeing the mathematics in play and understanding the ethical implications of the issues involved.

As a mathematician myself, someone who has devoted his professional life to furthering the teaching of mathematics in school and university, I might be expected to be among the last to question the value of mathematics. However, I believe that both the place of mathematics in the world and the benefits it brings are strengthened through looking at mathematics as a critical friend. This entails not only lauding and feeling pride at the benefits mathematics brings but also recognising the harm it can do, and not shying away from it. Acknowledging that there are negative outcomes opens the doors to solutions, to possible means of ameliorating and rectifying the possible damage brought about through mathematics.

Is Mathematics an Untrammelled Good?

In this chapter I wish to challenge the myth that mathematics is an untrammelled good, and that promoting and learning mathematics leads solely to beneficial outcomes and never causes harm. The received wisdom dominating the institutions of mathematics, mathematics education and society in general, is that mathematics of itself is a wonderful boon for all of humankind, and in areas where its positive benefits are not remarked it is simply neutral (Gowers, n.d.). Even stronger, Burnyeat (2000) argues that studying mathematics is good for the soul, basing his claims on the arguments of Plato. By contrast, a web searches linking mathematics to harm or damage reveals nothing that challenges the claim that mathematics is an untrammelled good.[1]

In place of the generally uncritical plaudits that mathematics receives I wish to ask what are or might be the actual outcomes and potential costs of elevating and privileging mathematics in education and society, including any unintended outcomes? Looking at such outcomes, does mathematics cause any harm or evil? To mathematicians and many others even asking this question, let alone answering it in the affirmative, might seem unthinkable, a ridiculous questioning of what has hitherto been unquestionable. To educationists it is not so difficult to ask this question, or even to answer it in the affirmative, when the impact on disadvantaged students and society is considered (Stanic, 1989).

Before I address the potential harm that mathematics may do, let me begin by affirming that mathematics has great value. The overall value of mathematics comprises the benefits and goods it offers to humanity as a whole. There are two types of value that mathematics possesses. First, there is the intrinsic value that mathematics has as a discipline or area of knowledge, the value of mathematics purely for its own sake. My claim is that mathematics is one of the great intellectual products of human culture. Thus teaching mathematics is enabling learners to encounter and engage with this great cultural product. Second, there is extrinsic value, the general social value of mathematics on the basis of its applications and uses in society. It is the language of all scientific and technological achievements. It is a universal language that allows us to understand and share our understanding of all physical forces from the sub atomic to the cosmic. Teaching about this aspect of mathematics opens up the world of mathematical applications to learners allowing them to appreciate its immense practical power as well as to participate in using such applications themselves. In addition to the social benefits of its applications mathematics also has personal value. This is the value of mathematics for learners and for other persons more widely as it plays out in terms of individual benefit. Such benefits will vary across individuals according to personal circumstances, experiences, social contexts and so on. For many students the learning of mathematics results in great personal power, manifested in increased social, professional and study opportunities, as well as enhanced feelings of mathematical self-efficacy and overall self-worth.

The Intrinsic Value of Mathematics

Mathematics has intrinsic value, and as I argue elsewhere the furthering of mathematics for its own sake is an ethical good for humankind (Ernest, 2016b). Mathematics expands the human intellect, broadening our conceptual horizons and

[1]The one exception that I have found lies in feminist critiques of mathematics as oppressive and patriarchal, see, e.g., Burton (1995) and Shelley (1995).

opening up vast areas of pure thought. Mathematics is a powerful exploration of pure thought, truth and ideas for their intrinsic beauty, intellectual power and interest. In its development mathematics creates and describes wondrous worlds of beauty, populated by linked crystalline forms that stretch off to infinity in richly etched exquisiteness, like the vision of the net of Indra. In addition to their intrinsic value, these forms make up the language of structure that frames virtually all possible abstract conceptual relationships, including those of the sciences and computing. Part of the intrinsic value of pure mathematics is its widely appreciated beauty (Ernest, 2016a). "Like painting and poetry mathematics has permanent aesthetic value" (Hardy, 1941, p. 14). "Mathematics possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture" (Russell, 1919, p. 60).

These virtues and values are not only appreciated by those initiated into the most exclusive inner sanctum of mathematics, the area occupied by the ground-breaking creative mathematicians. They also elicit wonder from the public. We are often confronted with complex and fascinating mathematics-based images in the media, for example multi-coloured pictures of fractals, complex tessellations and other beautiful representations. These contribute to the public perception that mathematics can be both beautiful and intriguing, and has an intrinsic value.

The Extrinsic and Social Value of Mathematics

First, with regards to science, mathematics is known as both the queen and servant of science (Bell, 1952). As its servant mathematics provides the language through which modern science is formulated. Models, laws, theories and predictions, going back over 2000 years ago to the Ptolemaic model of the universe, could not be expressed without mathematics. Since the industrial revolution, scientific applications based in mathematics have underpinned engineering, technology and the whole material basis of modern life.

Second, computing and the information and communication technologies that form the language and basis for all our modern media, knowledge systems and control mechanisms, rest solely on mathematics and logic. Both the knowledge representations and the programmed instructions upon which information and communication technology depends can only be expressed by means of the coding and logic supplied by mathematics.

Third, and far from least, finance, economics, trade, business, and through them, social organisation, rest on a mathematical foundation. Money, the intangible embodiment of economics, is the lifeblood that circulates throughout these bodies and activities. The commercial basis of modern society simply would not be possible without money, and thus, without arithmetic. Money is number that utilises one possible type of unitisation, a quantification of exchange value. This is not surprising given the evidence that tax, tribute and trade and the associated needs for systematic recording is what gave birth to written mathematics five thousand years ago (Høyrup, 1994).

Each of these domains of application has undoubtedly many great benefits in terms of human flourishing, including improvements in health, nutrition, housing, transport, agriculture, manufacturing, education, leisure, communications and wealth. More human beings than ever live longer, healthier, better educated, more comfortably and wealthier as a consequence of the mathematics-led developments in the sciences, technology and engineering, especially over the past two centuries.

In addition to these social benefits shared by so many, mathematics has great personal value. Learners and persons in general benefit from mathematics as:

1. an enlarging element of human culture,
2. a means of personal development and growth,
3. a valuable tool for use socially, both as workers, and citizens in society
4. a means of gaining certification for entry to employment or further education.

We live in a mathematized social world. The immense utility of mathematics must be acknowledged as a great strength and virtue. Without it not only would we have to forego many of the tools we as individuals and society rely on, but many of the necessities of life we enjoy and much of our prosperity would disappear. Mathematics is arguably the most generally applicable of all human knowledge fields and many if not most of the good qualities of modern living depend on it.

Features and Characteristics of Mathematics

An immediate question is what are the components of mathematics that contribute to its great intrinsic and extrinsic value? The most obvious dimensions are that of number and calculation. Calculation is central to mathematics, and it dominates both history and schooling. Mathematics as a scientific discipline is claimed to originate around 3000 years BCE (Høyrup, 1980). Thus it was already halfway through its history, around 500 years BCE, before proof entered into mathematics. Prior to that number recording and calculation, plus some geometric measurement, constituted pretty much the totality of mathematics. Even since then, numbers and calculation have dominated both the practical uses of mathematics and its educational content, with Euclidean geometry playing a minor role, and that just in elite education.

At the heart of calculation are rule-based general procedures. In these, the overall meaning of numerals, especially the place-value meaning signified through the relative positioning of the constituent digits, is largely ignored during most of the algorithmic processes. Further, largely as a result of Islamic contributions, algebra emerged in the Middle Ages. This provides the abstract language of mathematics upon which all modern developments depend. Algebra is generalized arithmetic in origin and as such is subject to generalized arithmetical procedures and rules, and its strength is that specific numerical meanings are detached. This was explicitly noted over 300 years ago by Bishop Berkeley.

> … in Algebra, in which, though a particular quantity be marked by each letter, yet to proceed right it is not requisite that in every step each letter suggest to your thoughts that particular quantity it was appointed to stand for. (Berkeley, 1710, p. 59)

At its heart, algebra is variable based, thus forcing the linguistic move in the language away from specific values and meanings to general rules and procedures concerning variables. This move has some great benefits. It enables the miracle of electronic computing in which mathematical rules and procedures are wholly automated and no reference to or comprehension of the meaning of mathematical expressions is required.

Overall, during the application of algorithms and other permitted procedures in arithmetic and algebra the meaning of expressions can largely be neglected with no detriment to the efficacy of the procedures. Meaning is dispensable.

A further characteristic of school, university and research mathematics is that they are represented in the symbolism and language of mathematics, and this is fundamentally in sentences. Mathematical sentences, although often containing symbols, conform to the usual subject-verb form, or more generally, to the terms-relation form, where a relation is equivalent to a generalised verb. In a detailed analysis Rotman (1993) found that although there is some limited use of the indicative mood, the predominant verb form in mathematical language is the imperative mood. Imperatives are orders that instruct or direct actions either inclusively, such as: let us …, consider …, or exclusively, such as: add, count, solve, prove, etc. Imperatives occur more frequently in mathematics than in any other academic school subject (Ernest, 1998; Rotman, 1993). In addition, mathematical operations require rigid rule following. At its most creative mathematics allows choices among multiple strategies and representations, but each of the lines of choice pursued involves strict rule following. Consequently mathematics is very unforgiving. There is no redundancy in its language and any error in rule following derails the procedures and processes. Thus students of mathematics must learn to use its language and follow its rules with great precision. The net result of extended exposure to and practice in mathematics is a social training in obedience, an apprenticeship in strict subservience to the text, be it printed or spoken. Mathematics is not the only subject that plays this role but it is by far the most important in view of its imperative rich and rule-governed character. Furthermore, the rule following is done without any need for attention to the meaning of the signs being worked on and transformed.

One of the most important ways that a social training in obedience is achieved is through the universal teaching and learning of mathematics from a very early age and throughout the school years. The central and universal role of arithmetic in schooling provides the symbolic tools for quantified thought, including not only the ability to conceptualize situations quantitatively, but a compulsion to do so. This compulsion first comes from without, but is appropriated, internalized and elaborated as part of the postmodern citizen's identity. We cannot stop calculating and assigning quantified values to everything, in a society in which what matters is what *counts* or is *counted*.

The teaching and learning of mathematics in schools, and thus the development of mathematical identity requires that, from the age of five or soon after, depending on the country, children will (Ernest, 2015):

1. Acquire an object-oriented language of objects and processes,
2. Learn to conduct operations on and with them without any intrinsic reasons or sense of value, thus operating with deferred meaning,
3. Decontextualise their world of experience and replace it by a deliberately unrealistic and very stylized model composed of simplified static objects and reversible processes,
4. Suppress subjectivity, experiential being and feelings in their mathematical operations on objects, processes and models,
5. Learn to prioritize and value the outcomes of such modelling above any personal or connected values and feelings, and apply these outcomes irrespective of such subjective dimensions to domains including the human "for [your] their own good" (Miller, 1983).

King (1982) researched the mathematics taught and learnt in 5–6 year old infant classrooms. He found that mathematics involves and legitimates the suspension of conventional reality more than any other school subject. People are coloured in with red and blue faces. "A class exercise on measuring height became a histogram. Marbles, acorns, shells, fingers and other counters become figures on a page, objects become numbers" (King, 1982, p. 244). Further, in the world of school mathematics even the meanings of the simplified representations of reality that emerge are dispensable.

> Most teachers were aware that some children could not read the instructions properly, but suggested they "know how to do it (the mathematics) without it." … Only in mathematics could words be left meaningless. (King, 1982, p. 244)

In the psychology of mathematics education instrumental understanding, defined as knowing how to carry out procedures without understanding, versus relational understanding, which includes in addition knowing how and why such procedures work, is much discussed as a problem issue (Mellin-Olsen, 1987; Skemp, 1976). It is no coincidence that what is termed instrumental understanding is also a form of the instrumental reasoning critiqued by the Frankfurt School, and which is discussed in the sequel.

In summary, many procedures on signs are carried out with abstracted or deferred meanings, and many mathematical texts, be they calculations, derivations or proofs, involve the reader following rule-governed sequences or orders. In education. mathematics is the subject most divorced from everyday or experienced meaning, and the objectification and dehumanisation of the subject are a necessary part of its acquisition.

However, I need to qualify these claims. Although mathematical signs and procedures are detached from meaningful referents in the world, engagement with mathematics can create an inner world of meanings. Successful mathematicians work within richly populated conceptual universes that are very meaningful to themselves. Success at mathematics at most levels is often associated with having a meaningful domain of interpretation of mathematical signs and symbols, often within the closed world of mathematics. In addition, applied mathematicians interpret mathematical models in the world around us so in applications meanings are reattached. Likewise, although mathematical language is very rich in imperatives, successful users of mathematics at all levels have certain degrees of freedom available to them, such as which methods and procedures to apply in solving problems, as is acknowledged above.

These qualifications notwithstanding, the study of mathematics instils both the capacity to, and the expectation of, meaning detachment during reasoning and calculative procedures. Likewise, it prepares its readers to follow the imperatives in the text during the technical and instrumental reasoning involved in mathematics.[2]

Mathematical Thinking as Detached Instrumental and Calculative Reasoning

My claim is that the linguistic characteristics and moves indicated above have costs, including unanticipated negative outcomes when extended and applied beyond mathematics. For as I have argued, the mathematical way of thinking promotes a mode of reasoning in which there is a detachment of meaning. Reasoning without meanings provides a training in ethics-free thought. Values neutrality and ethical irrelevance is presupposed because meanings, contexts and their associated purposes and values are stripped away and discounted as irrelevant to the task in hand. Furthermore, as I have argued elsewhere, there is a widespread perception of mathematics as timeless, universal and imbued with absolute certainty, and hence it is viewed as an objective, value-neutral and ethics-free domain of thought (Ernest, 1998, 2016a, 2016b). Such reasoning and perspectives contribute to a dehumanized outlook. For without meanings, values or ethical considerations reasoning can become mechanical and technical and 'thing' or object-orientated. These modes of thinking foster what have been termed separated values.

Gilligan (1982) proposes a theory of separated and connected values that can usefully be applied to mathematical and other types of reasoning. Her theory distinguishes separated from connected values positions and places them in opposition. The separated position valorises rules, abstraction, objectification, impersonality, unfeelingness, dispassionate reason and analysis, and tends to be atomistic and thing-centred in focus. The connected position is based on and valorises relationships, connections, empathy, caring, feelings and intuition, and tends

[2]In addition, in more advanced study of mathematics in high school or university, students learn to reason and draw inferences from assumptions and postulates that are not necessarily true. Such hypothetical reasoning adds yet another level of detachment from the world we live in, weakening the bonds to reality, values and ethics.

to be holistic and human-centred in its concerns. These two value positions can be seen as oppositions, with separated values (first) contrasted with connected values (second, respectively), providing the following oppositional pairs: rules versus relationships, abstraction versus personal connections, objectification versus empathy, impersonal versus human, unfeeling versus caring, atomistic versus holistic, dispassionate reason versus feelings, analysis versus intuition.

The separated values position applies well to mathematics. Mathematical objects are entities resulting from objectification and abstraction and are naturally impersonal and unfeeling. Mathematical structures are constituted by abstract and rule-based sets of objects and their structural relationships. The processes of mathematics are atomistic and object-centred, based on dispassionate analysis and reason in which personal feelings play no direct contributing part. Thus separated values fit mathematics very well and indeed can be said to be an essential part of mathematics. Mathematics both embodies and transmits these values.

Separated values and the associated outlooks are necessary, indeed essential, by the very nature of mathematics, and their acquisition constitute assets and are undoubtedly beneficial for thinking in mathematics. A separated scientific outlook is also useful in reasoning in other inanimate domains, such as in physics and chemistry, where atomistic analysis, strictly causal relationships and structural regularities yield high levels of knowledge. However, thinking exclusively in the separated mode can lead to problems and abuses when applied outside mathematics and the physical sciences to society. In the human sphere exclusively separated values are unnecessary and potentially harmful, since they factor out the human and ethical dimensions. In seeing the world mathematically, the richness of nature and human worlds, with all their beauty, contextual complexity and linkages, and ethical responsibilities, are replaced by simplified, abstracted and objectified structural models. The outcome parallels Wilde's (1907, p. 116) dictum about the outlook "that knows the price of everything and the value of nothing". Although mathematical perspectives and models are powerful and useful tools for actions in the world, including the improvement of human life conditions, when overextended they risk becoming a threat to our very humanity. Inculcating these values can lead to a dehumanized outlook if applied to social and human worlds. Furthermore, separated values extended too far beyond mathematics can also lead to the view that mathematics and its applications have no ethical or social responsibility. While there are legitimate philosophical arguments that pure mathematics is ethically neutral, although I argue the opposite (Ernest, 2016b), it is near universally agreed that mathematical applications bear full social responsibility for their impacts on the world, just as do the applications of science and technology.

My claim is that subjection to mathematics in schooling from halfway through one's first decade, to near the end of one's second decade, and beyond if one so chooses, structures and transforms our modes of thought in ways that may not be wholly beneficial. I do not claim that mathematics itself is harmful. But the manner in which the mathematical way of seeing is integrated into schooling, society and above all into the interpersonal and power relations in society results in the transformation of the human outlook. This is a contingency, an historical construction. It results from the way that mathematics has been recruited into systems thinking instead of empathising (Baron-Cohen, 2003) and separated values instead of connected values (Gilligan, 1982) that dominate western bureaucratic thinking. It also results from the way mathematics serves a culture of objectification, termed a culture of *having* rather than *being* by the critical theorist Fromm (1978).

One framework that acknowledges these aspects of the application of mathematics is the critique of instrumental reason and rationality provided by the critical theory of the Frankfurt School. Instrumental reason is the objective form of action or thought which treats its objects simply as a means and not as an end in itself. It focuses on the most efficient or most cost-effective means to achieve a specific end, without reflecting on the value of that end (Blunden, n.d.). Instrumental reason has been subjected to critique by a range of philosophers from Weber to Habermas (Schecter, 2010). This includes Heidegger, who argues that instrumental reason and what he terms *calculative thinking* lead us into enclosed systems of thought with no room for considering the ends, values and indeed ethical dimensions of our actions (Haynes, 2008). As Heidegger puts it, even "the world now appears as an object open to the attacks of calculative thought" (Dreyfus, 2004, p. 54). The central argument that means must never trump or eclipse ends, when human beings are the ends, can be found in Kant's *Grounding for the Metaphysics of Morals* of 1785. There he derives his Categorical Imperative from first principles, with the following as one of his conclusions. "Act in such a way that you treat humanity, whether in your own person or in the person of any other, never merely as a means to an end, but always at the same time as an end." (Kant, 1993, p. 36)[3]

A broader-based critique comes from the Critical Theorists of the Frankfurt School (including Adorno, Fromm, Habermas, Horkheimer and Marcuse) who see instrumental reason as the dominant form of thought within modern society (Bohman, 2005; Corradetti, n.d.). By focussing on technical means and not on the ends of their actions, persons, governments and corporations risk complicity in the treatment of human beings as objects to be manipulated, in actions that threaten social well-being, the environment and nature. This outlook underpins the behaviours of some governments and multinational corporations in reducing costs and chasing profits without regards for the human costs. Such actions by corporations have been termed psychopathic (Bakan, 2004). We are now so used to the economic, instrumental model of life and human governance that most persons see it as

[3]This chapter is intended as a contribution to the philosophy of mathematics education and does not summarise the range of ethical theories positions available beyond those of Kant (and the Critical Theorists). These philosophers provide a powerful basis for my argument, but others can also be cited. For example, Emmanuel Levinas is another philosopher who argues that we must treat other humans with infinite respect (Ernest, 2012). According to Levinas another person, the 'Other', is infinitely complex and not fully knowable or reducible to an object that can be known (Levinas, 1978).

an unquestionable practical reality, a necessary evil, and are not shocked or outraged by corporations or governments treating persons as objects with no concern for their well-being.

Much of the Frankfurt School critique was prompted by the rise of Nazism in Germany, with its authoritarian leaders (Adorno, Frenkel-Brunswik, Levinson, & Sanford, 1950) and the heartless complicity of ordinary citizens in Germany and occupied territories before and during World War 2. The capture, transportation, enslavement and murder of millions of fellow citizens was not simply undertaken by monsters. These wholesale activities would not have been possible without many ordinary citizens unquestioningly doing their everyday jobs as part of this monstrous programme. Arendt (1963) terms this ordinariness, from the actions of Eichmann downward, the 'banality of evil'. The fact that many ordinary citizens were highly educated did not prevent them from complicity in mass murder. As Dr. Haim Ginott, a school principal who survived a Nazi concentration camp, wrote in his advice to his teachers:

> I am a survivor of a concentration camp. My eyes saw what no man should witness: gas chambers built by learned engineers, children poisoned by educated physicians, infants killed by trained nurses, women and babies shot and burned by high school and college graduates. So I am suspicious of education. My request is: help your students to become human. Your efforts must never produce learned monsters, skilled psychopaths, educated Eichmanns. Reading, writing and arithmetic are important only if they serve to make our children more humane. (Ginott, 1972, p. 317)

My argument is that mathematics plays a central role in normalizing instrumental and calculative ways of seeing and thinking. From the very start of their education children are schooled in these ways of seeing and being. As I have argued, the detachment of meaning and the following of imperatives in mathematical texts provides the central platform for instrumental thought.[4]

There is a further factor too. Among philosophers, mathematicians, as well as in school and more generally, in society, mathematics has the image of objectivity, of unquestionable certainty, with claims being settled decisively as either true or false as well as being ethically neutral (Ernest, 1998; Hersh, 1997). Thus a training in mathematics is also a training in accepting that complex problems can be solved unambiguously with clear-cut right or wrong answers, with solution methods that lead to unique correct solutions. Within the domain of pure mathematical reasoning, problems, methods and solutions may be value-free and ethically neutral. But carrying these beliefs beyond mathematics to the more complex and ambiguous problems of the human world leads to a false sense of certainty, and encourages an instrumental and technical approach to daily problems. This is damaging, for when decision making is driven purely by a separated, instrumental rationality, then ethics, caring and human values are neglected, if not left out of the picture

[4]Of course the right social circumstances are needed too. A society with values of strong social-conformity and a culture of obedience to authority is needed, as Milgram (1974) showed in his experiments. However, as I have argued, subjection to thousands of hours of school mathematics and schooling in general will contribute to this.

altogether. Kelman (1973) observes that ethical considerations are eroded when three conditions are present: namely, standardization, routinization, and dehumanization. Since mathematics is the essence of instrumental reason, with its focus on means to ends and not on underlying values, and its procedures require standardization, routinization, and dehumanization, the concomitant erasure of ethics is no surprise. Thus a training in mathematical thinking, when misapplied beyond its own area of validity to the social domain, is potentially damaging and harmful.

Qualifying the Critique of Instrumental Thinking

However, I need to qualify the above critique of instrumental thinking and the role mathematics plays in it, so as not to be one-sided in my evaluation. It would be naive not to recognise that not only are reason and rationality essential for the fair running of complex modern societies, but also that depersonalized and objectivised thinking is necessary for all modern management. Complex modern societies and institutions cannot be run ethically or fairly, let alone effectively, without abstracted, depersonalized and objectivised thinking. Central to modern governance is the accumulation, allocation and distribution of resources. Whatever the political and ideological orientation of a government it needs to calculate where resources will most benefit society according to their values. It is the essence of democracy that the priorities for the distribution of resources varies with different elected governments. Whatever are the priorities and values of legitimate governments, and the social goals which are aspired to, resources need to be allocated to fulfil these goals effectively. We would not be able to pursue the practical meaning of our ethics and principles without working out their rational implications. In addition, systematic and rational record keeping is another necessary for fairness and equity. Thus calculative reasoning and instrumental thinking in the service of societal values and goals is a modern necessity. However, this is not an alibi for the blind following of orders from ‘above’. In a good society there must always be a place for ethical objections and whistle-blowing, where individual conscience can be exercised, when values or laws are transgressed, or unfairness or injustices are perceived.

Above I emphasized the values of connectedness and caring in contrast with separatedness (Gilligan, 1982) and rationality, because of their absence from mathematics. However, when it come to the fair running of society impersonal reason and rationality are essential. Only giving benefits to those one cares about or empathises with leads to inequality, favouritism and nepotism. As Pinker (2012) argues, against the dictum of the 1960s, love is not all you need, if you seek to be fair or just. In a just society you must treat strangers and other citizens as having equal rights and deserving equal treatment irrespective of any personal feelings towards them. This is also the basis of all legal systems. The law is based on a set of principles or precedents from which applications are deduced or reasoned in its practical applications to cases. Fairness and impartiality of reasoning underpin the dispensation of justice in the courts. Thus depersonalized objectivised thinking is necessary for all modern management of resources, human or material, within an overall framework based on ethical principles. It also underpins the law. Thus the rational and impersonal reasoning inculcated through mathematics makes a positive contribution to a just and fair society.[5]

The Social Impacts of Mathematics and Its Application

One of the key areas where instrumental modes of thinking are widespread lies within the applications of mathematics. I have described some of the broad range of applications of mathematics in society and their widespread benefits. Alongside these beneficial outcomes it is also possible to use mathematics in ways that are hurtful or harmful. My argument is that applied mathematicians should endeavour to be aware of the uses to which their applications are put, and if these are potentially hurtful or harmful should at least consider the consequences and their own involvement as facilitators. Applied mathematicians should assume some responsibility for the applications and technological innovations they help to create. It has been suggested that there should be a Hippocratic Oath for mathematicians (Davis, 1988). Given the widespread views of the neutrality of mathematics, even of applied mathematics, this would seem to be an unlikely development. Although there is a British Society for the Social Responsibility of Science, and even a group called Radical Statistics concerned with social responsibility in statistics, there is no society for the social responsibility of mathematics (pure or applied). Indeed the very idea of the social responsibility of pure mathematics will seem to many a contradiction in terms.

There is an outstanding use of mathematics that is not often counted among its applications. This is the role of mathematics as basis of money and finance. Money and thus mathematics is the tool for the distribution of wealth. It can therefore be argued that as the key underpinning conceptual tool mathematics is implicated in the global disparities in wealth and life chances manifested in the human world. It is not an exaggeration to claim that many current forms of capitalism distort equality in and across global societies to the detriment of social justice, as well as promoting consumerism in ways not wholly beneficial. Of course this is a hot political issue. My argument is not that we should oppose the western capitalist system like the Anti-Globalization and Occupy movements (Wikipedia, n.d. a, b). In the successful mixed economies of the West well regulated capitalism is the vital source of wealth and meaningful employment, and provides work, goods and the services we rely on for good living. Instead, my proposal is that we should foster an ethical and in

[5]Note that the terms justification and justice have the same roots. From the 14th century CE on justification has meant the action of justifying and the administration of justice, and justice is the quality of being fair and just—the exercise of authority in vindication of what is right (Harper, n. d.). Justification draws on rationality and impersonal reasoning, which therefore cannot be decoupled from values and fairness.

particular a critical, social justice oriented attitude towards mathematical applications alongside mathematical skills, so that students and citizens in our democracies can make up their own minds. There is a substantial literature on critical mathematics education that promotes this goal (Ernest, 1991; Ernest, Sriraman, & Ernest, 2016; Powell & Frankenstein, 1997; Skovsmose, 1994), Furthermore, the idea that our actions should be ethical and, in particular, promote social justice is now mainstream thinking, at least in Europe, for example the European Union Treaty stipulates that it shall promote social justice (European Union, n.d.).

The Social Impact of the Image of Mathematics

An indirect way through which mathematics impacts on society and individuals is through its images, which for the purposes of discussion can be divided into social and personal images. Social images of mathematics include public images, including representations in the mass media, such as film, cartoon, pictorial, and computer representations of mathematics and mathematicians. They also include school images which incorporate classroom posters, equipment, textbook, teacher presentations, and school mathematical activities as experienced by the learners. Parent, peer or others' narratives about mathematics also contribute to its social image. Personal images of mathematics include mental pictures, visual, verbal or other mental representations, and can be assumed to originate from past experiences and encounters with mathematics, as well as from social talk and other public representations. Personal images of mathematics comprise both cognitive and affective dimensions and effects. The types of mathematics as portrayed in its images can include research mathematics and mathematicians, school mathematics, and mathematical applications, both everyday and more complex. Social and personal images of mathematics are intimately related, as personal images must be assumed to result from the lived experiences of learning and using mathematics and from exposure to social images of mathematics. Likewise, social images of mathematics are constructed by individuals or groups drawing on their own personal images, which are then represented and made public. Both kinds of image can have implicit elements of which individuals are not explicitly aware. Thus, what is termed the hidden curriculum comprises those accidental or unplanned elements of knowledge representations and learning experiences within the school curriculum, which can include images of mathematics (Ernest, 2008).

One widespread public image of mathematics in Western countries, which may extend more widely, is of mathematics characterised as a difficult subject, viewed as cold, abstract, theoretical, ultra-rational, mainly masculine but nevertheless important (Buerk, 1982; Buxton, 1981; Ernest, 1995; Picker & Berry, 2000). Mathematics also has the image of being remote and inaccessible to all but a few super-intelligent beings with 'mathematical minds'. For many people the image of mathematics is also associated with anxiety and failure. For example, when Brigid Sewell was gathering data on adult numeracy for the Cockcroft Inquiry (1982) she asked a sample of adults on the street if they would answer some questions on the subject. Half of them refused to answer as soon as they understood the subject was mathematics, suggesting negative attitudes, or even mathephobia (Maxwell, 1989). While attitudes to and images of mathematics may have improved in the past few decades, following the increase of student-centred mathematics teaching approaches, a recent review of the literature reports the persistence of negative images and attitudes toward mathematics (Belbase, 2010).

Some of the problems associated with widespread social and personal images of mathematics follow from the perceptions that it is a masculine subject, much more accessible to males than females; and that it is a difficult subject only accessible to a small and gifted minority. The effect of these images, coupled with the negative learning experiences reported by some students, is to foster negative personal images of and attitudes to mathematics often incorporating poor confidence, lack of mathematical self-efficacy beliefs, and dislike of and even anxiety with respect to mathematics. One of the contributors to the negative images of mathematics can be the absolutist image of mathematics as objective, superhuman and value-free (Ernest, 1995, 1998). For many this contributes to a sense of alienation and exclusion from mathematics (Buerk, 1982; Buxton, 1981).

However, it needs to be mentioned that in contrast to these problems, for a different and more successful minority this absolutist image is part of the attraction of mathematics. Mathematics can be seen as unchanging, perfect, and a safe haven from the chaos and uncertainties of everyday life, and for this reason and others making it attractive to this successful minority. Thus no simple generalization can express the complex and varied effects of the public images of mathematics. The same dimensions or perceptions of mathematics may simultaneously attract and repel different groups of students.

One of the persistent myths of the twentieth century has been that females are 'naturally' less well equipped mathematically than males (Burton, 1990; Rogers & Kaiser, 1995). So two of the detrimental effects of images of mathematics that I shall foreground here are first the negative impact on female students following on from the masculine image of mathematics. Second, the negative impact of mathematics related experiences and images on the attitudes and self-esteem of a minority, including many girls and women. The problem with these negative impacts is that mathematics is a highly esteemed and valued subject in schools and universities, perhaps even overvalued. Mathematics examinations are used as a sifting and filtration device in society. Life chances and social rewards are disproportionately correlated with success at mathematics, even within many areas of study and work in which mathematics plays little part. Sells (1973, 1978) has termed mathematics the 'critical filter' in determining life-chances.

In addition, success in school mathematics is strongly correlated with the socio-economic status or social class background of students. Although this is true with virtually all academic school subjects, mathematics has a privileged status. It is the examinations in mathematics in particular that serve as a fractional distillation device that, to a significant extent, is class reproductive. Talented mathematicians from any background may be successful in life, but the net effect of mathematical examinations is the grading of students into a hierarchy with respect to life chances.

This hierarchy doubly correlates with socio-economic status and social class, understood in terms of both the social origins and the social destinations of students. So it is not merely raw mathematical talent that is reflected in mathematical achievement. It is also partially mediated by cultural capital (Bourdieu, 1986; Zevenbergen, 1998).

While mathematical knowledge has important uses and applications in modern societies, the status and value of mathematical achievement is elevated beyond its actual utility. Mathematics is increasingly hidden from citizens in modern society behind complex systems including information and communication technology applications, and the vast computerised control and surveillance systems. These regulate and monitor modern societies for the purposes governance, security and commerce. Advanced mathematical skills are not needed by the many that operate these systems, for such persons can do so successfully without awareness of their mathematical foundations (Niss, 1994; Skovsmose, 1988). It is the much smaller number of mathematicians, programmers and information technologists that design, implement and test the systems who need advanced specialist mathematical skills.

My claim is that the social image of mathematics as experienced by learners contributes to their personal image of mathematics and that this is an important factor in their success in mathematics. Personal images of mathematics include attitudes to mathematics and these play a key role in success at mathematics via multiplying mechanisms which I call the success and failure cycles (Ernest, 2013).

The mechanisms are as follows. Some students suffer from negative attitudes to mathematics, including poor confidence and poor mathematical self-concept, and in a minority possible mathematics anxiety (Buxton, 1981). Based on Maslow's (1954) hierarchy of needs theory, it can be said that persons will do a great deal to avoid risks including the risk of failure in a socially esteemed activity, with its concomitant threat to personal self-esteem. So negative attitudes lead to reduced persistence, and even mathematics avoidance in some cases, resulting in reduced learning opportunities. A consequence of this is lack of success in mathematics, which in the strong case is failure. Students who experience an overall lack of success and repeated failure at mathematical tasks and tests develop or strengthen their negative attitudes to mathematics, completing a self-reinforcing cycle, leading to a downward spiral in all three of its components, illustrated in Fig. 1.

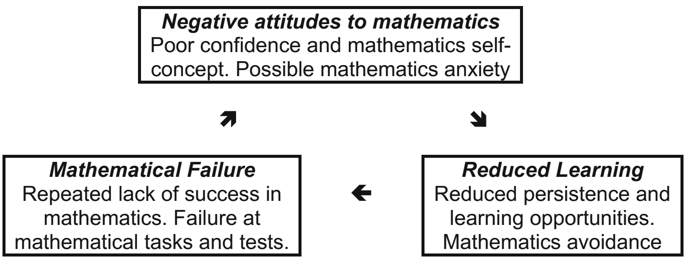

Fig. 1 The failure cycle (adapted from Ernest, 2013)

In this, as in any proper cycle, there is no identifiable beginning point. All three elements develop together, and any one of them could be nominated as a starting point. Thus, outcomes shape attitudes, and in particular failure often leads to poor attitudes. Negative attitudes impact upon behaviours, such as disengagement and low effort. Disengagement in turn reduces the chances of success. So once the cycle is started it becomes self-reinforcing and self-perpetuating, a vicious cycle.

In contrast, positive student attitudes to mathematics, including confidence, a sense of mathematical self-efficacy, pleasure in and motivation towards mathematics lead to increased effort, persistence, and the choice of more demanding tasks. This is because of the intrinsic rewards gained, such as intellectual satisfaction and pleasure gained through success. The increased efforts and engagement in turn lead to students' improved learning, as well as their experience of further success at mathematical tasks and mathematics overall. Consequently, positive student attitudes to mathematics are reinforced, completing a success cycle, in an enhancing upward spiral.

Psychologists including Howe (1990) have shown that a mechanism like that shown in Fig. 2 is an important factor in the development of exceptional abilities among gifted and talented students. Students who demonstrate some giftedness and talent at around the age of 10 are often very significantly further ahead of their peers at the age of 20 precisely because of the factors shown in the figure. Early success and the attitudes it breeds lead to much greater effort, persistence, and choice of more demanding tasks which lead to the flowering of the later manifested exceptional abilities. Howe found that the exceptionally talented invested an extra 5000 hours in practice of their skills and abilities. This is double the time spent by their capable but less outstanding peers. This finding has been popularized as the '10,000 hour rule' by Gladwell (2008). This rule proposes that 10,000 hours of practice in any activity or skill leads to expertise and mastery.

Figure 2 contains within it practical means of overcoming or ameliorating the failure cycle shown in Fig. 1. Students need to be given tasks and support in learning mathematics so that they experience success. This needs to be real success at tasks that fall within their zone of proximal development (ZPD), that is tasks that lie beyond what a learner can do without help, but which are within their grasp with guidance and support from teachers, adults or peers (Vygotsky, 1978). Relatively low level tasks that use knowledge and skills that students have already mastered

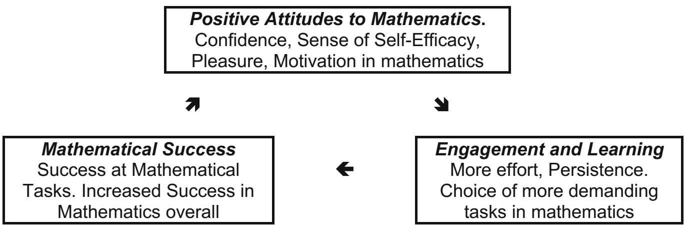

Fig. 2 The success cycle (adapted from Ernest, 2013)

may seem too easy to them, and may not contribute to a sense of success. However, when students experience repeated success at more demanding mathematical tasks, and become used to increased success in mathematics overall, one can expect improved attitudes to mathematics. Over time, students should come to ‘own’ their success and feel pleasure in it. They will therefore grow in confidence and develop their sense of mathematical self-efficacy.[6] They should gain motivation in engaging in and completing mathematical tasks. Following on from their increased engagement we would expect them to make more effort on mathematical tasks, become more persistent in solving mathematical problems, sometimes even choosing more demanding tasks. Thus they are now following a success cycle (Fig. 2). With this in mind the teacher’s role is to set tasks that fall within the learner’s ZPD, and to offer support and encouragement so they experience success and develop positive attitudes. If a student is firmly in the grip of the failure cycle (Fig. 1) it may take some time and significant personal support to initiate the success cycle. But once initiated there should be an upward spiral in terms of gains in attitudes, engagement and success.

Another impact of the social image of mathematics is in sex-differences in mathematical achievement and participation. Traditionally Western females have had lower levels of achievement in school mathematics and lower levels of participation in advanced mathematical study and in mathematical careers than males. Although school level achievements in mathematics have more or less evened out between the sexes in the 21st century, research shows that females continue to have, on average, more negative attitudes to mathematics than males, and this continues to be reflected in continuing lower levels of participation after the age of 16 years (Forgasz, Becker, Lee, & Steinthorsdottir, 2010). It is widely claimed that the social image of mathematics is a significant causal factor in these sex-differences (Mendick, 2006). Thus widespread gender stereotyped social images of mathematics include the view that mathematics is a male domain and is incompatible with femininity (Ernest, 1995). This contributes to gender stereotyped school images of mathematics which are manifested in a lack of equal opportunities, such as in classroom interactions in learning mathematics (Walkerdine, 1988, 1998). Social images, as well as these school factors lead to gender-stereotyping in females’ individual images of mathematics and impact negatively on their confidence and perceptions of their own mathematical abilities (Isaacson, 1989). The disadvantaging effects of these factors result in underachievement and lower participation rates in mathematics post-school.

However, in the past two decades, female underachievement has been balanced out by male underachievement due to a separate set of factors, such as many young men’s disengagement from school, especially in Anglophone countries such as

[6]Students with positive mathematical self-efficacy attitudes often attribute their success in school mathematics to stable and intrinsic causes such as their own skill and ability, while attributing their failures to extrinsic and unstable causes such bad luck, or a lack of effort (Weiner, 1972).

United Kingdom (Forgasz et al., 2010).[7] However, rather than meaning that equality between the sexes has been achieved, it means that there are now two different gender-related problems concerning school mathematics, and that these partially cancel out by negatively impacting differentially on both boys and girls. Furthermore, the lower female participation in higher mathematics post-school remains a significant problem.

Of course I have reported this in a primarily Anglocentric way, and many countries do not follow this pattern. For example in West Indian, Pacific Island states and some Middle Eastern countries girls have been outperforming boys in all subjects, including mathematics. In Latin American countries and Southern European countries the stereotypically male pattern of success in mathematics and science related studies and careers has fallen away. Furthermore, in many Eastern countries mathematical success is seen to be due to student effort and not due to inherited ability, including that associated with sex. However, where such problems persist, as they do in the most populous English speaking countries, images of mathematics are regarded as making a significant contribution.

Summary and Provisional Solutions

I have critiqued the idea that mathematics is an untrammelled force for good. Instead I offer the metaphor that mathematics has two faces, the good and bad faces. The good face displays the benefits and value of mathematics. I have argued that mathematics is intrinsically a force for good, a creative development of the human spirit and imagination. It is also good in its utility, for it has many benefits in its social applications and personal value that benefit human flourishing. But, more controversially, I also claim that mathematics has a bad face. It does harm through dehumanized thinking which fosters instrumentalism and ethics-free governance and social practices. Also, because of its over-valuation in the modern world through education it facilitates social reproduction and the perpetuation of class-based social injustice. Through its social image (coupled with school learning experiences) it aids the development of negative attitudes in some learners, and its gender-biased image maintains social disadvantage for females, especially in the English speaking world.

There are of course, in addition, ethically questionable and harmful applications of mathematics, as there are of any scientific and technological subject. Thus, for example, mathematics, science and technology are used in the manufacture of guns, explosives, nuclear and biological weapons, battlefield computer systems, tobacco products, and other potentially destructive artefacts and tools. But, there is a well known and legitimate argument that it is only in the choice of applications of mathematics in such activities that ethical considerations and violations emerge.

[7]This problem is particularly acute for boys from lower socio-economic status groups, who are often less engaged with most academic school subjects including the sciences (Banerjee, 2016).

My critique is independent of such deliberate applications, and perhaps even precedes them. I question whether mathematics itself, even before its wider applications beyond schooling, is solely a force for good, incapable of detriment and social harm. This view, which I might term a myth, hides the fact that mathematics through its actions on the mind is already implicated in some potentially harmful outcomes even before it is deliberately applied in social, scientific and technological applications.

However, some caveats to this argument are required. First of all, from the perspectives that I term absolutist philosophies of mathematics (Ernest, 1991), the image of mathematics that I have criticised follows as a necessary feature of mathematics emanating from its very nature. Although I and some others reject the associated absolutist epistemologies and ontologies, these remain legitimate philosophies of mathematics. Secondly, the fact that the mindset fostered by mathematical thinking can lead to harm when it is misapplied to social and other philosophical issues is a defect of human or social thinking, and not an intrinsic weakness of mathematics. Thirdly, such instrumentalist and abstracted modes of reasoning are necessary in modern governance and management, and provided the background values are humane and directed at human flourishing should do no harm. Fourth, the damage done by social images of mathematics is mediated by interpretations of mathematics, that is, socially and personally constructed images of mathematics. These images are not inescapable logical consequences of mathematics itself, for they can, are, and have been different in different societies and at different historical times. Thus the force of my critique is not directed at mathematics itself, but at the social institutions of mathematics, including training in mathematics, and the false social images of mathematics that they can legitimate and project. The harm that I am highlighting comes from what are largely unconscious misapplications of mathematics, including the modes of thought it generates, and from the image of mathematics that many find excluding and off-putting, as well as the current overvaluation of mathematical achievement in school and society.

Thus mathematics is not intrinsically bad or harmful, but as I have argued, its applications, both conscious and unconscious can be detrimental to many. This provokes the question: how can we prevent, ameliorate, or rectify this? In the space here I can only sketch a few possibilities for addressing these problems. I have already sketched how the personal damage done to some via the teaching of mathematics can be ameliorated or rectified. My further proposal is that we should include elements of the philosophy of mathematics and of the ethics of mathematics and its social responsibility in the teaching mathematics at all levels from school to university.

Teaching the Philosophy of Mathematics

My proposal is that we should include selected aspects of the philosophy of mathematics in the school mathematics curriculum and in university mathematics degree courses. Students at all levels should have some idea of proof and how mathematical knowledge is validated. This includes knowing that no finite number of examples can prove a generalisation, whereas a single counterexample can falsify it. Students need to understand the limits of mathematical knowledge, including the following: the certainties of mathematics do not apply to the world, there is always a margin of error in any measurement; no mathematical application or scientific theory can ever be proved true with certainty, and this applies to any mathematical model of the world. Likewise we need to teach the limits of mathematical thinking: the true/false dichotomies we find in mathematics do not apply to the world, where matters are almost never so clear cut. In addition, students need to be aware that there are controversies in the philosophy of mathematics over the nature of mathematics, especially the basis of mathematical knowledge and the status of mathematical objects; that there are controversies over whether mathematical knowledge is absolute, superhuman with an existence that predates humanity, and over whether the objects of mathematics exist in a superhuman Platonic space. A recent issue concerns whether humanly unsurveyable computer proofs, such as that of the 4-colour theorem, are indeed legitimate proofs. Strong disagreements rage over whether mathematics is intrinsically value- and ethics-free or value laden, and over whether it is invented or discovered. I believe that elements of the history of mathematics and mathematics in history can serve to make some of the above recommended points and to humanize mathematics. This can be reinforced by illustrating the ubiquity of mathematics in culture, art and social life. I have just picked out here some philosophical questions and issues that mathematics raises, and many more could be added.

Overall, my proposal is that students should see mathematics as more than just a set of tools, and instead be shown that it is long-standing discipline with its own philosophical issues and controversies, including human and ethical dilemmas. They should learn that mathematics is not an isolated and discrete area of knowledge, which despite having a distinct identity has rich connections with all other dimensions of human activity, practice and knowledge. The importance of grasping aspects of the complex interrelationships between mathematics and the human world is that some of the misunderstandings arising from an isolated and separated view may be obviated. By exploring some of the basic philosophical issues and presuppositions underpinning mathematics, as well the nature, validity and limitations of its knowledge, some of the ills that I have described can be reduced or avoided.

Teaching the Ethics and Social Responsibility of Mathematics

Although there is a widespread misperception, from my perspective, that mathematics is neutral and bears no social responsibility, clearly its uses and applications are value-laden. We should, in my view, add the ethics of mathematics to all university mathematics degree courses so that mathematicians gain a sense of its social responsibility. We need to teach that mathematics must be applied responsibly and with awareness, and that it is wrong to ignore or label its negative social impacts as 'incidental' outcomes or as 'collateral damage', thus allowing them to be viewed as outside of the responsibilities of mathematicians. In addition to teaching the ethics of explicit mathematical applications we also need to teach that mathematics has unintended ethical consequences. Thus, we need to teach the limits and dangers of instrumental thinking which mathematics can foster, and how it can lead to dehumanized perspectives in which people are both viewed and treated as objects.

Part of the social responsibility of mathematics is to foster public understanding. Mathematicians, including the wider professional mathematics community, have the responsibility both to promote the understanding of mathematics and to counter misconceptions and misunderstandings about the meanings and significance of its uses and applications in the public domain, especially in the media. Modern citizens need to be critically numerate, able to understand the everyday uses of mathematics in society. As citizens, they need to be able to interpret and critique the uses of mathematics in social, commercial and even political claims in advertisements, newspaper and other media presentations, published reports, and so on. Mathematical knowledge needs to be critical in the sense that citizens can understand the limits of validity of any uses of mathematics, what decisions are conveyed or concealed within mathematical applications, and to question and reject spurious or misleading claims made to look authoritative through the use of mathematics. Citizens need to be able to scrutinize financial sector and government systems and procedures for objectivity, correctness and uncover hidden assumptions. Ideally they should able to identify the ethical implications of applications of mathematics to guard against the instrumentalism and dehumanization that can be hidden behind technical decisions. My claim is that every citizen needs these capabilities to defend democracy and the values of humanistic and civilised societies, and it is part of the social responsibility of mathematics to help provide them. This responsibility begins with school teaching, where all students spend thousands of hours studying mathematics. The critical mathematics education movement has over the past quarter century provided both theoretical analyses and practical examples of what teaching critical mathematics means (D'Ambrosio, 1998; Ernest et al., 2016; Frankenstein, 1990; Skovsmose, 1994).

A purist objection to such additional teaching targets is, first of all, that they would steal valuable time and thus detract from the teaching of mathematics, and second that these not the responsibilities of mathematicians. With respect to the first objection it can be said that what I am proposing is not intended to take up even 2%

of the time devoted to mathematics teaching in schools and universities. At school, such issues can be brought up within the mathematics curriculum periodically but without taking even a whole lesson. A discussion of examples, models and applications can lead to the issues being raised ‘naturally’, provided mathematics teachers have been well prepared to do this. Furthermore, using problems concerning the environment, international trade, world development issues, for example, will motivate and engage learners in their mathematical studies as well as highlighting the social responsibility of mathematics. At university a small, time limited course on ethics and social responsibility of mathematics could easily be added as a mandatory course alongside pure, applied or service courses in mathematics. Thus, the costs in time could be very small, meeting this objection, although the positive impacts, in terms of mathematicians’ and other mathematics users’ awareness of the social responsibility of mathematics, could be significant.

With regard to the second objection on lack of social responsibility of mathematics, it is first interesting to contrast the received views about the responsibilities of mathematics and mathematicians with parallel views about the social responsibilities of science and scientists. Unlike the case in mathematics, there is widespread acknowledgement of the social responsibility of science. Many have argued that what they term the Promethean power of modern science and technology warrants an extended ethic of social responsibility on the part of the scientists and technologists (Bunge, 1977; Cournand, 1977; Jonas, 1985; Lenk, 1983; Luppicini, 2008; Moor, 2005; Sakharov, 1981; Weinberg, 1978; Ziman, 1998). In particular, The Russell-Einstein Manifesto called for scientists to take responsibility for developing weapons of mass destruction and urged them to “Remember your humanity, and forget the rest” (Russell & Einstein, 1955). This manifesto initiated the Pugwash meetings which emphasised “the moral duty of the scientist to be concerned with the ethical consequences of his (sic) discoveries” (Khan, 1988, p. 258). When accepting The Nobel Peace Prize on behalf of himself and the Pugwash conferences Joseph Rotblat stated “The time has come to formulate guidelines for the ethical conduct of scientist, perhaps in the form of a voluntary Hippocratic Oath. This would be particularly valuable for young scientists when they embark on a scientific career.” (Rotblat, 1995). Thus Rotblat and his colleagues propose that ethics needs to be included in the training of young scientists, a call that is echoed by many others including Bird (2014), Evers (2001) and Frazer and Kornhauser (1986). This call has been taken up authoritatively by UNESCO which emphasizes the theme “Ethics of Science and Technology” (UNESCO, n.d.), and according to which “The ethics and responsibility of science should be an integral part of the education and training of all scientists”. UNESCO (1999, section 3.2.71). Beyond this, Ziman claims that what is needed is what he calls ‘metascience’, an educational discipline extending “beyond conventional philosophy and ethics to include the social and humanistic aspects of the scientific enterprise" (Ziman, 2001, p. 165). He argues that metascience should become an integral part of scientific training in order to help equip scientists of the future with the skills necessary to tackle ethical dilemmas as they arise (Small, 2011).[8]

The situation is rather different in mathematics with the exception of the Radical Statistics group (n.d.), which publishes analyses of social problem topics with the aim of demystifying technical language and promoting the public good. Generally, very few mathematicians acknowledge the ethical and social responsibilities of mathematics, although there is some acknowledgement of the social responsibility of mathematicians, as I recounted above. Hersh (1990, 2007) discusses ethics for mathematicians, Davis (1988) proposes a Hippocratic oath, and the American Mathematical Society (2005) provides Ethical Guidelines for mathematicians. However, the content of these recommendations is primarily about professional conduct in research and teaching for professional mathematicians. Davis (1988) goes beyond this and argues that mathematics should not be put in the service of war or other harmful applications, and mathematicians should exercise their consciences. Ernest (1998, 2007), Davis (2007) and Johnson (2015) argue that mathematics needs to acknowledge its social responsibility, with Davis (2007) arguing for the need for ethical training throughout schooling for mathematicians and non-mathematicians alike. These, however, represent marginal voices in the mathematical and philosophical communities of scholars.

If one looks beyond mathematicians and philosophers to the area of mathematics education, there are many voices asserting the social responsibility of mathematics. Of course it is uncontroversial to claim that education is a value-laden and ethical activity, since it concerns the welfare of students and society, and the objectivity, purity and neutrality of mathematics itself is not at stake. In consequence, there is a very large literature comprising many thousands of publications on social justice and social responsibility in mathematics teaching, the first theme to be mentioned here.[9] Some of the main dimensions in this literature are mathematics and exclusion based on race and ethnic background (Powell & Frankenstein, 1997), gender and female disadvantage (Rogers & Kaiser, 1995; Walkerdine, 1988, 1998), low 'ability' and handicap as obstacles (Ernest, 2011; Ruthven, 1987), and disadvantages correlated with or caused by social class and its correlated cultural capital or other factors (Cooper & Dunne, 2000). A second theme is the role mathematics plays in critical citizenship and the public understanding of mathematics (Frankenstein, 1990). A related third theme is the Mathematics Education and Society (Mukhopadhyay & Greer, 2015), Critical Mathematics Education (Ernest et al. 2016; Skovsmose, 1994) and Ethnomathematics (D'Ambrosio, 1985; Powell & Frankenstein, 1997) movements which consider both the role mathematics plays in society and how it impacts on the first two themes. The Critical Mathematics

[8]The inclusion of metascience in science teaching loosely corresponds with my proposal to include the philosophy of mathematics in or alongside the teaching of mathematics.

[9]A very partial bibliography of mathematics education published 20 years ago has over 800 mathematics education entries concerning the issues of society and diversity (Ernest, 1996).

Education movement also looks critically at mathematical knowledge and the institutions of mathematics and their role in denying the relevance of ethics and values to mathematics, and thus denying its social responsibility (Skovsmose, 1994). It shares this concern with the Philosophy of Mathematics Education movement (Ernest, 1991, 2012, 2016a, 2016b), to which the present chapter and indeed this entire volume represents a contribution. However, within the mathematics education research community, beyond any commitment to the teaching of mathematics in a socially just way, the idea that ethics needs to be taught alongside mathematics remains a minority opinion, except perhaps within research in the third theme distinguished here.

Conclusion

In this chapter I question and challenge the idea that mathematics is an unqualified force for good. I acknowledge the traditional argument that like any other instrument, mathematics can be applied in both helpful and harmful ways, and I acknowledge the many benefits it brings. But I nevertheless endorse the minority view that mathematicians and other students of mathematics need to be taught the ethics of mathematical applications to question and limit harmful applications. They also need to be taught to think critically, understanding the uses of mathematics in society and in arguments justifying political claims, social policies and commercial interests. However, my main argument is more radical. I argue that in addition to the explicit and intended applications of mathematics, the nature of mathematical thought and the role mathematics plays in education and society can lead to collateral damage; some unintended but nevertheless harmful consequences. Mathematics has a hidden role in shaping our thought and society that is rarely scrutinised for its social effects and impacts, some of which are negative.

First of all, there is the harm caused by the overvaluation of mathematics in society and education, with its negative impacts on the confidence and self-esteem of groups of student including females and lower attainers in mathematics. These unintended outcomes of mathematics in school in leaves some students feeling inhibited, belittled or rejected by mathematical culture and perhaps even rejected by the educational system and society overall. In sorting and labelling learners and citizens in modern society, mathematics reduces the life chances of those labelled as mathematical failures or rejects (Ruthven, 1987). This is a hidden impact of mathematics that is usually brushed over as the fault of the individuals that suffer, rather than as a direct responsibility of the role accorded to mathematics in education and society.

Second, even for those successful in mathematics, in shaping thought in an amoral or ethics-free way, mathematics supports instrumentalism and ethics-free governance. Instrumental thinking leading to the objectification and dehumanisation of persons in business, society and politics, has the potential to cause great hurt and harm. This is manifested in warfare, in the psychopathic actions of some corporations, the exploitation of humans and the environment, and in all acts that treat persons as objects rather than moral beings deserving respectful and dignified treatment throughout (Marcuse, 1964).

I do not claim that mathematics is intrinsically harmful, but that without more careful thought about its role in society and thought it leads to harmful, albeit unintended, outcomes. The way we teach and how we use mathematics and its impact on our thinking are what can be harmful. My proposal is that to obviate or prevent the potential harm done by mathematics as well as to improve the teaching of mathematics we need to teach the philosophy and especially the ethics of mathematics alongside mathematics itself. Part of this teaching is needed to overcome the idea that mathematics, unlike any other domain of human knowledge bears no social responsibility for its roles in society, science and technology. All human activities should contribute to the enhancement of human life and general well-being and no domain can stand apart from such ethical scrutiny, although this should never be used as a reason for limiting advances within pure mathematics itself. However, the intended and unintended applications of mathematics and their consequences do need to be scrutinised and held accountable within the court of human happiness and human flourishing.

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Part IV
Philosophical Theory in Mathematics Education Research

On the Need for Theory of Mathematics Learning and the Promise of ‘Commognition’

Anna Sfard

Abstract In this chapter, *research* in mathematics education is defined as a special type of *discourse* in which potentially useful stories about learning and teaching mathematics are being told. A consistent collection of stories coming from a given discourse is known as a *theory*. A commognitive version of theory of mathematics learning, made distinct by its foundational assumption about the unity of thinking and communicating, is then presented in accord with this discursive definition.

Keywords Research · Theory · Discourse · Mathematics education
Commognition

For several decades now, the idea of theory, which can be considered as one of the most unique and impressive invention of human intellect, has been falling out of favour, at least in human sciences. The fierce manifesto “Against theory”, issued more than three decades ago by literary theorists Knapp and Michaels (1982) has been echoing ever since then in other human sciences. Today, with the advent of computerized pattern detection, we seem to have yet another reason to be nonchalant about theories. Titles such as “Big data and the death of the theorist” (Steadman, 2013) herald the possibility of theory-less research. The harbingers themselves sound relieved, if not triumphant.

No, theories do not seem to be well, at least not in human sciences. Being very much theory-minded, I regret this state of affairs. I am also worried by what is happening around. Most Ph.D. students I know are quite desperate. They are in a constant quest after the holy grail of theory, but, depending on how they manage to look, all they can see is either a dazzling abundance of candidates or an almost total absence thereof. Either way, they feel helpless. Those of them who experience *embarras de richesse*, enquire about the possibility of theoretical “networking”; those who complain about the paucity of supply, start asking why they need theory at all.

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I wish to claim that these predicaments are the result of misunderstandings about what theory is, what it is good for, and what can or cannot be done when several theories compete for everybody's attention. In this chapter, after stating my position on these matters, I outline and justify the theoretical perspective that guides my research at this point in time.

What Is Theory and Why Do We Need It?

Let me say this at the outset: I feel strongly about theoretical thinking and am unable to do research in its absence. For me, theory is the ground to stand on while trying to move and the signpost to follow while looking for a direction. Without it, I feel like walking on a thin ice in the middle of night. My aim in this section is to explain why I feel this way. For this, I will now try to clarify what theory is for me, why it indispensable part of my research, and what it is that makes me prefer one theory to another.

Because theories arise in *research*, it is natural to begin with a definition of this latter concept. Probably the simplest way to describe research is to say that it is the activity of telling stories about some aspects of reality. Thus, research in history produces stories about past events, research in physics tells us about what happens to material objects (think, for instance, about the formula $S = ½gt^2$, which with a properly interpreted symbols turns into a story of the free fall), and research in mathematics education narrates the processes of learning and teaching mathematics. All these are not just-so stories. Most of them are meant to mediate and improve those human activities around which they evolve, and let me leave for later the question of what counts as improvement and who is to tell. Thus, for instance, the researcher in the field of mathematics education strives to tell teachers and parents things about teaching and learning mathematics that may be useful for them to know, but across which they would probably not come themselves simply because of their being too preoccupied with precisely those activities around which the researcher's stories evolve. What makes the researcher's narration different from other types of storytelling is the systematic use of specially designed discursive tools with which she can reach beyond the commonplace and obvious and then communicate her insights in an unambiguous, precise way.

Different types of research are definable by how they tell their stories and by the stories themselves. The first of these characteristics is the special form of communication, that is, the *discourse* that the researchers employ in order to tell their stories. The other feature—the set of stories produced within a given discourse and endorsed by the researchers because of the general agreement among them that what is said presents reality in an accurate and useful way—is what we call *theory*.

Discourses differ from each other along four dimensions, the first three of which are their *keywords*, and their usages, the *visual mediators* storytellers employ to make clear what their stories are all about, and the *routine actions* of their participants. This latter feature, routine, is a discursive pattern that may be presented as a set of rules competent storytellers implicitly follow in implementing different tasks. Perhaps the most conspicuous among the special features of research discourses are the relative strictness of their routines and the explicitness of many of the routine-defining rules. Those of these meta-rules that regulate the use of words are called *definitions*. Ideally, the stringent rules guard the narratives against ambiguity and make it possible for the storyteller to defend these stories against doubts and criticism. The researcher, therefore, may be expected to be much more accountable for what she says than is usually possible in informal colloquial discourses.

Given a particular vocabulary, set of mediators and collections of routines, endorsed narratives forged with their help are the fourth and last defining feature of the resulting discourse. The distinctive feature of those sets of stories that deserve the name "theory" is the tight interrelatedness of their components. Thus, for instance, together with any sub-set of stories already endorsed, the theory includes also all those narratives that can be derived from this sub-set with the help of rules known as laws of logic. Theories begin their life as small sets of stories specifying some basic properties of a given discourse's focal objects. These are known as *theoretical assumptions*. Once established with the help of these assumptions, theories start growing by absorbing new narratives constructed either by logical derivation from previously endorsed narratives or on the basis of observation and in concert with the rest of the theory. The required property of theories is their overall internal consistency: there must be no pair of narratives that would mutually exclude each other's endorsability. Of course, this is a highly idealized picture of theories, and it was presented here as but a signpost one must follow in spite of the unattainability of the ultimate destination.

Having distinguished between discourses and theories, I should now correct myself and recast my own declared need for a theory as the need for a properly constructed research discourse. Indeed, the difference between those among us who are "theoretically-minded" and those who doubt the indispensability of theory is not, in fact, in the question of whether to have a theory or not. After all, according to the definitions just given, all researchers are storytellers, and thus theory-builders. The difference is in the strength of their commitment to properly constructed research discourses. I now have no difficulty explaining my own deeply sensed need for such discourse. Indeed, only the form of talk defined as precisely and explicitly as possible lets me arrive the level of accountability and communicability which I feel obliged to sustain as researcher. At the risk of sounding a bit dramatic, I would thus say that for me, working toward a well delineated research discourse is a matter of the researcher's professional ethics. For the same reason, I believe in the necessity of making my basic assumptions explicit. In research in mathematics education, these initial stories only too often remain tacit, whereas the researchers themselves claim to be starting their study "with an open mind" and "with no theory". This, however, cannot be true. To say the least, the very choice of things to investigate implies the assumption about these things' possible significance for educational practice, and this indicates, recursively, the presence of tacit assumptions about some underlying causal relations! To sum up, I feel obliged to spell out explicitly both the definitions that guide my use of words and those primary stories that, for one reason or another, I endorse as the point of departure to my research. This is exactly what I will try to do in the rest of this chapter, when I present my current research discourse.

First, however, let me say a few words about the status of our *theories as a discursive activity*, implicated in the just outlined idea of research. This requires some arguing, because even today, more than fifty years since this idea's inception (see e.g. Foucault, 1972), the followers of the still powerful positivist approach to scientific research view some entailments of the discursive vision as unacceptable, and thus reject this vision itself. Below, after presenting these entailments, I will show that this rejection is grounded in a logical fallacy.

The discursive approach implies that there is no such thing the "ultimate" theory and that the theoretical plurality is the inherent feature of research. Different discourses may be incommensurable, that is, may differ in their use of words (which in traditional terms may mean differing ontological foundations) and in the rules according to which their component stories are being endorsed (which some would say is the issue of diverse epistemologies). Theories coming from such discourses, even if they appear as contradicting each other, are not necessarily mutually exclusive. Indeed, assuming that one of them must be "untrue" would be as wrong as claiming that only one of the known geometries, Euclidean and non-Euclidean, can be true. In fact, the adjective "true" does not really apply to theories, and if we wish to choose between them, we should rather ask about their *usefulness*. As to the adjective *useful*, it is clearly a matter of what those in the position to decide consider as such rather than of any universal, rigidly defined criteria. Indeed, one's vision of usefulness depends on this person's values and her conception of the needs of those for whom research stories are told. All this implies the inherent impossibility of ordering all the theories along a single line. We thus seem to have no choice but to let thousand theories bloom.

On the other hand, it would be a mistake to deduce from here, as some people do, that "any theory goes". Claiming this would be as weird as asserting that the proposition "No single pair of trousers can be regarded as absolutely and forever the best one for Mr. X" entails that for Mr. X, any pair of trousers is as good as any other. Yes, some theories may appear to us better than some others, and some may be even seen as superior to all the alternatives we know. One must, however, to view this superiority as inherently provisional and to treat any current "winner" as a permanent candidate for dismissal. This is exactly how I see the *commognitive* approach, which I am about to outline now (the sources of the name will be explained later): although this discourse answers the current needs of my research better than any other I am aware of, I do not delude myself about its ability to retain this status for much longer (or, for that matter, about its being as attractive to others as it is to me!).

The rest of this chapter is devoted to the introduction to commognitive research. This is done according to the definitions of research and theory given above. In the next section, I take care of commognitive keywords; I then proceed to perceptual mediators, equivalent in this case to what is known in research as data; I follow with a section on routines, also known as methods of analysis; I conclude with a brief survey of stories generated so far by the commognitive research, and in so doing I outline the commognitive theory of learning and teaching mathematics in its present version.

The Commognitive Keywords and Their Use (Basic Concepts)

With the research in question, which deals with mathematic learning, the main keywords that have to be defined here are *mathematics* and *learning*. The first of these terms has already been implicitly defined in the former section: being a type of research or a "domain of knowledge", mathematics is a discourse. Of course, school mathematics is not the same as mathematics of research mathematicians. These two types of discourse differ in all four characteristics, but above all, in nature and strictness of their meta-rules. But although in this respect school mathematics may seem closer to colloquial discourses, it is still tightly related to the discourse of mathematicians. The talk and stories about shapes, numbers, sets and functions that the members of contemporary societies meet in school and are obliged to master can be considered as a product of "customization" of the mathematicians' mathematics to the needs and capacities of the young learners, as these needs and capacities are understood by whoever is responsible for the customization.

From here it follows that *learning* mathematics is the process in which students extends their discursive repertoire by individualizing the historically established discourse called mathematics. To *individualize* a discourse means to become able to communicate according to its rules, and to do so not only in conversations with other people and possibly with their help, but also while "talking" to oneself and solving one's own problems. Thus, to say that a person individualized mathematical discourse means that this discourse became a discourse of her thinking.

This last sentence obliges me to explain the commognitive interpretation of the word *thinking* as well. The way this term has just been used makes it clear that thinking is considered as tantamount to communicating with oneself, and not necessarily in words. This explains the source of the neologism *commognition*, coined from *communication* and *cognition* so as to serve as a constant reminder that communicating with others and thinking "in one's head" belong to one ontological category. All this implies that in spite of differences in these two activities' visibility, we can use a single set of tools to investigate them both.

Uniting the thinking-communicating divide, the idea in which the commognitive discourse takes its roots, has been inspired, among others, by philosophical writings of Ludwig Wittgenstein and by psychological musings of Lev Vygotsky. Both these thinkers repeatedly stressed, in one way or another, the inseparability of thought and speech (in those cases, in which thinking was done in words). Wittgenstein debunked the view of thinking as 'incorporeal process which lends life and sense to speaking, and which it would be possible to detach from speaking' (§339, Wittgenstein, 1953/2003, p. 109). Vygotsky's (1978) insistence that any uniquely human competency originates in a historically established, collectively executable activity implied that thinking, arguably the most unique of human activities, must also had a developmental predecessor in the form of some historically established, collectively implementable activity. Since communication is the most obvious candidate, one cannot but conclude that to think means to communicate with oneself. As will be explained in the next section, the feature of non-duality has multiple consequences for how researchers identify cases of mathematical learning and for how they subsequently interpret these events.

Commognitive Mediators (Data)

Mediators are the generators of the perceptual experiences that help us in getting to know the objects around which our stories evolve. In research, this includes all those things, known as *data*, which we are looking at or listening to while crafting our narratives. Because the objects of commognitive research are discourses, its mediators are mainly, although not uniquely, recordings and transcripts of learning-teaching events, such as classroom interactions or research interviews.

This type of data is not unique to commognitive research. Its less common feature is the researcher's insistence on recording sight as well as sound and the uncompromising rigorousness of transcribing. Indeed, the researcher tries to document anything that can impact communication. Of course, since nothing short of the event itself can provide the observer with information about all potentially relevant features of an interaction, she must choose which aspects to follow, and this task that requires considering the needs of upcoming analyses. Two features, though, should never be compromised. One is the *comprehensiveness* of the transcript: Because the commognitive researcher studies discursive processes rather than mental structures in the heads of the learners, she needs an access to the exchange in its entirety, with no participant's part being dismissed as of lesser importance, not even that of "unobtrusive" observer. The other necessary feature of transcripts is their *verbal fidelity*. Since the non-dualistically minded researcher rejects the word-meaning and form-content dichotomies, she needs verbatim records of things said, with no word changed and no pause omitted. Because there is no such thing as absolute precision, she must live with the fact that the work of transcribing is never fully done.

For all the emphasis on records of communicational events, commognitive research does not dismiss written questionnaires that can be submitted to quantitative analyses. These are welcome, provided they come as an integral part of a study in which records of interactions constitute the primary corpus of data. Each type of data has its own unique role: only videos and transcripts can give rise to conjectures about hitherto unknown phenomena; the quantitative follow-up helps to decide whether what was observed can really count as *phenomenon*, a pattern frequent enough to be considered as a possible explanation for any comparable occurrences.

Commognitive Routines (Method of Analysis)

Commognitive methods for crafting and endorsing new narratives, also known as *methods of analysis*, benefit from the operationality of the commognitive vocabulary. The fact that researcher is studying discourse, the activity that may be public (as in the case of classroom discussions) or private (as in the case of thinking), but is always describable with the help of the same operationally defined set of characteristics, lets the researcher be fully accountable for what she claims on the basis of her analyses. True, some important parts of the discourse under study may be inaccessible to direct inspection. And yet, researcher who makes conjectures about the learner's inner dialogues may be compared to the *archaeologist* who reconstructs an ancient bowl by complementing the excavated pieces with her own additions. In both cases, the observable pieces and those that have been added are of the same kind, whereas the latter ones are products of the researcher's familiarity with the relevant context and of her ability to decide what in this context would count as reasonable. In this respect, commognitive research differs from cognitivist studies, in which thinking is treated as a different, and more basic kind of phenomenon than interpersonal communication. Of several further differences between commognitive analyses and more traditional methods let me mention just two.

First, for commognitive researcher, it is the participants' discourse as such and not anything that can be seen "through" it that constitutes the object of study. Thus, for instance, if she is interested in the participant's learning, she makes clear that her findings are about *the learner's stories* about mathematical objects and not about some entities in their heads she was able to detect through these stories. Similarly, the interviewer who inquires students about their experience of mathematics presents her findings as a story *about her interviewees' stories* rather than her own direct testimony about the interviewee's experience. To those who wonder about the significance of stories about stories, as opposed to stories about the reality itself, let me remind that we live by the stories we tell, that is, by our *perception* of reality, rather than by the reality as such.

In her analyses, commognitive researcher must constantly alternate between the *insider's* and *outsider's perspectives*: to be able to do the work of 'archaeologist', she needs to act as an insider and use her own interpretations of words to begin making sense of what other participants are saying; in parallel, however, she must be able to act as an outsider to her own discourse and, by suspending her own understanding of words, allow herself to think about other possible uses of these words. Thanks to the insider's perspective, her stories take care of those aspects of learning processes that are, as a rule, left out from behaviourist accounts. Thanks to the outsider's perspective, her narratives are anything but "stories of deficit", so common in traditional research on learning: commognitive accounts of learning are about what and why the learners actually do rather than about the tasks they are "still" unable to do. This last feature makes commognitive narratives particularly useful to the teacher.

Commognitive Stories of Mathematics Learning (Theory)

The commognitive theory has been evolving for 25 years now, and the stories on the development of mathematical discourses told on the way cannot be summarized on the remaining pages. Below, I signal in italics some of the best developed commognitive storylines, adding a few words about the ongoing process of fleshing them out with additional detail, new insights and further evidence.

But let me first remark that in tune with the commognitive rejection of dualisms, *all the objects around which mathematical stories revolve must be understood as* metaphors *coming from discourses about material objects.* This means, among others, that those who tell stories of *numbers* or *functions* cannot experience the heroes of their narratives directly, the way they experience tables or stars. Instead, they must "communicate these objects into being" as they go. All the storylines listed below attend closely to different aspects of this unique process of objectification, that is, of turning stories about processes into narratives about as-if independently existing objects.

Storyline 1: Historically, mathematical discourses evolved hand in hand with practical activities, which these discourses helped to expand. Each such expansion, in turn, called for an additional growth of the discourses, and this led to yet another development of practical activities, and so on, up to a point where further development of mathematical discourse began coming from inside this discourse. The initial co-dependence of cultural practices and discourses has been corroborated in the study recently conducted in the Polynesian country called Tonga, in which the particular difficulty with fractions and probability experienced by otherwise successful mathematics learners has been explained by the fact that no native Tongan activity could be found that would benefit from any of these discourses (Morris, 2014).

Storyline 2: Mathematical thinking is developmentally secondary to interpersonal communication about mathematical objects. Unlike objects studied in, say, biology, at least some of which can be experienced by the child directly before she is able to speak, mathematical objects make their first appearance in one's life as words or symbols used by other people. Only the insistent use of these signifiers in mathematical conversations with others can eventually turn them into a signifier-signified pair. In our empirical studies, we had ample opportunities to see children grappling with the inherent dilemmas of the task. In result, we managed to present in much detail early versions of discourses of numbers (Sfard & Lavie, 2005; this study is ongoing), of algebra (Caspi & Sfard, 2012), and of functions (Nachlieli & Tabach, 2012). We believe to have outlined trajectories travelled by most mathematics learners before they reach canonical forms of these discourses.

Storyline 3: Routines are the basic building blocks of discourses, and they make their first appearance in the life of a child as rituals. This storyline comprises narratives on individual routines that develop as precursors to discourses of which they will eventually become a part. Before the child created the objects that such routine is supposed to involve, she can only performed this routine in a ritualized way. In the process of objectification, such separate rituals will coalesce into a full-fledged discourse and will lose their ritualized character. This process and those occurrences that interfere with its completion have been investigated in great detail in the studies mentioned above.

Storyline 4: Objectification, the key occurrence in the process of turning rituals into explorations, requires some struggle and happens in leaps rather than through gradual change. This claim was spelled out already in the earliest version of commognitive theory, and was based mainly on mathematicians' testimonies about their *aha* experiences. I was subsequently convinced by some critics that there was practically no chance for catching such events on camera. I am thus happy to report that Shai Caspi, in his study completed in 2015 and yet to be published, has been able to actually record such occurrence more than once, and with unexpected clarity.

Storyline 5: Learning-teaching interactions are a special variety of discourse, with its own routines. Among the rules that govern the "cocoon" interactions within which children's first mathematical routines incubate let me count *learning-teaching agreement* that requires all the participants to play their respective roles of teachers and learners. We became aware of the need for this agreement in studies in which it was violated (Sfard, 2007). In our current research we keep studying the evolution of learning-teaching routines, asking what can help the child "talking mathematics" in contexts other than such cocoon interaction.

Storyline 6: Mathematical achievement of an individual is a product of collective efforts. In her recent studies, Heyd-Metzuyanim (2013) was able to describe in finest detail how students' families, teachers, friends and, in a sense, their entire social environment, collaborate in creating and sustaining these students' identities as either successful or failing learner of mathematics. The inherently discursive processes of identity building has been shown to result from some common, but usually ignored classroom occurrences: the interplay between *mathematising*—talking about mathematical objects, and *subjectifying*—telling stories about participants of the process.

By way of conclusion, let me express the hope that some of these still-evolving commognitive stories will count as original contributions to our understanding of learning mathematics. To those who view some of these narratives as an attempt to sell old wine in new bottles, let me say that the similarity of the italicized bottom lines to some claims widely acknowledged as truths about mathematics learning conceals an important difference in the stories themselves. The added value of commognitive narratives is that except for stating *that* something is the case, they bring operational, detailed insights into the questions of *how* and *why* things happen. This makes commognitive stories particularly well suited for their role of practice changers. These narratives point their audience to specific occurrences that result in specific outcomes, and as such, yield advice specified at the level of the participants' elementary moves. This seems to be the kind of detailed, operational and reliable guidance that we need if we wish to change learning-teaching processes effectively, responsibly, and in a consistent way.

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Anna Sfard conducts research in the domain of learning sciences. In it, she focuses on the relation between thinking and communication. Results of her theoretical and empirical studies have been summarized in the monograph *Thinking as communicating: Human development, the growth of discourses, and mathematizing* (Cambridge, 2008) and in a number of subsequent edited volumes. She is a *Professor Emerita* of Education at the University of Haifa, Israel and the recipient of the 2007 Freudenthal Award.

On the Roles of Language in Mathematics Education

Ladislav Kvasz

Abstract The language of mathematics attracted recently the attention of philosophers, historians of mathematics, and researchers in mathematics education (see Dutilh Novaes in Formal languages in logic. A philosophical and cognitive analysis. Cambridge University, Cambridge, UK, 2012, Hoyrup in The development of algebraic symbolism. College Publications, London, 2010, Kvasz in Communication in the mathematical classroom. Wydawnictwo Uniwersitetu Rzeszowskiego, Rzeszów, pp. 207–228, 2014, Lakoff and Nunez in Where mathematics comes from. Basic Books, New York, 2000, Macbeth in Realizing reason. A narrative of truth and knowing. Oxford University Press, Oxford, 2014, Serfati in La Révolution Symbolique. La Constitution De L'Ecriture Symbolique Mathematique. Editions Petra, Paris, 2005, or Sfard in Thinking as communicating. Human development, the growth of discourses, and mathematizing. Cambridge University Press, Cambridge, UK, 2008). There exist a considerable number of approaches, each of which studies one particular aspect of the language of mathematics that is relevant in the particular context. Nevertheless, what is lacking is a theoretical framework that would make it possible to integrate these different approaches to the study of the language of mathematics and use their potential for a deeper theoretical foundation of mathematics education. The aim of the present paper is to argue for the need and to outline the possible structure of an integrative theoretical framework for the study of the language of mathematics. The author is convinced that such an integrative initiative can arise from the philosophy of mathematics that is historically grounded and educationally motivated.

Keywords Theory change · Diagrammatic thinking · Reification
Epistemological analysis

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Introduction—Language and Change

A casual look at mathematical texts of the past indicates that the language of mathematics has undergone substantive changes. Many terms changed their meanings—Euclid's straight lines were finite, our straight lines stretch to infinity; Euler considered a function continuous if it was given by a single formula, we define continuity differently. New terms emerge—Descartes introduced the term "curve" (in its modern sense); Leibniz the term "function"; Galois the term "group". These are some conspicuous changes that occurred at the '*surface*' *of the language of mathematics*. If a teacher wants her students to understand her, she must be aware of and take into account the differences between her linguistic usage and that of her students. The evolution of mathematics involves, however, also several changes of the *deeper structure of the language of mathematics*. We will focus on three kinds of them.

During its development the language of mathematics has undergone several *reifications of its entire layers* (see Sfard & Linchevski, 1994). So for instance in algebra during the 15th and 16th century the *reification of algebraic operations* led to the creation of the first layer of algebraic objects—powers and roots—just like the *reification of the transformations of equations* led during the 17th century to the introduction of a new layer of algebraic objects—polynomials (and later other bilinear forms or matrices); and the *reification of the symmetries of equations* under the transformations of their roots led during the 19th century to the reification of an even more abstract layer of algebraic objects—*groups* (and other analogous structures like fields or rings). Thus the majority of objects studied in algebra appeared in the process of reification of operations, transformations, or symmetries. A parallel development in geometry led to the *reification of space* during the Renaissance (which led to the understanding of geometrical objects as situated in space); then to the *reification of transformations* (in projective geometry); and later to the *reification of the infinitely remote points* (in the form of the *absolute*). Thus language makes it possible to turn different procedures and operations, into independent objects. The reifications (there were many of them) are much deeper and much more complex changes of language than the introduction or change of meaning of single notions. A theoretical analysis of the process of reifications in geometry and in algebra can be found in (Kvasz, 2008a, pp. 107–200). A teacher must have a good understanding of the different layers of reification in order to help her students in passing from one layer to the next if some (epistemological) obstacles occur.

But reifications are not the most radical form of linguistic change in mathematics. As fractal geometry shows, language of mathematics is able to *create an entirely new universe of forms* and enable us to see order where without it we were unable to see any regularities (Peitgen, Jürgens, & Saupe, 1992; Peitgen & Richter, 1986). The discovery of fractal geometry is only an example, nevertheless, an important one, because it makes it possible to step in the shoes of mathematicians, who witnessed the creation of analytic geometry in the 17th century, or of the calculus in the 18th century. In the case of analytic geometry or calculus, just like in the case of fractal geometry, in places where mathematicians were previously able to discern only half a dozen of curves (such as the *cissoid* of Diocles or the *conchoid* of Nicomedes), or a small number of functions (like logarithm or sinus) an entirely new universe of curves or functions emerged. Thus we see that language of mathematics is able not just turn particular procedures into new objects and thus to complement the existing universe by some new of objects (as in the case of *reifications*), but it is able to create universes formed of entirely new kinds of objects. The creation of the universe of curves by means of analytic geometry, or of functions by means of the infinitesimal calculus, are just two examples illustrating the point. Fractal geometry is important because it emerged relatively recently and so we have a feeling of the radical character of the change it has introduced. In the case of analytic geometry or the calculus we are situated on the other side of the break (on the side where analytic curves or functions are a matter of course) and so we simply cannot imagine mathematics, in which there are no analytic curves or no functions. Nevertheless, most of our pupils and students live in such worlds and we have to disclose the new universes of curves of analytic geometry, of functions of the calculus or of shapes of fractal geometry for them. The case of fractal geometry enables us to understand how radical cognitive changes this requires.

Nevertheless, even this explosive power of language to create entirely new universes is not the end of the story. The language of mathematics is able to introduce the notion of proof and thus to mediate the contact with the absolute, the feeling that we can know something with absolute certainty, the feeling that besides of *doxai* there is the *epistémé*. One of the great themes of philosophy of mathematics during the last hundred years was the criticism of the absolutistic view of mathematics starting with C. S. Pierce (Dörfler, 2005) through Ludwig Wittgenstein (Diamond, 1975) and Imre Lakatos (Lakatos, 1976) to social constructivism (see Ernest 1998 for a discussion and further references). But such criticism should not hinder us to recognize, that the absolutistic view of mathematics is in a sense a reflection of the *invention of proof*, which was a major step in cultural evolution of mankind. The exact details of this invention are still debated, but the central role of linguistic tools in this invention is obvious (see Netz, 1999). Mathematics is simply that part of human knowledge, where proving is the way of establishing objective validity of propositions and where the quest for certainty is an important constituent of its practice. In teaching mathematics we must convey an understanding of the meaning of and perhaps also the fascination by the intellectual beauty of proving (Hanna, Jahnke, & Pulte, 2010).

It seems that a theory of language that we need in mathematics education should integrate at least these four layers—it should be able to describe and explain changes on all of them. Thus it should be able to describe:

– *how new terms are introduced and how older terms change their meaning*;
– *the different successive layers of reification*;
– *the mechanism of creating entirely new universes of objects*;
– *how proof and deductive rigor are introduced.*

But even more important is that this theory of language should be able to describe the various interactions among these layers, thus to describe the relations between these four kinds of linguistic change. To create a first sketch of such a theory was the aim of the book *Patterns of Change, Linguistic Innovations in the Development of Classical Mathematics* (Kvasz, 2008a). Let me present a section of this historical sketch.

A Glimpse into History—The Layers of Reification in Geometry

As an illustration of the method of historical reconstruction of the changes of the language of mathematics I will present two layers of reification of space in geometry. But first I will introduce some ideas from Wittgenstein's *Tractatus*:

> A picture cannot, however, depict its pictorial form: it displays it. (Wittgenstein, 1922: 2.172)

Let us take as an example a Renaissance painting. In order to find the *pictorial form* of the painting we must look for all aspects which the painting *does not depict* (i.e. does not contain as a result of an intentional stroke of the brush), but only *displays* (i.e. contains unintentionally). After some time we realize that the pictorial form contains *the horizon*—if I would go into the countryside depicted in the painting to see what is there, where the painting displays the horizon, I would find nothing particular; as well as—*the point of view* of the painter. In a drawing from *De Pictura* from 1495 Leon Battista Alberti gives a clue, how to find the viewpoint from which the painting is painted; see (Fig. 1).

Thus in a painting we can identify two aspects—the *horizon* and the *point of view* from which it was painted—which the picture does not depict, but in spite of this are present.

> The boundaries of my language are the boundaries of my world. (Wittgenstein, 1922: 5.6)

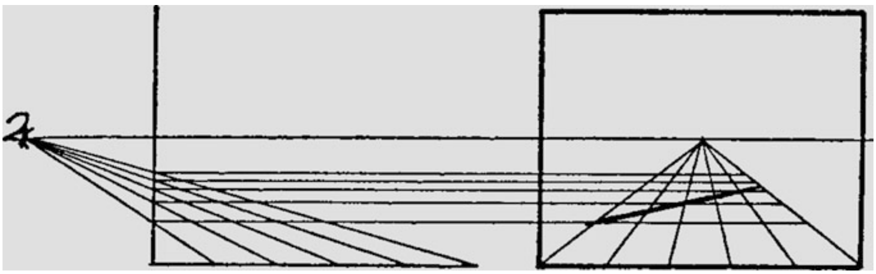

Fig. 1 Alberti's analysis of perspective in 1495

Thus what we find in paintings as a *horizon* is according to Wittgenstein not specific to paintings but it is a typical feature of every linguistic representation. Everyone who knows the history of geometry knows the important role of objects of this sort. The *circle* representing the boundary of Beltrami's model, the *absolute* in Cayley's theory or in Klein's *Erlanger Program* are only some examples.

The subject does not belong to the world. (Wittgenstein, 1922: 5.632)

A viewpoint exists not only in paintings, but in all linguistic representations. I turned to paintings, because there we can gain some understanding of what is meant by a viewpoint. From the drawing of Alberti we see that the viewpoint is at some distance in front of the canvas. Thus it is not depicted on the canvas; it does not belong to the world represented by the painting. Nevertheless, Alberti shows, that although not depicted, the viewpoint somehow accompanies the representation; its position can be unequivocally reconstructed. This kind of presence is meant by the verb "to display" by Wittgenstein.

In history of mathematics when we study a representational tool (for instance the figures contained in geometric texts) we can study the development of its pictorial form. Thus we can identify all aspects of the representations which are not *depicted* but only *displayed*. This means to look for the boundary of the language and for its epistemic subject. In history we can discover an interesting process that was not mentioned by Wittgenstein, namely the *incorporation of the pictorial form into the language*. To see what I mean, let us turn to a drawing of Albrecht Dürer from his *Underweysung der messung mit dem zirckel vn richt scheyt* reproduced in (Fig. 2).

Here Dürer explains us how a perspectivist painting is created by means of a transparent foil and a rope with its one end fixed on the wall. Gérard Desargues, the founder of projective geometry, introduced into this representation several important changes. He:

replaced the object by its picture
created a representation of a representation

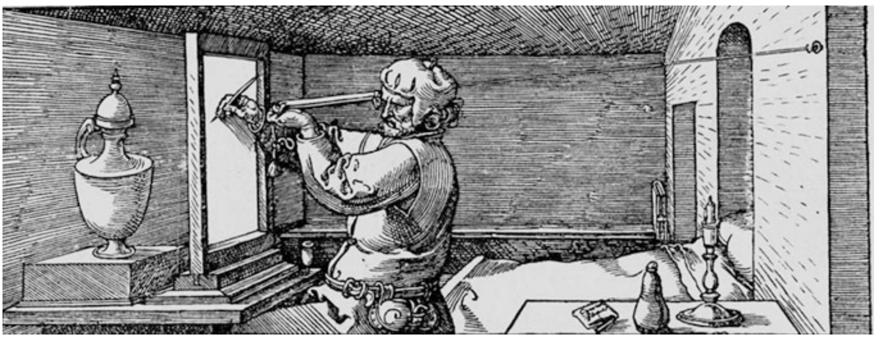

Fig. 2 Dürer's analysis of perspective in 1525

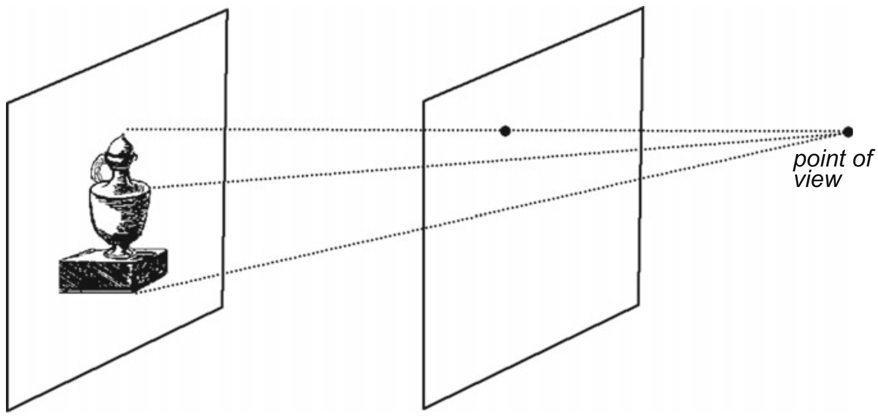

Fig. 3 Desargues' analysis of perspective in 1636

included the viewpoint into the picture as the center of projection. All these three changes can be seen in (Fig. 3).

Desargues created a new pictorial form. I call the first form (created by the painters), in which space was turned into an object, *the perspectivist form*; and the second form, (the one created by Dürer and Desargues), in which the point of view and the horizon were reified, *the projective form*. The development consisting in the incorporation of the pictorial form of language into the language itself (and in this way, of course, creating a new pictorial form) is rather universal mechanism of development of the language of mathematics. As a result of a historical reconstruction of the development of the language of mathematics altogether eight pictorial forms were identified. Their description and consequences for mathematics education can be found in (Kvasz, 2008a, pp. 107–200). I hope this short historical detour is sufficient to get some feeling about the methods of historical reconstructions and I will now turn to epistemology.

Language as a Tool of Epistemological Analysis

The link that has the potential to connect the theory of the historical evolution of the language of mathematics (containing as an important part a theory of linguistic change) with the theory of mathematics education (containing as an important ingredient a theory of cognitive change) is epistemology. To start with a historical reconstruction of the development of the language of mathematics before turning to its epistemological analysis is crucial. We must first separate the different historical layers of linguistic change and only then are we prepared for their philosophical analysis. Without such a separation we could easily make a mistake similar to that made by Piaget and Garcia (1989), who in describing the development of geometry constructed an evolutionary line leading from Euclid (the so called *intra-figurative stage*), through Descartes (*inter-figurative stage*), to Felix Klein's Erlanger Program (*trans-figurative stage*). But in reality there existed no such line of development. The line starting with Euclid and Descartes, having its first two stages in *synthetic geometry* (Euclid) and *analytic geometry* (Descartes) led further to *fractal geometry*. This is the line of *creating entirely new universes* described in the first section of the present paper (see Kvasz, 2008a, pp. 14–84). Its third stage is Felix Hausdorff (who created the notion of fractional dimension), or Benoit Mandelbrot (who coined the term fractal) and not Felix Klein. This developmental line has an entirely different dynamics than the one leading to Klein. This latter line contains after Euclid as stages the creation of *projective geometry* by Desargues; the discovery of the *non-Euclidean geometries* by Gauss, Bolyai and Lobachevski; the introduction of the *concept of a model* by Beltrami, and finally Klein's *Erlanger Program* (see Kvasz, 2008a, pp. 111–160).

Therefore I consider the reconstruction of the history of geometry by Piaget and Garcia, and first of all, their thesis that the ontogenesis recapitulates "*the historical development in reversed order*" (Piaget & Garcia, 1989, p. 113) as mistaken. If true, this thesis would have enormous consequences for mathematics education. But fortunately it is not true (for details see Kvasz, 2008a, pp. 249–251). This example clearly illustrates the necessity to precede the epistemological reconstruction of the language of mathematics by a historical one. We have already sketched a historical reconstruction in the previous part of the paper, so we are ready to turn to the epistemological analysis of the *different layers* of the language of mathematics.

Like the historical reconstruction of the development of the language of mathematics has to take into account the different levels, at which linguistic change occurs, in the epistemological reconstruction of a particular one of these levels we have to distinguish between several methods of epistemological reconstruction. The first can be called *intentional reconstruction*. Mathematics is a human activity, therefore to understand the development of mathematics (and of its language) at a particular level requires first of all understanding the *intentions*, the *motivations*, and the *aims* of the particular actors. This is partially an exercise in psychology of mathematics, but not entirely, because these intentions have often a public form of a *problem* (like Fermat's problem), or a *program* (like Klein's *Erlanger Program*). Several mathematicians can identify with the same intention, work on the solution of the same problem, or cooperate on the same program. So the subjective dimension of mathematics is connected with the social one.

The next method of reconstruction of the development of the language of mathematics is the *reconstruction of the linguistic innovations and deficiencies* of the particular contributions. It turns out, that mathematicians in the process of solving a problem or working on a program introduce some *linguistic innovations*, like Descartes introduced the coordinate system in order to solve Pappus' problem, or Fermat introduced derivation in order to find the law of refraction. We can speak about innovations and deficiencies, and not solely about changes, because the common intention enables us to compare the different solutions of the problem. We can judge one change as *innovative* compared to another, if it helps better to fulfill the original intention. Similarly a particular aspect of the language can be judged as a *deficiency*, and not solely its characteristic feature, when it hindered the progress towards the fulfillment of the intention. The *reconstruction of linguistic innovations and deficiencies* is important, because it enables us to see some objective features of the particular contributions proposed by individual mathematicians and by means of these innovations we can often explain, why a particular solution was more successful than another one and we can also understand the way how a program developed or degenerated.

Of course what we are interested in are not isolated linguistic innovations but rather the formation of a new linguistic framework (like that, which characterizes the reification of polynomial algebra or that, by means of which analytic geometry generates its curves). Therefore we turn from the analysis of the particular innovations to the *reconstruction of the process of merging of separate linguistic innovations into a single linguistic framework*. Language is social and not private; therefore the different linguistic innovations introduced by individual mathematicians must undergo the process of social negotiation. As the historian Michael Crowe said "*Multiple independent discoveries of mathematical concepts are the rule, not the exception*" (Crowe, 1975, p. 164).

A linguistic framework is created by merging of different innovations introduced by the authors of these independent discoveries. So for instance the linguistic framework of elementary algebra emerged from combining the linguistic innovations of several mathematicians, such as *Regiomontanus* (born 1436, who introduced the symbol R for root turning thus the process of root extraction into an object), *Nicolas Chuquet* (born 1445, who introduced arithmetical symbols for powers and a first idea of brackets—by underlining the expression that should be in brackets), *Johannes Widmann* (born 1462, who introduced the symbols + and − for the operations of addition and subtraction), *Michael Stifel* (born 1487, who replaced the R introduced by Regiomontanus by the symbol $\sqrt{\,}$ what allowed him to insert a further symbol into the symbol of a root), *François Viète* (born 1540, who introduced letters for coefficients of equations), and finally *Descartes* (born 1594, who integrated all these innovations into a comprehensive system of algebraic symbolism, that we still use today). We see a process of creation of a linguistic framework lasting for more than 150 years. The algebraic symbolism we use today contains several innovations introduced by at least 6 different mathematicians (for details see Kvasz, 2008b).

This process is of great importance for mathematics education, because it makes it possible to trace the causes of the difficulties our students encounter in learning algebraic symbolism to the particular linguistic innovations and their integration into the linguistic framework of algebra. It often happens that a symbolic innovation taken in isolation is well motivated, intuitively accessible and can be easily semantically grasped. Nevertheless, when it becomes incorporated into the general framework, the meaningful symbolic innovation is turned into an empty syntactic convention; its motivation gets hidden, intuitive accessibility lost and semantic grasp hampered. A detailed exposition of the *intentional reconstruction*, the *reconstruction of linguistic innovations and deficiencies* and the *reconstruction of the* *merging of different fragments into a linguistic framework* can be found in Kvasz (2012).

After a linguistic framework is created the evolution of language does not stop. On the contrary, we can witness a process of evolution of the entire framework. This evolution has, nevertheless a very interesting form, which I suggest to call *bipolarity*. If we take the evolution of geometry, along the line *synthetic geometry*, *analytic geometry*, and *fractal geometry*, we discover that these developmental stages of our linguistic tools for the representation of geometric forms were separated by developmental stages of the tools of symbolic representation. Thus synthetic and analytic geometry were separated by the birth of algebra and similarly analytic and fractal geometry were separated by the creation of the infinitesimal calculus. And it was not a mere historical coincidence. In the process of creation of analytic geometry Descartes made a substantial use of algebraic symbolism—the particular algebraic curves that he introduced were all defined by means of their algebraic equations. And similarly in the definition of the objects of fractal geometry the limit transition, which was actually introduced in the infinitesimal calculus, is used in a fundamental way. So we see that the evolution of the linguistic framework has a bipolar character. The new developmental stage in the development of the iconic language of geometry is reached by means of an intermediate symbolic stage and vice versa. The *reconstruction of the bipolar process of the evolution of language* is, after the methods of reconstruction of the *intentional structure*, of the *linguistic innovations*, and of the *process of their merging*, the fourth method of epistemological reconstruction of the mathematical language. Bipolarity is important from the educational point of view, because understanding of mathematics requires the ability to connect these poles, i.e. the ability to find a geometric interpretation of some aspect of a symbolic formula or to express in symbolic form a particular aspect of a geometric figure.

The bipolar evolution of the language of mathematics can be studied from the perspective of particular aspects of language. It is possible to introduce six such aspects. They are objective characteristics of the language of mathematics and can be called *potentialities of language*. We can introduce *logical power*—how complex formulas the language allows us to prove; *expressive power*—what new terms, predicates and relations can the language express, which were inexpressible in the previous stages; *explanatory power*—how the language can explain the failures which occurred in the previous stages; *integrative power*—what sort of unity and order the language enables us to conceive there, where we perceived just unrelated particular cases in the previous stages; *logical boundaries*—that are marked by occurrences of unexpected paradoxical expressions; and *expressive boundaries*—that are marked by failures of the language to describe some complex situations. As a fifth method of epistemological reconstruction of the language of mathematics we can therefore introduce the *reconstruction of the potentialities of the language of mathematics*. The evolution of the language of mathematics consists in the growth of its logical and expressive power—the later stages of development of the language make it possible to prove more theorems and to describe a wider range of phenomena. The explanatory and the integrative power of the language also gradually increases—the later stages of development of the language enable deeper understanding of its methods and offer a more unified view of its subject. To overcome the logical and expressive boundaries, more and more sophisticated and subtle techniques are developed. It is important to realize that the above mentioned potentialities of language are objective in the sense that it is an objective feature of the language of synthetic geometry, that by its means a trisection of an arbitrary angle cannot be achieved (as was proven by Pierre Wantzel in 1837), just like it is an objective feature of the language of algebra that Euler's number *e* cannot be expressed (as was proven by Charles Hermite in 1873).

After we have identified the potentialities of language the question arises how are they constituted. As they are aspects of language, there has to be a particular structure of language, the change of which causes the increase of its *logical*, *expressive*, *explanatory*, and *integrative* power. These structures of language cannot be connected to a particular subject matter. They must be formal, to allow the increase of the corresponding potentialities. I suggest calling them *formal aspects* of the language of mathematics. In the case of *reifications* we can introduce the following formal aspects: *the epistemic subject of the language* from the point of view of which the theory is formulated; *the horizon of the language* i.e. the boundary of the world that can be represented by the theory; *the individual of the language* i.e. the elementary constituents, that the language is able to distinguish; *the fundamental categories of the language* i.e. the most general notions the linguistic framework allows us to introduce; *the ideal objects of the language* i.e. objects that are introduced in order to make the universe of discourse complete; and *the background of the language* i.e. a neutral medium, onto which all the individual are situated. Their reconstruction is the sixth and so far final method of epistemological reconstruction of the language of mathematics—the *reconstruction of the formal aspects of the language of mathematics*. The last three methods of epistemological reconstruction are explained in Kvasz (2008a).

I mentioned these six methods of epistemological reconstruction of the language of mathematics in order to indicate the complexity of the epistemological problems we (and our students as well) are facing when we are dealing with linguistic change. In teaching and learning of mathematics all these six epistemological layers are implicitly present—we (and our students) have to understand the *intentions* that led to the creation of the particular theory we teach; we have to understand the *linguistic innovations* which were introduced; we have to become familiar with the *linguistic framework* in which the theory is formulated; we have to integrate the opposite poles of the *bipolarity* that occurs in the language; we must master the *potentialities of language* and get used to its *formal aspects*.

Taken together these six epistemological layers form a theoretical framework that is able to *reconstruct the relation between the subjective mathematical knowledge and the objective mathematical knowledge*. At the level of the *intentional reconstruction* we are dealing with the sphere of the subjective realm of meanings, goals, and motivations. Passing through the *reconstruction of the linguistic innovations* towards the reconstruction of their *merging into a single linguistic framework* we reach the sphere of the intersubjective realm of rules, norms, and conventions. Going further to the reconstruction of the *potentialities of the language of mathematics* and its *formal aspects* we are finally reaching the sphere of the objective knowledge. I consider this bridging of the subjective and objective knowledge an important advantage of the present approach when compared for instance with *social constructivism* (which cannot explain the objective nature of mathematics in a satisfying way). In the present approach the objectivity of mathematics is constituted by linguistic means, namely by means of the formal aspects of the language of mathematics, which are fully objective features of the language (as we have seen in the case of the horizon and the point of view of the pictorial form). Nevertheless, this *objective* pole of the formal aspects of the language of mathematics is not isolated from the *subjective* pole of the intentions, motivations, and goals by some insurmountable gulf. On the contrary, the two poles are connected by means of the four intermediate epistemological layers: the layer of linguistic innovations, linguistic frameworks, bipolarity, and potentialities of language.

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The Separation of Mathematics from Reality in Scientific and Educational Discourse

Uwe Schürmann

Abstract This contribution aims to illustrate that the separation between mathematics and reality is an outcome of several shifts in historic mathematics discourse. Therefore, Foucault's method of problematisation and Deleuze's distinction between axiomatic and problematic formalisation in mathematics are used to emphasize these shifts. Afterwards, a special attention is paid to mathematics teaching in German history. Here, it will be considered how reality was treated in the mathematics classroom due to the socio-political circumstances at a particular time.

Keywords Deleuze · Foucault · Frege · Kant · Mathematics education
Reality · Separation

Introduction

To analyse the separation of mathematics and reality against the background of historic and current discourses, the method of choice is the problematisation, a concept that Foucault describes in his late works and which combines his methods archaeology and genealogy (Dits et Écrits: IV/350, Koopman, 2014). While archaeology is related to the question how particular utterances became possible, a genealogy asks for the connection between a discourse and political power. Both methodological strands focus on the relations between knowledge, subjectification and power in a discourse. Foucault's genealogy is an analysis that will clarify the strategies of power as well as which forms of subjectification and which specific form of thinking are responsible for the expansion of a discourse. The starting point is a current question. Here the question might be: why is mathematics separated from reality?

Given the large number of publications in which applications and modelling are defined as interplays between mathematics and reality one may speak of a

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distinction that arranges the modelling discourse. With a glance at the history of mathematics it is evident that this is not a necessary distinction because of its logic, rather it is a historically developed distinction. There had been distinct opinions about the ontological status of mathematics and in particular about the question whether numbers and geometrical objects belong to the world or exist somewhere outside of it. For the late Pythagoreans it seemed to be clear that the entire structure of the world (cosmos) follows a numerical order. In contrast, the Platonic ontology that distinguishes an empirically accessible world and a world of ideas seems to justify the separation of mathematics and reality (Plato, 2000: 514a–517a).

According to Foucault 'knowledge' is not an outcome of rational subjects, but discursive structures prior to any kind of subjectivity lead to accepted knowledge in a particular epoch. That implies that different epochs are characterized by different types of knowledge regimes which Foucault calls 'episteme'. In "*the Order of Things*" (Foucault, 2002) his aim is to describe the different epistemes of the Renaissance, the classical epoch and modern era. The Episteme of modernity is characterized by two aspects: primarily by the search for origins and ultimate justifications and secondly by the role of man for all kinds of knowledge. Since the human being forms the transcendental basis of all knowledge in the modern epoch and is entirely dependent on the empirical realities of his times, the human being was going to be an empirical-transcendental doublet. It is the subject and object of knowledge at the same time. For human sciences in general and didactics in particular this way of thinking will be of fundamental importance. For mathematics the modern conception of knowledge has far-reaching consequences, too. To some extend, binding knowledge to the human cognition means to undermine the claim of absolute truth within mathematics.

Below, a shift in mathematical discourse, from Kant to Frege to Hilbert, which moved mathematics more and more away from reality will be traced. It is illustrated that shifts in mathematical discourse led to the separation of mathematics from reality, a powerful narrative which was not even harmed by Russell or Gödel and is now fundamental to the discourse on modelling in the classroom. Against this background, i.e. the tension between the claim of absolute truth within mathematics and the episteme of modernity, firstly the importance of Kant's "*Critique of Pure Reason*" (1787) will be highlighted in a nutshell. Secondly, Frege's logicism and Hilbert's formalism will be discussed in the context of Foucault's modern episteme.

The Separation of Mathematics from Reality

For Kant knowledge is entirely bound to the faculty of cognition of man (Erkenntnisvermögen). He turned the question of the nature of things to the question of the faculty of cognition. In this respect, Kant's Copernican revolution can be understood as a subjectivistic turn. Thus, his thinking is consistent with the modern episteme. Kant divides propositions into those which are analytic and those which are synthetic. In analytical propositions something is derived from the intension of a concept (e.g. 'the bachelor is unmarried') (Kant, 1787: B10 and B192). Such propositions are a priori true, so no experience is needed to confirm them. However, according to Kant these propositions will not broaden our horizon of understanding. We are only able to derive knowledge which had been part of the concept already before. In contrast, synthetic propositions are assumed to be empirical. They predicate something which is not already included in a concept (e.g. 'the bachelor is 34 years old') (Kant, 1787: B11). Such propositions actually expand our horizon of understanding. However, it cannot be said with certainty whether they are true because they always depend on experience. But what if there were propositions which are true prior to any kind of experience and yet enhance our horizon of understanding? Thus, the question is if there can be synthetic a priori propositions. Kant concludes that synthetic a priori propositions are possible if the range of possible experience is related to the pure conceptions of the understanding (categories) and the pure forms of intuition (space and time). To Kant the propositions of mathematics are examples of synthetic a priori propositions (Kant, 1787: B14–B16).

In this respect, mathematical knowledge is subjective as it is linked to man's faculty of cognition. Nevertheless, because the categories of understanding and the pure forms of intuition are shared by all rational beings mathematics possesses objective validity. From this point, Kant could have been seen as an advocate within mathematical discourse, an advocate who was able to defend the mathematical claim of absolute truth within the modern episteme. Probably this is one of the reasons why Kant's epistemology is of outstanding importance for the mathematical and scientific discourses of that time.

Nevertheless, in the early 19th century, when initial works on non-Euclidean geometry have been started the question arose whether Kant's conception of mathematical propositions had to be reconsidered. If the propositions of Euclidean geometry are not synthetic a priori propositions, the status of all mathematical propositions can be questioned. Again, it was asked if mathematical propositions are synthetic and therefore empirical facts or if they are analytic and therefore a priori true. To emphasize the ambivalent status of mathematics in the 19th century Frege's "*Foundation of Arithmetic*" (1884) have to be considered. Therein, he tries to base arithmetic only on logical conclusions, he refers to a variety of well-known mathematicians and philosophers and he expresses an inner need to save arithmetic's truth from any kind of relativism. In one respect, Frege's thinking is altogether related to the thinking within the modern episteme: he is searching for origins and ultimate justifications. At the same time, Frege rejects the conception of binding truth to the faculty of cognition which came along with the episteme of modernity. In his foundations the risk of relativizing mathematical truth by modern thinking is always present. Frege's approach can be seen as an act of resistance against this relativism.

So to defend arithmetic from any kind of relativisation, Frege must present arithmetic as independent from subjectivity and experience. For that reason, he opposed Kant (Frege, 1884: § 87–91) and Mill (Frege, 1884: § 7). To Mill propositions of arithmetic are empirical facts. They "*are not true by definition; they* *are, in Kantian terms, synthetic. But that implies, for Mill, against Kant, that they are* a posteriori*, inductive rather than* a priori" (Wilson, 2016). Frege rejects this view on numbers, and he rejects Kant's view on arithmetic as well. To Frege arithmetical propositions are analytic. But how to deal with Kant's problem that analytic propositions do not enhance our cognition horizon? Unlike Kant his aim was to show that analytic propositions can enhance our cognition horizon. On conclusions derived from analytical propositions he writes, "*The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house*" (Frege, 1884: § 88). The metaphor of propositions seen as plants growing from a seed illustrates the way axiomatic mathematicians would think about their own work. Here, every mathematical truth must already be contained in axioms. Nevertheless, new proven propositions widen the field of mathematical knowledge. Also another contemporary view on mathematical thinking, indicated by the upcoming human sciences, was heavily opposed by Frege. On the question whether numbers originate in human psychologic conditions he writes, "*For number is no whit more an object of psychology or a product of mental processes than, let us say, the North Sea is*" (Frege, 1884: § 26). Frege treated numbers as objects. Of course, they are not physical objects or derived from physical objects but objects established by mental processes.

At this point, Frege's considerations mark an intermediate stage in mathematical discourse. On the one hand, he is deeply convinced that arithmetical propositions are analytic. He rejects any kind of relation between arithmetical truth and human cognition and he also rejects the view that arithmetical objects have their origin in real world experience. His work signifies a shift, a shift turning mathematics, here arithmetic, away from reality, from mental processes and real world experience. On the other hand, he denies a pure formal interpretation of mathematical propositions. So it was not an absolute shift, Frege proposed. To understand his objections against a pure formal interpretation of mathematical propositions, one should have a look at his distinction between 'sense' (Sinn) and 'reference' (Bedeutung) of proper names, and 'thoughts' and 'truth values' of propositions (Frege, 1892). To Frege geometrical propositions are not only treating truth values but thoughts as well (Frege, 1903, 1905). From that point of view, terms used in propositions do make sense if they are related to an (also immaterial) object in a specific context. It seems that there is still a string between mathematics and reality.

This intermediate stage had been left and the string between mathematics and reality had been cut off when Hilbert published the "*Foundations of Geometry*" in 1899, a formal axiomatic system of geometry. Hilbert does not define objects like points and lines in respect to their geometrical content but in certain mutual relations. "*The complete and exact description of these relations follows as a consequence of the axioms of geometry*", Hilbert says (Hilbert, 1899: 2). In addition to this formal conception of axioms as a description of relations between unspecified things, Hilbert's aim is to prove the relative consistency of geometry. Here, relatively denotes that geometry is led back to a different mathematical axiom system whose consistency is required. Hilbert's proof deals with analytic geometry as a model of Euclidian geometry. So his axiomatic system of geometry is consistent if the axioms of real numbers are consistent.

To Frege things are entirely different. He turns decidedly against Hilbert's formal conception of geometry, where geometrical objects do not mediate a sense, and expresses doubts on Hilbert's evidence for consistency. To assert the relative consistency of an axiomatic system based on formal similarities of two systems is considered as inadequate by Frege. He objected that by a purely formal conception of axioms thoughts would not be expressed. As we know, Hilbert's conception of consistency prevailed in mathematical discourse. In the Stanford Encyclopedia of Philosophy it says, "Hilbert is clearly the winner in this debate, in the sense that roughly his conception of consistency is what one means today by consistency in the context of formal theories" (Blanchette, 2014). Hilbert did not address Frege's view on consistency anymore and expanded his projects from the early 20th century initially to the arithmetic, Hilbert's second problem (Hilbert, 1900), and then to all mathematical subareas from 1920. In Hilbert's formalistic program the consistency of each mathematical domain should be ensured by placing it on a formalistic axiomatic foundation. As we know, Frege's project came to an end due to Russell's antinomy (1902), and Gödel's incompleteness theorems gave the evidence that Hilbert's formalistic program, an outcome of his second problem, was not accomplishable (Smith, 2007). Nevertheless, these results did not harm the narrative of separation of mathematics and reality.

While this narrative of separation is very present in mathematics discourse, there is a minor strand in mathematics history which can be highlighted against the background of so-called 'major mathematics'. Deleuze's concepts of problematic and axiomatic formalisation in mathematics history (Deleuze, 1994; Deleuze & Guattari, 1987) can be used here. Problematic formalisation is a minor strand in history of mathematics or, as Deleuze calls it, a "*nomad*" science. The axiomatic way of formalisation is the major strand or as Deleuze calls it the "*royal*" science in history of mathematics.

> Nomadic mathematics, according to Deleuze and Guattari, disrupted the regime of axiomatic through its emphasis on the event-nature of mathematics. In particular, nomadic mathematics attended to the accidents that condition the mathematical event or encounter, while the axiomatic attended to the deduction of properties from an essence of fundamental origin. (de Freitas, 2013: 583)

The difference between axiomatic and problematic approaches in mathematics can be found at several points in history of mathematics. Three of them are going to be considered below. Firstly, the ancient Greek Euclidean geometry could be seen in contrast to Archimedes' geometry. The following example might illustrate this. While Euclid defines a straight line as a static object (a line which lies evenly with the points on itself), Archimedes characterizes the straight line as the shortest distance between two points. Archimedes' way of thinking is not focused on an essence of a straight line but on solving the problem how to connect two points on a plane. Here many solutions might be possible (curves, loops, etc.) and the straight line is the shortest solution above all others. The Euclidean geometry which goes from axioms to theorems and the Archimedean geometry which goes from problems to solutions can be contrasted to each other in other aspects as well. To Archimedes the circle is an outcome of a continuous process of rounding, the square is the result of the process of quadrature, etc. (Smith, 2006: 148). As we know, Euclidean geometry, the axiomatic way of thinking, prevailed in history of mathematics. It became the major geometry.

Secondly, by the turn of 17th century the tension between axiomatic and problematic geometry had shifted to a more general tension between geometry itself on the one side and algebra and arithmetic on the other side. Here, Fermat's and Descartes's analytic geometry shifted geometry to arithmetic relations which could be expressed in algebraic equations. This kind of geometry stood in opposition to a Desargues' projective geometry like where no algebra had to be used at all (Smith, 2006: 149). Since then arithmetisation became a parent trend in geometry and in other mathematics' disciplines; and resulted, for example, in Hilbert's geometry where the consistency of axioms is derived from the axioms of real numbers. By this means the process of arithmetisation within mathematics is related to a second parent trend, i.e. the axiomatization. It is tried to trace back as many areas of mathematics as possible to an ideal origin, i.e. an assumed essence of numbers.

Thirdly and finally, in the late 19th century an additional shift took place when the calculus was formulated in the epsilon-delta criterion by Weierstrass. In its origins, the calculus was an ideal example of problematic mathematics. The differential calculus dealt with the problem how to determine a tangent line to a point of a given curve. For Newton the calculus referred to a dynamic and infinite geometric process. That means that analysis was very different compared to arithmetic. While analysis was about the infinite geometrical processes, arithmetic dealt with discrete numbers. By the epsilon-delta method dynamic processes were reformulated in a static way. Of course, the calculus can still be interpreted in a dynamic way; yet it is not necessary anymore. By the epsilon-delta criterion, the calculus only refers to the axioms of real numbers (Smith, 2006: 152).

So, what does the distinction between axiomatic and problematic mathematics tells us about the separation between mathematics and reality? First of all, it seems that on the problematic side there is no need to separate mathematics from reality. On this minor side mathematics deals with (real-world) problems, e.g. Desargues's projective geometry was used in optics and the calculus already solved physical problems long before the epsilon-delta criterion was invented. In contrast, due to predominance of the major side there is an ongoing shift from problems to axioms, from intuitive geometry to arithmetic and algebra, and from the infinite and dynamic to the finite and static. All that can be seen as way to separate mathematics from reality. Nevertheless, it has to be taken into account that even the axiomatic attempts could not separate mathematics from reality completely. Additionally, positive outcomes of axiomatic mathematics influenced reality or at least our view on reality. The non-Euclidean geometry is to some extent the result of a pure axiomatic problem. Here, the question was, if Euclid's parallel postulate is in fact a theorem that can be derived from the rest of Euclid's axioms/postulates. Then, the non-Euclidean geometry provided many tools to Einstein's theory of relativity.

While this example might exemplify how an axiomatic problem led to a different view on reality, another filed of examples illustrates how reality itself is changed by axiomatic mathematics. At a glance, it can be seen that computer technology, based on axiomatic mathematics that Hilbert would have had appreciated, is changing our reality day by day. The examples that could be mentioned here are numerous. So, it seems that even between axiomatic mathematics and reality there is no gap which cannot be closed. More likely, the structure of the relationship between axiomatic formalisation and reality is similar to an ancient Greek tragedy. Like Oedipus who ran away from his parents trying to avoid the fulfilment of the prophecy, axiomatic mathematicians tried to avoid any confusion of mathematics by reality. However, for this means they are faced with reality even more, like Oedipus who made the prophecy really happen just when he tried to avoid it.

Reality in Mathematics Classrooms

Hitherto, some shifts which led to the separation between mathematics and reality in scientific discourse have been exemplified. The Foucauldian perspective illustrated the tendency of formalisation in mathematics discourse as a way of reacting to the modern episteme which binds all knowledge to the conditions of human ratio. By Deleuze's distinction between axiomatic and problematic formalisation in mathematics' history the possibility was given to think about the relation between mathematics and reality in a more complex way. However, the separation between mathematics and reality not only plays an important role in scientific discourse on mathematics, but also in educational research and in mathematics school curricula. Unsurprisingly, the role which reality plays in school mathematics had changed several times in history. Against the background of mathematics teaching in Germany, some of these changes are going to be outlined in regard to particular socio-political circumstances.

For Foucault's studies of power relations, the term dispositif (also translated as apparatus) is of crucial importance. With this term Foucault gets a close look at the varied connections between discursive and non-discursive practices. He says:

> What I'm trying to pick out with this term [dispositif, US] is, firstly, a thoroughly heterogeneous ensemble consisting of discourses, institutions, architectural forms, regulatory decisions, laws, administrative measures, scientific statements, philosophical, moral and philanthropic propositions–in short, the said as much as the unsaid. Such are the elements of the apparatus. The apparatus [i.e. the dispositif, US] itself is the system of relations that can be established between these elements. […]
>
> I understand by the term 'apparatus' [i.e. the dispositif, US] a sort of – shall we say – formation which has as its major function at a given historical moment that of responding to an urgent need. The apparatus thus has a dominant strategic function. This may have been, for example, the assimilation of a floating population found to be burdensome for an essentially mercantilist economy: there was a strategic imperative acting here as the matrix for an apparatus which gradually undertook the control or subjection of madness, sexual illness and neurosis. (Foucault, 1977: 194–195)

The network between the elements which are crucial for the educational discourse should be defined as the 'education dispositif'. With the concept of the education dispositif the ever-changing role of reality in the mathematics classroom can be analysed in regard to the multiple relations between pedagogic and scientific discourses, urgent needs of economy and political circumstances in a particular period of time. For the modelling discourse that means that you have to explore why mathematics sometimes was taught as a tool to solve real world problems, and sometimes as a pure formalistic science. For this purpose some crucial changes in the history of German mathematics classrooms are going to be outlined now.

Due to the structured school system in German countries, the goals of mathematics teaching may differ depending on the particular type of school, and so the relation between mathematics teaching and reality may vary from time to time and from school-type to school-type. For example, in the 19th century, goals of mathematics teaching in humanistic secondary schools (Gymnasium) were very different from those of the elementary schools (Volksschule). While elementary school tried to prepare students for labour and so mathematics was taught using realistic tasks, applications were nearly totally replaced by the training of logical thinking in mathematics classroom in secondary schools. Nevertheless, at the beginning of the 20th century, the will to change teaching of mathematics even in secondary schools was expressed. The growing industries' demand was to get well educated students suitable for engineer professions. Prussia politics' demand was to compete with other industrial countries. So, these demands resulted in the Meran syllabus initiated by Felix Klein who had already attempted to connect mathematics, mathematics teaching and industry on university level. What can be seen here is a shift in discourse on teaching mathematics as an outcome of the education dispositive, even if it is not that clear how successful the implementation of the Meran syllabus really was (Kaiser-Messmer, 1986: 31–42).

In any case, school curricula and mathematical discourse changed during the times of National Socialism. At this time, Hilbert's formalism has been defamed as "*Jewish mathematics*" by several mathematicians. These mathematicians begun to publish their works in the journal "*German Mathematics*" (Vahlen, 1936) whereby they tried to establish a racially founded typology of scientific research and a return to intuition in mathematics. So, in Nazi Germany the fundamental dispute between mathematics formalists and intuitionists was resumed and forcibly decided in favour of the latter. Teaching mathematics was highly characterized by racism, too. That means application tasks which were intended to educate students within the meaning of National Socialist ideology made their way into the classroom. The following task illustrates that:

> On average, the annual expense of the state for a mentally disordered person are 776 RM; costs for a deaf or a blind person are about 615 RM, a cripple costs 600 RM. In closed institutions there are: 167,000 mental, 8300 deaf and blind persons, 20,600 cripples. How many millions of RMs do these infirm people cost annually? (Bewersdorff & Sturhann, 1936: 111, translated by US)

Numerous of those tasks can be found in mathematics textbooks of these times.

A third moment, when the formation of the education dispositif can be seen at a glance is the reaction to the so-called Sputnik shock. The launch of the first artificial satellite by the Soviet Union in 1957 was followed by several educational policy responses in Europe and the US. Due to the Sputnik shock, the formalistic ‘new math’ found its way into textbooks and curricula. To get an impression, even primary school children were confronted with set theory. It is worth to note that teaching formalistic mathematics was promoted by university mathematicians. Oriented to Hilbert’s formalism mainly French authors formed the group which is known as the pseudonym Nicolas Bourbaki. Their textbooks “*Elements of Mathematics*”, developed since 1934 (Bourbaki, 1939), serves to illustrate the main content of (axiomatic) mathematics in a stringent structure. Here, all applied mathematics was excluded or rejected as unessential.

Intuition-free, purely formal mathematics in the classroom had soon proved to be problematic. So, only a few years later the introduced reforms have been undone. Since then, proposals have been made how to involve more applications into teaching mathematics. In this case, the discussion also focused on the nature of applications in mathematics teaching. Criticism was expressed that application tasks often use contexts only to illustrate a pre-given pure mathematical content. In contrast, modelling tasks are promoted because they provide openness and take reality more seriously. Also due to the unsatisfactory results of German pupils in the first PISA surveys, the so-called PISA shock, mathematical modelling is justified by several authors (Blum et al., 2004: 47) and was implemented in German mathematics curricula. Again, relations between different fields of power and knowledge —politics, media, research on mathematics teaching, and economy—have led to changes in mathematics curricula. In both cases, Sputnik and PISA, we can clearly see an assumed or actual urgent need as an opportunity to implement certain education policies.

Final Remarks

In this contribution Foucault’s method of problematisation and Deleuze’s concepts of axiomatic and problematic mathematics were used to illustrate that the distinction between mathematics and reality is not self-evident but an outcome of the mathematics discourse. By linking different levels of discourse (mathematics, education, politics) some of the connections between power and mathematical knowledge have been disclosed. It was outlined how mathematical endeavours in teaching and science follow economical and political imperatives, e.g. Felix Klein’s attempt to connect research, education to the demands of economy, or Nazi ideology introduced into the mathematics classroom by means of applied mathematical tasks. It has been illustrated that the image of apparently impersonal mathematics is the result of shifts in mathematics discourse.

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Mathematics Education Actualized in the Cyberspace: A Philosophical Essay

Maria Aparecida Viggiani Bicudo

Abstract In this article it is assumed that the role of the Philosophy of Mathematics Education is to analyze and ponder about reality, or a vision of the world, a conception of knowledge and of the human being, shedding light on the meanings and senses that emerge in the works by Mathematics Education authors. In order to comply with the clarity required in every Philosophical text, more specifically the Philosophy of Mathematics Education, virtual and real ontological aspects are specifically addressed aiming to expose comprehensions about cyberspace, understood as an important aspect of our world reality. In the present *essay*, the following aspects are covered: The ontological aspects of cyberspace; the epistemological aspects in cyberspace and the anthropological aspects. They are seen as intertwined. They are to be considered an entanglement that unveils the complexity of the everyday reality in which we live in, and in which cyberspace becomes. It is in this entanglement that Mathematics Education actualizes itself and that the Philosophy of Mathematics Education endeavours to analyze, to critique, and to reflect on.

Keywords Philosophy of mathematics education · Cyberspace
Ontology · Epistemology

I understand that it is the role of the Philosophy of Mathematics Education to analyze and ponder about reality, or vision of the world, conception of knowledge and of the human being, shedding light on the meanings and senses that emerge in works of Mathematics Education authors. These are questions that have been treated throughout the History of Western Philosophy, but today this subject demands more studies, because we are facing a new horizon open by the presence of the technologies, particularly in Mathematics Education.

Concerned with that theme, I have been trying to understand the ontological and epistemological aspects that, mandatorily, open up into anthropological questions. These are aspects that have to be appropriately captured, when our attention falls on

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the reality called "virtual" by authors that investigate and work with cyberspace reality, as, for example, Lévy (1994, 2005). To define the way of the cyberspace being as virtual, in order to differentiate it from the real, I begin to wonder how to question cyberspace reality, its ways of being and its modes of working in it and with it on Mathematics Education sceneries.

In order to comply with the clarity required in every Philosophical text, more specifically the Philosophy of Mathematics Education, I will disclose my thoughts about virtual and reality, Epistemology and Anthropology.

The Ontological Aspects of Cyberspace

Here, the discussion on the theme depicted in the title of this section is a comprehensive summary of the topic discussed in the book by Bicudo and Rosa (2010). In order to understand the questions that are specific to Ontology, we must inquire into the *where* in which the subjects intentionally attentive to the informational screen meet the computer program, as supported on the same screen. We can recognize that in cyberspace people get involved with one another from different perspectives, like the emotional, the cognitive, and the commercial standpoints, forming an intersubjective community. When we inquire into *where* this meeting takes place, our attention lies, deliberately, on the spatial question. The *where* in which subjective and intersubjective experiences occur, either mediated or side by side with media, is considered by virtual reality authors (Castells, 2003; Lévy, 1999, 2005; Lopes, 2005; Likauskas, 2005) not as real but as the virtual. They talk about the de-territorialization of space, the *atemporality* of time.

What is shown is that the concept of time and of space, such as in the Classical Physics model, does not explain what is seen happening in cyberspace. Classical Physics deals with the concept of *real* as what is objectively given and exists in its own right, what is possible to be measured in space (in the three dimensions: height, depth, and width), and time. It is a spatial and temporal totality, where all people and things are placed, and where history and social facts happen. The Cartesian space, with two input variables (space and time), affords to locate precisely where the event or the object are.

However, in the cyber world, the *where* does not fit in this Cartesian space. This is due to a variety of reasons. In it, we are unable to point to the locations people or ideas meet, the intersection of two axes—space and time—since this *where* unfolds along fast and dynamic connections that branch out to yet more and more unpredictable connections. This concept of space-time is gradually experienced in daily life, as far as we are interwoven in complex events whose triggers are not detectable, unless we resort to sophisticated investigative software, which in turn have to be handled by specialists. We sense ourselves in the shift of events, we perceive that we prompt actions that are transformed into messages and that certainly leave their own cultural imprint. However, at the same time we wake up to the fact that we are moving across a historical-cultural ground.

To my understanding, it is necessary to open up to the concept of space-time as used in Contemporary Physics so as to begin to comprehend the life-world[1] as we experience it today, and to see that cyberspace manifests side by side with the physicality of nature.

According to the Theory of Relativity, space and time vary with the speed of a given reference system. Here, time does not flow continuously; rather, it depends on the reference system. Space is conceived as the place where a gravitational field works.

The Theory of Relativity tells us that space is a material reality. Such space is four-dimensional; this means that it is a curved space that may be defined as finite and limitless simultaneously. Instead of accepting, as in Modern Physics, that space is a location where we find ourselves in the universe, we today see that the universe creates matter, as it expands. The measure of the mass in the universe reveals the infinitude of space, and of the future. It is this question—of finitude and infinitude—that carries along the problem of time. Space and time are, in this approach, inseparable, and form a four-dimensional continuum.

The notions instituted by Modern Science, actually help us understand reality from distinct perspectives. Space and time can no longer be treated separately; they have become part of the action. In turn, action creates reality. This is the trigger of the event, and therefore it expands space. When we collect these ideas together in cyberspace, we understand that cyberspace reality is better apprehended in terms of the four-dimensional continuum in which space-time are inseparable and where the action carried out by the subject and enabled by the computer creates spatialities and temporalities. Hence what is considered virtual in cyberspace is, merely, the *real* (Bicudo & Rosa, 2010).

Understanding Cyberspace as Real

The interrogation surrounding the virtual and the real has been addressed in the History of Philosophy for quite a while now. The *virtual* transcends the pragmatic aspects inherent to focusing on the real as a mere location, bestowed with geophysical characteristics and palpable and practical concreteness. Such is an ontological issue, when the question that emerges is *what is it, then, the real*?

[1]Life-world is a reality constructed in the historical and cultural moment that brings together the present, the past, and the future. It is not a vessel in which we are placed or in which we drop knowledge, theories, etc., as if these were objects in their own empiricism. Rather, these are the spatiality and the temporality in whose dimensions we live with others, whether human or not, whose reality we in turn weave using articulated comprehensions, subjectively and intersubjectively, that are materialized in available forms and contents. What is intersubjectively understood and is kept via the repetition of successful actions forms itself, gradually, through the intertwining of senses and meanings in objectualities. Objectualities are objectivities built on the shift of subjectivity-intersubjectivity and, therefore, do not concern objectivity separately from this shift.

By invoking the Greek philosophers we see that Aristotle explains the real as a constant movement of potency and act, form and matter (Brehier, 1962). Two are the pairs: *potency and act*, and *form and matter*; however, these are not synonyms or similar in their characteristics involving the "being"; instead, they intermingle in the occurrence of the real. The real oscillates between pure potency, which it is not, since it is not actualized, and pure form, which has nothing of matter (Mora, 1994).

This complexity is also addressed by Granger (1995), who, in the 1990s, studies the Philosophy of Science based on Aristotle's ideas. Back then the author develops the concept of *present*, understood as the actualized, and of *non-present*, which includes the virtual, the possible, and the probable, that is, what may happen, but has not yet. This occurrence may be understood from a number of standpoints. The author discusses several ideas, some of which, as I understand them, are important to comprehend information technology, especially those in the context of Physics and other sciences.

Granger considers science as it is today, in the Western World, and asks: what is science about? He goes on and answers: it is about the real. This answer refers to the progressive and constant refinement of mathematical, scientific, technical, and technological instrumentation. His explanations clarify that science contributes a representation of the real, according to its perspective and methods. Yet, science does not account for the experience sensed by the person, since it does not address the imaginary, the expectations and other aspects of sensitivity. His findings also show that Mathematics is the ground on which Modern Western Science roots itself. Hence the status of this Science as the pillar of Exact and Technological Sciences.

The author understands Mathematics as virtual, since, through serial abstraction processes, the forms with which Mathematics works are *forms in general*, in an ontology of forms that are not directly abstracted from the empirical experience. This science covers a wide domain, and encompasses invariants that are not accountable to actualization of *forms in general* and, at some point, Mathematics covers also the forms of empirically actualizable objects. For Granger, the *forms in general* with which Mathematics works are, therefore, virtual, possible, and probable, and may actualize themselves in actions (acts) triggered and intertwined to the materialities and techniques (matter) available, as well as in particular applications that are approximate probabilistic explanations of what is empirically presented, and so on. Reality in Mathematics is virtual in the sense that it does not depend on empirical contents to be shown, though it depends on formal contents.

The connection between the virtual aspect of Mathematics and the empiricism of Natural Sciences is effected through the scientific-theoretical system of references that supports modes of applicability. If we take an object in the realm of Mechanical Physics as an example, we see that its reality is determined by the theoretical referents in its coordinates and that, for that reason, its actualization is intertwined in a finite number of elements. In this sense, the *actual* of the product of this Science is determined, albeit incompletely, since it is more than the general form (virtual) of Mathematics, given that it realized the materialization of its product with the actualizing acts (acts) and with the technical-scientific-technological materiality (matter) available, being, at the same time, less than that general form (virtual), because it does not present it completely. The realized product, therefore, carries the virtual, the possible, and the probable.

This complexity is called *informational screen*, which sustains the scientific-technological actualization. It is not an inflexible screen, which would determine the invariants of actualization, by lodging the acts and the available materiality. The impossibility of completely realizing the virtual of the general form in Mathematics is transcended by the pluralism and multiplicity of possibilities in Natural Science.

The concept of *non-actual* and of *actual* (Bicudo & Rosa, 2010) led us to understand the reality of cyberspace as a complexity in which the virtual (general form of Mathematics), the possible, and the probable (the scientific-technological apparatus), the act (the actualizations triggered by the actions of the people who work with the informational screen) are present. The actualization is realized by the acts of the people who act according to their own traits, whether they are imaginative, emotive, cognitive, or judgmental, when they operate with the informational screen. In light of the scientific-technological apparatus that backs computers and other media, we see that a networked actualization takes place, branching out smoothly and quickly, connecting people that communicate through a specific language determined by a reference system (computational programs) and their own acts, laden with their own traits.

To me, the virtual only begins to exist when human subjectivity enters the system's cycle (Lévy, 2005), and this event is a crucial issue in philosophical thinking, since it requires us to take the pathways of epistemology and of anthropology. Next, I will discuss philosophical thought about these aspects.

The Interweaving of Anthropological and Epistemological Aspects

These aspects concern the ways the human being is and knows. It is important to make clear that I do not refer to an abstract and generalized concept of man, let alone of a theory of knowledge that may support possible explanations. I am framing questions and explanations about the different ways of the understanding of what it is to be with the informational screen, knowing oneself, knowing the other and producing knowledge.

I see that we live in the life-world, understood as the ground where we are, where we actualize our experiences, which are interleaved like a network. So, this is where we find ourselves, the moment we are with the other, with the computer or other media. This involves what there is in the spatiality/temporality in the actual and non-actual modes and, therefore, also involves the cyberspace. It is in the wake of this comprehension that I will frame questions and discuss the understandings about the way of being and knowing ourselves at and with the computer and other media.

I agree with the reasoning and the notion present in the concept "'beings-in-the-world' is the basic and indivisible unit" (Borba & Villarreal, 2005, p. 32). From this point on, I will rethink the meaning and the sense of *being-in-the-world*, as explained in *Being and Time*, by Heidegger (1962) in order to underline the complexity of this issue.

My intention is to develop a thought about the possible ways of being-with-the-computer and other media, with emphasis on the specificity of cyberspace. Here, I will address questions about the reorganization of thought, which carries the idea of the reorganization of knowledge, of language, of dialectics, of intersubjectivity, as well as the constitution of objectualities.

In Being and Time when exploring the unity being-with-the-other, Heidegger (1962), points to an existential totality, in which the Dasein (the human being seen as each one of us, in each case) is always in spatiality, that is, it is in the *there*, and it always is with the other (that is, anything, a person or otherwise). According to this way of thinking, the *with* is a *determination* of the Dasein, whose possibilities of becoming diversify, in line with what is, and in line with the way it is, since, given that its character of *being-there-in-the-world-with*, it does not require processes that are otherwise used to establish relationships in order to place itself together with, being with, though it is always with. It was from this standpoint that I understood the phrase humans-with-media, which I henceforth call being-with-the-computer and other media.

An initial, naïve examination may reveal a way of being made explicit in what Heidegger calls *occupation*, which means that the person uses this as a contrivance to do something, that is, the person uses the computer as a tool. It is possible to be with the computer, also, in this way. However, by zooming in we see that even when the computer is indeed used as a tool, the person triggers an intentional action[2] to realize something that he or she aims at, and, in this process, uses the available informational apparatus, establishing dialectic with the computer program. The logic and the language of the program used define the way the person exposes what is understood in this dialectic. *Being-with-the-computer and other media* acquires more complex shades, and is not trapped in the way of being of *occupation*, since the computer and media, supported by the informational screen, place themselves with the person, somehow. But how?

Heidegger, in the work cited, also states that we can make ourselves busy with what we are in the modes of *occupation*, described above, and *pre-occupation.* In the mode of pre-occupation, the care with the being of the other is present, now seen as the being that is to be, in being; there is an anticipation of what may take place inside, which makes the one who pre-occupies place himself or herself ahead of what will happen. The mode of being worried with the becoming of the other may take place in two ways: when the one who worries places him/herself ahead of the becoming of the other, preventing him/her from realizing actions when he/she

[2]Phenomenologically, intentional action is carried out by intentionality of the conscience that perceives what it is doing and of the experience of this realization.

chooses his/her ways,[3] and, thus preventing him/her from running into danger. In this case, he/she precludes the is-to-become of the other. It may also become as a guiding care, when he who worries directs the possible choices that he/she foresees as contributing factors for him/her being healthy and free.

As for the being-with-the-computer, we see that the person does not place him/herself with it, the computer, as in *being-with* in preoccupation. There is a preponderance of the way of *being-with*, as an occupation. Nevertheless, it is not possible to use Heidegger's explanation (Heidegger, 1962) about being with tools available. The reason is that, as said above, there is a dialectic that supports an exchange between the person and the computer and that accepts the idea of reorganization of thought and, to some extent, of dialogue too. Also, proceeding along this train of comprehensions, it is possible to invoke intersubjectivity.[4] In these articulations, the complexity of the reality in which we live becomes clear.

In the philosophical line explained here, it is understood that there is no humans-computer divide. Rather, they are united to the extent in which *humans-with-computers* is stated. But how are they united? To me, we may understand this relationship by beginning to focus on the mode in which the computer reorganizes human thought as put by Tikhomirov (1979).

But let us focus on the *being-with-the-computer*. If we ask, *what is the computer*?[5] the answer could be given in terms of *what one is with* or *who one is with*, pointing to ways of being-with-it, conducting an action.

We should note that the term that expresses the idea articulated is *action*, understood primarily as an intentional action that is triggered by the subject, effected with the computer when being with it, and therefore, it is an action between subjectivity and the computer; that is, an *inter-action.* Action, as understood in the framework of phenomenological thought, is always intentional, and is connected with a unit of meaning and that, by being expressed in different languages, produces meanings. Certainly, these meanings are arranged in materialities that are possible and present in the life-world. Does the computer act? Indeed, we do know that it carries out logical operations. But, do these operations constitute an action? An operation takes place according to a logical structure predefined by a program. In this dimension of logical operations, can the human being interact-with-the-computer? Certainly it can. But in what horizon of acting modes does this interaction occur?

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[3]In Being and Time (1962) Heidegger states that the person is a being of possibilities that realize themselves as the person makes choices, in the pathway of his/her life. He explains that, in this pathway, we always run into crossroads that require that we chose one. When we do, horizons of possible events open up that, in turn, also split continuously, demanding further choices. At the same time, the pathways that were not chosen end any possibility. It is thus that each of us defines our own historicity when making choices.

[4]This subject will be addressed again later.

[5]This question means, what are the characteristics of a computer's structure and of operations it may carry out?

Here, a field of themes spreads open, calling for more detailed investigation in order to evaluate the complexity of the mode of *being-with-the-computer and other media in the cyberspace.*

What strikes me as interesting is, essentially, that the computer is not an intentional conscience that aims at what it wants to know or any other mode of *being-with*. I also clearly perceive that it has a logical structure that allows the inter-action that takes place in terms of the comprehensions of language, as a logically structured phenomenon, having its own semantics, and, also, that this *being-with* is involved by the intentional conscience in the dialectics of the see-seen.[6]

Tikhomirov analyzes the role of computers concerning human activities, relying on and taking over the tradition of the historical view of these activities, which is born in the works of Vygotsky and followers. Tikhomirov states that he sees the computer and other machines as organs of the human brain, developed by human hands. He admits that they are tools of human intellectual activity and that they act as mediators for the very structure of mental activity; hence the meanings of such activities being new. Therefore, a new form of activity emerges. Moreover, when he analyzes this process, the author shows that there is no distinction between man and computer anymore, that he sees them as a system, *man-computer*, which, in its role of system, has two ends that are interdependent. In search of clarity about this system, he focuses on the meaning of knowledge and the meaning of thought, studying even the theory of artificial intelligence. Tikhomirov states that *knowledge*, in the context of research on artificial intelligence, is *the ability to answer questions* (*op. cit.*, p. 267), while thinking is the ability to solve problems. But, as he says, beyond the ability to solve problems, the human thought carries along the skill to also frame these problems.

This problem solving includes, as I see it, the context in which questions are framed, apart from the repositioning of reasoning towards sifting the components implicated in the questions and towards framing them in a logical manner, so that they may be logically solved, revealing knowledge itself, as well. In this problem solving, the meeting of an objective is also considered.

With this clarification, I question once again whether the discussion presented affords to solve the complexity sensed in the life-world, in which cyberspace is

[6]Conscience is understood as an intentional move carried out by the living being when it reaches towards what is focused on in the life-world as a figure, which it detached off the background and brings into itself as the perceived; it is, also, understood as the maker of acts that elaborate this perceived, expressing it as comprehensions that may be expressed. Therefore, it is a move that stretches towards and brings the perceived nearby, and may carry out acts of comparison, analysis, and reflection. The perceived is brought by not as what is objectively given, but as what is seen from the perspective of the own body, which, in phenomenology, is considered the ground zero from where seeing and perceiving spring. This move of perceive-perceived, or, as others prefer, of see-seen, is what Husserl calls noesis-noema, where noesis is the see, and noema is the seen.

present. I understand that it does not and, to confirm this opinion of mine, I call in the understanding that phenomenology makes of this subject.[7]

In Husserl (1970, 2002), I understood that the modes in which knowledge becomes include the senses given in the soma, considered as the first comprehensions nurtured in the dialectic perceive-perceived, and the constitution of the object by the subject. That is, this construction covers the articulation of subjectivity-intersubjectivity-objectivity.

The constitution of knowledge, from this perspective, occurs in a process of articulation of the subjectivity-intersubjectivity-objectivity (Bicudo, 2010). These three members form a totality; they do not occur in a linear process of inter-relationship that would unite them; on the contrary, they intertwine in mounting complexities. It is understood, already, in what is shown in the pre-predicative comprehension of the process, evolving into the predicative mode of depicting what is understood. The pre-predicative covers perceived comprehensions that have not yet been addressed theoretically, and that may be expressed by logically structured language. The predicative is seen as what has been addressed, investigated, and expressed in an intelligible language.

From the pre-predicative standpoint, the priority of subjectivity is understood through reductions[8] of what is given in the life-world, so as to focus on the senses that become to the subject, in the corporality of his living-body[9] (Husserl, 2002). This is what emerges as the main nucleus of the constitution of knowledge, since it reveals the *I feel*. It is in it that localized sensations (Husserl, 2002) take place, which, based on successive interweaving moves, lead to the constitution of (i) subjectivity, manifesting the perception of his own singularity to the subject; (ii) intersubjectivity, whose nucleus is the act of perceiving, the perception of the other as an equal, which is called intropathy, and language; and (iii) objectivity, through the constitution of the object by the subject (Bicudo, 2012). What becomes evident in the assessment of this reduction in the life-world (Husserl, 1970) towards what is nuclear in the living-body is its soma, which, through the organs responsible for the sensory traits such as tact, sight, and hearing, for instance, as well as other more sophisticated senses, such as the kinesthetic ones, reveal specific aspects of the thing perceived and, on top of that, the complexity of the sense *in* and *by* the *living-body*. All sensations are localized, and distinguished from one another in the

[7]Phenomenology is introduced here because it is the philosophy that I study and work with, though I do not intend to state that it is the only philosophy to present another way to understand and explain this topic—knowledge and thought.

[8]The reduction carried out in phenomenological procedures does not mean a summary of ideas, with the simplification of the text to the extent that the most significant ideas are shown. This reduction concerns the meanings and senses expressed in the language of the text, obtained through descriptions and investigations about what is being said about the topic analyzed, constituting ever more complex nuclei of ideas.

[9]Living-body is understood as the carnal body that intends to carry out actions. Husserl sees it as *Leib*, which differentiates it from *Köpper*, the body seen as the physical body. It is also called own body and carnal body.

somatic areas they affect. They intertwine so as to afford the perceptions of wholeness of the perceived, which manifests in the life-world where the living-body is. Therefore, perception involves the perceived and its historical ground[10]; that is, together with what is around. The other is given to the living-body also as an own body that, like it, is understood by feeling and perceiving what is around, perceiving it, also, as a carnal body.

From the predicative standpoint, the dialectic of the perceive-perceived, or of the see-seen, is involved by conscience,[11] which, through its psychological and spiritual acts, that is, acts of judgment, articulates the perceived and expresses it through language, whatever it is. The retention of the perceived is an objectification in the general sense and may, at the same time, effect an act of reflection, therefore realizing a mode of theorization at the level of subjective comprehension. This is the process of constitution of the object, or, rather, of objectuality, by the subject. Yet, the constitution of objectivity historically present in the life-world demands the constitution of intersubjectivity.

The nucleating act of intersubjectivity is the experience of the other that covers, beyond perception, intropathy and language. Intropathy is an act realized in the experiences in which the other is given (brought, exposed) to the *I* in its corporeity, forming a bond that in turn is enabled by the understanding of the "same as me", or, putting it differently, the other to whom I may reveal myself, since the other may understand with me what I understand of my own experience, as felt. Yet, this act, though it is the founder of intersubjective communication, does not accomplish it, since in itself it expresses nothing. It is necessary to resort to signs that indicate and to the word that, through the voice, says. The intertwining of acts of perception and of intropathy, together with language, seen as a structuring aspect of the articulations of what is understood and as a communicator of this articulation, constitute intersubjectivity.

Language expresses the intertwining between what is intended, what is expressed through voice, what is kept in historicity by tradition (since it may be written down), and, more than that, it affords a characteristic *logical* activity specifically connected with language, and the ideal cognoscitive configuration is specifically generated in it (Husserl, 1970). The objectifying act may therefore be consummated. It becomes ideally objective, that is, an objectuality and, as such, susceptible to be transmitted and recovered, when necessary.

Heidegger addresses the question of language, which he understands as the dwelling and the revelation of the being, and reveals thought as an articulation of what is understood in an intelligible discourse, so that the comprehension of the perceived, the articulation of the expressed in language, and the communication that tells the other about the understood and the articulated form a non-linear wholeness,

[10]Perception is an intentional act that takes place in the present, in the now, when the seen is understood immediately, with no previous theoretical reference systems, as it shows itself to the one who sees. The seen stands out against a backdrop, the historical ground brought in the historicity of the life-world.

[11]Conscience was defined in footnote 9.

which nevertheless supports the move of understanding-interpreting-communicating. The articulation cited is realized by the logos, which "lets something be seen, namely, what the discourse is about; and it does so either *for* the one who is doing the talking *(the medium)* or for persons who are talking with one another, as the case may be" (Heidegger, 1962, p. 56).

In the approach presented, that is, the phenomenological approach, thought and knowledge transcend the purely logical characteristic, understood as the Aristotelian mode that refers to logos as affirmation of a predicatively expressed rationality as truth, and cover the senses, the perception, the cognition, and the judgments that realize themselves in the embodied subjectivity, the living-body, and intertwine with intersubjectivity, whose constituent acts, intropathy, and perception articulated with language constitute objectivity, that is, the knowledge culturally present in the life-world. Therefore, thought is embodied, it is an expression of the intelligible articulation of what is understood and expressed through language.

The quest for this comprehension cannot ignore or do without the works by other authors, who take upon a *theory of activity*, even though this theory is part of the first works cited and that are representative in the theoretical construct that they elaborate. It is also impossible to ignore other modes of conceiving human thought. I also understand that this is a complexity that unfolds in the entanglement of explanations of the theory of activity, of the reorganization of thought, of the Heideggerian *being-with*, and of the Husserlian being in the life-world. In this entanglement there are modes of dialogue and of establishing intersubjectivity between subjects that presentify themselves, becoming present to one another, when they are-with-the-computer and other media and there are also existential ways of living temporality and spatiality. It demands focusing on the duration of the act performed, since it points to the living experience of time/space.

The Modes of Dialogue Present in the Humans-with-Computers

If we consider the computer and investigate thoroughly what constitutes it, beyond its interfaces, we will find a sort of skeleton that is structured by logic (Figueiredo, 2014) and whose characteristic is the interactivity that is realized in thought, revealed as the form[12] of human logical thought, and machine, formed by a logical skeleton, which establish a dialogue. This dialogue evolves in making sense of the answer, as expected by the mode of thinking of the person that triggers the action. The computer answers by changing, transforming, moving something that has been sketched and expressed as formal logical language. This is the doing of language.

[12]This is one of the concepts expressed by Figueiredo.

From the paper written by Figueiredo (2014), I understand that the principle of the interaction human-computer is manifested in the person's effort and adaptation to the skeletal form, therefore devoid of characters and movements, when the person fills this form with his or her ways of feeling, imagining, fantasizing, wanting, reasoning, in sum, with his or her way of being directed to what he or she is realizing when they are with the computer. The many applications of a program reveal interfaces that dress the skeleton of the machine and are at the user's disposal, offering the best conditions to express himself or herself, which in turn are enabled by the logical form of the computer's skeleton, and changing it. The interfaces fully unleashes the person's mode of being and of understanding of the need to deal with the codes in the logical skeleton of the computer. Then, the great turning point takes place: *thought* takes flight, accompanied and materialized by the informatics toolbox.

It should be noted that the human *interacts* with the computer, with the machine, starting commands; yet, we are not anywhere near a simple, mechanical action, since the modes in which the computer is with others and with the world, as well as the expectations, feelings, and modes of understanding one another and the world are all intertwined with the binary rationality that supports the construction of a computer program. By being with the computer, in the communication move, the actions are effected by the person with the keyboard, which operates numbers; however, the actions are transmitted, according to the complexity of the life-world where subjects and co-subjects, the beings-there, live and accept themselves, and accept the other with whom they dialogue.

Here, I resort to the term *dialogue*, which, from the phenomenological perspective, covers language and intropathy, discussed above. This interaction, which is interactive, is exposed through the construct humans-with-medias, which I borrowed from Borba and Villarreal (2005) due to its consistency, since it allows moving through the complexity of the determination of the mode the Dasein is in the *there*. I understand that, in this zone that spreads open, in the horizon of the there, modes of interacting with the computer take place, reorganizing thought.

When we look at the theories put forward by philosophers, psychologists, and educators, we see that dialogue may occur merely as a conversation in which information is linguistically communicated. It may also take place as a relationship from being to being, who, in an environment of mutual respect and authenticity, reveal themselves in their own mode of being and display their thoughts. The attitude is that of looking attentively to the other and to what he or she says. This second possibility is the concept accepted by philosophers such as Buber (1958) and by educators like Freire (1996), for instance.

In this flow of argumentations, as I see it, the complexity of the dialogue that may be established manifests in different perspectives and, to the extent that we deepen this understanding, the complexity of the lived reality today involves us step by step, many times preventing us from realizing ourselves, in this entanglement.

In this sense, dialogue comprises the hues in the modes of being. It may become in the interactive dialectic human-with-computer, when the pole *machine* answers the commands given by the human through the logic of the computer program, imposing modes of reasoning to the other pole of the system, the human. The human reasons logically, though his or her expectations are not determined by the program's logic, since he or she infuses his or her actions with human modes of being, with intentions, expectation, projects, and all his or her historicity. Therefore, the computer skeleton dresses up in human form and, more than that, it is filled with human information. In this sense, a mode of dialogue takes place.

However, there are other modes that are susceptible to occur in the realm of human-with-the-computer, when we analyze the networks that emerge quickly and dynamically, this time involving humans-computer-humans or, in lieu of these, avatars.

Based on these notions, it is possible to look, in more detail, into the *humans-with-computer and other media*, trying to understand the dialogic relationships that form between people who communicate with and through the informational screen, constituting intersubjectivities. It is also important to clarify the comprehensions about the dialogue with avatars positioned like people in the dialogicity person-computer-person/avatar.

As it was said above, from the perspective of the Husserlian phenomenology, the primacy of intersubjectivity is the inter-subject communication that finds support in the realization of intropathic intentional actions, and in language. The actions surrounding language have been addressed from different standpoints, notably those of logic and, therefore, of grammar and semantic as well. Yet, language brings along meanings manifested in the living-body that, in its intentional carnality, catches sensations that reach it through the sensory organs. These sensations also intertwine, in a move that gradually constitutes perceptions of the world. These meanings fill what is said in words and, in this way, carry the grammatical and semantic aspects of language, meaning, and sense.[13]

In this flow of argumentations, as I see it, the complexity of the dialogue that may be established manifests in different perspectives and, to the extent that we deepen this understanding, the complexity of the lived reality today involves us step by step, many times preventing us from realizing ourselves, in this entanglement.

Let us examine the situation of the connection person-computer-person, when both people already know each other in the physical-temporal ground of the life-world. In this case, there is a historicity that interconnects the modes of one person exposing himself or herself to the other, whose styles are linked with the position and roles played in the social organization whose dialogues are maintained

[13]Sense covers the whole noematic sphere, that is, the sphere of sensory perceptions that are given in the living-body and that wrap around one another, so as to, gradually, constitute a wholeness that may be given in the act of perception, as understood as logos being born (Merleau-Ponty, 1945), apart from including the perception of acts of consciousness that articulate the perceived in its non-expressive layer, which refers to the moving noematic acts. This movement runs towards the interpretation—which includes the logos not being born anymore, that is, given in the perception, but as some articulation with or with an articulation at an advanced stage—and, also, it runs towards the expression afforded by language, including, therefore, communication. Meaning is confined to the content of the ideal sense of the verbal expression, of the spoken and written discourse, present in the historicity of the life-world.

and that, when they connect through the computer, may enjoy the speed of communication, the fluidity of spatiality and of temporality, establishes in the synchronous or asynchronous communication, memory of information, forms of revealing themselves through the program being used, feedback of answers. However, the intersubjectivity constituted in the historicity of the experiences of both people prevails. What takes place is a *being-with-in-the-presence*, which is not stopped by physical distance and is susceptible to being interconnected through the computational apparatus, though it also is guaranteed by the spatiality that "does not state itself as extemporaneous to the constitution of my being, and my Dasein is the main mode through which I state what I am" (Detoni, 2014, p. 107). In other words, my spatiality gradually constitutes itself in my modes of being-in-the-world-with-the-other, *ex-posing* me, when I communicate.

Mathematics Education Realized in the Cyberspace

I understand that when we are teaching and learning Mathematics in the cyberspace we are living intersubjectively with-other (human beings and computers and other media). The computer language is present, as it is present the intropathic intentional actions. But we must be attentive to the way the dialogue occurs, because the logical language that sustains the computer skeleton is always on a kind of a familiar given soil to which we do not pay attention, and so it speaks loud under the interfaces of the programs. As it was said, the dialogue evolves in making sense of the answer, as expected by the mode of thinking of the person that triggers the action.

In a recent study, Barreto and Nascimento (2014) observed that children in the early education stage (between 3 and 6 years of age, in Brazil), whom they called digital naives, are familiar with digital resources. However, the interest manifested by these children proposed by teachers depends on the degree of novelty in the activities developed using software, apart from the response time, since, if the connection is slow, these subjects will start another activity. These children get involved in several activities at the same time, absorbing information quickly. The authors also frame an important question about the experience of time that digital naives exhibit:

> […] they are used to high speed; therefore, when they click on the desired icons, they expect an immediate answer. If that does not happen, they become impatient and, at times, they want to give up the activity initially intended. […] The children of this age are quite dynamic and attentive to sounds, colors, and movements. This characteristic was enhanced by the access to new technologies. (Barreto & Nascimento, 2014, pp. 268–269)

What becomes evident in the process of change is the way of living temporality: there is no tolerance, and waiting is not accepted. Time spent waiting is time to be spent doing something else. The flow of time is not lived: present, past, and future are not experienced. Everything is about the *now*, the instantaneous present. Places, scenarios, situations are replaced within a click. Spatiality is also experienced with-the-computer and other media, at the speed of a trip.

Rosa and Seidel (2014) aware of the distance established between digital naives, youngsters familiarized with digital technologies, and Mathematics teachers, investigated the "training of Mathematics teachers online", which they called cyberformation. This is a concept of form/action, addressed by Bicudo (2003), connected to the idea presented by Rosa (2008) of being-with, thinking-with, knowing-how-to-do-with-technologies, and the Heideggerian view of being-in-the-world-with.

Form/action is a concept that brings together form, which imprints itself on action, and action, which realizes form according to the mode of being of the one who triggers, with the materiality available. Therefore, it is not about a prior idea about what it means to form the teacher, but formation becomes in actions and materialities available, aiming at a *form* constituted in the historicity of culture. So, it is not about a pragmatic doing, but the formation move brings along this historicity.

Cyberformation aims to work with Mathematics teachers who are in the training process to teach in distance learning courses. It involves the learning of the way to place themselves with-the-computer and other media, perceiving themselves in action, that is, realizing activities with the logic of the informational screen, which supports these media when these teachers work with mathematical contents. The authors note that these teachers also have the concept of Mathematics teacher that moves from one concept that repeats modes of their own teachers, when they themselves were teaching us, to one concept that demands other modes of teaching and learning, which nevertheless are not clarified yet.

Rosa, Vanini, and Seidel (2011) address the questions concerning the production of mathematical knowledge online. The authors ask: What is the Mathematics that happens with the cyberspace, then? What are the aspects that afford Cybermathematics, as we conceive it, to become another and yet conserve itself? They state that these are aspects of the transformation of a Mathematics seen from a culture that is specific, but that is already consolidated in each computer connected to a network, in each home, school or university around the world. In the argumentation that they develop in support of their statements about the transformation undergone by Mathematics that is studied, perceived and understood by being-together-with-cyberspace, the authors quote Turkle (1995, p. 111), who declares: "When we cross the screen and enter virtual communities, we rebuild our identity at the other side of the mirror. This reconstruction is our cultural work in progress".

It is important to stress that, when the authors state that the transformation of Mathematics that takes place when we are-with-the-computer and other media, they are referring to the aspects of this transformation seen from the perspective of a specific culture, when they pave the way for the comprehension of the constitution of an ethnomathematics in the context of the culture of cyberspace. The work proposal is clearly defined as the existence of the production of mathematical knowledge in a particular context (the cyberspace) and maintain this existence (of the production of mathematical knowledge) from the perspective of Ethnomathematics, that is, from the standpoint of “doing Mathematics”, in a given cultural group. This group, called generation @ and/or interleaved in the “net” culture experiences the possibility of building mathematical concepts in the relationship of immanence planes and conceptual characters.

Concluding…

I understand that it is very important that we, mathematics educators, become watchful when referring to computer programs and to computer games as mediators of the communication process as well as the teaching and learning process. It is important to understand that these programs are not simple instruments to be deployed in order to obtain fast and correct answers. Likewise, they should not be naively taken as resources that allow people working together when they are spatially-temporally distant. We need to become conscious about what transcends the computing environment and the facilities available, and we need also to understand the way in which we humans are when we are next to the informational screen.

With this essay, my intention was to discuss the complexity of the life-world we live in. I underscored the possible modes of being-in-with-the-computer and of dialoguing, which affect teaching and learning issues, in this case, in Mathematics. The philosophical questions presented are like a backdrop, and were briefly addressed, with the intention of placing them in a perspective and affording the advancement of studies and argumentations that lead us to the comprehension of the life-world in which we are. With this essay, I hope I have contributed with the discussion about the theme covered.

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Part V
Philosophy of/in Teaching, Learning and Doing Mathematics

Making Distinctions: A Phenomenological Exploration in Mathematics Education

John Mason

Words, words, words, all I get is words.
I get words all day long, first from him, now from you,
Is that all you blighters can do?
(My Fair Lady)

From words we can learn only words. Indeed we learn only their sound and their noise …
We learn nothing new when we know the words already,
and when we don't know them we cannot say we have learned anything
unless we also learn their meaning.
And their meaning we learn not from hearing their sound when they are uttered,
but from getting to know the things they signify.
St. Augustine (389/1938) p. 46

Abstract After a brief remark about methods, readers are invited to consider two related phenomena, and then to connect their experience with my comments about them. Two mathematical tasks follow, both intended to highlight particular features of attention and how it shifts. After a brief commentary about attention linked to pedagogical actions that might be relevant, further comments on the task-exercises are offered which bring out more distinctions and associated actions. The chapter ends with discussion of some philosophical and pedagogical underpinnings, and some consequences of this way of working.

Keywords Attention · Phenomenological stance · Noticing · Experiential

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Introduction

An exploration of the making and use of distinctions in mathematics education is pursued from a phenomenological stance. This means starting with inviting the reader to enter or re-enter phenomena, often through engaging in work on some mathematical task-exercises. They are tasks in the sense that they are intended to initiate activity and so provide specific things that are worth noticing, that is, specific distinctions that have proved to be useful. They are exercises because they can be used to remind oneself in the future about things worth noticing. Commentary on the task-exercises is designed to resonate with participant's experience and to illustrate, instantiate and exemplify distinctions. They provide the closest thing possible in text to negotiating meaning for distinctions.

Distinctions by themselves can become merely cognitive-candy. They stimulate the intellect because having labels gives a certain amount of pleasure associated with classification, but they are otherwise ineffectual unless they are used to some purpose. To be useful, distinctions in mathematics education need to be linked to actions, whether mathematical, pedagogical or both. Only then can discerning similar phenomena in the future make those actions available, and so enrich the repertoire of mathematical and pedagogical choices available in the moment. I suggest that this is what Gattegno meant by *educating awareness* (Gattegno, 1970a, b, 1987 preface).

Methods

I eschew the word *knowledge* because it seems to imply some static and stable 'thing'. I much prefer the gerund *knowing* because it is dynamic and situation dependent, and hence fits better with my observations of myself and others, and with the human psyche as I experience it. So my method of enquiry is to start with my own experience, then try to generate or stimulate something in and for others, in the form of task-exercises (Mason, 2002). I deliberately delay theoretical discussions until after there has been some shared experience.

Two Phenomena

Consider the following two phenomena.

Phenomenon when you ask your children "what did you learn at school today?", if you get anything at all, it is generalities about particular events, and nothing about learning.

Phenomenon when you ask yourself 'how you taught' a particular lesson, what comes to mind are particular episodes, which when articulated tend to be expressed as vague generalities and superficial actions, such as "we discussed", "I asked them to …", "they didn't … in the way I expected" and the like.

Immediate Comment

Parents and researchers soon learn that these questions are not productive. Indeed, in neither case can the question actually be answered. Few if any people, much less young children, are aware of 'learning' while at school, and rarely are teachers aware of very specific actions that they initiate. Their attention is usually fully absorbed by what they are doing, what they are imagining, what they are feeling. They are caught up in the flow (Csikszentmihalyi, 1997) with their inner witness asleep. If something breaks into or disturbs that flow, then the inner witness wakes up, there may be a flood of emotions (emotional energy) and some sort of reaction (habit driven) or response (considered), together with cognitive story telling about what happened (often to explain some negative aspect away). Both teachers and learners act-as-if (James, 1890) their perception of the didactic contract (Brousseau, 1984), although rarely if ever articulated, is that learners are to engage with the tasks they are set by teachers, and, implicitly, it is assumed by both that the required or expected 'learning' will then be taking place.

Anyone who has tried to elicit from a teacher how they teach soon discovers that the things that can be said are mostly intentions rather than actions, and that it is not even clear that the notion of 'how you teach' makes any sense. What is regular or usual? What is common or invariant across different sessions on the same topic with the same learners, across different topics, and across different learners? Creative teaching takes place in those spontaneous unpredictable moments when something fresh is enacted: a fresh mathematical action, a fresh pedagogic action, a fresh mode of interaction between teacher, student and mathematics. Even when the question is made more precise, along the lines of 'what you actually do in the classroom', the scope of influences, the range of awarenesses, the array of sensitivities to notice, far exceeds the vocabulary available (Andrews, 2015).

Extended Comment

As Umberto Maturana (1988) noted, "everything that is said, is said by an observer": what is said by a practitioner about their practice is not 'the experience' itself but rather a description of (some aspects of) that experience, 'recollected in tranquillity' (Wordsworth, 1800: preface) with possibly the poetic transformation in which "the emotion is contemplated till by a species of reaction the tranquillity gradually disappears, and an emotion, similar to that which was before the subject of contemplation, is gradually produced, and does itself actually exist in the mind". Descriptions are necessarily incomplete because "to express is to over stress" (Mason, 2002, p. 198). What is said necessarily stresses some aspect(s), namely what is referred to. Consequently other aspects are de-emphasised, and consequently ignored and unsaid, certainly for the speaker, and doubly so for the hearer. This stressing and consequent ignoring has the effect of creating a property that characterises the reference or meaning of the descriptors, because stressing and ignoring is what creates generalisation (Gattegno, 1970a, p. 136).

Thus it is that educational researchers in general, and mathematics education researchers in particular turn to classifying what they observe on the basis of distinctions, whether through being present, through watching video or listening to audio recordings of a lesson, through interviewing students and-or teachers, or some combination of these. Such descriptions are effective when they resonate with readers, as in Wordsworth ‘poetical turn’. The distinctions people make are based on their interpretation of technical terms either drawn from the literature or constructed afresh, which are themselves rarely definitive as to meaning.

Hearing or reading descriptions of practices is a bit like viewing holiday photos: the person who took the photos is aware of what is just out of shot, with emotions, images, and actions being resonated from memory and framed by the boundary of the photo. By contrast, the viewer sees only the photo itself. Anything else stimulated by the picture arises from the viewer’s past experience and personal associations. This is what I think St. Augustine was getting at in the opening quotation. The reader-hearer has only the description to work with, whereas the describer has resonances with unspoken but experienced aspects. Both the describer and the hearer will have metonymic associations and metaphoric resonances with their own past experience, so that the same ‘words’ may generate quite different associations. As someone contributed to an art gallery installation “your perception of me is a reflection of you”, and as Hyemeyohsts Storm (1985) put it “the world is a mirror for the people”. What can be attended to is only what someone is attuned or sensitised to attend to.

An extra difficulty with descriptions is that when they are made up of words which include or require judgement and interpretation, the reader-hearer cannot distinguish between, on the one hand, what was observed, and on the other hand, judgements and theorising about what was observed. They cannot pick out what is evidence and what is deduction, association, explanation or judgement. Consequently they cannot decide whether they agree with the describer’s ‘analysis’ or whether there might be alternative interpretations. This is why it is necessary to take care to distinguish between an *account-of* an incident (minimising emotional commitments, judgements and interpretations) and *accounting-for* it by reference to judgements and use of technical terms and generalisations (Mason, 2002, pp. 40–42).

The act of observing is further complicated because discerning detail requires some frame(s) of reference, some prior enculturated experience as the basis for making a distinction, and these frames of reference may or may not be explicit and articulated. Like frames placed around a painting, or like looking through binoculars, the effect is to eliminate or reduce context, to create a border or boundary. Implicit frames can be dangerous but are part of what is meant by a ‘shared culture’. Even when frames are explicitly articulated, it remains open to question whether the interpretation put on the situation is appropriate. My own preference is for multiple interpretations so as to indicate and remain with the complexity which is inevitable when human interaction is involved.

Enquiring

In order to negotiate meaning it seems to me to be necessary to have some potentially shared experience. This is the basis of my *phenomenological* stance. I refer to *task-exercises* as tasks which a group of people can undertake so that they have some immediate shared experience, however varied according to the frames or collections of distinctions that they bring to the situation. They can then negotiate the use of labels so as to try to reach some 'taken-as-shared' meaning (Cobb, Wood, Yackel, & McNeal, 1992). Over time labels become technical terms. These terms refer to distinctions (recall the Sanskrit '*neti neti*' interpreted as "not this, not that") which can collect into potentially useful frameworks. They accrete a variety of examples or instances which enrich both the 'sense' of the label and the potential for resonating with or triggering access to actions associated with those examples. A theory is then a collection of frameworks, that is a collection of related distinctions with associated actions (or predictions).

However, labels as technical terms have to be re-negotiated with each encounter with a wider community, and the epistemologically and philosophically sound way to avoid falling into solipsism and idiosyncrasy is to take every opportunity to work with fresh people. In this way the distinction gains in potency for the individual, being part of a living culture deserving of the term *knowing*. The word *exercise* is included because such tasks can be used again and again, possibly with variations, as a reminder, as experience by means of which to refresh sensitivity to the associated distinctions and hence to regain access to associated actions.

Two Task-Exercises

The following two task-exercises are designed in the first instance to provide experience in which it may be possible to catch something of how attention shifts, and how there are different ways of attending to the same thing. Later, additional commentary is offered with further distinctions associated with both mathematical and pedagogical actions.

Task-Exercise 1: Productive Difference A
Notice that $82 \times 54 - 84 \times 52 = 60$ and that $73 \times 45 - 75 \times 43 = 60$

Initial 'Negotiating' Comments on Attention

What struck you during your activity?

The words of the task-exercise already invite *discerning of details*. However it is very possible that your eyes drifted past the numerals without paying much heed. In other words, you may have briefly gazed at the arithmetic without engaging with it, *holding it as a whole* without immediately probing further. You are aware of the presence of arithmetical

82 x 54 – 84 x 52 = 60	
73 x 45 – 75 x 43 = 60	

82 x 54 – 84 x 52 = 60	
	73 x 45 – 75 x 43 = 60

Fig. 1 Two grids

statements, but not *present to* them. Alerted by this comment, or perhaps earlier, you may have begun to discern some details (two two-digit numbers multiplied together and then subtracted from two other two-digit numbers multiplied together).

Reading the statements out loud or sub-vocally may be enough to draw attention to some repetitions. These details are more likely to emerge when you compare the two statements than simply from considering only one of them, because your brain is alert to invariance in the midst of change, a ubiquitous mathematical theme. For example, when the two 'facts' are displayed adjacently as on the left in the next figure, or provocatively as part of a grid as on the right, there is likely to be an immediate sense of relationship, even of 'filling in' or further instantiating that relationship (Fig. 1).

You may have *recognised a relationship* between the numbers (the digits in the second numerals are re-arrangements of the digits in the first two numerals), again enhanced by comparing the two arithmetical statements. On closer inspection, possibly alerted by the re-occurrence of 60 as the purported answer (at what point did you actually check the arithmetic?) you may have looked more carefully at relationships between the details of the two calculations.

It is possible that you 'saw' (re-cognised) that the digits, being transformed from the first part of the calculation to the second, meant that some parts of the multiplications were not necessary. This is a local form of reasoning, working with the particular numbers, and can be done with no thought whatsoever about possible generalisations.

If at any time you wondered whether this was an instance of some more general property, the nature of your attention shifted or was on the edge of shifting to seeking or *perceiving a property* as being instantiated, and either reached or justified by *reasoning on the basis of agreed properties* (properties of arithmetic), as will be developed later.

Distinguishing Forms of Attention

The initial comments make use of a number of phrases which I use as labels for distinctions between different ways to attend to something (Mason, 1998, 2003). These are

Holding Wholes or Gazing

Discerning Details

Recognising relationships and reasoning about them

Perceiving Properties as being instantiated

Reasoning on the basis of agreed properties.

Being confined to text, the best I can do is to use those terms as labels for experiences that readers may have had, or may recognise in retrospect when reading my commentary, even if they had not specifically noticed them at the time. Even if those labels help make sense of past experience or are 'recognisable' as distinctions (notice the self-referent or recursive nature of what is happening at the moment), they have yet to be associated with pedagogical actions.

The idea, borne out in my experience, is that becoming aware not only of what I am attending to, but of how I am attending to it, enables me to try to work at discerning both what learners are attending to, and how. If there is a mismatch, then what I am saying or doing may simply pass learners by, whereas if I pause and take care over my language, gestures and other ways of pointing and referring, I may get into sync with how they are attending, or better, attract their attention to take the form that seems to me to be mathematically productive.

For example (Huang & Leung, 2017) observed a teacher who, intending to work on the use of *vertical angles (vertically opposite angles) being equal* and *alternate angles being equal* spent time with learners first drawing their attention to the presence of vertical and alternate angles without even considering their equality, and then to finding instances of opposite and alternate angles in increasingly complex diagrams. This seemed to enable the learners to recognise, and later to apply and use the 'fact' about opposite angles being equal. Enabling learners to internalise the construct fully, integrating it into their functioning and sensitising them both to notice it and to act upon it was in contrast to a situation I observed in a different country in which even though learners could complete the chant "opposite angles are …", they neither recognised opposite angles in a diagram, nor thought to apply the 'fact'. Notice how this description of a pedagogical choice mirrors the phenomenological approach being taken in this writing.

The shift from recognising a relationship to perceiving a property as being instantiated is the fundamental shift of attention that underpins mathematical thinking, not to say thinking in other domains as well. I conjecture however, that many learners never really experience that shift sufficiently often for it to become part of their repertoire of actions, part of their ways of attending to things.

Before commenting more fully on Task-Exercise 1, consider Task-Exercise 2, perhaps alerted to shifts of attention.

Task-Exercise 2: Productive Difference B

In the grid on the right:
Multiply the numbers along the rows and then add the results;
Multiply the numbers down the columns and then add the results
Subtract the sum of the column-products from the sum of the row-products
Now construct a similar two-by-two grid but for which the answer is 7

8	4
2	5

Initial 'Negotiating' Comments on Attention

> I deliberately did not provide a 'worked example', in order to increase the possibilities for catching shifts in the focus of your attention, and in the nature of your attention, that is, of how you were thinking.
>
> What struck you during your activity?
>
> You may have noticed that whatever your propensity in the first task concerning actually carrying out the arithmetic, here the calculations are much simpler and so you are quite likely to have carried them out. Even so there was quite possibly an initial brief moment in which you gazed, *holding the whole* of the two-by-two grid without yet acting upon it. You probably shifted your gaze back to the words in order to recognise the relationship between the instructions and the actual grid itself. Perhaps then you calculated.
>
> You may or may not have noticed, perhaps alerted by the similarity in the titles of the task-exercises, that the digits used are the same as in task 1, or you may have gone further and recognised the row and column products as one tenth of the cross terms in the first arithmetical statement in task 1. This might have led you to construct a similar grid corresponding to the second arithmetical statement in Task 1, an action which would entail a shift from *recognising relationships* to *perceiving a property being instantiated*.
>
> You won't be surprised then that the answer is 6 in task 2 compared to 60 in task 1. If you are confident in algebraic manipulation you may even have *reasoned* out algebraically what is going on (more on this later).

It is worth while pausing and reviewing your sense of shifts both in the focus of your attention, and in the form or nature of your attention. Opportunity for people who are trying to make sense of what is said and done, for example, learners in classrooms, to pause and re-narrate what they have just seen and heard for themselves is immensely valuable as a contribution to learning.

Part of the second task was a reversal, setting a target for the result and asking for a two-by-two grid that would provide that result. The very notion of a functional connection between a two-by-two grid and a 'result' is a form of perceiving a property which can be instantiated, and is a different way of thinking compared to the initial encounter with the grid, where it is not clear what structural relationships might link a grid and its 'result'.

Associated Actions and Predictions

Recognising something of what the commentary so far has tried to highlight, namely shifts in the focus and nature of attention, is an instance of making distinctions. But as already mentioned, distinctions alone are simply distinctions. For a teacher they do not in and of themselves lead to actions, and for a researcher they do not in and of themselves contribute to 'analysis' of data.

I maintain that for teachers it is essential, if distinctions are to be of any use, that there be associated pedagogic actions (and for researchers, research actions) which become available in similar situations, that is, in what are deemed to be the 'same phenomena'. I include within pedagogic actions specifically *didactic actions* which are topic specific. How might a teacher, alerted to the issue of attention focus and to different ways of attending to something, act so as to reveal something of what students are attending to, and how? What can a teacher do when they recognise that students are not attending to the same thing, or not attending in the same way?

The first thing is to be aware of what they themselves are attending to, and how. For example, embarking on a worked example, it may not be sufficient to work through the example correctly and explicitly step by step. As someone familiar with the procedure(s) being used, the teacher is likely to be experiencing the example as an instance of something more general; numbers and other objects may be attended to as parameters, as place holders whose values are changeable and so are not in sharp focus, whereas students may not be distinguishing what is changeable and what is structurally part of the procedure. They are likely to focus sharply on the numbers themselves, not seeing them as place holders. This is articulated in the *principle of variation* (Huang & Lee, 2016; Marton, 2015; Marton & Booth, 1997): what is available to be learned is what has been varied in proximal space and time, or possibly, what has been relatively invariant while other aspects are varying in proximal space and time. That is why several worked examples may be required so that students become aware of aspects that can change (dimensions of possible variation). Students also need support in working out the range of permissible change for various parameters: not all values are likely to be possible, and there may be constraints among various parameters. For example, the coefficients of quadratic polynomials which have real roots must satisfy a constraint.

It is also the case, and well researched, that what matters in a worked example is not only 'what to do next' at each stage, but 'how you know what to do next' (Chi & Bassok, 1989). It may require on-going effort to integrate this awareness into a teacher's functioning and it may require additional effort to bring it back to the surface so as to be articulated. Yet it is extremely important that student attention is informed by attending in appropriate ways as well as to appropriate details.

Being aware oneself of what can change and over what range makes it possible to construct sequences of examples or exploratory tasks so that learners might experience what can change and in what ways. However, experience alone is not always sufficient. Most students require more than this. Just because something is available to be learned does not mean that it is indeed learned. Some form of explicit reflection, whether simply prompted by the teacher, or rehearsed by them, may be necessary. So associated pedagogic actions include listening-to what students are saying and watching what they are doing, not simply listening-for what you would like to hear and looking-for what you want to see (Davis, 1996), and prompting withdrawal from action and reflection upon that action in the form f personal narratives or 'own-explanations' (Chi & Bassok, 1989): what was effective, what ineffective; what can change and in what way.

These distinctions between different ways to attend to something may help make sense of moments in classrooms when students become 'lost'. If they are not attending to what the teacher is attending to, then communication between teacher and students is likely to be disrupted. For example, if the teacher is talking about the second calculation but students are concentrating on the first, there is likely to be confusion. Even when students and teacher are focusing on the same detail, if the teacher is talking about or acting on the basis of recognising a relationship between elements but students are simply trying to discern which elements are being talked about, it is likely that students will not be appreciating 'what the teacher is talking about, or doing'. Similarly, if the teacher is acting on the basis of perceiving a general property as being instantiated but learners are thinking in terms of trying to recognise what relationships are involved, students are unlikely to appreciate the generality and hence the implications of the class of similar examples, and so there is likely to be a breakdown in communication.

Extended Commentary

In order to be as clear as possible, it is worth underlining the process so far. Having offered a pair of task-exercises which from experience I know are likely to provide access to shifts between forms of attention, participants are invited to bring to mind what struck them during the exercise, and then to link it to any ensuing commentary (here, my text).

What someone notices while engaging in these two tasks is an indication of the sorts of things they are currently sensitised to notice, sensitised to become aware of, when doing arithmetic, at least in the context of reading a chapter in a philosophy book. That *awareness* is what enables action, whether mathematical or pedagogical. So engaging in reflection, preferably with others, and as a last resort, with my commentary, is a way of educating awareness so as to enable possible actions to become available in the future in similar situations. That is the core of the discipline of noticing (Mason, 2002).

There are several other things to notice when engaging with the two tasks, so in case attention was not your principal experience, here are some other things that might have struck you, though this is by no means an exhaustive list.

Having conjectured or seen a connection between the first arithmetical statement in task 1 and the grid in task 2, there may have been a temptation or inclination to construct a corresponding grid for the second arithmetical statement. This could have come from innate curiosity, or from a desire to check out a conjecture that was beginning to form, or perhaps simply to see if there was some underlying relationship. I see this as a form of *specialising* (Mason, 2012; Mason, Burton, & Stacey, 1982/2010; Pólya, 1962). It takes one of several forms: it may be in order to check a conjecture or to seek out underlying relationships, or it may be experienced as a form of tentative exploration based on a disposition to seek underpinning structural relationships, in contrast to a propensity simply to shrug shoulders when 'things happen' in arithmetic.

The use of the 'undoing' task of constructing a grid with a specified result may draw attention to the possibility of trying to work out what is going on. It is an example of a pedagogic action that can be initiated when designing a task sequence for students. Often participants try grids more or less at random, until they are stopped and invited to pay particular attention to the form of the calculation instead of doing the calculation itself.

Consider the first task-exercise again, depicted geometrically:

There is something about the diagram which invites ignoring the specific numbers, influenced at least partly by the schematic rather than pictorial nature of the diagrams. Concentrating on the fact that the two shaded areas in each figure are the same, you can 'see' (in the fullest sense) that the difference depends only on the comparison of the two white rectangular areas. Using a diagram to depict known relationships is another pedagogic action, one which could be taken up by students for themselves, and integrated into their functioning as a mathematical action. Contemplating a diagram can provide access to relationships that are not immediately discernible symbolically. However it is also the case that some relationships require symbolic processing of a diagram. For example, it is not immediately obvious from the diagram in Fig. 2, nor is it easy to augment the diagram so as to make it visually apparent, that $\frac{AZ}{ZB} + \frac{AY}{YC} = \frac{AP}{PX}$, that $\frac{AP}{PX} + \frac{BP}{PY} + \frac{CP}{PZ} = 2$, or that $\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = 1$, among other properties.

Arithmetically the diagram in Fig. 3 can be seen as depicting

$$
\begin{aligned}
& 82 \times 54 - 84 \times 52 \\
= & (80 + 2) \times (50 + 4) - (80 + 4) \times (50 + 2) \\
= & 80 \times 50 + 80 \times 4 + 2 \times 50 + 2 \times 4 - 80 \times 50 - 80 \times 2 - 4 \times 50 - 4 \times 2 \\
= & 80 \times 4 + 2 \times 50 - 80 \times 2 - 4 \times 50 \\
= & 80 \times (4 - 2) + (2 - 4) \times 50 \\
= & 80 \times (4 - 2) - (4 - 2) \times 50 \\
= & (80 - 50)(4 - 2) \\
= & (8 - 5)(4 - 2) \times 10
\end{aligned}
$$

Fig. 2 How can ratios be depicted?

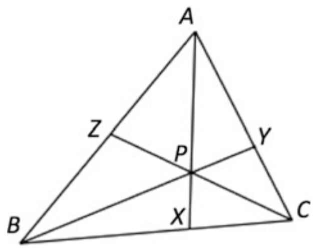
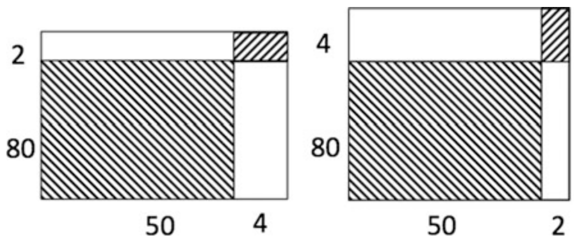


Fig. 3 Depicting products as areas 82 × 54 and 84 × 52



The calculation need go no further! The last line expresses the underpinning structural relationship between the starting digits and the result. The format of the calculation is an example of the mathematical act of *tracking arithmetic* (Mason, 2016; Mason, Johnston-Wilder, & Graham, 2005) which was inspired by an observation of Mary Boole (1909; see also Tahta, 1972) that if you can check whether a number is an answer to a problem, you can "acknowledge your ignorance" by denoting the answer with a symbol and then use that symbol to carry through the calculations to end up with an equation or constraint. After all, a numeral is also a symbol, not the 'thing itself'. To acknowledge your ignorance with a symbol, you track arithmetic, that is you isolate some numbers and simply express calculations in which they are involved without actually doing those calculations. Thus here the initial digits remain isolated until the last moment.

Algebraically, the 8, 2, 5 and 4 can be seen as placeholders for any digits. By tracking arithmetic in this way it is easier to "see the general through the particular" (Mason & Pimm, 1984). When the last calculation is seen as $(8 - 5) \times (4 - 2) \times 10$, the full generality appears, because in the second statement, $73 \times 45 - 75 \times 43 = 60$ is similarly presented as $(7 - 4) \times (5 - 3) \times 10$.

In the second task, carrying through the calculations while 'tracking' the digits leads to

$$
\begin{aligned}
& (8 \times 4 + 2 \times 5) - (8 \times 2 + 4 \times 5) \\
= \; & 8 \times (4 - 2) + (2 - 4) \times 5 \\
= \; & 8 \times (4 - 2) - (4 - 2) \times 5 \\
= \; & 8 \times (4 - 2) - 5 \times (4 - 2) \\
= \; & (8 - 5) \times (4 - 2).
\end{aligned}
$$

Now you can 'see' what an algebraist would 'see' having used letters in place of the grid-cell entries: what matters is the differences on the diagonals of the grid. Seeing through the particular to the general raises questions about the range of permissible change in the numbers … any numbers can be used. Using numbers as place-holders, or using more familiar algebraic reasoning is a fifth form of attention, *reasoning on the basis of agreed properties*. One of the things which makes proof so difficult to teach is that until learners are familiar and confident with expressions of generality, with the notion that it is only when things are known (characterised) by their properties that you can reason with them, learners have little with which to make sense of reasoning.

The link with Task 2 is likely to be strikingly evident by now. Furthermore, this structural relationship makes it easy to construct not simply one but all possible two-by-two grids with a given result, not even confined to single digit entries: notice that in Task 2 there is no requirement that the entries be single digits, though the way the task was set might have induced this assumption. Only by drawing attention in some way to the range of permissible change is the extended generality likely to emerge. A useful question to ask however is what the corresponding statements in task 1 would have to be if the digits were permitted to be numerals larger than 9, leading potentially to a deeper appreciation of the nature and role of place-value.

Juxtaposing the two arithmetic statements one above the other facilitates comparison, the natural arising of the potent question 'what is the same and what is different'. This is both a pedagogical action, and a mathematical action, and as a mathematical action it is worthwhile for students to internalise and make use of it for themselves. How to achieve this pedagogically is an instance of *scaffolding* and *fading* (Brown, Collins, & Duguid, 1989) with its associated pedagogical actions of using increasingly indirect prompts so as to encourage students to become aware of it, and thus to internalise it as a mathematical action (Love & Mason, 1992). The notion of *Zone of Proximal Development* (ZPD) introduced by Vygotsky (1978) and brought to the west by Wood, Bruner, and Ross (1976; see also Bruner, 1966) was, according to van der Veer and Valsiner (1991), intended to highlight actions which learners can carry out when cued, and are on the edge of being able to initiate for themselves, which is the mark of internalisation and integration.

The invitation to construct your own grid with a given answer can be seen as an instance of the ubiquitous mathematical theme of *doing and undoing* (Gardiner, 1992; Mason, 1988; Mason et al., 1982/2010). There is an opportunity to explore which other grids will give the same answer, and given an answer, how to construct a corresponding grid. These questions are mathematical actions which can contribute to deeper appreciation and comprehension of mathematics in any topic. Furthermore, being stimulated to construct their own examples, their own instances, can enrich learners' example spaces (Watson & Mason, 1998, 2005) as well as contributing to a shift from passively *assenting* towards actively *asserting* or conjecturing (Mason, 2009).

Of course, someone with more experience will have immediately recognised the arithmetical statements as connected with the awareness that when the sum of two numbers is constant, the product increases as the two numbers become closer to being equal. This task could be used as an introduction to that awareness. The action associated with the awareness, with recognising that the sum of the two numbers in the products is the same, is turning to the pair with the smallest difference in order to find the largest product.

Another possibility concerns what you noticed about how your recent experience with the first task framed or influenced your thinking in the second task. You may have found that you quickly moved to algebra, or perhaps a form of tracking arithmetic, encouraged by a sense of its recent success in the first task and supported by even a faint recognition of the same digits being used. Or you may have found that you treated the two tasks independently. This is of course rather hard to trap because it is very difficult to be aware of an uninterrupted flow.

Whatever personal experience has been resonated (or perhaps dissonated) by my comments, you may have been freshly re-minded of the power of 'Same and Different' as both a mathematical and a pedagogical strategy (Brown & Coles, 2000). This is really another version of the mathematical theme of *invariance in the midst of change*, which is entirely natural because it is the basis for how all of our senses work: contrast between what is changing and what is (relatively) not changing, or put another way, with stressing and consequently ignoring. This is the basis for variation theory both in its Gothenberg form (Marton, 2015) and its

Confucian Heritage form associated with the *I Ching* (Wilhelm, 1950). Sun (2011 p. 68) notes that the *I Ching* philosophy embraces the notion of "abstracting invariant concepts from a varied situation and applying these invariant concepts to the varied situations".

Theoretical Considerations

The previous section used two task-exercises to generate experience which could then be referred to in commentary. Distinctions were made between five different forms or structures of attention. Those distinctions were made using words whose purport was imagined by me as author to refer to what readers might have noticed or to what might have been available to notice in retrospect. That is, to become aware of. I have deliberately not provided specific definitions. Meanings arise from usage, and this works in a community as people offer what they conjecture may be instances or examples, and others discuss this, so that over time a rich collection of examples accrue around a label.

The distinctions between different ways to attend, or different ways of thinking while attending to some 'thing', were not at first associated with actions. This is also what is missing in most research reports in mathematics education. People make finer and finer distinctions, delineating differences between what children, teachers and teacher educators say and do, but rarely are these linked to effective pedagogic actions. By themselves, distinctions are 'academic'. They may assist in analysing 'what is going on, according to the observer', but they are only of use if they can be used either to predict outcomes of actions, or to inform future actions. In both cases this means linking distinctions, things that may be noticed, with both possible actions as response, and consequences of not acting. Such consequences may be positive, negative or neutral, and situation dependent. In the case of forms of attention, the distinctions can be used to predict or explain breakdown in communication.

In the extended commentary I drew upon the literature to invoke other distinctions, being careful to associate them with possible actions. I chose to be explicit about these in order to underline the importance of associating actions and consequences with those distinctions. Some actions were specifically pedagogic, being something a teacher could choose to initiate. Others were both pedagogic and mathematical, in the sense that it could be to students' advantage to integrate those actions into their own functioning, raising the question of how to go about promoting such integration, which brings us back to scaffolding and fading, and the ZPD.

Classifying

To classify is to make a distinction, to discern a difference. What really matters is to focus on "differences that make a difference" (Bateson, 1973). First some event, some action, some episode has to be discerned, separated from the flow of sense impressions and associations. What is and what is not part of the episode, the phenomenon, the action has to be discerned. Discernment is an ontological act for it brings what is discerned into existence as a 'thing' in itself. Boundaries are drawn, however fuzzy and imprecise, between this and not this, between this and that. Its principle features which make it 'what it is' may be experienced, reviewed, or expressed. This is what is meant by *holding wholes* or *gazing*, and is closely aligned with the van Hiele level 0 (van Hiele, 1986).

The very act of stressing certain aspects and consequently ignoring others, which Gattegno (1970a, b) identified as the precursor to generalising, requires the discerning of details, and is aligned with van Hiele level 1. The specific act or episode involves or is constituted by relationships amongst discerned details. Recognising these relationships as being important aligns with van Hiele level 2. Those relationships become an example of a phenomenon which might occur again in the future, when and only when there is a shift of attention from recognising relationships in the particular situation, to perceiving those relationships as an instance of a phenomenon, or in mathematics, as properties that are being instantiated, in alignment with van Hiele level 3. This calls upon the natural action of recognising what is the same and what different about two or more 'things', which often takes place below the surface of consciousness, because it is the basis for sense impressions.

Sensations depend on something changing against a background of (relative) invariance, or something relatively invariant against a background of change. However 'change' already requires a sense of relatedness in contrast to a sequence of random unrelated events. It is important to note that whereas van Hiele levels are seen by researchers as developmental levels, considered as forms of attention they are not levelled: attention can remain in one form for very short or for extended periods of time, and shifts back and forth between forms are common, necessary, and to be expected. There is no 'developmental' aspect, since expert and novice attention make such shifts, though perhaps in different ways.

A *phenomenon* is an abstracted 'sense' of some event, incident, thought, emotion, form or shift of attention: some action whose essence is likely to be, or has been recognised as having been, repeated. Identification of a phenomenon as a generality is enriched by using a label. There is often a quick slip between the salient details of a specific incident and the sense of a phenomenon, in which certain elements of the particular are stressed and taken as the defining characteristics of the phenomenon. Having a suitable label enhances and enriches this process and makes it more accessible for reinforcement in the future (Mason, 1999; Mason & De Geest, 2010). This is in essence the articulation of the principle of variation by Marton (2015). Variation sometimes draws attention to what is varying against a background of relative invariance, and sometimes to what is relatively invariant against a background of what is changing. It is no surprise therefore that most mathematical theorems (apart perhaps from pure existence theorems) can be cast in the form of proving that some properties must hold, must be invariant when other relationships are changing in specified ways.

When details are discerned which generate some 'sense' of repetition or repeatability, a phenomenon can emerge. A succinct way to put this is that "Thinging leads to is-ness, and is-ness leads to thinging" (Mason, 1987). In other words discerning details isolates certain features which are experienced as features of 'a thing', and promoting a label as if it referred to some 'thing' brings that 'thing' into existence. Its ontological status is established.

As with other powerful constructs, there is an almost paradoxical circularity. In order to recognise some 'thing' as an example or instance of some property, it is necessary to appreciate and comprehend that property; but in order to appreciate and comprehend a property it is usually necessary to have experienced some examples, some instances. Similarly, in order to recognise relationships which underpin a phenomenon and thus to discern a phenomenon as such, it is necessary to appreciate and comprehend the phenomenon as a phenomenon, but in order to do this it is necessary to have encountered the phenomenon in particular instances. There is a great difference between being told of a distinction by someone, and coming to make that distinction for yourself, in order to be sensitive to it in the future.

The opening two 'phenomena' at the beginning of this chapter may not be phenomena for readers who have not experienced either situation. Nevertheless they may be recognisable as possible if not probable, or they may be challenged by a sense of "it need not always be like this" or "I had (or would have) a different experience".

How then is the ontological status to be challenged and-or verified or validated? A great deal of mathematics education research literature which claims to be 'using such and such a theory' simply uses theoretical distinctions to discern details in the data, in the name of analysis. Verifying that others can make the same distinctions is a small contribution, perhaps, but if these distinctions are not connected to possible actions or consequences, and if there is no evidence for the effectiveness of those actions or the reliability of the consequences, short term or long term, then it is not at all clear what is actually being offered. In this I take the stance of the Discipline of Noticing (Mason, 2002) in that validation is a task for the reader, to sensitise themselves and to notice opportunities to consider acting in fresh ways.

Noticing and Acting

The etymology of noticing lies in 'becoming aware of, or making known' and hence 'making distinctions', and 'discerning'. Making fine distinctions may be a mark of sensitivity to notice, but a cycle of noticing and making ever finer distinctions becomes worthwhile only when the distinctions are associated with and make available particular actions that might otherwise have remained dormant, or inert (Whitehead, 1932).

It is tempting to say that 'actions come to mind', but this is only appropriate if 'mind' is interpreted very broadly. More precisely, actions become available to be enacted, often below the surface of consciousness, in what has been called *System 1* of a dualistic *dual systems theory* popularised by Kahneman (2012). Kahneman recognises Systems 1 and 2 (*S*1 and *S*2). *S*2 involves conscious consideration. A more nuanced analysis arising from ancient psychological observations suggests at least four 'systems' by means of which actions are initiated or enacted: *S*1 (automaticity), *S*1.5 (predominantly emotional), *S*2 (predominantly cognitively considered), and *S*3 (informed and inspired through holding oneself in front of some situation, being present to it, silently contemplating or meditating, or holding it in the Light). To develop these distinctions here would take me away from the focus of this chapter, but see Mason and Metz (2017).

In mentioning Csikszentmihalyi (1997) earlier I was bringing to mind that flow can be both positive and negative. It is possible to be so caught up in flow that you are unaware of the passing of time, oblivious to activity around you. You may be 'lost to the world'. Some philosopher-psychologists, close observers of human nature in both the East and the West, have suggested that most of the time we are all 'lost to the world' in the sense that our 'flow' is a sequence of habitual actions carried out mechanically. There is 'no-one actually present' (Raymond, 1972; Vaysse, 1979). It is also possible to be in flow but with an inner witness, an inner observer being present, so that we are "in the world but not of the world" (Rumi, 1999). Our "unsteady thoughts are guarded" (Dhammapada, quoted in Bennett, 1943, pp. 103–104). Gattegno (1987, p. 44) observed that "One may be aware for a split second that there is more than content to relate to [to be aware of]". The witness observes without acting, as described in the Rig Veda stanza:

> Two birds, close yoked companions,
>
> Both clasp the self-same tree.
>
> One eats of the sweet fruit;
>
> The other looks on without eating.
>
> (Translated in Bennett, 1964 p. 108; see also Rhadakrishnan, 1953 p. 623)

Here the second bird is the witness, the voice that suddenly asks "why are we doing this?" or "might there be a better way?", which has also been referred to as the *executive* (Schoenfeld, 1985) and the *monitor* (Mason et al., 1982/2010). When the witness is awake, when "thoughts are guarded", it is possible to act freshly, non-habitually. This is what the Discipline of Noticing (Mason, 2002) sets out to make possible.

The fact that actions are often enacted without cognitive processing and without the use of will or intention may be why Gattegno (1987) uses the term *awareness* to refer to subconscious as well as conscious sensitivities. Furthermore it is why I think he used the term to mean 'that which enables action'. For example, your soma is 'aware' of when the nostril that is handling the majority of inflow of air when you breath has become engorged, leading to the action of switching nostrils. The same applies to many aspects of your soma, such as pulse(s), skin pore contraction and so on. Similarly there is an awareness which may or may not be conscious that, for example, if you purchase several items then the total cost is the sum of the costs of the individual items (not including bulk discounts etc.): it seems so obvious, so natural that it does not impact on your consciousness. The action of 'adding', like that of switching nostrils, changing pulse or opening or closing pores, is enacted.

Along similar lines, Hudson (1968) used the metaphor of 'frames of mind' in his study of the psychology of schoolboys and their preferences for science or the arts, and Minsky (1975) used the metaphor of 'frames' in a cognitive science sense of an action which is 'fired' (enacted) when various parameters or inputs have been assigned values, some of which may have default values. Thus a frame may 'fire' without any conscious filling in of input parameters because all the parameters have been assigned values.

Sensitising and re-sensitising, and integrating associated actions into a repertoire of available actions is, I believe, what Gattegno (1970a, b, 1987; see also Young & Messum, 2011) referred to as *educating awareness*, which comes about by 'integration through subordination' (Gattegno, 1970a, b). Gattegno (1970a, b) coined the challenging statement: *only awareness is educable*, and I have come to the conclusion that what he was referring to was two-fold: that awareness, sensitivity to notice, is what can be educated, can be learned; and that educating awareness includes internalising associated actions so that when in the future you become aware of some situation, various associated actions from your repertoire become available to be enacted.

> Gattegno's claim that "only awareness is educable" gains in potency when it is contrasted with parallel assertions obtained from the ubiquitous chariot image for the human psyche, found in the Upanishads (Rhadakrishnan, 1953, p. 623) and other more recent writing (e.g. Gurdjieff, 1950; Ouspensky, 1950). These are "only behaviour is trainable" and "only emotion is harnessable". Training behaviour (reflected in the commonplace "practice makes perfect") makes an important contribution to personal development, but by itself is inflexible. Harnessing emotion is what is encompassed under motivating (note the common etymology) but also includes a shift from passively assenting to actively asserting (Mason, 2009), of course in a conjecturing manner (Mason et al., 1982/2010), and the development of a positive disposition to making use of one's natural powers in order to think mathematically (Kilpatrick, Swafford, & Findell, 2001). A more complex discernment of the human psyche augments the famous triple (cognition, affect and enaction) with attention, will and witness. The corresponding only might be "only attention can be directed", "only will initiates intentionally" and "only the witness is impartial".

Sensitivities to notice, and actions to be enacted, are integrated into someone's functioning not through mindless repetition but through subordinating cognition and affect to action, that is, through immersing oneself in an action and then withdrawing from that action to construct a personal narrative for, a self-explanation of, what was effective. It is important to bear in mind that consciousness itself is a "user illusion" (Norretranders, 1998), a 'tale told by an idiot …' (Shakespeare in Macbeth). Consciousness is always a late entrant into a situation when the witness is asleep, which is why people tend to "tell more than they know" (Nisbett & Wilson, 1977).

Although James (1890, p. 224ff) popularized the phrase "stream of consciousness", it seems that most often not only do we recall experience in disconnected fragments which we try to glue together into a continuous narrative, but experience itself is fragmentary (Mason, 1998, 2002). The stories we tell ourselves are usually post hoc retrospective accounts by cognition, anxious to assert the role of consciousness in enacting actions and so maintain the 'user illusion'.

From these considerations a phenomenological stance is for me the only coherent epistemological stance that contributes to professional development, to change and growth and yet takes into account the structure of the human psyche.

An Epistemological Stance

How do we come to experience that state which is commonly described as 'knowing'? My answer is that we discern details, we recognise relationships, and we perceive properties as being instantiated in the flow of experience. Experience itself is a succession of stimulated neurons. These initiate muscle contraction among many other somatic functions. Flows of energy are experienced as emotions and drives, usually through metonymic actions, generating thought patterns and images, usually through metaphoric resonance. Characteristic flows of energy activate *micro-identities* (Varela, 1999) also known as *frames of mind* (Hudson, 1968; Minsky, 1975), and as *polyphrenia*, otherwise known as *multiple selves* (Bennett, 1964) while Walt Whitman (1892 stanza 51) referred to a similar idea with the assertion "I contain multiplicities". To avoid unhelpful associations of mental disorders with *multiple selves*, it is convenient to think in terms of characteristic *adherences* and *co-ordinations* amongst enaction, affect, cognition, attention and will, as observed by the witness (Mason & Metz, 2017). All this is summarised in the expression "have come to mind", where *mind* includes all aspects of the psyche, not only behaviour, emotions and cognition, but also attention, will and witness (Mason & Metz, 2017); As Illeris (2003) points out there are three domain of influence and action: cognitive, emotional and social, or put another way, social sensitivities develop proclivities for certain patterns of energy flows, of certain selves or adherences, of micro-identities.

'Knowing' is then the experience of recognising, of perceiving, which means making a distinction between 'this and that', or in other words, discerning detail informed by and associated with past experiences. Labels (words and images) can act as axes around which experiences recognised as similar can accumulate (Mason, 1999). A possible image is of a supersaturated solution of salt or sugar with some introduced impurity (piece of string or a wire shape) around which crystals form.

Such an epistemological stance privileges experiences and associations, whether imagistic, kinaesthetic verbal or some combination thereof. It also values self-explanation (Chi & Bassok, 1989; Hodds, Alcock, & Inglis, 2014) and personal narrative construction. This can be summarised in the term 'phenomenological'.

A Phenomenological Stance

I am interested in my lived experience of posing and thinking about the mathematical problems, in developing my mathematical thinking. This leads me to be interested in the lived experience of other people when mathematics is involved, whether when thinking mathematically, when provoking and supporting others to think mathematically, or when working with educators on provoking and supporting others who are themselves working with learners. Furthermore, it is a mystery to me as to why this is not the stance espoused as well as enacted by everyone who works with others on mathematical thinking, because it seems evident that it is in the doing that there are opportunities to learn. People like to quote the adage that

> "I hear and I forget, I see and I understand, I do and I remember",
>
> often attributed incorrectly to Confucius, and possibly confused with
>
> "Not hearing is not as good as hearing,
>
> Hearing is not as good as seeing,
>
> Seeing is not as good as knowing,
>
> Knowing is not as good as acting;
>
> True learning continues until it is put into action."
>
> [Attributed to Xun Kuang].

Put another way, talking about how to think mathematically is about as useful as talking about how to ride a bicycle. Bicycle riding is based in bodily awarenesses not cognitive processing. The whole point is that the functions required to ride a bicycle are internalised through subordination of attention, allowing attention to prepare and direct where the bicycle is going rather than attending to the work of individual muscles and the act of maintaining a balance. As Idries Shah (1978) put it, "a solved problem is as useful as a broken sword on a battlefield", which I take to mean that it is through struggling to resolve a problem, it is through engaging in activity, it is in taking something on as a (personal) problem rather than keeping a task at arms length, that it is possible to develop one's mathematical thinking.

Reflection

Some parts of the previous abstractions may possibly resonate with some readers, but St. Augustine's quote is highly relevant and pertinent: from words you can learn only words. Words can be useful, but only when there is a negotiated common focus as to what is being referred to. To achieve this, examples seem to me to be essential, and these consist of some taken-as-shared experience (Wood, Cobb, Yackel, & Dillon, 1993). This is backed up in mathematics by the historical record, for from the earliest of written records that can be considered to be pedagogical, examples have always played a central role. But examples alone are not enough. "A succession of experiences does not add up to an experience of succession" (Mason, 1989, 1994), which turned out to be a variation on the observation by William James (1890 p. 628) that "a succession of feelings does not add up to a feeling of succession". Put another way, "One thing we do not seem to learn from experience, is that we do not often learn from experience alone. Something more is required." (Mason, 1998, 2002).

Research in mathematics education can be powerful and important when distinctions can be offered which lead to consequences or which inform future practice. Those distinctions are most effectively and efficiently communicated through a phenomenological stance in which experiences are shared, or taken-as-shared and negotiated as to pertinence. This process involves shifts not only in what is attended to (educating of sensitivities to notice, that is of awareness), but in how that is attended to, with associated potential actions or consequences. Sensitivities to your own attention provide indicators to possible actions which may promote others to attend to the same things in a similar way. This applies to researchers, educators, teachers and students. Education is about opening up suitable objects to attend to, and ways of attending to them.

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John Mason spent 40 years at the Open University, supporting students who wanted either to learn more mathematics, or to support the mathematical thinking of others. He continues in retirement to work on mathematical problems, to develop applets to help himself and others appreciate and comprehend mathematical concepts, and working with students, teachers and educators. He uses writing as a means for probing ever more deeply into the functioning of human beings in the context of mathematics teaching and learning, drawing on his own experience and observations and taking a phenomenological stance. John is perhaps best known as lead author of the classic text *Thinking Mathematically*, in print since 1982. He spent some 20 years articulating a way of working on oneself with colleagues (*The Discipline of Noticing*, Routledge, 2002), trying to capture the essence of Professional Development and the ways of working used in the Centre for Mathematics Education at the Open University, and based on what he learned from the Teacher, mathematician and mystic J. G. Bennett.

Using Rules for Elaborating Mathematical Concepts

Michael Meyer

Abstract This chapter focuses on an inferential view on elaborating concepts in mathematics classrooms. A framework is going to be presented and used, which helps to analyse and to reflect on the processes of teaching and learning mathematical concepts. The framework is based on Wittgenstein's theory of language-games and especially its core, the primacy of the use of words. Concerning the theory of inferentialism by Robert Brandom, the inferential use of words in language-games can be regarded as an indicator of the understanding of a concept. Together, the theoretical framework combines the role of judgements and their connections via rules in inferences in order to describe processes of concept formation.

Keywords Wittgenstein · Language game · Rules · Inferential use
Toulmin · Deduction

Introduction

A lot of research on communication in the mathematics classroom has been done. Mathematical interactions and especially processes of concept formation have been analysed from many different perspectives (e.g., Duval, 2006; Steinbring, 2006). By his theory of "language-games", Wittgenstein (1963) offers a pragmatic view on the introduction of concepts. Elements of his perspective have often been used to discuss aspects of communication in the mathematics classroom (e.g., Bauersfeld, 1995; Schmidt, 1998; Sfard, 2008).

Wittgenstein's concept of language-game is closely connected with the process of concept formation. It implies that words do not have a meaning by themselves. Therefore, a fixed, temporally lasting meaning of a word does not exist:

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> Naming is so far not a move in the language-game—any more than putting a piece in its place on the board is a move in chess. We may say: nothing has so far been done, when a thing has been named. It has not even got a name except in the language-game. (Wittgenstein, 1963: § 49)

Wittgenstein has a complex opinion on processes of concept formation, as he puts down the meaning of a word solely to its use:

> For a large class of cases—though not for all—in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language. (1963: § 43)

Thus, according to Wittgenstein, the expression of words does not constitute their meaning. Rather, it is the use of words which constitutes the meaning, and therefore, the use of words constitutes concepts.

The meaning of a word shows and manifests itself in using the word in language. This might be a reason for the fact that Wittgenstein does not define what exactly he understands as "language-games". He uses the word "language-game" by describing the use of this word (e.g., by giving examples). That way, he gives meaning to this word.

The theory of Wittgenstein of the attribution of meaning through the use of words is also closely connected with those of the language-game in another way. To elaborate on this, let us have a look at the concept of numbers: When students understand numbers as a quantitative aspect of objects, then they can use this understanding for calculating. But the handling of numerals is changing when numbers are regarded as ordinal numbers. Now, operations cannot be used in the same way anymore. When negative numbers are introduced, the comprehension of the cardinal aspect of numbers is not sufficient either. Each of these changes entails an alteration of the language-game. In the changing language-games, the same numbers can be used in different ways. The way of use determines the current meaning. However, a well-developed concept of numbers requires different kinds of comprehensions—that is different ways of use—which are connected by family resemblances (Wittgenstein, 1963: § 67; cf. Kunsteller, 2016).

> And for instance the kinds of number form a family in the same way. Why do we call something a 'number'? Well, perhaps because it has a—direct—relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres. (Wittgenstein, 1963: § 67)

The use of words in a language-game is by no means arbitrary. Rather, the use is determined by certain rules. These rules tell us how words can be applied:

> We can say that a language is a certain amount of activities (or habits) which are determined by certain rules, namely those rules that regulate all the different ways of use of words in language. (Fann, 1971: 74; my own translation)

Accordingly, observing the rules that determine the use of words, is a considerable feature of our linguistic acting. A rule has the function of a "sign-post"

(Wittgenstein, 1963: § 85), although, each rule can be interpreted in a different way. Within the mathematics education research, a lot of rules, which determine the language-game "mathematics education", have already been reconstructed. The patterns of interaction and routines, which were described by Voigt (1984), can also be counted as (combinations of) rules. For instance, the pattern of staged-managed everyday occurrences (in German: "Muster der inszenierten Alltäglichkeit") describes the "as if"-character of classroom situations, in which the students' extracurricular experiences are addressed: If the students make too much use of these experiences, the teacher is going to disregard this use and highlights the mathematical contents. Such rules make sure that the actions in class run smoothly by showing the agents, for instance, which actions they have to carry out, what they can achieve with them and where the limits of their actions are. Therefore, (some) rules are constitutive for the classes, particularly as they determine the use of words or rather sentences on the one hand and facilitate that the classes pass off smoothly on the other hand. In order to characterize classroom communication, Sfard (2008, p. 200) differentiates between "object-level", which regulate the behaviour of objects, and "meta-discoursive" rules, which regulate activities of the interacting persons (e.g., the pattern of staged-managed everyday occurrences).

Inferential Use of Words in Language Games

Following Wittgenstein, a concept can be developed, if different (rule-based) ways of using the relevant words are known. The definition of a word is just one possible way of using it. Knowing different ways of using a word includes, among other things, knowing and using sentences that go with them:

> "'Owning' a mathematical term requires to know more relations and to know more about the handling with the term than it is expressed in its definition. […] Proofs help to explain the terms' inner structures as well as to link concepts and with that to develop the purport of term." (Fischer & Malle, 2004: 189; my own translation)

We use words in situations of giving reasons for statements—also statements in which a particular word is used. For example, we can use "commutative law" to give reason for the similarity of 9 + 4 and 4 + 9. The aspect of reason of concept formation shows itself in the structure of the potential words' ways of use. Thus, every definition, for instance, has a conditional structure ("If…, then…"). Definitions are equivalence relations (or rather biconditional—"if and only if"), which are also used in arguments. In short: The words' meanings are arranged in an inferential way. The American philosopher Brandom (2000, p. 11) elaborated an inferential approach: "To talk about concepts is to talk about roles in reasoning.". The understanding of a word is described by Brandom as follows:

> Grasping the concept that is applied in such a making explicit is mastering its inferential use: knowing (in the practical sense of being able to distinguish, a kind of knowing how)

> what else one would be committing oneself to by applying the concept, what would entitle one to do so, and what would preclude such entitlement. (Brandom 2000, p. 11)

Brandom's approach focuses on commitments and entitlements concerning a concept in reasoning processes. This implicates that conceptual learning processes should not only be regarded in representational terms. Moreover the use of (elements of) concepts in inferences is essential (cf. Bakker & Derry, 2011). This view is in line with other common theoretical approaches in mathematics education, e.g.: By establishing and using the epistemological triangle Steinbring (2006) focuses on relations in the formation of concepts. A comparable approach, which highlights the role of relations in form of general rules has been elaborated by Meyer (2010), who presented the pattern of abduction in order to analyse processes of discovering mathematical coherences. In mathematics education Brandom's theory has been used several times in order to reconstruct learning processes (e.g.: Hussmann & Schacht, 2009; Meyer, 2014). Within the following paragraphs the role of rules in these processes is highlighted.

By combining Brandom and Wittgenstein we are able to say that an inferential use is carried out using arguments in situations of reasoning the use of words. To examine the students' corresponding arguments, the pattern of Toulmin, which is common in mathematical education research, is going to be used in this chapter. It helps to reconstruct also implicit contents of arguments. In accordance with this, an argument consists of several functional elements. Undisputable statements function as *datum* (Toulmin, 1996, 88). Coming from this, a *conclusion* (ibid.) can be inferred, which might have been a doubtful statement before. The *rule* shows the connection between datum and conclusion, and thus legitimizes the inference. If the rule's validity is questioned, the arguer could be forced to assure it. Within the reconstruction, such making safes are recorded as *backings* (ibid, p. 93) and can be realised, for instance, by giving further details about the field where the rule comes from. With regard to the theoretical considerations before, different important elements of the processes of concept formation can be recognized for the functional elements of an argument.

As an example for the analysis (cf. Meyer, 2010) by means of the pattern of Toulmin, the following fictitious remark of a student is reconstructed, which functions at the same time as an example for the inferential use of the concept *bigger*: "As 3 apples are more than 2 apples, 3 is bigger than 2." According to this statement that—talking about numbers of apples—there is a smaller-bigger relation (datum), it can be concluded that there is a relation of size between the relevant numbers (conclusion). The conclusion is legitimized by a rule which is only implicit and which can be supported by the reference to the aspect of cardinal numbers (backing). Accordingly, the following pattern can be reconstructed (Fig. 1).

Following Wittgenstein, by means of such an argument a relation between two concrete numbers is expressed. In certain language-games, such an argument is surely regarded to be valid. But introducing negative numbers at school means that such kind of use of the word 'bigger' is possibly no longer accepted. This change of

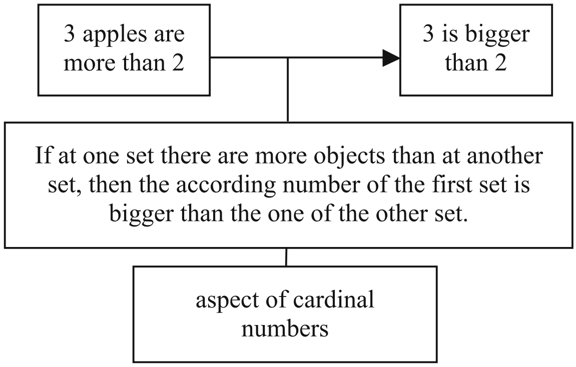

Fig. 1 Application of the pattern of Toulmin

the language-game causes a different use of numbers. Although the statement '3 apples are more than 2' is true, it does not mean that '−3 is bigger than −2'. If the rule is applied on negative numbers in this way, it loses its' validity.

With regard to the theoretical consideration above, different important elements of the processes of concept formation can be recognized:

- datum and conclusion consist of judgments, as a link between subjects and predicates,
- rules have a general character, in so far they connect general judgments in conditional or biconditional forms and
- the backings which are the basis for an argument.

In terms of entitlements and commitments the different functional elements can have different roles. E.g.: On the one hand, the rule can function as an entitlement insofar we can use it to infer the conclusion. On the other hand, we have to infer the conclusion by using this rule.

Accordingly, the enormous significance of concrete and (combinations of) general judgments for concept formation is shown: Concrete judgments (datum and conclusion) are linked via more general connections (rules). The possibility of this connection is based on the knowledge of a context, in which this connection is perceptible to the learners (backing).

Methodology

Following Wittgenstein and Brandom the use of words in arguments determines their meaning. Therefore, we have to analyse what kind of meaning a word gets in the classroom and, thus, we have to analyse social processes. Wittgenstein's philosophy enables a purely interactionist view on processes of concept formation which is a benefit for the interpretative researcher, particularly as we are not dependent on speculations concerning student's thoughts.

By analysing the students' "languaging" (Sfard, 2008) for mathematical concepts, the development and alteration of meaning by the use of the according words, we are able to reconstruct the social learning processes in the mathematics classroom. Therefore, the qualitative interpretation of the classroom communication is founded on an ethnomethodological and interactionist point of view (cf. Meyer, 2007; Voigt, 1984). Symbolic interactionism and ethnomethodology build the theoretical framework, which is going to be combined with the concepts of "language-game", "(inferential) use" and the functional elements of arguments. If the use gives meaning to words (in the interaction), then the (linguistic) action is the sole criterion for the reconstruction. Thus, we have to follow the ethnomethodological premise: The explication of meaning is the constitution of meaning.

The empirical data emerged from different classrooms in Germany: first a fourth grade (students aged from 9 to 10 years) and second a tenth grade (students aged from 15 to 16 years). Classroom communication has been videotaped and transcribed. The rules of transcription can be found in the appendix.

Using (Common Practice) Rules for Expressing Meaning

Teacher	Why do I fix such a picture on the blackboard? And why are these concepts written down on the blackboard? I have a reason to do so. Jonathan, it is your turn.
Jonathan	Because the painter has done everything in parallel, perpendicular and in right angles.
Teacher	You are right. You seem to know what parallel, perpendicular and right angle means. Maybe you can show it to us on the picture.
Jonathan	Perpendicular is this here (points first at a vertical, afterwards at a horizontal line). Parallel is this here (points at two vertical lines). A right angle is this (pursues two lines he previously called perpendicular).

By pointing to different things on the blackboard (Fig. 2), Jonathan makes use of the words "perpendicular", "parallel" and "right angle". He must have been in contact with practices of using them and thus with meanings of these words in a language-game outside of this classroom. In this situation of classroom interaction, the words get a meaning by him pointing at something. This use can be described as an exemplary use which is presented following a pretty common rule: If the teacher asks for the meaning of words, someone who knows them is going to point at examples on the blackboard. Right now, the presented extensions represent completely the meaning of the words. The use Jonathan makes of the words needs not to imply that those words could also be used in different ways, but this use, and respectively this meaning, get established in this classroom.

The teacher does not have any further questions and accepts the use of the words Jonathan must have known from another language-game. Thus, it seems that the exemplary use is an acceptable explanation and that the meaning of the words is "taken-to-be-shared" in the classroom (cf. Voigt, 1998, p. 203).

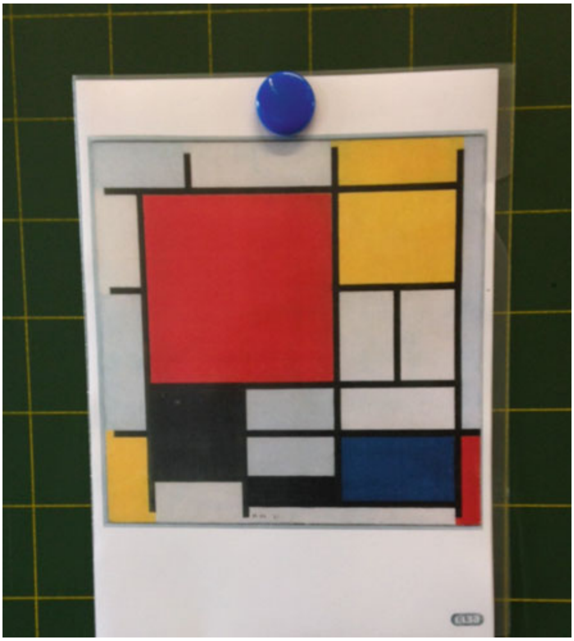

Fig. 2 Painting by Mondrian on the blackboard

After pointing at more examples (a longer interpretation of this scene can be found in Meyer, 2014) and introducing the symbol for right angles by the teacher, the scene continues:

Tim	Ah, this corner which is coming from the right side (*marks the angle with the teachers' sign*)
Teacher	Correct! Just make it a little bit thicker, so that the other ones can see it.
Tim	This is a left angle. (*points at the opposite side of the vertical line*)
Teacher	No!
Lisa	That is always a right angle.

Tim recognizes the examples as examples for the use of the word "right angle". He explains why John's example can be called "a right angle". Thus, he abstracts from the concrete example and presents a use of the word "right angle" by a kind of a "defining rule": The word "right angle" can be used, if a line for the angle comes from the right side. Tim tries to give an *explicit-definitional use* (cf. Winter, 1983) of the word: The student describes a general characteristic when and how the words "right angle" has to be used. He relates the words "right angle" to other words. Contrarily to the former use of "right angle", Tim tries to give a kind of definition and, thus, uses another ethnomethod to constitute meaning. Referring to the pattern of Toulmin Tim publishes the following inferential use of "right angle" (Fig. 3).

The concept of the word "left angle" is used by an implicit reference. It is implicit, because the pair of concepts "left-right" indicates that an orientation in space is considered—a relation between observer and object. Thus, the words "left

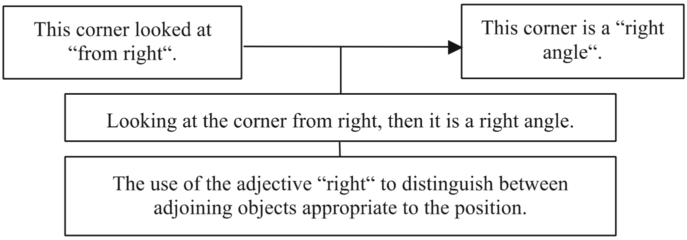

Fig. 3 The argument of Tim

angle" get an *implicit-definitional use*. The exemplary use Tim makes of "left angle" can be seen as a test of his proposal concerning the discovered meaning of "right angle". It can be regarded as a probable consequence of his first definition, following a hypothetic-deductive approach of verification (cf. Meyer, 2007, 2010). While the use of "left angle" is guided by a defining rule, which remains implicit, the way of testing the hypothesis can be reconsidered as being guided by a meta-rule (e.g., If one direction can be used to name a concept, also the opposite direction can be used to do so as well.).

Tim's use of "right angle" can be explained only because there is use of the word "right" in common practice. Here the word "right" can be used to show a certain relation between observer and object. So Tim was able to combine the uses of the words "right" and "angle" to establish a constructive meaning of the conglomerated word "right angle". The rather harsh comment of the teacher shows that the new language-game is not acceptable.

Generating Mathematical Rules for Expressing Meaning

In the scene above the students used common practice rules for elaborating concepts. The following scene illustrates the introduction of the power functions. Students' pre-knowledge comprised the parabola and calculation with x to the power of n. Now the students are asked to complete "$x \mapsto$ " for the dashed function (Fig. 4).

First, the students could not solve the task. The teacher suggests the following terms: $10 \cdot x^2$; $0.1 \cdot x^2$; $x^2 + 10$; x^5; 2^x. After eliminating x^5 the following scene takes places:

Eva Maybe it could be something with ehm, let's say x^4 or something like that because, eh the, the eh (*softly spoken*) no it is nonsense x^4 (*teacher writes the symbolic form on the blackboard, Eva speaks up*) as x^4 because eh, if you insert, than eh, a negative number for x, eh there will come something positive out of eh, of the, of the function as ehm, on the, x-axis eh so if you are now right eh of the point of origin (she is going to correct this afterwards,

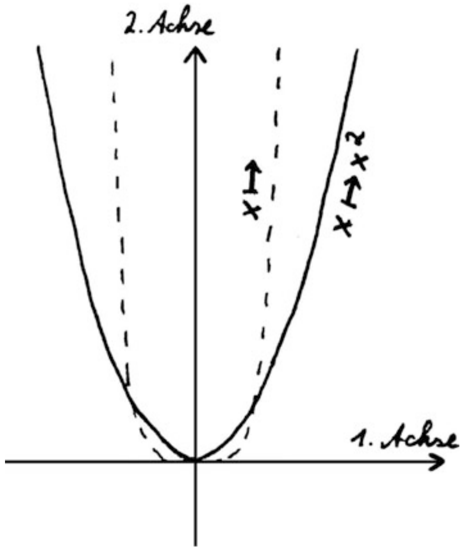

Fig. 4 The given functions on the working sheet ("Achse" means "axis")

> the author), (*teacher points at the point of origin*) then there still has to be a ehm, still ehm, thus still a positive eh result so that you actually can get high otherwise you have to go deeper if eh if there, is let's say x^5 you can, so minus times minus is plus, than another time minus times minus is again plus and then, another time minus times minus is yeah eh minus, so you would land in the range of the negative numbers and then the graph would drop down.

The scene can be interpreted as follows: Eva assumes that the dashed graph can be the graph of a function which refers to the term x^4. She refers her hypothesis to the course of the graph. In short, Eva expresses the following argument (Fig. 5):

As power function with exponents greater than 2 had not been discussed before, this scene is an example how a student sets ups a new rule by an argumentative expression of her former abduction (cf. Meyer, 2010). This is the reason why the rule in Fig. 5 can not have a backing. The first part of her following argument can be interpreted as a deductive proof of the (implicit) rule: Starting with the term x^n (n even) Eva deduces by multiplying x (supposing it is a negative number) successively with itself that the product has to be positive. In another step she infers that the corresponding graph has to run upwards. In detail three deductions can be reconstructed in this section (cf. Meyer, 2007, p. 197). Interpreting the second part of the transcript above, one can assume that Eva is also aware of the implicit contents of her argument. Here she expresses a nearly analogous argument for odd exponents of x^n. In other words: We can interpret the utterance of Eva as a generation of a rule in order to identify the term of a (power) function. As there is no former knowledge, which justifies the use of this rule, Eva is going to verify it.

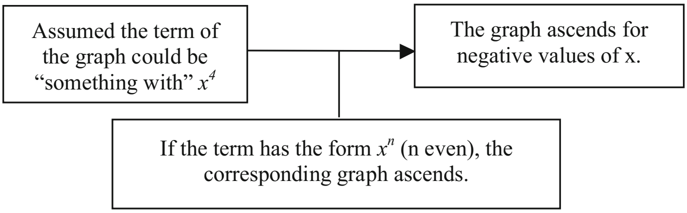

Fig. 5 One argument within the utterance of Eva

With her use of words, Eva makes explicitly traces back to her knowledge of calculations with x to the power of n. By generating the rule in Fig. 5 her contribution to the language game is the use of this knowledge in the context of functions. By using the rule, she explains the course of the graph and suggests a name for a new concept at the same time. Although Eva limits her contribution by saying "something with" and only regards the left part of the graph we can recognize an explicit-definitional way of naming a concept.

Final Remarks

Following Brandom and Wittgenstein we can understand conceptual learning as an inferential use of words. In inferences rules have a prominent role, insofar that they can be used several times in order to combine (more concrete) statements. Given rules entitle the students to infer their conclusions and also entitle them to make a specific conclusion. Thus, understanding (students') conceptual learning processes means to understand the rules they use in order to establish meaning.

The episodes—even though the established concept of Tim is wrong—indicate that language games in mathematics education undergo a continuous change: Rules of using words can be generated or modified and thus new forms of uses can be established. Compared to Tim, Eva's rule is a more mathematical one, which can be verified in the specific context.

In words of Sfard (2008, p. 200) we are able to differentiate between object-level rules and meta-discoursive rules. Object-level rules are used to find the meaning of words and/or to find the meaning of a representation of a concept. Nevertheless, Tim's rule can also be regarded as an object-level rule in so far it could be used for defining (mathematical) objects.

Both scenes show how mathematical concepts get established by generating rules which have not been a content of classroom communication before. Thus, the language-games developed. By interpreting the scene different forms of the rules which are orientating the use of words have been reconstructed:

- Rules concerning the exemplary use of words
- Rules concerning the explicit definitional use of words

- Rules concerning the implicit definitional use of words
- (Meta-)Rules concerning the hypothetic-deductive approach of verification.

The different rules orientate different forms of uses of words and, thus, the meaning of those words in the interaction: The exemplary use consists of pointing at examples to illustrate the meaning of a word. The explicit-definitional use consists of giving an explanation for words in relation to other concepts. Thus, it provides a deeper insight in mathematical coherences: The concept obtains a general character, not being linked to specific examples any more. An explicit-definitional use is also in need of a deeper mathematical insight, as it has to be known what counts as a definition. The implicit-definitional use concerning the scene "right angle" requires a common pair of concepts ("left-right") and an explicit-definitional use of the other word.

Appendix

Transcription

1. Paralinguistic signs

,	a short stop while speaking, max. one second
..	a short break, max. two seconds
sure-	the voice lingers on at the end of a word or a comment
sure	emphasis has been placed on this word
sure	word spoken with a drawl

2. Other characterizations

(..)	vague, but assumed words
(*shows*)	characterization of body language and facial expressions

A row starts at the end of the last word of the previous statement: Noticeable quick follow-up, e.g.:

M: why that
F: therefore

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Towards a Wider Perspective: Opening a Philosophical Space in the Mathematics Curriculum

Nadia Stoyanova Kennedy

Abstract The chapter argues that philosophical inquiry may have a place in the math classroom, helping to facilitate understandings that serve to complement and critically judge the inferences acquired in and with mathematics. In other words, philosophical inquiry may aid in the opening of a "wider horizon of interpretations" that includes a critical dimension. Such an opening represents a potential expansion of students' mathematical experience, and promises to provide bridges for establishing richer, critical, and more meaningful connections and interactions between students' personal experience and the broader culture.

Keywords Philosophical inquiry · Community of inquiry · School mathematics

Introduction

Mathematics offers a particular way of seeing and understanding the world. It represents a form of knowledge essential for people capable of functioning as critical citizens in contemporary societies, but it also imposes a particular worldview and epistemology—a way of seeing things, and a way of relating to the world of our experience. It is a "regime of truth," reinforced by public discourses, media, and schooling, that privileges what is perceived as objective, material and quantifiable over the subjective and experienced, facts and logic over values and feelings (Foucault, 2000). It aids in the structuring, ordering, regulating, and controlling of subjects and society that Foucault referred to as "biopower." In our postmodern world, mathematics has acquired "formatting power," and subjectivity and identity are deeply informed by meanings filtered through the conceptual lenses of number, measurement, calculation, probability, patterns and quantification (Skovsmose, 1994). Thus mathematics is not only a means of controlling the environment, but it

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becomes embodied in our thinking and action (Fischer, 2007). As such, it may lead to blind acceptance of the products of mathematization, and a lack of awareness of how mathematics operates in our culture.

Unless there is recognition of the cultural aspects of the operation of mathematics and its implicit epistemology, only narrow views of the discipline will result, and mathematics will be associated with the myth of objectivity, truth, and a value-free mathematical instrumentarium and form of judgment. Skovsmose (1994) draws attention to the "power" of mathematical language, which may help us in seeing patterns and relations, but may also result in overlooking what has been excised and deemphasized by this particular idiom. To become aware of this possibility, an "excavation" of unstated epistemological assumptions and processes that take place during mathematization is necessary.

In fact many proponents of critical mathematics education (e.g. Bishop, 1988; D'Ambrosio, 2006; Keithel, Kotzmann, & Skovsmose, 1993; Skovsmose, 1994; Vithal, 2003) do advocate for the development of critical mathematical competence in the school as one of the essential elements of democratic subjectivity. It is just here that opening a philosophical space in the mathematics curriculum may have a place in the math classroom, helping to facilitate understandings *about* mathematics that may serve to complement and critically judge our inferences acquired *in* and *with* mathematics in the process of mathematization. This space may contribute to the opening of a wider horizon of interpretations that includes a critical dimension.

Widening the Perspective: Bringing an Outsider View to the Mathematical World-Vision

British anthropologists and semiotician Gregory Bateson uses the phenomenon of binocular vision as a metaphor in order to advocate for bringing different perspectives together. He points to the fact that the stereoscopic effect—the perception of depth—is only produced due to the combination by the brain of the different perspectives of the two eyes on the world, which a single eye vision alone cannot produce. In Bateson's view, an 'extra depth' results in the enrichment that follows when two perspectives are combined, as long as the data informing those perspectives is "differently collected or differently coded" (Bateson, 2002, p. 66). Such enrichment can be particularly beneficial when the second perspective provides information on the context in which the first perspective was obtained. Philosophical inquiry in the mathematics classroom promises to give access to such an "outsider view" by way of furnishing a more global perspective regarding mathematics, its nature, its instrumentarium, its aesthetic and ethical dimensions, and the cultural and political implications of its uses in our society.

For example, a major portion of the current school curriculum is devoted to mathematical models and modelling. However, questions about how mathematical models correspond to the real world situations they model, and what the implications of those relationships are, are seldom if ever discussed in mathematics classrooms. Philosophical inquiry could supply the outsider perspective required in Bateson's view for achieving depth in understanding mathematical models, and facilitate (a) an understanding of the power and the limits of math models, (b) an understanding of what the modelling process involves, and (c) an awareness that the potential and the limits of math models are two sides of the same coin, and that we should not lose sight of both. In this regard, during a philosophical discussion concerning mathematical models we might discuss general questions about models, such as: What is a math model? How does a math model describe the 'real world'? What would a good metaphor for math model be: a map, a mirror, or something else? What happens during the translation process from the "real world" to the production of a math model? How does a math model represent a "real world" situation? Such questions and students' dialogical reflection on them promise to help students develop a better understanding of math models as products of mathematical abstraction, and provide an additional perspective in Bateson's terms.

Bateson's (2002) dictum, "a map is not the territory," needs to be considered in even wider perspective. Often, some of the judgments that mapmakers make are value judgments. Likewise, before a mathematical model is constructed, the model maker would need to consider questions such as, Who would benefit from it and who would not? Is it an ethical model, in the multiple senses of that term? A socially responsible application of the model would imply a critical evaluation of the criteria for making such value judgments. And this is just one example of how our understanding of math models and modelling and its effect on the individual and society would benefit from a counterpoint perspective, which creates a discursive space for an evaluation that is arrived at as a result of critical reflection on the explicit and implicit use of mathematics. It must be noted that such counterpoint perspectives are neither presented nor encouraged in school textbooks; only an "inside" perspective—a monocular vision—is offered.

A widened perspective could also allow for exploring epistemological assumptions, in examining the role of mathematics in social reproduction, in evaluating social practices, and thus in organizing everyday experience. Those determinations follow on the general blueprint of mathematics as a system and a method, and both prefigure and prescribe not only the mode of access to those products of mathematization, but also the mode of using, interpreting, and evaluating them (Skovsmose, 1994). Unless these practical aspects of mathematics are brought into the open and discussed, there is danger of students turning into uncritical consumers of mathematics with little or no understanding of the world-view it reinforces, and with no critical competence to judge mathematical productions and prescriptions. Finally, a widened perspective includes the examination of mathematics as a cultural product and an exploration of its aesthetic dimensions. In short, a combination of mathematical and philosophical perspectives promises to facilitate a deeper and more nuanced understanding of mathematics per se—its practical applications in the world, and its cultural and ethical implications for the individual and for society.

Philosophical Inquiry in the Classroom

By philosophical inquiry I understand, the process of arriving at critical judgments regarding philosophical questions or issues that have become a focusing point of a given group dialogue. These judgments are necessitated by an urge to resolve problematic experiences with philosophical dimensions, such as—to follow from the traditional categorization—ethical questions (e.g. What is fair to do?), ontological questions (e.g. What is the difference between a mathematical model and the real-world situation that it models?), epistemological questions (e.g. What does it mean to know something mathematically?) and aesthetic questions (e.g. What is an elegant solution to a math problem?). Here I use Mathew Lipman's approach to philosophical inquiry, which is the basis for his program Philosophy for Children (P4C). In the context of P4C, for a philosophical judgment to be reasonable it must be well-reasoned, rely on sound arguments and good evidence, and be well-informed and reflective of multiple and diverse perspectives (Gregory, 2006; Lipman, 2003). It must be capable of surviving the scrutiny of critical, communal dialogue, and be relevant to one's personal experience. Above all its most important pedagogical assumption is that philosophical inquiry is best carried on in the context of a community of philosophical inquiry (CPI) through a process of collaborative and dialogical deliberation.

The primary objective of deliberation in a CPI is the construction of meaningful arguments, not through transmission, individual reflection or debate, but through what is referred to as "building on each other's ideas"—that is through the practice of distributed thinking in a dialogical context. The ideal inquiry proceeds through a form of argumentation which, because it is inherently dialogical, is thus by implication a dialectical process, which is to say a process which moves forward through encountering and attempting to resolve tensions, ambiguities, or contradictions. The chief pedagogical significance of the constructive process of community of philosophical inquiry is that it operates in the collective zone of proximal development or ZPD (Vygotsky, 1978), which acts to scaffold concepts, skills and dispositions for each individual. The scaffolding process functions through subprocesses such as clarification, reformulation, summarization, and explanation, as well as through challenge and disagreement (Kennedy & Kennedy, 2011). Uncovering and analyzing assumptions are fundamental elements in the process of inquiry, which strives towards arriving at a collective judgment. It could be argued that such a dialogic space represents the ideal situation for the intrapersonal appropriation of the interpersonal—or "internalization"—not only on the conceptual but on the behavioural level, i.e. in the development of habits of both cognitive and behavioural self-control and self-regulation, all of which emerge within the group's collective ZPD.

A key point in Lipman's approach to philosophical inquiry is the notion, taken from Dewey, that inquiry should begin with a particular experience—in the case of the P4C program, a reading from a specially prepared philosophical novel, an exercise, or a list of questions—in order to provoke a unified cognitive event that is impregnated with conflicting ideas, that can prompt students to encounter uncertainty and perplexity, and that motivates them to inquire into the problematics of a situation and to search for its resolution. The agenda of the group discussion is guided by students' interests, not by the logical organization of the subject matter, which Dewey (1933) insisted was necessary for acquiring intrinsic meaning and understanding. The methodology is founded on the primacy of the students' questions, induced by the problematic of a presented stimulus that guides the agenda of inquiry, and the goal is that students' questions and interventions guide the inquiry process itself. In this case, what is seen and felt as problematic and perplexing in the situation presented by the stimulus must reflect the experiences of the group of students, or as Dewey puts it, the situation must "occasion" the inquiry. Finally, the communal deliberative discussions are natural outgrowths of previous discussions, strategically guided by the teacher-facilitator. The original P4C curriculum[1] represents a collection of children's novels, exercises, and discussion plans designed to provoke philosophical questions, but only insofar as the students can relate to them personally, i.e. through aspects of their own experience, which is based on the proposition that ethical, aesthetic, political, and other philosophical dimensions underlie most people's ordinary experience.

Community of Philosophical Inquiry: Methodology and Mathematical Practice

The model of the Philosophy for Children program is used here as a general framework for the design and performance of philosophical inquiry in the *mathematics* classroom. In this chapter, I will offer a few examples of curriculum that is designed for opening spaces for philosophical deliberation in a community of inquiry in school-based mathematical practices. The examples of stimuli given below are designed to foster critical understanding and promote Bateson's binocular perspective through engaged philosophical inquiry. They are inspired by Lipman's chief curricular innovation, which is the use of narrative as opposed to expository texts to introduce philosophical issues and to model the process of philosophical deliberation (Lipman, 2003). The short excerpts below are taken from a philosophical text-in-progress written in story form, associated discussion plans, and other activities that elaborate on the leading ideas in the narrative.

[1]The original P4C curriculum was developed by Mathew Lipman and Ann Sharp (see e.g. Lipman, Sharp, & Oscanyan, 1984). Other curriculum materials have been developed by theoreticians and practitioners in the field (see e.g. Daniel, Lafortune, Pallascio, & Sykes, 1999; Sprod, 2011).

Fragments from a Philosophical Text for Middle School Students

Pedagogically, the utilization of narrative, exercises, and discussion plans not only allows for the use of various stimuli, but also functions to "springboard" philosophical inquiry using already developed lists of questions, or asking students to generate their own questions, and then decide on one question to start the inquiry. The teacher might deliberately plan on designating time and space for philosophical inquiry, or it may arise spontaneously from the discussion of a mathematical problem.

The short narrative offered below[2] is a fragment of a larger work in progress, which is structured according to a set of themes and topics in mathematics that are broadly focused on and related to the middle school mathematics curriculum. These materials can be used as fruitful interruptions of the regular curriculum—opportunities to consider questions related to mathematics from a completely different perspective. They include discussion plans and other activities that elaborate on the leading ideas in the stories. The discussion plans are made up of questions, but questions that are of such a character that they model the questioning process itself for students. In line with the P4C methodology, the ultimate success of the narratives and the discussion plans depends on modelling for students a critical stance whereby they are encouraged to "interrogate mathematics" through critical questioning and reasoned discussion.

What Can Be Put in Numbers?

> "What do numbers matter to what you do, or how the world works? Trees, for example. I love trees, and I love to learn the names of them. It's gotten so I can recognize them right away, without thinking. But what can numbers tell me about trees?"
>
> "Well," Ted said, "you could probably describe lots of things about those trees with numbers. You could measure how tall they are, how old they are by counting the rings, how big around the trunk is, how much sap flows through it in how long a time, how many wood cells in one square inch of it, what kinds of shapes it takes—I don't know, all kinds of stuff, probably a lot more."
>
> I said, "Maybe you could describe everything that is in the world, and everything that happens in the world, in numbers. Can you Ms. Kwant?"
>
> Ms. Kwant looked thoughtful, but she didn't say anything.
>
> "Yeah!" said Philo. "That would be like, the laws of nature are written in the language of mathematics! Like gravity. Gravity is a law of nature, and you can write it in mathematics. And even if there isn't any connection between math and the world, it's just really cool—the way it works I mean. The way everything fits with everything else when you're in the number world."

[2]The curriculum materials described here have been developed in collaboration with David Kennedy from Montclair State University and IAPC (Institute for the Advancement of Philosophy for Children) at Montclair State University

Philo grinned at Antonio—he was just trying to cheer him up!–, but Antonio shook his head and stared at the floor. "But how can numbers match my feelings?" Nora said. "Can they describe how calm I feel, or measure how happy I am to see my friend Bessy after a whole year?"

"Of course they can" Francesca said. "People do all kinds of surveys on happiness, or satisfaction, or how much you like someone."

Marilena laughed. "I have a little robot in my pocket," she said. "He calls himself an 'affection measuring device'—AMD for short. I take him out of my pocket…" Marilena reached in her jeans and pretended to take something out and hold it against her heart. "Hmm, it says…" She made her voice sound like one of those machine voices on the telephone, when you call a big company. "'Affection level Sean—5. Affection level Tom—7'…So I say, hmm, that means I go play with my friend Tom today." She laughed again.

"Yeah," Nessim said, "I've got an app on my phone that does that!"

"No but really," Francesca said, "surveys do work you know. Like who's the most popular one running for president or things like that."

"They tell us something, but how much, really? And how do you know they aren't lying anyway?" Antonio said.

The bell rang. "Oh dear, said Ms. Kwant. I meant to spend the last 10 min talking about the test for tomorrow. Anyway, any questions, email me, OK?"

Philo stood up with that silly grin on his face. He started to make his whole body tremble. "Oh no!" he squawked. "Look at my fear-ometer! Oh my God! It's going up and up—and up. OMG! Through the roof!" And he hopped out of the room like a kangaroo. Go figure!

Usually the reading of the story is followed by students posing their own questions related to the story theme, and the facilitator initiates a discussion on the basis of one of them. The discussion can be preceded by work on an exercise such as the following:

Exercise 1:

Choose one thing from the list below and try to answer the two questions: 1) Can these things be described mathematically? 2) If so, how would one go about it?

A barn; A dance; A soccer game; A card game; The game of chess; A human life; A conversation; A rabbit warren; The economy; An election campaign; A thunderstorm; A dream; Personal "coolness"; Happiness; Anxiety.

The purpose of this exercise is to expose the differences between ways of mathematizing phenomena from the physical world such as a barn, or a dance, and psycho-social constructs like "coolness" and even happiness or anxiety. Its effect is to help us see that mathematics can be an appropriate tool for describing and studying the former, but not as good for studying the internal human world—human emotions, beliefs, knowledge, understanding, hopes, etc.

In addition to exercises such as the above, we have developed more general discussion plans, which can also be introduced either before or after reading the story, as well as during the group discussion itself, as a focusing mechanism.

Discussion Plan 1: Lost in Translation
Can all nature be expressed mathematically—turned into numbers?
Can architecture be translated into mathematics? Vice versa?

Can poetry be translated into mathematics? Vice versa?
Can music be translated into mathematics? Vice versa?
Can visual art be translated into mathematics? Vice versa?
Can dance be translated into mathematics? Vice versa?
What can mathematics tell us about how families get along, or whether something is fair, or how to heal mental illness, or how we should deal with crime?

Discussion Plan 2: The Mathematical Way of Knowing
Are mathematical descriptions always useful?
Is there anything they might miss?
Could a mathematical description be harmful? If so, how?
Can mathematical descriptions prevent other ways of knowing?
Can math description tell us about whether something is fair or beautiful?
Can math descriptions help us understand ourselves?

Finally, I offer a few examples of exercises designed to lead to philosophical inquiry in the areas of ontology and epistemology of mathematics. Such exercises raise and attempt to instantiate such questions as "Where is mathematics in the world? Where does it come from? Is it discovered or invented?" Posing such questions acts to transgress the artificial barriers that determine what is and what is not discussed in math class, and direct students' inquiry to the origins of the discipline, thus opening doors to even broader questions concerning mathematics and reality, such as those pursued by Pythagoras.

Exercise 2: The World and the Mind

Where are the following? 1) Just in the mind; 2) Just in the outside world; 3) In both the mind and the world; 4) Neither in the mind nor the world: Images; Words; Thoughts; Sounds; Smells; Tastes; Love; Cats; Mickey Mouse; Light; Numbers; Circles, Squares, Mathematical formulas.

Note that in this exercise we leave direct mathematical concepts like circles, squares, triangles, and formulae for the very end, in hopes that children would begin with more concrete examples, which act to ground and distinguish the more abstract concepts that followed.

Exercise 3: Are These Things Invented or Discovered?

Soap; The family; Human conflict; Your thoughts; Feelings; Electricity; A theory; Sports; History; Numbers; Circles, Squares and triangles; Graphs; Formulas; Mathematics.

Again, this exercise leads us to think about invention and discovery starting in the context of more concrete examples, and moves to more abstract terms such as numbers and mathematical formulas; the movement is always from the more obvious to the more "fuzzy" examples, where what appear to be category mistakes are in fact invitations to think more deeply. In the course of the deliberation provoked by this exercise, clarification of what "invention" and "discovery" mean inevitably becomes necessary, although most students do have an intuitive sense of what those terms mean.

The discussions that result from these exercises encourage students to critically examine viewpoints absorbed from parents, media and school, whose basic message is often: "Mathematics is everywhere and it is all-powerful. It can be used successfully in every single aspect of our lives." Participating in discussions that untangle multiple examples, perspectives and ideas voiced by peers can help students develop more personal views about mathematics, help them understand that mathematics has limits like any other human tool, and give them an opportunity to explore some of those limits.

One could go further. Does mathematical knowledge of the world and the adoption of a mathematical approach to studying and understanding the world actually inhibit or distract us from other, equally (or in some cases more) powerful forms of knowledge? Is mathematics always the one best way to understand the world? Is it a way to understand ourselves? Is there anything that it might miss? Mathematics students commonly question its practical usefulness, which is related, in turn, to its evident relation to the lived world that we all inhabit in more or less the same spatio-temporal way. My goal is to encourage questions that go deeper than the uses of mathematics for financial calculations or designing a new roof of a house—questions which stimulate inquiry that critically explores our culturally constructed and transmitted beliefs and assumptions about mathematics, and thus lead to greater awareness of its power and limitations. I believe that such inquiry acts to heighten awareness of the traps into which we may fall if we blindly trust someone else's uses of mathematics to make arguments for or against any number of policies or actions in the world.

Finally, the examples offered in this exercise act to span different disciplinary areas, and thereby not only bridge different dimensions of human experience, but encourage students to understand mathematics as a form of meaning-making with basic similarities to meaning- making in the physical and social sciences, history, sport and so on.

Conclusion

I have discussed the need for opening new spaces in mathematics classroom practice, where philosophical dialogue can act as an "epistemological wedge" for a counterpoint perspective in order to brings into view what is other and outside of the mathematical "single vision" of the world, and provoke a deeper "binocular vision" of the ontological and epistemological status of mathematics. Opening a space for philosophical inquiry in the context of communal, dialogical deliberation with peers may function to (a) bridge students' school mathematical experience and their experiences outside of school; (b) feed and regenerate students' curiosity towards mathematics; (c) enhance their understanding of the presence of mathematics across the disciplines; and (d) provide them with a metacognitive space that acts to enhance and even improve their classroom practice through enlarging their understanding of the field of mathematics as a particular discourse. School mathematical practices are notorious for producing student apathy in the math classroom, math phobias and negative attitudes among a significant number of students (see, e.g., Tobias, 1993). Research into this phenomenon suggests that it results from ignoring students' interests and perpetuating a pedagogy based on routine and the dominance of trivial activities and tasks, leading to an increasing disconnection of the subject matter from students' everyday knowledge and experience, and the perceived irrelevance of math to students' lives (see, e.g., Boaler, 2002; Wells, 1999). The goal of communal philosophical dialogue is to foster what John Dewey called "intellectual curiosity"—which Freire (1998) termed "epistemological curiosity"—and which may be described as an interest, not just in what one knows, but also in how one knows it. On both Dewey's and Freire's accounts, students' natural curiosity requires stimulation and nurturance in order to develop in the direction of ongoing inquiry, not just about the world and their place in it, but about the powers and limits of their own thinking.

Engaged philosophical inquiry in the classroom, regularly undertaken, promises to create a shared space between the culture of learning and the student, where she, together with others in a structured and disciplined collaborative environment of ongoing deliberation, can make sense of the relation between the material and the social worlds and the world of ideas that she encounters in the classroom. In this transitional space, students are afforded the cognitive leisure necessary to play with ideas apart from their production values (whether of test scores or of identified "skills"), and in the process become more aware that mathematics is at its base a sense-making process; that it is one language among many; that mathematizing require interpretation of a given situation; and that any inferences made in the process of mathematization are based on implicit or explicit assumptions that call for examination.

Without such a critical reading of mathematics—an "outsider" view that complements the mathematical descriptions that we use or produce to understand the world—there is an obvious danger that our students become uncritical and passive users and consumers of mathematics, with little or no understanding of the world-view it reinforces, and with no critical competence to judge mathematical productions, many of which lead to the creation of misguided and dysfunctional material and social productions in the real world—whether buildings, dams, infrastructure or other forms of large scale human organization. A narrow reading of mathematics can facilitate what Bourdieu (1977) calls "misrecognitions" about the objectivity of mathematics and the role it plays in society. Such misrecognitions are far from innocent, for they become accomplices in the development of student identities that, as they struggle to figure out the world they participate in, orient them towards positions of disempowerment and passivity. In order to counter this process of culturally induced learned helplessness, there is a clear need for opening new spaces in mathematics classroom practice (Maier & Seligman, 1976). Philosophical dialogue can act as an "epistemological wedge" for a counterpoint perspective, in order to bring into view what is other and outside of the strict mathematical vision of the world, and thus encourage a "binocular vision" through which to understand the meanings emerging in the encounter with mathematics as a form of knowledge. Through locating this encounter in the communal, dialogical, deliberative setting of Lipman's community of philosophical inquiry, the discourses of mathematics and philosophy are not set in opposition to each other, but rather enter into conversation, in the interest of achieving a wider perspective, and of empowering students to, following Kant's dictum, think "for themselves and with others" (Kennedy, 1999).

This undoubtedly renders the roles of teachers and students more ambiguous and complex, and the task of delivering the curriculum more complex as well. Sometimes it means welcoming emergent meanings and discussion trajectories that follow no predetermined ends, in the collective search for a shared mathematical practice that is more connected and relevant to students' and teachers' life experience and broader philosophical understanding. For teachers, it means learning to foster what Belenky, Clinchy, Goldberger, and Tarule (1986) call "connected knowing," and to encourage the development of students' agency through empowering them to ask questions, construct arguments, and engage in reasoned discussion. Philosophical and mathematical dialogue are bound to refer to each other and become entangled in complicated encounters and recursive iterations. Their boundaries are permeable, and in the process of group dialogue, the two can interact, both as a result of constructed transitions and of spontaneous slippages between the two. This anyway is one way to imagine the conversation between them, which is a complicated one but which is, in my view, a more enriching, empowering, and ecologically sound approach to mathematical learning and knowing than now prevails.

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Creativity Research in Mathematics Education Simplified: Using the Concept of Bisociation as Ockham's Razor

Bronislaw Czarnocha, William Baker and Olen Dias

Abstract This chapter proposes that bisociation, the Koestler theory of the creativity of the "Aha!" Moment, is the Ockham Razor for creativity research in mathematics education. It shows the power of bisociation in simplifying unnecessary components and in the synthesis of the fragmented ones. The discussion leads through the relationship of bisociation with Piaget's reflective abstraction, proposes cognitive/affective duality of the "Aha!" Moment and enriches Mason's theory of attention by the new structure of simultaneous attention necessary for the Eureka experience.

Keywords Ocham razor · Bisociation · Aha! Moment · Restructuring Creativity

Introduction

Ockham Razor (OR) is one of the oldest formal tools used to streamline the progress of knowledge and understanding. It is one of the principles underlying the philosophical discourse within Economy of Thought, a subject developed by E. Mach in his Science of Mechanics (1893), among others. His views were based on the principle "that science had the purpose of saving the mental effort." Devised by William Ockham (c. 1287–1347), OR has guided Oresme, Galileo, Copernicus, Newton, and its impact continues till the present day. In John Punch's formulation of 1639, it is stated that "*Entities must not be multiplied beyond necessity*," or in modern terms that "*other things being equal, simpler explanations are generally*

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better than more complex ones"; it's a quest for simplicity of scientific expression. This however was wisely constrained by Einstein, who pointed out "*Everything should be kept as simple as possible, but no simpler*."

Einstein's point is significant for the discussion below. It suggests that the process of saving mental effort involves two related edges. One is to make the minimal number of necessary assumptions, "shaving off" the remainder with Ockham Razor. The other is accomplished through the quality of the structure within which a given concept is formulated. The quality is often expressed by the precision and clarity of the relationships defining the schema of the concept in question, and has an independent aesthetic value. Consequently, a modern OR is comprised of these two different edges. One precisely and clearly explains the concepts and their related relationships only, as unnecessary concepts have been "shaved off." Simultaneously, the other edge is to integrate the concepts into irreducible wholes out of fragmented, oversimplified domain. Einstein's advice leads then to a perfect completion of the Ockham razor, by pointing to the irreducible complexity present in reality and our understanding of it. Yet at the same time, Einstein, Podolsky, and Rosen (1935), in his famous Einstein–Podolsky–Rosen paper *Is Quantum Mechanical Description of Reality Complete?* attempted to introduce the unnecessary hypothesis of "hidden variables" into QM. It was Bohr's (1935) structural argument that suggested that Einstein's notion of reality is too limited to account for the observed effects of Quantum Mechanics (QM) theory. We see here the degree to which the notion of simplicity is anchored in deep personal convictions of scientists.

The quest for simplicity in understanding increases in importance, whenever we have many approaches to the subject through many different theories of the phenomena, which ultimately introduce confusion as to its ontological existence. In science, Ockham's Razor is used as a heuristic technique (a discovery tool) to guide scientists in the development of theoretical models, rather than as an arbiter between published models (Gauch, 2003).

State of the Creativity Research—Brief Summary

There is an increase of interest recently in the classroom creativity of mathematics students. Lamon (2003) emphasizes the need for creative critical thinking. Mann (2005) asks for the explicit introduction of creativity as the component for learning in general. However, the conceptualization of creative learning varies due to the diversity of the proposed definitions of creativity (Kattou, Kontoyianni, Pantazi, & Christou, 2011). There is no single, authoritative perspective, or definition of creativity (Kattou et al., 2011; Mann, 2005; Sriraman, 2005) in the field of mathematics education. Mann (2005) found around 100 different definitions of creativity in the field. Main competition however comes from the Gestalt based theory of four stages of creative process (preparation-incubation-illumination and verification), proposed by Wallas (1926) and Hadamard (1996) and from Torrance (1974), with a product oriented theory of fluency, flexibility and originality as indicators of creativity. We note the recent appearance of two excellent collections of papers dealing with creativity in mathematics education (Leikin et al., 2009) that bears witness to the increased importance of the subject. However, we also point out that both volumes' central focus is the relationship between creativity and giftedness. In the light of the (Sriraman, Yaftian, & Lee, 2011) comment that "*There is almost little or no literature related to the synthetic abilities of 'ordinary' individuals, except for literature that examines polymathy*" (p. 120) there is a clear bias in the efforts of creativity research. In response, Prabhu and Czarnocha (2014) call for the democratization of research on creativity based on Arthur Koestler's (Koestler, 1964) definition of bisociation, formulated in *The Act of Creation*.

Furthermore, simultaneous with the focus on giftedness there is a tendency to describe creativity through the characteristics of insight, rather than to provide a mechanism for such creative thought—i.e., creativity being synonymous with novel, flexible, fluent or divergent thinking. Gestalt theory describes a pathway towards insight within problem solving. This begins when the individual in the preparation-incubation stages attempts a solution strategy that does not work. Then a 'discontinuity' occurs during the illumination stage, from which a new strategy with a different analytical process spontaneously occurs in an "Aha!" Moment of insight. The result of this 'productive thinking' is 'restructuring' the solver's problem representation (Weisberg, 1995). Koestler's mechanism of bisociation provides precision and clarity to both the "Aha!" Moment and the restructuring of an individual's problem representation, that Koestler would refer to as a matrix or frame of reference governed by rules or codes. The precision and clarity of this mechanism allows Koestler to generalize the bisociative "Aha!" Moment domains outside mathematics, such as humour and literacy. However the relevant question for mathematics education is how much does the mechanism of bisociation and the "Aha!" Moment add to our understanding of how students create meaning for themselves and for mathematics.

Koestler Theory of Creativity as Ockham Razor

Bisociation is "*a spontaneous leap of insight which connects previously unconnected matrices of experience*" (p. 45)—known also as an "Aha!" Moment, or Eureka experience. A bisociative framework is the framework composed of "two unconnected matrices of experience," where one may find a "hidden analogy"—the content of insight. The definition asserts that the presence of the bisociative framework is the necessary condition for the "Aha!" Moment to occur. It directs our attention towards the theory of schema formation. Before we proceed to explore the insight brought by the Koestler's theory of "Aha!" Moment into schema formation, one should add that the theory of bisociation is the generalization of three separate investigations, into the nature of humour, the nature of discovery and the nature of art. Within each, Koestler was able to find the same mechanism of bisociation as the root of the creative "Aha!" Moment experience. The Koestler triptych (Fig. 1) shows the process of the horizontal merging of humour with discovery and discovery with art, though they have fundamentally different languages of expression on both cognitive and affective planes. He points out how the emotional, expressive content of the panels change, while moving from left to right, as the structure of the bisociation mechanism remains the same throughout the different domains. Consequently, bisociation solves the standard division of creativity into domain specific and domain general by positing simultaneous presence of both components underlying creativity's structure whenever it appears: the general principle of bisociation and its particular formulation together with its expression depending on the domain.

The presence of bisociation suggests that the division into domain specific or domain general is inessential because they always take place together. Equally deep integrative processes explicated from Koestler's definition, and described below, allow us to see human creativity in a more coherent fashion. Guarding the coherence of the intellectual domain is, in our opinion, one of the fundamental tasks of the philosophy of mathematics education.

Koestler provides a series of supporting examples such as Gutenberg's discovery of the printing press from the bisociation framework between the wine press and a seal; or Poincare's note describing formulation of hidden analogy between Fuchsian functions and hypergeometric series. We see a close connection created between bisociative creativity and synthesis, reflected by the recent change in Bloom's

Fig. 1 The Koestler triptych

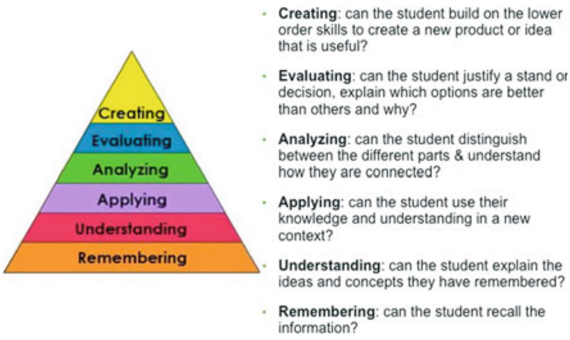

Fig. 2 Revised Bloom's taxonomy

Theory of Taxonomy. Here the last stage of synthesis has been altered to that of creativity (Krathwohl, 2002) (Fig. 2).

Mathematics Education: Bisociation, Schema Accommodation and Reflective Abstraction

The definition of bisociation as the leap of insight that synthesizes previously unconnected/separate frames of reference/matrices suggests the process of schema formation to be relevant in describing how schema are restructured to fit new problem situations. Piaget used the term 'accommodation' to describe the process for which significant modification, or restructuring, of existing schema is required, because the problem situation cannot be assimilated into any existing schemata. Piaget disliked the Gestalt notion that restructuring was a spontaneous act of creation. He debated with linguists, such as Noam Chomsky, who maintained that, primarily language acquisition was somehow innate to the individual. To reconcile his view that learning in a new problem situation involves accommodation of existing schema with the learning paradox, Piaget proposed the mechanism of reflective abstraction to explain how existing schema are so dramatically modified as to be considered restructured, and thus significantly different than its original state.

The description of reflective abstraction to describe accommodation in Piaget and Roland (1987) is that of a two step process in which an existing schema is seen as relevant. The existing schema is first projected into the new problem situation. Then the coordination between the original schema and problem situation bring about reflection to generalize, or abstract, one's existing knowledge and construct the new accommodated schema through 'constructive generalization.'

The OR second edge integrates here mutually reinforcing the network of two theories: bisociation and the Piaget theory of conceptual development. The mechanism of bisociation focuses on an affective, as well as a cognitive aspect, of the

"Aha!" Moment. In the cognitive aspect, concepts are existing simultaneously in two previously independent matrices of schema. Through this linking of concepts, Koestler would say a hidden analogy is brought to light, and the result is a synthesis of the codes or rules from the two matrices. The emphasis on synthesis as an act generating new concepts goes back, accordingly to Ilyenkov (1974), to Kant, who stated that "*By synthesis, in its most general sense, I understand the act of putting different representations together and of grasping what is manifold in them in one act of knowledge*"— an act of intellectual creation.

The insightful grasping of what is relevant to and underlies the matrix of the problem situation and the matrix of acquired relevant knowledge (the two previously unconnected matrices of thinking), results from the bisociation of concepts. It can be seen as shedding light on the process inherent in the constructive generalization phase, after the existing schema is projected into the problem situation (Baker, 2016).

The Bisociative Edge of OR as a Heuristic Guide to Question, Challenge and Critique Certain Claims in Mathematics Education Research

Paul Ernest asserts in the introduction to this volume that "*the role of the philosophy of mathematics education is to analyse, question, challenge, and critique the claims of mathematics education practice, policy and research.*" We will use this opportunity to show how the bisociative edge of OR helps to question, challenge and critique several research claims.

Cognitive-Affective Duality of Aha! Moment

The emphasis on the schema construction, together with the empirically observed positive impact upon thinker's affect (Liljedahl, 2013), allows us to formulate the principle of the cognitive/affective duality of the "Aha!" Moment (Czarnocha, 2014). While we appreciate the scope of the work in Liljedahl (2013), we are questioning here the absence of the cognitive dimension as the distinctive characteristic of the "Aha!" Moment in Liljedahl's (2013) approach that asserts:

> …what sets the phenomenon of illumination apart from other mathematical experiences is the affective component of the experience, and ONLY the affective component" and continues: "while the affective component of illumination is consequential to the differentiation of it from other mathematical experiences, the cognitive component is not. (p. 264)

The bisociation that is the theory of "Aha!" Moment suggests that its cognitive content is the construction of a mathematical schema—an experience very different from the process of calculation, although this process does enter into the construction. It is different than problem solving, although problem solving enters into the construction, as well. It is the process of a very different nature than several other mathematical experiences—it is the process of synthesis. This is where the cognitive uniqueness of "Aha!" Moment lies. In fact, Sfard (1995) suggests that the energy of the "Aha!" experience is the energy released during the cognitive process of schema construction, i.e., "compression of the discourse," during that experience.

Thus we suggest the cognitive/affective duality as the central quality of the "Aha!" Moment. Koestler's assertion: "*The creative act…is an act of liberation-the defeat of habit by originality*" allows one to exploit cognitive/affective duality in combat with negative habits hampering students' success in mathematics, especially those among "underserved" populations (Prabhu, 2016). The second edge of OR brings forth the cognitive component as its characteristic and creates the dynamic cognitive connection with affect.

Bisociation and Simultaneity of Attention

As bisociation is obtained by connecting simultaneously "previously unconnected matrices of experience", simultaneity of attention to two different "matrices of experience" is necessitated. This is significant because, in the history of modern physics, the notion of simultaneity being necessitated appeared twice at the very foundation of two physical theories, Theory of Relativity and Quantum Mechanics. One can surmise that its appearance in the theory of creativity will be equally, if not more fundamental, especially considering that quite possibly, creativity is incompatible with habit. Therefore, the measurement of creativity maybe similarly constrained, as are measurements of incompatible observables of QM.

Focus on simultaneous attention leads us to Mason's theory of attention and its shifts. Mason (2008) distinguishes macro and micro structures of attention (p. 4). The macro structure of attention can vary, according to the author, in multiplicity, locus, focus and sharpness. It also can vary in the degree of simultaneity of attention. Simultaneity of attention is related to what Mason (2008) calls parallel awareness and illustrates by the following type of situations: "…while reading this text you can form an image of where you partake of your next meal…." The type is characterized by at least two foci of attention thoroughly unrelated to each other. However, if we instead imagine a NYC taxi driver, whose goal is to "hunt" for a fare in an environment of heavy traffic and fierce competition, the driver has to simultaneously pay attention to driving and to searching for the next fare. Both of these activities, and therefore both attentions, are tightly related to the goal of obtaining a fare. Both are necessary for the purpose. Similarly, the attention needed for the occurrence of the "Aha!" Moment, that is a spontaneous leap of insight connecting—in the moment—at least two habitually non connected frameworks, or matrices of experience, has to, in general, be simultaneous to both matrices.

The micro structure of attention that is "ways of attending," according to Mason (2008) are: holding wholes, discerning details, recognizing relationships, perceiving properties, and reasoning on the basis of properties. Simultaneous attention is, of course, related to recognizing relationships. However recognizing relationships can be done, according to Koestler, in at least two ways related to two different processes of understanding:

> progress in understanding-the acquisition of new insights, and the exercise of understanding at any given stage of development. Progress in understanding is achieved by the formulation of new codes through the modification and integration of existing codes by methods discussed: empirical induction, abstraction and discrimination, bisociation. The exercise or application of understanding - the explanation of particular events - then becomes an act of subsuming the particular event under the codes formed by past experience. To say that we have understood a phenomenon means that we have recognized one or more of its relevant relational features as particular instances of more general or familiar relations, which have been previously abstracted and encoded. (p. 96)

Consequently, only one of these two processes of recognizing relationships, the progress of understanding requires simultaneous attention to both concepts being connected.

On the basis of this discussion, we postulate the existence of the/simultaneity structure of attention necessary for such pairs of processes to reach the goal common for both. Simultaneous attention is then one of the ways to recognize relationships through the "Aha!" Moment.

"Simultaneous attention" has received increased attention from psychological research since the end of the previous century. For Naglieri and Das (Naglieri & Das, 1997), "*simultaneous processing is engaged when the relationship between items and their integration into whole units of information is required*." The work of Riquelie and de Schonen (1997) on simultaneous attention in the two visual fields of 20, 24 and 26 month old infants demonstrated that "*simultaneous attention to the two visual fields and the production of a unified response emerge very late in development at about the age of 24 months*." Their result is interesting from the point of view of bisociation theory, when noting that the formation of the first bisociative framework between speech and thought takes place, according to Vygotsky 1987, around 2 years of age.

The introduction of the structure of simultaneity of attention raises some new research questions, such as what is the possible scope of simultaneous attention both in content and in time. One could also ask whether the scope of simultaneous attention is the same as the scope of the attention focused on a single object. If it is not the same, what are the dynamics and nature of the shift through which attention focused on single objects transforms into the simultaneous attention on two at once.

Bisociation and Teaching—Research

The second area of creative coherence is identified by bisociation in the area of classroom teaching and learning. The central point of bisociation as the new definition of creativity in mathematics education, emphasized by Prabhu and Czarnocha

(2014), has been the possibility to democratize research on creativity. Since the "Aha!" Moment is a popular experience, experienced essentially by everyone, the investigation and related facilitation of this experience means expansively focusing on the "creativity of all," without the limiting emphasis on the primarily gifted population as is prevailing custom, but of course, including the gifted as well.

"Aha!" Moments are common experiences for the general population, and accordingly to Koestler, are often encountered in conditions of "untutored learning." What is more, we have the important Hadamard (1996) statement: "Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only the difference of degree, the difference of a level, both works being of similar nature (1945, p. 104)." These three different observations suggest investigations into Koestler's definition will democratize research, redirecting it into everyone's creativity (including gifted learners). It will explore the role of the Discovery Method of teaching as the best classroom approximation to conditions of "untutored learning," and together with its cognitive/affective duality, it will provide the foundation to bridge the Achievement Gap. Again, we witness the integrative power of bisociative creativity in placing several different, separated components into the construction of the whole for a creative learning environment (Prabhu, 2016).

Similar coherence appears on the teaching spectrum of the classroom ethos. The methodology of teaching-research as connecting two separate, in general, disciplines of teaching and research turns out to be a bisociative framework, albeit a very creative one confirming the role of bisociativity.

Awareness of the bisociative coherence of Teacher as Researcher (TR)/NY City model has been facilitated by the work of Eisenhart (1991), who identified three frameworks of inquiry present in research of Mathematics Education: theoretical, practical, and conceptual (Lester, 2010). Following Eisenhart, Lester (2010) posits three types of frameworks used in Math Education. First, the theoretical framework based upon theory, i.e., the constructivist, radical constructivist, and social constructivist theories discussed. Second, a practical framework, "which guides research by using 'what works' … this kind of research is not guided by formal theory but by the accumulated practice knowledge of practitioners and administrators, the findings of previous research, and often the viewpoints offered by public opinion" (p. 72). The third is a conceptual framework that can pull from various theories as well as educational practice. We argue that among the three frameworks for research present in philosophy of education research only the conceptual framework allows for the possibility of bisociative synthesis between teaching and research through Stenhouse (1975) TR acts.

> a conceptual framework [that] is an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation. Like theoretical frameworks, conceptual frameworks are based on previous research, but conceptual frameworks are built from an array of current and possibly far-ranging sources. The framework used may be based on different theories and various aspects of practitioner knowledge. (Lester, 2010)

Of special importance in working with conceptual frameworks is the notion of *justification*. A conceptual framework is an argument including different points of view and culminating in a series of reasons for adopting some points and not others. The adopted ideas or concepts then serve as guides—to collecting data, and/or to ways in which the data from a particular study will be analysed and explained (Eisenhart, 1991).

The Stenhouse act is both a teaching act and a research act. Therefore, it is a bisociative act par excellence, and by definition, an intrinsic teaching-research act. This bisociative par excellence quality of the Stenhouse act has become the criterion that tells us which classroom methods are intrinsically teaching-research (TR) methods. We identified the Discovery method, the method of teaching-research interviews, and a facilitation of the "Aha!" Moment as distinctive bisociative TR methods.

Another whole was then created by the second OR edge. Bisociative teaching-research, as the classroom investigation, pairs with the Discovery method of teaching, again particularly conducive for the facilitation of the "Aha!" Moment.

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Teaching of Velocity in Mathematics Classes—Chances for Philosophical Ideas

Regina Dorothea Möller

Abstract The way the concept of velocity is presented in elementary mathematics textbooks builds on text problems and nowadays also on the use of tables. However the genesis of the velocity concept refers to a centuries-old search within the context of motion underlined with philosophical ideas. The concept that stands behind modern school mathematics refers back to Newton who himself relies on the work of Galileo Galilei. The historical development of mathematical education has shown that both mathematics and physics classes have their respective characteristic manner using this term. However, the mathematical potential for teaching this concept is by far not exhausted and asks for a philosophical background knowledge on the part of the teachers.

Keywords Concept of velocity · Historical development · Philosophical aspects in math class

Introduction

The causes to investigate the historical development of the concept of velocity and the potential for philosophical ideas in classrooms linked to it are mathematical problems that have lately appeared in German textbooks of fourth grade classes. Here is an example Fig. 1.

Several pictures show people or machines such as cars or trains, each with an information about the respective velocity: e.g. the train covers 150 km in an hour. The assigned task is to fill out tables in which the pupils write the distances for different time spans: 1, 2 h and so forth. Above the pictures and the assigned tasks for the pupils the problem gets the heading “velocities” with the request “apply tables” (Das Zahlenbuch, Klett, 2008, p. 73).

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Fig. 1 Velocity problem

Considering the given problem, its presentation and its tasks for the pupils, one starts pondering over the obvious aspects. It attracts attention because surely Galileo Galilei and Newton did not use tables to compute velocities. Also the used presentation within the given table is rather unusual: first the time and then the respective covered distance. This way could have been chosen by the authors of this textbook, because pupils could verbalize, e.g.: "In an hour the distance of 4 km is covered." This succession of the parameters time and distance does not prepare the equation $v = s/t$. It is also not intended according to German curricula for the elementary level. In the following paragraphs different aspects of the problem are analyzed.

Didactical Orientated Analysis

Within German mathematics education the didactical orientated analysis (in German terms: Didaktisch orientierte Sachanalyse) serves as a tool to prepare mathematical lessons following the principle of analyzing environmental relationships (in German terms: Umweltbezüge). The goal is to see mathematics in lessons not as an end in itself but to enable the pupils to construct mathematical concepts through examination with everyday life experiences, followed by discussing them. The goal of the didactical orientated analysis is to distinguish the didactical strategies for the lessons. Central for the didactical orientated analysis was the genetic, application orientated structure of the number system and later on the comparison theory of measuring (Griesel, 1996; Kirsch, 1970). It is apparent that the velocity concept is a composition of two quantities, length and time, which happens in class only when mathematically fractions have been introduced. In the following paragraphs an analysis is outlined.

At the stage of elementary fourth graders it is not yet clear what velocity is all about. In everyday life fast cars are fascinating for boys but very slow animals like snails are not understood as having velocity. In textbooks there is no definition or explanation—and with this specific textbook under consideration, pupils are expected to fill out tables. Like in other cases in which an introduction into the field is intended, there is a start with a quantified aspect of the concept. What could have been an approach to a first understanding of the concept of velocity? Probably looking at the series of different observable motions that give the opportunity for upcoming questions on the part of the pupils. After describing the fact, that some cars or trains show quicker or slower movements, they could ask themselves how to quantify their observations. Instead, there is an emphasis on the computational aspect with no apparent need. The idea of velocity disappears almost behind the computational activities, an observation that has been done before within other scholar fields like calculus (Doorman & van Maanen, 2009). Instead of approaching the idea of velocity by exercising better and better descriptions and verbalizing their observations, there are tables to fill out by the pupils.

This observation and the accompanying problem can be considered as an anticipation. There are at least three other kinds of anticipation tasks on the elementary level: one refers to the ratios such as a half, one quarter and three quarters and the second refers to the decimals within the context of quantities. The third is the appearance of tables in application problems. For example, in these problems prices of products are given—like 1 kg of apples costs 75c—and the question is how much is to pay for 2 kg/3 kg/5 kg. These three kinds of anticipations occur in math classes because of the application principle. Pupils experience these kinds of questions and these numbers in their everyday life and math classes respond to this phenomenon by introducing these numbers and tables without giving a didactical and methodical setting and a rigid mathematical reasoning.

What kind of anticipation is it, when students "solve" the kind of velocity problem given above? Since they are to fill out tables, one could assume that functions have arrived in elementary math classes and the chosen form should stress the functional relationship between the time and the covered distance. Tables are one type of a functional representation among others like graphs and equations. Normally they are introduced on the secondary level. Tables are often used for the first time to find the dependent variables given the independent. The fact that pupils fill out tables without knowing yet the mathematical impact of it, can be observed in other fields also (e.g. Möller, 1997). One could deduce from this situation that the idea of Klein (1905), to make functions a subject matter in school, has succeeded even on the elementary level. But this is probably not within the meaning that Felix Klein had in mind at the beginning of the last century.

The "velocity problem" or the way the problem is shown in textbooks can be seen as an example of an (anti-) didactical inversion (Freudenthal, 1983, p. 305). He argues that no mathematical ideas have been published the way they have been developed. Techniques often follow the new concepts and are used whenever a respective problem has to be solved. Further, Freudenthal says if there is a larger complex of statements and theories, definitions turn into propositions and vice versa. If it becomes a teaching matter the anti-didactical inversion happens. He argues that the learner is entitled to recapitulate the thinking process that stands behind a concept. The problem in consideration shows his thoughts well: It is obvious that this is not following the historical development and it is a short cut done from a later point in historical mathematical development.

Since this kind of velocity problem is not really an application task, as only tables need to be filled out, one could argue that it provides a computational way to compute velocities. Needless to say that today's pupils have much more contact with the phenomenon of velocities than their predecessors in the last century. Insofar there is a need to explain the observations that can be done in everyday life. Our modern world presents this phenomenon and students need explanations for it.

Observing the way velocity is represented mathematically, we find an example of a quantity: "He rides his bike with 20 miles an hour is written 20 mph." Having in mind the way quantities are introduced in math classes on the elementary level one finds the following steps: finding representatives, studying comparisons with chosen measurement objects and thereafter recognize the agreed upon conventional, standardized measurement units. At last the solving of application problems round up this subject matter. The way the issue of velocity is addressed does not show any such procedure although velocity is the first composed magnitude the pupils meet.

Looking upon the problem as one that introduces a new mathematical concept, one would expect a series of steps that lead to a definition. A possible approach would be the didactical triangle of Bruner (1960), pointing at the enactive, iconic and symbolic levels within a teaching process that is especially important on the elementary level. At the enactive level the students could operate themselves with little toy cars or they would observe their own walk or their run during sports classes of school competitions. The iconic level could be dedicated to the description of the observations or even measurement procedures. The last symbolic level could be discussed by referring to the experiences of the students.

Vollrath (1984) gives another didactical theory of learning concepts. He outlines in general what kind of different steps lead to an understanding of mathematical issues. Roughly we could expect examples of the concept in class. Sometimes counterexamples help clarify what is meant. Then we could inspect properties of the concept in focus. Referring to our problem above we have a couple of examples given by a picture.

Since it is clearly an application problem, one could expect experiments in class. They could demonstrate the range which velocity encompasses: very slow to very fast. These experiments would not only exercise pupils' observation capacities but would also bring into classroom the nature of todays' culture of natural science (Toretti, 2012).

Another concern is the fact that the concept of velocity can be looked upon as a real mathematical modelling procedure (e.g. Blum & Leis, 2005). Observable movements can be measured in two dimensions: length and time span. It could be arranged as a project for students in which the definition of velocity is the end product of their investigative endeavour. This point leads to the question why a fundamental phenomenon like the concept of velocity lacks any historical approach in textbooks. For example the students could measure the free fall of objects getting an idea of how scientists in the middle ages approached problems of velocity.

Relying on the idea of anti-didactical inversion of Freudenthal we look into the historical development of the concept of velocity with the idea that Newton might not have thought primarily of it as a function since he was still following Galilei's proportional theory. The development starts with some inherent philosophical considerations since in times of Newton the subject matter still belonged to the so-called natural philosophy.

Historical and Philosophical Aspects

The concept of velocity is one with a long tradition, similar to the history of calculus (e.g. Doorman & van Maanen, 2009). Embedded in the concept of motion already Greek mathematicians, especially Aristotle (384–322 BC), had ideas about velocity. He combined his observations of velocity with the ones of the spheres and of the free fall of objects. Within his texts one finds expressions like "quicker than" or "slower than", meaning, same distance in a shorter time or vice versa. Before historically Galilei (1564–1642) did the next step, Nicole Oresme (1360) used graphic representation of changing quantities. Later Galilei used experiments to argue for the statement that there is a quadratic dependency between the distance travelled and the falling time. Even later Newton (1642–1727) defined velocity using the concept of force that initiates the change of motion. Leibniz (1646–1716) developed the differential and integral calculus also considering the idea of (planetary) movements.

In the following sections, the focus is on two aspects that play a major part observing the historical development and looking at the philosophical impact of it: Firstly, the formal definition of the concept of velocity was preceded by a long struggle for a clarification of the concept of motion. Secondly, the concept of velocity embodies a circular reasoning.

Although a lot of the work of Archimedes (287–212 BC) concerning mechanics is transmitted and gives an idea of his far-reaching mathematical understanding, we have no clear idea what he understood by velocity. Aristotle (384–322 BC) left us his investigations and conclusions about the phenomenon of motion qualitatively and verbally (Aristotle, Physics, 1829). He distinguished three types of motion: motion in undisturbed order, such as the celestial spheres, the "earthy" motion, such as the concept of the rise and fall, and the violent motion of bodies that needs an impulse (cf. Hund, 1996, p. 29). Although his remarks touched the phenomenon of velocity, his conceptions proved wrong later on: "Aristotle came close to the concept of velocity when in the sixth book the words 'faster' (longer distance in the same time, same route in shorter time) and the 'same speed' are explained (Hund, 1996, p. 30)".

Galilei (1564–1642) succeeded in a better understanding of the concept of velocity, as he did not rely on his direct perception. Galilei was the first whose insight relied on inspection of how nature "behaves". On his early experiments "to ask nature" later physicists build their theories: "The means of scientific evidence was invented by Galilei and used for the first time. It is one of the most significant achievements, which boasts our intellectual history […]. Galilei showed that one cannot always refer to intuitive conclusions based on immediate observation because they sometimes lead to the wrong track" (Einstein & Infeld, 1950, p. 17). This intellectual turnaround from earlier thinking to later developments is important enough to be pointed out when it comes to scientific reasoning.

Weisheipl (1985) draws the attention to Galilei who struggled with the Aristotelian concept of nature. Aristotle considered nature as an active principle. "Nature is a source not only of activity but also of rest" (op. cit., p. 22) which has an impact on the understanding of motion (op. cit., p. 49.). He also still pondered over the idea of Parmenides "All change is illusion" and the one of Heraclitus "Everything is flux". Galilei can be seen as being the scientist at the brink of Aristotelian sight of nature and the one later proposed in Newton's Principia and Descartes' Principia. Without him, the work of Newton had no basis for further development. Newton formulated (in Latin) the principle of inertia like this: "Every body preserves in its state of rest or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it (Weisheipl, 1985, p. 69).

Another author, Palmerino (2004), argues for a new sight of the reception of Galilei's theory since like any other theory it was not at once accepted in Galilei's time. And certainly not known in the way textbooks nowadays present e.g. the free fall and projectile motion (Palmerino, 2004, p. 140). During the last decades it became apparent that the ideas that Galilei, Descartes and Newton presented at their times were different from each other. Also the European continent was not aware of Galilei's and Newton's theories. Only later Leibniz set a distinct focus on his functional approach using variables. That is, the way the theory of motion and velocity is presented nowadays in textbooks is much more formed by Leibniz than by Galilei and Newton although it was mostly their ideas that established eventually.

Only Newton (1643–1726) introduced the concepts of absolute space and absolute time. This new setting, or new modelling of reality ended the long history of relative space and relative time and opened up the exact relationship between force and motion: The power does not get the motion upright (Aristotle), but it causes its change (acceleration). While Aristotle argued by inspection, Newton made an abstraction as he looked upon length and time as not necessarily bounded materially. This abstraction formed the basis of further development, in Mathematics and in Physics and also constitutes the point in time to leave natural philosophies behind and turn into direction of modern physics with its mathematical language to express physical conditions.

In today's linguistic usage, we understand motion as a change in position in the (Euclidean) space during a certain period of time. Lengths and time periods are conditions for the quantification of such motions. On this basis, the (average) velocity is defined as the quotient of the distance travelled and the time required.

We come to the second point of close inspection of the concept of velocity. A circular argument is obvious on a closer look: Time depends on a movement and vice versa, because time is measured using motion (Mauthner, 1997, Vol. 3,

p. 438). For example, in the hourglass sand runs through, in an analogous clock the pointer moves, and for the period of a year, we follow the cycle of the earth around the sun.

Likewise, the idea of space is necessarily connected to motion, for only through movement we perceive the space. Mauthner said: "His [The people, note of the author] language makes it impossible for him to understand the metaphorical tautology of the preposition 'in'. Only rigorous reflection will enable him to understand the metaphorical aspect of the preposition (in time). In space means something like 'in the space of the room', 'for the time' as much as 'in the space of time' (ibid, p. 443)". Even Piaget refers to this circular argument: "Speed is defined as a relationship between space and time—but time can be measured solely on the basis of a constant velocity" (Piaget, 1973, p. 69). Also due to him one is aware that the concepts of space, time and speed are mutually dependent.

These two aspects already throw a distinct light on the long lasting development of the conceptualisation of velocity. It also shows how long the journey was to create a step out of the understanding of the concept of motion in Greek times. The earlier was observable, even the spheres, the latter involved thinking and experimenting. It is by no means self-evident and asks for explanations in class.

Some Mathematical Aspects of the Velocity Concept

The reasoning of Aristotle and Galilei is based on their observations of linear motions. However, both had also planetary motions in mind. For motions on a curved path you need two different aspects: the direction and the magnitude of a velocity vector. It is this distinction which led, in modern terms, to a vectorial description and thus to a further clarification of the concept of velocity, which is thus a generalization of the concept of velocity on a straight line. Bodies on a straight line have the same speed, in the same direction and the same absolute value. Since then, the following statement is true: The changes in force and velocity are vectors with the same direction. We observe an idea of permanence, because all statements that apply to velocities along curved paths must also apply to linear trajectories.

The cause of this observation is obvious by an idealized thought experiment, which confirmed the theory (Einstein & Infeld, 1950); this is yet another idea that came to an effect only at the time of Galilei. Since then, the mathematical language is used in physics to reason for not only qualitative but also quantitative conclusions.

In mathematics classes one has to recognize another concept when applications are involved. As soon as one engages in quantitative calculations applied to the real world, one deals with quantities ("Grössen"). With respect to the concept of velocity you have the dimension (the quotient of distance and time) and the measured value (an element of the real numbers), which is a composite physical quantity.

Griesel (1973) analysed the subject matter of quantities on the primary level (length, weight, time periods) as a technical background for the didactics of quantities. This presentation does not fit the quantity of velocity (and is not mentioned there) because it requires a description as an element of a vector space, which can be higher than one-dimensional.

However, Freudenthal (1977) argues that one can interpret measure indications in terms of function symbols; an idea that was not previously addressed in classes. Another functional aspect occurs in two ways with the concept of velocity: The distance-time-function leads by considering of the difference quotients to the average speed and the transition to differential quotients to instantaneous velocities that are themselves again functions, namely, the velocity-time-functions.

Some Philosophical Aspects in Mathematics Class

It has taken over 2000 years for the concept of velocity to be defined consistently out of the concept of motion—in today's view within mathematics education a process of mathematical modelling. This is therefore a prime example of a concept belonging to mathematics as well as to physics. Its conceptual development was accompanied by mathematical and physical-mechanical representations throughout history. The use of tables in modern textbooks does not show its historical impact.

The intuitive conclusions drawn by Aristotle led to difficulties later in history and proved much later as untenable. Only an idealized thought experiment by Newton led eventually to an accepted and verifiable physical theory. The change in thought happened as Galileo and Newton started asking nature and prepared experiments. This is also the start of the separate development of mathematics and physics.

In contemporary mathematics education the concept of velocity is an example of a mathematical application and possibly of a modelling process of general motion. One could look upon it as a qualitative knowledge with additional potentials as there are qualitative and quantitative statements as well. The knowledge of such phenomena, the resulting misconceptions that even adults still carry with them and the trodden paths of knowledge are essential components of a mathematical education that should exemplify scientific processes.

The genesis also shows a potential for didactical perspectives of mathematics education. Despite the scarce representation of this topic in mathematics classes (in many curricula of the German federal states the concept of velocity is mentioned only one time on the secondary level), there is a diversity of ideas which can be reflected in class.

Instead of giving students tables at once, a long process could be undertaken before the computational form is addressed. The idea of velocity needs to be in the centre of classroom discussions (Vollrath, 1978) long before one starts to compute. These discussions could centre around the following questions: Where can we observe motion? How do we perceive motion? What can be called quick or quicker (slow, slower)? How could we measure observations? What measurements could we use? … This kind of endeavour invites the pupils to think and to experiment for their arguments and can lead to a thorough understanding of how and why tables could be used.

The above considerations show on the basis of the concrete example of velocity what Ernest (2016) is saying about the limits of human understanding and knowledge: "However, a closer examination also reveals that mathematics, viewed as the most certain and infallible of all the disciplines, is beset with limitations and uncertainties that strikingly show up the limits of human understanding" (Ernest, 2016). And the genesis of the velocity concept makes also apparent that the mathematical knowledge develops which is notably insightful when it comes to applications.

It is also obvious with the example of velocity that mathematics has two roles: mathematics as a science of its own and a complementary, describing science, here to model a physical phenomenon.

Conclusions

The concept of velocity has a long history in mathematics and in physics and roots within the Greek natural philosophy. Its genesis shows different perspectives and roles of the two fields and different philosophical ideas evolved during the rich historical development. It also shows clearly that some ideas became "thinking obstacles" that had to be thought over again to remodel reality.

It is therefore an excellent example for the historical development of the Natural Sciences and the role that mathematics and physics play in it. Additionally it also shows the role that technical applications play in our modern world. While the Greeks could only observe spheres moving, our daily life nowadays gives a lot of ‚moving' objects to observe and to manipulate (Torreti, 2012).

This observation offers opportunity for a lot of experiments which can be done and describe by the pupils in classroom. They learn to ask questions, to set up experiments, to make sense out of the data they receive and learn to interpret them.

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Time for Work: Finding Worth-While-Ness in Making Mathematics

Hilary Povey, Gill Adams and Colin Jackson

Abstract In this chapter we are concerned to understand the connection that can occur for primary school children between relevant practical, "hands-on" engagement with the material world in partnership with others and the development of mathematical commitment, enthusiasm and understanding. We draw on our personal experiences of children who, as part of a theory driven intervention, prepared for a mathematical exhibition by working on extended tasks which they then displayed and explained. We use the writings of David Jardine on time and Peter Applebaum on work to aid our thinking. We point to how absorbing, extended practical work recognises and calls forth our humanity and can provide purposeful spaces for making mathematics.

Keywords Practical work · Time · Work · Artefacts · Primary

This chapter draws on and slightly extends Povey, Adams, Jackson, and Ughi (2016a).

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Introduction

In this chapter we begin by introducing a primary school mathematics intervention which was based in five European Union (EU) countries—England, Germany, Hungary, Italy and Portugal.[1] We, the authors, participated in this project in England as researchers, curriculum developers and teacher educators; and we contributed to the theoretical underpinnings of the project. The classroom activities which were generated as part of the project extended over time and involved practical "hands-on" work leading to mathematical artefacts of one kind or another which would be exhibited and have their meaning explained by the children in a public exhibition.

Our experience of the exhibition in England and the commitment, enthusiasm and mathematical meaning-making of the children surprised us. It led us to ask ourselves broadly philosophical questions about how using our hands in practical activities generates space to think, reflect and understand. Here, we draw on the writings of David Jardine and Peter Applebaum to aid our thinking, considering how absorbing, extended practical "work" (Arendt, 1958) provides respite from the fragmented and alienating piece-work undertaken in the clock-time of much contemporary mathematics in primary schools in England and gives instead a space for meaning-making. We conclude that such experience also acknowledges and underscores our humanity.

Mathematics in the Making (MiMa)

In this section, we give the background to the intervention and discuss its theoretical underpinnings. We then describe and illustrate the project itself from the perspective of England, including something about the classroom activities devised, the model of teacher professional development employed and the experience of the public exhibition in England with which the schools' engagement with the project culminated.

Background to the Intervention

The intervention was the brain child of Emanuela Ughi, Professor of Mathematics at the University of Perugia. She had noticed over the years that many of her

[1] This was an EU project, *Mathematics in the Making (MiMa)* (http://www.mathematicsinthemaking.eu), (Project no. 539872-LLP-1-2013-1-IT-COMENIUS–CMP) funded through the Comenius Lifelong Learning Programme. The partners were Università degli Studi di Perugia, P3 Poliedra Progetti in Partenariato, Eotvos Lorand University, Mathematikum, Universidade Nova de Lisboa and Sheffield Hallam University.

undergraduate mathematics students, despite being successful products of school mathematics, had very limited mental images of mathematical objects and found mathematical visualisation difficult. This impeded their understanding of the mathematical concepts they were studying. Emanuela and her students built mathematical models—for example, articulated systems of interconnected levers to explore surfaces or designing cylindrical mirror anamorphoses to explore the laws of reflection—and discussed them as part of the students' studies, deepening the students' grasp of key mathematical ideas. Emanuela realised that, had they had concrete, "hands-on", visceral, practical experience of mathematical objects earlier in their lives and the opportunity to make, handle and "play" with them, it would have enriched their previous learning.

The second key driver for the project was the recognition that for many young people mathematics is thought of as a "meaning-less" subject only accessible to the gifted, a subject in which one is not expected to make meaning but only to carry out routine and mechanical procedures (Organisation for Economic Co-operation and Development, 2009). The combination of these two concerns produced the *MiMa* intervention project.

Theoretical Underpinnings for MiMa

The recognition of the need for practical, "hands-on" experience in learning mathematics is not new. In the 19th Century, Pestalozzi (1801) argued that learning followed from *hand* to *heart* to *head*: knowledge is gained by first providing motivating mathematical work for the hands. Such activity fosters joy and thus enters the heart of the child and from there a pathway opens up to the head where deep cognition can then take place. At this point, following Pestalozzi, we understood the significance of working with the hands to be about an active rather than a passive engagement with the learning process, with the child having direct experience of the world and using all the senses for learning. We also followed him in wanting to offer cross-curricular opportunities for learning.

We saw these ideas as being developed further by Bruner (1966). He argued that understanding develops from the *enactive* to the *iconic* and thence to the *symbolic* and that our approach to teaching should reflect this. The enactive is concerned with concrete and material engagement with a concept with physical objects being observed or manipulated or the body itself being used to make and experience phenomena. Extended experience of the enactive allows us to move to the iconic in which we abstract from direct experience and use representational forms—pictorial, graphical and so on. Further abstraction leads to the symbolic where the concept becomes understood as and located within systems of symbols.

In our view (Fig. 1), the *hand*, the *heart* and the *head* and the *enactive*, *iconic* and *symbolic* are not to be understood as disconnected nor progress through them to be thought of as rigidly linear. For example, learners return to the enactive at all levels of mathematical thinking to scrutinise and deepen existing knowledge and to

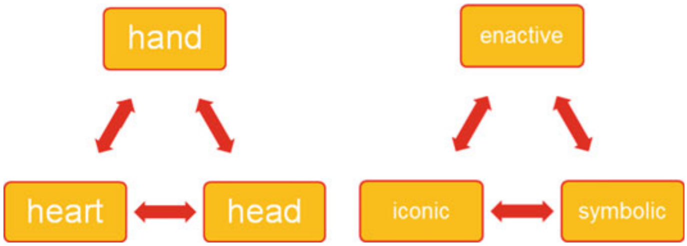

Fig. 1 Two of the theoretical underpinnings for *MiMa*

resolve new uncertainties. Both the enactive and the iconic remain as metaphors in the mind to support symbolic thinking. Similarly, deep analytic thinking also arouses the heart and can become embodied and be communicated through the hand.

The *MiMa* project also embraced the idea that learning should be understood as fundamentally social: Vygotsky (1962, 1978) stressed the role of social interaction in the development of cognition, with children and their partners co-constructing knowledge within their zone of proximal development. Alongside this we saw effective learning as dialogic (Alexander, 2008) and considered that dialogic learning is fostered through practical, "hands-on" activities: these enable learning to be *collective* with children engaged together in shared tasks; *reciprocal* with children listening to and experiencing each other's ideas; and *supportive* with a multiplicity of approaches and solutions being encouraged. Lastly, the project acknowledged that worthwhile learning is democratic (Dewey, 1949). We know that knowledge is shaped by the pedagogical context in which it is acquired (Povey & Burton, 2004). Practical, "hands-on" activities allow children to make mathematics itself whilst, together, they make mathematical artefacts; and learning becomes open and consensual which accords with a democratic way of life.

The Intervention

Working together and sharing ideas, the *MiMa* partners produced ten sets of activities (MiMa, 2015a, b) with teacher notes and explanatory introductory videos. The activities included, for example, making giant logic mazes in the playground; building a variety of polyhedra including a whole class beehive; creating a model of the solar system and taking the scaled "Neptune" two miles from the metre "sun" in the school playground; constructing frieze patterns in clay; and making and calibrating a variety of sundials.

In England, participating teachers from five local primary schools experienced the activities themselves through two day-long professional development workshops (Fig. 2) in which they were able to work "hands-on" with concrete materials. They made some of the mathematical artefacts and were able to discuss their own

Fig. 2 Participants in the *MiMa* teacher professional development workshops in Sheffield, UK

mathematical understandings. They then ran mathematical "laboratories" based on the activities with their children in school.

A distinctive feature of *MiMa* was that, throughout the laboratories, the teachers and children knew that they were preparing their objects and activities for public exhibition,[2] with the schools often having a "dry-run" in their schools for parents and the rest of the school community. Thus a fundamental aspect of the *MiMa* methodology was that children would display and explain their mathematics to others—their parents, other children and members of the general public (Fig. 3). The physical models and activities produced in the project represented the concrete output of the children's mathematical thinking and acted as a supportive bridge to their reasoning and explanations. The reflection on their own learning and the mathematical strategies they had employed (which the process of explanation required) contributed to the metacognitive strength of the experience.

Visitors to the exhibition were encouraged to join in the children's mathematics and children and teachers were full of enthusiasm and praise for *Mima*, planning to continue to use the activities and to involve neighbouring schools.

Reflections 1: Time and Worth-Whiling

We were at the exhibition ourselves and were very much surprised by what we saw and heard. We had not anticipated the commitment and the depth of understanding that the children displayed towards their mathematics. This led us to revisit our thinking about the theoretical underpinnings and to seek to think more deeply about

[2]The teachers all noted how important the exhibition had been for the children's learning. The five themes which emerged during the initial analysis (the opportunity to exercise responsibility and autonomy; explaining and communicating mathematics clearly to other people; becoming successful and more confident learners; experiencing effort and engagement; and being part of a wider community) are reported in Povey, Adams, Jackson and Ughi (2016c).

Fig. 3 The *MiMa* exhibition in the Winter Gardens, Sheffield, UK

the nature of the "hands-on" experiences of the children and their teachers. Our interest in exploring further was awoken:

> Something awakens our being in the middle (*inter*) of things (*esse*), and we find that there is a story already underway, one in which we are already moving and living. (Jardine, 2013, p. 1)

The first thing we noticed was that the sense of *time* experienced in the project: it made a place for extended, full-filled time very different from the clock time of contemporary schooling (Jardine, 2013). The practical, extended "hands-on" tasks sometimes seemed to have created "worth-whiling" time through the opportunity opened up by the device of the public exhibition which made the tasks 'worth lingering over, meditating upon, remembering, and returning to' (Jardine, 2012, p. 174). We have written more fully about this elsewhere.[3]

Further, however, reading Jardine prompted us both to think again about the practical, "hands-on", even visceral and embodied *nature of the activity itself* which was exemplified by the *MiMa* tasks and also to reflect on the way in which this contributed to mathematical meaning making amongst the children. As well as providing tools to think with and metaphoric representations to revisit when developing mathematical understanding, the *MiMa* activities also provided the opportunity (sometimes, it has to be said, not taken up) to become absorbed and thoughtful, to work like an artisan creating a hand-crafted product, be this a series of scaled models of the planets or a gigantic carpet of tessellated shapes or, from a

[3]We have explored this further in Povey, Adams and Boylan (2016a).

Fig. 4 Children's work from Portugal

cardboard box, a decorated lion with head, mane and tail and visiting flea (Fig. 4). Opportunities were there for 'conversing, debating, investigating … exploring, studying, changing your mind, illustrating … demonstrating what you know and giving a public account of that knowledge' (Jardine, 2012, p. 126).

We know from introspection that, as we cut or paste or colour or construct or mould or … the mind relaxes and expands, we become absorbed; and a space is created for thinking. (Hazlitt (2005) powerfully evokes a similar effect from a long country walk taken on one's own.)

This is in sharp contrast to the fragmented mechanical tasks—the piece-work—of usual school mathematics (Boylan, 2004) which are reminiscent of the Tayloresque world of factory production where 'no one of these isolated bits or pieces *requires* any prolonged attention' (Jardine, 2013, p. 9). Tasks are utterly fragmented in school mathematics for most English schoolchildren, with atomised "learning objectives" against which "progress" must be made in a single lesson (or, even worse, within twenty minutes, if the school inspectorate is around). Giving attention and devotion to task at hand

> is very often not simply *unnecessary* but *impossible* because the school-matters at hand have been stripped of the very memorability and relatedness … that might require and sustain and reward such attention and devotion. (Jardine, 2012, p. 175, initial emphasis)

The mathematical activity becomes meaning-less and alienating rather than allowing for 'the unity of thought and action, conception and execution, hand and mind' (Marx, 1975, p. 285) which occurs with meaning-full work.

Reflections 2: Worth-While Work

Peter Applebaum reminds us of Hannah Arendt's distinction between "labor" "work" and "action" (Arendt, 1958) and suggests these as a way to think about teaching and learning mathematics:

> Mathematics educators and mathematics education researchers might use the distinctions across types of activity from Arendt to analyze the forms that are manifested in their practices. They would be interested in the relationships and potential for mathematics to be a catalyst of action rather than merely work, and hope to encourage forms of activity that avoid the alienating effects of what would be described as labour. (2016, p. 4).

For Arendt, inasmuch as we only "labor", without the possibility of experiencing "work" or "action", we are being denied the right to our humanity (1958, p. 84). We would describe usual school mathematics as "labor", defined by its remorseless repetition of isolated units of task, the task of the assembly line, which carry no meaning or purpose for the one "laboring". We speculate that, occasionally, at times, the *MiMa* children and their teachers experienced instead a re-imagined school mathematics in which "work"—the interaction between the material world and human artisanship—produced whole and meaning-full artefacts. For the children these would be their mathematical objects and for the teachers, perhaps, lessons that fully made sense in human terms. From our observations and conversations at the public exhibition, we believe that from time to time—though, we think, at best, infrequently during the *MiMa* intervention—the "work" became "action", that is, purposeful, worthwhile activity with others with whom a common goal was shared and which motivated the endeavour. For Applebaum, "action" involves meeting as a group to work on the material world for some purpose, 'maybe even to specifically learn mathematics, or to use mathematics to learn something else' (2016, pp. 3–4). He envisages classrooms where learners:

> come together to create mathematics problems that they need to solve, rather than provide answers on tests that others score for no apparent purpose, disconnected from the classroom experience. (Applebaum, 2016, p. 4)

When we take hold of such practical "work"—or even, "action"—it in turn, reciprocally, takes hold of us:

> This possibility of, shall we say, "absorption" and being moved and addressed and, shall we say, summoned or beckoned by the work itself, is phenomenologically familiar. (Jardine, 2013, p. 23)

In the process of taking hold of and absorbing us, the task and our engagement with it have the capacity to open up possibilities for seeing aspects of the world and its mathematics differently—and thus to create a space for mathematical meaning-making.

Conclusion

We have described and discussed a primary mathematics intervention which was based on practical, "hands-on", extended mathematical activities leading to a public exhibition of the children's work at which the children were called upon by the visitors to show and explain their work. Our experience of the exhibition and the children's engagement and their mathematical authority surprised us and prompted us to think more deeply about the connection between the nature of the activities with which the children were engaged and the mathematical meaning-making they showed. We have suggested that, sometimes, the *MiMa* activities and the preparation for the exhibition allowed children and their teachers to escape from "labor", from piece-work and from clock-time to "work" on tasks which commanded the children's attention and with which they were able to be absorbed.

The recognition that there is something to be understood about the relationship between the practical and embodied and the development of the intellect is not new—for example, Comenius, Locke, Rousseau and Franke (Olafsson & Thorsteinsson, 2011) all wrote about education and the role within it of artisanal craft. However, it is a link that has never occupied a central place in English schooling; and the understanding that there is such a link has been very largely lost generally in England school education and, even more so, in the teaching and learning of mathematics. In these neo-liberal times, this disconnection has been exacerbated by the performativity culture of schooling (Ball, 2003) which requires that teaching and learning becomes fragmented into measurable and auditable "bite size" pieces, thus precluding extended tasks which absorb us and for which we may not have a fixed and final end-product pre-determined.

Interrupting such ways of thinking is very far from easy and we are very far from wanting to suggest any sort of "magic bullet". Nevertheless our experience of children engaged in practical, "hands-on" mathematics in preparation for a public exhibition where their mathematical artefacts were viewed and mathematical explanations invited in genuine acts of communication—in other words, in response to enquiries from those who do not already know the answers—suggests that this is one way to provide some respite: being purposefully absorbed in meaning-full worth-whiling "work" acknowledges our humanity and allows us, sometimes, to make some mathematics.

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Hades—The Invisible Side of Mathematical Thinking

Walther Paravicini, Jörn Schnieder and Ingrid Scharlau

Abstract The chapter presents a philosophical approach to teaching and learning mathematics in which five philosophical methods are translated into methods of reading and writing mathematical texts. The philosophical approaches are Hermeneutics, Analytical philosophy, Dialectics, Experience based (phenomenological) and Speculative philosophy. We use the acronym HADES for the combination of these approaches. For each of them we present reading and writing material which can be used for teaching peer tutors and by them in their interaction with students.

Keywords Reading mathematics · Writing mathematics · Philosophical approaches · Teaching and learning mathematics

Introduction

Mathematics is a language of written text. Doing mathematics thus involves the close reading of mathematical texts. Mathematical language often consists of an intricate, strictly regulated interplay of prosaic and formal language. Importantly, however, the meaning of mathematical texts is not restricted to their deductive or logical consistency. It also results from additional historical aspects of theory and research and aspects of discourse which often go unnoticed—such as what is left out of a text or how a question is framed. Thus, we view mathematics—as all disciplines—as practice. Under the perspective of their purposes and means, a

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conclusive understanding of mathematical contents, theorems and even theories presupposes comprehensive knowledge and appraisal of the interests which govern research—as well as those interests which have been supplanted.

One very important task for student tutors is to encourage first-year students to independently learn mathematical thinking. Among the obstacles for learning are mathematical texts, especially the economical style of presenting mathematical content in introductory textbooks, which abandons the context of discovery in favour of the context of justification, as well as the very common focus on a correct solution. In the present chapter, we develop a systematic approach for dealing with and understanding mathematical texts. This educational approach combines specific mathematics-related writing, reading, and understanding-related tasks with meta-reflection. In the short run, the methods help tutors to richly understand mathematical thinking; in the long run, they help to question and analyse how mathematical content is typically and appropriately presented, that is to engage in critical self-examination. We expect that the approach we have developed (termed HADES—explained below) will firstly support tutors to make content intelligible both by means of enhancing their reading and writing skills and by increasing their understanding how mathematics is done. They will secondly be able to integrate (parts of) the methods into their own teaching.

The basic idea is to develop philosophical methods of thinking and working into reflective reading and writing strategies for mathematical texts. We assume that comprehensive understanding of mathematical texts can result from the interplay of five philosophical or "elementary" methods of thinking (Martens, 2003). These methods are based on the idea that rational scholarship can be explained by and construed of elementary thinking tools which can be learned and taught. These are,

- H: hermeneutic methods of understanding and interpreting texts and images
- A: analytical methods of precise definitions
- D: dialectical methods for analysing (apparent) contradictions and structured dialogs
- E: experience-based or phenomenological methods of exact descriptions
- S: speculative methods of (thought) experiments.

These methods can be considered the didactic essence of five important schools of philosophy (Nida-Rümelin, Spiegel, & Tiedemann, 2015) and were developed in the context of a didactic of philosophy with the goal of fostering comprehensive text understanding. They were derived by simplifying classical philosophical positions to their essential features for educational purposes (Schnieder, 2013) and combined with appropriate tools for reflective and exploratory reading and writing.

In the present chapter, we describe these five elementary methods. We provide examples of how the methods can be turned into mathematical reading and writing tasks. Our main focus is on peer tutors and our aim is to provide them with tools for rich and reflective communication about mathematical content or, more precisely, mathematical discourse.

Hermeneutics

As a theory and practice of understanding texts, hermeneutics provides methods to incorporate what has been said by others, especially what has been passed on in writing, into one's own theoretical thinking. Hermeneutics thus relates a text systematically to an individual horizon of understanding. Besides objective understanding, which is constituted by historical reasons and effects, this approach accentuates subjective understanding: gaining access to meaning by starting from prior knowledge in order to subsequently relate "objective and subjective interpretations" to each other. Thus, hermeneutics addresses problems that arise because the language of the arguments in a text is not or is only partly one's own language. At its core is the hermeneutic circle (Gadamer, 2010), a process of prior understanding (Vorentwurf), text understanding (Textverstehen), and the fusion of horizons (Horizontverschmelzung). Readers inevitably move in a hermeneutic circle because they read texts with expectations and understand individual statements in a general context and, conversely, the general context from single statements.

The hermeneutic circle allows focused work with mathematical texts, structured reading, slow reading, and self-clarification. Within this circle, students establish connections to their prior knowledge. The main operation is translating text into one's own language. Specifically, the hermeneutic circle requests readers to check the suitability of the concepts used in translation by making them explicit and showing how a term is a suitable translation or why its meaning cannot (yet) be confirmed with the help of the text. Naming these differences and similarities of text and translation allows the reader to put forward new interpretations and test hypotheses which might reduce the differences.

The hermeneutic circle has two dimensions, *depth* combining the historical development of a mathematical proof with its critical reconstruction, and *breadth* containing aspects of prior knowledge and assumptions which can be used for developing hypotheses. Following Martens (2003), interesting aspects besides the logical structure of a text could be its cultural or historical context—particularly in combination with its scientific claims—, its structure, ruptures, gaps, and fringes and, last, but not least, its effect on the reader.

Rhetorically speaking, the primary goal of the hermeneutics approach in HADES is to engage in reflecting and making systematic and rich connections to personal thinking and prior knowledge. This goal can be pursued by a *writing circle method* of which we give an example: Tutors are presented with different historical texts which are the starting point of a central mathematical idea, e.g., Barrow's (1976, Lectio X, Prop. 11) preliminary version of the fundamental theorem of calculus together with the corresponding picture (Jahnke, 2009, p. 87). Firstly, the tutors translate the texts according to their actual, unaided mathematical understanding, similar to the situation of a student. Secondly, they work through the hermeneutic circle and note their thoughts as inner monologue. For *prior understanding*, they collect all their impressions of the text, including subjective impressions. In a preliminary translation they then hypothesize what the text is about, including an analysis of the 'prejudices' governing their current understanding. In the *text interpretation* stage, they test how far their hypothesis summarizes its content and whether it deviates from the latter. They validate their impressions with the help of the text. Finally, in the *fusion of horizons*, they summarize their thoughts by enumerating similarities between the topics of the text and the translation. They then start the circle a second time, developing a deeper prior understanding. They compare their inner monologue with the translation from the first step which was made without help. Finally, they discuss the advantages and disadvantages of the hermeneutic circle and how it can be presented in their classes. The intensive step-by-step analysis of the text puts the recipients into the role of reflecting producers who, in a written inner monologue, describe their thoughts while translating, making them explicit and, in turn, comprehensible. The writers emphasize translation and conceptual issues, compare meanings, evaluate translations and make the evaluation comprehensible. This resembles methods from teaching writing, e.g. the exploratory methods developed by Bean (2011). Although the task itself already encourages reflection, a necessary final step is *meta-reflection*: describing and evaluating experiences and discussing transfer of methods.

Analytics

Analytic philosophy developed in close interplay with modern fundamental research in mathematics (Frege, 1953). Its methods therefore well match the analysis of mathematical concepts and theory. Analytic philosophy focuses on methodological aspects of scientific thought and especially on establishing clear rules of how to define concepts, how to argue and how to criticize. Consequently, epistemological, aesthetic, ethic, metaphysical problems are all put down to questions of language, concept, or argumentation analysis (Russell, 1948).

More precisely, analytic philosophy discusses norms and procedures for establishing concepts and relations such as explicit and implicit, predicative and impredicative definitions, abstraction and ideation. It furthermore develops methods for analysing arguments, for instance by uncovering hidden implications, and testing internal consistency. One important claim underlying this approach is that these procedures serve to clarify central elements of everyday and scientific language without taking any stand content-wise.

It goes without saying that mathematical texts and in particular mathematical proofs follow strict rules. Notwithstanding they are only partly deductive. Actually *doing* mathematics, solving a specific problem, cannot be reduced to applying the rules of formal logic. Consequently, formal logic cannot capture the complexity and creativity of mathematical research with its sudden insights and the personal involvement of researchers.

What is more: Mathematical proofs consist of an intricate interplay of prose and formal language in which the true claims are neither easy to find nor easy to validate —especially for beginners. For instance, arguments in proofs may be highly condensed: premises, background knowledge or warrants may remain implicit and tacit. This is where analytic procedures come into play: Their point is to uncover and highlight the formal structure behind the verbal form of mathematical texts.

Turned into a didactic approach, this means to introduce and practise the (classical) concept of argument and how to complete an argument by making its premises and warrants explicit. Teachable and learnable tools can be developed which aim at precise and detailed analysis and evaluation of arguments by deductive reconstruction (Toulmin, 1958).

To instruct students to pay close attention to the deductive structure of a mathematical argument when reading it line by line has been shown to be a simple and effective way to enhance student comprehension, at least to a certain degree (Hodds, Alcock, & Inglis, 2014). With the following example task we want to sensitise tutors to how complex the analytic procedure is that expert mathematicians seem to carry out automatically when reading mathematical argumentations.

The tutors are given an elementary, but non-trivial, textbook proof showing, for example, the existence of Feuerbach's nine-point circle or some number theoretic fact. They now extract, individually and by reading it line by line, every single argument that is given in the course of the proof, where an argument consists, quite classically, of a premise, a warrant and a conclusion. In some steps of the proof, a premise or a warrant might be implicit, and the tutors are asked to fill in these gaps. The individual three-step arguments are written down, each in a separate box, and the boxes are arranged into a tree-like diagram that reflects the global deductive structure of the argumentation. The tutors are also asked to add arguments that they feel are missing to the proof. What each tutor ends up with should be a diagram that represents the argumentation to a degree of completeness satisfying to her (see Fig. 1 for an example—with the actual arguments removed from the boxes). Note that similar diagrams are used for research purposes in the argumentation analysis literature (e.g. Knipping & Reid, 2014; Krummheuer, 2003).

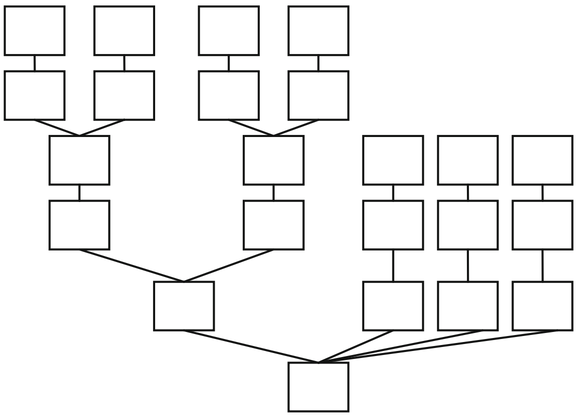

Fig. 1 The global deductive structure of a proof of the existence of the nine-point circle

Now the tutors are asked to compare their diagrams among each other and to reflect on how the ideals of completeness and exactness are met in academic mathematical writing—and on what (other) didactical tools might be suitable for helping novice readers to understand the argumentative structure of mathematical texts.

One central advantage of this approach is that it can provide students and tutors with a (low-threshold) search strategy that helps them find gaps in argumentations. It thus allows for the fact that mathematical proofs are composed as written texts which—at least on the surface—only gradually differ from other scientific or everyday texts although they are geared to the ideal of deductive conclusiveness. Note also that all arguments which can be correctly reconstructed with this method can also be reconstructed as deductively valid. In principle, thus, the method is independent of the teacher's demands and thus supports student autonomy.

Dialectics

Dialectics does not have a good standing in academia. It is often associated with pointless talk, formalistic debate, hair-splitting, and even manipulation. It might therefore surprise to see methods for reading and writing derived from such an approach. However, an essential goal of dialectics is and always was to develop methods which help to reason with specific addressees and in an open and undogmatic way (Martens, 2003; Rohbeck, 2008).

Mutual dialogs with concrete persons and within historical contexts play a prominent role in mathematics. Doing mathematical research does not happen in a silent inner dialog or purely rational thinking. Quite the reverse, argumentation that cuts off experiences, moods, or spontaneous ideas and that views itself as context-independent is only the final result of a highly communicative process. This process is both dependent on and oriented towards comprehensibility and communication. Talking with a concrete person is therefore expected to aid conceptual and argumentative clarity and thus problem solving and idea and knowledge generation.

Thus, mathematical arguments do not result from well-aimed manipulation. Yet, this notion can help to test the logical validity of an argument by systematically searching all possible criticisms which are then put into words and invalidated. Strictly speaking, a mathematical proof is understood only if all potential objections have been disproved. Thus, it is beneficial to be provided with methods for scrutinizing proofs from all perspectives in claims and replies.

This can be achieved by a reduced form of dialogical logic as developed by the Erlangen school of logical constructivism (Lorenzen, 1969): Mathematical proving and arguing is schematised as a dialog. In this dialog, a proponent and an opponent *stage a dialog* in order to attack and defend a thesis. This staging uses few predefined rules which suffice to formulate logically relevant objections (Lorenzen & Lorenz, 1978).

A further advanced writing method is a *dialog with the author* in which students take one part and the tutor the other. This approach dedicates special attention to how authors and their (assumed) readers are present in all kinds of texts and negotiate content. In order to foster this, attention is first drawn to the linguistic means by which this presence is realised: There is always crosstalk between authors and readers (or metadiscourse) in texts, even though readers (or writers) may not be aware of it. Linguistic signs of metadiscourse are rare or invisible in mathematical texts, but not absent. An important approach to this crosstalk is to *identify instances of metadiscourse* such as stance and engagement in a text (Hyland, 2010) and re-write it with a different amount of both (possibly more). So-called interactive means guide readers by transitions, for instance by frame markers, references, or code glosses, that is, words which help readers grasp meanings of material. Interactional means involve the reader in the text, for instance by hedges and boosters which withhold or emphasise the writer's commitment to a statement, attitude or engagement markers. The reading/writing task consists of (1) learning the different linguistic means, (2) identifying signs of interactive and interactional means in a mathematical text, including their absence where they might be helpful, and (3) rewriting the text with more (or even less) of those means. A variant of (3) would be to create a dialog of the author with another person, the core of the dialectic approach. It should follow the text structure, but complement it by including thoughts of the persons which can be deduced from stance and engagement markers and are turned into questions and the author's possible answers.

Experience/Phenomenology

Phenomenology as founded by Edmund Husserl (1970) is one of the most influential trends of current philosophy. Phenomenological methods uncover, in a complex analysis, how things present themselves. The phenomenological approach neither defines nor puts forwards theoretical propositions, but focuses on experience: What something is results from how different aspects form its actual experience. The question "What is X?" thus turns into the question "How does something present itself as X (to me, us)?" (Waldenfels, 1992).

Phenomenological methods foster deep understanding of mathematical concepts and ideas by illuminating the interplay between definition/concept and example—an interplay which is very important for learning. They provide processes by which one can approach the ideas or essence of mathematical concepts, that is the motivation behind their invention and study. Phenomenological writing methods should therefore encourage variation and support reduction to the essential. Such methods can be adapted from creative writing. For instance, concepts—imagine something like *continuity*—could be described with several of the following instructions:

> Consider an example for continuous functions of your own choice and its graphic representation. Then imagine you are standing on a ladder and look at the graph from high above. What becomes almost invisible in this perspective, what presents itself clearly? Or imagine yourself lying on the floor with your nose almost in the dust and looking at the graph with a magnifying glass. What becomes difficult to grasp? What is very present?

The students write down answers to the questions of at least a few sentences, capitalizing on the figurative aspect of the task, but without bothering much about what they suspect or imagine is correct (variation). The exercise ends with a first attempt at distilling the core of the concept by discussing which of the aspects visible in each perspective are essential and which are not (reduction).

This task provides a structured procedure to patiently and richly describe—for the time being without comparison without reference to formal definitions or statements. The example is described unhurriedly until ideas are gained which can be developed in a more theoretical manner. For instance, all examples of continuous functions found might be differentiable. Is that coincidence? Is there a counterexample?

The observations are then compared to the formal definition or scientific facts. Thereto, the descriptions are formalised and compared to the exact definition: Can the descriptions be identified as necessary or sufficient conditions for continuity? Can the claims be proved and illustrated with suitable examples and counterexamples? The opposite direction is possible as well: A concept could be illustrated by examples which are described precisely and in much detail, in what is visible as well as invisible. Again, the description is formalised and compared to the formal definition.

After the comparison, generation of examples continues (variation). Are there other examples—possibly more extreme ones? An important aspect of this procedure is that it can be executed a number of times. That means that it is not necessary that the first examples are good or exciting—better or richer examples will turn up in the process. A point of this method is that it conveys understanding as a process of sharpening of a concept. Thereby it allows to independently explore and understand mathematical content.

The task can be transferred to concepts that do not or only barely lend themselves to visualization. Instead of the very imaginative perspectives of the ladder scenario, other perspectives should be chosen, for instance logical, symbolical, pattern-related perspectives as well as metaphorical, spatio-temporal, social, communicative and even emotional models of presentation of a concept. In order to attenuate the possible strangeness of the task, a set of specified formulations might be used. Again, meta-reflection constitutes the last step, here with a focus on the question how perspective enables aspects of a concept to become visible.

Speculation

Speculative philosophy provides methods which support finding or devising new and relevant approaches to solve difficult problems—within philosophy itself, everyday life and not least science (Bloch, 1963; Peirce, 1997). Being able to have good ideas is an essential part of doing mathematics and an important starting point for developing mathematical ideas.

There are various methods of speculation in philosophy. As an example for the didactics of mathematics, we introduce an adapted version of freewriting. Freewriting as advocated, for example, by Elbow (1973) is a standard method in the teaching of academic writing. At its core is fast, uninterrupted and uncensored writing. To this end, all thoughts are written down without reflecting on them, evaluating them or revising them, even if they are only fragments. Writing uninterruptedly is meant to prevent reflection stopping writing flow.

Speculation draws on a combination of favourable attitudes, methods, sensitivity and art rather than a strictly methodical, step-wise procedure. Thinking speculatively, especially in science, often means to detach oneself from familiar perspectives, see through apparent necessities and constraints, but without giving way to arbitrariness. Newness as such has not value; it requires a purpose which arises from the particular context of research.

The method presented below meets these claims in its first two steps: Pre-set material helps the student tutor because it can be combined into sentences in a playful, non-committal way and thus with little risk. At the same time, they ensure that the sentences refer to a structured field and thus meet the need that new ideas are interesting insofar as they relate to current research. The method postpones the goal of refuting or finally proving and thus provides more space for thinking in everyday language or in a preformal or semiformal way: It provokes tentative deliberation of pros and cons, speculation and experimentation. Thinking is driven by questioning, raising objections, as an experiment holding a view which is not one's own and putting the results down on paper. The way towards an exact proof is staged as a slow approach. It is not the formalism alone which is the measure for correct work.

In the first step, writers are provided with concepts and (few) standard formulations (a variant of focused freewriting). Such a list might contain elementary concepts relating to functions—*injective*, *surjective*, *bijective*, *composition*, *range*, *domain*, *identity function*, *inverse function*, *group*, *unique*, *linear*, *homomorphic*, *continuous*, *iterated*, *and*, *or*, *not*, *if then, for all* and *there are*. The writers then produce as many linguistically and mathematically appropriate, complete sentences as possible, but they do not bother about the correctness of the sentences. This task has a given time, e.g. five or ten minutes.

In the second step, the claims are examined. In a second round of freewriting, first impressions, presumptions and thoughts about these sentences are put down as a stream of thought. This can be supported by a list of hedges such as “maybe”, “possibly”, “it is possible that”, “one could argue/imagine/point out” etc. Providing the list and encouraging to hedge one's own thoughts makes speculation easier for the students.

Two aspects alternate in this kind of writing: Firstly, ideas are generated by making use of intuition and imagination and thus gaining insights or raising risky presumptions. This is the core of the first step. The second step focuses on testing the insights and guesses with respect to their relevance and logical or argumentative conclusiveness. These steps can be repeated several times, for instance if no appropriate material has been gained in the first step or the second step makes it necessary to focus more closely on a certain area.

An optional (but fruitful and important) third step is meta-reflection. This can be done in pairs who think about attitudes and strategies which help them to think speculatively.

Scientific progress and creativity cannot be enforced; they cannot be put down into general instructions or trained. Planned creativity is a contradiction in terms. On the other hand, mathematical creativity and speculative thinking are more than a simply irrational, incomprehensible event which somehow leads to sudden insights. And even if creative insights cannot be produced, the ground for creativity can be prepared, for instance by helpful attitudes, methods and strategies, and fostered by internal maxims. Conversely, anxiety or little willingness to take chances in thinking can block creativity. How to develop the courage to take risks and the stamina to take novel and incalculable ways towards a solution, how to support composure facing the uncertainty of not yet having the solution and not being certain how to develop it, are all productive questions for the last step.

Outlook

We understand HADES as a descriptive as well as a normative system: We suggest that the five modes of thinking sketched above are central to how mathematical experts read and write texts, but also that the modes are usually chosen and applied unconsciously. We further suggest that tutors can use the methods we have elaborated above to improve how they read and write mathematical texts and, perhaps more importantly, to productively reflect on how they teach how to do mathematics. It might even prove to provide a reasonable scheme to plan their teaching.

In the future, we plan to improve the HADES tasks sketched above (and develop further tasks) by analysing how student tutors interact with them and how tutors incorporate them in their teaching. We suppose that specific aspects of (proof) comprehension can be improved in students by training them in one or the other of the five thinking modes—and that this compares favourably to a general training like the SET (Hodds, Alcock, & Inglis, 2014). We also consider the categories suggested by HADES to be a useful tool to analyse how mathematical experts read texts and solve problems.

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Developing Rules Due to the Use of Family Resemblances in Classroom Communication

Jessica Kunsteller

Abstract In mathematics education research the recognition and utilization of similarities is highlighted in various ways, for example in the context of learning by analogies or metaphors. In this article Wittgenstein's concept of "family resemblances" will be presented and illustrated by empirical examples. Based on this concept an overarching perspective on learning by similarities will be presented. Furthermore, different types of family resemblances will be exposed.

Keywords Family resemblences · Wittgenstein · Language game
Abduction

Family Resemblances

Ludwig Wittgenstein (1889–1851) used the term "family resemblance" in his later philosophy. He did not define this term but he illustrated it with examples, like the use of the word "number":

> And for instance the kinds of number form a family in the same way. Why do we call something a 'number'? Well, perhaps because it has a—direct—relationship with several things that have hitherto been called 'number'; […]. (Wittgenstein, 2009, §67)

While comparing signs like "35", "12 kg" and "π" one is able to recognize that they all have something in common—they or respective parts of them can be called "numbers". "35" and "12 kg" are natural numbers, while "35" can be used as cardinal number and "12 kg" as an index. "π" can be arranged as an irrational number, without having the shape of a "real" number included. These similarities, which are based upon notations, shapes or uses and can be regarded as "family resemblances" following Wittgenstein:

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> I can think of no better expression to characterize these similarities than 'family resemblances'; for the various resemblances between members of a family: build, features, colour of eyes, gait, temperament, etc. etc. overlap and criss-cross in the same way.—And I shall say: 'games' form a family. (ibid., §66)

Family members resemble each other in facial features or in character traits, as well as the mentioned numbers belong to the same family in different ways. Likewise words or (language) games can resemble each other due to their use in the language:

> [The] meaning of a word is its use in in the language. And the meaning of a name is sometimes explained by pointing to its bearer. (ibid., §43)

For instance the words "add" and "plus" can resemble each other concerning their use in a mathematics classroom in order to describe the same sort of operation. Meyer (2016) exposes the meaning of words according to Wittgenstein as well. In this article other kinds of (family) resemblances will also be presented.

Wittgenstein compares several games with each other. Games like tennis or chess resemble each other in different ways: both are sports, both have a winner etc. Wittgenstein concludes:

> And the upshot of these considerations is: we see a complicated network of similarities overlapping and criss-crossing: similarities in the large and in the small. (Wittgenstein, 2009, §66)

This quotation indicates a variety of similarities, especially the fact that the degree of similarities can be different ("similarities in the large and in the small", ibid., §66). In the scientist discussion there is a debate about the already stated "network of similarities" (starting with Bambrough, 1961). Like Wittgenstein they compare several features of games with each other. Kellerwessel (2009, p. 126) uses an application to demonstrate similarities between games.

In the horizontal row random sets of games (e.g. "1", "2", "3", …) are displayed and in the vertical row random features of games (e.g. "ABCD"). For example the games "1", "2", "3" and "4" correspond concerning feature "A", while game "5" does not own this feature. For instance the "division", "subtraction" and "multiplication" have the feature in common that they can invert another operation. In this way they are connected in a "direct relationship" (Wittgenstein, 2009, §67). As this example denotes similarities can be analysed from a normative point of view. However, according to the use of language similarities can be also observed in an empirical perspective when individuals explicate and use similarities.

Some of the games or words in Fig. 1 correspond in several features while they do not have to be identical. Even though they only conform in one single feature they are games or rather language games:

> […] there is no fixed list of family characteristics, nor fixed number required for admission, nor sharp border for the individual characteristics themselves. (Hallett, 1977, p. 150)

	1	2	3	4	5	...
Distribution of features	ABCD	ABCE	ABDE	ACDE	BCDE	....

Fig. 1 Application for (family) resemblances according to Kellerwessel (2009, p. 126)

Methodology and Theoretical Framework

The research concern of my Ph.D. study is to reconstruct and categorize (family) resemblances in discovering and reasoning processes. Therefore four lessons in mathematics (each 90 min long) about three different episodes of workings sheets, which have three different thematic topics from a normative point of view. The lessons were videographed and afterwards transcribed in three fourth (students at the age of 9–10) and two sixth grades (students at the age of 12–13) in Germany.

The qualitative analysis is based on an interactionist perspective (cf. Voigt, 1995). The interactionist perspective assumes that words are ambiguous and that the meaning of words is negotiated in interaction. The participants in classroom interaction obtain a "taken-as-shared understanding" (ibid., p. 164) of words. They negotiate a "thematic coherence" (ibid., p. 164) which contains the teacher's as well as the students' contributions. The "thematic coherence" does not depend on the content. Thus, every utterance is interpreted for itself, its implications and its consequences on following utterances.

In this article the students' discovering processes will be reconstructed as abductions wherefore the pattern of abduction will be presented in brief according to Meyer (2010). From a constructivist point of view learners create their knowledge themselves. They observe "facts" and form hypotheses, in order to explain their hypotheses (cf. Meyer, 2010, p. 189). Meyer (ibid.) elaborates a theoretical framework named "abduction" to analyse processes of discovering knowledge conforming to the theory of the philosopher C. S. Peirce (cf. Fig. 2). Peirce considers the abduction as third inference apart from deduction and induction. According to Meyer (2010, p. 189), abduction is the only (elementary) inference, which allows the reconstruction of new knowledge.

Result: $R(x_0)$	Result: $R(x_0)$
Rule: $\forall i: C(x_i) \Rightarrow R(x_i)$	Rule: $\forall i: C(x_i) \Rightarrow R(x_i)$
Case: $C(x_0)$	Case: $C(x_0)$

Fig. 2 General patterns of abduction according to Meyer (2010, p. 189) [in the left: "the cognitive 'flash of genius'" (ibid.); in the right: "abduction as process of making a hypothesis plausible" (ibid.)]

The abduction as a cognitive process begins with noticing a "(surprising) fact" (Meyer, 2010, p. 189) on the one hand, which we are going to explain by a (general) rule and a (concrete) fact on the other hand. The relationship of rule and case is a complex one: We infer the case by the rule. But the case is implicated in a concrete form in the rule's antecedent. Thus, if we know the rule we would already know the case.

Altogether, rule and case have to be inferred at the same time in order to explain the facts. However, if we explicate our abduction we (usually) use the rule to infer to the case. Thus, in the process of publishing hypotheses the rule gets a role as a premise. In the context of empirical utterances 'public abductions' will be reconstructed (cf. Fig. 2 in the right). Another rule might also explain the given facts and would go hand in hand with another case. Thus, abduction is not a necessary inference.

Empirical Examples

Subsequently, empirical data will be presented in order to reconstruct the use of (family) resemblances in learning processes. Different kinds of (family) resemblances will be exposed. Hereby, the stated concept of (family) resemblances will be elaborated. Therefore, an excerpt of a lesson in a fourth class in Germany will be demonstrated. The learners (aged 9–10) worked on a working sheet concerning balancing equations in addition (cf. Fig. 3).

At first the scene is displayed by presenting and by interpreting selected utterances along to the student's contributions (Section "Platonism 1"). Those abductions that are relevant for the subsequent analysis (cf. Section "Platonism 2") will be reconstructed. In the following section the pupils' expressions as well as their

Fig. 3 Excerpt of the working sheet, translated in English

(a) Solve the following calculations.

$20 + 2 =$ ____	$16 + 7 =$ ____
$18 + 4 =$ ____	$3 + 20 =$ ____
$16 + 6 =$ ____	$17 + 6 =$ ____
$14 + 8 =$ ____	$7 + 16 =$ ____
$12 + 10 =$ ____	$20 + 3 =$ ____
$10 + 12 =$ ____	$13 + 10 =$ ____

(b) Formulate two general rules.

abductions will be investigated concerning their use of (family) resemblances. Therefore, different kinds of (family) resemblances will be exposed.

Description and Interpretation of the Scene

After the learners have edited the working sheet the sums are noted on the board.[1] It is found that the sums are always *22* or *23* (part a). Afterwards the learners are asked to read out their rules (part b).

Anke's First Contribution (T 26–28)
Anke (T 26) reports:

> […] we had eh so at first to the first […] when the first number gets smaller by two the second number gets higher by two.. the outcome remains the same.[2]

Anke explains that the numbers *20, 18, 16, 14, 12, 10* each decreases by two and that the numbers of the second summand "rise up by two" (T 28) correspondingly. The discovery of this relationship can be reconstructed as abduction (cf. Fig. 4). Anke describes cases wherefore the sums always stay the same (result) by explicating the exchange by *+2* and *−2* along the corresponding summands (case). Anke's rule describes a general relationship between the equality of the sums and the change of the first and second summands without mentioning concrete numbers. Thus, her rule can be regarded as the abduction's rule.

Tanja's Contribution (T 72)
Another rule for the left calculations is uttered by Tanja (T 72): "if the second number minus the outcome […] is calculated ehm then it gives the first number." The teacher names this rule as "inverse operation" (T 73).

Matteo's Contribution (T 76–82)
Thereupon, Matteo raises his hand and states that there are always tasks that contain the "commutative property" (T 76) at the first calculations (T 76):

T 80 Matteo	eighteen plus four, and fourteen plus eight […]
T 81 Teacher	[…] show me what you mean. (*Matteo goes to the board, 4 s*)
T 82 Matteo	so, this was the number. *(points at "12", task "12 + 10")* […] that *(points at "10", task "12 + 10")* is changed […] and *(points at "10", task "10 + 12")* stays in the beginning, […] (*points at "12", task "10 + 12"*) this is the second number

[1]The calculations are noted on the board.

[2]The students' quotations were translated and transcribed by the author (rules of transcription cf. Meyer, 2010, p. 203).

Result: The sums of the first calculations are always 22.
Rule: „when the first number gets smaller by two the second number gets higher by two .. the outcome remains the same.“
Case: $(20-2)+(2+2)=18+\ 4$, $(18-2)+(4+2)=16+\ 6$, $(16-2)+(6+2)=14+\ 8$, $(14-2)+(8+2)=12+10$

Fig. 4 Anke's abduction (T 26–28)

Matteo's utterances indicate that the pairs of tasks *12 + 10* and *10 + 12* as well as *18 + 4* and *14 + 8* are tasks that contain the commutative law. From a normative perspective, the tasks *18 + 4* and *14 + 8* are not convenient concerning the commutative law. However, Matteo establishes this term when summands and ones are changed. He announces new cases that the sums remain *22* (abduction's result). The cases refer on the one hand to the changing of the addends (*12 + 10* and *10 + 12*) and to the changing of the ones with equal tens (*18 + 4* and *14 + 8*) on the other hand. The discovery, which may be the basis of his statement, can be reconstructed as abduction (cf. Fig. 5). The abduction's rule follows from the generalization of the case and the result due to the pattern of abduction (cf. Fig. 2).

Jakob's Contribution (T 86–90)

The teacher then asks to express rules concerning the right calculations. Jakob says that the tasks are "actually tasks that contain the commutative property and the inverse operation" (T 86). "Actually" can be read as attenuation to Matteo's utterances. Since the tasks *18 + 4* and *4 + 18* are not tasks that imply the commutative law in a normative perspective. Jakob shows which tasks are "for example" (T 80) "actually tasks that contain the commutative property and the inverse operation" (T 86) on the board:

> […] here *(goes along the tasks with his finger, here: "20 + 3")* and here, *(here: "3 + 20")* here *(here: "17 + 6")* and here, *(points along "16 + 7")* and then also here (*here: "7 + 16*") and here. *(here: "16 + 7")…* (Jakob T 90)

According to Jakob, the tasks *20 + 3* and *3 + 20*, *17 + 6* and *16 + 7*, *7 + 16* and *16 + 7* are "actually tasks that contain the commutative property and the

Result: The tasks 12 + 10 and 10 + 12, 18 + 4 and 14 + 8 have the same sum.
Rule: If „numbers“ are changed or the addends' ones (with equal tens) are changed, then the sums are the same.
Case: The addends are changed in the tasks 12 + 10 and 10 + 12. The ones are changed in the tasks 18 + 4 and 14 + 8.

Fig. 5 Matteo's abduction (T 76–82)

Result:	20 + 3 and 3 + 20, 7 + 16 and 16 + 7, 17 + 6 and 16 + 7 have the same sum.
Rule:	If „numbers“ are changed or the addends’ ones (with equal tens) are changed, then the sums are the same.
Case:	The addends are changed in the tasks 20 + 3 and 3 + 20, 7 + 16 and 16 + 7. The ones are changed in the tasks 17 + 6 and 16 + 7.

Fig. 6 Jakob’s abduction (T 86–90)

inverse operation” (T 86). Jakob discovers similar explanations as Matteo to explain that the sums are always the same (result): the changing of addends (*20 + 3* and *3 + 20*, *7 + 16* and *16 + 7*) as well as the changing of ones (with equal tens) (*17 + 6* and *16 + 7*) (case). The process of discovering can be reconstructed as abduction (cf. Fig. 6). The abduction’s rule follows from the generalization of the case and the result due to the pattern of abduction (cf. Fig. 2).

Possibly, Jakob takes the term “inverse operation” out of Tanja’s rule (T 72). According to the above interpretation, Tanja’s rule appealed to inverse the addition. From a normative point of view Jakob does not show tasks that include the inverse operation. One possible interpretation could be that “inverse” (T 86) gets new meaning insofar that the ones are “inversed”. In this perspective, tasks that contain the “commutative property” (T 86) are *20 + 3* and *3 + 20*, *7 + 16* and *16 + 7* and tasks that contain the “inverse operation” (T 86) are *17 + 6* and *16 + 7*.

Anke’s Second Contribution (T 92)

After Jakob was at the board, the teacher asks if the learners agree, whereupon Anke (T 92) puts her hand up:

> … so actually to one task there is not only the commutative property but rather th- there are always f- four tasks actually always […] of one sort, always because now it is seventeen plus six. and above it is not exactly the commutative property but rather sixteen plus seven, but this so to say includes as well, it is one less’ […] but seven is one more than six, and that’s why, […] there are always four tasks. (Anke T 92)

One possible reading is that Anke points out that *16 + 7* (“above”) is not the exact task that includes the commutative law of *17 + 6*. She explains what Jakob named as “actually tasks that contain the commutative property and the inverse operation” (T 86). Probably, she explains that these tasks are not tasks that imply the commutative law. Since she explains the changing of the first summand by -1 and the changing of the second summand with $+1$: $(17 - 1) + (6 + 1) = 16 + 7$. The summands’ changing could suggest the use of Anke’s first rule in a more general way (cf. Fig. 4).

Moreover, Anke states that every task consists out of four tasks “of one sort” (T 92). One possible interpretation could be that three different tasks can be formed out of the task *17 + 6*. Due to the summands’ changing the task *6 + 17* contains the “commutative property” (T 92) and according to the “actual commutative property” the task *16 + 7* can be formed. Out of task *16 + 7* another task that implies the “commutative property” (T 92) can be built: *7 + 16*.

Anke's Third Contribution (T 95)

Hereupon, the teacher notes that the tasks *3 + 20* and *17 + 6* are not tasks that include the commutative law and asks why the sums still remain the same. Anke comments this as follows (T 95):

> *ehm* at *ehm* for example the thirteen plus ten includes as well three plus twenty and twenty plus- three, actually, and then- no no.

From an interactionist point of a view it is possible that Anke describes two "sorts" (Anke T 92) of tasks: The already mentioned tasks above (*17 + 6*, *6 + 17*, *16 + 7*, *7 + 16*) belong to the task *17 + 6* and the tasks *13 + 10* und *3 + 20* belong to a second sort. On the first sort the addends and ones are changed. However, on the second sort the tens are changed, insofar as the tens are shifted out of *13* to *3* and the tens of *20* are shifted to *10*: $(13 - 10) + (10 + 10) = 3 + 20$. In this perspective, Anke's rule is indicated again in a general way (cf. Fig. 4). But Anke does not refer to it.

Paul's Contribution (T 97)

Subsequently, the teacher asks again for an explanation why the sums of the tasks *3 + 20* and *17 + 6* are the same. Paul (T 97) answers:

> […] actually- they are almost the same, so I formulated it like this, the first number changes always alternating higher or deeper, and the second number changes inversely but by the same number […] one can take it minus, so at the first number it is eh sixteen minus thirteen […] seven plus thirteen is twenty and there it is like in the first, what is taken away […] is put on the other one […] so it changes- always.. so by the same but inversely. so when it is plus at the first, it is minus at the other one, and inversely, when it is minus at the first it is plus at the second.

Paul's utterance can be interpreted as meaning that all tasks in the right calculations "are almost the same" (T 97). He explains it by formulating a general relationship between the sums' equality and the addends' exchanging without mentioning concrete numbers. Thus, his rule can be seen as the abduction's rule. Additionally, he includes the commutative law that the sums stay the same even if an undetermined number is added to the addend and this number is then subtracted from the second addend. Due to this interpretation, he uses the word "inversely" to explain the commutativity of the addends' exchanging. He specifies his rule at the tasks *16 + 7* and *3 + 20* by exchanging them with -13 and $+13$ (case). The exchanging with -13 and $+13$ can be read as the concretion of the rule's antecedent. Due to this perspective, Paul expects the exchanging of the addends from one task to the other as causal for the equality of the sums (result). His discovering process can be reconstructed by the following abduction (cf. Fig. 7).

Furthermore, Paul probably compares the first calculations ("there it is like in the first") to Anke's rule (T 26) respectively. Anke already explained the exchanging of the addends by two for the first calculations (T 26–28). Paul's rule describes the sums' remaining without mentioning concrete numbers of the exchange.

Result:	The sums of the right calculations are the same.
Rule:	„the first number changes always alternating higher or deeper, and the second number changes inversely but by the same number (that is why the sum remains the same, J.K.)“
Case:	$(16 - 13) + (7 + 13) = 3 + 20$

Fig. 7 Paul's abduction (T 97)

The Use of (Family) Resemblances in the Scene

In this section, those (family) resemblances that the learners use and explicate are presented along to the student's utterances. In addition, different kinds of (family) resemblances due to the learners' use in the scene are presented.

According to the above interpretation, Matteo (T 76–82) creates (family) resemblances between pairs of tasks *18 + 4* and *14 + 8* as well as *10 + 12* and *12 + 10*. These pairs of tasks are similar to each other insofar that the summands and the ones are changed.

While explicating tasks, which contain the commutative law, Matteo emphasises the ones: "eighteen plus four, and fourteen plus eight" (T 80). In general the structure of numbers resemble each other insofar as they have comparable digits and that they are similar in their pronunciation as "four" and "fourteen", "eight" and "eighteen" as well as "fourteen" and "eighteen". If single words or parts of them match in terms of sound patterns, *(family) resemblances in a phonetic way* can be reconstructed. Furthermore, the words "second number" by Matteo (T 82) and Tanja (T 72) resemble each other and (family) resemblances in a phonetic way can be reconstructed.

Matteo and Tanja use the term "second number" to name the second summand. Thus, they resemble each other concerning their use in language. *(Family) resemblances in a semantic way* describe similar meanings of words and (parts of) sentences in language use. The expressions' use of "first number" and "second number" of Anke (T 26) and Tanja (T 72) can also be included. Since both word pairs relate to operands. Additionally, the structure of the mentioned numbers resembles each other in their use.

Jakob (T 90) establishes (family) resemblances between the left and the right calculations. In the above perspective Matteo (T 76–82) explicates tasks that imply the commutative law within the left calculations. Jakob focuses those in the right calculations. Like Matteo, Jakob considers the addends' changing and the ones' changing as causal that the sums remain the same. Thus, the reconstructed rules are the same (cf. Figs. 5 and 6). Correspondingly, the cases and the rules resemble each other (cf. Figs. 5 and 6) and only differ in the considered numbers (for example the equality of the sums *22* and *23* in the abduction's results). When elements like the abductions' results, cases or rules are similar to each other one can speak of *(family) resemblances in an inferential way*.

Matteo already established family resemblances within the first calculations due to the addends' changing and the ones' changing. Jakob creates (family) resemblances between the left and right calculations. Since Jakob uses the same rule as Matteo to explain the equality of the sums (cf. Figs. 5 and 6). One can suppose that Jakob's discovering process is guided by (family) resemblances because he states similar cases for the equality of the sums like Matteo.

Moreover, (family) resemblances can be observed concerning the approaches to explain observed facts. Jakob and Matteo look for equalities between the several tasks to explain that the sums are the same. Furthermore, they use the same rules to explain that the sums remain the same (cf. Figs. 5 and 6). When approaches concerning the finding of explanations for observed facts resemble each other this is called *(family) resemblances to orientate the solution process*.

Jakob's use of the term "inverse operation" establishes (family) resemblances to Tanja's rule (T 72) whereas Jakob does not explicate terms with the "inverse operation" from a normative perspective. Due to the above interpretation, Jakob may use "inverse" in a different meaning insofar as Tanja refers to the addition's inversion and Jakob to the inversion of ones. Thus, (family) resemblances in a phonetic way do not implicate semantic ones necessarily. It is possible that the several (family) resemblances in a phonetic and semantic way lead the formulation of explanations and rules.

Anke (T 92) creates (family) resemblances between Jakob (T 90) and Matteo (T 76–82). Presumably, she explains that Matteo and Jakob did not use the term "commutative property" correctly. Additionally, (family) resemblances between the left and right calculations can be regarded due to the exchanging by one and the exchanging by two in Anke's rule (T 26). The exchanging by one and Anke's rule can be reconstructed as (family) resemblances in an inferential way. These (family) resemblances can be seen again in T 95 when the tens are shifted. But Anke does not refer to her rule.

Paul's rule (T 97) is similar to Anke's rule at the beginning (T 26). In contrast to Anke's rule Paul's rule describes a general relationship between the sums and the inverse arranging of the addends in a more general. Insofar the cases and results are similar too (cf. Figs. 4 and 7). All in all, one can speak of (family) resemblances in an inferential way. Besides, Anke's explanations in Turn 93 and 95 are similar to Paul's rule. Since she describes the addends' exchanging by one and by ten.

Moreover, (family) resemblances in a phonetic and semantic way can be observed between Paul, Anke (T 26), Tanja (T 72) and Matteo (T 82) due to their use of the terms "first" and "second number". Since they use these terms to name the summands. Paul also uses the term "number" to describe the addends' exchanging so that (family) resemblances in a phonetic way can be reconstructed to Anke, Matteo and Tanja but not in the semantic.

Furthermore, (family) resemblances in a semantic way can be reconstructed in regard to the word pairs "higher"–"deeper" (Paul) and "smaller"–"higher" (Anke T 26). Since the terms describe the change along the summands.

Moreover, Anke's and Paul's approaches are characterized by (family) resemblances to orientate the solution process. While they both consider the changes of the numbers along the summands in order to find reasons for the equality of the sums. From an interactionist perspective one can suppose that these (family) resemblances as well as the (family) resemblances in an inferential way concerning Paul's and Anke's abduction lead the development of Paul's rule.

Further (family) resemblances can be reconstructed in the context of written elements (cf. Kunsteller, 2016). If only written elements like the tasks, the writings on the board or the pupil's writings are regarded one can speak of *(family) resemblances in an iconic and/or symbolic way*. For instance the numbers *8* and *18* as well as *3* and *8* resemble each other.

The presented ways of (family) resemblances can overlap from time to time. But they do not have to occur simultaneously, like the use of the term "inverse" shows: In the above interpretations, Tanja's rule (T 72) explicates the inversion of addition, Jakob (T 86–90) "inverses" the ones of tasks and Paul (T 97) uses the term "inversely" to explain the commutativity of the addends' exchanging. Even if they use the same word they do not explain the same. Thus, (family) resemblances in a phonetic way do not have to imply (family) resemblances in a semantic way necessarily.

Conclusions and Outlook

The presented scene illustrates how (family) resemblances can influence learning processes. The learners used several (family) resemblances to formulate explanations for observed facts and rules within and between the left and right calculations. The analysis shows that different roles can be attributed to the use of (family) resemblances: The learners perceived previously created (family) resemblances and applied them to other tasks within or between the calculations. Furthermore, it seems that the recognition and use of (family) resemblances can orientate the discovering of relationships and rules. The learners observed similar facts, which could serve as a possible starting point for a forthcoming abduction. In order to find possible explanations, the learners seemed to use similar approaches or similar views on the tasks (family resemblances to orientate the solution process). In this sense the formulated relationships and rules are of a heuristic value, in so far as they can also be used for other tasks.

Based on an interpretative framework, the empirical part of this paper indicates that the concept of (family) resemblances can be used to reconstruct learning processes and thus to understand learning processes. Different types of (family) resemblances due to the students' use were presented:

- (Family) Resemblances in a phonetic way
- (Family) Resemblances in a semantic way
- (Family) Resemblances in an inferential way
- (Family) Resemblances to orientate the solution process
- (Family) Resemblances in an iconic and/or symbolic way

In Wittgenstein's words, the comparison of the utterances and rules illustrates a complex "network of similarities" (Wittgenstein, 2009, §66), which the learners seemed to make use of in different ways. The various ways of (family) resemblances allow analysing verbal and written student comments more precisely.

The focus on the different (family) resemblances allows the orientation of processes of understanding, insofar as similarities in the discussion of mathematics education research are assigned an important role (cf. introduction). The presented types of (family) resemblances enable further structuring the theories of analogical reasoning (cf. English & Sharry, 1996; Goswami, 2004) and of metaphors (cf. Pimm, 1987; Sfard, 2008). English and Sharry (1996, p. 138) pointed out that reasoning by analogy is generally defined by transferring structural information from one system (the base) to another system (the target). For instance one can transfer the commutative law of the term $3 + 5 = 5 + 3$ (base) to the term $\frac{1}{4} + \frac{1}{2} = \frac{1}{2} + \frac{1}{4}$ (target). Hence, (family) resemblances in a semantic way like the equality of terms are transferred to other numbers and number ranges. Moreover, the inferential structure (commutative law) is transferred and expanded to other number ranges. Thus, (family) resemblances in an inferential way can be reconstructed.

The situation is different in metaphors: Presmeg (1998, p. 26) stated that a metaphor connects two domains—"one domain of experience" (ibid.) and "another seemingly disparate domain" (ibid.). For example the metaphor "slope" (Pimm, 1987) was borrowed from the daily use of the term "slope" and now it is used in analysis. In this example (family) resemblances in a semantic way go hand in hand with (family) resemblances of another way (e.g. phonetic, iconic and/or symbolic).

In further publications the role of (family) resemblances in other discovering and reasoning processes will be exposed. Furthermore, the relations between analogies, metaphors and (family) resemblances will be elaborated and described by the theory of abduction.

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