Chapter 6
Stability Definitions: Degrees of Preference

In a water quality dispute, an environmental agency may greatly prefer that an industrial enterprise does not seriously pollute a nearby river into which it discharges wastes. The purpose of this chapter is to present a formal methodology that can handle this type of "degree", "strength", or "level" of preference, which often arises in practice, in order to determine its strategic consequences. More specifically, a multiple-degree preference structure is developed within the paradigm of the Graph Model for Conflict Resolution (GMCR) in conjunction with associated stability definitions for determining individual stability of each state from a given decision maker's (DM's) viewpoint as well as the overall equilibria (Hamouda et al. 2004, 2006, Xu et al. 2009, 2010, 2011). Within this structure, a DM may have multiple degrees of preference when comparing pairs of states. For example, if state a is preferred to state b, it may be mildly preferred at degree 1 ($d = 1$), more strongly preferred at degree 2 ($d = 2$), ..., or maximally preferred at degree r ($d = r$), where $r > 0$ is a fixed parameter. The number of degrees, r, is unrestricted in this system, thereby extending the earlier simple preference structure having two types of preferences consisting of equally preferred (degree zero) and more preferred (degree one) in Chap. 4 and the special case of three kinds of preferences (equally preferred, mildly preferred, and greatly more preferred) discussed in detail in this chapter.

The main properties of the preference structure according to degree are introduced in Sect. 6.1 in this chapter. Because DMs make moves and countermoves when interacting with one another under conflict, reachable lists are defined in Sect. 6.2 to keep track of the possible unilateral movements in one step from a given state for a particular DM with respect to multiple types of preference. When considering stability definitions for more than two DMs, coalition moves are defined since two or more DMs can participate in blocking a unilateral improvement by another DM. Subsequently, multiple-degree versions of four stability definitions consisting of Nash

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stability, general metarationality, symmetric metarationality, and sequential stability, are defined for the graph model with this extended preference structure and the relationships among them are investigated. Additionally, in this chapter, matrix representations of the four stabilities are presented for graph models having a preference structure of up to degree 3.

6.1 Multiple Degrees of Preference

The simple preference structure discussed in Chaps. 3 and 4 contains two types of preferences: indifference, in which a DM is indifferent between, or equally prefers, two states, and strict preference, in which a DM prefers one state more than another. The third kind of preference can be added by allowing a DM to greatly prefer one state over another. Hence, an expanded preference structure for a given DM can have two states being equally preferred (called preference of degree zero, or simply $d = 0$), one state being more or mildly preferred over another (degree $d = 1$), or one state being greatly more preferred than another ($d = 2$). In fact, one can extend two degrees of preference to an unlimited number. Below, preference structures having preferences of up to two degrees and the general case of having any number of degrees are discussed in Sects. 6.1.1 and 6.1.2, respectively.

6.1.1 Three Types of Preference

A triplet relation on S that expresses strength of preference according to indifferent, mild, or strong preference, was developed by Hamouda et al. (2004, 2006). For states $s, q \in S$, the preference relation $s \sim_i q$ indicates that DM i is indifferent between states s and q, the relation $s >_i q$ means that DM i mildly prefers s to q, and $s \gg_i q$ denotes that DM i strongly prefers s to q. Similar to the properties for simple preference given in Sect. 3.2.4, the characteristics of the preference structure, $\{\sim_i, >_i, \gg_i\}$, containing three kinds of preference for each DM $i \in N$, are as follows:

(i) $\sim_i$ is reflexive and symmetric;
(ii) $>_i$ and $\gg_i$ are asymmetric; and
(iii) $\{\sim_i, >_i, \gg_i\}$ is strongly complete.

Note that $\{\sim_i, >_i, \gg_i\}$ is strongly complete. Hence, if $s, q \in S$, then exactly one of the following relations holds: $s \sim_i q$, $s >_i q$, $s \gg_i q$, $q >_i s$, or $q \gg_i s$. Also, it is assumed that, for any $s, q \in S$, $s >_i q$ is equivalent to $q <_i s$. The preference type "$\gg_i$" has similar properties to "$>_i$".

Table 6.1 Subsets of S with respect to three degrees of preference for DM i

Subsets of S	Descriptions
$\Phi_i^{++}(s) = \{q : q \gg_i s\}$	States strongly preferred to state s by DM i
$\Phi_i^{+m}(s) = \{q : q >_i s\}$	States mildly preferred to state s by DM i
$\Phi_i^{=}(s) = \{q : q \sim_i s\}$	States equally preferred to state s by DM i
$\Phi_i^{-m}(s) = \{q : s >_i q\}$	States mildly less preferred than state s for DM i
$\Phi_i^{--}(s) = \{q : s \gg_i q\}$	States strongly less preferred to state s by DM i

The set of feasible states, S, can be partitioned or divided into a set of non-overlapping or disjoint subsets based on the types of preference relative to a specific state $s \in S$. These categorizations of preferences are needed for carrying out stability analyses according to different kinds of human behavior under conflict as explicitly defined in Sect. 6.3. For example, a DM may be tempted to unilaterally move to a mildly preferred state which can be blocked by another DM moving to a state which is greatly less preferred by the original DM. The descriptions of these different classifications of preferences are presented in Table 6.1.

Let $s \in S$ and $i \in N$. Based on different structures of preferences, DM i can identify different subsets of S. For simple preference, DM i can identify three subsets of S with respect to a state s: the set of states more preferred by DM i than state s (denoted by $\Phi_i^{+}(s)$); the set of states equally preferred to state s by DM i ($\Phi_i^{=}(s)$); and the set of states less preferred by DM i to state s ($\Phi_i^{-}(s)$) (see Sect. 4.1 for details). For the three types of preference, DM i can identify five subsets of S: $\Phi_i^{++}(s)$, $\Phi_i^{+m}(s)$, $\Phi_i^{=}(s)$, $\Phi_i^{-m}(s)$, and $\Phi_i^{--}(s)$, which are explained in Table 6.1. Notice that in this table that the set of states mildly preferred to state s by DM i, given by $\Phi_i^{+m}(s)$, have an "m" in the superscript in order to distinguish this set from $\Phi_i^{+}(s)$ for the case of a simple preference structure in which $\Phi_i^{++}(s)$ does not exist. Therefore, all states that are more preferred to state s by DM i would be included in $\Phi_i^{+}(s)$ for a simple preference structure. Similar comments hold for the set $\Phi_i^{-m}(s)$ in Table 6.1.

In Sect. 6.3, a given DM can levy a sanction against a unilateral improvement by DM i from state s if the sanctioning DM can put DM i in either a less preferred or equally preferred state relative to state s. Therefore, the set of states given by $\Phi_i^{--,-,=}(s) = \Phi_i^{--}(s) \cup \Phi_i^{-m}(s) \cup \Phi_i^{=}(s)$, where $\cup$ denotes the union operation, is important in various stability definitions. Note that in the graph model with strength of preference, $s \succ_i q$ iff either $s >_i q$ or $s \gg_i q$. Hence, the three types of preference structure expand simple preference.

The simple preference structure having the set of binary relations given as $\{\sim, \succ\}$, and the expanded preference structure with strength of preference, which has the set of binary relations $\{\sim, >, \gg\}$, are referred to as having two types of preferences

Table 6.2 Degree of relative preference

Degree of strength	Description	Notation
$d = r$	Preferred at degree r	$\overbrace{> \cdots >}^{r}$
……	……	……
$d = 3$	Very strongly preferred	$\ggg$
$d = 2$	Strongly preferred	$\gg$
$d = 1$	Moderately preferred	$>$
$d = 0$	Equally preferred	$\sim$

and three kinds of preferences, respectively. The existing two preference structures in the graph model are extended to the general case of multiple types of preference structures with any specified degree in the next section (Xu et al. 2009).

6.1.2 *Multiple Degrees of Preference*

A set of new and more general binary relations $\overbrace{> \cdots >}^{d}$ for $d = 1, 2, \ldots, r$, as listed in Table 6.2, are introduced in this section to represent DM i's preference at each degree d. With the introduction of these new binary relations, the three types of preference structures in the graph model are extended from a triplet of relations, to an $r + 1$ types of preference relations for DM i over the set of states, which is expressed as $\{\sim_i, >_i, \gg_i, \ldots, \overset{r}{\succ}_i\}$ on S, where $\overset{r}{\succ}_i$ denotes $\overbrace{> \cdots >}^{r}_i$, i.e., DM i has preference at degree r for comparing states with respect to preference. For instance, $s \ggg_i q$ means that DM i very strongly prefers state s to state q. Similar to the case for simple preference as described in Sect. 3.2.4, it is assumed that the preference relations of each DM $i \in N$ have the following properties:

(i) $\sim_i$ is reflexive and symmetric (i.e., $\forall s, q \in S$, $s \sim_i s$, and if $s \sim_i q$, then $q \sim_i s$);

(ii) $\overset{d}{\succ}_i$ for $d = 1, 2, \ldots, r$, is asymmetric (i.e., $s \overset{r}{\succ}_i q$ and $q \overset{r}{\succ}_i s$ cannot occur simultaneously); and

(iii) $\{\sim_i, >_i, \gg_i, \ldots, \overset{r}{\succ}_i\}$ is strongly complete (i.e. if $s, q \in S$, then exactly one of the following relations holds: $s \sim_i q$, $s \overset{d}{\succ}_i q$, or $q \overset{d}{\succ}_i s$ for $d = 1, 2, \ldots, r$).

Preference information can be either transitive or intransitive. For states $k, s, q \in S$, if $k \overset{d}{\succ}_i s$ and $s \overset{d}{\succ}_i q$ imply $k \overset{d}{\succ}_i q$, then the preference $\overset{d}{\succ}_i$ is transitive. Otherwise, the preferences are called intransitive. Note that the assumption of transitivity of preferences is not required in the following definitions so that the results in this chapter hold for both transitive and intransitive preferences. When all of the preferences for

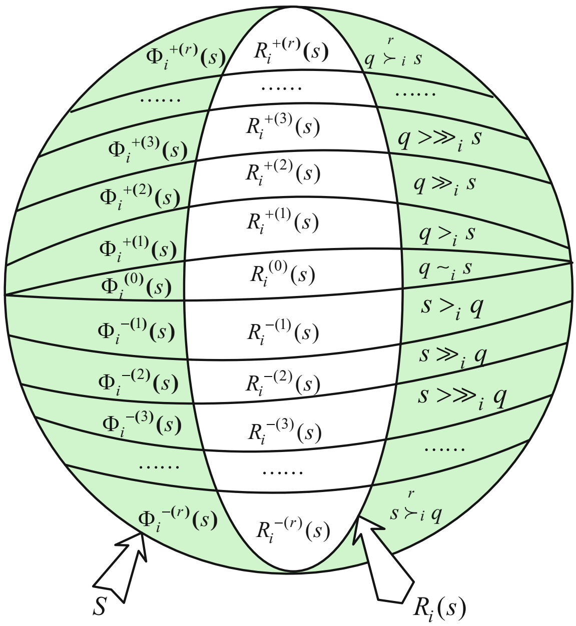
Table 6.3 Subsets of S for DM i with respect to multiple degrees of preference

Degree of strength	Subsets of S	Description
$d = r$	$\Phi_i^{+(r)}(s) = \{q : q \overbrace{> \cdots >_i}^{r} s\}$	States preferred to state s at degree r by DM i
	$\Phi_i^{-(r)}(s) = \{q : s \overbrace{> \cdots >_i}^{r} q\}$	States less preferred to state s at degree r by DM i
$\vdots$		
$\vdots$		
$d = 3$	$\Phi_i^{+(3)}(s) = \{q : q \ggg_i s\}$	States very strongly preferred to state s by DM i
	$\Phi_i^{-(3)}(s) = \{q : s \ggg_i q\}$	States very strongly less preferred to state s by DM i
$d = 2$	$\Phi_i^{+(2)}(s) = \{q : q \gg_i s\}$	States strongly preferred to state s by DM i
	$\Phi_i^{-(2)}(s) = \{q : s \gg_i q\}$	States strongly less preferred to state s by DM i
$d = 1$	$\Phi_i^{+(1)}(s) = \{q : q >_i s\}$	States moderately preferred to state s by DM i
	$\Phi_i^{-(1)}(s) = \{q : s >_i q\}$	States moderately less preferred to state s by DM i
$d = 0$	$\Phi_i^{(0)}(s) = \Phi_i^{=}(s) = \{q : q \sim_i s\}$	States equally preferred to state s by DM i

a given DM i are transitive, the preferences are said to be ordinal and, hence, the states in a conflict can be ordered or ranked from most to least preferred, where ties are allowed. Sometimes this ranking of states according to preference is referred to as a "preference ranking".

A list and associated descriptions for the range of subsets of S with respect to multiple types of preference are presented in Table 6.3. Starting at the bottom of the table at degree 0, the notation for the states equally preferred to state s by DM i is given as $\Phi_i^{(0)}(s)$ or $\Phi_i^{=}(s)$. Notice that for degree of strength $d = 1, \ldots, r$, two subsets of states are given for each degree as $\Phi_i^{+(d)}(s)$ and $\Phi_i^{-(d)}(s)$, to indicate subsets of states preferred to state s at degree d by DM i, and states less preferred to state s at degree d by DM i, respectively. Hence, overall there is a total of $2r + 1$ subsets of S when considering multiple degrees of preference. A diagram displaying these degrees of preference for DM i is furnished later in Sect. 6.2.2 as the left side of Fig. 6.1.

Fig. 6.1 Relationships among subsets of S and reachable lists from s



6.2 Reachable Lists of a Decision Maker

In addition to preference, one must be aware of the moves DMs control when ascertaining stability. Accordingly, in this section, moves unilaterally controlled by a DM in one step are defined as reachable lists for the cases of three types and multiple kinds of preferences in Sects. 6.2.1 and 6.2.2, respectively. Potential moves by a DM in the face of simple preference are defined using reachable lists in Sect. 4.1.1. In the upcoming two subsections, let $i \in N$, $s \in S$, and $m = |S|$ be the number of the states in S. The notation given by $\cap$ denotes the intersection operation while $\cup$ is the union operation. Recall that each arc of $A_i \subseteq S \times S$ indicates that DM i can make a unilateral move (in one step) from the initial state to the terminal state of the arc.

6.2.1 *Reachable Lists for Three Degrees of Preference*

The reachable lists of a DM for three types of preference are defined as follows:

(i) $R_i^{++}(s) = \{q \in S : (s, q) \in A_i \text{ and } q \gg_i s\}$ stands for DM i's reachable list from state s by a strong unilateral improvement. This set contains all states q which are strongly preferred by DM i to state s and can be reached in one step from s;

Table 6.4 Unilateral movements for DM i in the three types of preference structure

Type of movements	Description
$R_i^{++}(s) = R_i(s) \cap \Phi_i^{++}(s)$	All strong unilateral improvements from state s for DM i
$R_i^{+m}(s) = R_i(s) \cap \Phi_i^{+m}(s)$	All mild unilateral improvements from state s for DM i
$R_i^{=}(s) = R_i(s) \cap \Phi_i^{=}(s)$	All equally preferred states reachable from state s by DM i
$R_i^{-m}(s) = R_i(s) \cap \Phi_i^{-m}(s)$	All mild unilateral disimprovements from state s for DM i
$R_i^{--}(s) = R_i(s) \cap \Phi_i^{--}(s)$	All strong unilateral disimprovements from state s for DM i

(ii) $R_i^{+m}(s) = \{q \in S : (s, q) \in A_i \text{ and } q >_i s\}$ denotes DM i's reachable list from state s by a mild unilateral improvement;
(iii) $R_i^{-m}(s) = \{q \in S : (s, q) \in A_i \text{ and } s >_i q\}$ denotes DM i's reachable list from state s by a mild unilateral disimprovement;
(iv) $R_i^{--}(s) = \{q \in S : (s, q) \in A_i \text{ and } s \gg_i q\}$ is DM i's reachable list from state s by a strong unilateral disimprovement;
(v) $R_i^{+,++}(s) = R_i^{+m}(s) \cup R_i^{++}(s) = \{q \in S : (s, q) \in A_i \text{ and } q >_i s \text{ or } q \gg_i s\}$ denotes DM i's reachable list from state s by a mild unilateral move or strong unilateral move.

From the above definitions, these reachable lists from state s by DM i can be summarized as presented in Table 6.4. As discussed in Sect. 4.1.1, DM i's reachable list from state s, $R_i(s)$, represents DM $i's$ unilateral moves (UMs). $R_i(s)$ is partitioned according to the three kinds of preference structure as $R_i(s) = R_i^{++}(s) \cup R_i^{+m}(s) \cup R_i^{=}(s) \cup R_i^{-m}(s) \cup R_i^{--}(s)$.

6.2.2 Reachable Lists for Multiple Degrees of Preference

The set $R_i(s)$ denotes the unilateral moves (UMs) of DM i from $s \in S$, and is also called i's reachable list from s. It contains all states to which DM i can move, unilaterally and in one step, from state s. Similarly, the set $R_i^{+}(s) = \{q \in S : q \in R_i(s) \text{ and } q \overset{d}{\succ}_i s \text{ for } d = 1, 2, \ldots, r\}$ contains DM i's unilateral improvements (UIs) from state s for all degrees of preference. Note that although the same notation "$R_i^{+}(s)$" is used in Sect. 4.1.1 to represent DM i's unilateral improvements from state s at degree 1, the meaning of $R_i^{+}(s)$ here differs from that: there, it denotes all unilateral improvements, which can only be of degree 1 from s by DM i, whereas here, it includes all unilateral improvements, no matter what degree. All reachable lists from state s at each degree of preference for DM i are expressed by $R_i^{+(r)}(s), \ldots,$

Table 6.5 Reachable lists of DM i at some degree of preference

Type of movement	Description
$R_i^{+(d)}(s) = R_i(s) \cap \Phi_i^{+(d)}(s)$ $(d = 1, 2, \ldots, r)$	All unilateral improvements of degree d from state s for DM i
$R_i^{(0)}(s) = R_i^{=}(s) = R_i(s) \cap \Phi_i^{=}(s)$ $(d = 0)$	All equally preferred states reachable from state s by DM i
$R_i(s)^{-(d)}(s) = R_i(s) \cap \Phi_i^{-(d)}(s)$ $(d = 1, 2, \ldots, r)$	All unilateral disimprovements of degree d from state s for DM i

$R_i^{+(1)}(s)$, $R_i^{(0)}(s)$, $R_i^{-(1)}(s)$, ..., and $R_i^{-(r)}(s)$. Let $R_i(s) = \bigcup_{d=0}^{r}(R_i^{-(d)}(s) \cup R_i^{+(d)}(s))$ and $R_i^{+}(s) = \bigcup_{d=1}^{r} R_i^{+(d)}(s)$, where $R_i^{+(d)}(s)$ and $R_i^{-(d)}(s)$ for $d = 0, 1, \ldots, r$, are described in Table 6.5. Additionally, the relations among the subsets of S, $\Phi_i^{+(d)}(s)$ and $\Phi_i^{-(d)}(s)$ for $d = 0, 1, \ldots, r$, and the corresponding reachable lists from state s for DM i, $R_i^{+(d)}(s)$ and $R_i^{-(d)}(s)$ for $d = 0, 1, \ldots, r$, are depicted in Fig. 6.1.

Incorporating these extended multiple kinds of preference into the Graph Model for Conflict Resolution results in multi-degree versions of the four basic solution concepts presented in Sect. 6.4. The stability definitions for three types of preference are presented in next section.

6.3 Logical Representation of Stabilities for Three Types of Preference

Three types of preference including strength of preference are integrated into the Graph Model for Conflict Resolution to extend the four basic solution concepts in order to ascertain their strategic impacts. Recall that the three types of preference are equally preferred ($\sim$), mildly preferred ($>$), and strongly preferred ($\gg$) which together form the preference structure denoted as $\{\sim, >, \gg\}$. The four stability definitions given in the next subsection recognize two cases in which the degree of strength in the three kinds of preference are distinguished. Firstly, general stabilities are defined, and then the two subclasses, strong and weak, are determined. Stabilities of the first kind are referred to as general because they are in essence the same as the stability definitions using simple preference, as defined in Sect. 4.2. Stability definitions are called strong or weak stabilities in order to reflect the additional preference information contained in the strength of the preference relation. These more sophisticated definitions furnish expanded strategic insights into a conflict model that handles strength of preference. Sections 6.3.1 and 6.3.3 furnish the above stability definitions for 2-DM and n-DM ($n \geq 2$), respectively, while Sect. 6.3.2 presents definitions for reachable lists of a coalition of DMs required in the n-DM stability definitions.

6.3.1 Two Decision Maker Case

In order to calculate the stability of a state for a given DM $i \in N$, it is necessary to examine possible responses by all other DMs $j \in N \setminus \{i\}$. In a two-DM model, the only opponent of DM i is the remaining DM j. For all of the definitions given in next section, assume that $N = \{i, j\}$ and $s \in S$.

6.3.1.1 Logical Representation of General Stabilities

Four general solution concepts are given below in which strength of preference is not considered in sanctioning. However, the general stabilities are different from those defined in Sect. 4.2 for simple preference, because the stability definitions for simple preference do not directly take into account degree or strength of preference.

Definition 6.1 State s is Nash stable for DM i, denoted by $s \in S_i^{Nash}$, iff $R_i^{+,++}(s) = \emptyset$.

Definition 6.2 State s is general GMR (GGMR) for DM i, denoted by $s \in S_i^{GGMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j(s_1)$ such that $s_2 \in \Phi_i^{--,-,=}(s)$.

Definition 6.3 State s is general SMR (GSMR) for DM i, denoted by $s \in S_i^{GSMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j(s_1)$, such that $s_2 \in \Phi_i^{--,-,=}(s)$ and $s_3 \in \Phi_i^{--,-,=}(s)$ for any $s_3 \in R_i(s_2)$.

Definition 6.4 State s is general SEQ (GSEQ) for DM i, denoted by $s \in S_i^{GSEQ}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j^{+,++}(s_1)$ such that $s_2 \in \Phi_i^{--,-,=}(s)$.

6.3.1.2 Logical Representation of Strong and Weak Stabilities

When strength of preference is introduced into the graph model, stability definitions can be strong or weak, according to the degree of sanctioning. For three kinds of preference, stabilities are divided into strongly and weakly stable with respect to the strength of possible sanctions. Hence, if a particular state s is general stable, then s is either strongly stable or weakly stable. Strong and weak stabilities only include GMR, SMR, and SEQ because Nash stability does not involve sanctions.

Definition 6.5 State s is strongly GMR (SGMR) for DM i, denoted by $s \in S_i^{SGMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.

Definition 6.6 State s is strongly SMR (SSMR) for DM i, denoted by $s \in S_i^{SSMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j(s_1)$, such that $s_2 \in \Phi_i^{--}(s)$ and $s_3 \in \Phi_i^{--}(s)$ for all $s_3 \in R_i(s_2)$.

Definition 6.7 State s is strongly SEQ (SSEQ) for DM i, denoted by $s \in S_i^{SSEQ}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_j^{+,++}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.

Definition 6.8 Let $s \in S$ and $i \in N$. State s is weakly stable for DM i iff s is general stable, but not strongly stable for some stability definition.

Example 6.1 (Stabilities for the Extended Sustainable Development Model under Three-degree Preference) The sustainable development conflict is introduced in Example 3.1. Here, this conflict is expanded to include three degrees of preference for the two-DM case. Specifically, the conflict consists of two DMs: an environmental agency (DM 1: E) and a developer (DM 2: D); and two options: DM 1 controls the option of being proactive (labeled P) and DM 2 has the option of practicing sustainable development (labeled SD) for properly treating the environment. The two options are combined to form four feasible states: s_1, s_2, s_3, and s_4. These results are listed in Table 6.6, where a "Y" indicates that an option is selected by the DM controlling it and an "N" means that the option is not chosen.

The preference information for each DM among the four states is provided at the bottom of Table 6.6. As can be seen for the case of DM 1, this DM prefers s_1 over s_3, greatly prefers s_3 to s_2, which is equally preferred to s_4. Notice that DM 2 greatly prefers s_1 to s_4. The graph model for the extended sustainable development conflict is presented in Fig. 6.2. One can see, for instance from DM 1's directed graph on the left side of Fig. 6.2, that this DM controls the movement between states s_1 and s_3 as well as s_2 and s_4.

The extended sustainable development model with three degrees of preference is used to illustrate how to determine general and strong stabilities under the three-degree version using Definitions 6.1–6.8. In particular, first consider analyzing

Table 6.6 Extended sustainable development game in option form under three-degree preference

DM 1: Environmental agency				
1. Proactive (P)	Y	Y	N	N
DM 2: Developer				
2. Sustainable development (SD)	Y	N	Y	N
States	s_1	s_2	s_3	s_4

Preferences $s_1 >_1 s_3 \gg_1 s_2 \sim_1 s_4$ for DM 1 and
$s_3 >_2 s_1 \gg_2 s_4 \sim_2 s_2$ for DM 2

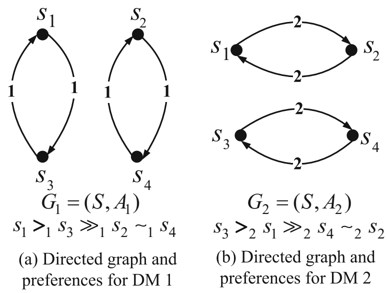


Fig. 6.2 Graph model for the extended sustainable development conflict under three-degree preference

state s_1 with respect to general Nash stability for DM 1. From Fig. 6.2, DM 1 has a unilateral move (UM) from s_1 to s_3. However, s_1 is mildly more preferred to s_3 by DM 1 because the move from s_1 to s_3 does not fall into the category of a mild or strong unilateral improvement. Therefore, state s_1 is general Nash stable for DM 1 according to Definition 6.1. Moreover, s_1 is also general GMR, SMR, and SEQ stable for DM 1.

Next, one can assess whether s_3 is general GMR stable for DM 1. From Fig. 6.2, DM 1 has a mild unilateral improvement from s_3 to s_1 and DM 2 has a unilateral move from s_1 to s_2. However, since s_2 is strongly less preferred than s_3 for DM 1, state s_3 is general GMR stable for DM 1 according to Definition 6.2. The stabilities of the other three states for the two DMs can be determined in a similar fashion.

Now, consider analyzing state s_3 from DM 1's viewpoint for GSMR stability using Definition 6.3. As can be seen from DM 1's directed graph in Fig. 6.2a, DM 1 has a mild UI from s_3 to s_1 and DM 2 has a UM from s_1 to s_2, from which DM 1 has only a UM from s_2 to s_4. Because DM 1 is indifferent between s_2 and s_4, which are greatly less preferred to state s_3, s_3 is GSMR stable for DM 1 using Definition 6.3. General SMR stability for other states can be calculated in a similar way.

To explain how general SEQ stability is calculated, consider state s_3 from DM 1's perspective. Because DM 2's possible countermove from s_1 to s_2 is, in fact, a move to a greatly less preferred state, this DM has no credible sanction to stop DM 1 from taking advantage of its UI from s_3 to s_1. Accordingly, state s_3 is not general SEQ stable for DM 1.

One could also provide an explanation for determining strong or weak stabilities using Definitions 6.5–6.8 for the extended sustainable development conflict. The discussion would be quite similar to the general stabilities.

The stable states and equilibria for the extended sustainable development conflict under three-degree preference are summarized in Table 6.7, in which "$\surd$" for a given state means that this state is general, strong, or weak stable for DM 1 or DM 2 and "Eq" is an equilibrium for an appropriate solution concept. The results provided by Table 6.7 show that state s_1 is a strong equilibrium for the four basic stabilities. State s_3 is strongly stable for GMR and SMR for all DMs. Hence, s_1 and s_3 are better choices for decision makers.

Table 6.7 Stability results of the extended sustainable development conflict under three-degree preference

State	Nash			GGMR			GSMR			GSEQ			SGMR			SSMR			SSEQ			WGMR			WSMR			WSEQ		
	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq
s_1	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√									
s_2	√			√			√			√			√			√			√											
s_3		√		√	√	√	√	√	√		√		√	√	√	√	√	√		√										
s_4	√			√			√			√			√			√			√											

6.3.2 Reachable Lists of a Coalition of Decision Makers

To extend the definitions of the reachable lists for a coalition to take three kinds of preference ($\sim, >, \gg$) into account, a legal sequence of coalitional mild or strong unilateral improvements (MSUIs) must be defined first. The reachable lists of coalition H from state s by the legal sequences of UMs and UIs are defined in Sect. 4.2.2 for simple preference. The reachable lists of coalition H are expanded to three kinds of preference in this section. A legal sequence of MSUIs is a sequence of allowable mild unilateral improvements or strong unilateral improvements by a coalition, with the same restriction that any member in the coalition may move more than once, but not twice consecutively. The formal definition for reachable lists of coalition H by the legal sequence of MSUIs is presented as follows.

Definition 6.9 Let $s \in S$, $H \subseteq N$, and $H \neq \emptyset$. A mild or strong unilateral improvement (MSUI) by H is a member of $R_H^{+,++}(s) \subseteq S$, defined inductively by

(1) assuming $\Omega_H^{+,++}(s, s_1) = \emptyset$ for all $s_1 \in S$;
(2) if $j \in H$ and $s_1 \in R_j^{+,++}(s)$, then $s_1 \in R_H^{+,++}(s)$ and $\Omega_H^{+,++}(s, s_1) = \Omega_H^{+,++}(s, s_1) \cup \{j\}$;
(3) if $s_1 \in R_H^{+,++}(s)$, $j \in H$, and $s_2 \in R_j^{+,++}(s_1)$, then, provided $\Omega_H^{+,++}(s, s_1) \neq \{j\}$, $s_2 \in R_H^{+,++}(s)$ and $\Omega_H^{+,++}(s, s_2) = \Omega_H^{+,++}(s, s_2) \cup \{j\}$.

Definition 6.9 is similar to Definition 4.7 for simple preference in Sect. 4.2.2 and Definition 5.18 for unknown preference in Sect. 5.2.2. It is also an inductive definition. By (2) in Definition 6.9, the states reachable from s are identified and added to the set $R_H^{+,++}(s)$; then, using (3), all states reachable from those states are identified and added to $R_H^{+,++}(s)$; afterwards the process is repeated in finitely many steps until no further states are added to the coalitional reachable list by legal sequences of mild or strong unilateral improvements, $R_H^{+,++}(s)$. For $\Omega_H^{+,++}(s, s_1)$, if $s_1 \in R_H^{+,++}(s)$, then $\Omega_H^{+,++}(s, s_1) \subseteq H$ is the set of all last DMs in legal MSUI sequences from s to s_1. Suppose that $\Omega_H^{+,++}(s, s_1)$ contains only one DM $j \in N$. Then any move from s_1 to a subsequent state s_2 must be made by a member of H other than j; otherwise DM j would have to move twice in succession.

6.3.3 n-Decision Maker Case

Within an n-DM model ($n \geq 2$) for three degrees of preference structure, DM i's opponents, $N \setminus \{i\}$, consist of a group of one or more DMs. In order to analyze the stability of a state for DM $i \in N$, it is necessary to take into account possible responses by all other DMs $j \in N \setminus \{i\}$. The key components in stability definitions for three degrees of preference are reachable lists of coalition $N \setminus \{i\}$ from state s, $R_{N \setminus \{i\}}(s)$ and $R_{N \setminus \{i\}}^{+,++}(s)$, discussed above. The stability definitions for two DM cases presented in Sect. 6.3.1 are extended to general n-DM models next.

6.3.3.1 Logical Representation of General Stabilities

Four standard solution concepts are given below in which strength of preference is not considered in sanctioning. However, the general stabilities are different from those defined in Sect. 4.2.3 for simple preference, because stability definitions for simple preference cannot analyze conflict models having strength of preference. Let $i \in N$ and $s \in S$ for the following definitions.

Definition 6.10 State s is Nash stable for DM i, denoted by $s \in S_i^{Nash}$, iff $R_i^{+,++}(s) = \emptyset$.

Nash stability definitions are identical for both the 2-DM and the n-DM models because Nash stability does not consider opponents' responses.

Definition 6.11 State s is general GMR (GGMR) for DM i, denoted by $s \in S_i^{GGMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $s_2 \in \Phi_i^{--,-,=}(s)$.

Definition 6.12 State s is general SMR (GSMR) for DM i, denoted by $s \in S_i^{GSMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,-,=}(s)$ and $s_3 \in \Phi_i^{--,-,=}(s)$ for any $s_3 \in R_i(s_2)$.

Definition 6.13 State s is general SEQ (GSEQ) for DM i, denoted by $s \in S_i^{GSEQ}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^{+,++}(s_1)$ such that $s_2 \in \Phi_i^{--,-,=}(s)$.

Similar to 2-DM case, general stabilities for n-DM models are partitioned into strong or weak stabilities according to the level of sanctioning. Strong and weak stabilities only include GMR, SMR, and SEQ because Nash stability does not involve sanctions.

6.3.3.2 Logical Representation of Strong and Weak Stabilities

Definition 6.14 State s is strongly GMR (SGMR) for DM i, denoted by $s \in S_i^{SGMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.

Definition 6.15 State s is strongly SMR (SSMR) for DM i, denoted by $s \in S_i^{SSMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--}(s)$ and $s_3 \in \Phi_i^{--}(s)$ for all $s_3 \in R_i(s_2)$.

Definition 6.16 State s is strongly SEQ (SSEQ) for DM i, denoted by $s \in S_i^{SSEQ}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^{+,++}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.

The important components, $R_{N\backslash\{i\}}(s_1)$ and $R_{N\backslash\{i\}}^{+,++}(s_1)$, in Definitions 6.14–6.16 are defined in Sects. 4.2.2 and 6.3.2, respectively, for $H = N\backslash\{i\}$. The definition of weak stability is presented next.

Definition 6.17 Let $s \in S$ and $i \in N$. State s is weakly stable for DM i iff s is general stable, but not strongly stable for some stability definition.

6.4 Logical Representation of Stabilities for Multiple Degrees of Preferences

The following stability definitions for multiple kinds of preference are analogous to the concepts for three types of preference presented in Sect. 6.3. The multiple-degree preference is included into the Graph Model for Conflict Resolution resulting in multilevel versions of the four basic solution concepts, $Nash_k$, GMR_k, SMR_k, and SEQ_k for $k = 0, 1, \ldots, r$. The stability definitions in a 2-DM conflict model are presented next.

6.4.1 Two Decision Maker Case

6.4.1.1 Logical Representation of General Stabilities

Definition 6.18 State s is **general Nash stable** (GNash) for DM i, denoted by $s \in S_i^{GNash}$, iff $R_i^+(s) = \emptyset$.

Definition 6.19 State s is **general GMR** (GGMR) for DM i, denoted by $s \in S_i^{GGMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$.

Definition 6.20 State s is **general SMR** (GSMR) for DM i, denoted by $s \in S_i^{GSMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$.

Definition 6.21 State s is **general SEQ** (GSEQ) for DM i, denoted by $s \in S_i^{GSEQ}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j^+(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$.

Note that in this section the meaning of $R_i^+(s)$ differs from that in Sect. 4.1.1 to represent DM i's UI from state s for simple preference; there, it denotes all one-degree unilateral improvements from s by DM i, whereas here, it includes all unilateral improvements, no matter how many degrees of preference. For three degrees of preference discussed above, general stabilities are divided into strongly and weakly stable according to the strength of the possible sanction, i.e., if a particular state s is general stable, then s is either strongly stable or weakly stable. Within multiple degrees of preference, the general stabilities are constituted by stabilities at each level of preference.

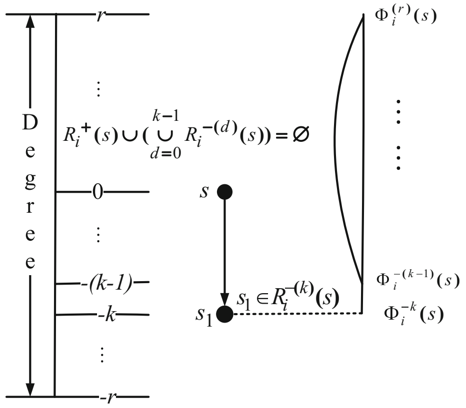


Fig. 6.3 Nash stability at degree k for DM i

6.4.1.2 Logical Representation of Stabilities at Degree k

Firstly, definitions are now given for different strengths of Nash stability. Even though unilateral improvements do not exist under Nash stability, the idea of strength of stability can still be captured using the degree of preference for the most preferred states to which the DM could unilaterally move. All these states must be less preferred than the initial state. A special definition is required for the case in which no movements of any type exist for the DM. In particular, if DM i has no unilateral move at all degrees of preference from state s, state s is extremely stable. The stability is proposed next.

Definition 6.22 If $R_i(s) = \emptyset$, then state s is **super stable** for DM i at any degree of preference, denoted by $s \in S_i^{Super}$.

Definition 6.23 State s is **Nash stable** ($Nash_0$) at degree 0 for DM i, denoted by $s \in S_i^{Nash_0}$, iff $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$.

Notice in the definition of $Nash_0$ that no unilateral improvements by DM i from state s exist but an equally preferred state must be present.

Definition 6.24 For $1 \leq k \leq r$, state s is **Nash stable** ($Nash_k$) at degree k for DM i, denoted by $s \in S_i^{Nash_k}$, iff $R_i^+(s) \cup (\bigcup_{d=0}^{k-1} R_i^{-(d)}(s)) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$.

For $Nash_k$ stability, the most preferred state to which DM i can unilaterally move from s is located at degree $-k$ (below degree 0). The kth degree Nash stability is depicted in Fig. 6.3. The super stability is referred to as Nash stability at the highest degree, because no unilateral moves exist for DM i from s.

When multiple-degree preference is incorporated into the graph model, GMR, SMR, and SEQ stabilities at different degrees can be distinguished according to the strength of the sanctions.

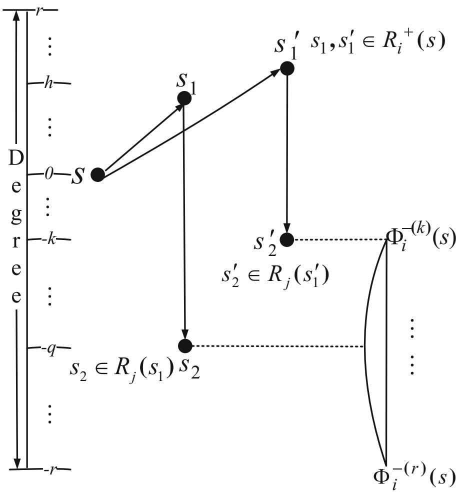


Fig. 6.4 GMR stability at degree k for DM i

Definition 6.25 State s is **general metarational** (GMR_0) at degree 0 for DM i, denoted by $s \in S_i^{GMR_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_j(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_j(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

Based on Definition 6.25, when DM i has no UIs from state s and it is $Nash_0$ stable, as in Definition 6.24, then state s is also GMR stable at degree 0.

Definition 6.26 For $1 \leq k \leq r-1$, state s is **general metarational** (GMR_k) at degree k for DM i, denoted by $s \in S_i^{GMR_k}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_j(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_j(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

Figure 6.4 contains a specific example to explain the meaning of Definition 6.26. Notice that DM i has UIs from state s to states s_1 and s_1', each of which can be at any degree from 1 to r. From state s_1, DM j, who is DM i's opponent, has one unilateral move to state s_2 (labeled $R_j(s_1)$), which is as shown on the degree axis to be of degree $-q$, where q can range from k to r relative to s. With respect to state s_1', DM j can move to state s_2', which is only located at degree $-k$ relative to state s. Therefore, state s for DM i possesses general meterational stability at degree k for which $0 < k < r$ according to Definition 6.26.

If all of DM i's UIs from a state are sanctioned at the highest degree r (exactly r levels below the state), then the state is called general metarational at degree r. Its formal definition is given below.

Definition 6.27 State s is **general metarational** (GMR_r) at degree r for DM i, denoted by $s \in S_i^{GMR_r}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$.

For DM i, if a UI from a state is sanctioned at degree k below the state and all other UIs from the particular state are sanctioned at a degree of at least k below the state, and these corresponding sanctions cannot be avoided by any counterresponse, then the state is called SMR stable at degree k. Its formal definition is given below.

Definition 6.28 State s is **symmetric metarational** (SMR_0) at degree 0 for DM i, denoted by $s \in S_i^{SMR_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_j(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_j(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$, as well as $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for any $s_3 \in R_i(s_2) \cup R_i(s_2')$.

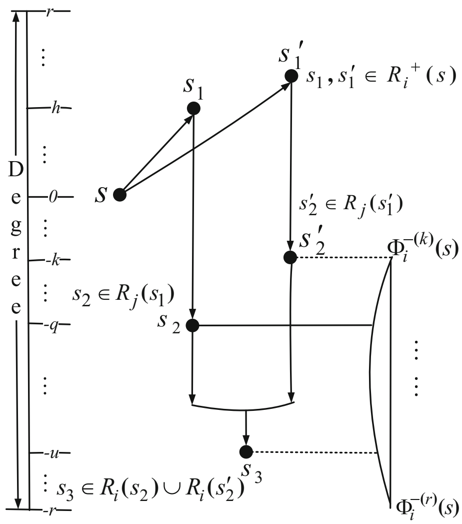
Symmetric metarationality at degree k ($0 < k \leq r$) for DM i consists of SMR_{k^+} and SMR_{k^-} that are defined next.

Definition 6.29 For $1 \leq k \leq r-1$, state s is **symmetric metarational** (SMR_{k^+}) at degree k for DM i, denoted by $s \in S_i^{SMR_{k^+}}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_j(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_j(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$, as well as $s_3 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ for any $s_3 \in R_i(s_2) \cup R_i(s_2')$.

Figure 6.5 vividly illustrates the SMR stability at k^+ for DM i. Stability SMR_{k^-} is defined by $S_i^{SMR_{k^-}} = S_i^{GSMR} \cap S_i^{GMR_k} - S_i^{SMR_{k^+}}$. Equivalently,

Definition 6.30 For $1 \leq k \leq r-1$, state s is **symmetric metarational** (SMR_{k^-}) at degree k for DM i, denoted by $s \in S_i^{SMR_{k^-}}$, iff $s \in S_i^{GMR_k}$ and $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$, as well as there exists $s_1' \in R_i^+(s)$ and

Fig. 6.5 SMR stability at degree k^+ for DM i



for every $s'_2 \in R_j(s'_1) \cap (\bigcup_{d=k}^{r} \Phi_i^{-(d)}(s))$, $R_i(s'_2) \cap \Phi_i^{(-d)}(s) \neq \emptyset$ for at least one $d \in \{0, \ldots, (k-1)\}$.

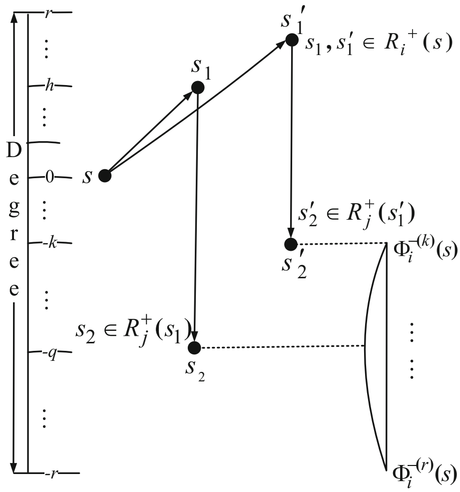
Definition 6.31 State s is **symmetric metarational** (SMR_{r^+}) at degree r for DM i, denoted by $s \in S_i^{SMR_{r^+}}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$ and $s_3 \in \Phi_i^{-(r)}(s)$ for any $s_3 \in R_i(s_2)$.

Definition 6.32 State s is **symmetric metarational** (SMR_{r^-}) at degree r for DM i, denoted by $s \in S_i^{SMR_{r^-}}$, iff $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$, as well as there exists $s'_1 \in R_i^+(s)$ and for every $s'_2 \in R_j(s_1) \cap \Phi_i^{-(r)}(s)$, $R_i(s'_2) \cap \Phi_i^{(-d)}(s) \neq \emptyset$ for at least one $d \in \{0, \ldots, (r-1)\}$.

Sequential stability at degree k is similar to the stability of GMR at the same degree. The only modification is that all DM i's UIs are subject to credible sanctions by DM i's opponent. Its formal definition is given below.

Definition 6.33 State s is **sequentially stable** (SEQ_0) at degree 0 for DM i, denoted by $s \in S_i^{SEQ_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j^+(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s'_1 \in R_i^+(s)$ and $s'_2 \in R_j^+(s'_1)$ such that $s'_2 \in \Phi_i^{(0)}(s)$ and $R_j^+(s'_1) \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

Fig. 6.6 SEQ stability at degree k for DM i



Definition 6.34 For $1 \leq k \leq r-1$, state s is **sequentially stable** (SEQ_k) at degree k for DM i, denoted by $s \in S_i^{SEQ_k}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j^+(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_j^+(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_j^+(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

Figure 6.6 can be used to explain the meaning of Definition 6.34. In fact, Definition 6.34 is similar to Definition 6.29. The only difference is that DM j, who is DM i's opponent, has one unilateral improvement to state s_2 at a degree ranged from k to r relative to s. With respect to state s_1', DM j has a unilateral improvement s_2', which is only located at degree $-k$ relative to state s. Therefore, state s for DM i is sequentially stable at degree k for which $0 < k < r$ according to Definition 6.34.

Definition 6.35 State s is **sequentially stable** (SEQ_r) at degree r for DM i, denoted by $s \in S_i^{SEQ_r}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j^+(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$.

In an n-DM model, where $n \geq 2$, the opponents of a DM can be thought of as a coalition of one or more DMs. To extend the graph model stability definitions to stability definitions in n-DM models with multiple degrees of preference, the definitions of a legal sequence of moves for three degrees of preference presented in Sect. 6.3.2 must first be extended to take multiple degrees of preference into account.

6.4.2 Reachable Lists of a Coalition of Decision Makers

A legal sequence of UMs in a graph model with multiple degrees of preference for a coalition of DMs is a sequence of states linked by unilateral moves controlled by members of the coalition, in which a DM may move more than once, but not twice in succession. As explained in Sect. 4.2.2 before Definition 4.6, this rule allows the GMCR methodology to handle intransitive moves, in addition to transitive moves. When $H = \{i\}$, a legal sequence of UMs for the coalition H reduces to a unilateral move of DM i.

Let the coalition $H \subseteq N$ satisfy $|H| \geq 2$ and let the status quo state be $s \in S$. Define $R_H(s) \subseteq S$, the reachable list of coalition H from state s by a legal sequence of UMs in a graph model with multiple degrees of preference. The following definitions are adapted from Fang et al. (1993) and Hamouda et al. (2006):

Definition 6.36 Let $s \in S$, $H \subseteq N$, and $H \neq \emptyset$. Here, $R_j(s) = \bigcup_{d=0}^{r}(R_j^{-(d)}(s) \cup R_j^{+(d)}(s))$ for any $j \in H$. A unilateral move by H is a member of $R_H(s) \subseteq S$, defined inductively by:

(1) if $j \in H$ and $s_1 \in R_j(s)$, then $s_1 \in R_H(s)$ and $\Omega_H(s, s_1) = \Omega_H(s, s_1) \cup \{j\}$;
(2) if $s_1 \in R_H(s)$, $j \in H$ and $s_2 \in R_j(s_1)$, then, provided $\Omega_H(s, s_1) \neq \{j\}$, $s_2 \in R_H(s)$ and $\Omega_H(s, s_2) = \Omega_H(s, s_2) \cup \{j\}$.

Note that Definition 6.36 is analogous to Definition 4.6, but, here, unilateral moves include the states that are reachable from state s by multiple degrees of preference (may have more than three degrees) listed in Table 6.5.

In a graph model with multiple degrees of preference, a legal sequence of UIs for coalition H is a sequence of states linked by unilateral improvements including each-degree UIs controlled by members of the coalition H with the usual restriction that a member of the coalition may move more than once, but not twice consecutively. The formal definition is given below.

Definition 6.37 Let $R_j^+(s) = \bigcup_{d=1}^{r} R_j^{+(d)}(s)$ for any $j \in H$. A unilateral improvement by H is a member of $R_H^+(s) \subseteq S$, defined inductively by:

(1) if $j \in H$ and $s_1 \in \bigcup_{d=1}^{r} R_j^{+(d)}(s)$, then $s_1 \in R_H^+(s)$ and $\Omega_H^+(s)(s, s_1) = \Omega_H^+(s)(s, s_1) \cup \{j\}$;
(2) if $s_1 \in R_H^+(s)$, $j \in H$ and $s_2 \in \bigcup_{d=1}^{r} R_j^{+(d)}(s_1)$, then, provided $\Omega_H^+(s)(s, s_1) \neq \{j\}$, $s_2 \in R_H^+(s)$ and $\Omega_H^+(s, s_2) = \Omega_H^+(s, s_2) \cup \{j\}$.

Definition 6.37 is identical to Definition 6.36 except that each move is to a state strictly preferred with some degree of preference by the mover to the current state. Similarly, $\Omega_H^+(s, s_1)$ includes all last movers in a legal sequence of UIs by coalition

H from state s to state s_1. Specifically, this definition is inductive: first, using (1), the states reachable by a single DM in H from s by one step UIs in multiple levels of preference are identified and added to $R_H^+(s)$; then, using (2), all states reachable from those states are identified and added to $R_H^+(s)$; afterwards the process is repeated until no further states are added to $R_H^+(s)$ by repeating (2). Because $R_H^+(s) \subseteq S$, and S is finite, this limit must be reached in finitely many steps.

6.4.3 n-Decision Maker Case

6.4.3.1 Logical Representation of General Stabilities

Super stability and Nash stability definitions are identical for both the 2-DM and the n-DM models because these stabilities do not consider the opponents' responses. Let $i \in N$ and $s \in S$ for the following definitions.

Definition 6.38 State $s \in S$ is GGMR for DM i, denoted by $s \in S_i^{GGMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$.

Definition 6.39 State $s \in S$ is GSMR for DM i, denoted by $s \in S_i^{GSMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$.

Definition 6.40 State $s \in S$ is GSEQ for DM i, denoted by $s \in S_i^{GSEQ}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^+(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$.

6.4.3.2 Logical Representation of Stabilities at *k* Degree

Similar to 2-DM conflicts, solution concepts for n-DM conflicts can be defined as different-degree stabilities, according to degrees of preference. Nash stability definitions in multiple DM conflicts are the same as those in 2-DM cases. Therefore, only the extended GMR, SMR, and SEQ are defined here. For DM i, if a UI from state s is sanctioned by the legal sequence of UMs of i's opponents in exactly k degrees below s and all other UIs from state s are sanctioned in at least k degrees below s, then the status quo s is called general metarational at degree k. The process is portrayed in Fig. 6.7 and the formal definition is given below.

Definition 6.41 State s is GMR_0 for DM i, denoted by $s \in S_i^{GMR_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists

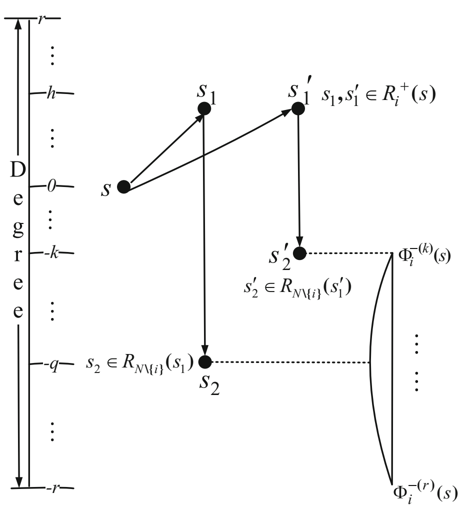


Fig. 6.7 General metarationality at degree k for DM i

at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\backslash\{i\}}(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_{N\backslash\{i\}}(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

Definition 6.42 For $1 \leq k \leq r-1$, state s is GMR_k for DM i, denoted by $s \in S_i^{GMR_k}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\backslash\{i\}}(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_{N\backslash\{i\}}(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

If all of DM i's UIs from a state are sanctioned at exactly r degrees below the state, then the state is called general metarational at degree r. Its formal definition is given below.

Definition 6.43 State s is GMR_r for DM i, denoted by $s \in S_i^{GMR_r}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$.

For DM i, if a UI from a state is sanctioned by the legal sequence of UMs of DM i's opponents at degree k and all other UIs from the particular state are sanctioned at degree at least k, and these corresponding sanctions cannot be avoided by any counterresponse, then the state is called symmetric metarational at degree k. The

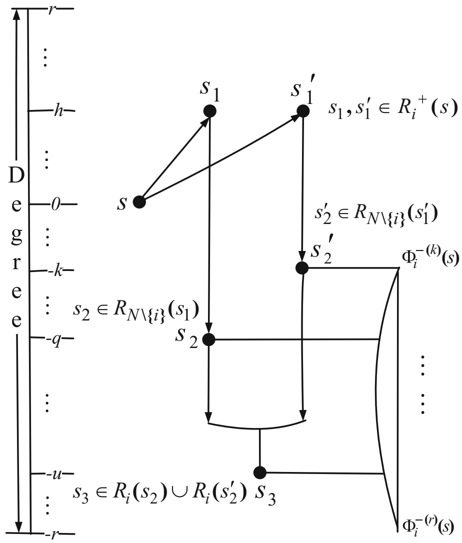


Fig. 6.8 Symmetric metarationality at degree k^+ for DM i

stability of SMR at degree k is portrayed in Fig. 6.8 and the formal definition is given below.

Definition 6.44 State s is SMR_0 for DM i, denoted by $s \in S_i^{SMR_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\setminus\{i\}}(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_{N\setminus\{i\}}(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$, as well as $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for any $s_3 \in R_i(s_2) \cup R_i(s_2')$.

Symmetric metarationality at degree k ($0 < k \leq r$) for DM i consists of SMR_{k^+} and SMR_{k^-} that are defined next.

Definition 6.45 For $1 \leq k \leq r-1$, state s is SMR_{k^+} for DM i, denoted by $s \in S_i^{SMR_{k^+}}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\setminus\{i\}}(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_{N\setminus\{i\}}(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$, as well as $s_3 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ for any $s_3 \in R_i(s_2) \cup R_i(s_2')$.

Stability SMR_{k^-} is defined by $S_i^{SMR_{k^-}} = S_i^{GSMR} \cap S_i^{GMR_k} - S_i^{SMR_k}$. Equivalently,

Definition 6.46 For $1 \le k \le r-1$, state s is SMR_{k^-} for DM i, denoted by $s \in S_i^{SMR_{k^-}}$, iff $s \in S_i^{GMR_k}$ and $R_i^+(s) \ne \emptyset$, and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$, as well as there exists $s_1' \in R_i^+(s)$ and for every $s_2' \in R_{N\setminus\{i\}}(s_1') \cap (\bigcup_{d=k}^{r} \Phi_i^{-(d)}(s))$, $R_i(s_2') \cap \Phi_i^{(-d)}(s) \ne \emptyset$ for at least one $d \in \{0, \ldots, (k-1)\}$.

Definition 6.47 State s is SMR_{r^+} for DM i, denoted by $s \in S_i^{SMR_{r^+}}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(r)}(s) \ne \emptyset$, or $R_i^+(s) \ne \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$ and $s_3 \in \Phi_i^{-(r)}(s)$ for any $s_3 \in R_i(s_2)$.

Definition 6.48 State s is SMR_{r^-} for DM i, denoted by $s \in S_i^{SMR_{r^-}}$, iff $R_i^+(s) \ne \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$, as well as there exists $s_1' \in R_i^+(s)$ and for every $s_2' \in R_{N\setminus\{i\}}(s_1) \cap \Phi_i^{-(r)}(s)$, $R_i(s_2') \cap \Phi_i^{(-d)}(s) \ne \emptyset$ for at least one $d \in \{0, \ldots, (r-1)\}$.

The only modification between GMR_k and SEQ_k is that all DM i's UIs are subject to credible sanctions by the legal sequence of UIs of DM i's opponents. Figure 6.9 depicts sequential stability at degree k. Its formal definition is given below.

Definition 6.49 State s is **sequentially stable** (SEQ_0) at level 0 for DM i, denoted by $s \in S_i^{SEQ_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \ne \emptyset$, or $R_i^+(s) \ne \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^+(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\setminus\{i\}}^+(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_{N\setminus\{i\}}^+(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

Definition 6.50 For $1 \le k \le r-1$, state s is **sequentially stable** (SEQ_k) at level k for DM i, denoted by $s \in S_i^{SEQ_k}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(k)}(s) \ne \emptyset$, or $R_i^+(s) \ne \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^+(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\setminus\{i\}}^+(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_{N\setminus\{i\}}^+(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

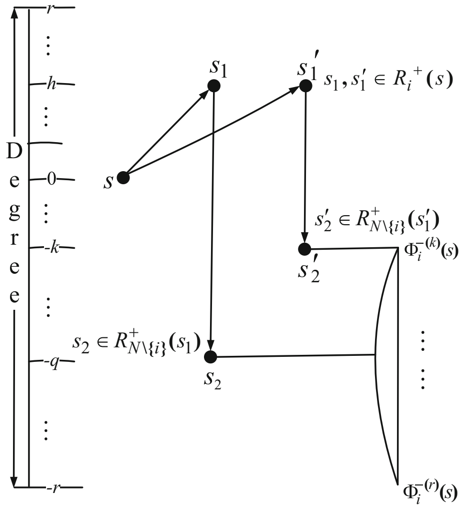


Fig. 6.9 Sequential stability at degree k for DM i

Definition 6.51 State s is **sequentially stable** (SEQ_r) at level r for DM i, denoted by $s \in S_i^{SEQ_r}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R^+_{N\setminus\{i\}}(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$.

When $n = 2$, the DM set N becomes $\{i, j\}$ in Definitions 6.41–6.51, and the reachable lists for $H = N \setminus \{i\}$ by legal sequences of UMs and UIs from s_1, $R_{N\setminus\{i\}}(s_1)$ and $R^+_{N\setminus\{i\}}(s_1)$, degenerate to $R_j(s_1)$ and $R_j^+(s_1)$, DM j's corresponding reachable lists from s_1. Obviously, Definitions 6.25–6.35 are special cases of Definitions 6.41–6.51, so the same notation is used for two DM cases and n-DM situations.

6.4.4 Interrelationship Among Stability Definitions for Multiple Degrees of Preference

In Sect. 4.2.4, relationships among the four basic stabilities consisting of Nash, GMR, SMR, and SEQ are presented for two types of preference (or simple preference). Within Sect. 6.3.3, stabilities under three kinds of preference are defined. In Sect. 6.4.1.2, the four stability definitions at degree k are formally defined. In the following five theorems, a range of theoretical relationships among and within stability definitions for different degrees of preference are proven.

Theorem 6.1 *The interrelationships among the four basic stabilities at degree k are*

$$S_i^{Nash_k} \subseteq S_i^{SMR_{k+}} \subseteq S_i^{GMR_k},\ S_i^{SMR_{k-}} \subseteq S_i^{GMR_k},\ \textit{and}\ S_i^{Nash_k} \subseteq S_i^{SEQ_k} \subseteq S_i^{GMR_k},$$

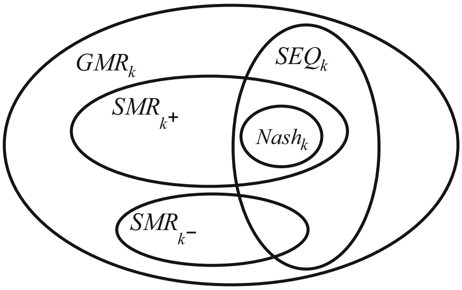


Fig. 6.10 Interrelationships among four stabilities at level k

for $0 \leq k \leq r$.

Proof When $k = 0$, the results are obvious, since there are no unilateral improvements by DM i relative to state s, but there exist equally preferred states. Assume that $0 < k \leq r$. If $s \in S_i^{Nash_k}$, then $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^{+}(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$. This implies that state $s \in S_i^{SMR_{k^+}}$ using Definitions 6.45 and 6.47. Hence, if $s \in S_i^{Nash_k}$ for $0 \leq k \leq r$, then $s \in S_i^{SMR_{k^+}}$, which implies $S_i^{Nash_k} \subseteq S_i^{SMR_{k^+}}$.

Using Definitions 6.41–6.47, if $s \in S_i^{SMR_{k^+}}$, it is obvious that $s \in S_i^{GMR_k}$ for $0 \leq k \leq r$. Therefore, the inclusion relations $S_i^{Nash_k} \subseteq S_i^{SMR_{k^+}} \subseteq S_i^{GMR_k}$ now follow.

Based on Definitions 6.46 and 6.48, the relation $S_i^{SMR_{k^-}} \subseteq S_i^{GMR_k}$ is obvious. Relations $S_i^{Nash_k} \subseteq S_i^{SEQ_k} \subseteq S_i^{GMR_k}$ can be similarly verified. □

Let $0 \leq k \leq r$. The inclusion relationships presented by Theorem 6.1 are depicted in Fig. 6.10. One should keep in mind that these relationships among stabilities are valid for the situations in which all stabilities being compared have the same degree. As can be clearly seen in this diagram, for example, if a state is $Nash_k$, it is also GMR_k, SMR_{k^+}, and SEQ_k, which is similar to the finding in Fig. 4.4 for the case of simple preference.

The next theorem confirms the relationship that exists for a specific stability definition at two different degrees. In particular, for each of the stability definitions, there are no common stable states when the preferences are different degrees.

Theorem 6.2 *Let* $0 \leq h, q \leq r$. *When* $h \neq q$, *the relationships between stabilities at h degree and at q degree are*

$$S_i^{Nash_h} \cap S_i^{Nash_q} = \emptyset, \tag{6.1}$$

$$S_i^{GMR_h} \cap S_i^{GMR_q} = \emptyset, \tag{6.2}$$

$$S_i^{SMR_{h^+}} \cap S_i^{SMR_{q^+}} = \emptyset,\ S_i^{SMR_{h^-}} \cap S_i^{SMR_{q^-}} = \emptyset,\ S_i^{SMR_{h^+}} \cap S_i^{SMR_{h^-}} = \emptyset, \textit{and} \tag{6.3}$$

$$S_i^{SEQ_h} \cap S_i^{SEQ_q} = \emptyset. \tag{6.4}$$

Proof First, Eq. 6.1 is proven. Assume that $h > q$. If there exists $s \in S_i^{Nash_h} \cap S_i^{Nash_q}$, then $s \in S_i^{Nash_h}$ and $s \in S_i^{Nash_q}$. Therefore, $R_i^+(s) \cup (\bigcup_{d=0}^{h-1} R_i^{-(d)}(s)) = \emptyset$ and $R_i^{-(h)}(s) \neq \emptyset$ as s is $Nash_h$ stable. Since $h-1 \geq q$, $R_i^{-(q)}(s) = \emptyset$. This contradicts the hypothesis that s is $Nash_q$ stable. Therefore, Eq. 6.1 holds.

Now, Eq. 6.2 is verified. If $s \in (S_i^{Nash_h} \cup S_i^{Nash_q})$, Eq. 6.2 is obvious. Assume that $h > q$ and $s \notin (S_i^{Nash_h} \cup S_i^{Nash_q})$. If there exists $s \in S_i^{GMR_h} \cap S_i^{GMR_q}$, then $s \in S_i^{GMR_h}$ and $s \in S_i^{GMR_q}$. Because s is GMR_q stable, $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=q}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\backslash\{i\}}(s_1')$ such that $s_2' \in \Phi_i^{-(q)}(s)$ and $R_{N\backslash\{i\}}(s_1') \bigcap (\bigcup_{d=q+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$. This implies that for all $s_2' \in R_{N\backslash\{i\}}(s_1')$, $s_2' \in \bigcup_{d=0}^{q} \Phi_i^{-(d)}(s)$ which means $s_2' \notin \bigcup_{d=h}^{r} \Phi_i^{-(d)}(s)$ as $h > q$. This contradicts with the hypothesis that s is GMR_h stable. Therefore, Eq. 6.2 follows.

Finally, the verification of Eqs. 6.3 and 6.4 can be similarly carried out using contradiction. □

The interrelationships among general stabilities, super stability, and stabilities at each degree are presented in the following theorem. Specifically, for each of the stability definitions, the set of stable states over the general stabilities is the same as the union of all of the stable states over all of the degrees of preference plus the super stable states.

Theorem 6.3 *The interrelationships among general stabilities, super stability, and stabilities at each level are*

$$S_i^{GNash} = (S_i^{Super}) \cup \left(\bigcup_{d=0}^{r} S_i^{Nash_d} \right), \tag{6.5}$$

$$S_i^{GGMR} = (S_i^{Super}) \cup \left(\bigcup_{d=0}^{r} S_i^{GMR_d} \right), \tag{6.6}$$

$$S_i^{GSMR} = (S_i^{Super}) \cup \left(\bigcup_{d=0}^{r} (S_i^{SMR_{d+}} \cup S_i^{SMR_{d-}}) \right), \textit{and} \tag{6.7}$$

$$S_i^{GSEQ} = (S_i^{Super}) \cup \left(\bigcup_{d=0}^{r} S_i^{SEQ_d} \right). \tag{6.8}$$

Proof Equation 6.5 is derived directly from Definitions 6.22–6.24. Now consider the proof for Eq. 6.6. The inclusion relation $S_i^{GGMR} \supseteq (S_i^{Super}) \cup (\bigcup_{d=0}^{r} S_i^{GMR_d})$ is

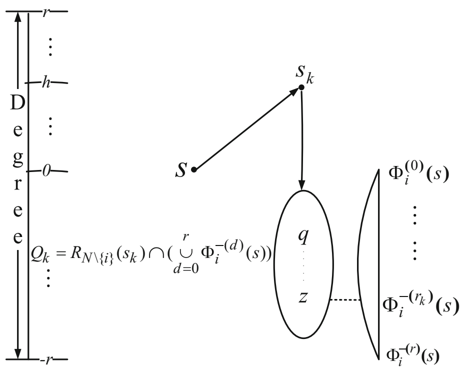


Fig. 6.11 The legal sequence of UM from state s_k

obvious from Definitions 6.41–6.43. It will be proved that the inclusion relation $S_i^{GGMR} \subseteq (S_i^{Super}) \cup (\bigcup_{d=0}^{r} S_i^{GMR_d})$ holds. The two cases will be respectively proved when $s \in (S_i^{Super} \cup S_i^{GNash})$ and $s \notin (S_i^{Super} \cup S_i^{GNash})$. For any $s \in S_i^{GGMR}$, based on Definition 6.38, if $s \in (S_i^{Super} \cup S_i^{GNash})$, then the above inclusion relation must be true.

Next, assume that $s \notin (S_i^{Super} \cup S_i^{GNash})$. Let $|R_i^+(s)| = l$ denote the cardinality of $R_i^+(s)$. Then, for any $s \in S_i^{GGMR}$, $R_i^+(s) \neq \emptyset$ and for every $s_k \in R_i^+(s)$ ($k = 1, \ldots, l$), there exists at least one $s'_k \in R_{N\setminus\{i\}}(s_k)$ with $s'_k \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$. Let $Q_k = \{q : q \in R_{N\setminus\{i\}}(s_k) \cap \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)\}$. It is obvious that $s'_k \in Q_k$. Hence, $Q_k \neq \emptyset$. Let $z \in Q_k$ and be DM i's least preferred in the state set Q_k. Since $z \in R_{N\setminus\{i\}}(s_k) \cap (\bigcup_{d=0}^{r} \Phi_i^{-(d)}(s))$, there exists $0 \leq r_k \leq r$ such that $z \in \Phi_i^{-(r_k)}(s)$ for $k = 1, \ldots, l$. Therefore, either $r_k = r$ or $R_{N\setminus\{i\}}(s_k) \cap (\bigcup_{d=r_k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$. This process is portrayed in Fig. 6.11.

Let $r_m = \min\{r_k : k = 1, \ldots, l\}$. Then, $0 \leq r_m \leq r$. It is easy to follow that if $s \in S_i^{GGMR}$ and $R_i^+(s) \neq \emptyset$, then $s \in S_i^{GMR_{r_m}}$. In fact, for every $s_k \in R_i^+(s)$, there exists at least one $s'_k \in R_{N\setminus\{i\}}(s_k)$ with $s'_k \in \Phi_i^{-(r_k)}(s)$. Since $0 \leq r_m \leq r_k$, then $s'_k \in \bigcup_{d=r_m}^{r} \Phi_i^{-(d)}(s)$, and $s'_m \in R_{N\setminus\{i\}}(s_m)$ with $s'_m \in \Phi_i^{-(r_m)}(s)$. Based on the rule of selecting r_m, either $r_m = r$ so that $s \in S_i^{GMR_r}$, or $R_{N\setminus\{i\}}(s_m) \cap (\bigcup_{d=r_m+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$ so that $s \in S_i^{GMR_{r_m}}$. From the above discussion, Eq. 6.6 is proven.

Equations 6.7 and 6.8 can be proven in a fashion similar to that just presented for Eq. 6.6. □

Let S_i^{Nash}, S_i^{GMR}, S_i^{SMR}, and S_i^{SEQ} denote the set of stable states for DM i for Nash, GMR, SMR, and SEQ stability, respectively, in the graph model for simple preference presented in Sect. 4.2.3. When $r = 1$, stabilities having multiple-degree preference degenerate to the stabilities presented in Sect. 4.2.3, which includes two types of preference. Specifically,

Theorem 6.4 *For the multiple levels of preference, when* $r = 1$, $S_i^{Super} \cup S_i^{Nash_0} \cup S_i^{Nash_1} = S_i^{Nash}$, $S_i^{Super} \cup S_i^{GMR_0} \cup S_i^{GMR_1} = S_i^{GMR}$, $S_i^{Super} \cup S_i^{SMR_0} \cup S_i^{SMR_{1+}} \cup S_i^{SMR_{1-}} = S_i^{SMR}$, *and* $S_i^{Super} \cup S_i^{SEQ_0} \cup S_i^{SEQ_1} = S_i^{SEQ}$.

Stability calculations for the preference structure for $r = 2$ in the graph model for multiple degrees of preference produces the same stability findings as found for the three types of preference or strength of preference framework. More specifically, let S_i^{SGMR}, S_i^{SSMR}, and S_i^{SSEQ} denote the set of strong stable states for strongly GMR, SMR, and SEQ stability, respectively, presented in Sect. 6.3.3.2. The stabilities at degree 2 in the graph model with three kinds of preference degenerate to the corresponding strong stabilities presented in Sect. 6.3.3.2, except for the states that are Nash stable, because Nash stable states are not considered in strong GMR, SMR, and SEQ stability in Sect. 6.3.3.2. Formally, this is expressed in the next theorem.

Theorem 6.5 *For the multiple degrees of preference, when* $r = 2$, $S_i^{GMR_2} \backslash S_i^{Nash_2} = S_i^{SGMR}$, $S_i^{SMR_{2+}} \backslash S_i^{Nash_2} = S_i^{SSMR}$, *and* $S_i^{SEQ_2} \backslash S_i^{Nash_2} = S_i^{SSEQ}$.

The previous two theorems can be easily proven using the appropriate stability definitions.

6.5 Matrix Representation of Stability Definitions for Three Degrees of Preference

The matrix representations for conflict resolution for simple preference and unknown preference are presented in Sects. 4.3 and 5.3, respectively. It is natural to extend the logical form for conflict resolution under the three types of preference presented in Sect. 6.3 to the matrix representation. Following definitions for preference matrices and a reachability matrix in Sect. 6.5.1, matrix representation for various stability definitions for the two DM and n-DM cases for both general and strong stabilities are presented in Sects. 6.5.2 and 6.5.4, respectively, for three degrees of preference.

6.5.1 Preference Matrices Including Strength of Preference

Preference information is an important component in the Graph Model for Conflict Resolution under the three types of preference. Preference matrices corresponding to the preference information are constructed now.

Let $m = |S|$ denote the number of states. For DM i, a mild or strong unilateral improvement matrix (MSUI matrix) $J_i^{+,++}$ is an $m \times m$ matrix defined by

$$J_i^{+,++}(s,q) = \begin{cases} 1 & \text{if } q \in R_i^{+,++}(s), \\ 0 & \text{otherwise,} \end{cases} \tag{6.9}$$

where $R_i^{+,++}(s)$ stands for DM i's reachable list from states s by a MSUI as defined at the start of Sect. 6.2.1. Because the elements of the vector e_s are assigned a value of zero except for the element connected to state s where a value of 1 is given, $R_i^{+,++}(s) = e_s^T \cdot J_i^{+,++}$, if $R_i^{+,++}(s)$ is written as a 0-1 row vector, where a "1" at the jth element indicates DM i has a MSUI from s to s_j. The MSUI matrix $J_i^{+,++}$ depicts DM i's mild or strong unilateral improvements in one step. To carry out a stability analysis, a set of matrices corresponding to strength of preference is constructed next. Specifically,

$$P_i^{++}(s,q) = \begin{cases} 1 & \text{if } q \gg_i s, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P_i^{--}(s,q) = \begin{cases} 1 & \text{if } s \gg_i q, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $(P_i^{++})^T = P_i^{--}$, where T denotes the transpose of a matrix.

$$P_i^{--,-,=}(s,q) = \begin{cases} 1 & \text{if } q \ll_i s, q <_i s, \text{ or } (q \sim_i s \text{ and } q \neq s), \\ 0 & \text{otherwise.} \end{cases}$$

and

$$P_i^{+,++}(s,q) = \begin{cases} 1 & \text{if } q >_i s \text{ or } q \gg_i s, \\ 0 & \text{otherwise.} \end{cases}$$

For three-degree preference, $P_i^{--,-,=}(s,q) = 1 - P_i^{+,++}(s,q)$ for $s, q \in S$ and $s \neq q$.

Based on the aforementioned definitions, for DM i, a set of adjacency matrices, J_i and $J_i^{+,++}$, and preference matrix $P_i^{+,++}$ have the following relationship between them:

$$J_i^{+,++} = J_i \circ P_i^{+,++},$$

where "∘" denotes the Hadamard product given in Definition 3.15.

6.5.2 Two Decision Maker Case

6.5.2.1 Matrix Representation of General Stabilities

Equivalent matrix representations of the aforementioned logical definitions for Nash stability, GGMR, GSMR, and GSEQ in a two-DM graph model can be determined directly by using the matrices containing information regarding possible moves such as $J_i^{+,++}$ and those keeping track of preferences. Let $i \in N$, $|N| = 2$, and $m = |S|$.

Let E denote an $m \times m$ matrix with each entry equal to 1. Define the $m \times m$ Nash matrix M_i^{Nash} as

$$M_i^{Nash} = J_i^{+,++} \cdot E.$$

Theorem 6.6 *State $s \in S$ is Nash stable for DM i iff $M_i^{Nash}(s, s) = 0$.*

Note that Theorem 6.6 provides a matrix method to assess whether state s is Nash stable for DM i by identifying the Nash matrix's diagonal entry $M_i^{Nash}(s, s)$.

For the case of general GMR stability, define the $m \times m$ matrix M_i^{GGMR} as

$$M_i^{GGMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--,-,=})^T\right)],$$

where E is an $m \times m$ matrix with each entry having a value of 1.

Theorem 6.7 *State s is general GMR (GGMR) for DM i iff $M_i^{GGMR}(s, s) = 0$.*

Proof Since $M_i^{GGMR}(s, s) = (e_s^T \cdot J_i^{+,++}) \cdot [\left(E - sign\left(J_j \cdot (P_i^{--,-,=})^T\right)\right) \cdot e_s]$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s, s_1)[1 - sign\left((e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{--,-,=})^T\right)],$$

then $M_i^{GGMR}(s, s) = 0$ iff $J_i^{+,++}(s, s_1)[1 - sign\left((e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{--,-,=})^T\right)] = 0$, for $\forall s_1 \in S$. This implies that $M_i^{GGMR}(s, s) = 0$ iff

$$(e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{--,-,=})^T \neq 0, \forall s_1 \in R_i^{+,++}(s). \tag{6.10}$$

From Eq. 6.10, for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector $e_{s_1}^T \cdot J_j$ has a value of 1 for the s_2th element and the m-dimensional column vector $(P_i^{--,-,=})^T \cdot e_s$ has an entry of 1 for the s_2th element.

Therefore, $M_i^{GGMR}(s, s) = 0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$. □

In order to consider the general SMR stability, define the $m \times m$ matrix M_i^{GSMR} as $M_i^{GSMR} = J_i^{+,++} \cdot [E - sign(F)]$, with

$$F = J_j \cdot [(P_i^{--,-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+,++})^T\right)\right)].$$

Theorem 6.8 *State s is general SMR (GSMR) for DM i iff* $M_i^{GSMR}(s,s) = 0$.

Proof Let $G = (P_i^{--,-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+,++})^T\right)\right)$.

Since $M_i^{GSMR}(s,s) = (e_s^T \cdot J_i^{+,++}) \cdot [(E - sign(F)) \cdot e_s]$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s,s_1)[1 - sign\,(F(s_1,s))]$$

with

$$F(s_1,s) = \sum_{s_2=1}^{m} J_j(s_1,s_2) \cdot G(s_2,s),$$

and $G(s_2,s) = P_i^{--,-,=}(s,s_2)[1 - sign\left(\sum_{s_3=1}^{m}\left(J_i(s_2,s_3)P_i^{+,+}(s,s_3)\right)\right)]$, thus, $M_i^{GSMR}(s,s) = 0$ holds iff $F(s_1,s) \neq 0$, $\forall s_1 \in R_i^{+,++}(s)$, which is equivalent to the statement that, $\forall s_1 \in R_i^{+,++}(s)$, $\exists s_2 \in R_j(s_1)$ such that

$$P_i^{--,-,=}(s,s_2) \neq 0, \tag{6.11}$$

and

$$\sum_{s_3=1}^{m}\left(J_i(s_2,s_3)P_i^{+,++}(s,s_3)\right) = 0. \tag{6.12}$$

Obviously, for $\forall s_1 \in R_i^{+,++}(s)$, $\exists s_2 \in R_j(s_1)$, Eq. 6.11 holds iff $s_2 \in \Phi_i^{--,-,=}(s)$. For $\forall s_1 \in R_i^{+,++}(s)$, $\exists s_2 \in R_j(s_1)$, Eq. 6.12 holds iff for all $s_3 \in R_i(s_2)$, $P_i^{+,++}(s,s_3) = 0$ which implies $s_3 \in \Phi_i^{--,-,=}(s)$.

Therefore, $M_i^{GSMR} = 0$ iff for every $s_1 \in R_i^{+,++}(s)$ there exists $s_2 \in R_j(s_1)$ such that $s_2 \in \Phi_i^{--,-,=}(s)$ and $s_3 \in \Phi_i^{--,-,=}(s)$ for all $s_3 \in R_i(s_2)$. □

In order to analyze general SEQ stability using matrix approach, define the $m \times m$ matrix M_i^{GSEQ} as

$$M_i^{GSEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--,-,=})^T\right)].$$

Theorem 6.9 *State s is general SEQ (GSEQ) for DM i iff* $M_i^{GSEQ}(s,s) = 0$.

Proof Since $M_i^{GSEQ}(s,s) = (e_s^T \cdot J_i^{+,++}) \cdot [\left(E - sign\left(J_j^{+,++} \cdot (P_i^{--,-,=})^T\right)\right) \cdot e_s]$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot J_j^{+,++}) \cdot (e_s^T \cdot P_i^{--,-,=})^T\right)],$$

then $M_i^{GSEQ}(s,s) = 0$ iff for any $s_1 \in S$,

$$J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot J_j^{+,++}) \cdot (e_s^T \cdot P_i^{--,-,=})^T\right)] = 0.$$

This implies that $M_i^{GSEQ}(s,s) = 0$ iff

$$(e_{s_1}^T \cdot J_j^{+,++}) \cdot (e_s^T \cdot P_i^{--,-,=})^T \neq 0, \forall s_1 \in R_i^{+,++}(s). \tag{6.13}$$

By Eq. 6.13, for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector $e_{s_1}^T \cdot J_j^{+,++}$ has the s_2th element 1 and the m-dimensional column vector $(P_i^{--,-,=})^T \cdot e_s$ has the s_2th element 1.

Therefore, $M_i^{GSEQ}(s,s) = 0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R_j^{+,++}(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$. □

6.5.2.2 Matrix Representation of Strong Stabilities

Corresponding to the logical representation of strong stabilities for three degrees of preference, matrix representation of strong GMR, SMR, and SEQ stabilities are presented below according to the degree of sanctioning. For three kinds of preference, these stabilities are divided into strongly and weakly stable with respect to the strength of possible sanctions. Hence, if a particular state s is general stable, then s is either strongly stable or weakly stable. Strong and weak stabilities only include GMR, SMR, and SEQ because Nash stability does not involve sanctions.

In the upcoming theorems, let $i \in N$, $|N| = 2$, and $m = |S|$. To consider strong GMR stability, define the $m \times m$ matrix M_i^{SGMR} as

$$M_i^{SGMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--})^T\right)].$$

Theorem 6.10 *State $s \in S$ is strong general metarational $(SGMR)$ for DM i iff $M_i^{SGMR}(s,s) = 0$.*

In order to analyze strong SMR stability, define the $m \times m$ matrix M_i^{SSMR} as

$$M_i^{SSMR} = J_i^{+,++} \cdot [E - sign(J_j \cdot F)],$$

with

$$F = (P_i^{++}) \circ [E - sign\left(J_i \cdot (E - P_i^{++})\right)].$$

Theorem 6.11 *State $s \in S$ is strong symmetric metarational $(SSMR)$ for DM i iff $M_i^{SSMR}(s,s) = 0$.*

In order to calculate strong SEQ, define the $m \times m$ matrix M_i^{SSEQ} as

$$M_i^{SSEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--})^T\right)].$$

Theorem 6.12 *State $s \in S$ is strong sequentially stable $(SSEQ)$ for DM i iff* $M_i^{SSEQ}(s, s) = 0$.

The proofs of Theorems 6.10–6.12 are similar to those of the three general stabilities presented in Theorems 6.7–6.9, respectively, in Sect. 6.5.2.1.

Let GS denote a graph model stability, GMR, SMR, or SEQ. The symbols GGS, SGS, and WGS respectively represent a general stability, GGMR, GSMR, or GSEQ, the strong stability, SGMR, SSMR, or SSEQ, and the weak stability, WGMR, WSMR, or WSEQ, under three degrees of preference. M_i^{GGS} and M_i^{SGS} denote DM i's general stability matrix, M_i^{GGMR}, M_i^{GSMR}, or M_i^{GSEQ}, and DM i's strong stability matrix, M_i^{SGMR}, M_i^{SSMR}, or M_i^{SSEQ}, respectively. Based on the notation, one has the following theorem.

Theorem 6.13 *State $s \in S$ is weak stable (WGS) for DM i iff $M_i^{GGS}(s, s) = 0$, but* $M_i^{SGS}(s, s) \neq 0$.

Theorem 6.13 means that if s is general stable, but not strong stable for a GS stability, then s is weak stable for the GS stability.

Example 6.2 (Stabilities for the Extended Sustainable Development Model under Three-degree Preference by using Matrix Representation) The sustainable development conflict is explained in Example 3.1. In this illustration, this conflict is expended to include three degrees of preference for the two-DM case. Specifically, the conflict consists of two DMs: an environmental agency (DM 1: E) and a developer (DM 2: D); and two options: DM 1 controls the option of being proactive (labeled P) and DM 2 has the option of practicing sustainable development (labeled SD) for properly treating the environment. The two options are combined to form four feasible states: s_1, s_2, s_3, and s_4. These results are listed in Table 6.8, where a "Y" indicates that an option is selected by the DM controlling it and an "N" means that the option is not chosen.

From Table 6.8, DM 1 and DM 2's preference information includes strength. The graph model for the extended sustainable development conflict is presented in Fig. 6.12.

From the graph model, the UM adjacency matrices for each DM are constructed by

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Table 6.8 Extended sustainable development game in option form

DM 1: Environmental agency				
1. Proactive (P)	Y	Y	N	N
DM 2: Developer				
2. Sustainable development (SD)	Y	N	Y	N
States	s_1	s_2	s_3	s_4

Preferences $s_1 >_1 s_3 \gg_1 s_2 \sim_1 s_4$ for DM 1 and
$s_3 >_2 s_1 \gg_2 s_4 \sim_2 s_2$ for DM 2.

The preference matrices for the DMs 1 and 2 are given by

$$P_1^{++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \; P_1^{+,++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$P_2^{++} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{ and } P_2^{+,++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, $J_i^{+,++} = J_i \circ P_i^{+,++}$, $P_i^{--,-,=} = E - I - P_i^{+,++}$, and $P_i^{--} = (P_i^{++})^T$ for $i = 1, 2$.

The stability matrices used by Theorems 6.6–6.13 are included in Table 6.9, which are employed to calculate the general stabilities of Nash, GMR, SMR, and SEQ, as well as the strong stabilities of SGMR, SSMR, and SSEQ for two-DM conflicts, respectively.

The stable states and equilibria for the sustainable development conflict are summarized in Table 6.10, in which "$\surd$" for a given state means that this state is stable for DM 1 or DM 2 and "Eq" is an equilibrium for an appropriate solution concept.

The results provided by Table 6.10 show that state s_1 is strong equilibrium for the four basic stabilities. State s_3 is strongly stable for GMR and SMR. Hence, s_1 and s_3 are better choices for decision makers.

6.5.3 Reachability Matrix Under Strength of Preference

An important matrix corresponding to the reachable list under three degrees of preference is now defined. Fix coalition $H \subseteq N$ such that $|H| \geq 2$, and let $s \in S$. In order to construct the reachability matrix corresponding to $R_H^{+,++}(s)$ presented in Definition 6.9, the reachable list of H from s by legal sequences of MSUIs, the t-step reachability matrix is defined as follows.

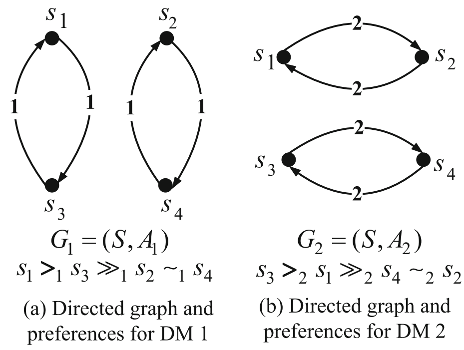


Fig. 6.12 Graph model for the extended sustainable development conflict under three-degree preference

Table 6.9 Stability matrices under three degrees of preference in two decision maker case

Category	Stability matrices
General stabilities	$M_i^{Nash} = J_i^{+,++} \cdot E$
	$M_i^{GGMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--,-,=})^T\right)]$
	$M_i^{GSMR} = J_i^{+,++} \cdot [E - sign(F)]$ with $F = J_j \cdot [(P_i^{--,-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+,++})^T\right)\right)]$
	$M_i^{GSEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--,-,=})^T\right)]$
Strong stabilities	$M_i^{SGMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--})^T\right)]$
	$M_i^{SSMR} = J_i^{+,++} \cdot [E - sign(J_j \cdot F)]$ with $F = (P_i^{++}) \circ [E - sign\left(J_i \cdot (E - P_i^{++})\right)]$
	$M_i^{SSEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--})^T\right)]$
Weak stabilities	$S_i^{WGS} = S_i^{GGS} - S_i^{SGS}$

Definition 6.52 For $i \in H$, $H \subseteq N$, and $t = 1, 2, 3, \ldots$, define the $m \times m$ matrix $M_i^{(H,t,+,++)}$ with (s, q) entries as follows:

$$M_i^{(H,t,+,++)}(s,q) = \begin{cases} 1 & \text{if } q \in S \text{ is reachable from } s \in S \text{ in exactly} \\ & t \text{ legal MSUIs by } H \text{ with last mover DM } i, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to Lemma 5.1, one has

Lemma 6.1 *For $i \in H$ and $H \subseteq N$, the matrix $M_i^{(H,t,+,++)}$ satisfies that*

$$for\ t = 2, 3, \ldots, M_i^{(H,t,+,++)} = sign[(\bigvee_{j \in H - \{i\}} M_j^{(H,t-1,+,++)}) \cdot J_i^{+,++}].$$

Table 6.10 Stability results of the sustainable development conflict with strength of preference

State	Nash			GGMR			GSMR			GSEQ			SGMR			SSMR			SSEQ			WGMR			WSMR			WSEQ		
	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq
s_1	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√									
s_2	√			√			√			√			√			√			√											
s_3		√		√	√	√	√	√	√		√		√	√	√	√	√	√		√										
s_4	√			√			√			√			√			√			√											

with $M_i^{(H,1,+,++)}(s,q) = J_i^{+,++}(s,q)$.

The proof of this lemma is similar to that of Lemma 5.1.

The UM and UI reachability matrices are given in Definition 4.19 in Chap. 4. The MSUI reachability matrix is now similarly defined for a graph model having three degrees of preference.

Definition 6.53 For the graph model G, the MSUI reachability matrix for H is the $m \times m$ matrix $M_H^{+,++}$ with (s,q) entry

$$M_H^{+,++}(s,q) = \begin{cases} 1 & \text{if } q \in R_H^{+,++}(s), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $R_H^{+,++}(s) = \{q : M_H^{+,++}(s,q) = 1\}$. If $R_H^{+,++}(s)$ is written as a 0-1 row vector, then

$$R_H^{+,++}(s) = e_s^T \cdot M_H^{+,++},$$

where e_s^T denotes the transpose of the sth standard basis vector of the m-dimensional Euclidean space. Therefore, the MSUI reachability matrix for coalition H, $M_H^{+,++}$, can be used to calculate the reachable lists of H from state s by the legal sequence of MSUIs, $R_H^{+,++}(s)$.

Let $L_4 = |\bigcup_{i \in H} A_i^{+,++}|$, where $A_i^{+,++}$ is DM i's MSUI oriented arcs, representing mild or strong unilateral improvements by DM i in coalition H. Then the following theorem can be derived using Lemma 6.1.

Theorem 6.14 *Let* $L_4 = |\bigcup_{i \in N} A_i^{+,++}|$, $s \in S$, $H \subseteq N$, *and* $H \neq \emptyset$. *The MSUI reachability matrix* $M_H^{+,++}$ *by* H *can be expressed as*

$$M_H^{+,++} = \bigvee_{t=1}^{L_4} \bigvee_{i \in H} M_i^{(H,t,+,++)}. \tag{6.14}$$

Proof To prove Eq. 6.14, assume that $C = \bigvee_{t=1}^{L_4} \bigvee_{i \in H} M_i^{(H,t,+,++)}$. Using the definition for matrix $M_H^{+,++}$, $M_H^{+,++}(s,q) = 1$ iff $q \in R_H^{+,++}(s)$. Using Definition 6.9, $q \in R_H^{+,++}(s)$ implies that there exists $1 \leq t_0 \leq L_4$ and $i_0 \in H$ such that $M_{i_0}^{(H,t_0,+,++)}(s,q) = 1$. This implies that matrix C has (s,q) entry 1. Therefore, $M_H^{+,++}(s,q) = 1$ iff $C(s,q) = 1$. Since $M_H^{+,++}$ and C are 0-1 matrices, it follows that $M_H^{+,++} = C$. □

6.5.4 n-Decision Maker Case

6.5.4.1 Matrix Representation of General Stabilities

Matrix representations of solution concepts with three degrees of preference for 2-DM cases presented in Sect. 6.5.2 are now extended to n-DM situations. Nash stability definitions are identical for both the 2-DM and the n-DM models, because Nash stability does not consider opponents' responses. In this subsection, let $i \in N$ and $|N| = n$.

In order to consider general GMR stability for n-DMs, define the $m \times m$ general GMR matrix M_i^{GGMR} as

$$M_i^{GGMR} = J_i^{+,++} \cdot [E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{--,-,=})^T\right)].$$

In order to avoid using a complex notation, the symbol used for the general GMR matrix representation for the n-DM situation is the same as that employed for the 2-DM case in Sect. 6.5.2.1. The context in which the definition is being utilized will clearly indicate whether it is for the 2-DM or n-DM situation. The same comments hold for the other definitions given in this section as well as Sect. 6.5.4.2. Because the proofs of the next three theorems are similar to the GMR, SMR, and SEQ stabilities presented in Sect. 6.5.2.2 for 2-DM models, the proofs are not given for the n-DM case.

Theorem 6.15 *State s is general GMR for DM i iff $M_i^{GGMR}(s, s) = 0$.*

The above matrix method, called matrix representation of general GMR stability, is equivalent to the logical representation for general GMR stability given in Definition 6.11. To analyze general GMR stability of state s for DM i, one only needs to check if the entry, $M_i^{GMR}(s, s)$, in the GMR matrix is zero. If so, state s is general GMR stable for i; otherwise, s is general GMR unstable for DM i. Note that all information about general GMR stability is contained in the diagonal entries of the general GMR matrix.

To analyze general SMR stability, define the $m \times m$ general SMR matrix M_i^{GSMR} as $M_i^{GSMR} = J_i^{+,++} \cdot [E - sign(Q)]$, with

$$Q = M_{N\backslash\{i\}} \cdot [(P_i^{--,-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+,++})^T\right)\right)].$$

Theorem 6.16 *State s is general SMR for DM i iff $M_i^{GSMR}(s, s) = 0$.*

Theorem 6.16 indicates that the matrix representation of general SMR stability is equivalent to the logical representation for general SMR stability presented in Definition 6.12. To calculate general SMR stability of state s for DM i, one only has to assess whether the diagonal entry, $M_i^{GSMR}(s, s)$, of DM i's general SMR matrix is zero. If so, state s is general SMR stable for i; otherwise, s is general SMR unstable for DM i.

General sequential stability is similar to general GMR stability, but includes only those sanctions that are "credible". Define the $m \times m$ general SEQ matrix M_i^{GSEQ} as

$$M_i^{GSEQ} = J_i^{+,++} \cdot [E - sign\left(M_{N\backslash\{i\}}^{+,++} \cdot (P_i^{--,-,=})^T\right)].$$

Theorem 6.17 *State s is general SEQ for DM i iff* $M_i^{GSEQ}(s, s) = 0$.

Similar to the previous two theorems, the matrix representation of SEQ stability is equivalent to the logical version given in Definition 6.13. When the diagonal entry at (s, s) is zero, the state s under consideration is SEQ stable for DM i.

6.5.4.2 Matrix Representation of Strong Stabilities

Similar to the two-DM case, matrix representations of general stabilities under the three degrees of preference for n-DMs include matrix versions of strong or weak stability. First, construct matrices J_i and $J_i^{+,++}$ using Definition 4.13 and Eq. 6.9. The matrices M_H and $M_H^{+,++}$ are calculated utilizing Theorems 4.9 and 7.5, for which $H = N\backslash\{i\}$. For convenience, the same notation employed for the two-DM situation in Sect. 6.5.2.2 is used for the n-DM case.

Define the $m \times m$ strong GMR matrix M_i^{SGMR} for DM i as

$$M_i^{SGMR} = J_i^{+,++} \cdot [E - sign\big(M_{N\backslash\{i\}} \cdot (P_i^{--})^T\big)].$$

Theorem 6.18 *State* $s \in S$ *is strong GMR (SGMR) for DM i, denoted by* $s \in S_i^{SGMR}$, *iff* $M_i^{SGMR}(s, s) = 0$.

Proof Since $M_i^{SGMR}(s, s) = (e_s^T \cdot J_i^{+,++}) \cdot [\big(E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{--})^T\right)\big) \cdot e_s]$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s, s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T\right)],$$

then

$$M_i^{SGMR}(s, s) = 0 \Leftrightarrow J_i^{+,++}(s, s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T\right)] = 0, \forall s_1 \in S.$$

This implies that $M_i^{SSGM}(s, s) = 0$ iff

$$(e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T \neq 0, \forall s_1 \in R_i^{+,++}(s). \tag{6.15}$$

By Eq. 6.15, for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector, $e_{s_1}^T \cdot M_{N\backslash\{i\}}$, has the s_2th element 1 and the m-dimensional column vector, $(P_i^{--})^T \cdot e_s$, has the s_2th element 1.

Therefore, $M_i^{SGMR}(s,s)=0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s \gg_i s_2$. □

For strong SMR, the n-DM model is similar to the two-DM case. The only modification is that responses to block improvements by DM i can come from more than one of DM i's opponents instead of from a single DM.

If $F=(P_i^{++})\circ[E-sign\big(J_i\cdot(E-P_i^{++})\big)]$, then one can define the $m\times m$ strong SMR matrix M_i^{SSMR} for DM i as

$$M_i^{SSMR}=J_i^{+,++}\cdot[E-sign(M_{N\setminus\{i\}}\cdot F)].$$

Theorem 6.19 *State $s\in S$ is strong SMR (SSMR) for DM i, denoted by $s\in S_i^{SSMR}$, iff $M_i^{SSMR}(s,s)=0$.*

Proof Let $Q=M_{N\setminus\{i\}}\cdot F$. Since $M_i^{SSMR}(s,s)=(e_s^T\cdot J_i^{+,++})\cdot[(E-sign(Q))\cdot e_s]$

$$=\sum_{s_1=1}^{m}J_i^{+,++}(s,s_1)[1-sign(Q(s_1,s))]$$

then $M_i^{SSMR}(s,s)=0$ iff $J_i^{+,++}(s,s_1)[1-sign(Q(s_1,s))]=0$, for any $s_1\in S$. This means that $M_i^{SSMR}(s,s)=0$ iff

$$(e_{s_1}^T\cdot M_{N\setminus\{i\}})\cdot(F\cdot e_s)\neq 0,\ \forall s_1\in R_i^{+,++}(s). \tag{6.16}$$

Since $(e_{s_1}^T\cdot M_{N\setminus\{i\}})\cdot(F\cdot e_s)=\sum_{s_2=1}^{m}M_{N\setminus\{i\}}(s_1,s_2)\cdot F(s_2,s)$, then Eq. 6.16 holds iff for any $s_1\in R_i^{+,++}(s)$, there exists $s_2\in R_{N\setminus\{i\}}(s_1)$ such that $F(s_2,s)\neq 0$.

Because $F(s_2,s)=P_i^{++}(s_2,s)\cdot[1-sign(\sum_{s_3=1}^{m}J_i(s_2,s_3)(1-P_i^{++}(s_3,s)))]$, $F(s_2,s)\neq 0$ implies that for $s_2\in R_{N\setminus\{i\}}(s_1)$,

$$P_i^{++}(s_2,s)\neq 0 \tag{6.17}$$

and

$$\sum_{s_3=1}^{m}J_i(s_2,s_3)(1-P_i^{++}(s_3,s))=0. \tag{6.18}$$

Equation 6.17 is equivalent to the statement that, $\forall s_1\in R_i^{+,++}(s)$, $\exists s_2\in R_{N\setminus\{i\}}(s_1)$ such that $s\gg_i s_2$. Equation 6.18 is the same as the statement that, $\forall s_1\in R_i^{+,++}(s)$, $\exists s_2\in R_{N\setminus\{i\}}(s_1)$ such that $P_i^{++}(s_3,s)\neq 0$, for $\forall s_3\in R_i(s_2)$. Based on the definition of the $m\times m$ preference matrix P_i^{++}, one knows that $P_i^{++}(s_3,s)\neq 0 \Leftrightarrow s\gg_i s_3$.

Therefore, one can conclude from the above discussion that $M_i^{SMR}(s,s)=0$ iff for any $s_1\in R_i^{+,++}(s)$, there exists at least one $s_2\in R_{N\setminus\{i\}}(s_1)$ with $s\gg_i s_2$ and $s\gg_i s_3$ for all $s_3\in R_i(s_2)$. □

Strong sequential stability examines the credibility of the sanctions by DM i's opponents, in the sense that opponents will not move to less preferred situations to block improvements by DM i.

First, find matrix $M^{+,++}_{N\backslash\{i\}}$ using Theorem 7.5 for $H = N\backslash\{i\}$. Define the $m \times m$ strong SEQ matrix M_i^{SSEQ} for DM i as

$$M_i^{SSEQ} = J_i^{+,++} \cdot [E - sign\big(M^{+,++}_{N\backslash\{i\}} \cdot (P_i^{--})^T\big)].$$

Theorem 6.20 *State $s \in S$ is strong SEQ (SSEQ) for DM i, denoted by $s \in S_i^{SSEQ}$, iff $M_i^{SSEQ}(s,s) = 0$.*

Proof Since $M_i^{SSEQ}(s,s) = (e_s^T \cdot J_i^{+,++}) \cdot [\Big(E - sign\left(M^{+,++}_{N\backslash\{i\}} \cdot (P_i^{--})^T\right)\Big) \cdot e_s]$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s,s_1)[1 - sign\left(e_{s_1}^T \cdot M^{+,++}_{N\backslash\{i\}} \cdot (e_s^T \cdot P_i^{--})^T\right)],$$

then

$$M_i^{SSEQ}(s,s) = 0 \Leftrightarrow J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M^{+,++}_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T\right)] = 0, \forall s_1 \in S.$$

This implies that $M_i^{SSEQ}(s,s) = 0$ iff

$$(e_{s_1}^T \cdot M^{+,++}_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T \neq 0, \forall s_1 \in R_i^{+,++}(s). \tag{6.19}$$

By Eq. 6.19, for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector, $e_{s_1}^T \cdot M^{+,++}_{N\backslash\{i\}}$, has the s_2th element 1 and the m-dimensional column vector, $(P_i^{--})^T \cdot e_s$, has the s_2th element 1.

Therefore, $M_i^{SSEQ}(s,s) = 0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R^{+,++}_{N\backslash\{i\}}(s_1)$ with $s \gg_i s_2$. □

Theorems 6.18–6.20 indicate that the matrix representation of strong solution concepts are equivalent to the strong stability definitions in the logical forms presented in Sect. 6.3.3.2. When $n = 2$, Theorems 6.18–6.20 are reduced to those theorems presented in Sect. 6.5.2.2.

6.6 Application: The Garrison Diversion Unit (GDU) Conflict

In this section, the four-degree version of stability definitions presented in Sect. 6.4 is applied to the Garrison Diversion Unit (GDU) conflict to illustrate how the procedure works. For combination with a brief overview of this international environmental dispute between Canada and United States, a conflict model in terms of DMs, options,

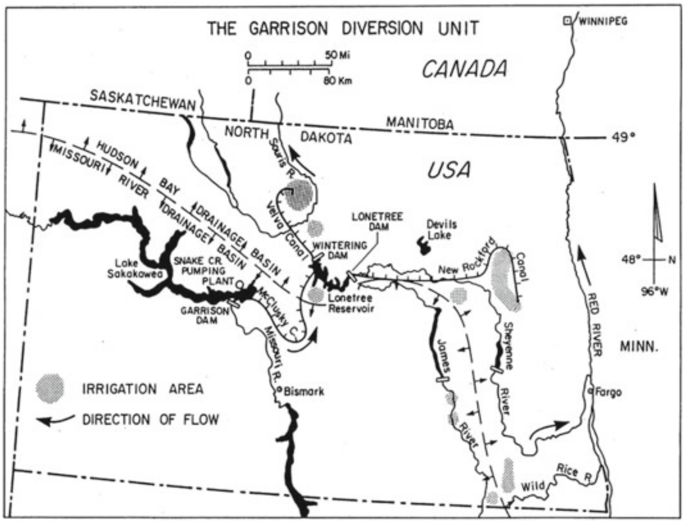


Fig. 6.13 Garrison Diversion Unit (GDU)

and preferences is constructed in the next subsection. Subsequently, a stability analysis is executed for four degrees of preference utilizing the calibrated model and insights regarding the stability results are discussed.

6.6.1 Model of the GDU Conflict

The history of the GDU conflict dates back to the nineteenth century. In order to irrigate land in the northeastern region of the American State of North Dakota, an irrigation project was proposed by the **United States Support (USS)** regarding construction of the McClusky Canal to transfer an immense amount of water from the Missouri River Basin to the Hudson Bay Basin as depicted in Fig. 6.13, which originally appeared in Fraser and Hipel (1984). From the Lonetree Reservoir, water can be conveyed to the planned irrigation areas marked on the map. Eventually, the irrigation runoff would flow into the Canadian province of Manitoba via the Red and Souris rivers. This irrigation initiative is called the Garrison Diversion Unit project. Among other problems, biologists were concerned that foreign biota from the Missouri River Basin could adversely affect biological species in the Hudson Bay Drainage Basin and could, for example, decimate fish species in Lake Winnipeg and thereby destroy the fishing industry. The GDU conflict was strategically analyzed using metagame analysis, conflict analysis and the graph model by Hipel and Fraser (1980), Fraser and Hipel (1984), and Fang et al. (1993) using two-degree preference. Later, Hamouda et al. (2006) examined a simplified version of the GDU dispute for three degrees of preference. In this simpler conflict, the **Canadian Opposition**

Table 6.11 Feasible states for the GDU model

USS									
1. Proceed	Y	Y	N	Y	N	Y	N	Y	N
2. Modify	N	N	Y	N	Y	N	Y	N	Y
CDO									
3. Legal	N	N	N	Y	Y	N	N	Y	Y
IJC									
4. Completion	N	Y	Y	Y	Y	N	N	N	N
5. Modification	N	N	N	N	N	Y	Y	Y	Y
State number	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9

(CDO) was considering whether or not to oppose the project because of the potential negative environmental impacts that Canada would suffer. Based on the Boundary Water Treaty of 1909 between Canada and the United States, the **International Joint Commission (IJC)** consisting of representatives from the governments of the USA and Canada was called upon by both nations to carry out unbiased studies and make recommendations regarding the proposed GDU project.

The graph model for the simplified GDU conflict is comprised of three DMs: 1. **USS**, 2. **CDO**, and 3. **IJC**; and five options: 1. **Proceed**−Proceed with the project regardless of Canada's concerns; 2. **Modify**−Modify the project to reduce impacts on Canada; 3. **Legal**−Legal action by CDO based on the Boundary Waters Treaty; 4. **Completion**−IJC recommends completion of the project as originally planned; and 5. **Modification**−IJC stipulates modification of the project to reduce environmental impacts on Canada. Each of these three DMs followed by the option or options under its control are listed as the left column in Table 6.11. As explained in Sect. 3.1.2, when using the option form, a state is defined as a selection of options for each DM. Since there are five options in the GDU dispute, a total of $2^5 = 32$ states is mathematically possible. However, some states can be removed because they cannot possibly occur in reality. For instance, because options 4 and 5 are mutually exclusive, these two options cannot be selected together. Likewise, options 1 and 2 are mutually exclusive for USS. Moreover, it is assumed that the USS will do something and thereby choose one of its options. After all of the infeasible states are eliminated, only nine states are identified as being feasible. The feasible states are designated as columns of Ys and Ns on the right side of Table 6.11 in which a "Y" indicates that an option is selected by the DM controlling it and an "N" means that the option is not chosen.

The integrated graph model of the GDU conflict is shown in Fig. 6.14, in which a label on an arc indicates the DM who controls the move. Notice, for instance, that USS controls movement between states s_2 and s_3. From Table 6.11, one can see that for states s_2 and s_3 the option selections for USS change while the option choices by the other DMs, consisting of CDO and IJC, remain the same. All that is still required for a graph model is knowledge of each DM's preferences over the feasible states for the situation of four-degree preference in the GDU conflict. The preference

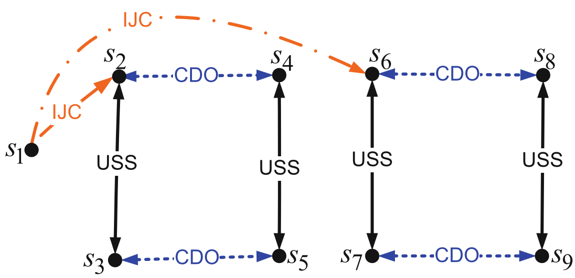


Fig. 6.14 The integrated graph model for movement in the GDU conflict

Table 6.12 Four-degree preferences for DMs in the GDU conflict

DM	Preference
USS	$s_2 > s_4 > s_3 > s_5 > s_1 > s_6 > s_9 > s_7 \ggg s_8$
CDO	$\{s_3 \sim s_7\} > \{s_5 \sim s_9\} > \{s_4 \sim s_8\} \ggg \{s_1 \sim s_2 \sim s_6\}$
IJC	$\{s_2 \sim s_3 \sim s_4 \sim s_5 \sim s_6 \sim s_7 \sim s_8 \sim s_9\} \gg s_1$

information for this conflict over the feasible states is given in Table 6.12, where $>$, $\gg$, and $\ggg$ mean more preferred, strongly preferred, and very strongly preferred, respectively, and equally preferred states are given in brackets and connected using the symbol $\sim$. The fact that states are ranked from most preferred on the left to least preferred on the right, where ties are allowed, indicates that the preferences are transitive for this application. One can see that state s_8 is very strongly less preferred to all other states for USS, because at state s_8 the USS is proceeding to construct the full project while IJC recommends a modified version and CDO is taking legal action based on the Boundary Waters Treaty. The DM CDO considers states s_1, s_2, and s_6 to be equally preferred and very strongly less preferred relative to all other states. Note that this representation of preference information presented in Table 6.12 implies that the preferred relations, $>$, $\gg$, and $\ggg$ are transitive. For instance, since $s_9 > s_7$ and $s_7 \ggg s_8$, then $s_9 \ggg s_8$ for USS. However, in general, the preference structure presented in this book does not require the transitivity of preference relations, and hence can handle intransitive preferences.

6.6.2 Stability Analysis Under Four-Degree Preference

Formally, in a stability analysis, one determines the stability of each state for each DM for various solution concepts. Here, four-degree versions of five stability definitions consisting of super stability; Nash stability, $Nash_k$; general metarationality, GMR_k; symmetric metarationality, SMR_k; and sequential stability, SEQ_k, for $k = 0, 1, 2,$ and 3, are employed to obtain stability results for the GDU conflict. An equilibrium for degree k for a specific solution concept represents a likely resolution to the conflict, since it is stable for every DM according to the stability definition under consideration. Note that the super stable states are treated as Nash stable at the highest level when determining an equilibrium in the graph model with multiple degrees of preference.

To explain how a stability calculation is carried out, consider SMR_k stability for state s_5 from DM 2's viewpoint for $k = 0, 1, 2,$ and 3. Using the definition of a reachable list presented in Sect. 6.3.2 and Table 6.12, $R_2^+(s_5) = \{s_3\}$ and $R_{N\backslash 2}(s_3) = \{s_2\}$ with $s_5 \ggg_2 s_2$ and $s_5 >_2 s_4$ for $R_2(s_2) = \{s_4\}$. Therefore, according to Definition 6.48 state s_5 is stable for SMR_{3^-}. Other cases can be analyzed similarly. The stability results for the GDU conflict are summarized in Table 6.13, in which "$\surd$" for a given state under a DM means that this state is stable at a given degree for the particular DM; "$\surd^{k^+}$" and "$\surd^{k^-}$" for a given state under a DM means that this state is SMR_{k^+} or SMR_{k^-} stable for the specified DM; and "$\surd^k$" for a state under "Eq" signifies that this state is an equilibrium for a corresponding solution concept at degree k. Note that U, C, and I displayed in Table 6.13 denote the three DMs, USS, CDO, and IJC, respectively.

A state that is not an equilibrium has no long-term stability because there is at least one individual DM who has an incentive to move to a more preferred state and thereby not permit an equilibrium to form. Table 6.14 provides stability results for different versions of preference. In particular, when stabilities are analyzed using two degrees of preference introduced in Sect. 4.2, states s_4, s_7, and s_9 are equilibria; if preference information is provided using three degrees of preference, then states s_7 and s_9 are equilibria using stability definitions presented in Sect. 6.3; there is only one equilibrium state s_9 for four degrees of preference. If state s_4 were the resolution for the GDU conflict, this would mean that IJC recommends to complete the GDU project regardless of Canada's concerns, so USS proceeds with this project. It is obvious that this resolution cannot settle this conflict in the long term. State s_7 means that the USS follows the IJC recommendation to modify this project, but Canada does not take legal action based on the Boundary Waters Treaty. State s_9 is the same as state s_7 except that Canada chooses legal procedures. When comparing states s_7 and s_9, equilibrium s_9 is a more reasonable resolution for solving this conflict. Therefore, a multiple-degree version of a stability analysis provides new insights and valuable guidance for decision analysts.

Although the example of the GDU conflict shown in Table 6.11 and Fig. 6.14 is a relatively small model having three DMs, five options, and nine feasible states, a graph model structure can handle any finite number of states and DMs, each of whom can control any finite number of options. As pointed out by Fang et al. (2003a, b), an available decision support system (DSS) for stability analysis of a graph model with two degrees of preference can work well. Theorem 6.4 reveals the relationship of stabilities between two degrees of preference presented in Chap. 4 and multiple degrees of preference. This theorem indicates the possibility of developing an effective algorithm to implement the multilevel versions of the four stabilities within a DSS, which would be essential if the proposed stability analysis were applied to larger practical problems. In fact, a DSS based on the matrix version will be designed in Chap. 10.

Table 6.13 Stability results of the GDU conflict for the graph model with four levels of preference

State	Super				Level(k)	Nash				GMR				SMR				SEQ			
	U	C	I	Eq		U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq
s_1	√	√			0																
					1																
					2																
					3	√	√			√	√			$\surd^{3+}$	$\surd^{3+}$			√	√		
s_2			√		0																
					1	√				√				$\surd^{1+}$				√			
					2																
					3			√				√				$\surd^{3+}$				√	
s_3			√		0																
					1		√				√				$\surd^{1+}$				√		
					2																
					3			√				√				$\surd^{3+}$				√	
s_4			√		0																
					1	√				√				$\surd^{1+}$				√			
					2																
					3		√	√			√	√			$\surd^{3+}$	$\surd^{3+}$			√	√	
s_5			√		0																
					1																
					2																
					3			√			√	√			$\surd^{3-}$	$\surd^{3+}$			√	√	
s_6			√		0																
					1	√				√				$\surd^{1+}$				√			
					2																
					3			√				√				$\surd^{3+}$				√	
s_7			√		0																
					1		√				√				$\surd^{1+}$				√		
					2																
					3			√		√		√				$\surd^{3+}$		√		√	
s_8			√		0																
					1																
					2																
					3		√	√			√	√			$\surd^{3+}$	$\surd^{3+}$			√	√	
s_9			√		0																
					1																
					2																
					3	√		√		√	√	√	$\surd^{3}$	$\surd^{3+}$	$\surd^{3-}$	$\surd^{3+}$		√	√	√	$\surd^{3}$

Table 6.14 The comparison of stability results for three versions of preference

Version of preference	Equilibria	Analysis method
Two degrees of preference	s_4, s_7, s_9	See Sect. 4.2
Three degrees of preference	s_7, s_9	See Sect. 6.3
Four degrees of preference	s_9	See Sect. 6.4.3.2

6.7 Important Ideas

In this chapter, a multiple-degree preference framework is developed for the graph model methodology to handle multiple degrees of preference, which lie between relative and cardinal preferences in terms of information content. Multilevel versions of the four solution concepts consisting of Nash, GMR, SMR, and SEQ are defined in the graph model for multiple degrees of preference. Specifically, solution concepts at degree k are defined for $Nash_k$, GMR_k, SMR_k, and SEQ_k for $k = 1, 2, \ldots, r$, where r is the maximum number of degrees of preference between two states. The proposed stability definitions extend existing definitions based on two degrees and three degrees of preference, so that more practical and complicated problems can be analyzed at greater depth.

The algebraic system to ease the coding of logically-defined stability definitions proposed in Chaps. 4 and 5 for simple preference and unknown preference, respectively, is extended in this chapter in a similar way to handle three degrees of preference. The algebraic method is developed to represent general, strong, and weak graph model stability definitions based on strength of preference using explicit matrix formulations instead of graphical or logical representations. These explicit algebraic formulations allow algorithms to assess rapidly the stabilities of states, and to be applied to large and complicated conflict models, using an advanced decision support system (DSS) like the one designed in Chap. 10. Because of the flexible nature of these explicit expressions, the matrix representations introduced here can be used as a solid framework for incorporating new solution concepts reflecting human behavior and novel theoretical constructs for handling different kinds of conflict situations, into the basic GMCR paradigm.

6.8 Problems

6.8.1 The concept of degree of preference constitutes a procedure for internalizing the psychological phenomenon of emotion. For instance, an environmentalist greatly prefers that a company not pollute the surrounding environment via discharges of gas, liquid and solid wastes. Describe two types of real-world disputes in which emotions are present and hence must be taken into account.

6.8.2 Attitudes can play a role in how people may behave in a conflict situation. Based on the research of Inohara et al. (2007) and Bernath Walker et al. (2009, 2012a), outline how attitude is operationalized within GMCR. Qualitatively, what connections do you see between degrees of preference and attitudes? Do you think they could be combined?

6.8.3 The concept of dominating attitudes within GMCR is put forward by Bernath Walker et al. (2012b). Briefly explain how this approach works and discuss its links to degrees of preference.

6.8.4 In the Prisoner's Dilemma conflict described in Problem 3.5.1, suppose that both DMs greatly prefer state s_1, in which they both cooperate, over state s_4, in which they do not cooperate with one another. Carry out a complete stability analysis following a logical interpretation using the four solution concepts utilized in this chapter as well as other chapters in the text. Did you gain additional strategic insights using this approach over the situation in which degree of preference is not present? Justify your response.

6.8.5 For the question involving Prisoner's Dilemma in Problem 6.8.4, execute the stability analysis using the matrix or algebraic formulation rather than the logical form.

6.8.6 For the game of Chicken in Problem 3.5.4, in which both drivers who are driving at high speed towards one another in their cars, carry out a stability analysis in which both drivers greatly prefer not to have a head-on crash over all of the other scenarios. Use the logical form of the four stability definitions for calculating individual stability and the associated equilibria. Explain why your findings make sense.

6.8.7 For the game of Chicken in Problem 3.5.4, in which both drivers are racing towards each other at high speed, execute a stability analysis for which one of the two drivers greatly prefers the situation in which they do not crash over all of the others states. Comment about the strategic meaning of your stability results.

6.8.8 For the Elmira conflict described and modeled in option form in Sect. 1.2.2 and analyzed for the case of simple preference in Sect. 4.5, suggest a reasonable model containing strength of preference such as when the Ministry of the Environment greatly prefers that Uniroyal not close down its chemical plant over situations in which it does terminates its operations. Carry out a complete stability analysis in which the DMs behave according to Nash or SEQ stability. Explain whether or not your findings make strategic sense.

6.8.9 Three degrees of preference often occur in practice. However, four degrees of preference may not take place as often. Explain a conflict situation in which it makes sense to entertain four degrees of preference in a conflict investigation. Provide references to support your claim.

6.8.10 The Gisborne conflict arising over the export of fresh water in bulk quantities is studied in Sect. 5.4. Provide a version of this conflict model in which it is reasonable to consider three degrees of preference for at least one of the DMs. Carry out a stability analysis of this conflict model for the case of Nash and SEQ stability. Discuss interesting stability results that you found.

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