Chapter 4
Stability Definitions: Simple Preference

Strategic conflicts, or situations in which two or more decision makers (DMs) with different objectives interact, occur often in the real-world. As discussed in Chap. 3, many models are available to represent strategic conflicts, such as the normal-form conflict model, the option-form conflict model, and the graph model. Conflict resolution has been investigated within many disciplines (Hipel 2009) including international relations, psychology, and law, as well as from mathematical and engineering perspectives (Saaty and Alexander 1989, Howard et al. 1992, Fang et al. 1993, Bennett 1995). Among the formal methodologies that handle strategic conflict, the graph model (or Graph Model for Conflict Resolution (GMCR)) (Kilgour et al. 1987, Fang et al. 1993) provides a remarkable combination of simplicity and flexibility.

The main goal of this chapter is to define stabilities in graph models with simple preference structure, based on a strict preference and an indifference relation, to be discussed in Sect. 4.1. As explained in Sect. 4.2, when determining the stability of a state for a given DM, a logical structure is employed for tracking the moves and countermoves that could take place if the DM decides to improve his or her situation. If the DM perceives that he or she will end up in a less preferred situation as a result of these potential interactions with others, the state is deemed to be stable. However, these logical representations of stabilities often require complex calculations and are difficult to code. In particular, the construction of reachable lists of a coalition having two or more DMs is a complicated process. The restriction that no DM may move twice consecutively does not constrain a coalition in the way that it limits an individual DM. For example, if there are only two DMs in a model, then a response to a unilateral improvement (UI) by one of them is necessarily a single move. But if there are more than two DMs in the model, a response to one DM's UI may consist of a sequence of many moves, provided no specific DM moves twice consecutively. The subset of DMs levying the moves under the control of group members is called a coalition. The sequence of actions by members of a coalition may constitute an action to sanction a UI by another DM or the coalition members may be moving to

H. Xu et al., *Conflict Resolution Using the Graph Model: Strategic Interactions in Competition and Cooperation*, Studies in Systems, Decision and Control 153,
https://doi.org/10.1007/978-3-319-77670-5_4

a state which is more preferred by all members of the coalition which is referred to as a coalition improvement.

The foregoing types of situations led to the development of matrix representations of a graph model and explicit matrix calculations to determine the stabilities introduced in Sect. 4.3. Because the graph model consists of several interrelated graphs, well-known results of graph theory can help to analyze a graph model. This analysis involves searching paths in a graph, subject to the important restriction that no DM can move twice in succession along any path. Therefore, a graph model must be treated as an edge colored digraph in which each arc represents a unilateral move and distinct colors refer to different DMs. An algebraic approach to searching colored paths in a colored digraph is presented in Sect. 4.3. The computational complexity of employing the matrix formulation of the graph model is investigated in Sect. 4.4. The sustainable development conflict is used throughout this chapter to illustrate how stability calculations are executed for both the logical and matrix formulations of the graph model. In Sect. 4.5, the Elmira dispute is employed to demonstrate how stability calculations are carried out using the matrix representation. Finally, part of the presentation appearing in this chapter is based upon research published earlier (Xu et al. 2007, 2009, 2010a, b, 2011, 2014).

4.1 Simple Preference

In the original form, a graph model could be calibrated using only a relative preference relation, "$\succ$ preferred", and an "equality" relation, "$\sim$ indifferent", to represent a DM's preference for one state with respect to another. The features and properties of this type of preference, called a simple preference structure, were discussed in Sect. 3.2.4. Specifically, simple preference of DM i is represented by a pair of relations $\{\succ_i, \sim_i\}$ on S, where $s \succ_i q$ indicates that DM i prefers s to q and $s \sim_i q$ means that DM i is indifferent between s and q (or equally prefers s and q). Note that, for each i, $\succ_i$ is assumed irreflexive and asymmetric, and $\sim_i$ is assumed reflexive and symmetric. Also, it is assumed that, for any $s, q \in S$, either $s \succ_i q$, $s \sim_i q$, or $q \succ_i s$. The conventions that $s \succeq_i q$ is equivalent to either $s \succ_i q$ or $s \sim_i q$, and that $s \prec_i q$ is equivalent to $q \succ_i s$, are convenient. Based on such preference information, DM i's reachable lists from a status quo state along the arcs of the directed graph, important components of stability analysis, can be defined for a graph model, as will be accomplished next.

4.1.1 Reachable Lists of a Decision Maker

Let S and N denote the state set and the DM set. The state set S can be partitioned into subsets based on preference relative to a fixed state $s \in S$. These subsets, which are essential in stability analysis, are described as follows:

- $\Phi_i^+(s) = \{q : q \succ_i s\}$, the states preferred to state s by DM i;
- $\Phi_i^=(s) = \{q : q \sim_i s\}$, the states indifferent to state s by DM i;
- $\Phi_i^-(s) = \{q : s \succ_i q\}$, the states less preferred than state s for DM i.

Let $i \in N$ and $s \in S$ be arbitrary. Denote the intersection operation by $\cap$. Recall that each arc of $A_i \subseteq S \times S$ indicates that DM i can make a unilateral move (in one step) from the initial state to the terminal state of the arc. DM i's reachable lists from state $s \in S$ for simple preference are defined as follows:

Definition 4.1 For a graph model G, A_i denotes the arcs controlled by DM i for $i \in N$. DM i's reachable lists from $s \in S$ are subsets of S as follows:

(i) $R_i(s) = \{q \in S : (s, q) \in A_i\}$ is DM i's reachable list from s by unilateral moves (UMs);
(ii) $R_i^+(s) = \{q \in S : (s, q) \in A_i \text{ and } q \succ_i s\}$ is DM i's reachable list from s by unilateral improvements (UIs);
(iii) $R_i^=(s) = \{q \in S : (s, q) \in A_i \text{ and } q \sim_i s\}$ is DM i's reachable list from s by equally preferred moves; and
(iv) $R_i^-(s) = \{q \in S : (s, q) \in A_i \text{ and } s \succ_i q\}$ is DM i's reachable list from s by unilateral disimprovements.

From the above definitions, the relationships among the subsets of S and the corresponding reachable lists from state s for DM i are depicted in Fig. 4.1. For ease of use, some additional notation is defined by $\Phi_i^{-,=}(s) = \Phi_i^-(s) \cup \Phi_i^=(s)$.

Example 4.1 A graph model with two DMs $N = \{1, 2\}$ and four feasible states $S = \{s_1, s_2, s_3, s_4\}$ is depicted in Fig. 4.2. The labels on the arcs of the graph indicate the DM who can make the move. Preference information about the states is given below the directed graph. If $s = s_1$ is selected as the status quo state, the subsets of S separated by DM i, $\Phi_i^+(s)$, $\Phi_i^=(s)$ and $\Phi_i^-(s)$, and DM i's reachable lists from s, $R_i(s)$, $R_i^+(s)$, $R_i^=(s)$ and $R_i^-(s)$ for $i \in N$, can be calculated easily.

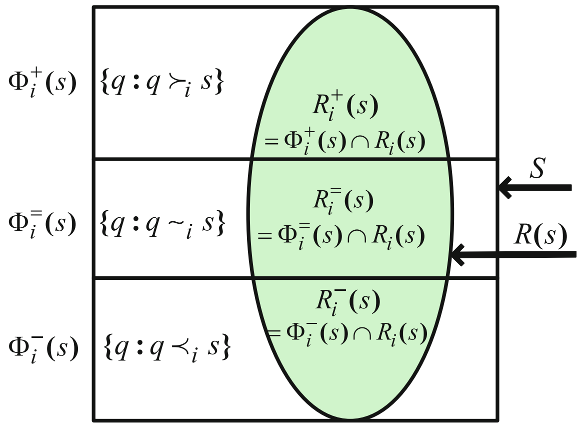


Fig. 4.1 Relations among the subsets of S and the corresponding reachable lists

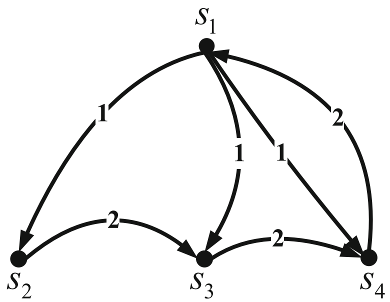


Fig. 4.2 Graph model for a two DM model

Note that the preference information

$$s_2 \succ_1 s_3 \succ_1 s_1 \succ_1 s_4 \text{ and } s_1 \succ_2 s_4 \succ_2 s_3 \succ_2 s_2$$

in Fig. 4.2 implies that the preference relations $\succ_1$ and $\succ_2$ are transitive. According to the descriptions of the subsets of S,

- $\Phi_1^+(s_1) = \{q : q \succ_1 s_1\} = \{s_2, s_3\}$ and $\Phi_2^+(s_1) = \{q : q \succ_2 s_1\} = \emptyset$;
- $\Phi_1^=(s_1) = \{q : q \sim_1 s_1\} = \emptyset$ and $\Phi_2^=(s_1) = \{q : q \sim_2 s_1\} = \emptyset$; and
- $\Phi_1^-(s_1) = \{q : s_1 \succ_1 q\} = \{s_4\}$ and $\Phi_2^-(s_1) = \{q : s_1 \succ_2 q\} = \{s_2, s_3, s_4\}$.

Clearly, DM 1's arc set is $A_1 = \{(s_1, s_2), (s_1, s_3), (s_1, s_4)\}$ and DM 2's arc set is $A_2 = \{(s_2, s_3), (s_3, s_4), (s_4, s_1)\}$. According to Definition 4.1, the DMs' reachable lists from s_1 are

- $R_1(s_1) = \{q \in S : (s_1, q) \in A_1\} = \{s_2, s_3, s_4\}$ and $R_2(s_1) = \{q \in S : (s_1, q) \in A_2\} = \emptyset$;
- $R_1^+(s_1) = \{q \in S : (s_1, q) \in A_i \text{ and } q \succ_1 s_1\} = \{s_2, s_3\}$ and $R_2^+(s_1) = \emptyset$;
- $R_1^=(s_1) = \{q \in S : (s_1, q) \in A_i \text{ and } q \sim_1 s_1\} = \emptyset$ and $R_2^=(s_1) = \emptyset$; and
- $R_1^-(s_1) = \{q \in S : (s_1, q) \in A_i \text{ and } s_1 \succ_1 q\} = \{s_4\}$ and $R_2^-(s_1) = \emptyset$.

As shown in Fig. 4.1 for Example 4.1, the relations among the subsets of the state set and the reachable lists are

- $R_i^+(s) = R_i(s) \cap \Phi_i^+(s)$;
- $R_i^=(s) = R_i(s) \cap \Phi_i^=(s)$; and
- $R_i^-(s) = R_i(s) \cap \Phi_i^-(s)$.

DM i's oriented arcs A_i are related to reachable lists as follows:

- $A_i = \{(p, q) : q \in R_i(p)\}$ is DM i's UM arc set;
- if $s \in S$ is fixed, then $A_i(s) = \{(s, q) \in A_i : q \in R_i(s)\}$ are DM i's UM arcs from s.

DM i's UM arcs, A_i, can be partitioned as follows:

- $A_i^+ = \{(p, q) \in A_i : q \succ_i p\}$ is DM i's UI arc set;
- $A_i^= = \{(p, q) \in A_i : q \sim_i p\}$ is DM i's equally preferred arc set; and
- $A_i^- = \{(p, q) \in A_i : p \succ_i q\}$ is DM i's less preferred arc set.

According to the completeness property of preference, $A_i = A_i^+ \cup A_i^= \cup A_i^-$. Now fix $s \in S$. Then

- $A_i^+(s) = \{(s, q) \in A_i : q \in R_i^+(s)\}$ are DM i's UI arcs from s;
- $A_i^=(s) = \{(s, q) \in A_i : q \in R_i^=(s)\}$ are DM i's equally preferred arcs from s; and
- $A_i^-(s) = \{(s, q) \in A_i : q \in R_i^-(s)\}$ are DM i's less preferred arcs from s.

Note that $A_i(s)$ is a subset of the arc set A_i while $R_i(s)$ is a subset of the state set S.

4.2 Logical Representation of Stability Definitions

In a graph model, a stability definition (solution concept) is a procedure for determining whether a state is stable for a decision maker (DM), and identifying a situation in which the DM would have no incentive to move away from the state unilaterally. An equilibrium of a graph model, or a possible resolution of the conflict it represents, is a state that all DMs find stable under an appropriate stability definition. Many solution concepts have been formulated to represent various decision styles and contexts. In this book, four basic solution concepts—Nash stability (Nash 1950, 1951), general metarationality (GMR) (Howard 1971), symmetric metarationality (SMR) (Howard 1971), and sequential stability (SEQ) (Fraser and Hipel 1979)—are emphasized. Recently, Li et al. (2004) extended these four solution concepts to models having preference uncertainty, which will be introduced in Chap. 5. As well, Hamouda et al. (2004, 2006) proposed new stability definitions that take strength of preference (strong or mild) into account, which will be discussed in Chap. 6.

The logical representations of Nash, GMR, SMR, and SEQ stabilities in the graph model with simple preference are given below. The four stability definitions for two-DM models are introduced first.

4.2.1 *Two Decision Maker Case*

Let $N = \{i, j\}$ and $s \in S$ in the following definitions.

Definition 4.2 State s is Nash stable for DM i, denoted by $s \in S_i^{Nash}$, iff $R_i^+(s) = \emptyset$.

For Nash stability (Nash 1950, 1951), DM i expects that DM j will stay at any state DM i moves to, and consequently that any state that i moves to will be final state.

Table 4.1 Nash stability of the sustainable development game with simple preferences

State	$R_i^+(s)$		Nash stability		Equilibrium
	DM 1	DM 2	DM 1	DM 2	
s_1	$\emptyset$	$\emptyset$	s	s	Eq
s_2	$\emptyset$	$\{s_1\}$	s	u	
s_3	$\{s_1\}$	$\emptyset$	u	s	
s_4	$\{s_2\}$	$\{s_3\}$	u	u	

Example 4.2 (*Nash Stability for the Sustainable Development Model*) The sustainable development game was presented in normal form, option form, and graph form in Tables 3.2 and 3.3, and Fig. 3.2, respectively. The graph model of this model is shown in Fig. 3.2 with the state set $S = \{s_1, s_2, s_3, s_4\}$ and the DM set $N = \{1, 2\}$. The letter on a given arc indicates which DM controls the movement while the arrowhead shows the direction of movement. The two DMs' preference information is presented underneath the digraph.

State s_1 is now analyzed to ascertain if it is Nash stable for DM i. From Fig. 3.2, DM 1 has a unilateral move from s_1 to s_3. However, $s_1 \succ_1 s_3$ based on the preference information, so the move by DM 1 from s_1 to s_3 is not a unilateral improvement and, therefore, state s_1 is Nash stable for DM 1 according to Definition 4.2. Next, consider the Nash stability of s_1 for DM 2. Clearly, DM 2 has a unilateral move from s_1 to s_2. Because $s_1 \succ_2 s_2$, the move by DM 2 from s_1 to s_2 is not a unilateral improvement and, therefore, state s_1 is Nash stable for DM 2. Accordingly, s_1 is an equilibrium in the sense of Nash stability. Similarly, the other three states can be assessed for Nash stability.

Nash stability results are listed in Table 4.1 in which $R_i^+(s)$ denotes DM i's UIs from $s \in S$ for $i \in N$. The letter "s" indicates that a state is Nash stable for a given DM, whereas "u" denotes that the state is Nash unstable. The letter "Eq" means that the state is an equilibrium, that is Nash stable for both DMs.

State $s \in S$ is GMR for DM i iff, whenever DM i makes any UI from s, then i's opponent can move to sanction i (that is, hurt i) in response. (A "sanction" must be an opponent's move.) The formal definition is given next.

Definition 4.3 State s is GMR stable (or, simply, GMR) for DM i, denoted by $s \in S_i^{GMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \Phi_i^{-,=}(s)$ (or $s \succeq_i s_2$).

For GMR, DM i expects that its opponent j will respond by hurting i, so s is GMR stable for i iff DM j can hurt i if i takes any UI.

Example 4.3 (*GMR Stability for the Sustainable Development Model*) From Definitions 4.2 and 4.3, one can see that if $R_i^+(s) = \emptyset$, then s is Nash stable and GMR stable for DM i. Hence, for instance, s_3 is GMR stable for DM 2 for the sustainable development model. Let us assess whether s_3 is GMR for DM 1. DM 1

Table 4.2 GMR stability of the sustainable development game with simple preferences

State	$R_i^+(s)$		$R_i(s)$		GMR stability		Equilibrium
	DM 1	DM 2	DM 1	DM 2	DM 1	DM 2	
s_1	$\emptyset$	$\emptyset$	$\{s_3\}$	$\{s_2\}$	s	s	Eq
s_2	$\emptyset$	$\{s_1\}$	$\{s_4\}$	$\{s_1\}$	s	u	
s_3	$\{s_1\}$	$\emptyset$	$\{s_1\}$	$\{s_4\}$	s	s	Eq
s_4	$\{s_2\}$	$\{s_3\}$	$\{s_2\}$	$\{s_3\}$	u	u	

has a unilateral improvement from s_3 to s_1 and DM 2 has a unilateral move from s_1 to s_2. However, s_2 is less preferred than s_3 for DM 1, hence, s_3 is GMR for DM 1 according to Definition 4.3. The stabilities of other three states for the two DMs can be determined, similarly. GMR stability results are listed in Table 4.2, where, as usual, $R_i^+(s)$ denotes DM i's UIs from $s \in S$, "s" indicates GMR stable, "u" indicates GMR unstable, and "Eq" indicates a GMR equilibrium.

SMR is a similar but more restrictive stability definition compared to GMR. Under SMR, DM i expects to have a chance to counterrespond to its opponent's response to i's original move.

Definition 4.4 State s is SMR stable for DM i, denoted by $s \in S_i^{SMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ such that $s_2 \in \Phi_i^{-,=}(s)$ (or $s \succeq_i s_2$) and $s_3 \in \Phi_i^{-,=}(s)$ (or $s \succeq_i s_3$) for every $s_3 \in R_i(s_2)$.

Example 4.4 (*SMR Stability for the Sustainable Development Model*) By comparing Definitions 4.2–4.4, one can see that if $R_i^+(s) = \emptyset$, then s is Nash stable, GMR stable, and SMR stable for DM i. Therefore, for instance, s_3 is SMR stable for DM 2 for the sustainable development model. Let us determine whether s_3 is SMR stable for DM 1. DM 1 has a unilateral improvement from s_3 to s_1 and DM 2 has a unilateral move from s_1 to s_2, then DM 1 has only a unilateral move from s_2 to s_4. Because s_2 and s_4 are less preferred than s_3 for DM 1 and, hence, s_3 is SMR stable for DM 1 by Definition 4.4. The stabilities of other three states for the two DMs can be assessed, similarly. SMR stability results are obtained and listed in Table 4.3, where, similarly, "s" indicates SMR stable, "u" indicates SMR unstable, and "Eq" indicates a SMR equilibrium.

SEQ is similar to GMR, but includes only sanctions that are "credible". A credible action is a unilateral improvement.

Definition 4.5 State s is SEQ stable for DM i, denoted by $s \in S_i^{SEQ}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j^+(s_1)$ with $s_2 \in \Phi_i^{-,=}(s)$ (or $s \succeq_i s_2$).

Example 4.5 (*SEQ Stability for the Sustainable Development Model*) Similar to GMR stability, if $R_i^+(s) = \emptyset$, then s is SEQ stable for DM i. Therefore, for the

Table 4.3 SMR stability of the sustainable development game with simple preferences

State	$R_i^+(s)$		$R_i(s)$		SMR stability		Equilibrium
	DM 1	DM 2	DM 1	DM 2	DM 1	DM 2	
s_1	$\emptyset$	$\emptyset$	$\{s_3\}$	$\{s_2\}$	s	s	Eq
s_2	$\emptyset$	$\{s_1\}$	$\{s_4\}$	$\{s_1\}$	s	u	
s_3	$\{s_1\}$	$\emptyset$	$\{s_1\}$	$\{s_4\}$	s	s	Eq
s_4	$\{s_2\}$	$\{s_3\}$	$\{s_2\}$	$\{s_3\}$	u	u	

Table 4.4 SEQ stability of the sustainable development game with simple preferences

State	$R_i^+(s)$		$R_i(s)$		SEQ stability		Equilibrium
	DM 1	DM 2	DM 1	DM 2	DM 1	DM 2	
s_1	$\emptyset$	$\emptyset$	$\{s_3\}$	$\{s_2\}$	s	s	Eq
s_2	$\emptyset$	$\{s_1\}$	$\{s_4\}$	$\{s_1\}$	s	u	
s_3	$\{s_1\}$	$\emptyset$	$\{s_1\}$	$\{s_4\}$	u	s	
s_4	$\{s_2\}$	$\{s_3\}$	$\{s_2\}$	$\{s_3\}$	u	u	

sustainable development model, s_3 is SEQ stable for DM 2. Let us analyze SEQ stability of s_3 for DM 1. DM 1 has a unilateral improvement from s_3 to s_1, but DM 2 has no any unilateral improvement from s_1. Hence, s_3 is GMR and SMR stable for DM 1 rather than SEQ stable. Similarly, one can assess whether other three states are SEQ stable for the two DMs by Definition 4.5. SEQ stability results are listed in Table 4.4 in which "s" indicates SEQ stable, "u" indicates SEQ unstable, and "Eq" indicates a SEQ equilibrium.

4.2.2 Reachable Lists of a Coalition of Decision Makers

Any nonempty subset H of DMs, $H \subseteq N$ and $H \neq \emptyset$, is called a coalition. If $|H| = 1$, then the coalition H is trivial; if $|H| > 1$, then the coalition H is non-trivial. (Here, $|H|$ denotes the cardinality of H.) Within an n-DM model ($n \geq 2$), an important coalition is the set of opponents of a fixed DM i, namely $N \backslash \{i\}$, where $\backslash$ refers to "set subtraction". In order to analyze the stability of a state for DM $i \in N$, it is necessary to take into account possible responses by all other DMs $j \in N \backslash \{i\}$. The essential inputs of stability analysis are reachable lists of group $N \backslash \{i\}$ from state s, $R_{N \backslash \{i\}}(s)$ and $R_{N \backslash \{i\}}^+(s)$ for simple preference. For a two-DM model, DM i has only an opponent, DM j, so i's opponent's reachable lists from s are the states reachable by DM j's moves. In an n-DM model ($n > 2$), the opponents of a DM constitute a coalition of two or more DMs and the determination of their reachable lists is more subtle. The definition of a legal sequence of UMs for a nontrivial coalition is given first.

A legal sequence of UMs for a coalition of DMs is a sequence of states linked by unilateral moves by members of the coalition, in which a DM may move more than once, but not twice consecutively. In general, a DM's directed graph can be transitive or intransitive within the GMCR paradigm. When, for example, a DM can move from s_1 to s_2 and s_2 to s_3 in one step, moves are transitive if the DM can also move in one step from s_1 to s_3. If this is not possible, the move is intransitive. Hence, the restriction of non-successive-moves by the same DM means that GMCR can handle intransitive moves, in addition to transitive moves. Let the coalition $H \subseteq N$ satisfy $|H| \geq 2$ and let the status quo state be $s \in S$. Let $R_H(s) \subseteq S$ (defined formally below) denote the set of states that can be reached by any legal sequence of UMs, by some or all DMs in H, starting at state s. If $s_1 \in R_H(s)$, then $\Omega_H(s, s_1)$ (also defined formally below) denotes the set of all last DMs in legal sequences from s to s_1. The formal definition of $R_H(s) \subseteq S$ and $\Omega_H(s, s_1) \subseteq H$ for $s_1 \in R_H(s)$ is given as follows:

Definition 4.6 A unilateral move by H is a member of $R_H(s) \subseteq S$, defined inductively by

(1) assuming $\Omega_H(s, s_1) = \emptyset$ for all $s_1 \in S$;
(2) if $j \in H$ and $s_1 \in R_j(s)$, then $s_1 \in R_H(s)$ and $\Omega_H(s, s_1) = \Omega_H(s, s_1) \cup \{j\}$;
(3) if $s_1 \in R_H(s)$, $j \in H$, and $s_2 \in R_j(s_1)$, then, provided $\Omega_H(s, s_1) \neq \{j\}$, $s_2 \in R_H(s)$ and $\Omega_H(s, s_2) = \Omega_H(s, s_2) \cup \{j\}$.

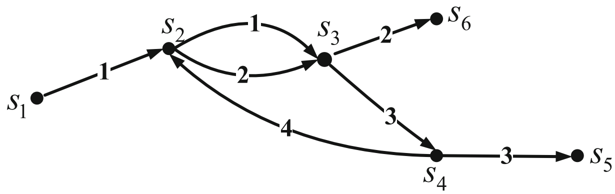
Note that this definition is inductive: first, using (2), the states reachable from s are identified and added to $R_H(s)$; then, using (3), all states reachable from those states are identified and added to $R_H(s)$; then the process is repeated until no further states can be added to $R_H(s)$ and there is no change in $\Omega_H(s, s_2)$ for any $s_2 \in R_H(s)$. Because $R_H(s) \subseteq S$ and S is finite, this limit must be reached in finitely many steps.

To interpret Definition 4.6, note that if $s_1 \in R_H(s)$, then $\Omega_H(s, s_1) \subseteq H$ is the set of all last DMs in legal sequences from s to s_1. (If $s_1 \notin R_H(s)$, it can be assumed that $\Omega_H(s, s_1) = \emptyset$.) Suppose that $\Omega_H(s, s_1)$ contains only one DM, say $j \in N$. Then any move from s_1 to a subsequent state, say s_2, must be made by a member of H other than j; otherwise DM j would have to move twice in succession. On the other hand, if $|\Omega_H(s, s_1)| \geq 2$, any member of H who has a unilateral move from s_1 to s_2 may exercise it.

A legal sequence of UIs for a coalition can be defined similarly. Let $R_H^+(s) \subseteq S$ (defined formally below) denote the set of states that can be reached by any legal sequence of UIs, by some or all DMs in H, starting at state s. If $s_1 \in R_H^+(s)$, then $\Omega_H^+(s, s_1)$ (also defined formally below) denotes the set of all last DMs in legal sequences from s to s_1 by UIs. The formal definition of $R_H^+(s) \subseteq S$ and $\Omega_H^+(s, s_1) \subseteq H$ for $s_1 \in R_H^+(s)$ is given as follows:

Definition 4.7 Let $s \in S$, $H \subseteq N$, and $H \neq \emptyset$. A unilateral improvement by H is a member of $R_H^+(s) \subseteq S$, defined inductively by

(1) assuming $\Omega_H^+(s, s_1) = \emptyset$ for all $s_1 \in S$;
(2) if $j \in H$ and $s_1 \in R_j^+(s)$, then $s_1 \in R_H^+(s)$ and $\Omega_H^+(s, s_1) = \Omega_H^+(s, s_1) \cup \{j\}$;



$$s_2 \succ_1 s_1 \succ_1 s_3 \succ_1 s_4 \succ_1 s_6 \succ_1 s_5$$

$$s_3 \succ_2 s_2 \succ_2 s_6 \succ_2 s_5 \succ_2 s_1 \succ_2 s_4$$

$$s_5 \succ_3 s_4 \succ_3 s_3 \succ_3 s_2 \succ_3 s_1 \succ_3 s_6$$

$$s_5 \succ_4 s_2 \succ_4 s_3 \succ_4 s_4 \succ_4 s_1 \succ_4 s_6$$

Fig. 4.3 Graph model with four DMs and six states

(3) if $s_1 \in R_H^+(s)$, $j \in H$, and $s_2 \in R_j^+(s_1)$, then, provided $\Omega_H^+(s, s_1) \neq \{j\}$, $s_2 \in R_H^+(s)$ and $\Omega_H^+(s, s_2) = \Omega_H^+(s, s_2) \cup \{j\}$.

Definition 4.7 is identical to Definition 4.6 except that all moves are required to be UIs, i.e. each move is to a state strictly preferred by the mover to the current state. Similarly, $\Omega_H^+(s, s_1)$ includes all last movers in UIs by H from state s to state s_1. An example that shows the procedures to construct the reachable lists of a group is presented as follows:

Example 4.6 (*Constructing Reachable Lists of a Coalition*) Figure 4.3 shows a graph model with DM set $N = \{1, 2, 3, 4\}$ and state set $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. The labels on the arcs of the graph indicate the controlling DMs. Preference information is given below the directed graph. If $s = s_1$ is selected as the status quo state, then the reachable lists of $H = N$ from s, $R_N(s_1)$ and $R_N^+(s_1)$, can be constructed according to Definitions 4.6 and 4.7.

Constructing $R_N(s_1)$:

1. s_2 can be reached by DM 1 from s_1 by one step UM, so $s_2 \in R_N(s_1)$;
2. s_3 cannot be attained by DM 1 from s_2 since DM 1 cannot move twice consecutively;
3. s_3 can be reached by DM 2 from s_2 by one step UM, so $s_3 \in R_N(s_1)$;
4. s_6 cannot be attained by DM 2 from s_3 since DM 2 cannot move twice consecutively;
5. s_4 can be reached by DM 3 from s_3 by one step UM, so $s_4 \in R_N(s_1)$;
6. s_5 cannot be attained by DM 3 from s_4 since DM 3 cannot move twice consecutively;
7. s_2 is reachable again by DM 4 from s_4 by one step UM, then s_3 is reachable again by DM 1 from s_2, and s_6 is finally reachable by DM 2 from s_3, so $s_6 \in R_N(s_1)$.

Accordingly, $R_N(s_1) = \{s_2, s_3, s_4, s_6\}$.

Constructing $R_N^+(s_1)$:

From the preference information provided, $A_1^+ = \{(s_1, s_2)\}$, $A_2^+ = \{(s_2, s_3)\}$, $A_3^+ = \{(s_3, s_4), (s_4, s_5)\}$, and $A_4^+ = \{(s_4, s_2)\}$.

1. s_2 can be reached by DM 1 from s_1 by a UI, so $s_2 \in R_N^+(s_1)$;
2. s_3 can be reached by DM 2 from s_2 by a UI, so $s_3 \in R_N^+(s_1)$;
3. s_4 can be reached by DM 3 from s_3 by a UI, so $s_4 \in R_N^+(s_1)$;
4. s_5 cannot be attained by DM 3 from s_4 since DM 3 cannot move twice consecutively.

Therefore, $R_N^+(s_1) = \{s_2, s_3, s_4\}$.

The four basic stabilities of Nash, GMR, SMR, and SEQ with simple preference in two-DM models, described using logical representation in Sect. 4.2.1, can be extended to models including more than two DMs, which is the objective of the next subsection.

4.2.3 n-Decision Maker Case

In an n-DM model, where $n > 2$, the opponents of a DM can be thought of as the coalition of all other DMs. To calculate the stability of a state for DM $i \in N$, it is necessary to examine possible responses by this coalition, $N \setminus \{i\}$ from the states in $R_{N\setminus\{i\}}(s)$ or $R^+_{N\setminus\{i\}}(s)$. Let $i \in N$ and $s \in S$ in the following definitions.

Nash stability definition is identical for two-DM and n-DM models because this formal stability does not consider the opponents' responses.

Definition 4.8 State s is Nash stable for DM i, denoted by $s \in S_i^{Nash}$, iff $R_i^+(s) = \emptyset$.

For GMR stability, DM i expects that its opponents, $N \setminus \{i\}$, will respond to any unilateral improvement by i from s to s_1 with a sequence of legal unilateral moves to a state in $R_{N\setminus\{i\}}(s_1)$, so as to hurt i if possible. As before, i anticipates that the conflict will end after the opponents have responded.

Definition 4.9 State s is GMR for DM i, denoted by $s \in S_i^{GMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s \succeq_i s_2$.

As in the two-DM case, for SMR stability, DM i expects to have a chance to counterrespond (s_3) to its opponents' response (s_2) to i's original move.

Definition 4.10 State s is SMR for DM i, denoted by $s \in S_i^{SMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$.

A state is SEQ stable for a given DM iff the DM can be deterred by subsequent unilateral improvements by its opponents.

Definition 4.11 State s is SEQ for DM i, denoted by $s \in S_i^{SEQ}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^+(s_1)$ with $s \succeq_i s_2$.

SEQ stability indicates that all UIs of the focal DM are sanctioned by a subsequent group unilateral improvement by the DM's opponents.

Definitions 4.8–4.11 cover Nash stability, GMR, SMR, and SEQ in the graph model for multiple-decision-maker conflict models (or, simply, n-DM models) with simple preference. These definitions retain the features of Definitions 4.2–4.5, in the two-DM case, except that DM i's opponents are a subset of N, instead of a single opponent. When $n = 2$, the DM set N is $\{i, j\}$, so that the reachable list of coalition $N \setminus \{i\}$ from s_1, $R_{N\setminus\{i\}}(s_1)$, reduces to DM j's reachable list from s_1, $R_j(s_1)$.

4.2.4 Interrelationships Among Stability Definitions

Fang et al. (1989, 1993) established general relationships among Nash, GMR, SMR, and SEQ solution concepts as shown in Fig. 4.4. The following theorem demonstrates that the same relationships hold for these solution concepts in both two-DM and n-DM models.

Theorem 4.1 *Let $i \in N$, $|N| = n$, and $n \geq 2$. Then the stable states under the four basic stability definitions satisfy*

$$S_i^{Nash} \subseteq S_i^{SMR} \subseteq S_i^{GMR} \tag{4.1}$$

and

$$S_i^{Nash} \subseteq S_i^{SEQ} \subseteq S_i^{GMR}. \tag{4.2}$$

Proof The inclusion relations presented in Eq. 4.1 will be proven. The proof for Eq. 4.2 is similar.

If $s \in S_i^{Nash}$, then $R_i^+(s) = \emptyset$, so using Definition 4.10, $s \in S_i^{SMR}$. Hence, $S_i^{Nash} \subseteq S_i^{SMR}$.

For any $s \in S_i^{SMR}$, if $R_i^+(s) = \emptyset$, then $s \in S_i^{GMR}$. Otherwise, for any $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every

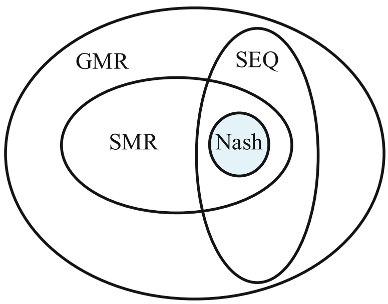


Fig. 4.4 Interrelationships among the solution concepts

Table 4.5 Summary of stability results for the sustainable development game with simple preferences

State number	Nash			GMR			SMR			SEQ		
	DM 1	DM 2	Eq	DM 1	DM 2	Eq	DM 1	DM 2	Eq	DM 1	DM 2	Eq
s_1	s	s	√	s	s	√	s	s	√	s	s	√
s_2	s	u		s	u		s	u		s	u	
s_3	u	s		s	s	√	s	s	√	u	s	
s_4	u	u		u	u		u	u		u	u	

$s_3 \in R_i(s_2)$, by Definition 4.10. Accordingly, for every $s_1 \in R_i^+(s)$ there exists $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s \succeq_i s_2$. This implies that $S_i^{SMR} \subseteq S_i^{GMR}$. Therefore, the inclusion relations $S_i^{Nash} \subseteq S_i^{SMR} \subseteq S_i^{GMR}$ hold. □

There is no necessary inclusion relation between S_i^{SMR} and S_i^{SEQ}, i.e., it may be true that $S_i^{SMR} \supseteq S_i^{SEQ}$, or that $S_i^{SMR} \subseteq S_i^{SEQ}$, or neither.

The sustainable development model is utilized to illustrate the interrelationships among the four stabilities, Nash, GMR, SMR, and SEQ. Stability results for the four solution concepts are summarized in Table 4.5, where "Eq" means "equilibrium", "s" indicates stable, "u" indicates unstable, and "√" indicates an equilibrium for some solution concept. In fact, one can also utilize Theorem 4.1 to help determine some stabilities. For example, $R_i^+(s_1) = \emptyset$ for $i = 1, 2$, so s_1 is Nash stable for the two DMs. Therefore, s_1 is GMR, SMR, and SEQ for the two DMs by Eqs. 4.1 and 4.2 in Theorem 4.1. Because s_2 is GMR unstable for DM 2, then it is certain that s_2 is SMR unstable for DM 2 from the relation Eq. 4.1 in Theorem 4.1. In this case, s_3 is SMR but not SEQ for DM 2.

Theorem 4.1 examines the relationships of individual stability definitions from a single DM's viewpoint. Recall that a possible resolution or equilibrium of a graph model is a state that all DMs find stable under appropriate stability definitions. Hence, Theorem 4.1 implies that the same relationships hold for equilibria. Let $S^{Nash} = \bigcap_{i\in N} S_i^{Nash}$, $S^{GMR} = \bigcap_{i\in N} S_i^{GMR}$, $S^{SMR} = \bigcap_{i\in N} S_i^{SMR}$, and $S^{SEQ} = \bigcap_{i\in N} S_i^{SEQ}$ denote the equilibrium sets under the four stability definitions, respectively. The following theorem is immediate.

Theorem 4.2 *Let $i \in N$, $|N| = n$, and $n \geq 2$. Then the equilibria under the four basic stability definitions satisfy*

$$S^{Nash} \subseteq S^{SMR} \subseteq S^{GMR} \tag{4.3}$$

and

$$S^{Nash} \subseteq S^{SEQ} \subseteq S^{GMR}. \tag{4.4}$$

4.3 Matrix Representation of Stability Definitions

Stability definitions in the graph model are traditionally defined logically, in terms of the underlying graphs and preference relations, as in Sect. 4.2. However, as noted in the development of the DSS GMCR II (Fang et al. 2003a, b), the nature of logical representations makes coding difficult. The new preference structure proposed by Li et al. (2004) to represent uncertainty in DMs' preferences included some extensions of the four stability definitions, and algorithms were outlined but they have not been coded. The work of Hamouda et al. (2004, 2006) integrated strength of preference information into these four solution concepts but, again, it proved difficult to code and has not been integrated into GMCR II. Then, difficulties in coding, mainly because of the logical formulation, were the primary motivation for the development of explicit matrix representations of the graph model with simple preference. In the following subsection, matrix expressions are used to capture the relative preferences and reachable lists of a single DM, both by UMs and by UIs.

4.3.1 Preference Matrices and UM and UI Matrices

Let $m = |S|$ denote the number of states. For a graph model, several matrices can represent relative preference relations between two states.

Definition 4.12 For a graph model G, the preference matrix and the indifference matrix for DM i are $m \times m$ matrices, P_i^+ and $P_i^=$, with (s, q) entries

$$P_i^+(s,q) = \begin{cases} 1 & \text{if } q \succ_i s, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P_i^=(s,q) = \begin{cases} 1 & \text{if } q \sim_i s \text{ and } q \neq s, \\ 0 & \text{otherwise.} \end{cases}$$

A nonzero entry $P_i^+(s,q) = 1$ in the preference matrix indicates that DM i prefers state q to state s, while zero entry $P_i^+(s,q) = 0$ indicates that DM i either prefers s to q or is indifferent between s and q (by the property of preference completeness). Similarly, $P_i^=(s,q) = 1$ implies that DM i is indifferent between s and q while $P_i^=(s,q) = 0$ denotes that DM i prefers either s to q or q to s. It is convenient to define

$$P_i^{-,=} = E - I - P_i^+,$$

where E is the $m \times m$ unit matrix (all entries 1) and I is the $m \times m$ identity matrix. Note that $P_i^{-,=}(s,q) = 1$ means that DM i does not prefer state q to state s.

For $i \in N$ and $s \in S$, DM i's unilateral moves (UMs) and unilateral improvements (UIs) can be represented as follows:

Definition 4.13 For a graph model G, DM i's UM and UI matrices are the $m \times m$ matrices, J_i and J_i^+, with (s, q) entries

$$J_i(s,q) = \begin{cases} 1 & \text{if } (s,q) \in A_i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$J_i^+(s,q) = \begin{cases} 1 & \text{if } (s,q) \in A_i \text{ and } q \succ_i s, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $J_i(s, q) = 1$ if and only if DM i can move from state s to state q (in one step). In other words, $(s, q) \in A_i$. Also, $J_i^+(s, q) = 1$ iff $J_i(s, q) = 1$ and DM i prefers q to s.

The set $R_i(s) = \{q \in S : J_i(s, q) = 1\}$ is DM i's *reachable list* from state s by UMs. It contains all states to which DM i can make unilateral moves from state s in one step. Similarly, $R_i^+(s) = \{q \in S : J_i^+(s, q) = 1\}$ is DM $i's$ reachable list from s by UIs. Clearly, $R_i^+(s)$ is identical to $R_i(s)$ except that all moves are required to be UIs. Note that, if $R_i(s)$ and $R_i^+(s)$ are written as 0–1 row vectors, then

$$R_i(s) = e_s^T \cdot J_i \text{ and } R_i^+(s) = e_s^T \cdot J_i^+,$$

where e_s^T denotes the transpose of the sth standard basis column vector of the m-dimensional Euclidean space, $\mathbb{R}^S$.

The definitions of DM i's UM matrix, J_i, UI matrix, J_i^+, and preference matrix, P_i^+, imply that

$$J_i^+ = J_i \circ P_i^+, \tag{4.5}$$

where "$\circ$" denotes the Hadamard product.

The objective of the next subsection is to develop an explicit algebraic version of a graph model, to facilitate stability calculations. The two-DM models are considered first. One will see that matrix representation of solution concepts (MRSC) is feasible for four graph model stability definitions in the two-DM graph model. Using explicit matrix formulations instead of graphical or logical representations makes MRSC more effective and convenient for calculating stabilities and identifying equilibria.

4.3.2 Two Decision Maker Case

The matrix representation of Nash, GMR, SMR, and SEQ stabilities in two-DM conflict models with simple preference is developed in this subsection. The system, called the MRSC method, incorporates a set of $m \times m$ matrices, M_i^{Nash}, M_i^{GMR},

M_i^{SMR}, and M_i^{SEQ}, to capture Nash, GMR, SMR, and SEQ for DM $i \in N$, where $m = |S|$. For now, $|N| = 2$.

From Definition 4.2, state s is Nash stable for DM i iff DM i cannot move from s to any state i prefers. Define DM i's Nash matrix as the $m \times m$ matrix $M_i^{Nash} = J_i^+ \cdot E$ ("E" denotes the $m \times m$ matrix with each entry being set to 1). The diagonal element of M_i^{Nash} matrix at (s, s) is

$$M_i^{Nash}(s, s) = e_s^T \cdot J_i^+ \cdot e, \tag{4.6}$$

for $s \in S$, and all off-diagonal entries zero. Here, e is the m-dimensional unit column vector (all elements 1). Then the following theorem shows how this matrix represents Nash stability.

Theorem 4.3 *State $s \in S$ is Nash stable for DM i iff $M_i^{Nash}(s, s) = 0$.*

Proof By Eq. 4.6, $M_i^{Nash}(s, s) = 0$ holds iff

$$e_s^T \cdot J_i^+ = 0^T.$$

According to the definition of DM i's UI matrix, $M_i^{Nash}(s, s) = 0$ iff $R_i^+(s) = \emptyset$, which is the definition of Nash stability for DM i given in Definition 4.2. □

Note that Theorem 4.3 provides a matrix method to assess whether state s is Nash stable for DM i by identifying the Nash matrix's diagonal entry $M_i^{Nash}(s, s)$. If the sth diagonal entry is zero, then s is Nash stable for DM i; otherwise s is Nash unstable for DM i. This matrix representation of Nash stability will be adapted to the other basic stability definitions.

Example 4.7 (*Matrix Representation of Nash Stability for the Sustainable Development Model*) The logical representation of Nash stability for the sustainable development game was presented in Example 4.2. For the graph model of the sustainable development conflict presented in Fig. 3.2, the UM matrices for DM 1 and DM 2 are

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.7}$$

According to Definition 4.12, the preference matrices are

$$P_1^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \text{ and } P_2^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{4.8}$$

Accordingly, one uses $J_i^+ = J_i \circ P_i^+$, for $i = 1, 2$, to obtain

$$J_1^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } J_2^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.9}$$

Then, from Eq. 4.6, the Nash matrices are

$$M_1^{Nash} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } M_2^{Nash} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $M_1^{Nash}(1, 1) = M_1^{Nash}(2, 2) = 0$ and $M_1^{Nash}(3, 3) = M_1^{Nash}(4, 4) = 1$, then s_1 and s_2 are Nash stable, and s_3 and s_4 are Nash unstable for DM 1, according to Theorem 4.3. Similarly, because $M_2^{Nash}(1, 1) = M_2^{Nash}(3, 3) = 0$ and $M_2^{Nash}(2, 2) = M_2^{Nash}(4, 4) = 1$, s_1 and s_3 are Nash stable, and s_2 and s_4 are Nash unstable, for DM 2. The results are identical to those in Example 4.2 obtained by logical representation.

A state $s \in S$ is general metarational for DM i iff whenever DM i makes any UI from s, then its opponent can hurt i in response. Define DM i's $m \times m$ GMR matrix as

$$M_i^{GMR} = J_i^+ \cdot [E - sign\left(J_j \cdot (P_i^{-,=})^T\right)], \tag{4.10}$$

where $j \in N$, $j \neq i$. The following theorem establishes the matrix method to assess whether state s is GMR stable for a DM.

Theorem 4.4 *State $s \in S$ is GMR for DM i iff $M_i^{GMR}(s, s) = 0$.*

Proof Since

$$M_i^{GMR}(s, s) = (e_s^T \cdot J_i^+) \cdot [\left(E - sign\left(J_j \cdot (P_i^{-,=})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} J_i^+(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{-,=})^T\right)],$$

then $M_i^{GMR}(s, s) = 0$ holds iff

$$J_i^+(s, s_1) \cdot [1 - sign\left((e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{-,=})^T\right)] = 0, \tag{4.11}$$

for every $s_1 \in S$. It is clear that Eq. 4.11 is equivalent to

$$(e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{-,=})^T \neq 0,$$

for every $s_1 \in R_i^+(s)$. Therefore, for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R_j(s_1)$ with $s \succeq_i s_2$. According to Definition 4.3, $M_i^{GMR}(s, s) = 0$ implies that s is GMR stable for DM i. □

Theorem 4.4 proves that this matrix method, called matrix representation of GMR stability, is equivalent to the logical representation for two-DM GMR stability in Definition 4.3. To analyze GMR stability at s for DM i, one only needs to identify whether the diagonal entry $M_i^{GMR}(s, s)$ of i's GMR matrix is zero. If so, s is GMR stable for i; otherwise, s is GMR unstable for DM i. Note that all information about GMR stability is contained in the diagonal entries of the GMR matrix.

Example 4.8 (*Matrix Representation of GMR Stability for the Sustainable Development Model*) The logical representation of GMR stability for the sustainable development game was illustrated in Example 4.3. First, one uses $P_i^{-,=} = E - I - P_i^+$ for $i = 1, 2$, to obtain

$$P_1^{-,=} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.12}$$

and

$$P_2^{-,=} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.13}$$

From Eq. 4.10, DM i's GMR matrix is

$$M_i^{GMR} = J_i^+ \cdot \left(E - sign\left(J_j \cdot (P_i^{-,=})^T\right)\right),$$

where $i, j = 1, 2$. Therefore,

$$M_1^{GMR} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} - sign\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T\right)\right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Similarly, DM 2's GMR matrix is calculated by

$$M_2^{GMR} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} - sign \left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}^T \right) \right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Since $M_1^{GMR}(1,1) = M_1^{GMR}(2,2) = M_1^{GMR}(3,3) = 0$ and $M_1^{GMR}(4,4) \neq 0$, then s_1, s_2, and s_3 are GMR stable, while s_4 is GMR unstable, for DM 1, according to Theorem 4.4. Similarly, because $M_2^{GMR}(1,1) = M_2^{GMR}(3,3) = 0$ and $M_2^{GMR}(2,2) = M_2^{GMR}(4,4) = 1$, s_1 and s_3 are GMR stable, and s_2 and s_4 are GMR unstable, for DM 2. These results are identical to those in Example 4.3 obtained by logical representation.

Symmetric metarationality is similar to general metarationality except that DM i expects to have a chance to counterrespond to its opponent j's response to i's original move. Define DM i's SMR $m \times m$ matrix as

$$M_i^{SMR} = J_i^+ \cdot [E - sign(Q)]$$

in which

$$Q = J_j \cdot [(P_i^{-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^+)^T\right)\right)],$$

for $j \in N$, $j \neq i$. The following theorem establishes the matrix method to determine whether state s is SMR stable for a DM.

Theorem 4.5 *State $s \in S$ is SMR for DM i iff $M_i^{SMR}(s,s) = 0$.*

Proof Since

$$M_i^{SMR}(s,s) = (e_s^T \cdot J_i^+) \cdot [(E - sign(Q)) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} J_i^+(s, s_1)[1 - sign\left(Q(s_1, s)\right)]$$

with

$$Q(s_1, s) = \sum_{s_2=1}^{m} J_j(s_1, s_2) \cdot W,$$

and

$$W = P_i^{-,=}(s, s_2) \cdot \left[1 - sign\left(\sum_{s_3=1}^{m} \left(J_i(s_2, s_3) \cdot P_i^{+}(s, s_3)\right)\right)\right],$$

then $M_i^{SMR}(s, s) = 0$ holds iff $Q(s_1, s) \neq 0$, for every $s_1 \in R_i^{+}(s)$, which is equivalent to the statement that, for every $s_1 \in R_i^{+}(s)$, there exists $s_2 \in R_j(s_1)$ such that

$$P_i^{-,=}(s, s_2) \neq 0, \tag{4.14}$$

and

$$\sum_{s_3=1}^{m} J_i(s_2, s_3) \cdot P_i^{+}(s, s_3) = 0. \tag{4.15}$$

Obviously, for every $s_1 \in R_i^{+}(s)$, there exists $s_2 \in R_j(s_1)$ such that Eqs. 4.14 and 4.15 hold iff for every $s_1 \in R_i^{+}(s)$ there exists $s_2 \in R_j(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. □

Theorem 4.5 proves that this matrix method, called matrix representation of SMR stability, is equivalent to the logical representation for two-DM SMR stability in Definition 4.4. To calculate SMR stability at s for DM i, one only needs to assess whether the diagonal entry $M_i^{SMR}(s, s)$ of i's SMR matrix is zero. If so, s is SMR stable for i; otherwise, s is SMR unstable for DM i.

Example 4.9 (*Matrix Representation of SMR Stability for the Sustainable Development Model*) The logical representation of SMR stability for the sustainable development game was described in Example 4.4. First, one uses Eq. 4.7 for J_i, Eq. 4.8 for P_i^{+}, Eq. 4.9 for J_i^{+}, and Eqs. 4.12 and 4.13 for $P_i^{-,=}$, for $i = 1, 2$. DM i's SMR matrix is

$$M_i^{SMR} = J_i^{+} \cdot [E - sign(Q)]$$

in which

$$Q = J_j \cdot [(P_i^{-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+})^T\right)\right)],$$

where $i, j = 1, 2$. Therefore,

$$M_1^{SMR} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } M_2^{SMR} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Since $M_1^{SMR}(1, 1) = M_1^{SMR}(2, 2) = M_1^{SMR}(3, 3) = 0$ and $M_1^{SMR}(4, 4) \neq 0$, then s_1, s_2, and s_3 are SMR stable, while s_4 are SMR unstable, for DM 1, according to Theorem 4.5. Similarly, because $M_2^{SMR}(1,1) = M_2^{SMR}(3,3) = 0$ and $M_2^{SMR}(2,2) = M_2^{SMR}(4,4) = 1$, s_1 and s_3 are SMR stable, and s_2 and s_4 are SMR unstable, for DM 2. These results are identical to those in Example 4.4 obtained by logical representation.

Sequential stability is similar to general metarationality, but includes only those sanctions that are "credible". Define DM i's SEQ $m \times m$ matrix as

$$M_i^{SEQ} = J_i^+ \cdot [E - sign\left(J_j^+ \cdot (P_i^{-,=})^T\right)],$$

for $j \in N, j \neq i$. The following theorem provides the matrix method to analyze whether state s is SEQ stable for a DM.

Theorem 4.6 *State $s \in S$ is SEQ for DM i iff $M_i^{SEQ}(s,s) = 0$.*

Proof Since

$$M_i^{SEQ}(s,s) = (e_s^T J_i^+) \cdot [\left(E - sign(J_j^+ \cdot (P_i^{-,=})^T)\right) e_s]$$

$$= \sum_{s_1=1}^{|S|} J_i^+(s,s_1)[1 - sign((e_{s_1}^T J_j^+) \cdot \left(e_s^T P_i^{-,=})^T\right)],$$

then $M_i^{SEQ}(s,s) = 0$ holds iff

$$J_i^+(s,s_1)[1 - sign\left((e_{s_1}^T J_j^+) \cdot (e_s^T P_i^{-,=})^T\right)] = 0, \forall s_1 \in S. \tag{4.16}$$

It is clear that Eq. 4.16 is equivalent to

$$(e_{s_1}^T J_j^+) \cdot (e_s^T P_i^{-,=})^T \neq 0, \forall s_1 \in R_i^+(s).$$

It implies that for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R_j^+(s_1)$ with $s \succeq_i s_2$. □

Note that the SEQ matrix is identical to the GMR matrix except that DM j's UM matrix J_j is replaced by the UI matrix J_j^+.

Theorem 4.6 proves that this matrix method, called matrix representation of SEQ stability, is equivalent to the logical representation for two-DM SEQ stability in Definition 4.5. To identify DM i's SEQ stability at s for DM i, one only needs to determine whether the diagonal entry $M_i^{SEQ}(s,s)$ of i's SEQ matrix is zero. If so, s is SEQ stable for i; otherwise, s is SEQ unstable for DM i.

Example 4.10 (*Matrix Representation of SEQ Stability for the Sustainable Development Model*) The logical representation of SEQ stability for the sustainable development game was presented in Example 4.5. First, one uses Eq. 4.9 to obtain J_i^+, and Eqs. 4.12 and 4.13 for $P_i^{-,=}$, for $i = 1, 2$. DM i's SEQ matrix is

$$J_i^+ \cdot [E - sign\left(J_j^+ \cdot (P_i^{-,=})^T\right)],$$

where $i, j = 1, 2$. Therefore,

$$M_1^{SEQ} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } M_2^{SEQ} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Since $M_1^{SEQ}(1,1) = M_1^{SEQ}(2,2) = 0$ and $M_1^{SEQ}(3,3) = M_1^{SEQ}(4,4) = 1$, then s_1 and s_2 are SEQ stable, while s_3 and s_4 are SEQ unstable, for DM 1. Similarly, because $M_2^{SEQ}(1,1) = M_2^{SEQ}(3,3) = 0$ and $M_2^{SEQ}(2,2) = M_2^{SEQ}(4,4) = 1$, s_1 and s_3 are SEQ stable, and s_2 and s_4 are SEQ unstable, for DM 2. These results are identical to those in Example 4.5 obtained by logical representation.

4.3.3 *Matrices to Construct Reachable Lists of a Coalition*

The aim of a stability analysis is to find the states of a graph model that are stable for all DMs, under appropriate stability definitions, or equilibria. As discussed in Sect. 4.2.2, the reachable lists of coalition H by sequences of the legal UMs and the legal UIs, $R_H(s)$ and $R_H^+(s)$, are essential ingredients for stability analysis and the construction of these two sets is a complicated process. In this section, the reachability matrices M_H and M_H^+ are proposed to provide an algebraic method of constructing $R_H(s)$ and $R_H^+(s)$ (Xu et al. 2010b).

4.3.3.1 Several Extended Definitions in the Graph Model

First, the adjacency matrix and the incidence matrix of a graph (Godsil and Royle 2001) are extended to a graph model. Let $m = |S|$ and $l = |A|$.

Definition 4.14 For a graph model G, the **UM adjacency matrix** and the **UI adjacency matrix** for $H \subseteq N$ and $H \neq \emptyset$ are $m \times m$ matrices J_H and J_H^+ with (s, q) entries

$$J_H(s,q) = \begin{cases} 1 & \text{if } d_i(s,q) \in A \text{ for some } i \in H, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$J_H^+(s,q) = \begin{cases} 1 & \text{if } d_i(s,q) \in A^+ \text{ for some } i \in H, \\ 0 & \text{otherwise,} \end{cases}$$

for $s, q \in S$ in which $d_i(s,q)$ denotes arc (s,q) controlled by some DM i.

The adjacency matrix for any coalition H has been defined. For example, if $H = i$, then $J_H(s,q)$ reduces to $J_i(s,q)$ that represents the adjacency relation between s and q in DM i's graph.

Definition 4.15 For the graph model, the **UM incidence matrix** and the **UI incidence matrix** are $m \times l$ matrices B and B^+, with (s,a) entries

$$B(s,a) = \begin{cases} -1 & \text{if } a = d_i(s,x) \in A \text{ for some } i \in N \text{ and some } x \in S, \\ 1 & \text{if } a = d_i(x,s) \in A \text{ for some } i \in N \text{ and some } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B^+(s,a) = \begin{cases} -1 & \text{if } a = d_i(s,x) \in A^+ \text{ for some } i \in N \text{ and some } x \in S, \\ 1 & \text{if } a = d_i(x,s) \in A^+ \text{ for some } i \in N \text{ and some } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

where $s \in S$ and $a \in A$.

The extension of incidence matrix has two versions, both including and excluding preference information.

According to the signs of the entries, the UM incidence matrix can be separated into the UM in-incidence and out-incidence matrices.

Definition 4.16 For a graph model G, the **UM in-incidence matrix** and the **UM out-incidence matrix** are the $m \times l$ matrices B_{in} and B_{out} with (s,a) entries

$$B_{in}(s,a) = \begin{cases} 1 & \text{if } a = d_i(x,s) \in A \text{ for some } i \in N \text{ and some } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B_{out}(s,a) = \begin{cases} 1 & \text{if } a = d_i(s,x) \in A \text{ for some } i \in N \text{ and some } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

where $s \in S$ and $a \in A$.

It is obvious that $B_{in} = (B + |B|)/2$ and $B_{out} = (|B| - B)/2$, where $|B|$ denotes the matrix in which each entry equals the absolute value of the corresponding entry of B. The **UI in-incidence matrix** B_{in}^+ and the **UI out-incidence matrix** B_{out}^+ can be defined similarly.

The relationships among the UM, UI adjacency matrices and the UM, UI in-incidence and out-incidence matrices are described as follows:

Theorem 4.7 *In a graph model G, J_H and J_H^+ denote the UM and the UI adjacency matrices for H, B_{in} and B_{out} denote the UM in-incidence and out-incidence matrices, and B_{in}^+ and B_{out}^+ indicate the UI in-incidence and out-incidence matrices. Then*

$$J_H = B_{out} \cdot I_H \cdot (B_{in})^T \text{ and } J_H^+ = B_{out}^+ \cdot I_H \cdot (B_{in}^+)^T,$$

where I_H is the $l \times l$ diagonal matrix in which $I_H(k,k) = 1$ if $a_k = d_i(s,q)$ and $i \in H$, otherwise $I_H(k,k) = 0$. Note that the diagonal matrix I_H has 1's as the (k,k) entry if and only if the arc a_k is controlled by DM i; otherwise all diagonal entries, and, of course, all non-diagonal entries are zeros.

From algebraic graph theory (Godsil and Royle 2001), Theorem 4.7 can follow easily. Two important matrices to link conflict evolution that will be introduced in Chap. 9 and conflict resolution in the graph model are proposed as follows:

Definition 4.17 For the graph model G, the **legal UM arc-incidence matrix** LJ_H and the **legal UI arc-incidence matrix** LJ_H^+ for coalition H are the $l \times l$ matrices with (a,b) entries

$$LJ_H(a,b) = \begin{cases} 1 & \text{if edge } a \text{ is incident on edge } b \text{ in } IG(G) \text{ for } a,b \in A, \\ & \text{and } a \text{ and } b \text{ controlled by different DMs in } H, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$LJ_H^+(a,b) = \begin{cases} 1 & \text{if edge } a \text{ is incident on edge } b \text{ in } IG(G) \text{ for } a,b \in A^+, \\ & \text{and } a \text{ and } b \text{ controlled by different DMs in } H, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $H = N$, then LJ_N and LJ_N^+ are written as LJ and LJ^+, respectively.

Let D_i and D_i^+ denote the $l \times l$ diagonal matrices with (k,k) entries

$$D_i(k,k) = \begin{cases} 1 & \text{if } a_k = d_i(s,q) \text{ for } s,q \in S \text{ and } a_k \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$D_i^+(k,k) = \begin{cases} 1 & \text{if } a_k = d_i(s,q) \text{ for } s,q \in S \text{ and } a_k \in A^+, \\ 0 & \text{otherwise.} \end{cases}$$

Based on Definitions 4.16 and 4.17, the legal UM and the legal UI arc-incidence matrices can be obtained by the following theorem.

Theorem 4.8 *For the graph model G, let B_{in} and B_{out} be the UM in-incidence and the UM out-incidence matrices, and B_{in}^{+} and B_{out}^{+} denote the UI in-incidence and the UI out-incidence matrices. Then, the legal UM arc-incidence matrix LJ_H and the legal UI arc-incidence matrix LJ_H^{+} for coalition H satisfy that*

$$LJ_H = \bigvee_{i,j \in H, i \neq j} [(B_{in} \cdot D_i)^T \cdot (B_{out} \cdot D_j)],$$

and

$$LJ_H^{+} = \bigvee_{i,j \in H, i \neq j} [(B_{in}^{+} \cdot D_i^{+})^T \cdot (B_{out}^{+} \cdot D_j^{+})].$$

Proof Let $M = \bigvee_{i,j \in H, i \neq j} [(B_{in} \cdot D_i)^T \cdot \big(B_{out} \cdot D_j\big)]$. Thus, any entry (a_k, a_h) of matrix M can be expressed as

$$M(a_k, a_h) = sign[\sum_{i,j \in H, i \neq j} \sum_{q=1}^{m} (B_{in}(q, a_k) \cdot D_i(k,k) \cdot B_{out}(q, a_h) \cdot D_j(h,h))],$$

for $a_k, a_h \in A$ and $q \in S$.

Hence, $M(a_k, a_h) \neq 0$ iff $B_{in}(q, a_k) \cdot B_{out}(q, a_h) \neq 0$ for some $q \in S$ such that $a_k = d_i(s, q)$ and $a_h = d_j(q, u)$ for $s, u \in S$, and $i, j \in H$ and $i \neq j$. This implies that $M(a_k, a_h) \neq 0$ iff edge a_k is incident on edge a_h in $IG(G)$ and a_k and a_h are controlled by different DMs in H (see Fig. 4.5). Therefore, based on the definition of the matrix LJ, $M(a_k, a_h) \neq 0$ iff $LJ_H(a_k, a_h) \neq 0$. Since M and LJ_H are 0–1 matrices, then, $LJ_H = \bigvee_{i,j \in H, i \neq j} [(B_{in} \cdot D_i)^T \cdot (B_{out} \cdot D_j)]$ follows.

The proof of $LJ_H^{+} = \bigvee_{i,j \in H, i \neq j} [(B_{in}^{+} \cdot D_i^{+})^T \cdot (B_{out}^{+} \cdot D_j^{+})]$ is similar. □

It is obvious that unilateral moves on the branches of paths will end when the same arc appears twice. Generally, if there is no new appropriate arc produced, then the corresponding joint moves will stop. Therefore, the following Lemma 4.1 is obvious. Let $l = |A|$, $l^{+} = |A^{+}|$ in the following lemma.

Lemma 4.1 *For the graph model G, let $H \subseteq N$. $R_H(s)$ and $R_H^{+}(s)$ are the reachable lists of H by the legal sequences of UMs and UIs from s. The δ_1 and δ_2 symbols are the numbers of iteration steps required to find $R_H(s)$ and $R_H^{+}(s)$, respectively. Then*

$$\delta_1 \leq l \quad and \quad \delta_2 \leq l^{+}.$$

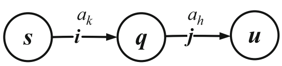


Fig. 4.5 a_k incident on a_h in $IG(G)$

Lemma 4.2 *For a graph model G, let t be a nonnegative integer, and fix* $a, b \in A$*. Then,* $(LJ_H)^t(a, b)$*, the* (a, b) *entry of matrix* $(LJ_H)^t$ *equals the number of legal UM arc-by-arc paths of length t in G for H, from edge a to edge b.*

Proof This Lemma is proved using induction on t.

When $t = 1$, the result is obvious. Note that $(LJ_H)^1(a, b)$ equals the number of arcs from a to b.

Assume that when $t = r$, the result holds. Then, when $t = r + 1$, $(LJ_H)^{r+1}(a_k, a_h) = \sum_{w=1}^{l} [(LJ_H)^r(a_k, a_w) \cdot LJ_H(a_w, a_h)]$. By the induction hypothesis, $(LJ_H)^r(a_k, a_w)$ denotes the number of legal paths by UMs from a_k to a_w with length r, and $LJ_H(a_w, a_h)$ equals the number of legal paths by UMs from a_w to a_h with length 1. Thus, $(LJ_H)^r(a_k, a_w) \cdot LJ_H(a_w, a_h)$ denotes the number of legal paths from a_k to a_h through a_w with length $r + 1$. Therefore, $\sum_{w=1}^{l} [(LJ_H)^r(a_k, a_w) \cdot LJ_H(a_w, a_h)]$ is the total number of legal paths from a_k to a_h by UMs with length $r + 1$.

Therefore, $(LJ_H)^t(a, b)$ equals the number of legal UM arc-by-arc paths for H from edge a to edge b with length t. □

Note that if $a = d_i(u, s)$ and $b = d_j(q, v)$ for $u, s, q, v \in S$ and $i, j \in H$, then the number of legal UM state-by-state paths for H from state u to state v of length $t + 1$ is at least $(LJ_H)^t(a, b)$. In fact, $(LJ_H)^t(a, b)$ is the number of legal paths of length t from u to v with initial edge a and terminal edge b. Similarly, $(LJ_H^+)^t(a, b)$ denotes the number of legal UI arc-by-arc paths for H in the G from edge a to edge b with length t. For example, Fig. 4.6a depicts an arc-by-arc path from arc a_1 to arc a_4 with length 5 in the graph model G presented in Fig. 4.3, where $a_1 = d_1(s_1, s_2)$ and $a_4 = d_2(s_3, s_6)$. Figure 4.6b presents the corresponding state-by-state path from s_1 to s_6 with initial edge a_1 and terminal edge a_4, which is of length 6.

The UM incidence matrix B and the UI incidence matrix B^+ depict unilateral move and unilateral improvement in one-step. The legal UM arc-incidence matrix LJ and the legal UI arc-incidence matrix LJ^+ can trace all evolutionary paths of length greater than 1 by UMs and UIs in a strategic conflict, respectively. The details of the evolution of a conflict will be discussed in Chap. 9.

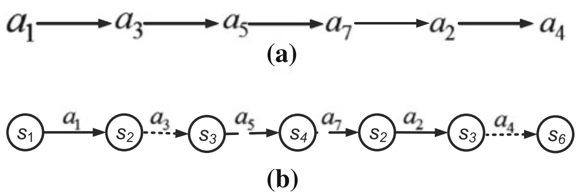


Fig. 4.6 The arc-by-arc and the state-by-state UM paths

Example 4.11 Determine the legal UM and the legal UI arc-incidence matrices for the graph model G presented in Fig. 4.3 (Xu et al. 2010b).

Based on Fig. 4.3, the UM incidence matrix B is

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$B_{in} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mathrm{B_{out}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then Theorem 4.8 implies that the legal UM arc-incidence and the legal UI arc-incidence matrices are

$$LJ = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mathrm{LJ}^{+} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Searching the nonzero entries of matrix LJ produces the UM arc-by-arc evolutionary path from a_1 to a_4 as presented in Fig. 4.7. Since there are two nonzero entries in the seventh row of matrix LJ, as seen in Fig. 4.7, branches a_2 and a_3 appear following a_7. However, arc a_3 has been passed in the path, so the branch with a_3 ends. Similarly, the branch following arc a_2 with arc a_5 stops. However, because a_4 is not a UI arc, a_4 cannot be reached by the legal UI paths so that s_6 is not reachable by the legal sequence of UIs from s_1.

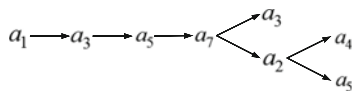


Fig. 4.7 The arc-by-arc evolutionary paths from a_1 to a_4

4.3.3.2 Reachability Matrices to Construct Reachable Lists of a Coalition

Definition 4.18 For the graph model G, the t-**UM reachability matrix** and the t-**UI reachability matrix** for H, where $t = 1, 2, 3, \cdots$, are the $m \times m$ matrices with (s, q) entries

$$M_H^{(t)}(s,q) = \begin{cases} 1 & \text{if } q \in S \text{ is reachable by } H \text{ from } s \in S \text{ in exactly } t \text{ legal UMs,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$M_H^{(t,+)}(s,q) = \begin{cases} 1 & \text{if } q \in S \text{ is reachable by } H \text{ from } s \in S \text{ in exactly } t \text{ legal UIs,} \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $M_H^{(1)} = J_H$ and $M_H^{(1,+)} = J_H^+$. The t-UM and the t-UI reachability matrices for coalition H can be constructed by the following lemma.

Lemma 4.3 *For the graph model G, let B_{in} and B_{out} denote the in-incidence and out-incidence matrices, respectively. LJ_H and LJ_H^+ are the legal UM and the legal UI arc-incidence matrices for H. Then, for $t \geq 2$, the t-UM reachability and the t-UI reachability matrices for H can be expressed as*

$$M_H^{(t)} = sign[B_{out} \cdot (LJ_H)^{t-1} \cdot B_{in}^T] \text{ and } M_H^{(t,+)} = sign[B_{out}^+ \cdot (LJ_H^+)^{t-1} \cdot (B_{in}^+)^T].$$

Proof Based on Definition 4.18, $M_H^{(t)}(u, v) = 1$ iff state v is reachable by coalition H from state u in exactly t legal unilateral moves. Let $(LJ_H)^{t-1} = Q$ and $W = sign[B_{out} \cdot Q \cdot B_{in}^T]$. Then $W(u, v) \neq 0$ iff there exist $Q(a, b) \neq 0$ such that $a, b \in A$, $a = d_i(u, s)$, and $b = d_j(q, v)$ for $i, j \in H$, where $s, q, u, v \in S$. Using Lemma 4.2, $Q(a, b) \neq 0$ implies that state v can be attained by H from state u in exactly t legal UMs. Therefore, $M_H^{(t)}(u, v) = 1$ iff $W(u, v) \neq 0$. Since $M_H^{(t)}$ and W are 0–1 matrices, $M_H^{(t)} = sign[B_{out} \cdot (LJ)^{t-1} \cdot B_{in}^T]$.

The proof of $M_H^{(t,+)} = sign[B_{out}^+ \cdot (LJ_H^+)^{t-1} \cdot (B_{in}^+)^T]$ is similar. □

Definition 4.19 For the graph model G, the UM reachability matrix and the UI reachability matrix for H are the $m \times m$ matrices M_H and M_H^+ with (s, q) entries

$$M_H(s,q) = \begin{cases} 1 & \text{if } q \in R_H(s), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$M_H^+(s,q) = \begin{cases} 1 & \text{if } q \in R_H^+(s), \\ 0 & \text{otherwise,} \end{cases}$$

respectively.

It is clear that $R_H(s) = \{q : M_H(s,q) = 1\}$ and $R_H^+(s) = \{q : M_H^+(s,q) = 1\}$. If $R_H(s)$ and $R_H^+(s)$ are written as 0–1 row vectors, then

$$R_H(s) = e_s^T \cdot M_H \text{ and } \mathrm{R_H^+(s) = e_s^T \cdot M_H^+},$$

where e_s^T denotes the transpose of the sth standard basis vector of the m-dimensional Euclidean space. Therefore, the reachability matrices for coalition H, M_H and M_H^+, can be used to construct the reachable lists of H from state s, $R_H(s)$ and $R_H^+(s)$.

The reachability matrices for coalition H can now be obtained by the following lemma.

Lemma 4.4 *For the graph model, let $M_H^{(t)}$ and $M_H^{(t,+)}$ be the t-UM and the t-UI reachability matrices. Then, the UM and the UI reachability matrices for H satisfy that*

$$M_H = \bigvee_{t=1}^{l} M_H^{(t)} \text{ and } M_H^+ = \bigvee_{t=1}^{l^+} M_H^{(t,+)}.$$

Proof Let $C = \bigvee_{t=1}^{l} M_H^{(t)}$. Based on the definition of M_H, $M_H(u,v) \neq 0$ iff v is reachable by H from u with a sequence of legal UMs. By Lemma 4.1, $l \geq \delta_1$. Hence, $M_H(u,v) \neq 0$ iff there exists $1 \leq t_0 \leq \delta_1 \leq l$ such that v is reachable by H from u with t_0 legal UMs. Based on Definition 4.18, this implies that $M_H^{(t_0)}(u,v) = 1$. Therefore, $M_H(u,v) \neq 0$ iff $C(u,v) \neq 0$. Since M_H and C are 0–1 matrices, $M_H = \bigvee_{t=1}^{l} M_H^{(t)}$ holds. The proof of $M_H^+ = \bigvee_{t=1}^{l^+} M_H^{(t,+)}$ can be carried out similarly. □

Lemma 4.5 *For the graph model G, LJ_H and LJ_H^+ denote the legal UM arc-incidence matrix and the legal UI arc-incidence matrix for H, respectively. Then*

(1) $(LJ_H + I)^n = \sum_{t=0}^{n} C_n^t \cdot (LJ_H)^t$,

(2) $(LJ_H^+ + I)^n = \sum_{t=0}^{n} C_n^t \cdot (LJ_H^+)^t$,

where the constant $C_n^t = \binom{n}{t} = \frac{n\cdot(n-1)\cdots(n-t+1)}{t!}$, $(LJ_H)^0 = (LJ_H^+)^0 = I_H$, *and I is the identity matrix.*

The above lemma is an obvious result of matrix theory. Based on the above discussions, the relations among the reachability matrices and the legal arc-incidence matrices for coalition H can now be established by the following theorem.

Theorem 4.9 *For the graph model G, LJ_H and LJ_H^+ are the legal UM and the legal UI arc-incidence matrices for H, respectively. The UM and UI reachability matrices for H, M_H and M_H^+, can be obtained by*

$M_H = sign[B_{out} \cdot (LJ_H + I)^{l-1} \cdot B_{in}^T]$ *and* $M_H^+ = sign[B_{out}^+ \cdot (LJ_H^+ + I)^{l^+-1} \cdot (B_{in}^+)^T]$,

where I *is the identity matrix.*

Proof Let $Q = sign[B_{out} \cdot (LJ_H + I)^{l-1} \cdot B_{in}^T]$. By Lemma 4.5 and $C_{l-1}^t > 0$, then

$$Q = sign[\sum_{t=0}^{l-1} C_{l-1}^t \cdot B_{out} \cdot (LJ_H)^t \cdot B_{in}^T] = (B_{out} \cdot I_H \cdot B_{in}^T) \bigvee sign[\sum_{t=1}^{l-1} B_{out} \cdot (LJ_H)^t \cdot B_{in}^T].$$

Based on Lemma 4.3 and Theorem 4.7,

$$Q = J_H \bigvee \left(\bigvee_{t=1}^{l-1} M_H^{(t+1)} \right) = \bigvee_{t=1}^{l} M_H^{(t)}.$$

Based on Lemma 4.4, $M_H = sign[B_{out} \cdot (LJ_H + I)^{l-1} \cdot B_{in}^T]$ follows.

The proof of $M_H^+ = sign[B_{out}^+ \cdot (LA^+ + I)^{l^+-1} \cdot (B_{in}^+)^T]$ is similar. □

The aim of a stability analysis is to find the equilibria of a graph model that are stable for all DMs under appropriate stability definitions. The reachable lists of coalition H by the sequences of the legal UMs and the legal UIs, $R_H(s)$ and $R_H^+(s)$, are essential components for stability analysis and the construction of the two state sets is a complicated process (Fang et al. 1993). An algebraic method for constructing $R_H(s)$ and $R_H^+(s)$ using the reachability matrices M_H and M_H^+ based on the incidence matrix B is developed here. In Chap. 5, another algebraic approach based on the adjacency matrix J is presented in Theorem 5.20.

Example 4.12 Fig. 4.3 shows a graph model with DM set $N = \{1, 2, 3, 4\}$ and state set $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. The UM reachability matrix M_N is calculated according to Theorem 4.9 by

$$M_N = sign[B_{out} \cdot (LJ + I)^{l-1} \cdot B_{in}^T] = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where B_{out}, B_{in} and LJ are provided by Example 4.11. Similarly, the UI reachability matrix M_N^+ is obtained by

$$M_N^+ = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $s = s_1$ is selected as the status quo state, then the reachable lists of $H = N$ from s_1, $R_N(s_1)$ and $R_N^+(s_1)$, can be constructed by $R_H(s_1) = \{q : M_H(s_1, q) = 1\}$ and $R_H^+(s_1) = \{q : M_H^+(s_1, q) = 1\}$. Therefore, $R_N(s_1) = \{s_2, s_3, s_4, s_6\}$. $R_N^+(s_1) = \{s_2, s_3, s_4\}$.

Theorem 4.9 provides an algebraic method to construct the reachable lists of a coalition. The matrix representation of stability definitions can be extended to models including more than two DMs, which is the objective of the next subsection.

4.3.4 n-Decision Maker Case

Equivalent matrix representations of the logical definitions for Nash stability, GMR, SMR, and SEQ can be determined directly by using the relationship that has been established between matrix elements and the state set of a graph model, and by using preference relation matrices among the states.

Let $i \in N$, $|N| = n$, and $|S| = m$ in the following theorems. Nash stability in n-DM models is identical to two-DM cases because Nash stability does not consider opponents' responses.

It should be pointed out that the following stability matrices for n-DMs use the same notation as that presented in Sect. 4.3.2 for two-DMs. For general metarationality, DM i will take into account the opponents' possible responses, which are the legal sequence of UMs by members of $N\backslash\{i\}$. For $i \in N$, find DM i's UI adjacency matrix J_i^+ and the UM reachability matrix $M_{N\backslash\{i\}}$ using Theorem 4.9 for which $H = N\backslash\{i\}$. Define the $m \times m$ matrix M_i^{GMR} as

$$M_i^{GMR} = J_i^+ \cdot [E - sign\big(M_{N\backslash\{i\}} \cdot (P_i^{-,=})^T\big)].$$

Theorem 4.10 *State $s \in S$ is GMR for DM i, denoted by $s \in S_i^{GMR}$, iff M_i^{GMR} $(s, s) = 0$.*

Proof Since the diagonal element of matrix M_i^{GMR}

$$M_i^{GMR}(s, s) = \langle (J_i^+)^T e_s, \big(E - sign\big(M_{N\backslash\{i\}} \cdot (P_i^{-,=})^T\big)\big) e_s \rangle$$

$$= \sum_{s_1=1}^{m} J_i^+(s, s_1)[1 - sign\big(\langle (M_{N\backslash\{i\}})^T e_{s_1}, (P_i^{-,=})^T e_s \rangle\big)],$$

then $M_i^{GMR}(s, s) = 0$ iff

$$J_i^+(s, s_1)[1 - sign\big(\langle (M_{N\backslash\{i\}})^T e_{s_1}, (P_i^{-,=})^T e_s \rangle\big)] = 0, \forall s_1 \in S.$$

This implies that $M_i^{GMR}(s,s)=0$ iff

$$(e_{s_1}^T M_{N\backslash\{i\}})\cdot(e_s^T P_i^{-,=})^T \neq 0, \forall s_1 \in R_i^+(s). \tag{4.17}$$

Equation 4.17 means that, for any $s_1 \in R_i^+(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector, $e_{s_1}^T \cdot M_{N\backslash\{i\}}$, with s_2th element 1 and the m-dimensional column vector, $(P_i^{-,=})^T \cdot e_s$, with s_2th element 1.

Therefore, $M_i^{GMR}(s,s)=0$ iff for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$. □

For symmetric metarationality, the n-DM model is similar to the two-DM model. The only modification is that responses come from DM i's opponents instead of from a single DM. Let

$$G=(P_i^{-,=})^T \circ [E - sign\big(J_i \cdot (P_i^+)^T\big)],$$

then define the $m \times m$ matrix M_i^{SMR} as

$$M_i^{SMR} = J_i^+ \cdot [E - sign(M_{N\backslash\{i\}} \cdot G)].$$

Theorem 4.11 *State $s \in S$ is SMR for DM i, denoted by $s \in S_i^{SMR}$, iff $M_i^{SMR}(s,s)=0$.*

Proof Since the diagonal element of matrix M_i^{SMR}

$$M_i^{SMR}(s,s) = \langle (J_i^+)^T \cdot e_s, \big(E - sign(M_{N\backslash\{i\}}\cdot G)\big)e_s\rangle$$

$$= \sum_{s_1=1}^{m} J_i^+(s,s_1)[1 - sign\big(\langle (M_{N\backslash\{i\}})^T \cdot e_{s_1}, G\cdot e_s\rangle\big)],$$

then $M_i^{SMR}(s,s)=0$ iff

$$J_i^+(s,s_1)[1 - sign\big(\langle (M_{N\backslash\{i\}})^T \cdot e_{s_1}, G\cdot e_s\rangle\big)] = 0, \forall s_1 \in S.$$

This means that $M_i^{SMR}(s,s)=0$ iff

$$(e_{s_1}^T \cdot M_{N\backslash\{i\}})\cdot(G\cdot e_s) \neq 0, \forall s_1 \in R_i^+(s). \tag{4.18}$$

Let $G(s_2,s)$ denote the (s_2,s) entry of matrix G. Since

$$(e_{s_1}^T M_{N\backslash\{i\}})\cdot(G\cdot e_s) = \sum_{s_2=1}^{m} M_{N\backslash\{i\}}(s_1,s_2)\cdot G(s_2,s),$$

then Eq. 4.18 holds iff for any $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $G(s_2,s) \neq 0$.

Because $G(s_2, s) = P_i^{-,=}(s, s_2)[1 - sign(\sum_{s_3=1}^{m} J_i(s_2, s_3)P_i^{+}(s, s_3))]$, then $G(s_2, s) \neq 0$ implies that for $s_2 \in R_{N\backslash\{i\}}(s_1)$,

$$P_i^{-,=}(s, s_2) \neq 0 \tag{4.19}$$

and

$$\sum_{s_3=1}^{m} J_i(s_2, s_3)P_i^{+}(s, s_3) = 0. \tag{4.20}$$

Equation 4.19 is equivalent to the statement that, $\forall s_1 \in R_i^{+}(s)$, $\exists s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $s \succeq_i s_2$. Equation 4.20 is the same as the statement that, $\forall s_1 \in R_i^{+}(s)$, $\exists s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $P_i^{+}(s, s_3) = 0$ for $\forall s_3 \in R_i(s_2)$. Based on the definition of $m \times m$ matrix P_i^{+}, one knows that $P_i^{+}(s, s_3) = 0 \Longleftrightarrow s \succeq_i s_3$.

Therefore, the above discussion is concluded that $M_i^{SMR}(s, s) = 0$ iff for any $s_1 \in R_i^{+}(s)$, there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$ and $s \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. □

Sequential stability examines the credibility of the sanctions by DM i's opponents. For $i \in N$, find the UI reachability matrix $M_{N\backslash\{i\}}^{+}$ using Theorem 4.9. Define the $m \times m$ matrix M_i^{SEQ} as

$$M_i^{SEQ} = J_i^{+} \cdot [E - sign\big(M_{N\backslash\{i\}}^{+} \cdot (P_i^{-,=})^T\big)].$$

Theorem 4.12 *State $s \in S$ is SEQ for DM i, denoted by $s \in S_i^{SEQ}$, iff M_i^{SEQ} $(s, s) = 0$.*

Proof Since the diagonal element of matrix M_i^{SEQ}

$$M_i^{SEQ}(s, s) = \langle (J_i^{+})^T \cdot e_s, \big(E - sign\big(M_{N\backslash\{i\}}^{+} \cdot (P_i^{-,=})^T\big)\big)e_s \rangle$$

$$= \sum_{s_1=1}^{m} J_i^{+}(s, s_1)[1 - sign\big(\langle (M_{N\backslash\{i\}}^{+})^T \cdot e_{s_1}, (P_i^{-,=})^T \cdot e_s \rangle\big)],$$

then $M_i^{SEQ}(s, s) = 0$ iff $J_i^{+}(s, s_1)[1 - sign\big(\langle (M_{N\backslash\{i\}}^{+})^T \cdot e_{s_1}, (P_i^{-,=})^T \cdot e_s \rangle\big)] = 0, \forall s_1 \in S$. This implies that $M_i^{SEQ}(s, s) = 0$ iff

$$(e_{s_1}^T M_{N\backslash\{i\}}^{+}) \cdot (e_s^T \cdot P_i^{-,=})^T \neq 0, \forall s_1 \in R_i^{+}(s). \tag{4.21}$$

Equation 4.21 means that, for any $s_1 \in R_i^{+}(s)$, there exists $s_2 \in S$, such that the m-dimensional row vector, $e_{s_1}^T \cdot M_{N\backslash\{i\}}^{+}$, with s_2th element 1 and the m-dimensional column vector, $(P_i^{-,=})^T \cdot e_s$, with s_2th element 1.

Therefore, $M_i^{SEQ}(s, s) = 0$ iff for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R^+_{N\setminus\{i\}}(s_1)$ with $s \succeq_i s_2$. □

When $n = 2$, the DM set N becomes to $\{i, j\}$ in Theorems 4.10–4.12, and the reachable lists for $H = N \setminus \{i\}$ by legal sequences of UMs and UIs from s_1, $R_{N\setminus\{i\}}(s_1)$ and $R^+_{N\setminus\{i\}}(s_1)$, degenerate to $R_j(s_1)$ and $R_j^+(s_1)$, DM j's corresponding reachable lists from s_1. Thus, Theorems 4.10–4.12 are reduced to Theorems 4.4–4.6.

4.4 Computational Complexity

The proposed matrix method raises the question of computational complexity, which is the number of steps or arithmetic operations required to solve a computational problem. In this section, the computational complexities of MRSC and the graph model stability definitions are compared using general metarationality as an example.

4.4.1 Two Decision Maker Case

Recall the logical representation of GMR stability. State s is GMR for DM i iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s \succeq_i s_2$. Let $m = |S|$. The following procedures are utilized to calculate DM i's GMR stability.

- It takes at most m operations (or comparisons) to determine the state set $R_i^+(s)$;
- for every $s_1 \in R_i^+(s)$, it takes at most m operations (comparisons) to find $R_j(s_1)$;
- for $s_2 \in R_j(s_1)$, it makes at most $m - 1$ preference comparisons about states s and s_2.

Hence, using the logical definition to calculate DM i's GMR stability for state s will take at most $m + (m - 1)(m + m - 1) = 2m^2 - 2m + 1$ comparisons.

Recall DM i's matrix representation of GMR stability for state s. State s is GMR for DM i iff $M_i^{GMR}(s, s) = 0$, where $M_i^{GMR} = J_i^+ \left(E - sign\left(J_j \cdot (P_i^{-,=})^T\right)\right)$. By the proof of Theorem 4.4, one knows that $M_i^{GMR}(s, s) = 0$ iff

$$\sum_{s_1=1}^{m} J_i^+(s, s_1) \cdot \left(1 - sign((e_{s_1}^T \cdot J_j) \cdot (e_s^T \cdot P_i^{-,=})^T)\right) = 0. \tag{4.22}$$

It is easy to see that Eq. 4.22 takes $2m^2$ multiplication and addition operations. Comparing the computational complexities of these two methods to calculate GMR stability, one finds that their computational complexities are $O(m^2)$ at the same level. Note that the computational complexity of GMR stability is considered for the worst case. For logical representation of the GMR stability, the actual number of comparisons required is often smaller than $2m^2 - 2m + 1$. For matrix representation, the standard multiplication of two $m-$dimensional vectors is used, so it requires $O(m^2)$ arithmetic operations.

4.4.2 n-Decision Maker Case

In n-DM models, GMR stability is also selected as an example for analysis of the computational complexity of the proposed matrix method. According to Theorem 4.10, GMR stability definition is formulated using matrices as follows. State s is GMR for DM i iff $M_i^{GMR}(s, s) = 0$, where $M_i^{GMR} = J_i^+ \cdot [E - sign(M_{N\setminus\{i\}} \cdot (P_i^{-,=})^T)]$. By Theorem 4.9, one can estimate the computational complexity of the UM reachability matrix M_N. It is less than $\delta \cdot (n-1) \cdot O(m^3)$, where δ is the number of iterations, $n = |N|$ is the number of DMs, and m is the number of states. Let $l = |\bigcup_{i \in N} A_i|$. Then $\delta \leq l$ using Lemma 4.1. Therefore, the computational complexity to calculate DM i's GMR stability for state s in n-DM models is less than

$$l \cdot (n-1) \cdot O(m^3) + O(m^2) = l \cdot (n-1) \cdot O(m^3),$$

which presents a polynomial-time effective algorithm.

Many researchers are now attempting to develop faster algorithms for matrix operations. For example, for the multiplication of two $m \times m$ matrices, the standard method requires $O(m^3)$ arithmetic operations, but the Strassen algorithm (Strassen 1969) requires only $O(m^{2.807})$ operations. Coppersmith and Winograd's work (1990) shows that the computational complexity of matrix multiplication was decreased to $O(m^{2.376})$. In fact, some researchers believe that an optimal algorithm for multiplying $m \times m$ matrices will reduce the complexity to $O(m^2)$ (Cohn et al. 2005). Table 4.6 shows that the computational complexity of MRSC can be reduced using the Strassen or Coppersmith–Winograd algorithm. So far, the matrix representation of solution concepts has been established in multiple decision maker graph models for simple preference. As shown above, the matrix method for calculating the individual stability and equilibria is attractive from a computational point of view. Therefore, the proposed matrix method not only is propitious for theoretical analysis, but also has the potential to deal with large and complicated conflict problems.

In Sect. 4.3, matrix expressions are used to develop an explicit algebraic form conflict model that facilitates stability calculations. In following section, the efficiency of the matrix approach is illustrated using the Elmira conflict.

Table 4.6 The computational complexity of GMR stability using MRSC

Input	Output	Algorithm	Complexity
J_i, J_i^+, E, and $P_i^{-,=}$	$M_i^{GMR}(s, s)$	Standard matrix multiplication	$O(\delta \cdot n \cdot m^3)$
		Strassen algorithm	$O(\delta \cdot n \cdot m^{2.807})$
		Coppersmith–Winograd algorithm	$O(\delta \cdot n \cdot m^{2.376})$

4.5 Application: Elmira Conflict

As an introduction on how to formally investigate conflict taking place in the real-world, the Elmira groundwater contamination dispute was utilized in Sects. 1.2.2 (Modeling), 1.2.3 (Stability Analysis) and 1.2.4 (Follow-up Analysis). Here, as well as other sections in the book, this interesting dispute is utilized to explain and demonstrate technical definitions and concepts.

Briefly, Elmira, a small agricultural town renowned for its annual maple syrup festival, is located in southwestern Ontario, Canada. In 1989, the Ontario **Ministry of Environment (MoE)** tested the underground aquifer supplying water to Elmira, and determined that it was polluted by N-nitroso demethylamine (NDMA). A local pesticide and rubber manufacturer, **Uniroyal Chemical Ltd. (UR)**, was identified as the prime suspect, since NDMA was a by product of its production process. A Control Order was issued by MoE requiring UR to take expensive measures to remedy the contamination. UR immediately appealed the control order. The **Local Government (LG)**, consisting of the Regional Municipality of Waterloo and the Township of Woolwich, sided with MoE, but sought legal advice from independent consultants on its possible role in resolving this conflict (see Hipel et al. (1993) and Kilgour et al. (2001) for more details).

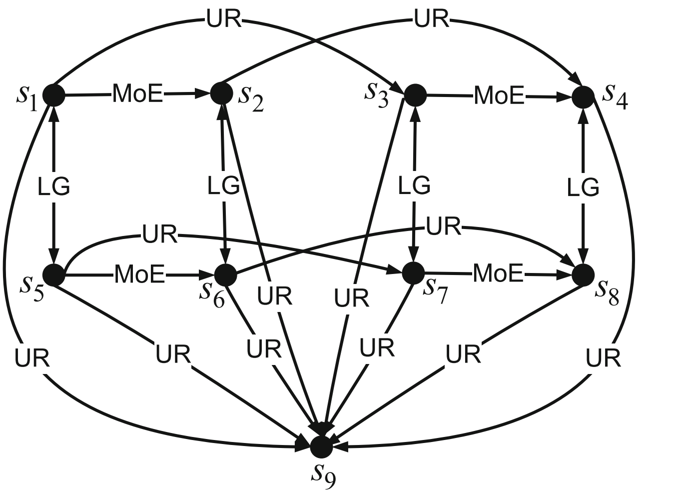
Hipel et al. (1993) established a graph model for this conflict, comprised of three DMs and five options, as follows:

- Ministry of Environment (MoE): its only option is to **modify** the Control Order to make it more acceptable to UR;
- Uniroyal Chemical Ltd. (UR): its options are to **delay** the appeal process, **accept** the Control Order in its current form, or **abandon** the Elmira operation; and
- Local Government (LG): its only option is to **insist** that the original Control Order be applied.

Given the five options in the model, there are 32 mathematically possible states. But many of them are infeasible for a variety of reasons; the nine feasible states are listed in Table 4.7 (where a "Y" indicates that an option is selected by the DM controlling it, an "N" means that the option is not chosen, and a dash "–" denotes that the entry

Table 4.7 Options and feasible states for the Elmira model

MoE									
1. Modify	N	Y	N	Y	N	Y	N	Y	–
UR									
2. Delay	Y	Y	N	N	Y	Y	N	N	–
3. Accept	N	N	Y	Y	N	N	Y	Y	–
4. Abandon	N	N	N	N	N	N	N	N	Y
LG									
5. Insist	N	N	N	N	Y	Y	Y	Y	–
State number	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9



MoE: $s_7 \succ_1 s_3 \succ_1 s_4 \succ_1 s_8 \succ_1 s_5 \succ_1 s_1 \succ_1 s_2 \succ_1 s_6 \succ_1 s_9$

UR: $s_1 \succ_2 s_4 \succ_2 s_8 \succ_2 s_5 \succ_2 s_9 \succ_2 s_3 \succ_2 s_7 \succ_2 s_2 \succ_2 s_6$

LG: $s_7 \succ_3 s_3 \succ_3 s_5 \succ_3 s_1 \succ_3 s_8 \succ_3 s_6 \succ_3 s_4 \succ_3 s_2 \succ_3 s_9$

Fig. 4.8 Integrated graph model for the Elmira conflict

may be "Y" or "N"). The integrated graph model of the Elmira conflict is shown in Fig. 4.8, in which labels on the arcs indicate the DM controlling the move and preference information over the states is below the integrated graph.

4.5.1 Procedures for Calculating Stability

4.5.1.1 Finding Stable States from the Definitions

Let $N = \{1, 2, 3\}$ be the set of DMs (1 = MoE, 2 = UR, and 3 = LG). As an example, DM 3's SMR stability for state s_1 is analyzed using the logical representation presented in Definition 4.10. The procedures are as follows:

1. DM 3's reachable list from s_1 by UIs is $R_3^+(s_1) = \{s_5\}$;
2. The reachable list of coalition $H = N \setminus \{3\}$ from s_5 by UMs is $R_H(s_5) = \{s_6, s_7, s_8, s_9\}$;
3. $s_8 \in R_H(s_5)$ satisfies $s_1 \succ_3 s_8$; also $R_3(s_8) = \{s_4\}$ and $s_1 \succ_3 s_4$;
4. Therefore, s_1 is SMR stable for DM 3 by Definition 4.10.

Other cases can be analyzed similarly. Since the Elmira conflict is modeled as a standard graph model with simple preference, its stabilities can also be analyzed using DSS GMCR II (Fang et al. 2003a, b). The stability results of the Elmira conflict are presented in Table 4.8 in which "$\surd$" denotes that this state is stable for DM 1 (or

Table 4.8 Stability results of the Elmira conflict

State number	Nash				GMR				SMR				SEQ			
	MoE	UR	LG	Eq	MoE	UR	LG	Eq	MoE	UR	LG	Eq	MoE	UR	LG	Eq
s_1	√	√			√	√	√	√	√	√	√	√	√	√		
s_2	√				√		√		√		√		√		√	
s_3	√				√		√		√		√		√		√	
s_4	√	√			√	√	√	√	√	√	√	√	√	√		
s_5	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
s_6	√		√		√		√		√		√		√		√	
s_7	√		√		√		√		√		√		√		√	
s_8	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
s_9	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√

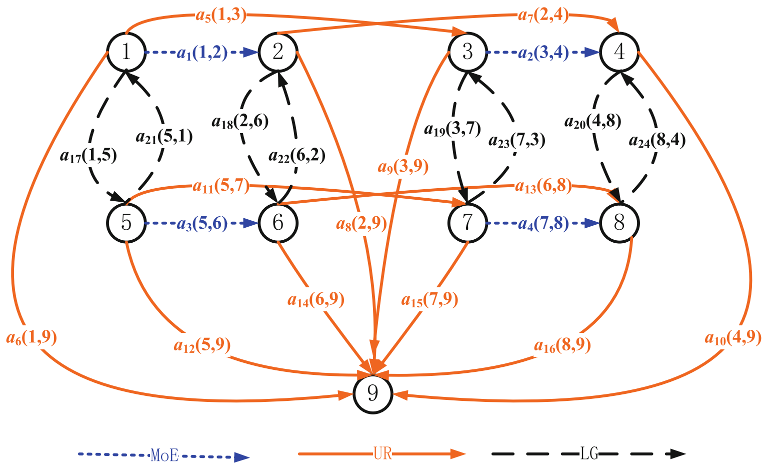


Fig. 4.9 The labeled graph for the Elmira conflict

MoE), DM 2 (or UR), or DM 3 (or LG) under the appropriate stability definitions, and "Eq" means an equilibrium that is stable for the three DMs.

4.5.1.2 Finding Stable States from Matrix Representation

The labeled graph of the Elmira conflict, determined according to the Rule of Priority presented in Sect. 3.3.2, is depicted in Fig. 4.9. The procedures to calculate the stabilities for the Elmira model using the matrix method are as follows:

1. For $i = 1, 2,$ and 3, using Fig. 4.8, determine DM i's adjacency matrix J_i and preference matrix P_i^+ as presented in Tables 4.9 and 4.10;
2. For $i = 1, 2,$ and 3, using $J_i^+ = J_i \circ P_i^+$, calculate the UI adjacency matrices;
3. For $i = 1, 2,$ and 3, using $P_i^{-,=} = E - I - P_i^+$, calculate the preference matrices $P_i^{-,=}$;
4. Construct the UM in-incidence and out-incidence matrices B_{in} and B_{out}, and the UI in-incidence and out-incidence matrices B_{in}^+ and B_{out}^+, based on the labeled graph in Fig. 4.9 and Definition 4.16;
5. Determine the UM arc-incidence and the UI arc-incidence matrices for H, LJ_H and LJ_H^+ using Theorem 4.8 by

$$LJ_H = \bigvee_{i,j \in H, i \neq j} [(B_{in} \cdot D_i)^T \cdot (B_{out} \cdot D_j)] \text{ and } LJ_H^+ = \bigvee_{i,j \in H, i \neq j} [(B_{in}^+ \cdot D_i^+)^T \cdot (B_{out}^+ \cdot D_j^+)];$$

6. Calculate the reachability matrices M_H and M_H^+ using Theorem 4.9 by

$$M_H = sign[B_{out} \cdot (LJ_H + I)^{l-1} \cdot B_{in}^T] \text{ and } M_H^+ = sign[B_{out}^+ \cdot (LJ_H^+ + I)^{l^+-1} \cdot (B_{in}^+)^T]$$

for $l = 24$ and $l^+ = 10$ as presented in Table 4.11;

Table 4.9 Adjacency matrices for the Elmira conflict

Matrix	J_1									J_2									J_3								
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_1	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
s_2	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0
s_3	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
s_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
s_5	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0
s_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0
s_7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
s_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0
s_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4.10 Preference matrices for the Elmira conflict

Matrix	P_1^+									P_2^+									P_3^+								
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_1	0	0	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0
s_2	1	0	1	1	1	0	1	1	0	1	0	1	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0
s_3	0	0	0	0	0	0	1	0	0	1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	1	0	0
s_4	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1	1	0
s_5	0	0	1	1	0	0	1	1	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
s_6	1	1	1	1	1	0	1	1	0	1	1	1	1	1	0	1	1	1	1	0	1	0	1	0	1	1	0
s_7	0	0	0	0	0	0	0	0	0	1	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0
s_8	0	0	1	1	0	0	1	0	0	1	0	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	0
s_9	1	1	1	1	1	1	1	1	0	1	0	0	1	1	0	0	1	0	1	1	1	1	1	1	1	1	0

Table 4.11 Reachability matrices for the Elmira conflict

Matrix	$M_{N\backslash\{1\}}$									$M_{N\backslash\{2\}}$									$M_{N\backslash\{3\}}$								
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_1	0	0	1	0	1	0	1	0	1	0	1	0	0	1	1	0	0	0	0	1	1	1	0	0	0	0	1
s_2	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1
s_3	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	1	1	0	0	0	0	1	0	0	0	0	1
s_4	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
s_5	1	0	1	0	0	0	1	0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	1	1	1	1
s_6	0	1	0	1	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
s_7	0	0	1	0	0	0	0	0	1	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	1	1
s_8	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
s_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Matrix	$M^+_{N\backslash\{1\}}$									$M^+_{N\backslash\{2\}}$									$M^+_{N\backslash\{3\}}$								
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
s_2	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1
s_3	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
s_4	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
s_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
s_6	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
s_7	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
s_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
s_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4.12 Stability matrices for the Elmira conflict

Stability matrices
$M_i^{Nash} = J_i^+ \cdot E$
$M_i^{GMR} = J_i^+ \cdot [E - sign\left(M_{N \setminus \{i\}} \cdot (P_i^{-,=})^T\right)]$
$M_i^{SMR} = J_i^+ \cdot [E - sign(M_{N \setminus \{i\}} \cdot Q)]$ with $Q = (P_i^{-,=})^T \circ [E - sign\big(J_i \cdot (P_i^+)^T\big)]$
$M_i^{SEQ} = J_i^+ \cdot [E - sign\left(M_{N \setminus \{i\}}^+ \cdot (P_i^{-,=})^T\right)]$

Table 4.13 Diagonal entries of stability matrices for the Elmira conflict

State number	Nash			GMR			SMR			SEQ		
	MoE	UR	LG	MoE	UR	LG	MoE	UR	LG	MoE	UR	LG
s_1	0	0	1	0	0	0	0	0	0	0	0	1
s_2	0	1	1	0	1	0	0	1	0	0	1	0
s_3	0	1	1	0	1	0	0	1	0	0	1	0
s_4	0	0	1	0	0	0	0	0	0	0	0	1
s_5	0	0	0	0	0	0	0	0	0	0	0	0
s_6	0	1	0	0	1	0	0	1	0	0	1	0
s_7	0	1	0	0	1	0	0	1	0	0	1	0
s_8	0	0	0	0	0	0	0	0	0	0	0	0
s_9	0	0	0	0	0	0	0	0	0	0	0	0

7. Calculate the stability matrices using the mathematical formulations in Table 4.12 and present their diagonal entries in Table 4.13; and
8. Analyze the stabilities of the conflict using Theorems 4.3 and 4.10–4.12 based on the information in Table 4.13.

The stability results using the matrix approach are identical to those obtained using logical definitions and presented in Table 4.8.

4.5.2 *Analysis of Stability Results*

The reachability matrices, M_H and M_H^+, are analyzed first. Using Table 4.11 with $H = N \setminus \{1\}$, one has:

$$e_4^T \cdot M_H = (0, 0, 0, 0, 0, 0, 0, 1, 1).$$

This means that $R_H(s_4) = \{s_8, s_9\}$, i.e. states s_8 and s_9 can be reached from the status quo $s = s_4$ by legal sequences of UMs by DMs in $H = \{2, 3\}$. Similarly,

$$e_4^T \cdot M_H^+ = (0, 0, 0, 0, 0, 0, 0, 1, 0),$$

which indicates that $R_H^+(s_4) = \{s_8\}$, i.e. s_8 can be reached from status quo $s = s_4$ by legal UI sequences for $H = \{2, 3\}$. It is obvious that if $R_H(s)$ and $R_H^+(s)$ are written as 0–1 row vectors, respectively, then

$$R_H(s) = e_s^T \cdot M_H \text{ and } R_H^+(s) = e_s^T \cdot M_H^+.$$

After the reachability matrices have been determined, stability analysis can be carried out using the stability matrices shown in Table 4.12. For example, the diagonal vector of DM 2's GMR stability matrix, $diag(M_2^{GMR}) = (0, 1, 1, 0, 0, 1, 1, 0, 0)^T$ indicates that states s_1, s_4, s_5, s_8, and s_9 are GMR stable for DM 2.

4.6 Important Ideas

The Graph Model for Conflict Resolution is a powerful tool to model, analyze, and understand strategic conflicts. In this chapter, logical and matrix representations of four basic stability definitions for simple preference are introduced for two-DM and multiple-DM conflicts. The graph model solution concepts discussed in Sect. 4.2 are expressed logically, making them difficult for computer implementation. But the matrix representation of solution concepts discussed in Sect. 4.3 handles this problem efficiently. In particular, the matrix method

- facilitates the development of improved algorithms to assess the stabilities of states,
- is ideally suited for the theoretical study of conflict problems,
- has the advantage of easy calculation and computer implementation, compared with the logical representation of solution concepts,
- provides explicit algebraic expressions that may be adapted for new solution concepts, and
- can be readily integrated into a decision support system as mentioned in Sect. 2.3.3 and explained in detail in Chap. 10.

Because of the nature of its explicit expressions, the matrix representation is easy to employ with different kinds of preference structures and associated modified solution concepts. For example, it could be extended to represent models with preference uncertainty or with multiple degrees of preference, and to determine stabilities in the graph model with these preference structures. The details are discussed in Chaps. 5 and 6, respectively.

4.7 Problems

4.7.1 In the tourism industry, two airlines are competing with each other by reducing the price to obtain more market share. Each airline company can either reduce the price (R) or do not reduce (D). The normal form of this conflict is as shown in Table 4.14.

Table 4.14 The airline conflict in normal form

		Air Cloud: Reduce Price (R)	Air Cloud: Do Not Reduce (D)
Cloudways	Reduce Price (R)	s_1 2, 2 RR	s_2 4, 1 RD
	Do Not Reduce (D)	s_3 1, 4 DR	s_4 3, 3 DD

For this conflict, write the model in:

(a) Option form,
(b) Graph form, and
(c) Using logical form, calculate Nash, general metarational (GMR), symmetric metarational (SMR), and sequential (SEQ) stability for each state and DM. Indicate the equilibria and explain what they mean. Be sure to provide representative examples of stability calculations in normal, option and graph forms.

4.7.2 For the airline conflict provided in Problem 4.7.1, use the matrix formulation to carry out the stability calculations for each state and each DM for Nash, GMR, SMR, and SEQ stability. Determine the equilibria in this conflict and explain why they make sense.

4.7.3 The normal form of the game for Prisoner's Dilemma is given in Problem 3.5.1 in the previous chapter. Determine Nash stability for each of the four states and each of the two DMs. Does Nash stability predict a Nash equilibrium? What obvious equilibrium was missed? Howard (1971) as well as Fraser and Hipel (1979, 1984) refer to this as a breakdown of rationality. This breakdown provided the motivation for Howard to develop the solution concepts of GMR and SMR, and for Fraser and Hipel to propose the SEQ stability definition.

4.7.4 The normal form of the game of Chicken is presented in Problem 3.5.4. Determine which states are Nash stable for each of the two DMs. Are there any Nash equilibria? Which states do you think should be equilibria? The failure of not having a Nash equilibrium is referred to by Howard (1971) and also Fraser and Hipel (1979, 1984) as an example of the breakdown of rationality.

4.7.5 Using the logical form of the stability definitions, determine the stability of each of the four states and each of the two DMs in the game of Prisoner's Dilemma with respect to Nash, GMR, SMR, and SEQ stability. Which states are equilibria? Use the normal form of the game to explain your calculations and show the equilibria. How has the breakdown of rationality referred to in Problem 4.7.3 been resolved?

4.7.6 By employing the logical form of the four stability definitions given in this chapter, ascertain the stability of each of the four states for each of the two DMs in Prisoner's Dilemma. Show your calculations using the option form of the game. Point out which states are equilibria and explain why this is important.

4.7.7 Utilizing the logical form of the solution concepts consisting of Nash, GMR, SMR and SEQ stability, determine the stable states for each stability definition, DM and state for the game of Prisoner's Dilemma. Employ the graph form of the conflict to explain your calculations. Comment on the importance of the equilibria that you find.

4.7.8 In their 1984 book, Fraser and Hipel (1984) introduce Tableau Form to "graphically and intuitively" carry out stability calculations, especially for the case of Nash and SEQ stability. Recall that SEQ stability is especially well-designed because a DM will not harm himself or herself when levying a sanction against another DM's unilateral improvement (UI), since the move for the sanctioning DM must be a UI for him. Refer to Chaps. 2 and 3 in Fraser and Hipel's (1984) book to see how Tableau Form is written for the case of two and more than two DMs, respectively. Write Prisoner's Dilemma in Tableau Form and then carry out a stability analysis for Nash and SEQ stability. Notice the way the Tableau Form naturally portray how moves and countermoves work. How can Tableau Form be expanded to handle GMR and SMR stability?

4.7.9 Calculate Nash, GMR, SMR and SEQ stability using the matrix representation of GMCR for Prisoner's Dilemma for each DM and each of the four states.

4.7.10 For the game of Chicken shown in Problem 3.5.4, calculate Nash, GMR, SMR and SEQ stability for each state and DM using the matrix foumulation of GMCR. Which states are equilibria? Has the breakdown of rationality referred to in Problem 4.7.4 been overcome?

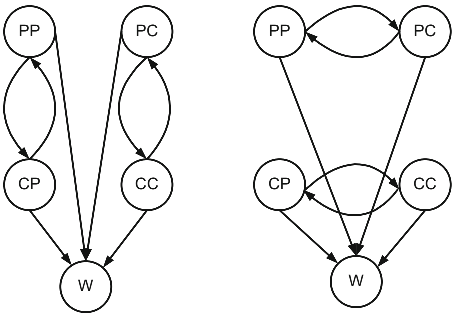
4.7.11 As is also described in Problem 3.5.10, a superpower nuclear confrontation (Fang et al. 1993) can be modeled using two DMs and six options shown in Table 4.15. These options determine the five feasible states as listed in Table 4.15. Note that state W represents a nuclear winter. The graph model for this dispute is displayed in Fig. 4.10.

(a) Analyze stabilities for this model using the logical representation of stability definitions;
(b) Analyze stabilities for this model employing the matrix representation of stability definitions.

4.7.12 The Rafferty-Alameda dams, in the Souris River basin in southern Saskatchewan, Canada, were planned for flood control, recreation and cooling the Shand generating plant (Roberts 1990). The **province of Saskatchewan** wanted to finish the project promptly, seeking a license from the Environment Minister of the

Table 4.15 Decision makers, options and feasible states for the superpower nuclear confrontation conflict

DM 1					
1. Peace (P)	Y	Y	N	N	N
2. Conventional attack (C)	N	N	Y	Y	N
3. Full nuclear attack (W)	N	N	N	N	Y
DM 2					
1. Peace (P)	Y	N	Y	N	N
2. Conventional attack (C)	N	Y	N	Y	N
3. Full nuclear attack (W)	N	N	N	N	Y
States	**PP**	**PC**	**CP**	**CC**	**W**



(a) Graph model for DM 1 (b) Graph model for DM 2

$$DM1 : PP \succ_1 CP \succ_1 CC \succ_1 PC \succ_1 W$$
$$DM2 : PP \succ_2 PC \succ_2 CC \succ_2 CP \succ_2 W$$

Fig. 4.10 The graph model of the superpower nuclear confrontation conflict

Federal Government. An **environmental group**, the Canadian Wildlife Federation, quickly petitioned against the license and argued that the provincial government had not respected regulations. The **federal court** sided with the environment group and ordered the suspension of the license, but later the license was reissued by a new federal environment minister. The environmental group petitioned again, and this time the federal court ordered the suspension of the license and the creation of a **review panel** to evaluate the project. However, construction of the dams continued during the review period, and the federal and provincial governments even reached an agreement that the project would continue while ten million dollars are set aside to alleviate any future environmental impacts. As the province had hoped, the project moved ahead at full speed, and the review panel resigned in protest. (See Hipel et al. (1991) for details.)

This conflict is modeled using four DMs: DM 1, **Federal (F)**; DM 2, **Saskatchewan (S)**; DM 3, **Groups (G)**; and DM 4, **Panel (P)**, each having some options. The following is a summary of the four DMs and their options:

- Federal Government (**Federal**): its options are to seek a court order to halt the project (Court Order) or to lift the license **(Lift)**,
- Province of Saskatchewan (**Saskatchewan**): its option is to go ahead at full speed **(Full speed)**,
- Environmental Groups (**Groups**): its option is to threaten court action to halt the project **(Court action)**, and
- Federal Environmental Review Panel (**Panel**): its option is to resign **(Resign)**.

Five options and ten feasible states of this model are presented in Table 4.16. The graph model of the Rafferty-Alameda dams conflict is shown in Fig. 4.11.

Table 4.16 Feasible states for the Rafferty-Alameda dams conflict

Federal										
1. Court order	–	N	Y	N	Y	N	Y	N	Y	N
2. Lift	–	N	N	N	N	N	N	N	N	Y
Saskatchewan										
3. Full speed	N	Y	Y	Y	Y	Y	Y	Y	Y	–
Groups										
4. Court action	–	N	N	Y	Y	N	N	Y	Y	–
Panel										
5. Resign	–	N	N	N	N	Y	Y	Y	Y	–
State number	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}

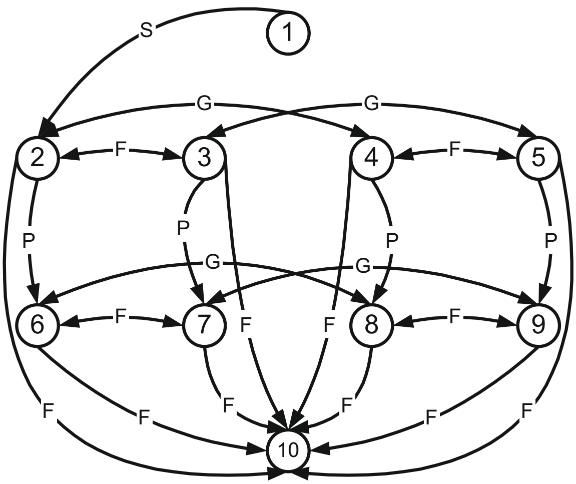


Fig. 4.11 The graph model of the Rafferty-Alameda dams conflict

The ordinal preferences for DMs 1, 2, 3, and 4 are

$$s_1 \succ_1 s_3 \succ_1 s_5 \succ_1 s_2 \succ_1 s_4 \succ_1 s_7 \succ_1 s_9 \succ_1 s_6 \succ_1 s_8 \succ_1 s_{10},$$

$$s_2 \succ_2 s_4 \succ_2 s_6 \succ_2 s_8 \succ_2 s_3 \succ_2 s_5 \succ_2 s_7 \succ_2 s_9 \succ_2 s_{10} \succ_2 s_1,$$

$$s_{10} \succ_3 s_1 \succ_3 s_7 \succ_3 s_3 \succ_3 s_6 \succ_3 s_2 \succ_3 s_9 \succ_3 s_5 \succ_3 s_8 \succ_3 s_4,$$

and

$$s_1 \succ_4 s_9 \succ_4 s_7 \succ_4 s_8 \succ_4 s_6 \succ_4 s_{10} \succ_4 s_5 \succ_4 s_3 \succ_4 s_4 \succ_4 s_2.$$

(a) Label the graph model in Fig. 4.11 according to the Rule of Priority and draw its labeled graph;
(b) Calculate the stabilities of Nash, GMR, SMR, and SEQ for the Rafferty-Alameda dams conflict using the matrix method.

References

Bennett, P. G. (1995). Modelling decision in international relations: Game theory and beyond. *Mershon International Studies Review*, *39*, 19–52. https://doi.org/10.2307/222691.

Cohn, H., Kleinberg, R., Szegedy, B., & Umans, C. (2005). Group-theoretic algorithms for matrix multiplication. *Proceedings of the 46th Annual Symposium on Foundations of Computer Science* (pp. 379–388). https://doi.org/10.1145/1145768.1145772.

Coppersmith, D., & Winograd, S. (1990). Matrix multiplication via arithmetic programming. *Journal of Symbolic Computation*, *9*(3), 251–280.

Fang, L., Hipel, K. W., & Kilgour, D. M. (1989). Conflict models in graph form: Solution concepts and their interrelationships. *European Journal of Operational Research*, *41*(1), 86–100. https://doi.org/10.1016/0377-2217(89)90041-6.

Fang, L., Hipel, K. W., & Kilgour, D. M. (1993). *Interactive decision making: The graph model for conflict resolution*. New York: Wiley. https://doi.org/10.2307/2583940.

Fang, L., Hipel, K. W., Kilgour, D. M., & Peng, X. (2003a). A decision support system for interactive decision making, part 1: Model formulation. *IEEE Transactions on Systems, Man and Cybernetics, Part C: Applications and Reviews*, *33*(1), 42–55. https://doi.org/10.1109/tsmcc.2003.809361.

Fang, L., Hipel, K. W., Kilgour, D. M., & Peng, X. (2003b). A decision support system for interactive decision making, part 2: Analysis and output interpretation. *IEEE Transactions on Systems, Man and Cybernetics, Part C: Applications and Reviews*, *33*(1), 56–66. https://doi.org/10.1109/tsmcc.2003.809360.

Fraser, N. M., & Hipel, K. W. (1979). Solving complex conflicts. *IEEE Transactions on Systems, Man, and Cybernetics*, *9*(12), 805–816. https://doi.org/10.1109/tsmc.1979.4310131.

Fraser, N. M., & Hipel, K. W. (1984). *Conflict analysis: Models and resolutions*. New York, North-Holland. https://doi.org/10.2307/2582031.

Godsil, C., & Royle, G. (2001). *Algebraic graph theory*. New York: Springer. https://doi.org/10.1007/978-1-4613-0163-9.

Hamouda, L., Kilgour, D. M., & Hipel, K. W. (2004). Strength of preference in the graph model for conflict resolution. *Group Decision and Negotiation*, *13*, 449–462. https://doi.org/10.1023/b:grup.0000045751.21207.35.

Hamouda, L., Kilgour, D. M., & Hipel, K. W. (2006). Strength of preference in graph models for multiple decision-maker conflicts. *Applied Mathematics and Computation*, *179*(1), 314–327. https://doi.org/10.1016/j.amc.2005.11.109.

Hipel, K. W. (2009). Conflict resolution: Theme overview paper in conflict resolution. *Encyclopedia of Life Support Systems (EOLSS)* (pp. 1–31). Oxford, U.K: EOLSS Publishers.

Hipel, K. W., Fang, L., & Kilgour, D. M. (1991). The graph model for conflicts in environmental planning. *Proceedings of the 1991 IEEE International Conference on Systems, Man, and Cybernetics* (Vol. 3, pp. 1997–2002). Charlottesville, Virginia, USA.

Hipel, K. W., Fang, L., Kilgour, D. M., & Haight, M. (1993). Environmental conflict resolution using the graph model. *Proceedings of the 1993 IEEE International Conference on Systems, Man, and Cybernetics* (Vol. 1, pp. 153–158). https://doi.org/10.1109/ICSMC.1993.384737.

Howard, N. (1971). *Paradoxes of rationality: Theory of metagames and political behavior*. Cambridge: MIT Press. https://doi.org/10.2307/1266876.

Howard, N., Bennett, P. G., Bryant, J. W., & Bradley, M. (1992). Manifesto for a theory of drama and irrational choice. *Journal of the Operational Research Society*, *44*, 99–103. https://doi.org/10.2307/2584447.

Kilgour, D. M., Hipel, K. W., & Fang, L. (1987). The graph model for conflicts. *Automatica*, *23*, 41–55. https://doi.org/10.1016/0005-1098(87)90117-8.

Kilgour, D. M., Hipel, K. W., Fang, L., & Peng, X. (2001). Coalition analysis in group decision support. *Group Decision and Negotiation*, *10*(2), 159–175.

Li, K. W., Hipel, K. W., Kilgour, D. M., & Fang, L. (2004). Preference uncertainty in the graph model for conflict resolution. *IEEE Transactions on Systems, Man, and Cybernetics, Part A, Systems and Humans*, *34*(4), 507–520. https://doi.org/10.1109/tsmca.2004.826282.

Nash, J. F. (1950). Equilibrium points in *n*-person games. *Proceedings of the National Academy of Sciences of the United States of America*, *36*(1), 48–49. https://doi.org/10.1073/pnas.36.1.48.

Nash, J. F. (1951). Noncooperative games. *Annals of Mathematics*, *54*(2):286–295. https://doi.org/10.1515/9781400884087-008.

Roberts, D. (1990). Rafferty-alameda: The tangled history of 2 dams projects. *The Globe and Mail, October 16* (pp. A4).

Saaty, T. L., & Alexander, J. M. (1989). *Conflict resolution: The analytic hierarchy approach*. New York: Praeger Publishers.

Strassen, V. (1969). Gaussian elimination is not optimal. *Numerische Mathematik*, *13*(4), 354–356. https://doi.org/10.1007/bf02165411.

Xu, H., Hipel, K. W., & Kilgour, D. M. (2007). Matrix representation of conflicts with two decision makers. *Proceedings of the 2007 IEEE International Conference on Systems, Man, and Cybernetics*, (pp. 1764–1769). Montreal, Canada. https://doi.org/10.1109/icsmc.2007.4413988.

Xu, H., Hipel, K. W., & Kilgour, D. M. (2009). Matrix representation of solution concepts in multiple decision maker graph models. *IEEE Transactions on Systems, Man, and Cybernetics, Part A: Systems and Humans*, *39*(1), 96–108. https://doi.org/10.1109/tsmca.2009.2007994.

Xu, H., Kilgour, D. M., & Hipel, K. W. (2010a). An integrated algebraic approach to conflict resolution with three-level preference. *Applied Mathematics and Computation*, *216*(3), 693–707. https://doi.org/10.1016/j.amc.2010.01.054.

Xu, H., Kilgour, D. M., Hipel, K. W., & Kemkes, G. (2010b). Using matrices to link conflict evolution and resolution in a graph model. *European Journal of Operational Research*, *207*, 318–329. https://doi.org/10.1016/j.ejor.2010.03.025.

Xu, H., Kilgour, D. M., & Hipel, K. W. (2011). Matrix representation of conflict resolution in multiple-decision-maker graph models with preference uncertainty. *Group Decision and Negotiation*, *20*(6), 755–779. https://doi.org/10.1007/s10726-010-9188-4.

Xu, H., Kilgour, D. M., Hipel, K. W., & McBean, E. A. (2014). Theory and implementation of coalitional analysis in cooperative decision making. *Theory and Decision*, *76*(2), 147–171. https://doi.org/10.1007/s11238-013-9363-6.