Chapter 8
Coalitional Stabilities

Stability definitions for simple preference, unknown preference, degrees of preference, and hybrid preference (unknown combined with degree of preference) are presented in Chaps. 4–7, respectively. A typical stability analysis is built upon a noncooperative framework, with the underlying assumption being that each DM acts independently in its own self interest, after calculating moves and countermoves by its opponents. On the other hand, a coalitional analysis takes place in a cooperative framework, and assesses whether individual DMs can jointly improve their positions by forming a coalition (Kilgour et al. 2001, Inohara and Hipel 2008a, b, Xu et al. 2010, 2011, 2014). In fact, as emphasized in this book, after determining how well a DM can fare on his or her own by carrying out individual stability analyses, one should ascertain if a DM can do even better by cooperating with others via executing coalitional stability analyses, which is the focus of this chapter. Outside of Chap. 8, discussions regarding the importance of coalition investigations are put forward in Sect. 1.2.3 and portrayed in Fig. 1.5. Moreover, coalition modeling and analysis should be embedded as a key function of a decision support system for GMCR as explained in Sect. 10.2 and depicted in Figs. 10.2 and 10.4.

Coalition formation and stability analysis have long been active research areas in game theory (Aumann and Hart 1994, van Deeman 1997). The coalitional analysis considered in this book is confined to the Graph Model for Conflict Resolution (GMCR) paradigm. It assesses whether a subset of self-interested and independent DMs can gain by forming a coalition and coordinating their choices. The rationale is that a nonequilibrium state is not sustainable, because at least one DM can deviate from it in its own interest. An equilibrium, on the other hand, is expected to be sustainable, as no DM is motivated to depart from it. However, when a subset of DMs forms a coalition, an equilibrium may be upset via a sequence of joint moves by the coalition. In this case, the target state must also be an equilibrium, as any nonequilibrium state is transient. In Kilgour et al. (2001), this process is referred to

H. Xu et al., *Conflict Resolution Using the Graph Model: Strategic Interactions in Competition and Cooperation*, Studies in Systems, Decision and Control 153,
https://doi.org/10.1007/978-3-319-77670-5_8

as an "equilibrium jump". Understandably the target state of an equilibrium jump should make all members in the coalition better off and cannot be achieved by any DM acting individually. Coalition analysis, therefore, aims to alert the analyst that such a coalition exists and, if so, which equilibria are vulnerable to equilibrium jumps and how these jumps can be achieved by coalitional joint moves.

Coalition movements under various preference structures are introduced in Sect. 8.1. Subsequently, the logical representations of the four coalitional stability definitions, coalitional Nash stability, coalitional general metarationality, coalitional symmetric metarationality, and coalitional sequential stability, are defined in Sects. 8.2–8.5 under simple preference, unknown preference, three-level preference, and hybrid preference, respectively. Additionally, in this chapter, matrix representations of coalitional stabilities are presented in Sects. 8.6–8.9 for the four types of preference structures.

8.1 Coalition Movement Definitions

To define coalitional stabilities, concepts of *coalitional improvement* under various preferences must be introduced.

Definition 8.1 For a status quo state s and a nonempty coalition $H \subseteq N$, a state $s_1 \in R_H(s)$ is **a coalitional improvement** for H under simple preference from s, denoted by $s_1 \in CR_H^+(s)$, iff $s_1 \succ_i s$ for every $i \in H$.

It is worth noting that $CR_H^+(s) \neq R_H^+(s)$, as $R_H^+(s)$ denotes all states that are attainable by coalition H via legal sequences of UIs from s (see Definition 4.7). Although each individual move is a UI for the mover, there is no guarantee that the terminal state is preferred to s by any DM in H. On the contrary, $CR_H^+(s)$ is the subset of the terminal states preferred to s by all DMs in the coalition, although any individual move in the sequence may not be a UI for the mover.

Xu et al. (2010) extend the definition of coalitional improvement to weak coalitional improvement by including uncertain preference in the definition. A weak coalitional improvement for a coalition is a state that is the result of a sequence of moves from the status quo by members of the coalition, where each move is a *coalition improvement or uncertain move (CIUM)*, defined as follows.

Definition 8.2 For a status quo state s and a nonempty coalition $H \subseteq N$, a state $s_1 \in R_H(s)$ is **a coalition improvement or uncertain move** for H from s, denoted by $s_1 \in CR_H^{+,U}(s)$, iff $s_1 \succ_i s$ or $s_1\ U_i\ s$ for every $i \in H$.

Here, $CR_H^{+,U}(s)$ differs from $R_H^{+,U}(s)$ in Definition 5.18 in that $R_H^{+,U}(s)$ reflects the steps of the process without taking into account the final result, while $CR_H^{+,U}(s)$ is the final result, instead of the process. In other words, $R_H^{+,U}(s)$ requires each move in a legal sequence to be a UIUM for the mover, but the relative preference of the final state and the status quo is not a concern. On the contrary, $CR_H^{+,U}(s)$ ensures that all coalition members prefer the terminal state to the status quo, or are uncertain about their preference between these two states, without examining the relative preference for each individual move along the legal sequence.

Similarly, a coalitional improvement can be extended to include strength of preference.

Definition 8.3 For a status quo state s and a nonempty coalition $H \subseteq N$, a state $s_1 \in R_H(s)$ is **a mild or strong coalitional improvement** for H from s under the three-degree preference, denoted by $s_1 \in CR_H^{+,++}(s)$, iff $s_1 >_i s$ or $s_1 \gg_i s$ for every $i \in H$.

This means that, under a model with three degrees of preference, a coalitional improvement is a state mildly preferred or strongly preferred to s by any DM in H and is reachable by the coalition H. As before, note that $CR_H^{+,++}(s) \neq R_H^{+,++}(s)$, because $R_H^{+,++}(s)$ (Definition 6.9) denotes the states attainable by coalition H via legal sequences of mild or strong unilateral improvements (MSUIs) from s. But there is no guarantee that every DM in H prefers the terminal state to state s. On the other hand, $CR_H^{+,++}(s)$ ensures that the terminal state is always mildly or strongly preferred to s by all DMs in H though any individual move in the sequence may not be an MSUI for the mover. The following definition of coalitional movement is for the combination of unknown preference with three degrees of preference.

Definition 8.4 For a status quo state s and a nonempty coalition $H \subseteq N$, a state $s_1 \in R_H(s)$ is **a mild or strong or uncertain coalitional improvement** for H from s under hybrid preference, denoted by $s_1 \in CR_H^{+,++,U}(s)$, iff $s_1 >_i s$, $s_1 \gg_i s$, or $s_1 \, U_i \, s$ for every $i \in H$.

Now that the important concept of coalitional improvement or coalitional uncertainty has been defined for various preference structures, the logical and matrix representations of coalitional stabilities can be presented as follows.

8.2 Logical Representation of Coalitional Stabilities Under Simple Preference

The logical representations of individual stabilities in the graph model for simple preference, unknown preference, three degrees of preference and hybrid preference are presented in Sects. 4.2, 5.2, 6.3 and 7.2, respectively. In this section, logical representations of coalitional stabilities are defined for the four kinds of preference structure.

Firstly, coalitional stabilities under Nash, GMR, SMR, and SEQ with simple preference are furnished.

Definition 8.5 Let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional Nash stable for H, denoted by $s \in S_H^{CNash}$, iff $CR_H^+(s) = \emptyset$.

From Definition 8.1, $CR_H^+(s)$ honors the rule of no-successive-moves by the same DM and, hence, this definition is applicable to both transitive and intransitive graph models. As mentioned earlier, an empty coalition has no meaning, so it is assumed hereafter that $|H| > 0$. If $|H| = 1$, then $H = \{i\}$ and $CR_H^+(s) = R_i^+(s)$. In this special case, Definition 8.5 reduces to individual Nash stability defined in Chap. 4. However, for a nontrivial coalition $H \subseteq N, |H| \geq 2$, coalitional Nash stability depends on the coalitional improvement list $CR_H^+(s)$, rather than coalition members' individual UI lists, $R_i^+(s)$, for $i \in H$.

If state $s \in S$ is Nash stable for every nonempty coalition $H \subseteq N$, it is called universally coalitional Nash stable. The formal definition is described as follows.

Definition 8.6 State $s \in S$ is **universally coalitional Nash stable**, denoted by $s \in S^{UCNash}$, iff s is coalitional stable for every nonempty coalition $H \subseteq N$.

Note that S_H^{CNash} in Definition 8.5 is different from S^{UCNash}. S_H^{CNash} is the set of coalitional Nash stable states for some coalition H, whereas S^{UCNash} contains all coalitional Nash stable states.

For notational convenience, the notation to represent a preference relation in coalition H is defined as follows.

Definition 8.7 For the graph model G, let $H \subseteq N$ be a coalition. $\Phi_H^{\precsim}(s) = \{t \in S : s \succsim_i t$ for at least one $i \in H\}$ in which $s \succsim_i t$ denotes $s \succ_i t$ or $s \sim_i t$.

It is apparent that $\Phi_H^{\precsim}(s)$ considers only preference relative to state s without regard to reachability from s.

Definition 8.8 Let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional general metarational (CGMR) for H, denoted by $s \in S_H^{CGMR}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\precsim}(s)$.

If $H = \{i\}$, this definition reduces to individual GMR, defined in Sect. 4.2.3. If a state is coalitional GMR for every coalition, it is called universally coalitional GMR stable, formally defined as follows.

Definition 8.9 State $s \in S$ is **universally coalitional GMR stable**, denoted by $s \in S^{UCGMR}$, iff s is coalitional GMR stable for every nonempty coalition $H \subseteq N$.

Definition 8.10 Let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional symmetric metarational (CSMR) for H, denoted by $s \in S_H^{CSMR}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\precsim}(s)$ and $s_3 \in \Phi_H^{\precsim}(s)$ for all $s_3 \in R_H(s_2)$.

As usual, if $H = \{i\}$, Definition 8.10 reduces to individual SMR, defined in Sect. 4.2.3. If a state is coalitional SMR for every coalition, it is called universally coalitional SMR stable, defined as follows.

Definition 8.11 State $s \in S$ is **universally coalitional SMR stable** , denoted by $s \in S^{UCSMR}$, iff s is coalitional SMR stable for every nonempty coalition $H \subseteq N$.

Normally, coalition H's opponents $N - H$ may be treated as a coalition or as individual DMs in the next two definitions.

Definition 8.12 Let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional sequentially stable ($CSEQ_1$) for H, denoted by $s \in S_H^{CSEQ_1}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in CR_{N-H}^+(s_1)$ such that $s_2 \in \Phi_H^{\precsim}(s)$.

Definition 8.13 Let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional sequentially stable ($CSEQ_2$) for H, denoted by $s \in S_H^{CSEQ_2}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}^+(s_1)$ such that $s_2 \in \Phi_H^{\precsim}(s)$.

Remark: By employing the subclass improvement list concept, the SEQ stability definition for coalition H introduced by Inohara and Hipel (2008a, b) considers credible sanctions by subcoalitions of opponents. But their result assumes that the rule of no consecutive moves by the same DM has been lifted for the sake of tractability. The implication is that the definition is applicable only to transitive graph models, so in this book, one retains this restriction for coalitional stabilities. Because the number of subcoalitions increases exponentially with the number of DMs in the opponents, making the calculation of subclass improvement lists prohibitively difficult, H's opponents $N - H$ are treated here as a coalition or individual DMs, as shown in Definitions 8.12 and 8.13, respectively.

As usual, when $H = \{i\}$, coalitional SEQ would be reduced to individual SEQ stability. Similarly, if state is coalitional SEQ for every coalition, it is called universally coalitional SEQ stable. Specifically,

Definition 8.14 State $s \in S$ is **universally coalitional** SEQ_1 **stable**, denoted by $s \in S^{UCSEQ_1}$, iff s is coalitional SEQ_1 stable for every nonempty coalition $H \subseteq N$.

Definition 8.15 State $s \in S$ is **universally coalitional** SEQ_2 **stable**, denoted by $s \in S^{UCSEQ_2}$, iff s is coalitional SEQ_2 stable for every nonempty coalition $H \subseteq N$.

From the discussions above, it is clear that coalitional stability analysis extends individual stabilities under simple preference. Next, the coalitional stabilities are extended to preference with uncertainty.

8.3 Logical Representation of Coalitional Stabilities Under Unknown Preference

DMs may exhibit different attitudes toward preference uncertainty when making choices. For instance, an optimistic DM tends to view uncertainty as a potential opportunity, while a pessimistic DM may regard an uncertain outcome as a risk. In addition, a DM's attitude towards uncertainty may change with the status quo state: a DM who has little to lose is more likely to take an aggressive attitude towards uncertainty and treat it as a potential gain. On the contrary, a DM who has little to gain is highly likely to regard uncertain outcomes as a risk and adopt a conservative stance.

To accommodate different attitudes toward preference uncertainty, Li et al. (2004) define individual Nash, GMR, SMR, and SEQ stabilities with preference uncertainty under four forms, a, b, c, and d (see Chap. 5). The purpose of these four extensions is to characterize a focal DM with diverse attitudes toward preference uncertainty, ranging from aggressive to mixed to conservative. When coalitional GMR, SMR, and SEQ stability definitions are extended from graph models with simple preference, as presented in Sect. 8.2, to those with unknown preference, these four extensions apply, depending on the focal coalition's attitude towards preference uncertainty.

First, the coalitional Nash, GMR, SMR, and SEQ stabilities with indices a, b, c, and d for unknown preference are described as follows. Let $l \in \{a, b, c, d\}$.

8.3.1 Logical Representation of Coalitional Stabilities Indexed l

(1) Logical Representation of Coalitional Stabilities Indexed a

Definition 8.16 State $s \in S$ is coalitional $Nash_a$ stable for $H \subseteq N$, denoted by $s \in S_H^{CNash_a}$, iff $CR_H^{+,U}(s) = \emptyset$.

Definition 8.17 State $s \in S$ is coalitional GMR_a for $H \subseteq N$, denoted by $s \in S_H^{CGMR_a}$, iff for every $s_1 \in CR_H^{+,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\preceq}(s)$.

Definition 8.18 State $s \in S$ is coalitional SMR_a for $H \subseteq N$, denoted by $s \in S_H^{CSMR_a}$, iff for every $s_1 \in CR_H^{+,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\preceq}(s)$ and $s_3 \in \Phi_H^{\preceq}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.19 State $s \in S$ is coalitional SEQ_a for $H \subseteq N$, denoted by $s \in S_H^{CSEQ_a}$, iff for every $s_1 \in CR_H^{+,U}(s)$, there exists $s_2 \in R_{N-H}^{+,U}(s_1)$ such that $s_2 \in \Phi_H^{\preceq}(s)$.

In extension a, the focal coalition members are conceived to be aggressive. They are willing to deviate from the status quo state for uncertain outcomes in that uncertainty is allowed at the incentive end for the focal coalition. Therefore, $s \in S_H^{CNash_a}$ is also said to be Nash stable for aggressive DMs in H. While assessing sanctions by opponents, at least one coalition member must end up in a no-better-off position in order to successfully block the focal coalition. Thus, uncertainty is not allowed at the sanction end for the focal coalition.

(2) Logical Representation of Coalitional Stabilities Indexed b

Definition 8.20 State $s \in S$ is coalitional $Nash_b$ stable for $H \subseteq N$, denoted by $s \in S_H^{CNash_b}$, iff $CR_H^{+}(s) = \emptyset$.

Definition 8.21 State $s \in S$ is coalitional GMR_b for $H \subseteq N$, denoted by $s \in S_H^{CGMR_b}$, iff for every $s_1 \in CR_H^{+}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\preceq}(s)$.

Definition 8.22 State $s \in S$ is coalitional SMR_b for $H \subseteq N$, denoted by $s \in S_H^{CSMR_b}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\precsim}(s)$ and $s_3 \in \Phi_H^{\precsim}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.23 State $s \in S$ is coalitional SEQ_b for $H \subseteq N$, denoted by $s \in S_H^{CSEQ_b}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}^{+,U}(s_1)$ such that $s_2 \in \Phi_H^{\precsim}(s)$.

Compared to the stability definitions for a coalition in extension a, this extension does not treat uncertain moves as sufficient incentive for the focal coalition to deviate from the status quo. The focal coalition under this extension presumably exhibits a mixed attitude towards preference uncertainty, conservative at the incentive end but aggressive at the sanction end (Li et al. 2004). Although Definitions 8.20–8.23, respectively, look the same as Definitions 8.5, 8.8, 8.10, and 8.13, they are in fact different in the sense that Definitions 8.20–8.23 assume preference uncertainty but uncertain moves are neither strong enough motivation for the focal coalition to deviate from the status quo nor allowed as valid sanctions to deter the focal coalition. On the other hand, Definitions 8.5, 8.8, 8.10, and 8.13 assume graph models with simple preference.

(3) Logical Representation of Coalitional Stabilities Indexed c

For convenience, let $\Phi_H^{\precsim,U}(s) = \{t \in S : s \succsim_i t \text{ or } s\ U_i\ t \text{ for at least one } i \in H\}$. As Nash stability does not examine countermoves by the opponents, similar to the individual stability case in Chap. 5, $S_H^{CNash_c} = S_H^{CNash_a}$.

Definition 8.24 State $s \in S$ is coalitional GMR_c for $H \subseteq N$, denoted by $s \in S_H^{CGMR_c}$, iff for every $s_1 \in CR_H^{+,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\precsim,U}(s)$.

Definition 8.25 State $s \in S$ is coalitional SMR_c for $H \subseteq N$, denoted by $s \in S_H^{CSMR_c}$, iff for every $s_1 \in CR_H^{+,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\precsim,U}(s)$ and $s_3 \in \Phi_H^{\precsim,U}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.26 State $s \in S$ is coalitional SEQ_c for $H \subseteq N$, denoted by $s \in S_H^{CSEQ_c}$, iff for every $s_1 \in CR_H^{+,U}(s)$, there exists $s_2 \in R_{N-H}^{+,U}(s_1)$ such that $s_2 \in \Phi_H^{\precsim,U}(s)$.

Extension c assumes that uncertain moves are allowed as sufficient incentives and sanctions for the focal coalition and is designed to characterize focal coalition members with mixed attitude towards preference uncertainty: aggressive at the incentive end but conservative at the sanction end.

(4) Logical Representation of Coalitional Stabilities Indexed d

Similar to the individual stability case, $S_H^{CNash_d} = S_H^{CNash_b}$.

Definition 8.27 State $s \in S$ is coalitional GMR_d for $H \subseteq N$, denoted by $s \in S_H^{CGMR_d}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\precsim,U}(s)$.

Definition 8.28 State $s \in S$ is coalitional SMR_d for $H \subseteq N$, denoted by $s \in S_H^{CSMR_d}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\preceq,U}(s)$ and $s_3 \in \Phi_H^{\preceq,U}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.29 State $s \in S$ is coalitional SEQ_d for $H \subseteq N$, denoted by $s \in S_H^{CSEQ_d}$, iff for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}^{+,U}(s_1)$ such that $s_2 \in \Phi_H^{\preceq,U}(s)$.

Coalitional stability definitions in extension d are devised for conservative focal coalitions: When contemplating incentives, they do not envision uncertain moves as opportunities (preference uncertainty is not allowed as incentives); while assessing sanctions, these DMs would view uncertain moves as potential harm (preference uncertainty is allowed as valid sanctions).

Let $l \in \{a, b, c, d\}$. As usual, if state $s \in S$ is coalitional Nash, GMR, SMR, or SEQ stable for each coalition $H \subseteq N$ under a particular extension l, it is called universally coalitional Nash, GMR, SMR, or SEQ stable indexed l, and denoted by $s \in S^{UCNash_l}$, $s \in S^{UCGMR_l}$, $s \in S^{UCSMR_l}$, or $s \in S^{UCSEQ_l}$. It is obvious that $S^{UCGMR_l} = \cap_{H \subseteq N} S_H^{CGMR_l}$, $S^{UCSMR_l} = \cap_{H \subseteq N} S_H^{CSMR_l}$, and $S^{UCSEQ_l} = \cap_{H \subseteq N} S_H^{CSEQ_l}$.

The logical representations of the coalitional stabilities for simple preference and unknown preference have been described in Sects. 8.2 and 8.3. The logical representation of coalitional stabilities when there are three degrees of preference are presented next.

8.4 Logical Representation of Coalitional Stabilities Under Three Degrees of Preference

Two-degree preference (simple preference) is often inadequate for modeling the complex strategic conflicts that arise in practical applications, so it is natural to explore how to expand coalitional stability from two-degree preference, presented in Sect. 8.2, to the three-degree version. The coalitional stability definitions given below for three degrees of preference recognize three distinct categories of stability that are general coalitional stability, strong coalitional stability, and weak coalitional stability. Coalitional stability definitions are called strong or weak to reflect the additional preference information contained in the strength of preference relation. General coalitional stabilities are defined first.

8.4.1 *General Coalitional Stabilities*

In order to analyze the coalitional stability of a state for a coalition $H \subseteq N$, it is necessary to take into account possible responses from the opponents of H, $j \in N - H$.

The reachable lists of coalition H from state s, $R_H(s)$ and $R_H^{+,++}(s)$, defined in Sects. 4.2.2 and 6.3.2, respectively, are used in this subsection for coalitional stability definitions for three degrees of preference . A mild or strong coalitional improvement from s for H, $CR_H^{+,++}(s)$, is presented in Definition 8.3. General coalitional stabilities are defined next.

Definition 8.30 For $H \subseteq N$, state $s \in S$ is general coalitional Nash stable for coalition H, denoted by $s \in S_H^{GCNash}$, iff $CR_H^{+,++}(s) = \emptyset$.

State s is general coalitional Nash stable for coalition H iff H has no coalitional improvements from state s. Nash stability takes no account of possible responses by the opponents of H for any move by H away from s.

To develop the coalitional versions of GMR, SMR, and SEQ, it is necessary to identify coalition H's UMs, $R_H(s)$, MSUIs, $R_H^{+,++}(s)$, and coalitional improvements, $CR_H^{+,++}(s)$, from state s.

Definition 8.31 For $H \subseteq N$, state s is general coalitional GMR (GCGMR) for coalition H, denoted by $s \in S_H^{GCGMR}$, iff for every $s_1 \in CR_H^{+,++}(s)$ there exists at least one $s_2 \in R_{N-H}(s_1)$ such that $s \gg_i s_2$, $s >_i s_2$, or $s \sim_i s_2$ for some DM $i \in H$.

Definition 8.32 State s is general coalitional SMR (GCSMR) stable for coalition H, denoted by $s \in S_H^{GCSMR}$, iff for every $s_1 \in CR_H^{+,++}(s)$ there exists $s_2 \in R_{N-H}(s_1)$, such that $s \gg_i s_2$, $s >_i s_2$, or $s \sim_i s_2$ for at least one $i \in H$ and $s \gg_i s_3$, $s >_i s_3$, or $s \sim_i s_3$ for all $s_3 \in R_H(s_2)$.

State s is general coalitional SMR stable for H iff, for every s_1 that H can attain from s, and that is mildly or strongly preferred to s by everyone in H, there exists s_2 that $N - H$ can reach from s_1 that someone in H finds no more preferable than s, and, moreover, every s_3 that H can attain from s_2 is no more preferable than s for some member of H. If the sanction imposed by the opponents on H's improvement cannot be mitigated by coalition H's counterresponse, then coalition H is better off staying at the original state. Coalitional SMR presumes one step more foresight than coalitional GMR.

Coalitional SEQ stability examines the credibility of sanctions of coalition H's improvements by its opponents. The legality of sequences of improvements by subcoalitions of $N - H$ is another issue. Similar to Sect. 8.2, H's opponents $N - H$ may be treated as a coalition or as individual DMs in the next two definitions.

Definition 8.33 For $H \subseteq N$, state s is general coalitional SEQ_1 ($GCSEQ_1$) for coalition H, denoted by $s \in S_H^{GCSEQ_1}$, iff for every $s_1 \in CR_H^{+,++}(s)$ there exists at least one $s_2 \in CR_{N-H}^{+,++}(s_1)$ such that $s \gg_i s_2$, $s >_i s_2$, or $s \sim_i s_2$ for some $i \in H$.

The state $s \in S$ is general coalitional SEQ_1 stable for H iff, for every s_1 that H can reach from s which everyone in H mildly or strongly prefers to s, there exists s_2 that $N - H$ can reach from s_1 such that everyone in $N - H$ mildly or strongly prefers s_2 to s_1 and someone in H finds s_2 no more preferable than s. (Note that s_2 may be reachable from s_1 by unilateral moves rather than unilateral improvements.)

This is the same as saying that, for every coalitional unilateral improvement by H from s, there is a response that can be achieved by $N - H$ such that at least one person in H finds the coalitional improvement sanctioned. In this case, at least one person in H would rather be at s than at s_2. This person therefore refuses to contribute to the move from s to s_1. (Of course, if this person is not essential to making the move from s to s_1 in the first place, then he or she could be dropped from the coalition.)

Alternatively, H's opponents can be treated as individual DMs, producing the general coalitional SEQ_2 stability, defined as follows:

Definition 8.34 For $H \subseteq N$, state s is general coalitional SEQ_2 ($GCSEQ_2$) for coalition H, denoted by $s \in S_H^{GCSEQ_2}$, iff for every $s_1 \in CR_H^{+,++}(s)$ there exists at least one $s_2 \in R_{N-H}^{+,++}(s_1)$ such that $s \gg_i s_2$, $s >_i s_2$, or $s \sim_i s_2$ for some $i \in H$.

8.4.2 *Strong or Weak Coalitional Stabilities*

When degree of preference is introduced into the graph model, general coalitional stability definitions can be strong or weak, according to the degree of sanctioning. For a risk-averse coalition H, if all of coalition H's improvements from a particular state are strongly sanctioned, then the status quo state possesses an extra degree of stability, called strong stability. A coalitional improvement of a focal H is sanctioned strongly if it could result in a greatly less preferred state relative to the initial state, and this sanction cannot be avoided by an appropriate counterresponse.

Definition 8.35 For $H \subseteq N$, state s is strong coalitional GMR (SCGMR) for coalition H, denoted by $s \in S_H^{SCGMR}$, iff for every $s_1 \in CR_H^{+,++}(s)$ there exists at least one $s_2 \in R_{N-H}(s_1)$ such that $s \gg_i s_2$ for some DM $i \in H$.

Under strong coalitional GMR stability, all H's coalitional improvements can be strongly sanctioned by the opponents.

Definition 8.36 State s is strong coalitional SMR (SCSMR) stable for coalition H, denoted by $s \in S_H^{SCSMR}$, iff for every $s_1 \in CR_H^{+,++}(s)$ there exists $s_2 \in R_{N-H}(s_1)$, such that $s \gg_i s_2$ for at least one $i \in H$ and $s \gg_i s_3$ for all $s_3 \in R_H(s_2)$.

If the strong sanction imposed by the opponents on H's improvements cannot be mitigated by coalition H's counterresponse, then at least one member of the coalition H is better off staying at the original state. Two following definitions are analogous to Definitions 8.33 and 8.34.

Definition 8.37 For $H \subseteq N$, state s is strong coalitional SEQ_1 ($SCSEQ_1$) for coalition H, denoted by $s \in S_H^{SCSEQ_1}$, iff for every $s_1 \in CR_H^{+,++}(s)$ there exists at least one $s_2 \in CR_{N-H}^{+,++}(s_1)$ such that $s \gg_i s_2$ for at least one $i \in H$.

Definition 8.38 For $H \subseteq N$, state s is strong coalitional SEQ_2 ($SCSEQ_2$) for coalition H, denoted by $s \in S_H^{SCSEQ_2}$, iff for every $s_1 \in CR_H^{+,++}(s)$ there exists at least one $s_2 \in R_{N-H}^{+,++}(s_1)$ such that $s \gg_i s_2$ for at least one $i \in H$.

For three-degree preference, general coalitional stabilities are classified as strong and weak according to the strength of the possible sanctions. Let GCGS and SCGS denote general coalitional graph model stability, GCNash, GCGMR, GCSMR, $GCSEQ_1$, or $GCSEQ_2$, and strong coalitional graph model stability, SCGMR, SCSMR, $SCSEQ_1$, or $SCSEQ_2$, respectively. Strong coalitional Nash stability is excluded because CNash stability does not involve sanctions. The symbol WCGS denotes weak coalitional graph model stability , WCGMR, WCSMR, or WCSEQ, under three-degree preference. Weak coalitional stability is defined as follows:

Definition 8.39 For $H \subseteq N$, state s is weak coalitional stable for coalition H, denoted by $s \in S_H^{WCGS}$, iff $s \in S_H^{GCGS}$ but $s \notin S_H^{SCGS}$.

A weak coalitional stable state means that it is general coalitional stable for some stability, but not strong coalitional stable for the corresponding stability. Hence, if a particular state s is general coalitional stable, then s is either strong coalitional stable or weak coalitional stable.

8.5 Logical Representation of Coalitional Stability with Hybrid Preference

The logical representations of coalitional stabilities under unknown preference and three-level preference have been defined in Sects. 8.3 and 8.4, respectively. The two types of preference are combined into the hybrid preference structure. The coalitional stabilities under the hybrid preference are discussed in this section.

8.5.1 General Coalitional Stabilities with Hybrid Preference

The hybrid preference is to combine three-level preference and unknown preference together. Therefore, general coalitional stabilities within hybrid preference expand the general coalitional stabilities under simple preference, unknown preference, and three-degree preference. Let $l \in \{a, b, c, d\}$.

8.5.1.1 General Coalitional Stabilities Indexed *l*

(1) General Coalitional Stabilities Indexed *a*

For coalitional stabilities indexed a, coalition H is willing to move to states that are mildly preferred or strongly preferred, as well as states having uncertain preference relative to the status quo but does not wish to be sanctioned by a strongly less preferred, mildly less preferred, or equally preferred state relative to the status quo. The definitions given below assume that $s \in S$ and $i \in N$.

Definition 8.40 For the graph model G, let $H \subseteq N$ be a coalition. Define $\Phi_H^{\ll,<,\sim}(s) = \{t \in S : s \gg_i t, s >_i t, \text{ or } s \sim_i \text{ for at least one } i \in H\}$ and $\Phi_H^{\ll,<,\sim,U}(s) = \{t \in S : s \gg_i t, s >_i t, s \sim_i t, \text{ or } s\ U_i\ t \text{ for at least one } i \in H\}$.

Note that $\Phi_H^{\ll,<,\sim}(s)$ and $\Phi_H^{\ll,<,\sim,U}(s)$ do not consider the reachability from s.

Definition 8.41 State $s \in S$ is general coalitional $Nash_a$ stable for coalition $H \subseteq N$, denoted by $s \in S_H^{GCNash_a}$, iff $CR_H^{+,++,U}(s) = \emptyset$.

Definition 8.42 State $s \in S$ is general coalitional GMR_a for coalition $H \subseteq N$, denoted by $s \in S_H^{GCGMR_a}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim}(s)$.

Definition 8.43 State $s \in S$ is general coalitional SMR_a for coalition $H \subseteq N$, denoted by $s \in S_H^{GCSMR_a}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim}(s)$ and $s_3 \in \Phi_H^{\ll,<,\sim}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.44 State $s \in S$ is general coalitional SEQ_a for coalition $H \subseteq N$, denoted by $s \in S_H^{GCSEQ_a}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim}(s)$.

(2) General Coalitional Stabilities Indexed *b*

Definition 8.45 State $s \in S$ is general $Nash_b$ stable for coalition $H \subseteq N$, denoted by $s \in S_H^{GCNash_b}$, iff $CR_H^{+,++}(s) = \emptyset$.

Definition 8.46 State $s \in S$ is general coalitional GMR_b for coalition $H \subseteq N$, denoted by $s \in S_H^{GCGMR_b}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim}(s)$.

Definition 8.47 State $s \in S$ is general coalitional SMR_b for coalition $H \subseteq N$, denoted by $s \in S_H^{GCSMR_b}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim}(s)$ and $s_3 \in \Phi_H^{\ll,<,\sim}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.48 State $s \in S$ is general coalitional SEQ_b for coalition $H \subseteq N$, denoted by $s \in S_H^{GCSEQ_b}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim}(s)$.

(3) General Coalitional Stabilities Indexed *c*

Definition 8.49 State $s \in S$ is general coalitional $Nash_c$ stable for coalition $H \subseteq N$, denoted by $s \in S_H^{GCNash_c}$, iff $CR_H^{+,++,U}(s) = \emptyset$.

Definition 8.50 State $s \in S$ is general coalitional GMR_c for coalition $H \subseteq N$, denoted by $s \in S_H^{GCGMR_c}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$.

Definition 8.51 State $s \in S$ is general coalitional SMR_c for coalition $H \subseteq N$, denoted by $s \in S_H^{GCSMR_c}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$ and $s_3 \in \Phi_H^{\ll,<,\sim,U}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.52 State $s \in S$ is general coalitional SEQ_c for coalition $H \subseteq N$, denoted by $s \in S_H^{GCSEQ_c}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$.

(4) General Coalitional Stabilities Indexed d

Definition 8.53 State $s \in S$ is general coalitional $Nash_d$ stable for coalition $H \subseteq N$, denoted by $s \in S_H^{GCNash_d}$, iff $CR_H^{+,++}(s) = \emptyset$.

Definition 8.54 State $s \in S$ is general coalitional GMR_d for coalition $H \subseteq N$, denoted by $s \in S_H^{GCGMR_d}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$.

Definition 8.55 State $s \in S$ is general coalitional SMR_d for coalition $H \subseteq N$, denoted by $s \in S_H^{GCSMR_d}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$ and $s_3 \in \Phi_H^{\ll,<,\sim,U}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.56 State $s \in S$ is general coalitional SEQ_d for coalition $H \subseteq N$, denoted by $s \in S_H^{GCSEQ_d}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$.

8.5.2 *Strong Coalitional Stabilities with Hybrid Preference*

The notation related to strong preference is defined within the hybrid preference framework.

Definition 8.57 For the graph model G, let $H \subseteq N$ be a coalition. $\Phi_H^{\ll}(s) = \{t \in S : s \gg_i t \text{ for at least one } i \in H\}$.

Definition 8.58 Let $l \in \{a, b, c, d\}$. Strong coalitional $Nash_l$ stable for coalition $H \subseteq N$ is identical with general coalitional $Nash_l$ stable for coalition $H \subseteq N$. In other words, $S_H^{SCNash_l} = S_H^{GCNash_l}$.

For example, when $l = a$, then $S_H^{SCNash_a} = S_H^{GCNash_a}$.

8.5.2.1 Strong Coalitional Stabilities Indexed *l*

(1) Strong Coalitional Stabilities Indexed a

Definition 8.59 State $s \in S$ is strong coalitional GMR_a for coalition $H \subseteq N$, denoted by $s \in S_H^{SCGMR_a}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll}(s)$.

Definition 8.60 State $s \in S$ is strong coalitional SMR_a for coalition $H \subseteq N$, denoted by $s \in S_H^{SCSMR_a}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\lll}(s)$ and $s_3 \in \Phi_H^{\lll}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.61 State $s \in S$ is strong coalitional SEQ_a for coalition $H \subseteq N$, denoted by $s \in S_H^{SCSEQ_a}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_H^{\lll}(s)$.

(2) Strong Coalitional Stabilities Indexed b

Definition 8.62 State $s \in S$ is strong coalitional GMR_b for coalition $H \subseteq N$, denoted by $s \in S_H^{SCGMR_b}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\lll}(s)$.

Definition 8.63 State $s \in S$ is strong coalitional SMR_b for coalition $H \subseteq N$, denoted by $s \in S_H^{SCSMR_b}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\lll}(s)$ and $s_3 \in \Phi_H^{\lll}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.64 State $s \in S$ is strong coalitional SEQ_b for coalition $H \subseteq N$, denoted by $s \in S_H^{SCSEQ_b}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_H^{\lll}(s)$.

(3) Strong Coalitional Stabilities Indexed c

Definition 8.65 State $s \in S$ is strong coalitional GMR_c for coalition $H \subseteq N$, denoted by $s \in S_H^{SCGMR_c}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\lll,U}(s)$.

Definition 8.66 State $s \in S$ is strong coalitional SMR_c for coalition $H \subseteq N$, denoted by $s \in S_H^{SCSMR_c}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\lll,U}(s)$, and $s_3 \in \Phi_H^{\lll,U}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.67 State $s \in S$ is strong coalitional SEQ_c for coalition $H \subseteq N$, denoted by $s \in S_H^{SCSEQ_c}$, iff for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_H^{\lll,U}(s)$.

(4) Strong Coalitional Stabilities Indexed d

Definition 8.68 State $s \in S$ is strong coalitional GMR_d for coalition $H \subseteq N$, denoted by $s \in S_H^{SCGMR_d}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\lll,U}(s)$.

Definition 8.69 State $s \in S$ is strong coalitional SMR_d for coalition $H \subseteq N$, denoted by $s \in S_H^{SCSMR_d}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\lll,U}(s)$, and $s_3 \in \Phi_H^{\lll,U}(s)$ for all $s_3 \in R_H(s_2)$.

Definition 8.70 State $s \in S$ is strong coalitional SEQ_d for coalition $H \subseteq N$, denoted by $s \in S_H^{SCSEQ_d}$, iff for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_H^{\lll,U}(s)$.

8.6 Matrix Representation of Coalitional Stability Under Simple Preference

Although the four basic coalitional stabilities are defined for simple preference in Sect. 8.2, unknown preference in Sect. 8.3, three degree-preference in Sect. 8.4 and hybrid preference in Sect. 8.5, they are represented logically, which make coding difficult. In order to develop algorithms to implement these coalitional stabilities more easily, matrix representation of coalitional stabilities under various preference structures is introduced in the following sections. The matrix version of coalitional stability under simple preference is presented first (Xu et al. 2014).

8.6.1 Coalitional Improvement Matrix

Let $m = |S|$ denote the number of states, E be the $m \times m$ matrix with each entry equal to 1, and e_s denote the sth standard basis vector of the m-dimensional Euclidean space, $\mathbb{R}^S$. Recall that the UM reachability matrix M_H is constructed using two approaches that are based on the incidence matrix B and the adjacency matrix J presented in Chaps. 4 and 5, respectively.

A matrix approach is proposed in this section to construct the coalitional improvements from state s, $CR_H^+(s)$, given in Definition 8.1 in logical form.

Definition 8.71 For the graph model G, let $H \subseteq N$ be a nonempty coalition. The coalitional improvement matrix for H is defined as the $m \times m$ matrix CM_H^+ with (s, q) entry

$$CM_H^+(s,q) = \begin{cases} 1 \text{ if } q \in CR_H^+(s), \\ 0 \text{ otherwise.} \end{cases}$$

It is clear that $CR_H^+(s) = \{q : CM_H^+(s,q) = 1\}$. Then

$$CR_H^+(s) = e_s^T \cdot CM_H^+,$$

if $CR_H^+(s)$ is written as 0–1 row vectors, where a "1" at the jth element indicates coalition H has a coalitional improvement from s to s_j. Note that e_s^T denotes the transpose of e_s, the sth standard basis vector of m-dimensional Euclidean space. Therefore, the coalitional improvement matrix for coalition H, CM_H^+, can be used to construct the coalitional improvements of H from state s, $CR_H^+(s)$.

Using Definition 8.1, the coalitional improvement matrix of H can be constructed by the following theorem. Recall that $P_H^{-,=} = \bigvee_{i \in H} P_i^{-,=}$ ("$\bigvee$" denotes the disjunction operator described in Definition 3.16).

Theorem 8.1 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. The coalitional improvement matrix for H is expressed as*

$$CM_H^+ = M_H \circ (E - P_H^{-,=}). \tag{8.1}$$

Proof To prove Eq. 8.1, assume that $C = M_H \circ (E - P_H^{-,=})$. Using the definition for matrix M_H given in Chaps. 4 and 5, $C(s,q) = 1$ iff $M_H(s,q) = 1$ and $P_H^{-,=}(s,q) = 0$, which together imply that there is $q \in R_H(s)$ such that $P_i^{-,=}(s,q) = 0$ for every DM $i \in H$. Therefore, $C(s,q) = 1$ iff there is $q \in R_H(s)$ with $q \succ_i s$ for every $i \in H$, so that $q \in CR_H^+(s)$, according to Definition 8.1. Thus, $CM_H^+(s,q) = 1$ using Definition 8.71. Hence, $CM_H^+(s,q) = 1$ iff $C(s,q) = 1$. Since CM_H^+ and C are 0–1 matrices, it follows that $CM_H^+ = M_H \circ (E - P_H^{-,=})$. □

8.6.2 Matrix Representation of Coalitional Stabilities

For a fixed state $s \in S$, let e_s be an m-dimensional vector with 1 as its sth element and 0 everywhere else and e be an m-dimensional vector with every entry 1. Let $(\overrightarrow{0})^T$ denote the transpose of $\overrightarrow{0}$.

Theorem 8.2 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional Nash stable for H, denoted by $s \in S_H^{CNash}$, iff $e_s^T \cdot CM_H^+ \cdot e = 0$.*

Proof Since $e_s^T \cdot CM_H^+ \cdot e = 0$ iff $e_s^T \cdot CM_H^+ = (\overrightarrow{0})^T$, then $CR_H^+(s) = \emptyset$ using Definition 8.71. Consequently, the proof of this theorem follows by Definition 8.5. □

Coalitional Nash stability extends individual Nash stability. For example, If $|H| = 1$, Theorem 8.2 reduces to the matrix representation of individual Nash stability presented in Theorem 4.3. Specifically,

Corollary 8.1 *For the graph model G, let $i \in N$. If $e_s^T \cdot CM_{\{i\}}^+ \cdot e = 0$, then s is Nash stable for DM i.*

From Corollary 8.1, coalitional Nash stability is a generalization of individual Nash stability.

Theorem 8.3 *For the graph model G, state $s \in S$ is universally coalitional Nash stable for every $H \subseteq N$, denoted by $s \in S^{UCNash}$, iff $\sum_{\forall H \subseteq N} e_s^T \cdot CM_H^+ \cdot e = 0$.*

Proof Since $\sum_{\forall H \subseteq N} e_s^T \cdot CM_H^+ \cdot e = 0$ iff for any $H \subseteq N$, $e_s^T \cdot CM_H^+ \cdot e = 0$. By Theorem 8.2, $e_s^T \cdot CM_H^+ \cdot e = 0$ iff $s \in S$ is coalitional Nash stable for H. Consequently, $\sum_{H \subseteq N} e_s^T \cdot CM_H^+ \cdot e = 0$ iff $s \in S$ is coalitional Nash stable for every coalition $H \subseteq N$. The proof is completed by Definition 8.6. □

Theorem 8.3 shows the matrix representation of universally coalitional Nash stability equivalent to logical representation stated in Definition 8.6.

Similar to the individual GMR stability, define coalitional GMR stability matrix as

$$M_H^{CGMR} = CM_H^+ \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,=})^T\right)], \tag{8.2}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is coalitional GMR stable for H.

Theorem 8.4 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional GMR stable for H, denoted by $s \in S_H^{CGMR}$, iff $M_H^{CGMR}(s, s) = 0$.*

Proof Since

$$M_H^{CGMR}(s, s) = (e_s^T \cdot CM_H^+) \cdot [\left(E - sign\left(M_{N-H} \cdot (P_H^{-,=})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^+(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=})^T\right)],$$

then $M_H^{CGMR}(s, s) = 0$ holds iff

$$CM_H^+(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=})^T\right)] = 0, \tag{8.3}$$

for every $s_1 \in S - \{s\}$. It is clear that Eq. 8.3 is equivalent to

$$(e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=})^T \neq 0,$$

for every $s_1 \in CR_H^+(s)$. Therefore, for a coalitional improvement from s, $s_1 \in CR_H^+(s)$, there exists at least one $s_2 \in R_{N-H}(s_1)$ with $P_H^{-,=}(s, s_2) = 1$ that is equivalent to $s \succeq_i s_2$ for some DM $i \in H$. According to Definition 8.8, $M_H^{CGMR}(s, s) = 0$ implies that s is coalitional GMR stable for H. □

Theorem 8.4 shows that this matrix method, called matrix representation of coalitional GMR stability, is equivalent to the logical version of the same stability given in Definition 8.8. To analyze the coalitional GMR stability at s for coalition H, one only needs to identify whether the diagonal entry $M_H^{CGMR}(s, s)$ of the coalitional GMR matrix is zero. If so, s is coalitional GMR stable for H; otherwise, s is coalitional GMR unstable for H. Similar to individual GMR stability, all information about coalitional GMR stability is contained in the diagonal entries of the coalitional GMR stability matrix.

If $|H| = 1$, Theorem 8.4 reduces to the matrix representation of individual GMR stability presented in Theorem 4.10. Specifically,

Corollary 8.2 *For the graph model G, let $i \in N$. if $M_{\{i\}}^{CGMR}(s, s) = 0$, then s is GMR stable for DM i.*

Coalitional SMR is similar to coalitional GMR except that coalition H expects to have a chance to counterrespond to its opponent $(N-H)$'s response to H's original move. Define the coalitional SMR stability matrix as

$$M_H^{CSMR} = CM_H^+ \cdot [E - sign(F)]$$

in which

$$F = M_{N-H} \cdot [(P_H^{-,=})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,=})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is coalitional SMR stable for H.

Theorem 8.5 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SMR for H, denoted by $s \in S_H^{CSMR}$, iff $M_H^{CSMR}(s,s) = 0$.*

Proof Since

$$M_H^{CSMR}(s,s) = (e_s^T \cdot CM_H^+) \cdot [(E - sign(F)) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^+(s,s_1)[1 - sign\,(F(s_1,s))]$$

with

$$F(s_1,s) = \sum_{s_2=1}^{m} M_{N-H}(s_1,s_2) \cdot W(s_2,s),$$

and

$$W(s_2,s) = P_H^{-,=}(s,s_2) \cdot \left[1 - sign\left(\sum_{s_3=1}^{m}\left(M_H(s_2,s_3) \cdot (1 - P_H^{-,=}(s,s_3))\right)\right)\right],$$

then $M_H^{CSMR}(s,s) = 0$ holds iff $F(s_1,s) \neq 0$, for every $s_1 \in CR_H^+(s)$, which is equivalent to the statement that, for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that

$$P_H^{-,=}(s,s_2) \neq 0, \tag{8.4}$$

and

$$\sum_{s_3=1}^{m} M_H(s_2,s_3) \cdot (1 - P_H^{-,=}(s,s_3)) = 0. \tag{8.5}$$

Equation 8.4 means that $s \succeq_i s_2$ for at least one DM $i \in H$, i.e., $s_2 \in \Phi_H^{\preceq}(s)$ that is given in Definition 8.7. Equation 8.5 is equivalent to

$$P_H^{-,=}(s,s_3) = 1 \text{ for any } s_3 \in R_H(s_2). \tag{8.6}$$

Obviously, for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that Eqs. 8.4 and 8.5 hold iff for every $s_1 \in CR_H^+(s)$ there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\preceq}(s)$ and $s_3 \in \Phi_H^{\preceq}(s)$ for all $s_3 \in R_H(s_2)$. Therefore, the proof of this theorem follows using Definition 8.10. □

Theorem 8.5 displays this matrix method, called matrix representation of coalitional SMR stability, which is equivalent to the logical version given in Definition 8.10. To calculate coalitional SMR stability at s for H, one only needs to assess whether the diagonal entry $M_H^{CSMR}(s, s)$ of coalitional SMR stability matrix is zero. If so, s is coalitional SMR stable for H; otherwise, s is coalitional SMR unstable for H.

Corollary 8.3 *For the graph model G, let $i \in N$. if $M_{\{i\}}^{CSMR}(s, s) = 0$, then s is SMR stable for DM i.*

Coalitional sequential stability is similar to coalitional GMR stability, but includes only those sanctions that are "credible". If H's opponents are treated as a coalition, the coalitional SEQ_1 stability matrix $M_H^{CSEQ_1}$ is defined as

$$M_H^{CSEQ_1} = CM_H^+ \cdot [E - sign\left(CM_{N-H}^+ \cdot (P_H^{-,=})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is coalitional SEQ_1 stable for H when H's opponents, $N - H$, are in a coalition.

Theorem 8.6 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SEQ_1 stable for H, denoted by $s \in S_H^{CSEQ_1}$, iff $M_H^{CSEQ_1}(s, s) = 0$.*

Proof Since

$$M_H^{CSEQ_1}(s, s) = (e_s^T CM_H^+) \cdot [\left(E - sign(CM_{N-H}^+ \cdot (P_H^{-,=})^T)\right) e_s]$$
$$= \sum_{s_1=1}^{|S|} CM_H^+(s, s_1)[1 - sign\left((e_{s_1}^T CM_{N-H}^+) \cdot (e_s^T P_H^{-,=})^T\right)],$$

then $M_H^{CSEQ_1}(s, s) = 0$ holds iff

$$CM_H^+(s, s_1)[1 - sign\left((e_{s_1}^T CM_{N-H}^+) \cdot (e_s^T P_H^{-,=})^T\right)] = 0, \forall s_1 \in S. \tag{8.7}$$

It is clear that Eq. 8.7 is equivalent to

$$(e_{s_1}^T CM_{N-H}^+) \cdot (e_s^T P_H^{-,=})^T \neq 0 \text{ for any } s_1 \in CR_H^+(s).$$

This implies that for any $s_1 \in CR_H^+(s)$, there exists at least one $s_2 \in CR_{N-H}^+(s_1)$ with $s \succeq_i s_2$ for some DM $i \in H$ that satisfies $s_2 \in \Phi_H^{\preceq}(s)$. The proof of this theorem follows using Definition 8.12. □

Note that the coalitional SEQ_1 stability matrix is identical to the coalitional GMR stability matrix except that the UM reachability matrix for H's opponents, M_{N-H}, is replaced by the coalitional improvement matrix CM^+_{N-H}.

Similar to the previous two theorems, the matrix representation of coalitional SEQ stability is equivalent to the logical version given in Definition 8.12. Once, when the diagonal entry at (s, s) is zero, the state s under consideration is coalitional SEQ_1 stable for H. The following theorem is equivalent to the coalitional SEQ_2 stability given in Definition 8.13. Define the coalitional SEQ_2 stability matrix $M_H^{CSEQ_2}$ is defined as

$$M_H^{CSEQ_2} = CM_H^+ \cdot [E - sign\left(M^+_{N-H} \cdot (P_H^{-,=})^T\right)].$$

Theorem 8.7 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SEQ_2 stable for H, denoted by $s \in S_H^{CSEQ_2}$, iff $M_H^{CSEQ_2}(s, s) = 0$.*

Corollary 8.4 *For the graph model G, let $i \in N$. Then, (1) $M_{\{i\}}^{CSEQ_1} = M_{\{i\}}^{CSEQ_2}$; (2) If $M_{\{i\}}^{CSEQ_1}(s, s) = 0$ or $M_{\{i\}}^{CSEQ_2}(s, s) = 0$, then s is SEQ stable for DM i.*

8.7 Matrix Representation of Coalitional Stabilities Under Unknown Preference

8.7.1 Matrix Representation of Coalitional Improvement or Uncertain Move

Let $m = |S|$ denote the number of states, E be the $m \times m$ matrix with each entry equal to 1, and e_s denote the sth standard basis vector of the m-dimensional Euclidean space, $\mathbb{R}^S$. Recall that the UM reachability matrix M_H is constructed using Theorem 4.9.

A matrix approach is presented in this section to construct the coalitional improvements and coalitional improvements or uncertain moves from state s, $CR_H^+(s)$ and $CR_H^{+,U}(s)$, given in Definitions 8.1 and 8.2, respectively, in logical form.

Definition 8.72 For the graph model G, let $H \subseteq N$ be a nonempty coalition. The coalitional improvement matrix for H is defined as the $m \times m$ matrix CM_H^+ with (s, q) entry

$$CM_H^+(s, q) = \begin{cases} 1 \text{ if } q \in CR_H^+(s), \\ 0 \text{ otherwise.} \end{cases}$$

Moreover, the coalitional improvement or uncertain move matrix for H is defined as the $m \times m$ matrix $CM_H^{+,U}$ with (s, q) entry

$$CM_H^{+,U}(s, q) = \begin{cases} 1 \text{ if } q \in CR_H^{+,U}(s), \\ 0 \text{ otherwise.} \end{cases}$$

It is clear that $CR_H^+(s) = \{q : CM_H^+(s,q) = 1\}$ and $CR_H^{+,U}(s) = \{q : CM_H^{+,U}(s,q) = 1\}$. Then

$$CR_H^+(s) = e_s^T \cdot CM_H^+ \text{ and } CR_H^{+,U}(s) = e_s^T \cdot CM_H^{+,U},$$

if $CR_H^+(s)$ and $CR_H^{+,U}(s)$ are written as 0–1 row vectors, where a "1" at the jth element indicates coalition H has a coalitional improvement from s to s_j and coalition H has a coalitional improvement or uncertain move from s to s_j, respectively. Note that e_s^T denotes the transpose of e_s, the sth standard basis vector of m-dimensional Euclidean space. Therefore, the coalitional improvement and coalitional improvement or uncertain move matrices for coalition H, CM_H^+ and $CM_H^{+,U}$, can be used to construct the coalitional improvements and the coalitional improvements or uncertain moves of H from state s, $CR_H^+(s)$ and $CR_H^{+,U}(s)$, respectively.

Theorem 8.8 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. The coalitional improvement matrix for H is expressed as*

$$CM_H^+ = M_H \circ (E - P_H^{-,=,U}). \tag{8.8}$$

Proof To prove Eq. 8.8, assume that $C = M_H \circ (E - P_H^{-,=,U})$. Using the definition for matrix M_H presented in Chap. 4, $C(s,q) = 1$ iff $M_H(s,q) = 1$ and $P_H^{-,=,U}(s,q) = 0$, which together imply that there is $q \in R_H(s)$ such that $P_i^{-,=,U}(s,q) = 0$ for every DM $i \in H$. Therefore, $C(s,q) = 1$ iff there is $q \in R_H(s)$ with $q \succ_i s$ for every $i \in H$, so that $q \in CR_H^+(s)$, according to Definition 8.1. Thus, $CM_H^+(s,q) = 1$ using Definition 8.72. Hence, $CM_H^+(s,q) = 1$ iff $C(s,q) = 1$. Since CM_H^+ and C are 0–1 matrices, it follows that $CM_H^+ = M_H \circ (E - P_H^{-,=,U})$. □

Note that $CM_H^+ \neq M_H \circ P_H^+$. Recall that matrix $P_H^+ = \bigvee_{i \in H} P_i^+$ ("$\bigvee$" denotes the disjunction operator described in Definition 3.16). $(M_H \circ P_H^+)(s,q) = 1$ iff $M_H(s,q) = 1$ and $P_H^+(s,q) = 1$, which means that there is $q \in R_H(s)$ such that $P_i^+(s,q) = 1$ for some DM $i \in H$. This is not consistent with the definition of CM_H^+.

It is worth to note that matrix CM_H^+ defined in Theorem 8.8 is different from the matrix specified in Theorem 8.1 that cannot be used to analyze conflict models with preference uncertainty. The matrix defined in Theorem 8.8 contains information about uncertain preference. Using Definition 8.2, the coalitional improvement or uncertain move matrix of H can be constructed by the following theorem. Recall that $P_H^{-,=} = \bigvee_{i \in H} P_i^{-,=}$.

Theorem 8.9 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. The coalitional improvement or uncertain move matrix for H is expressed as*

$$CM_H^{+,U} = M_H \circ (E - P_H^{-,=}). \tag{8.9}$$

Proof To prove Eq. 8.9, assume that $C = M_H \circ (E - P_H^{-,=})$. Using the definition for matrix M_H presented in Chap. 4, $C(s, q) = 1$ iff $M_H(s, q) = 1$ and $P_H^{-,=}(s, q) = 0$, which together imply that there is $q \in R_H(s)$ such that $P_i^{-,=}(s, q) = 0$ for every DM $i \in H$. Therefore, $C(s, q) = 1$ iff there is $q \in R_H(s)$ with $q \succ_i s$ or $q\ U_i\ s$ for every $i \in H$, so that $q \in CR_H^{+,U}(s)$, according to Definition 8.2. Thus, $CM_H^{+,U}(s, q) = 1$ using Definition 8.72. Hence, $CM_H^{+,U}(s, q) = 1$ iff $C(s, q) = 1$. Since $CM_H^{+,U}$ and C are 0–1 matrices, it follows that $CM_H^{+,U} = M_H \circ (E - P_H^{-,=})$. □

Theorems 8.8 and 8.9 provide a matrix approach to construct the coalitional improvements from state s by H, $CR_H^{+}(s)$, and coalitional improvements or uncertain moves for state s by H, $CR_H^{+,U}(s)$. After obtaining the two important components of coalitional stability definitions with unknown preference, the matrix representation of coalitional stabilities can be constructed as follows. Let $l \in \{a, b, c, d\}$.

8.7.2 Matrix Representation of Coalitional Stabilities Indexed *l*

(1) Matrix Representation of Coalitional Stabilities Indexed a

For a fixed state $s \in S$, let e_s be an m-dimensional vector with 1 as its sth element and 0 everywhere else and e be an m-dimensional vector with every entry 1. Let $(\overrightarrow{0})^T$ denote the transpose of $\overrightarrow{0}$.

Theorem 8.10 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional $Nash_a$ stable for H, denoted by $s \in S_H^{CNash_a}$, iff* $e_s^T \cdot CM_H^{+,U} \cdot e{=}0$.

Proof Since $e_s^T \cdot CM_H^{+,U} \cdot e = 0$ iff $e_s^T \cdot CM_H^{+,U} = (\overrightarrow{0})^T$, then $CR_H^{+,U}(s) = \emptyset$ using Definition 8.72. Consequently, the proof of the theorem follows by Definition 8.16. □

Define coalitional $CGMR_a$ stability matrix for coalition H as

$$M_H^{CGMR_a} = CM_H^{+,U} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,=})^T\right)], \tag{8.10}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is coalitional GMR_a stable for H.

Theorem 8.11 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional GMR_a stable for H, denoted by $s \in S_H^{CGMR_a}$, iff $M_H^{CGMR_a}$ $(s, s) = 0$.*

Proof Since

$$M_H^{CGMR_a}(s, s) = (e_s^T \cdot CM_H^{+,U}) \cdot [\left(E - sign\left(M_{N-H} \cdot (P_H^{-,=})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+,U}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=})^T\right)],$$

then $M_H^{CGMR_a}(s, s) = 0$ holds iff

$$CM_H^{+,U}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=})^T\right)] = 0, \tag{8.11}$$

for every $s_1 \in S - \{s\}$. It is clear that Eq. 8.11 is equivalent to

$$(e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=})^T \neq 0,$$

for every $s_1 \in CR_H^{+,U}(s)$. Therefore, for a coalitional improvement or uncertain move from s, $s_1 \in CR_H^{+,U}(s)$, there exists at least one $s_2 \in R_{N-H}(s_1)$ with $P_H^{-,=}(s, s_2) = 1$ that is equivalent to $s \succeq_i s_2$ for some DM $i \in H$. According to Definition 8.17, $M_H^{CGMR_a}(s, s) = 0$ implies that s is coalitional GMR_a stable for H. □

Theorem 8.11 shows that this matrix method, called matrix representation of coalitional GMR_a stability, is equivalent to the logical version of the same stability given in Definition 8.17.

Coalitional SMR_a is similar to coalitional GMR_a except that coalition H expects to have a chance to counterrespond to its opponent $(N-H)$'s response to H's original move. Define the coalitional SMR_a stability matrix as

$$M_H^{CSMR_a} = CM_H^{+,U} \cdot [E - sign(F)]$$

in which

$$F = M_{N-H} \cdot [(P_H^{-,=})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,=})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes a matrix method to determine whether state s is coalitional SMR stable for H.

Theorem 8.12 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SMR_a for H, denoted by $s \in S_H^{CSMR_a}$, iff $M_H^{CSMR_a}(s, s) = 0$.*

Proof Since

$$M_H^{CSMR_a}(s, s) = (e_s^T \cdot CM_H^{+,U}) \cdot [(E - sign(F)) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+,U}(s, s_1)[1 - sign\,(F(s_1, s))]$$

with

$$F(s_1, s) = \sum_{s_2=1}^{m} M_{N-H}(s_1, s_2) \cdot W(s_2, s),$$

and

$$W(s_2, s) = P_H^{-,=}(s, s_2) \cdot \left[1 - sign\left(\sum_{s_3=1}^{m} \left(M_H(s_2, s_3) \cdot (1 - P_H^{-,=}(s, s_3)\right)\right)\right],$$

then $M_H^{CSMR_a}(s, s) = 0$ holds iff $F(s_1, s) \neq 0$, for every $s_1 \in CR_H^{+,U}(s)$, which is equivalent to the statement that, for every $s_1 \in CR_H^{+,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that

$$P_H^{-,=}(s, s_2) \neq 0, \tag{8.12}$$

and

$$\sum_{s_3=1}^{m} M_H(s_2, s_3) \cdot (1 - P_H^{-,=})(s, s_3) = 0. \tag{8.13}$$

Equation 8.12 means that $s \succeq_i s_2$ for at least one DM $i \in H$. Equation 8.13 is equivalent to

$$P_H^{-,=}(s, s_3) \neq 0 \text{ for any } s_3 \in R_H(s_2). \tag{8.14}$$

Obviously, for every $s_1 \in CR_H^{+,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s \succeq_i s_2$ and Eq. 8.13 hold iff for every $s_1 \in CR_H^{+,U}(s)$ there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\preceq}(s)$ and $s_3 \in \Phi_H^{\preceq}(s)$ for all $s_3 \in R_H(s_2)$. Therefore, the proof of this theorem follows using Definition 8.18. □

Theorem 8.12 displays that this matrix method, called matrix representation of coalitional SMR stability, is equivalent to the logical version given in Definition 8.18. To calculate coalitional SMR_a stability at s for H, one only needs to assess whether the diagonal entry $M_H^{CSMR_a}(s, s)$ of coalitional SMR_a stability matrix is zero. If so, s is coalitional SMR_a stable for H; otherwise, s is coalitional SMR_a unstable for H.

Coalitional sequential stability is similar to coalitional GMR stability, but includes only those sanctions that are "credible". The coalitional SEQ_a stability matrix $M_H^{CSEQ_a}$ is defined as

$$M_H^{CSEQ_a} = CM_H^{+,U} \cdot [E - sign\left(M_{N-H}^{+,U} \cdot (P_H^{-,=})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is coalitional SEQ_a stable for H when H's opponents, $N - H$, are in a coalition.

Theorem 8.13 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SEQ_a stable for H, denoted by $s \in S_H^{CSEQ_a}$, iff $M_H^{CSEQ_a}(s, s) = 0$.*

Proof Since

$$M_H^{CSEQ_a}(s,s) = (e_s^T CM_H^{+,U}) \cdot [\Big(E - sign(M_{N-H}^{+,U} \cdot (P_H^{-,=})^T)\Big) e_s]$$
$$= \sum_{s_1=1}^{|S|} CM_H^{+,U}(s,s_1)[1 - sign\left((e_{s_1}^T M_{N-H}^{+,U}) \cdot (e_s^T P_H^{-,=})^T\right)],$$

then $M_H^{SEQ_a}(s,s) = 0$ holds iff

$$CM_H^{+,U}(s,s_1)[1 - sign\left((e_{s_1}^T M_{N-H}^{+,U}) \cdot (e_s^T P_H^{-,=})^T\right)] = 0, \forall s_1 \in S. \tag{8.15}$$

It is clear that Eq. 8.15 is equivalent to

$$(e_{s_1}^T M_{N-H}^{+,U}) \cdot (e_s^T P_H^{-,=})^T \neq 0 \text{ for any } s_1 \in CR_H^{+,U}(s).$$

It implies that for any $s_1 \in CR_H^{+,U}(s)$, there exists at least one $s_2 \in R_{N-H}^{+,U}(s_1)$ with $s \succeq_i s_2$ for some DM $i \in H$ that satisfies $s_2 \in \Phi_H^{\preceq}(s)$. The proof of this theorem follows using Definition 8.19. □

Note that the coalitional SEQ_a stability matrix is identical to the coalitional GMR_a stability matrix except that the UM reachability matrix for H's opponents, M_{N-H}, is replaced by the coalitional improvement or uncertain move matrix $CM_{N-H}^{+,U}$.

(2) Matrix Representation of Coalitional Stabilities Indexed *b*

The following theorems establish relationships between logical and matrix representations for coalitional stabilities indexed b under unknown preference. The extension indexed b excludes uncertainty in preferences when the focal coalition H considers incentives to leave a state and evaluates sanctions from its opponents. However, the following coalitional definitions are different from the coalitional stability definitions without preference uncertainty as discussed in Sect. 8.6, because the previous definitions cannot be used to analyze coalitional stabilities with uncertain preference.

Theorem 8.14 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional $Nash_b$ stable for H, denoted by $s \in S_H^{CNash_b}$, iff $e_s^T \cdot CM_H^+ \cdot e = 0$.*

Define coalitional $CGMR_b$ stability matrix for coalition H as

$$M_H^{CGMR_b} = CM_H^+ \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,=})^T\right)], \tag{8.16}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is coalitional GMR_b stable for H.

Theorem 8.15 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional GMR_b stable for H, denoted by $s \in S_H^{CGMR_b}$, iff $M_H^{CGMR_b}(s,s) = 0$.*

Theorem 8.15 shows that this matrix method, called matrix representation of coalitional GMR_b stability, is equivalent to the logical version of the same coalitional GMR_b stability given in Definition 8.21.

Define the coalitional SMR_b stability matrix as

$$M_H^{CSMR_b} = CM_H^+ \cdot [E - sign(Q)]$$

in which

$$Q = M_{N-H} \cdot [(P_H^{-,=})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,=})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is coalitional SMR_b stable for H.

Theorem 8.16 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SMR_b for H, denoted by $s \in S_H^{CSMR_b}$, iff $M_H^{CSMR_b}(s, s) = 0$.*

The coalitional SEQ_b stability matrix $M_H^{CSEQ_b}$ is defined as

$$M_H^{CSEQ_b} = CM_H^+ \cdot [E - sign\left(M_{N-H}^{+,U} \cdot (P_H^{-,=})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is coalitional SEQ_b stable for H when H's opponents, $N - H$, are in a coalition.

Theorem 8.17 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SEQ_b stable for H, denoted by $s \in S_H^{CSEQ_b}$, iff $M_H^{CSEQ_b}(s, s) = 0$.*

The proofs of the above theorems on coalitional stabilities indexed b are similar to the proofs for the matrix representation of coalitional stabilities indexed a. Therefore, these proofs are left as exercises.

(3) Matrix Representation of Coalitional Stabilities Indexed c

Coalitional Nash stability similar to the individual stability case in Chap. 5 does not examine countermoves by the opponents, so $S_H^{Nash_c} = S_H^{Nash_a}$.

Define coalitional $CGMR_c$ stability matrix for coalition H as

$$M_H^{CGMR_c} = CM_H^{+,U} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,=,U})^T\right)], \qquad (8.17)$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is coalitional GMR_c stable for H.

Theorem 8.18 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional GMR_c stable for H, denoted by $s \in S_H^{CGMR_c}$, iff $M_H^{CGMR_c}(s, s) = 0$.*

Theorem 8.18 shows that this matrix method, called matrix representation of coalitional GMR_c stability, is equivalent to the logical version of the same coalitional GMR_c stability given in Definition 8.24.

Define the coalitional SMR_c stability matrix as

$$M_H^{CSMR_c} = CM_H^{+,U} \cdot [E - sign(Q)]$$

in which

$$Q = M_{N-H} \cdot [(P_H^{-,=,U})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,=,U})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is coalitional SMR_c stable for H.

Theorem 8.19 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SMR_c for H, denoted by $s \in S_H^{CSMR_c}$, iff $M_H^{CSMR_c}(s, s) = 0$.*

The coalitional SEQ_c stability matrix $M_H^{CSEQ_c}$ is defined as

$$M_H^{CSEQ_c} = CM_H^{+,U} \cdot [E - sign\left(M_{N-H}^{+,U} \cdot (P_H^{-,=,U})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is coalitional SEQ_c stable for H when H's opponents, $N - H$, are in a coalition.

Theorem 8.20 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SEQ_c stable for H, denoted by $s \in S_H^{CSEQ_c}$, iff $M_H^{CSEQ_c}(s, s) = 0$.*

The proofs of the above theorems on coalitional stabilities indexed c are left as exercises.

(4) Matrix Representation of Coalitional Stabilities Indexed d

As mentioned before, similar to the individual stability case in Chap. 5 coalitional Nash stability does not examine countermoves by the opponents, so $S_H^{Nash_d} = S_H^{Nash_b}$.

Define coalitional $CGMR_d$ stability matrix for coalition H as

$$M_H^{CGMR_d} = CM_H^{+} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,=,U})^T\right)], \tag{8.18}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is coalitional GMR_d stable for H.

Theorem 8.21 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is coalitional GMR_d stable for H, denoted by $s \in S_H^{CGMR_d}$, iff $M_H^{CGMR_d}(s, s) = 0$.*

Proof Since

$$M_H^{CGMR_d}(s, s) = (e_s^T \cdot CM_H^{+}) \cdot [\left(E - sign\left(M_{N-H} \cdot (P_H^{-,=,U})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=,U})^T\right)],$$

then $M_H^{CGMR_d}(s, s) = 0$ holds iff

$$CM_H^{+}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=,U})^T\right)] = 0, \tag{8.19}$$

for every $s_1 \in S - \{s\}$. It is clear that Eq. 8.19 is equivalent to

$$(e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,=,U})^T \neq 0,$$

for every $s_1 \in CR_H^{+}(s)$. Therefore, for a coalitional improvement from s, $s_1 \in CR_H^{+}(s)$, there exists at least one $s_2 \in R_{N-H}(s_1)$ with $P_H^{-,=,U}(s, s_2) = 1$ that is equivalent to $s \succeq_i s_2$ or $s\ U_i\ s_2$ for some DM $i \in H$. According to Definition 8.27, $M_H^{CGMR_d}(s, s) = 0$ implies that s is coalitional GMR_d stable for H. □

Theorem 8.21 shows that this matrix method, called matrix representation of coalitional GMR_d stability, is equivalent to the logical version of the same coalitional GMR_d stability given in Definition 8.27.

Define the coalitional SMR_d stability matrix as

$$M_H^{CSMR_d} = CM_H^{+} \cdot [E - sign(F)]$$

in which

$$F = M_{N-H} \cdot [(P_H^{-,=,U})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,=,U})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is coalitional SMR_d stable for H.

Theorem 8.22 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SMR_d for H, denoted by $s \in S_H^{CSMR_d}$, iff $M_H^{CSMR_d}(s, s) = 0$.*

Proof Since

$$M_H^{CSMR_d}(s, s) = (e_s^T \cdot CM_H^{+}) \cdot [(E - sign(F)) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+}(s, s_1)[1 - sign\,(F(s_1, s))]$$

with

$$F(s_1, s) = \sum_{s_2=1}^{m} M_{N-H}(s_1, s_2) \cdot W(s_2, s),$$

and

$$W(s_2, s) = P_H^{-,=,U}(s, s_2) \cdot \left[1 - sign\left(\sum_{s_3=1}^{m} \left(M_H(s_2, s_3) \cdot (1 - P_H^{-,=,U}(s, s_3)\right)\right)\right],$$

then $M_H^{CSMR_d}(s, s) = 0$ holds iff $F(s_1, s) \neq 0$, for every $s_1 \in CR_H^+(s)$, which is equivalent to the statement that, for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that

$$P_H^{-,=,U}(s, s_2) \neq 0, \tag{8.20}$$

and

$$\sum_{s_3=1}^{m} M_H(s_2, s_3) \cdot (1 - P_H^{-,=,U})(s, s_3) = 0. \tag{8.21}$$

Equation 8.20 means that $s \succeq_i s_2$ or $s\ U_i\ s_2$ for at least one DM $i \in H$. Equation 8.21 is equivalent to

$$P_H^{-,=,U}(s, s_3) \neq 0 \text{ for any } s_3 \in R_H(s_2). \tag{8.22}$$

Obviously, for every $s_1 \in CR_H^+(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that $s \succeq_i s_2$ or $s\ U_i\ s_2$ and Eq. 8.22 hold iff for every $s_1 \in CR_H^+(s)$ there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\preceq,U}(s)$ and $s_3 \in \Phi_H^{\preceq,U}(s)$ for all $s_3 \in R_H(s_2)$. Therefore, the proof of this theorem follows using Definition 8.28. □

The coalitional SEQ_d stability matrix $M_H^{CSEQ_d}$ is defined as

$$M_H^{CSEQ_d} = CM_H^+ \cdot [E - sign\left(M_{N-H}^{+,U} \cdot (P_H^{-,=,U})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is coalitional SEQ_d stable for H when H's opponents, $N - H$, are in a coalition.

Theorem 8.23 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is coalitional SEQ_d stable for H, denoted by $s \in S_H^{CSEQ_d}$, iff $M_H^{CSEQ_d}(s, s) = 0$.*

Proof Since

$$M_H^{CSEQ_d}(s, s) = (e_s^T CM_H^+) \cdot [\left(E - sign(M_{N-H}^{+,U} \cdot (P_H^{-,=,U})^T)\right) e_s]$$

$$= \sum_{s_1=1}^{|S|} CM_H^+(s, s_1)[1 - sign\left((e_{s_1}^T M_{N-H}^{+,U}) \cdot (e_s^T P_H^{-,=,U})^T\right)],$$

then $M_H^{SEQ_d}(s, s) = 0$ holds iff

$$CM_H^+(s, s_1)[1 - sign\left((e_{s_1}^T M_{N-H}^{+,U}) \cdot (e_s^T P_H^{-,=,U})^T\right)] = 0, \forall s_1 \in S. \tag{8.23}$$

It is clear that Eq. 8.23 is equivalent to

$$(e_{s_1}^T M_{N-H}^{+,U}) \cdot (e_s^T P_H^{-,=,U})^T \neq 0 \text{ for any } s_1 \in CR_H^+(s).$$

It implies that for any $s_1 \in CR_H^+(s)$, there exists at least one $s_2 \in R_{N-H}^{+,U}(s_1)$ with $s \succeq_i s_2$ or $s\ U_i\ s_2$ for some DM $i \in H$ that satisfies $s_2 \in \Phi_H^{\preceq,U}(s)$. The proof of this theorem follows using Definition 8.29. □

8.8 Matrix Representation of Coalitional Stability with Three Degrees of Preference

The logical representation of coalitional stabilities under three-degree preference is discussed in Sect. 8.4. The matrix form of these coalitional stabilities is introduced as follows.

8.8.1 Matrix Representation of Mild or Strong Coalitional Improvement

Definition 8.73 For the graph model G, the mild or strong coalitional improvement matrix for coalition H is an $m \times m$ matrix $CM_H^{+,++}$ with (s, q) entry

$$CM_H^{+,++}(s,q) = \begin{cases} 1 & \text{if } q \in CR_H^{+,++}(s), \\ 0 & \text{otherwise.} \end{cases}$$

The mild or strong coalitional improvement matrix is equivalent to the coalitional reachable list, $CR_H^{+,++}(s)$, defined in Sect. 8.1. The matrix $CM_H^{+,++}$ can be constructed as follows.

To carry out coalitional stability analysis, recall a set of matrices corresponding to three-level preference defined in Chap. 6.

$$P_i^{++}(s,q) = \begin{cases} 1 \text{ if } q \gg_i s, \\ 0 \text{ otherwise,} \end{cases}$$

$$P_i^{--}(s,q) = \begin{cases} 1 \text{ if } s \gg_i q, \\ 0 \text{ otherwise,} \end{cases}$$

$$P_i^{+,++}(s,q) = \begin{cases} 1 \text{ if } q >_i s \text{ or } q \gg_i s, \\ 0 \text{ otherwise,} \end{cases}$$

and

$$P_i^{--,-,=}(s,q) = \begin{cases} 1 \text{ if } s >_i q, s \gg_i q, \text{ or } (s \sim_i q \text{ and } s \neq q), \\ 0 \text{ otherwise.} \end{cases}$$

Based on the above definitions, the UM adjacency matrix J_i, mild or strong unilateral improvement adjacency matrix $J_i^{+,++}$, and preference matrix $P_i^{+,++}$ for DM i have the relationship among them:

$$J_i^{+,++} = J_i \circ P_i^{+,++}.$$

Theorem 8.24 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. The mild or strong coalitional improvement matrix for H is expressed as*

$$CM_H^{+,++} = M_H \circ (E - P_H^{-,--,=}). \tag{8.24}$$

Proof To prove Eq. 8.24, assume that $C = M_H \circ (E - P_H^{-,--,=})$. Using the definition for matrix M_H presented in Chap. 4, $C(s,q) = 1$ iff $M_H(s,q) = 1$ and $P_H^{-,--,=}(s,q) = 0$, which together imply that there is $q \in R_H(s)$ such that $P_i^{-,--,=}(s,q) = 0$ for every DM $i \in H$. Therefore, $C(s,q) = 1$ iff there is $q \in R_H(s)$ with $q >_i s$ or $q \gg_i s$ for every $i \in H$, so that $q \in CR_H^{+,++}(s)$, according to Definition 8.3. Hence, $CM_H^{+,++}(s,q) = 1$ iff $C(s,q) = 1$. Since CM_H^{+} and C are 0–1 matrices, it follows that $CM_H^{+,++} = M_H \circ (E - P_H^{-,--,=})$. □

Note that $CM_H^{+,++} \neq M_H \circ P_H^{+,++}$. Recall that matrix $P_H^{+,++} = \bigvee_{i \in H} P_i^{+,++}$ ("$\bigvee$" denotes the disjunction operator described in Definition 3.16). $(M_H \circ P_H^{+,++})(s,q) = 1$ iff $M_H(s,q) = 1$ and $P_H^{+,++}(s,q) = 1$, which means that there is $q \in R_H(s)$ such that $P_i^{+,++}(s,q) = 1$ for some DM $i \in H$. This is not consistent with the definition of $CM_H^{+,++}$.

8.8.2 Matrix Representation of General Coalitional Stabilities

Let $m = |S|$ denote the number of states and E be the $m \times m$ matrix with each entry equal to 1. For a fixed state $s \in S$, let e_s be an m-dimensional vector with 1 as its sth element and 0 everywhere else and e be an m-dimensional vector with every entry 1. Let $(\overrightarrow{0})^T$ denote the transpose of $\overrightarrow{0}$. Recall that the UM reachability matrix M_H is constructed using Theorem 4.9. General coalitional stabilities are presented using matrix approach next.

Theorem 8.25 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is general coalitional Nash stable for H, denoted by $s \in S_H^{GCNash}$, iff $e_s^T \cdot CM_H^{+,++} \cdot e = 0$.*

Proof Since $e_s^T \cdot CM_H^{+,++} \cdot e = 0$ iff $e_s^T \cdot CM_H^{+,++} = (\overrightarrow{0})^T$, then $CR_H^{+,++}(s) = \emptyset$. Consequently, the proof of the theorem follows by Definition 8.30. □

Similar to the individual general GMR stability for the three-degree preference, define general coalitional GMR stability matrix as

$$M_H^{GCGMR} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,--,=})^T\right)], \tag{8.25}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is general coalitional GMR stable for H.

Theorem 8.26 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is general coalitional GMR stable for H, denoted by $s \in S_H^{GCGMR}$, iff $M_H^{GCGMR}(s, s) = 0$.*

Proof Since

$$M_H^{GCGMR}(s, s) = (e_s^T \cdot CM_H^{+,++}) \cdot [\left(E - sign\left(M_{N-H} \cdot (P_H^{-,--,=})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+,++}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=})^T\right)],$$

then $M_H^{GCGMR}(s, s) = 0$ holds iff

$$CM_H^{+,++}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=})^T\right)] = 0, \tag{8.26}$$

for every $s_1 \in S - s$. It is clear that Eq. 8.26 is equivalent to

$$(e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=})^T \neq 0,$$

for every $s_1 \in CR_H^{+,++}(s)$. Therefore, for a coalitional mild or strong improvement from s, $s_1 \in CR_H^{+,++}(s)$, there exists at least one $s_2 \in R_{N-H}(s_1)$ with $P_H^{-,--,=}(s, s_2) = 1$ that is equivalent to $s >_i s_2$, $s \gg_i s_2$ or $s \sim_i s_2$ for some DM $i \in H$. According to Definition 8.31, $M_H^{GCGMR}(s, s) = 0$ implies that s is general coalitional GMR stable for H. □

Theorem 8.26 shows that this matrix method, called matrix representation of general coalitional GMR stability, is equivalent to the logical version of the same stability given in Definition 8.31. To analyze the general coalitional GMR stability at s for coalition H, one only needs to identify whether the diagonal entry $M_H^{GCGMR}(s, s)$ is zero. If so, s is general coalitional GMR stable for H; otherwise, s is general coalitional GMR unstable for H.

General coalitional SMR stability is similar to general coalitional GMR except that coalition H expects to have a chance to counterrespond to its opponents' $(N-H)$ response to H's original move. Define the general coalitional SMR stability matrix as

$$M_H^{GCSMR} = CM_H^{+,++} \cdot [E - sign(F)]$$

in which

$$F = M_{N-H} \cdot [(P_H^{-,--,=})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,--,=})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is general coalitional SMR stable for H.

Theorem 8.27 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SMR for H, denoted by $s \in S_H^{GCSMR}$, iff $M_H^{GCSMR}(s, s) = 0$.*

Proof Since

$$M_H^{GCSMR}(s, s) = (e_s^T \cdot CM_H^{+,++}) \cdot [(E - sign(F)) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+,++}(s, s_1)[1 - sign\,(F(s_1, s))]$$

with

$$F(s_1, s) = \sum_{s_2=1}^{m} M_{N-H}(s_1, s_2) \cdot W(s_2, s),$$

and

$$W(s_2, s) = P_H^{-,--,=}(s, s_2) \cdot \left[1 - sign\left(\sum_{s_3=1}^{m} \left(M_H(s_2, s_3) \cdot (1 - P_H^{+,++}(s, s_3))\right)\right)\right],$$

then $M_H^{GCSMR}(s, s) = 0$ holds iff $F(s_1, s) \neq 0$, for every $s_1 \in CR_H^{+,++}(s)$, which is equivalent to the statement that, for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that

$$P_H^{-,--,=}(s, s_2) \neq 0, \tag{8.27}$$

and

$$\sum_{s_3=1}^{m} M_H(s_2, s_3) \cdot (1 - P_H^{-,--,=}(s, s_3)) = 0. \tag{8.28}$$

Equation 8.27 means that $s >_i s_2$, $s \gg_i s_2$, or $s \sim_i s_2$ for at least one DM $i \in H$. Equation 8.28 is equivalent to

$$P_H^{-,--,=}(s, s_3) = 1 \text{ for any } s_3 \in R_H(s_2). \tag{8.29}$$

Obviously, for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that Eqs. 8.27 and 8.28 hold iff for every $s_1 \in CR_H^{+,++}(s)$ there exists $s_2 \in R_{N-H}(s_1)$ such that $s >_i s_2$, $s \gg_i s_2$, or $s \sim_i s_2$ and $s >_i s_3$, $s \gg_i s_3$, or $s \sim_i s_3$ for some DM i with all $s_3 \in R_H(s_2)$. Therefore, the proof of this theorem follows using Definition 8.32. □

Theorem 8.27 displays that this matrix method, called matrix representation of general coalitional SMR stability, is equivalent to the logical version given in Definition 8.32. To calculate general coalitional SMR stability at s for H, one only needs to assess whether the diagonal entry $M_H^{GCSMR}(s,s)$ is zero. If so, s is general coalitional SMR stable for H; otherwise, s is general coalitional SMR unstable for H.

General coalitional sequential stability is similar to general coalitional GMR stability, but includes only those sanctions that are "credible". If H's opponents are treated as a coalition, the general coalitional SEQ_1 stability matrix $M_H^{GCSEQ_1}$ is defined as

$$M_H^{GCSEQ_1} = CM_H^{+,++} \cdot [E - sign\left(CM_{N-H}^{+,++} \cdot (P_H^{-,--,=})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is general coalitional SEQ_1 stable for H when H's opponents, $N-H$, are in a coalition.

Theorem 8.28 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SEQ_1 stable for H, denoted by $s \in S_H^{GCSEQ_1}$, iff $M_H^{GCSEQ_1}(s,s) = 0$.*

Proof Since

$$M_H^{GCSEQ_1}(s,s) = (e_s^T \cdot CM_H^{+,++}) \cdot [\left(E - sign(CM_{N-H}^{+,++} \cdot (P_H^{-,--,=})^T)\right) e_s]$$

$$= \sum_{s_1=1}^{|S|} CM_H^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T CM_{N-H}^{+,++}) \cdot (e_s^T P_H^{-,--,=})^T\right)],$$

then $M_H^{GCSEQ_1}(s,s) = 0$ holds iff

$$CM_H^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot CM_{N-H}^{+,++}) \cdot (e_s^T \cdot P_H^{-,--,=})^T\right)] = 0, \forall s_1 \in S. \tag{8.30}$$

It is clear that Eq. 8.30 is equivalent to

$$(e_{s_1}^T \cdot CM_{N-H}^{+,++}) \cdot (e_s^T \cdot P_H^{-,--,=})^T \neq 0 \text{ for any } s_1 \in CR_H^{+,++}(s).$$

It implies that for any $s_1 \in CR_H^{+,++}(s)$, there exists at least one $s_2 \in CR_{N-H}^{+,++}(s_1)$ with $s >_i s_2$, $s \gg_i s_2$ or $s \sim_i s_2$ for some DM $i \in H$. The proof of this theorem follows using Definition 8.33. □

Note that the general coalitional SEQ_1 stability matrix is identical to the general coalitional GMR stability matrix except that the UM reachability matrix for H's opponents, M_{N-H}, is replaced by the mild or strong coalitional improvement matrix $CM_{N-H}^{+,++}$.

Similar to the previous two theorems, the matrix representation of general coalitional SEQ_1 stability is equivalent to the logical version given in Definition 8.33. When the diagonal entry at (s, s) is zero, the state s under consideration is general coalitional SEQ_1 stable for H. The following theorem is equivalent to the coalitional SEQ_2 stability presented in Definition 8.34. Define the coalitional SEQ_2 stability matrix $M_H^{GCSEQ_2}$ as

$$M_H^{GCSEQ_2} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H}^{+,++} \cdot (P_H^{-,--,=})^T\right)].$$

Theorem 8.29 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SEQ_2 stable for H, denoted by $s \in S_H^{GCSEQ_2}$, iff $M_H^{GCSEQ_2}(s, s) = 0$.*

The proof of this theorem is similar to the proof of Theorem 8.28.

8.8.3 Matrix Representation of Strong Coalitional Stabilities

When three degrees of preference is introduced into the graph model, general coalitional stability definitions may be strong or weak, according to the strength of sanctioning. The following matrix representations of strong or weak coalitional stabilities are equivalent to the logical forms presented in Sect. 8.4.2.

Define the strong coalitional GMR stability matrix as

$$M_H^{SCGMR} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{--})^T\right)], \tag{8.31}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is strong coalitional GMR stable for H.

Theorem 8.30 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is strong coalitional GMR stable for H, denoted by $s \in S_H^{SCGMR}$, iff $M_H^{SCGMR}(s, s) = 0$.*

Proof Since

$$M_H^{SCGMR}(s, s) = (e_s^T \cdot CM_H^{+,++}) \cdot [\left(E - sign\left(M_{N-H} \cdot (P_H^{--})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+,++}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{--})^T\right)],$$

then $M_H^{SCGMR}(s,s)=0$ holds iff

$$CM_H^{+,++}(s,s_1)\cdot[1-sign\left((e_{s_1}^T\cdot M_{N-H})\cdot(e_s^T\cdot P_H^{--})^T\right)]=0, \tag{8.32}$$

for every $s_1 \in S-\{s\}$. It is clear that Eq. 8.32 is equivalent to

$$(e_{s_1}^T\cdot M_{N-H})\cdot(e_s^T\cdot P_H^{--})^T\neq 0,$$

for every $s_1\in CR_H^{+,++}(s)$. Therefore, for a mild or strong coalitional improvement from s, $s_1\in CR_H^{+,++}(s)$, there exists at least one $s_2\in R_{N-H}(s_1)$ with $P_H^{--}(s,s_2)=1$ that is equivalent to $s\gg_i s_2$ for some DM $i\in H$. According to Definition 8.35, $M_H^{SCGMR}(s,s)=0$ implies that s is strong coalitional GMR stable for H. □

Theorem 8.30 shows that this matrix method, called matrix representation of strong coalitional GMR stability, is equivalent to the logical version of the same stability given in Definition 8.35. To analyze the strong coalitional GMR stability at s for coalition H, the diagonal entry (s,s) of matrix M_H^{SCGMR} is identified whether it is zero. If so, s is strong coalitional GMR stable for H.

Define the strong coalitional SMR stability matrix as

$$M_H^{SCSMR}=CM_H^{+,++}\cdot[E-sign(F)]$$

in which

$$F=M_{N-H}\cdot[(P_H^{--})^T\circ\left(E-sign\left(M_H\cdot(E-P_H^{--})^T\right)\right)],$$

for $H\subseteq N$. The following theorem establishes the matrix method to determine whether state s is strong coalitional SMR stable for H.

Theorem 8.31 *For the graph model G, let $H\subseteq N$ be a coalition. State $s\in S$ is strong coalitional SMR for H, denoted by $s\in S_H^{SCSMR}$, iff $M_H^{SCSMR}(s,s)=0$.*

Proof Since

$$M_H^{SCSMR}(s,s)=(e_s^T\cdot CM_H^{+,++})\cdot[(E-sign(F))\cdot e_s]$$

$$=\sum_{s_1=1}^{m}CM_H^{+,++}(s,s_1)[1-sign\,(F(s_1,s))]$$

with

$$F(s_1,s)=\sum_{s_2=1}^{m}M_{N-H}(s_1,s_2)\cdot W(s_2,s),$$

and

$$W(s_2, s) = P_H^{--}(s, s_2) \cdot \left[1 - sign\left(\sum_{s_3=1}^{m} \left(M_H(s_2, s_3) \cdot (1 - P_H^{--}(s, s_3))\right)\right)\right],$$

then $M_H^{SCSMR}(s, s) = 0$ holds iff $F(s_1, s) \neq 0$, for every $s_1 \in CR_H^{+,++}(s)$, which is equivalent to the statement that, for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that

$$P_H^{--}(s, s_2) \neq 0, \tag{8.33}$$

and

$$\sum_{s_3=1}^{m} M_H(s_2, s_3) \cdot (1 - P_H^{--}(s, s_3)) = 0. \tag{8.34}$$

Equation 8.33 means that $s \gg_i s_2$ for at least one DM $i \in H$. Equation 8.34 is equivalent to

$$P_H^{--}(s, s_3) = 1 \text{ for any } s_3 \in R_H(s_2). \tag{8.35}$$

Obviously, for every $s_1 \in CR_H^{+,++}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that Eqs. 8.33 and 8.34 hold iff for every $s_1 \in CR_H^{+,++}(s)$ there exists $s_2 \in R_{N-H}(s_1)$ such that $s \gg_i s_2$ and $s \gg_i s_3$ for some DM i with all $s_3 \in R_H(s_2)$. Therefore, the proof of this theorem follows using Definition 8.36. □

Theorem 8.31 provides a matrix method, called matrix representation of strong coalitional SMR stability, which is equivalent to the logical version given in Definition 8.36. The following theorem displays the matrix method to identify whether state s is strong coalitional SEQ_1 stable for H when H's opponents, $N - H$, are in a coalition. Let the strong coalitional SEQ_1 stability matrix $M_H^{SCSEQ_1}$ be defined as

$$M_H^{SCSEQ_1} = CM_H^{+,++} \cdot [E - sign\left(CM_{N-H}^{+,++} \cdot (P_H^{--})^T\right)].$$

Theorem 8.32 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SEQ_1 stable for H, denoted by $s \in S_H^{SCSEQ_1}$, iff $M_H^{SCSEQ_1}(s, s) = 0$.*

Proof Since

$$M_H^{SCSEQ_1}(s, s) = (e_s^T \cdot CM_H^{+,++}) \cdot [\left(E - sign(CM_{N-H}^{+,++} \cdot (P_H^{--})^T)\right) e_s]$$

$$= \sum_{s_1=1}^{|S|} CM_H^{+,++}(s, s_1)[1 - sign\left((e_{s_1}^T CM_{N-H}^{+,++}) \cdot (e_s^T P_H^{--})^T\right)],$$

then $M_H^{SCSEQ_1}(s, s) = 0$ holds iff

$$CM_H^{+,++}(s,s_1)[1-sign\left((e_{s_1}^T\cdot CM_{N-H}^{+,++})\cdot(e_s^T\cdot P_H^{--})^T\right)]=0, \forall s_1\in S. \quad (8.36)$$

It is clear that Eq. 8.36 is equivalent to

$$(e_{s_1}^T\cdot CM_{N-H}^{+,++})\cdot(e_s^T\cdot P_H^{--})^T\neq 0 \text{ for any } s_1\in CR_H^{+,++}(s).$$

It implies that for any $s_1\in CR_H^{+,++}(s)$, there exists at least one $s_2\in CR_{N-H}^{+,++}(s_1)$ with $s \gg_i s_2$ for some DM $i\in H$. The proof of this theorem follows using Definition 8.37. □

The following theorem is equivalent to the strong coalitional SEQ_2 stability presented in Definition 8.38. The strong coalitional SEQ_2 stability matrix $M_H^{SCSEQ_2}$ is defined as

$$M_H^{SCSEQ_2}=CM_H^{+,++}\cdot[E-sign\left(M_{N-H}^{+,++}\cdot(P_H^{--})^T\right)].$$

Theorem 8.33 *For the graph model G, let $H\subseteq N$ be a coalition. State $s\in S$ is strong coalitional SEQ_2 stable for H, denoted by $s\in S_H^{SCSEQ_2}$, iff $M_H^{SCSEQ_2}(s,s)=0$.*

The proof of this theorem is left as an exercise.

8.9 Matrix Representation of Coalitional Stability with Hybrid Preference

After discussing matrix representations of coalitional stabilities with unknown preference and with three degrees of preference, respectively, it is nature to construct the matrix form of the coalitional stabilities under hybrid preference.

8.9.1 Matrix Representation of Coalitional Improvement Under Hybrid Preference

Definition 8.74 For the graph model G, the mild or strong or uncertain coalitional improvement matrix for coalition H is an $m\times m$ matrix $CM_H^{+,++,U}$ with (s,q) entry

$$CM_H^{+,++,U}(s,q)=\begin{cases}1 & \text{if } q\in CR_H^{+,++,U}(s),\\ 0 & \text{otherwise.}\end{cases}$$

The mild or strong or uncertain coalitional improvement matrix is equivalent to the coalitional reachable list $CR_H^{+,++,U}(s)$ given in Definition 8.4. The matrix $CM_H^{+,++,U}$ can be constructed as follows. To carry out coalitional stability analysis, recall a set of matrices corresponding to hybrid preference defined in Chap. 7.

The following $m \times m$ matrices are important in stability definitions under hybrid preference. Let E denote the $m \times m$ matrix with each entry 1 and let I be the $m \times m$ unit matrix. Then, $m \times m$ preference matrix $P_i^{+,++,U}$ is defined as

$$P_i^{+,++,U}(s,q) = \begin{cases} 1 \text{ if } q >_i s, \ q \gg_i s, \ \text{or } q \ U_i \ s, \\ 0 \text{ otherwise.} \end{cases}$$

For hybrid preference, $P_i^{--,-,=} = E - I - P_i^{+,++,U}$.

Theorem 8.34 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. The mild or strong coalitional improvement matrix, $CM_H^{+,++}$, and mild or strong or uncertain coalitional improvement matrix, $CM_H^{+,++,U}$, for H are expressed as*

$$CM_H^{+,++} = M_H \circ (E - P_H^{-,--,=,U}), \tag{8.37}$$

$$CM_H^{+,++,U} = M_H \circ (E - P_H^{-,--,=}), \tag{8.38}$$

respectively.

Proof Equation 8.37 is left as an exercise. To prove Eq. 8.38, assume that $C = M_H \circ (E - P_H^{-,--,=})$. Using the definition for matrix M_H given in Chap. 4, $C(s,q) = 1$ iff $M_H(s,q) = 1$ and $P_H^{-,--,=}(s,q) = 0$, which together imply that there is $q \in R_H(s)$ such that $P_i^{+,++,U}(s,q) = 1$ for every DM $i \in H$. Therefore, $C(s,q) = 1$ iff there is $q \in R_H(s)$ with $q >_i s$, $q \gg_i s$ or $q \ U_i \ s$ for every $i \in H$, so that $q \in CR_H^{+,++,U}(s)$, according to Definition 8.74. Hence, $CM_H^{+,++,U}(s,q) = 1$ iff $C(s,q) = 1$. Since $CM_H^{+,++,U}$ and C are 0–1 matrices, it follows that $CM_H^{+,++,U} = M_H \circ (E - P_H^{-,--,=})$. □

Note that $CM_H^{+,++}$ here is different from the matrix in Theorem 8.24 which cannot be used to analyze situations with uncertain preference. Furthermore, $CM_H^{+,++,U} \neq M_H \circ P_H^{+,++,U}$. Recall that matrix $P_H^{+,++,U} = \bigvee_{i \in H} P_i^{+,++,U}$ ("$\bigvee$" denotes the disjunction operator described in Definition 3.16). $(M_H \circ P_H^{+,++,U})(s,q) = 1$ iff $M_H(s,q) = 1$ and $P_H^{+,++,U}(s,q) = 1$, which means that there is $q \in R_H(s)$ such that $P_i^{+,++,U}(s,q) = 1$ for some DM $i \in H$. This is not consistent with the definition of $CM_H^{+,++,U}$.

8.9.2 Matrix Representation of General Coalitional Stabilities with Hybrid Preference

8.9.2.1 Matrix Representation of General Coalitional Stabilities Indexed *l*

(1) Matrix Representation of General Coalitional Stabilities Indexed *a*

Let $m = |S|$ denote the number of states and E be the $m \times m$ matrix with each entry equal to 1. For a fixed state $s \in S$, let e_s be an m-dimensional vector with 1 as its sth element and 0 everywhere else and e be an m-dimensional vector with every entry 1. Let $(\overrightarrow{0})^T$ denote the transpose of $\overrightarrow{0}$. Recall that the UM reachability matrix M_H is constructed using Theorem 4.9. General coalitional stabilities are presented using matrix approach next.

Theorem 8.35 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is general coalitional $Nash_a$ stable for H, denoted by $s \in S_H^{GCNash_a}$, iff $e_s^T \cdot CM_H^{+,++,U} \cdot e = 0$.*

Proof Since $e_s^T \cdot CM_H^{+,++,U} \cdot e = 0$ iff $e_s^T \cdot CM_H^{+,++,U} = (\overrightarrow{0})^T$, then $CR_H^{+,++,U}(s) = \emptyset$ using Definition 8.74. Consequently, the proof of the theorem follows by Definition 8.41. □

Similar to the individual general GMR_a stability for the hybrid preference, define general coalitional GMR_a stability matrix as

$$M_H^{GCGMR_a} = CM_H^{+,++,U} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,--,=})^T\right)], \tag{8.39}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is general coalitional GMR_a stable for H.

Theorem 8.36 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is general coalitional GMR_a stable for H, denoted by $s \in S_H^{GCGMR_a}$, iff $M_H^{GCGMR_a}(s, s) = 0$.*

Proof Since

$$M_H^{GCGMR_a}(s, s) = (e_s^T \cdot CM_H^{+,++,U}) \cdot [\left(E - sign\left(M_{N-H} \cdot (P_H^{-,--,=})^T\right)\right) \cdot e_s]$$
$$= \sum_{s_1=1}^{m} CM_H^{+,++,U}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=})^T\right)],$$

then $M_H^{GCGMR_a}(s, s) = 0$ holds iff

$$CM_H^{+,++,U}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=})^T\right)] = 0, \tag{8.40}$$

for every $s_1 \in S - \{s\}$. It is clear that Eq. 8.40 is equivalent to

$$(e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=})^T \neq 0,$$

for every $s_1 \in CR_H^{+,++,U}(s)$. Therefore, for a coalitional mild, strong, or uncertain improvement from s, $s_1 \in CR_H^{+,++,U}(s)$, there exists at least one $s_2 \in R_{N-H}(s_1)$ with $P_H^{-,--,=}(s, s_2) = 1$ that is equivalent to $s_2 \in \Phi_H^{\ll,<,\sim}(s)$, i.e., $s >_i s_2$, $s \gg_i s_2$, or $s \sim_i s_2$ for some DM $i \in H$. According to Definition 8.42, $M_H^{GCGMR_a}(s, s) = 0$ implies that s is general coalitional GMR_a stable for H. □

Theorem 8.36 shows that the matrix representation of general coalitional GMR_a stability is equivalent to the logical version of the same stability given in Definition 8.42. To analyze the general coalitional GMR_a stability at s for coalition H, one only needs to identify whether the diagonal entry $M_H^{GCGMR_a}(s, s)$ is zero.

General coalitional SMR_a stability is similar to general coalitional GMR_a except that coalition H expects to have a chance to counterrespond to its opponents' $(N - H)$ response to H's original move. Define the general coalitional SMR_a stability matrix as

$$M_H^{GCSMR_a} = CM_H^{+,++,U} \cdot [E - sign(F)]$$

in which

$$F = M_{N-H} \cdot [(P_H^{-,--,=})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,--,=})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is general coalitional SMR_a stable for H.

Theorem 8.37 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SMR_a for H, denoted by $s \in S_H^{GCSMR_a}$, iff $M_H^{GCSMR_a}(s, s) = 0$.*

Proof Since

$$M_H^{GCSMR_a}(s, s) = (e_s^T \cdot CM_H^{+,++,U}) \cdot [(E - sign(F)) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+,++,U}(s, s_1)[1 - sign\,(F(s_1, s))]$$

with

$$F(s_1, s) = \sum_{s_2=1}^{m} M_{N-H}(s_1, s_2) \cdot W(s_2, s),$$

and

$$W(s_2, s) = P_H^{-,--,=}(s, s_2) \cdot \left[1 - sign\left(\sum_{s_3=1}^{m}\left(M_H(s_2, s_3) \cdot (1 - P_H^{-,--,=}(s, s_3))\right)\right)\right],$$

then $M_H^{GCSMR_a}(s,s) = 0$ holds iff $F(s_1, s) \neq 0$, for every $s_1 \in CR_H^{+,++,U}(s)$, which is equivalent to the statement that, for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that

$$P_H^{-,--,=}(s, s_2) \neq 0, \tag{8.41}$$

and

$$\sum_{s_3=1}^{m} M_H(s_2, s_3) \cdot (1 - P_H^{-,--,=}(s, s_3)) = 0. \tag{8.42}$$

Equation 8.41 means that $s >_i s_2$, $s \gg_i s_2$, or $s \sim_i s_2$ for at least one DM $i \in H$. Equation 8.42 is equivalent to

$$P_H^{-,--,=}(s, s_3) = 1 \text{ for any } s_3 \in R_H(s_2). \tag{8.43}$$

Obviously, for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that Eqs. 8.41 and 8.42 hold iff for every $s_1 \in CR_H^{+,++,U}(s)$ there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim}(s)$ and $s_3 \in \Phi_H^{\ll,<,\sim}(s)$ with all $s_3 \in R_H(s_2)$. Therefore, the proof of this theorem follows using Definition 8.43. □

Theorem 8.37 displays the matrix representation of general coalitional SMR_a stability, which is equivalent to the logical version given in Definition 8.43. To calculate general coalitional SMR_a stability at s for H, one only needs to assess whether the diagonal entry $M_H^{GCSMR_a}(s,s)$ is zero.

General coalitional sequential stability is similar to general coalitional GMR stability, but includes only those sanctions that are "credible". The logical representation of two types of coalitional SEQ stability under hybrid preference was discussed, the matrix form is provided here for $CSEQ_2$ only. If H's opponents are treated as a coalition, the general coalitional SEQ_a stability matrix $M_H^{GCSEQ_a}$ is defined as

$$M_H^{GCSEQ_a} = CM_H^{+,++,U} \cdot [E - sign\left(M_{N-H}^{+,++,U} \cdot (P_H^{-,--,=})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is general coalitional SEQ_a stable for H when H's opponents, $N-H$, are in a coalition.

Theorem 8.38 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SEQ$_a$ stable for H, denoted by $s \in S_H^{GCSEQ_a}$, iff $M_H^{GCSEQ_a}(s,s) = 0$.*

Proof Since

$$M_H^{GCSEQ_a}(s,s) = (e_s^T \cdot CM_H^{+,++,U}) \cdot [\left(E - sign(M_{N-H}^{+,++,U} \cdot (P_H^{-,--,=})^T)\right) e_s]$$

$$= \sum_{s_1=1}^{|S|} CM_H^{+,++,U}(s, s_1)[1 - sign\left((e_{s_1}^T M_{N-H}^{+,++,U}) \cdot (e_s^T P_H^{-,--,=})^T\right)],$$

then $M_H^{GCSEQ_a}(s, s) = 0$ holds iff

$$CM_H^{+,++,U}(s, s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N-H}^{+,++,U}) \cdot (e_s^T \cdot P_H^{-,--,=})^T\right)] = 0, \forall s_1 \in S. \tag{8.44}$$

It is clear that Eq. 8.44 is equivalent to

$$(e_{s_1}^T \cdot M_{N-H}^{+,++,U}) \cdot (e_s^T \cdot P_H^{-,--,=})^T \neq 0 \text{ for any } s_1 \in CR_H^{+,++,U}(s).$$

It implies that for any $s_1 \in CR_H^{+,++,U}(s)$, there exists at least one $s_2 \in R_{N-H}^{+,++,U}(s_1)$ with $s_2 \in \Phi_H^{\ll,<,\sim}(s)$. The proof of this theorem follows using Definition 8.44. □

Note that the general coalitional SEQ_a stability matrix is identical to the general coalitional GMR_a stability matrix except that the UM reachability matrix for H's opponents, M_{N-H}, is replaced by the mild, strong or uncertain reachability improvement matrix $M_{N-H}^{+,++,U}$.

(2) Matrix Representation of General Coalitional Stabilities Indexed b

Theorem 8.39 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is general coalitional $Nash_b$ stable for H, denoted by $s \in S_H^{GCNash_b}$, iff $e_s^T \cdot CM_H^{+,++} \cdot e = 0$.*

This theorem is different from Theorem 8.25 presented in Sect. 8.8.2 though their representations are identical. Theorem 8.39 can analyze Nash stability with hybrid preference.

Define the general coalitional GMR_b stability matrix as

$$M_H^{GCGMR_b} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,--,=})^T\right)], \tag{8.45}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is general coalitional GMR_b stable for H.

Theorem 8.40 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is general coalitional GMR_b stable for H, denoted by $s \in S_H^{GCGMR_b}$, iff $M_H^{GCGMR_b}(s, s) = 0$.*

General coalitional SMR_b stability is similar to general coalitional GMR_b except that coalition H expects to have a chance to counterrespond to its opponents' $(N - H)$ response to H's original move. Define the general coalitional SMR_b stability matrix as

$$M_H^{GCSMR_b} = CM_H^{+,++} \cdot [E - sign(Q)]$$

in which

$$Q = M_{N-H} \cdot [(P_H^{-,--,=})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,--,=})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is general coalitional SMR_b stable for H.

Theorem 8.41 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SMR_b for H, denoted by $s \in S_H^{GCSMR_b}$, iff $M_H^{GCSMR_b}(s, s) = 0$.*

Although matrix representations of $GCNash_b$, $GCGMR_b$ and $GCSMR_b$ do not include uncertain preference, they may be used to analyze situations with preference uncertainty. If H's opponents are treated as a coalition, the general coalitional SEQ_b stability matrix $M_H^{GCSEQ_b}$ is defined as

$$M_H^{GCSEQ_b} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H}^{+,++,U} \cdot (P_H^{-,--,=})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is general coalitional SEQ_b stable for H when H's opponents, $N-H$, are in a coalition.

Theorem 8.42 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SEQ_b stable for H, denoted by $s \in S_H^{GCSEQ_b}$, iff $M_H^{GCSEQ_b}(s, s) = 0$.*

The proofs of the general coalitional stabilities indexed b are similar to the general coalitional stabilities indexed a. The proofs are left for readers.

(3) Matrix Representation of General Coalitional Stabilities Indexed c

Theorem 8.43 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. $S_H^{GCNash_c} = S_H^{GCNash_a}$.*

Let $M_H^{GCGMR_c}$ denote the general coalitional GMR_c matrix. It is defined by

$$M_H^{GCGMR_c} = CM_H^{+,++,U} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,--,=,U})^T\right)], \tag{8.46}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is general coalitional GMR_c stable for H.

Theorem 8.44 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is general coalitional GMR_c stable for H, denoted by $s \in S_H^{GCGMR_c}$, iff $M_H^{GCGMR_c}(s, s) = 0$.*

Proof Since

$$M_H^{GCGMR_c}(s, s) = (e_s^T \cdot CM_H^{+,++,U}) \cdot [\left(E - sign\left(M_{N-H} \cdot (P_H^{-,--,=,U})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+,++,U}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=,U})^T\right)],$$

then $M_H^{GCGMR_c}(s, s) = 0$ holds iff

$$CM_H^{+,++,U}(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=,U})^T\right)] = 0, \quad (8.47)$$

for every $s_1 \in S - \{s\}$. It is clear that Eq. 8.47 is equivalent to

$$(e_{s_1}^T \cdot M_{N-H}) \cdot (e_s^T \cdot P_H^{-,--,=,U})^T \neq 0,$$

for every $s_1 \in CR_H^{+,++,U}(s)$. Therefore, for a coalitional mild, strong, or uncertain improvement from s, $s_1 \in CR_H^{+,++,U}(s)$, there exists at least one $s_2 \in R_{N-H}(s_1)$ with $P_H^{-,--,=,U}(s, s_2) = 1$ that is equivalent to $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$, i.e., $s >_i s_2$, $s \gg_i s_2$, $s \sim_i s_2$, or $s\ U_i\ s_2$ for some DM $i \in H$. According to Definition 8.50, $M_H^{GCGMR_c}(s, s) = 0$ implies that s is general coalitional GMR_c stable for H. □

General coalitional SMR_c stability is similar to general coalitional GMR_c except that coalition H expects to have a chance to counterrespond to its opponents' $(N-H)$ response to H's original move. Define the general coalitional SMR_c stability matrix as

$$M_H^{GCSMR_c} = CM_H^{+,++,U} \cdot [E - sign(F)]$$

in which

$$F = M_{N-H} \cdot [(P_H^{-,--,=,U})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,--,=,U})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is general coalitional SMR_c stable for H.

Theorem 8.45 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SMR_c for H, denoted by $s \in S_H^{GCSMR_c}$, iff $M_H^{GCSMR_c}(s, s) = 0$.*

Proof Since

$$M_H^{GCSMR_c}(s, s) = (e_s^T \cdot CM_H^{+,++,U}) \cdot [(E - sign(F)) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} CM_H^{+,++,U}(s, s_1)[1 - sign\,(F(s_1, s))]$$

with

$$F(s_1, s) = \sum_{s_2=1}^{m} M_{N-H}(s_1, s_2) \cdot W(s_2, s),$$

and

$$W(s_2, s) = P_H^{-,--,=,U}(s, s_2) \cdot \left[1 - sign\left(\sum_{s_3=1}^{m}\left(M_H(s_2, s_3) \cdot (1 - P_H^{-,--,=,U}(s, s_3))\right)\right)\right],$$

then $M_H^{GCSMR_c}(s, s) = 0$ holds iff $F(s_1, s) \neq 0$, for every $s_1 \in CR_H^{+,++,U}(s)$, which is equivalent to the statement that, for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that

$$P_H^{-,--,=,U}(s, s_2) \neq 0, \tag{8.48}$$

and

$$\sum_{s_3=1}^{m} M_H(s_2, s_3) \cdot (1 - P_H^{-,--,=,U}(s, s_3)) = 0. \tag{8.49}$$

Equation 8.48 means that $s >_i s_2$, $s \gg_i s_2$, $s \sim_i s_2$, or $s\ U_i\ s_2$ for at least one DM $i \in H$. Equation 8.49 is equivalent to

$$P_H^{-,--,=,U}(s, s_3) = 1 \text{ for any } s_3 \in R_H(s_2). \tag{8.50}$$

Obviously, for every $s_1 \in CR_H^{+,++,U}(s)$, there exists $s_2 \in R_{N-H}(s_1)$ such that Eqs. 8.48 and 8.49 hold iff for every $s_1 \in CR_H^{+,++,U}(s)$ there exists $s_2 \in R_{N-H}(s_1)$ such that $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$ and $s_3 \in \Phi_H^{\ll,<,\sim,U}(s)$ with all $s_3 \in R_H(s_2)$. Therefore, the proof of this theorem follows using Definition 8.51. □

The general coalitional SEQ_c stability matrix $M_H^{GCSEQ_c}$ is defined as

$$M_H^{GCSEQ_c} = CM_H^{+,++,U} \cdot [E - sign\left(M_{N-H}^{+,++,U} \cdot (P_H^{-,--,=,U})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is general coalitional SEQ_c stable for H when H's opponents, $N-H$, are in a coalition.

Theorem 8.46 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SEQ_c stable for H, denoted by $s \in S_H^{GCSEQ_c}$, iff $M_H^{GCSEQ_c}(s, s) = 0$.*

Proof Since

$$M_H^{GCSEQ_c}(s, s) = (e_s^T \cdot CM_H^{+,++,U}) \cdot [\left(E - sign(M_{N-H}^{+,++,U} \cdot (P_H^{-,--,=,U})^T)\right) e_s]$$

$$= \sum_{s_1=1}^{|S|} CM_H^{+,++,U}(s, s_1)[1 - sign\left((e_{s_1}^T M_{N-H}^{+,++,U}) \cdot (e_s^T P_H^{-,--,=,U})^T\right)],$$

then $M_H^{GCSEQ_c}(s, s) = 0$ holds iff

$$CM_H^{+,++,U}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N-H}^{+,++,U}) \cdot (e_s^T \cdot P_H^{-,--,=,U})^T\right)] = 0, \forall s_1 \in S. \quad (8.51)$$

It is clear that Eq. 8.51 is equivalent to

$$(e_{s_1}^T \cdot M_{N-H}^{+,++,U}) \cdot (e_s^T \cdot P_H^{-,--,=,U})^T \neq 0 \text{ for any } s_1 \in CR_H^{+,++,U}(s).$$

It implies that for any $s_1 \in CR_H^{+,++,U}(s)$, there exists at least one $s_2 \in R_{N-H}^{+,++,U}(s_1)$ with $s_2 \in \Phi_H^{\ll,<,\sim,U}(s)$. The proof of this theorem follows using Definition 8.52. □

(4) Matrix Representation of General Coalitional Stabilities Indexed d

Theorem 8.47 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. $S_H^{GCNash_d} = S_H^{GCNash_b}$.*

Define the general coalitional GMR_d stability matrix as

$$M_H^{GCGMR_d} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{-,--,=,U})^T\right)], \quad (8.52)$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is general coalitional GMR_d stable for H.

Theorem 8.48 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is general coalitional GMR_d stable for H, denoted by $s \in S_H^{GCGMR_d}$, iff $M_H^{GCGMR_d}(s,s) = 0$.*

General coalitional SMR_d stability is similar to general coalitional GMR_d except that coalition H expects to have a chance to counterrespond to its opponents' $(N-H)$ response to H's original move. Define the general coalitional SMR_d stability matrix as

$$M_H^{GCSMR_d} = CM_H^{+,++} \cdot [E - sign(Q)]$$

in which

$$Q = M_{N-H} \cdot [(P_H^{-,--,=,U})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{-,--,=,U})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is general coalitional SMR_d stable for H.

Theorem 8.49 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SMR_d for H, denoted by $s \in S_H^{GCSMR_d}$, iff $M_H^{GCSMR_d}(s,s) = 0$.*

The general coalitional SEQ_d stability matrix $M_H^{GCSEQ_d}$ is defined as

$$M_H^{GCSEQ_d} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H}^{+,++,U} \cdot (P_H^{-,--,=,U})^T\right)].$$

The following theorem provides the matrix method to analyze whether state s is general coalitional SEQ_d stable for H when H's opponents, $N-H$, are in a coalition.

Theorem 8.50 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is general coalitional SEQ_d stable for H, denoted by $s \in S_H^{GCSEQ_d}$, iff $M_H^{GCSEQ_d}(s,s) = 0$.*

The proofs of the general coalitional stabilities indexed d are similar to the general coalitional stabilities indexed a. The proofs are left for readers.

8.9.3 Matrix Representation of Strong Coalitional Stabilities with Hybrid Preference

When hybrid preference is introduced into the graph model, general coalitional stability definitions indexed a, b, c or d may be strong or weak coalitional stability definitions indexed a, b, c or d, according to the degree of sanctioning. The following matrix representations of strong coalitional stabilities under hybrid preference are equivalent to the logical forms presented in Sect. 8.5.2.

Theorem 8.51 *For the graph model G, let $H \subseteq N$ be a nonempty coalition and $l \in \{a, b, c, d\}$. State $s \in S$ is general or strong coalitional $Nash_l$ stable for H, denoted by $s \in S_H^{GNash_l}$ or $s \in S_H^{SNash_l}$, respectively. Then $S_H^{SNash_l} = S_H^{GNash_l}$.*

8.9.3.1 Matrix Representation of Strong Coalitional Stabilities Indexed *l*

(1) Matrix Representation of Strong Coalitional Stabilities Indexed *a*

The strong coalitional GMR_a stability matrix is defined as

$$M_H^{SCGMR_a} = CM_H^{+,++,U} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{--})^T\right)], \tag{8.53}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is strong coalitional GMR_a stable for H.

Theorem 8.52 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is strong coalitional GMR_a stable for H, denoted by $s \in S_H^{SCGMR_a}$, iff $M_H^{SCGMR_a}(s,s) = 0$.*

Theorem 8.52 shows that the matrix representation of strong coalitional GMR_a stability is equivalent to the logical version of the same stability given in Definition 8.59. The diagonal entry (s, s) of matrix $M_H^{SCGMR_a}$ is identified whether it is zero. If so, s is strong coalitional GMR_a stable for H.

Define the strong coalitional SMR_a stability matrix as

$$M_H^{SCSMR_a} = CM_H^{+,++,U} \cdot [E - sign(Q)]$$

in which

$$Q = M_{N-H} \cdot [(P_H^{--})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{--})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is strong coalitional SMR_a stable for H.

Theorem 8.53 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SMR_a for H, denoted by $s \in S_H^{SCSMR_a}$, iff $M_H^{SCSMR_a}(s, s) = 0$.*

Theorem 8.53 provides a matrix method, which is equivalent to the logical version given in Definition 8.60. The following theorem displays the matrix method to identify whether state s is strong coalitional SEQ_a stable. Let the strong coalitional SEQ_a stability matrix $M_H^{SCSEQ_a}$ be defined as

$$M_H^{SCSEQ_a} = CM_H^{+,++,U} \cdot [E - sign\left(M_{N-H}^{+,++,U} \cdot (P_H^{--})^T\right)].$$

Theorem 8.54 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SEQ_a stable for H, denoted by $s \in S_H^{SCSEQ_a}$, iff $M_H^{SCSEQ_a}(s, s) = 0$.*

(2) Matrix Representation of Strong Coalitional Stabilities Indexed *b*

Define the strong coalitional GMR_b stability matrix as

$$M_H^{SCGMR_b} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{--})^T\right)], \tag{8.54}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is strong coalitional GMR_b stable for H.

Theorem 8.55 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is strong coalitional GMR_b stable for H, denoted by $s \in S_H^{SCGMR_b}$, iff $M_H^{SCGMR_b}(s, s) = 0$.*

Define the strong coalitional SMR_b stability matrix as

$$M_H^{SCSMR_b} = CM_H^{+,++} \cdot [E - sign(Q)]$$

in which

$$Q = M_{N-H} \cdot [(P_H^{--})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{--})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is strong coalitional SMR_b stable for H.

Theorem 8.56 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SMR_b for H, denoted by $s \in S_H^{SCSMR_b}$, iff $M_H^{SCSMR_b}(s, s) = 0$.*

The following theorem displays the matrix method to identify whether state s is strong coalitional SEQ_b stable. Let the strong coalitional SEQ_b stability matrix $M_H^{SCSEQ_b}$ be defined as

$$M_H^{SCSEQ_b} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H}^{+,++,U} \cdot (P_H^{--})^T\right)].$$

Theorem 8.57 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SEQ_b stable for H, denoted by $s \in S_H^{SCSEQ_b}$, iff $M_H^{SCSEQ_b}(s, s) = 0$.*

(3) Matrix Representation of Strong Coalitional Stabilities Indexed c

The strong coalitional GMR_c stability matrix is defined as

$$M_H^{SCGMR_c} = CM_H^{+,++,U} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{--,U})^T\right)], \tag{8.55}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is strong coalitional GMR_c stable for H.

Theorem 8.58 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is strong coalitional GMR_c stable for H, denoted by $s \in S_H^{SCGMR_c}$, iff $M_H^{SCGMR_c}(s, s) = 0$.*

Theorem 8.58 shows that the matrix representation of strong coalitional GMR_c stability is equivalent to the logical version of the same stability given in Definition 8.65. The diagonal entry (s, s) of matrix $M_H^{SCGMR_c}$ is identified whether it is zero. If so, s is strong coalitional GMR_c stable for H.

Define the strong coalitional SMR_c stability matrix as

$$M_H^{SCSMR_c} = CM_H^{+,++,U} \cdot [E - sign(Q)]$$

in which

$$Q = M_{N-H} \cdot [(P_H^{--,U})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{--,U})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is strong coalitional SMR_c stable for H.

Theorem 8.59 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SMR_c for H, denoted by $s \in S_H^{SCSMR_c}$, iff $M_H^{SCSMR_c}(s, s) = 0$.*

Theorem 8.59 provides a matrix method, which is equivalent to the logical version given in Definition 8.66. The following theorem displays the matrix method to identify whether state s is strong coalitional SEQ_c stable. Let the strong coalitional SEQ_c stability matrix $M_H^{SCSEQ_c}$ be defined as

$$M_H^{SCSEQ_c} = CM_H^{+,++,U} \cdot [E - sign\left(M_{N-H}^{+,++,U} \cdot (P_H^{--,U})^T\right)].$$

Theorem 8.60 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SEQ_c stable for H, denoted by $s \in S_H^{SCSEQ_c}$, iff $M_H^{SCSEQ_c}(s, s) = 0$.*

(4) Matrix Representation of Strong Coalitional Stabilities Indexed d

Define the strong coalitional GMR_d stability matrix as

$$M_H^{SCGMR_d} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H} \cdot (P_H^{--,U})^T\right)], \tag{8.56}$$

where $H \subseteq N$. The following theorem establishes the matrix method to assess whether state s is strong coalitional GMR_d stable for H.

Theorem 8.61 *For the graph model G, let $H \subseteq N$ be a nonempty coalition. State $s \in S$ is strong coalitional GMR_d stable for H, denoted by $s \in S_H^{SCGMR_d}$, iff $M_H^{SCGMR_d}(s, s) = 0$.*

Theorem 8.61 shows that the matrix representation of strong coalitional GMR_d stability is equivalent to the logical version of the same stability given in Definition 8.68. The diagonal entry (s, s) of matrix $M_H^{SCGMR_d}$ is identified whether it is zero. If so, s is strong coalitional GMR_d stable for H.

Define the strong coalitional SMR_d stability matrix as

$$M_H^{SCSMR_d} = CM_H^{+,++} \cdot [E - sign(Q)]$$

in which

$$Q = M_{N-H} \cdot [(P_H^{--,U})^T \circ \left(E - sign\left(M_H \cdot (E - P_H^{--,U})^T\right)\right)],$$

for $H \subseteq N$. The following theorem establishes the matrix method to determine whether state s is strong coalitional SMR_d stable for H.

Theorem 8.62 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SMR_d for H, denoted by $s \in S_H^{SCSMR_d}$, iff $M_H^{SCSMR_d}(s, s) = 0$.*

Theorem 8.62 provides a matrix method, which is equivalent to the logical version given in Definition 8.69. The following theorem displays the matrix method to identify whether state s is strong coalitional SEQ_d stable. Define the strong coalitional SEQ_d stability matrix $M_H^{SCSEQ_d}$ as

$$M_H^{SCSEQ_d} = CM_H^{+,++} \cdot [E - sign\left(M_{N-H}^{+,++,U} \cdot (P_H^{--,U})^T\right)].$$

Table 8.1 Options and feasible states for the Lake Gisborne conflict

Federal								
1. Continue	N	Y	N	Y	N	Y	N	Y
Provincial								
2. Lift	N	N	Y	Y	N	N	Y	Y
Support								
3. Appeal	N	N	N	N	Y	Y	Y	Y
States	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8

Theorem 8.63 *For the graph model G, let $H \subseteq N$ be a coalition. State $s \in S$ is strong coalitional SEQ_d stable for H, denoted by $s \in S_H^{SCSEQ_d}$, iff $M_H^{SCSEQ_d}$* $(s, s) = 0$.

The matrix representation of coalitional stabilities under hybrid preference presented in this section is identical with the logical form discussed in Sect. 8.5. However, the matrix form is more efficient for calculating coalitional stabilities than logical representation.

8.10 Application: Coalition Analysis for Lake Gisborne Conflict with Simple Preference

In this section, the matrix approach is used to analyze the coalitional stability for the Lake Gisborne conflict with simple preference. Recall from Sects. 5.4 and 7.5 that the graph model for the Lake Gisborne Conflict has the following DMs and options:

- Federal Government of Canada (**Federal**): its option is to continue a Canada-wide accord on the prohibition of bulk water export **(Continue**) or not,
- Provincial Government of Newfoundland and Labrador (**Provincial**): its option is to lift the ban on bulk water exports **(Lift)** or not, and
- Support groups (**Support**): their option is to appeal for continuing the Lake Gisborne project **(Appeal)** or not.

The three DMs and the options they control are listed on the left in Table 8.1. Together, the three options create eight possible states as listed on the right in Table 8.1, where a "Y" indicates that an option is selected by the DM controlling it and an "N" means that the option is not chosen. Each state, shown as a column of Ys and Ns in Table 8.1, represents a possible scenario as to what could occur. For instance, s_4 means that the Federal Government will continue prohibiting bulk water exports, the Provincial Government will lift the ban on bulk water exports, and the Support Groups will not appeal for implementing this project. The graph model capturing the possible moves by the three DMs in the Lake Gisborne conflict

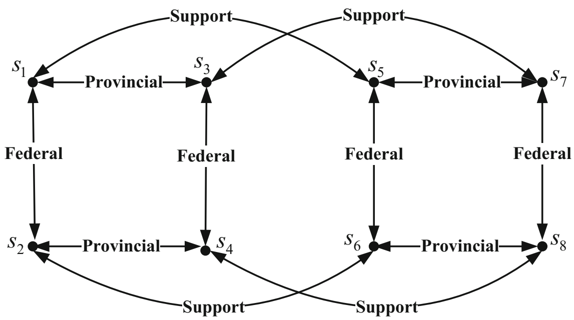

Fig. 8.1 Graph model of moves for the Lake Gisborne conflict

Table 8.2 Preference information for the Lake Gisborne model with low water price

DMs	Certain preferences
Federal	$s_2 \succ s_6 \succ s_4 \succ s_8 \succ s_1 \succ s_5 \succ s_3 \succ s_7$
Provincial	$s_2 \succ s_6 \succ s_1 \succ s_5 \succ s_4 \succ s_8 \succ s_3 \succ s_7$
Support	$s_3 \succ s_4 \succ s_7 \succ s_8 \succ s_5 \succ s_6 \succ s_1 \succ s_2$

Table 8.3 Preference information for the Lake Gisborne model with high water price

DMs	Certain preferences
Federal	$s_2 \succ s_6 \succ s_4 \succ s_8 \succ s_1 \succ s_5 \succ s_3 \succ s_7$
Provincial	$s_3 \succ s_7 \succ s_4 \succ s_8 \succ s_1 \succ s_5 \succ s_2 \succ s_6$
Support	$s_3 \succ s_4 \succ s_7 \succ s_8 \succ s_5 \succ s_6 \succ s_1 \succ s_2$

is shown in Fig. 8.1, where the labels on the arcs identify the DMs who control the relevant moves.

Besides the DMs, states, and potential moves, the other key component of a graph model is the relative preferences for each DM. Tables 8.2 and 8.3 provide the preferences for the situations in which the price of water is low and high, respectively. Notice that only the preferences for the Provincial Government are different for these two conflicts. In these tables, the symbol given by $\succ$ means more preferred. When the price of water is low, the Provincial most prefers state s_2 from Table 8.2. State s_2 indicates that the Provincial sides with the Federal for protecting the environment. With the increasing price of water, Table 8.3 shows that state s_3 is most preferred by the Provincial. It means that the economical-oriented provincial government will lift the ban on bulk water exports. Two attitudes of the Provincial will result in different coalitions and different coalitional stability resolutions. Due to the methods to analyze the two models for the Lake Gisborne conflict are similar, the following discussions will be based on the second case in which the Provincial sides with the Support Groups. The reachability matrices for the Lake Gisborne model is constructed using the algebraic approach next.

Table 8.4 UM reachability matrices by $N-\{i\}$ for $i=1,2,$ and 3 for the Lake Gisborne model with high water price

Matrix	$M_{N-\{1\}}$								$M_{N-\{2\}}$								$M_{N-\{3\}}$							
State	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
1	0	0	1	0	1	0	1	0	0	1	0	0	1	1	0	0	0	1	1	1	0	0	0	0
2	0	0	0	1	0	1	0	1	1	0	0	0	1	1	0	0	1	0	1	1	0	0	0	0
3	1	0	0	0	1	0	1	0	0	0	0	1	0	0	1	1	1	1	0	1	0	0	0	0
4	0	1	0	0	0	1	0	1	0	0	1	0	0	0	1	1	1	1	1	0	0	0	0	0
5	1	0	1	0	0	0	1	0	1	1	0	0	0	1	0	0	0	0	0	0	0	1	1	1
6	0	1	0	1	0	0	0	1	1	1	0	0	1	0	0	0	0	0	0	0	1	0	1	1
7	1	0	1	0	1	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	1	1	0	1
8	0	1	0	1	0	1	0	0	0	0	1	1	0	0	1	0	0	0	0	0	1	1	1	0

Table 8.5 UI reachability matrices by $N-\{i\}$ for $i=1,2,$ and 3 for the Lake Gisborne model with high water price

Matrix	$M^+_{N-\{1\}}$								$M^+_{N-\{2\}}$								$M^+_{N-\{3\}}$							
State	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
1	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0	0	0	1	1	1	0	0	0	0
2	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1
6	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
7	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0

8.10.1 Reachability Matrices in the Lake Gisborne Model

$N=\{1,2,3\}=\{Federal, Provincial, Support\}$ is the set of three DMs. Use the Lake Gisborne model as an example to demonstrate how the algebraic approach works for building UM, UI, and CI (Coalitional Improvement) reachability matrices (Xu et al. 2014). One can adhere to the following steps:

- Construct matrices, J_i, J_i^+, P_i^+, and $P_i^{-,=}$, for $i=1,2,$ and 3, using information provided in Fig. 8.1 and Table 8.3;
- Calculate the UM, UI, and CI reachability matrices, M_H, M_H^+, and CM_H^+ by $H=N-\{i\}$ for $i=1,2,$ and 3, respectively;
- Three reachability matrices are shown in Tables 8.4, 8.5, and 8.6.

For example, using Table 8.5, one has:

Table 8.6 CI reachability matrices by $N - \{i\}$ for $i = 1, 2,$ and 3 for the Lake Gisborne model with high water price

Matrix	$CM^+_{N-\{1\}}$								$CM^+_{N-\{2\}}$								$CM^+_{N-\{3\}}$							
State	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
1	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
2	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
6	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0

$$e_2^T \cdot M^+_{N-\{1\}} = (0, 0, 0, 1, 0, 1, 0, 1),$$

which means that the reachable list of $H = N - \{1\}$ by the legal UIs from state s_2, $R^+_H(s_2) = \{s_4, s_6, s_8\}$, i.e., states s_4, s_6, and s_8 can be reached by any legal UI sequence, by coalition $H = \{2, 3\}$, from the status quo $s = s_2$. However, from Table 8.6,

$$e_2^T \cdot CM^+_{N-\{1\}} = (0, 0, 0, 1, 0, 0, 0, 1),$$

which indicates that the coalitional improvements from s_2 by coalition $H = \{2, 3\}$ are $CR^+_H = (0, 0, 0, 1, 0, 0, 0, 1)$. As mentioned after Definition 8.1, normally, $R^+_H(s) \neq CR^+_H(s)$. It is clear from this example that $R^+_H(s_2) \neq CR^+_H(s_2)$ for $H = \{2, 3\}$. In fact, although $s_6 \in R^+_H(s_2)$, $s_6 \notin CR^+_H(s_2)$ for $H = \{2, 3\}$, since $s_2 \succ_2 s_6$.

8.10.2 Coalitional Stability Results in the Lake Gisborne Model

After obtaining three important components, UM, UI, and CI reachability matrices (M_H, M^+_H, and CM^+_H, respectively), coalitional stabilities, CNash, CGMR, CSMR, $CSEQ_1$, and $CSEQ_2$, can be calculated using Theorems 8.2 and 8.4–8.7 and are shown in Table 8.7.

Both of the foregoing water export conflicts were thoroughly analyzed using the algebraic methodology for coalitional analysis provided in this chapter. In the first dispute for which the price of water is low, the only equilibrium according to both noncooperative stability calculations and coalitional stability when the Federal and Provincial Governments form a coalition ($H = \{1, 2\}$) is s_6. From Table 8.1, state

Table 8.7 Coalitional stabilities of the Lake Gisborne model for various coalitions with high water price

State	Stability	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}
s_1	CNash						
	CGMR						
	CSMR						
	$CSEQ_1$						
	$CSEQ_2$						
s_2	CNash	√			√	√	
	CGMR	√			√	√	
	CSMR	√			√	√	
	$CSEQ_1$	√			√	√	
	$CSEQ_2$	√			√	√	
s_3	CNash		√	√	√	√	√
	CGMR		√	√	√	√	√
	CSMR		√	√	√	√	√
	$CSEQ_1$		√	√	√	√	√
	$CSEQ_2$		√	√	√	√	√
s_4	CNash	√	√	√	√	√	√
	CGMR	√	√	√	√	√	√
	CSMR	√	√	√	√	√	√
	$CSEQ_1$	√	√	√	√	√	√
	$CSEQ_2$	√	√	√	√	√	√
s_5	CNash			√		√	
	CGMR			√		√	
	CSMR			√		√	
	$CSEQ_1$			√		√	
	$CSEQ_2$			√		√	
s_6	CNash	√		√	√	√	
	CGMR	√		√	√	√	
	CSMR	√		√	√	√	
	$CSEQ_1$	√		√	√	√	
	$CSEQ_2$	√		√	√	√	
s_7	CNash		√		√		
	CGMR		√	√	√	√	√
	CSMR		√	√	√	√	√
	$CSEQ_1$		√		√		√
	$CSEQ_2$		√		√		√
s_8	CNash	√	√				
	CGMR	√	√	√	√	√	
	CSMR	√	√	√	√	√	
	$CSEQ_1$	√	√		√		
	$CSEQ_2$	√	√		√		

s_6 is the situation in which the Federal Government continues to promote a ban, the Provincial Government does not lift the ban, and the Support Groups appeal. In this case, the Provincial Government is environmentally oriented. For the second conflict, in which the price of water is high, the noncooperative stability results are listed in the 3, 4, and 5th columns of Table 8.7. Obviously, state s_4 is an equilibrium for all individual noncooperative stability definitions consisting of Nash, GMR, SMR, and SEQ; s_8 is also an equilibrium for GMR and SMR individual stabilities. However, notice from Table 8.7 that when the Provincial Government and the Support Groups form a coalition ($H = \{2, 3\}$), s_8 is coalitionally unstable for CGMR and CSMR. As can be seen from Table 8.1, at s_8, the Support Groups are appealing, which is not necessary because the Provincial Government and the Support Groups are cooperating. Therefore, state s_8 is not long-term stable. For nontrivial coalitions, the cooperative stabilities are listed in the three columns on the right of Table 8.7. Observe that s_4 is universally CNash, CGMR, CSMR, $CSEQ_1$, and $CSEQ_2$ stable, which means that at state s_4, the Federal Government continues with the ban, the Provincial Government lifts the ban and the Support Groups do not appeal. State s_4 is a resolution of the conflict when the price of water is high. In this case, the water export project will proceed.

8.11 Important Ideas

Coalition analysis should form a key component of every formal conflict resolution investigation. After determining what a given DM can accomplish on his or her own and in his own self-interest, one should determine if the DM can do even better by cooperating with others. The coalition ideas presented in this chapter provide a solid mathematical foundation for coalition modeling and analysis, which can be programmed into a decision support system (DSS) for GMCR, as explained in Sect. 10.2. Hence, an encompassing coalition approach to formal conflict studies can be fully operationalized for employment by researchers, teachers, students, and practitioners working in many fields. The logical representation of coalitional stability analyses for four key solution concepts are presented in this chapter for the four types of preference structures given in Chaps. 4–7. Moreover, the matrix representation of coalitional analysis under a range of preference framework given later in this chapter means that coalitional analysis can be readily incorporated into the construction of the engine for a DSS for GMCR, as explained in Sect. 10.2. Accordingly, coalitional analysis is now a fully mature decision technology within the paradigm of GMCR, which can be readily utilized as evidenced by the water export conflict application presented in Sect. 8.10.

8.12 Problems

8.12.1 Select a current conflict, such as an international trading dispute or negotiating a climate change agreement, which is of direct interest to you. Explain why you think coalition modeling and analysis may or may not be an important tool for better resolving this conflict.

8.12.2 In a coalition improvement given in Definition 8.1, a state is a coalition improvement for the members of a coalition with respect to another state if and only if the state to which the DMs are jointly moving is more preferred by all of the members of the coalition. For a conflict of your choice, provide an example of a coalition improvement. Explain how this move could be carried out in practice via appropriate communication among the coalition members.

8.12.3 The game of Prisoner's Dilemma is presented in Problem 3.5.1. If both DMs were to move together from state s_4 to state s_1 in this conflict, this constitutes an example of a coalition improvement. Write a short discussion about interpreting Prisoner's Dilemma as some type of typical or generic real-world dispute, such as a trading or environmental dispute. Explain sensible steps that could be taken in practice to ensure that both DMs move together from state s_4 to s_1 and, hence, no DM defects during this process. Why is the move from state s_4 to s_1 called an equilibrium jump?

8.12.4 In the game of Chicken in Problem 3.5.4, both DMs or drivers moving together from state s_1 to state s_4 is an example of a coalition improvement as presented in Definition 8.1. Furnish an example of a real-world interpretation of the game of Chicken. Explain how the two DMs could improve together from state s_1 to state s_4 via taking appropriate measures.

8.12.5 If a conflict consists of only two DMs, these two DMs can still participate in a coalition improvement as presented in Definition 8.1. However, when there are only two DMs in a conflict, there are no other DMs left in the conflict to block possible coalition improvements. Explain why the definitions for coalitional stabilities given in Sect. 8.2 for simple preference work when there are only two DMs. Why are these coalitional stability definitions identical to the stability definitions given in Chap. 4 for simple preference with no coalitions having two or more DMs?

8.12.6 The Lake Gisborne conflict over the proposed exportation of water is presented in Sect. 8.10. Apply the logical form of the coalitional stability definitions given in Sect. 8.2 to the Lake Gisborne example to show by hand how you calculate various coalitional stabilities. Be sure to present the special situation for which there are no coalitions for each stability definition.

8.12.7 For the Elmira groundwater contamination dispute first presented in Sect. 1.2.2 in the book, calculate by hand the coalitional stabilities for simple preference using the matrix formulation given in Sect. 8.6. Be sure to include sample calculations and the stability results for the special situation in which there are no coalitions.

8.12.8 For the coalition investigation approach presented in this chapter, it is assumed that DMs will form a coalition during a conflict when it is in their interest to do so, as reflected by the way a coalition improvement is defined in Definition 8.1. However, other ways to study coalitions exist. In particular, in some situations, such as a military alliance among nations during warfare, a coalition may last throughout the duration of the dispute. Accordingly, researchers developed a procedure for determining the preference of a coalition based on the preferences of the individual coalition members (Kuhn et al. 1983, Hipel and Fraser 1991, Meister et al. 1992, Hipel and Meister 1994). By referring to the research of these authors, outline how coalition preferences are ascertained. Explain how these authors identify possible coalition formation and how coalitional stability analyses are executed. Describe a specific actual dispute for which you think this approach could be useful for obtaining strategic insights.

8.12.9 Logical definitions of coalitional stabilities under unknown preference are presented in Sect. 8.3. By employing a real-world application of your choice, explain a situation in which you think this kind of coalitional analysis could prove to be informative.

8.12.10 For the case of three degrees of preference, logical definitions for coalitional stability are provided in Sect. 8.4. Describe an actual situation in which you think this kind of coalitional stability analysis could provide insightful strategic findings.

8.12.11 Hybrid preference coalitional stability definitions, in which both unknown preference and three degrees of preference are simultaneously taken into account, are presented in Sect. 8.5. Based on an actual dispute which is of direct interest to you, describe why you think this hybrid coalitional stability approach could provide insightful strategic findings.

8.12.12 The matrix representations of coalitional stability under simple, unknown, three degree, and hybrid preference are presented in Sects. 8.6–8.9, respectively. As explained in Sect. 10.2, these matrix representations are needed for designing and programming a flexible decision support system (DSS) especially for the analysis engine. Using diagrams, outline how you would design a DSS which can handle coalitional analyses.

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