A Criterion for Blow Up in Finite Time of a System of 1-Dimensional Reaction-Diffusion Equations

Eugenio Guerrero and José Alfredo López-Mimbela

Abstract We give a criterion for blow up in finite time of the system of semilinear partial differential equations $\frac{\partial u_i(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u_i(t,x)}{\partial x^2} + \frac{\varphi_i'(x)}{\varphi_i(x)}\frac{\partial u_i(t,x)}{\partial x} + u_j^{1+\beta_i}(t,x)$, $t > 0$, $x \in \mathbb{R}$, with initial values of the form $u_i(0,x) = h_i(x)/\varphi_i(x)$, where $0 < \varphi_i \in L^2(\mathbb{R}, \mathrm{d}x) \cap C^2(\mathbb{R})$, $0 \leq h_i \in L^2(\mathbb{R}, \mathrm{d}x)$, $\beta_i > 0$ and $i = 1, 2$, $j = 3 - i$. Moreover, we find an upper bound T^* for the blowup time of such system which depends both on the initial values f_1, f_2, and the measures $\mu_i(\mathrm{d}x) = \varphi_i^2(x)\,\mathrm{d}x$, $i = 1, 2$.

Keywords Semilinear system of PDEs · Local mild solution · Finite time blow up

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1 Introduction

Consider the semilinear partial differential equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\varphi'(x)}{\varphi(x)}\frac{\partial u(t,x)}{\partial x} + u^{1+\beta}(t,x), \quad t > 0, \quad x \in \mathbb{R}, \qquad (1)$$

where $\beta > 0$, $\varphi \in C^2(\mathbb{R})$ is a square-integrable, strictly positive function, and the initial value is of the form $u(0,x) = h(x)/\varphi(x)$ with $h \in L^2(\mathbb{R}, \mathrm{d}x)$ and $\varphi\check{S}(x) = d\varphi(x)/dx$. Setting $\varphi(x) = e^{-x^2/2}$ in (1) it becomes

$$\frac{\partial u(t,x)}{\partial t} = L^{\varphi}u(t,x) + u^{1+\beta}(t,x), \quad t > 0, \quad x \in \mathbb{R},$$

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where $L^{\varphi} := \frac{1}{2}\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x}$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{T_t,\ t \geq 0\}$. Using essentially Jensen's inequality and the fact that the measure $\mu(\mathrm{d}x) = \varphi^2(x)\,\mathrm{d}x$ is invariant for $\{T_t,\ t \geq 0\}$, in [8] we were able to prove that Eq. (1) exhibits blow up in finite time for any nontrivial initial value of the form $u(0,x) = h(x)/\varphi(x)$, $x \in \mathbb{R}$.

Motivated by this example, in this note we provide a criterion for explosion in finite time of positive mild solutions of the 1-dimensional semilinear system

$$\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= \frac{1}{2}\frac{\partial^2 u_1(t,x)}{\partial x^2} + \frac{\varphi_1'(x)}{\varphi_1(x)}\frac{\partial u_1(t,x)}{\partial x} + u_2^{1+\beta_1}(t,x), \quad t>0, \quad x\in\mathbb{R},\\
\frac{\partial u_2(t,x)}{\partial t} &= \frac{1}{2}\frac{\partial^2 u_2(t,x)}{\partial x^2} + \frac{\varphi_2'(x)}{\varphi_2(x)}\frac{\partial u_2(t,x)}{\partial x} + u_1^{1+\beta_2}(t,x), \quad t>0, \quad x\in\mathbb{R}, \quad (2)\\
u_i(0,x) &= f_i(x), \quad x\in\mathbb{R}, \quad i=1,2,
\end{aligned}$$

where $\beta_1, \beta_2 > 0$ are constants, f_1, f_2 are nonnegative functions and $\varphi_1, \varphi_2 \in C^2(\mathbb{R}) \cap L^2(\mathbb{R}, \mathrm{d}x)$ are strictly positive. Semilinear systems of this type have been investigated intensively in last years, starting with the pioneering work of Galaktionov et al. [4] (see also [2, 3, 5, 7, 9] and the review papers [1, 6]). This kind of systems arise as simplified models of the process of diffusion of heat and burning in a two-component continuous media, where u_1 and u_2 represent the temperatures of the two reactant components.

Recall that a pair (u_1, u_2) of measurable functions is termed *mild solution* of system (2) if it solves the system of integral equations

$$u_i(t,x) = T_t^i\left(f_i(x)\right) + \int_0^t T_{t-s}^i\left(u_j^{1+\beta_i}(s,x)\right)\mathrm{d}s, \quad t\geq 0, \quad x\in\mathbb{R}, \qquad (3)$$

where $i = 1, 2$, $j = 3 - i$ and $\{T_t^i,\ t \geq 0\}$ is the semigroup of continuous linear operators on $L^{\infty}(\mathbb{R}, \mathrm{d}x)$ having infinitesimal generator

$$L^{\varphi_i} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\varphi_i'}{\varphi_i}\frac{\partial}{\partial x}; \quad i = 1, 2.$$

If there exists $T \in (0,\infty)$ such that $\|u_1(t,\cdot)\|_{L^\infty(\mathbb{R},\mathrm{d}x)} = \infty$ or $\|u_2(t,\cdot)\|_{L^\infty(\mathbb{R},\mathrm{d}x)} = \infty$ for all $t \geq T$, then it is said that (u_1, u_2) *blows up (or explodes) in finite time*, and in this case the infimum of such T's is called the *blow up time* (or the *explosion time*) of (u_1, u_2).

Notice that for any $g \in L^\infty(\mathbb{R}, \mathrm{d}x)$ and $i = 1, 2$,

$$T_t^i(g(x)) = \mathbb{E}\left[g\left(X_t^{x,i}\right)\right], \quad t \geq 0, \quad x \in \mathbb{R},$$

where $\{X_t^{x,i}, t \geq 0\}$ is the unique strong solution of the stochastic differential equation

$$Y_t = x + B_t + \int_0^t \frac{\varphi_i'}{\varphi_i}(Y_s)\, \mathrm{d}s, \quad t \geq 0, \quad x \in \mathbb{R};$$

here $\{B_t, t \geq 0\}$ is a standard 1-dimensional Brownian motion. It turns out that under our assumptions both processes $\{X_t^{x,i}, t \geq 0\}$, $i = 1, 2$, are recurrent and, moreover, possess corresponding invariant measures

$$\mu_i(\mathrm{d}x) = \varphi_i^2(x)\, \mathrm{d}x, \quad i = 1, 2. \tag{4}$$

The intuitive explanation of the blow up phenomenon in non-linear heat equations of the archetype

$$\frac{\partial u}{\partial t} = \mathcal{A}u + u^{1+\beta}; \quad u(0) = f \geq 0,$$

where $\beta > 0$ and $\mathcal{A}$ is the generator of a strong Markov process on a locally compact space, is that if the initial value f is "small" then the tendency of the solution to blow up (which it would do if $u^{1+\beta}$ were the only term in the left-hand side of the equation) can be inhibited by the dissipative effect of the migration with generator $\mathcal{A}$; see e.g. [6, 9] or [10]. In view of the ergodicity of the processes $\{X_t^{x,i}, t \geq 0\}$, $i = 1, 2$, the mild solution of (2) should therefore blow up in finite time, at least for certain non-trivial positive initial values f_i, $i = 1, 2$.

In this work we give conditions which imply blow up in finite time of system (2) under the assumption that φ_1/φ_2 is a strictly positive bounded function such that $\inf_{x\in\mathbb{R}}\{\varphi_1(x)/\varphi_2(x)\} > 0$, and the initial values are of the form $f_i = h_i/\varphi_i$, where $h_i \in L^2(\mathbb{R}, \mathrm{d}x)$, $i = 1, 2$. We distinguish two cases: if $\beta_1 = \beta_2$ we show that any non-trivial positive mild solution of (2) blows up in finite time. If $\beta_1 \neq \beta_2$ we prove that a condition on the "sizes" of f_1 and f_2 and on the measures μ_1, μ_2 of the form

$$\int f_1\, \mathrm{d}\mu_1 + \int f_2\, \mathrm{d}\mu_2 > c_0,$$

(where the constant $c_0 > 0$ is determined by the system parameters) already implies finite time explosion of (2); see Theorem 2 below. Moreover, we find an upper bound T^* for the blowup time of system (2) which depends both on the initial values f_1, f_2, and the invariant measures (4). Our setting allows us to consider a wide range of choices for φ_1 and φ_2, for instance

$$\varphi_1(x) = (\sin(x) + 2)\,\varphi_2(x) \text{ with } \varphi_2(x) = e^{-x^2/2},$$

or else

$$\varphi_1(x) = \left(e^{-x^2/2} + 1\right)\varphi_2(x) \text{ with } \varphi_2(x) = 1/(1 + x^2).$$

In these two cases the functions h_i, $i = 1, 2$, can be chosen of the form $h_i(x) = P_i(|x|)/Q_i(|x|)$, where P_i, Q_i are polynomial functions with non-negative coefficients such that their degrees satisfy $2 \leq \deg(Q_i) - \deg(P_i)$, and $Q_i(0) > 0$.

In the next section we prove existence and uniqueness of local mild solutions of (2) using the classical fixed-point argument, adapted to our context. Our main result, Theorem 2, is stated and proved in Sect. 3.

2 Local Existence and Uniqueness of Mild Solutions

Our proof of existence, uniqueness and positiveness of mild solutions of system (2) is based on [14, Theorem 2.1], (see also [12, Theorem 2.1], [15, Theorem 3], [7, Theorem 2] or [11, Theorem 1]).

For each $\tau \in (0, \infty)$ we define the set

$$E_\tau := \left\{(u_1, u_2)\,|u_1, u_2 : [0, \tau] \to L^\infty(\mathbb{R}, \mathrm{d}x),\ |||(u_1, u_2)||| < \infty\right\},$$

where

$$|||(u_1, u_2)||| := \sup_{t \in [0,\tau]} \left\{\|u_1(t, \cdot)\|_{L^\infty(\mathbb{R},\mathrm{d}x)} + \|u_2(t, \cdot)\|_{L^\infty(\mathbb{R},\mathrm{d}x)}\right\}.$$

Then $(E_\tau, |||\cdot|||)$ is a Banach space and the sets

$$P_\tau := \{(u_1, u_2) \in E_\tau : u_1 \geq 0, u_2 \geq 0\} \quad \text{and}$$
$$B_R := \{(u_1, u_2) \in E_\tau : |||(u_1, u_2)||| \leq R\}$$

are closed subsets of E_τ for any $R \in (0, \infty)$. Therefore $(P_\tau \cap B_R, |||\cdot|||)$ is a Banach space for all $\tau, R \in (0, \infty)$.

Theorem 1 *There exist $\tau, R \in (0, \infty)$ such that system (2) has a unique positive mild solution in $P_\tau \cap B_R$.*

Proof We will prove that the operator $\Psi : P_\tau \cap B_R \to P_\tau \cap B_R$ defined by

$$\Psi \left(\left(u_1 \left(t, x \right), u_2 \left(t, x \right) \right) \right) = \left(T_t^1 \left(f_1 \left(x \right) \right) + \int_0^t T_{t-s}^1 \left(u_2^{1+\beta_1} \left(s, x \right) \right) \mathrm{d}s, \right.$$

$$\left. T_t^2 \left(f_2 \left(x \right) \right) + \int_0^t T_{t-s}^2 \left(u_1^{1+\beta_2} \left(s, x \right) \right) \mathrm{d}s \right),$$

is a contraction for certain $\tau, R \in (0, \infty)$. We start by verifying that Ψ is in fact an operator from $P_\tau \cap B_R$ onto $P_\tau \cap B_R$ for suitably chosen $\tau, R \in (0, \infty)$. Let $\tau_0, R_0 \in (0, \infty)$ be such that

$$R_0 > \left(\| f_1 \|_{L^\infty(\mathbb{R}, \mathrm{d}x)} + \| f_2 \|_{L^\infty(\mathbb{R}, \mathrm{d}x)} \right) \text{ and}$$

$$\tau_0 \le \frac{R_0 - \left(\| f_1 \|_{L^\infty(\mathbb{R}, \mathrm{d}x)} + \| f_2 \|_{L^\infty(\mathbb{R}, \mathrm{d}x)} \right)}{R_0^{1+\beta_1} + R_0^{1+\beta_2}}.$$

If $(u_1, u_2) \in P_{\tau_0} \cap B_{R_0}$ then $\Psi((u_1, u_2))$ has positive components due to the definition of Ψ and the fact that $u_1, u_2 \ge 0$. Hence

$$|||\Psi \left(\left(u_1, u_2 \right) \right)||| = \sup_{t \in [0, \tau_0]} \left\{ \left\| T_t^1 \left(f_1 \left(\cdot \right) \right) + \int_0^t T_{t-s}^1 \left(u_2^{1+\beta_1} \left(s, \cdot \right) \right) \mathrm{d}s \right\|_{L^\infty(\mathbb{R}, \mathrm{d}x)} \right.$$

$$\left. + \left\| T_t^2 \left(f_2 \left(\cdot \right) \right) + \int_0^t T_{t-s}^2 \left(u_1^{1+\beta_2} \left(s, \cdot \right) \right) \mathrm{d}s \right\|_{L^\infty(\mathbb{R}, \mathrm{d}x)} \right\}$$

$$\le \| f_1 \|_{L^\infty(\mathbb{R}, \mathrm{d}x)} + \| f_2 \|_{L^\infty(\mathbb{R}, \mathrm{d}x)} + \tau_0 \left(R_0^{1+\beta_1} + R_0^{1+\beta_2} \right),$$

where we have used the contraction property of the operators T_t^i, $i = 1, 2$, to obtain the last inequality. It follows that $|||\Psi((u_1, u_2))||| \le R_0$, i.e., Ψ is an operator from $P_{\tau_0} \cap B_{R_0}$ onto itself.

In order to prove the contraction property of Ψ we choose τ_0 as above in such a way that

$$\max_{i \in \{1,2\}} \left\{ (1 + \beta_i) R_0^{\beta_i} \right\} \tau_0 \in (0, 1). \tag{5}$$

Let $(u_1, u_2), (\hat{u}_1, \hat{u}_2) \in P_{\tau_0} \cap B_{R_0}$. Using again the contraction property of the operators T_t^i, $i = 1, 2$, and the well-known inequality $|a^p - b^p| \le p (a \vee b)^{p-1} |a - b|$, which holds for all $a, b > 0$ and $p \ge 1$, we obtain

$$\left| \left| \left| \Psi \left(\left(u_1, u_2 \right) \right) - \Psi \left(\left(\hat{u}_1, \hat{u}_2 \right) \right) \right| \right| \right|$$

$$= \sup_{t \in [0, \tau_0]} \left\{ \left\| \int_0^t T_{t-s}^1 \left(u_2^{1+\beta_1} \left(s, \cdot \right) - \hat{u}_2^{1+\beta_1} \left(s, \cdot \right) \right) \mathrm{d}s \right\|_{L^\infty(\mathbb{R}, \mathrm{d}x)} \right.$$

$$
\begin{aligned}
&\quad + \left\| \int_0^t T_{t-s}^2 \left(u_1^{1+\beta_2}(s,\cdot) - \hat{u}_1^{1+\beta_2}(s,\cdot) \right) \mathrm{d}s \right\|_{L^\infty(\mathbb{R},\mathrm{d}x)} \Bigg\} \\
&\le \sup_{t\in[0,\tau_0]} \int_0^t \left\| u_2^{1+\beta_1}(s,\cdot) - \hat{u}_2^{1+\beta_1}(s,\cdot) \right\|_{L^\infty(\mathbb{R},\mathrm{d}x)} \mathrm{d}s \\
&\quad + \sup_{t\in[0,\tau_0]} \int_0^t \left\| u_1^{1+\beta_2}(s,\cdot) - \hat{u}_1^{1+\beta_2}(s,\cdot) \right\|_{L^\infty(\mathbb{R},\mathrm{d}x)} \mathrm{d}s \\
&\le (1+\beta_1)\, R_0^{\beta_1} \int_0^{\tau_0} \left\| u_2(s,\cdot) - \hat{u}_2(s,\cdot) \right\|_{L^\infty(\mathbb{R},\mathrm{d}x)} \mathrm{d}s \\
&\quad + (1+\beta_2)\, R_0^{\beta_2} \int_0^{\tau_0} \left\| u_1(s,\cdot) - \hat{u}_1(s,\cdot) \right\|_{L^\infty(\mathbb{R},\mathrm{d}x)} \mathrm{d}s \\
&\le \max_{i\in\{1,2\}} \left\{ (1+\beta_i)\, R_0^{\beta_i} \right\} \tau_0 \left|\left|\left| (u_1,u_2) - \left(\hat{u}_1,\hat{u}_2\right) \right|\right|\right| .
\end{aligned}
$$

From the last inequality we conclude, due to (5), that Ψ is a contraction in $P_{\tau_0} \cap B_{R_0}$. It follows from the Banach fixed-point theorem that Ψ has a unique fixed point in $P_{\tau_0} \cap B_{R_0}$, which is the unique mild solution of system (2). □

3 A Condition for Blowup in Finite Time

Our main result is the following

Theorem 2 *Let $\varphi_i \in L^2(\mathbb{R}, \mathrm{d}x) \cap C^2(\mathbb{R})$ be a strictly positive function and assume that the initial value f_i admits the representation*

$$
f_i(x) := \frac{h_i(x)}{\varphi_i(x)} \ge 0, \quad x \in \mathbb{R}, \tag{6}
$$

for some positive nontrivial $h_i \in L^2(\mathbb{R}, \mathrm{d}x)$, $i = 1, 2$. Suppose in addition that there exist strictly positive constants k_1, k_2 such that

$$
k_1 \le \frac{\varphi_1(x)}{\varphi_2(x)} \le k_2, \quad x \in \mathbb{R}. \tag{7}
$$

1. *Assume that $\beta_1 = \beta_2$. Then any non-trivial positive mild solution (u_1, u_2) of system (2) blows up in finite time.*

2. Assume that $\beta_1 > \beta_2$. Let $A_0 := \left(\frac{1+\beta_2}{1+\beta_1}\right)^{\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1-\beta_2}{1+\beta_1}$ and suppose that

$$\int_{\mathbb{R}} f_1(x)\,\mu_1(\mathrm{d}x) + \int_{\mathbb{R}} f_2(x)\,\mu_2(\mathrm{d}x) > 2^{\frac{\beta_2}{1+\beta_2}} A_0^{\frac{1}{1+\beta_2}}. \tag{8}$$

Then any mild solution (u_1, u_2) of system (2) blows up in finite time.

Proof Let (u_1, u_2) be a mild solution of system (2). We denote

$$w_i(t,x) := \varphi_i(x)\,u_i(t,x), \quad t \geq 0, \quad x \in \mathbb{R}.$$

Multiplying both sides of (3) by φ_i yields

$$w_i(t,x) = \varphi_i(x)\,T_t^i\left(\frac{h_i}{\varphi_i}(x)\right) + \int_0^t \varphi_i(x)\,T_{t-s}^i\left(w_{3-i}^{1+\beta_i}(s,x)\,\varphi_{3-i}^{-(1+\beta_i)}(x)\right)\mathrm{d}s. \tag{9}$$

Since the function $g_i(x) := \varphi_i^2(x)$ satisfies the differential equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} g_i(x) - \frac{\partial}{\partial x}\left(g_i(x)\,\frac{\varphi_i'(x)}{\varphi_i(x)}\right) = 0, \quad x \in \mathbb{R},$$

it follows that $\mu_i(\mathrm{d}x) = \varphi_i^2(x)\,\mathrm{d}x$ is invariant for the semigroup $\left\{T_t^i,\ t \geq 0\right\}$. Let us write $\mathbb{E}^i[f] := \int_{\mathbb{R}} f(x)\,\varphi_i(x)\,\mathrm{d}x$. Due to (9) this implies that

$$\mathbb{E}^i[w_i(t,\cdot)] = \mathbb{E}^i[h_i(\cdot)] + \int_0^t \mathbb{E}^i\left[w_{3-i}^{1+\beta_i}(s,\cdot)\,\varphi_i(\cdot)\,\varphi_{3-i}^{-(1+\beta_i)}(\cdot)\right]\mathrm{d}s. \tag{10}$$

Define $a := \min\left\{k_1^2, \frac{1}{k_2^2}\right\}$. From assumption (7) we get $\frac{\varphi_i^2(x)}{\varphi_{3-i}^2(x)} \geq a$ for all $x \in \mathbb{R}$ and $i = 1, 2$. Therefore

$$\begin{aligned}
&\mathbb{E}^i\left[w_{3-i}^{1+\beta_i}(s,\cdot)\,\varphi_i(\cdot)\,\varphi_{3-i}^{-(1+\beta_i)}(\cdot)\right] \\
&= \int_{\mathbb{R}}\left(\frac{w_{3-i}(s,x)}{\varphi_{3-i}(x)}\right)^{1+\beta_i} \varphi_i^2(x)\,\mathrm{d}x \\
&\geq a\,\|\varphi_{3-i}\|^2_{L^2(\mathbb{R},\mathrm{d}x)} \int_{\mathbb{R}}\left(\frac{w_{3-i}(s,x)}{\varphi_{3-i}(x)}\right)^{1+\beta_i} \frac{\varphi_{3-i}^2(x)}{\|\varphi_{3-i}\|^2_{L^2(\mathbb{R},\mathrm{d}x)}}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\geq a \frac{\|\varphi_{3-i}\|^2_{L^2(\mathbb{R},\mathrm{d}x)}}{\|\varphi_{3-i}\|^{2+2\beta_i}_{L^2(\mathbb{R},\mathrm{d}x)}} \left(\int_{\mathbb{R}} \frac{w_{3-i}(s,x)}{\varphi_{3-i}(x)} \varphi^2_{3-i}(x)\,\mathrm{d}x \right)^{1+\beta_i} \\
&= a \|\varphi_{3-i}\|^{-2\beta_i}_{L^2(\mathbb{R},\mathrm{d}x)} \left(\mathbb{E}^{3-i}\left[w_{3-i}(s,\cdot)\right] \right)^{1+\beta_i}, \qquad (11)
\end{aligned}
$$

where we have used Jensen's inequality to obtain the last inequality. Plugging (11) into (10) renders

$$
\mathbb{E}^i\left[w_i(t,\cdot)\right] \geq \mathbb{E}^i\left[h_i(\cdot)\right] + a \|\varphi_{3-i}\|^{-2\beta_i}_{L^2(\mathbb{R},\mathrm{d}x)} \int_0^t \left(\mathbb{E}^{3-i}\left[w_{3-i}(s,\cdot)\right] \right)^{1+\beta_i} \mathrm{d}s. \qquad (12)
$$

Let $y_i(t)$ be the solution of the system

$$
\begin{aligned}
y_i'(t) &= a \|\varphi_{3-i}\|^{-2\beta_i}_{L^2(\mathbb{R},\mathrm{d}x)} y^{1+\beta_i}_{3-i}(t), \quad t > 0, \\
y_i(0) &= \mathbb{E}^i\left[h_i(\cdot)\right], \quad i = 1, 2.
\end{aligned}
$$

Putting $b := a \min\left\{ \|\varphi_1\|^{-2\beta_2}_{L^2(\mathbb{R},\mathrm{d}x)}, \|\varphi_2\|^{-2\beta_1}_{L^2(\mathbb{R},\mathrm{d}x)} \right\}$ we get the system of differential inequalities

$$
\begin{aligned}
y_i'(t) &\geq b y^{1+\beta_i}_{3-i}(t), \quad t > 0, \\
y_i(0) &= \mathbb{E}^i\left[h_i(\cdot)\right], \quad i = 1, 2.
\end{aligned}
$$

Let $(z_1(t), z_2(t))$ be the solution of the system of ordinary differential equations

$$
\begin{aligned}
z_i'(t) &= b z_j^{1+\beta_i}(t), \quad t > 0, \\
z_i(0) &= \mathbb{E}^i\left[h_i(\cdot)\right], \quad i = 1, 2, \quad j = 3 - i.
\end{aligned}
$$

By the Picard-Lindelöf theorem, this system with $(z_1(0), z_2(0)) = (0, 0)$ has a unique local solution $(w_1(t), w_2(t)) \equiv (0, 0)$ for all $t \in [0, \tau)$, for some $\tau \in (0, \infty]$. In our case $\mathbb{E}^i\left[h_i(\cdot)\right] \geq 0$. Therefore by a classical comparison theorem, $z_1(t), z_2(t) \geq 0$ for all $t \in [0, \tau)$.

Consider the new function

$$
E(t) := z_1(t) + z_2(t), \quad t \geq 0.
$$

We deal separately with the two cases in the statement of the theorem:

1. Case $\beta_1 = \beta_2$. Using the fact that

$$
x^{1+\beta_1} + y^{1+\beta_1} \geq 2^{-\beta_1} (x + y)^{1+\beta_1}, \quad x \geq 0, \quad y \geq 0, \qquad (13)
$$

we get

$$\begin{aligned} E'(t) &= z_1'(t) + z_2'(t) \\ &= b\left(z_1^{1+\beta_1}(t) + z_2^{1+\beta_1}(t)\right) \\ &\geq 2^{-\beta_1} b E^{1+\beta_1}(t), \quad t > 0, \\ E(0) &= \mathbb{E}^1[h_1(\cdot)] + \mathbb{E}^2[h_2(\cdot)]. \end{aligned}$$

Let $I(t)$ be the solution of the ordinary differential equation

$$\begin{aligned} I'(t) &= 2^{-\beta_1} b I^{1+\beta_1}(t), \quad t > 0, \\ I(0) &= \mathbb{E}^1[h_1(\cdot)] + \mathbb{E}^2[h_2(\cdot)]. \end{aligned}$$

Since I is a subsolution of E (see [13], Lemma 1.2.) and I explodes at time

$$T^* = \frac{2^{\beta_1}}{b\beta_1\left(\mathbb{E}^1[h_1(\cdot)] + \mathbb{E}^2[h_2(\cdot)]\right)^{\beta_1}} \in (0, \infty),$$

it follows that E explodes at some time $t_E \leq T^*$, and therefore, by a classical comparison theorem we get that

$$\begin{aligned} \mathbb{E}^1[w_1(t,\cdot)] &= \|u_1(t,\cdot)\|_{L^1(\mathbb{R},\mu_1)} = \infty \quad \text{or} \\ \mathbb{E}^2[w_2(t,\cdot)] &= \|u_2(t,\cdot)\|_{L^1(\mathbb{R},\mu_2)} = \infty \end{aligned}$$

for all $t \geq T^*$. Since $\|u_i(t,\cdot)\|_{L^1(\mathbb{R},\mu_i)} \leq \|u_i(t,\cdot)\|_{L^\infty(\mathbb{R},\mathrm{d}x)} \|\varphi_i\|^2_{L^2(\mathbb{R},\mathrm{d}x)}$ for all $t \in [0,\infty)$, $i = 1, 2$, we conclude that the mild solution (u_1, u_2) of system (2) blows up in finite time.

2. Case $\beta_1 > \beta_2$. Recall that for all $x, y \geq 0$, $\delta > 0$ and $p, q \in (1,\infty)$ such that $p^{-1} + q^{-1} = 1$ we have Young's inequality

$$xy \leq \frac{\delta^{-p} x^p}{p} + \frac{\delta^q y^q}{q}. \tag{14}$$

From the definition of A_0 it follows that

$$z_2^{1+\beta_1}(t) \geq z_2^{1+\beta_2}(t) - A_0, \quad \text{for all } t \geq 0.$$

In fact, it suffices to choose in (14)

$$x = 1, \quad y = z_2^{1+\beta_2}(t), \quad \delta = \left(\frac{1+\beta_1}{1+\beta_2}\right)^{\frac{1+\beta_2}{1+\beta_1}} \quad \text{and} \quad q = \frac{1+\beta_1}{1+\beta_2}.$$

Therefore we have

$$E'(t) \geq b\left(z_1^{1+\beta_2}(t) + z_2^{1+\beta_2}(t) - A_0\right).$$

Using again inequality (13) we conclude that

$$z_1^{1+\beta_2}(t) + z_2^{1+\beta_2}(t) \geq 2^{-\beta_2} E^{1+\beta_2}(t),$$

hence

$$E'(t) \geq b\left(2^{-\beta_2} E^{1+\beta_2}(t) - A_0\right).$$

Let $I(t)$ solve the ordinary differential equation

$$I'(t) = b\left(2^{-\beta_2} I^{1+\beta_2}(t) - A_0\right), \quad t > 0,$$
$$I(0) = \mathbb{E}^1[h_1(\cdot)] + \mathbb{E}^2[h_2(\cdot)].$$

It follows from the same comparison theorem as above that I is a subsolution of E. Using separation of variables we get, for $t \in (0, \infty)$,

$$t = \int_{E(0)}^{I(t)} \frac{\mathrm{d}x}{b\left(2^{-\beta_2} x^{1+\beta_2} - A_0\right)} \leq \int_{E(0)}^{\infty} \frac{\mathrm{d}x}{b\left(2^{-\beta_2} x^{1+\beta_2} - A_0\right)} =: T^*. \tag{15}$$

But the hypothesis (8) implies that $T^* < \infty$. Hence (15) cannot hold for sufficiently large t, which yields that I explodes at a finite time $T^{**} \in (0, T^*]$. Therefore E explodes no later than T^* as well. From here we proceed as in the case $\beta_1 = \beta_2$ to conclude that the mild solution (u_1, u_2) of system (2) blows up in finite time also in this case.

□

The following result is an immediate consequence of the previous theorem. Recall that $E(0) = \int_{\mathbb{R}} f_1 \, \mathrm{d}\mu_1 + \int_{\mathbb{R}} f_2 \, \mathrm{d}\mu_2$ and

$$A_0 = \left(\frac{1+\beta_2}{1+\beta_1}\right)^{\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1 - \beta_2}{1+\beta_1}, \quad b = \min\left\{k_1^2, \frac{1}{k_2^2}\right\} \min_{i\in\{1,2\}} \left\{\|\varphi_i\|_{L^2(\mathbb{R},\mathrm{d}x)}^{-2\beta_i}\right\}.$$

Corollary 3 *Under the assumptions of Theorem 2, if $\beta_1 = \beta_2$ then the explosion time of any non-trivial positive solution of (2) is bounded above by*

$$T^* = \frac{2^{\beta_1}}{b\beta_1 (E(0))^{\beta_1}}.$$

If $\beta_1 > \beta_2$ *and* (8) *holds, then the time of explosion of* (2) *is bounded above by*

$$T^* = \int_{E(0)}^{\infty} \frac{\mathrm{d}x}{b\left(2^{-\beta_2} x^{1+\beta_2} - A_0\right)}.$$

Remark Theorem 2 and Corollary 3 remain valid when $\beta_2 > \beta_1$, with the obvious changes in the correspondent statements.

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