A Note on Γ-Convergence of Monotone Functionals

Erick Treviño Aguilar

Abstract In this note we present a criterion under which a functional defined on vectors of non-decreasing functions is the Γ-limit of a functional defined on vectors of continuous non-decreasing functions. To this end, we present a separation principle in which a weakly converging sequence of continuous non-decreasing functions is decomposed in two parts, one converging to a non-decreasing function with a finite number of jumps and the other to the complementary jumps.

Keywords Γ-Convergence · Monotone functionals · Singular control · Skorokhod representation

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1 Introduction

For $\mathbb{T} > 0$ fixed, we denote by C the class of right-continuous with left-limits functions defined on the interval $[0, \mathbb{T}]$, which are non-negative and non-decreasing. We denote by C_{finite} the elements of C with a finite number of jumps and by C^0 the elements of C with no jumps. For $\mathsf{c} \in \mathsf{C}$, the jump at time $t \in [0, \mathbb{T}]$ is denoted by $\Delta\mathsf{c}(t)$ and is defined as the difference $\mathsf{c}(t) - \mathsf{c}(t-)$. If $\mathsf{c}(0) > 0$ then we consider a jump of size $\mathsf{c}(0)$ at time $t = 0$. Thus $\Delta\mathsf{c}(0) := \mathsf{c}(0)$. An element of C defines a unique positive measure in the interval $[0, \mathbb{T}]$ and we will consider the topology of weak convergence on C. Recall that a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}}$ converges weakly to a measure μ if for each continuous bounded function $f : [0, \mathbb{T}] \to \mathbf{R}$ we have $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$. An equivalent property to weak convergence is formulated in terms of the elements of C (which can be seen as "distribution functions"). A sequence $\{\mathsf{c}(n)\}_{n\in\mathbf{N}} \subset \mathsf{C}$ converges pointwise to an element $\mathsf{c} \in \mathsf{C}$

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for each continuity point of c if and only if the corresponding measures converges weakly. By a slight abuse of language we will say that the sequence $\{\mathsf{c}(n)\}_{n\in\mathbf{N}}$ converges weakly to c.

An important property of weak convergence is that it is metrizable on separable spaces. Indeed, the well-known Prokhorov distance is a metric which on separable spaces characterizes weak convergence; see e.g., Ethier and Kurtz [6, Section 3.1]. This property will be crucial for our results here.

Now consider a functional $\mathsf{J} : \mathsf{C}^0 \times \mathsf{C}^0 \to \mathbf{R}$ and suppose we are required to consider the functional in all of the space $\mathsf{C} \times \mathsf{C}$. One reason why we might need the functional in an enlarged space is related to the problem of minimizing the functional. Indeed, a minimizer may fail to exist in the class of continuous elements and we might need to consider an enlarged space. A classical method to construct a functional in an enlarged space is by density and approximation. In this method, we take a point $(\mathsf{c}^1, \mathsf{c}^2) \in \mathsf{C} \times \mathsf{C}$ and a sequence $\{(\mathsf{c}^1(n), \mathsf{c}^2(n))\}_{n\in\mathbf{N}} \subset \mathsf{C}^0 \times \mathsf{C}^0$ which componentwise converges weakly. We might define a functional J^* in the point $(\mathsf{c}^1, \mathsf{c}^2)$ by the limit:

$$\mathsf{J}^*(\mathsf{c}^1, \mathsf{c}^2) = \lim_{n\to\infty} \mathsf{J}(\mathsf{c}^1(n), \mathsf{c}^2(n)).$$

The method requires that the limit always exists and to be independent of the particular sequence. However, we will illustrate in Sect. 3 that for weak convergence, with a very simple functional one gets different limits and even oscillatory behaviors. Note also that even for elements of $\mathsf{C}^0 \times \mathsf{C}^0$ the functionals J^* and J does not necessarily coincide and J^* is not necessarily an extension of J. Thus, the method does not work in general and we might need to consider "envelopes" instead of extensions. A convenient solution still keeping in mind problems of minimization is that of Γ-convergence. The concept was introduced in the study of variational problems by De Giorgi [5]. It is systematically presented by Dal Maso [4] and its relevance in optimal control, which is our main motivation here, is presented by e.g., Buttazzo and Dal Maso [3]. The Γ-convergence is a far reaching concept providing a powerful framework covering a wide range of applications; see e.g., Braides [2] and its references. In Sect. 2 below, we give more detail on this concept for our specific setting. Let us at this point formulate on the relevance of Γ-convergence in optimal control. Consider two topological spaces U (the space of controls) and Y (the space of state variables), and a function $\mathsf{J} : U \times Y \to [0, +\infty]$. Given a set of "admissible control-states" $\mathcal{A} \subset U \times Y$, consider the minimization problem:

$$\min_{(u,y)\in\mathcal{A}} \mathsf{J}(u, y).$$

This general problem may be difficult to study directly and instead, it might be convenient to study related problems formulated with other sets $\mathcal{A}^h \subset U \times Y$ and other functions J^h for $h \in \mathbf{N}$. In principle, the minimization problem formulated in terms of the pair $(\mathcal{A}^h, \mathsf{J}^h)$ should be easier and provide information about the original minimization problem formulated in terms of $\mathcal{A}$ and J. A way in which the sequence of auxiliary minimization problems help to understand the original problem is that of convergence of minimal values and convergence of optimal controls for the auxiliary problems, possibly along a subsequence, to an optimal control of the original problem. This is one of the main properties of Γ-Convergence; see e.g., Buttazzo and Dal Maso [3, Theorem 2.1].

The construction of Γ-limits is a highly non trivial task and in this paper we obtain a substantial reduction based on an assumption of monotonicity.

Definition 1.1 A functional J is monotone if for each $\mathsf{c}^1, \mathsf{c}^2 \in \mathsf{C}^0$ and $\nu^1, \nu^2 \in \mathsf{C}^0$ we have

$$\mathsf{J}(\mathsf{c}^1 + \nu^1, \mathsf{c}^2 + \nu^2) \geq \mathsf{J}(\mathsf{c}^1, \mathsf{c}^2). \tag{1.1}$$

In this note, we prove the Γ-convergence in $\mathsf{C} \times \mathsf{C}$ for monotone functionals as a consequence of the property for elements of $\mathsf{C}_{finite} \times \mathsf{C}_{finite}$ which have a finite number of atoms. This is a non trivial reduction that makes use of Skorokhod's representation of weak convergence and depends strongly on the property of monotonicity of the functional.

After this introduction, the note is organized as follows. In Sect. 2, we elaborate on the concept of Γ-convergence in our specific setting. In Sect. 3, we illustrate the phenomenon of oscillatory behavior. In Sect. 4 we prove a separation principle for sequences of continuous distributions by making use of Skorokhod's representation of weak convergence. In Sect. 5 we prove the sufficient condition for Γ-convergence.

2 Γ-Convergence

The next definition can be seen as a special case of the concept systematically presented by Dal Maso [4].

Definition 2.1 For a functional $\mathsf{J} : \mathsf{C}^0 \times \mathsf{C}^0 \to \mathbf{R}$ we say that the functional $\mathsf{J}^* : \mathsf{C} \times \mathsf{C} \to \mathbf{R}$ is the Γ-limit of J if the following conditions are satisfied:

1. for each point $(\mathsf{c}^1, \mathsf{c}^2) \in \mathsf{C} \times \mathsf{C}$ and sequence $\{(\mathsf{c}^1(n), \mathsf{c}^2(n))\}_{n \in \mathbf{N}} \subset \mathsf{C}^0 \times \mathsf{C}^0$ which componentwise weakly-converges to $(\mathsf{c}^1, \mathsf{c}^2)$ we have:

$$\mathsf{J}^*(\mathsf{c}^1, \mathsf{c}^2) \leq \liminf_{n \to \infty} \mathsf{J}(\mathsf{c}^1(n), \mathsf{c}^2(n)),$$

2. there exists a sequence $\{(\mathsf{c}^{1*}(n), \mathsf{c}^{2*}(n))\}_{n \in \mathbf{N}} \subset \mathsf{C}^0 \times \mathsf{C}^0$ which weakly-converges component by component to $(\mathsf{c}^1, \mathsf{c}^2)$ with the property

$$\mathsf{J}^*(\mathsf{c}^1, \mathsf{c}^2) = \lim_{n \to \infty} \mathsf{J}(\mathsf{c}^{1*}(n), \mathsf{c}^{2*}(n)).$$

3 An Example of Oscillatory Behavior

In this section we illustrate the phenomenon of oscillatory behavior with the very simple functional J defined by

$$\mathsf{J}(\mathsf{c}^1, \mathsf{c}^2) := \int_0^{\mathbb{T}} d\mathsf{c}_s^2 \int_0^s d\mathsf{c}_z^1.$$

In particular this example illustrates the convenience of considering the concept of Γ-convergence.

For $\tau \in (0, \mathbb{T})$, let

$$\mathsf{c}_t^2 := \mathsf{c}_t^1 := 1_{[\tau,\mathbb{T}]}(t). \tag{3.1}$$

Now we define continuous approximations. Take $\epsilon > 0$ with $\tau + \epsilon < \mathbb{T}$ and for $\alpha \in (0, 1)$ let $\epsilon' := \alpha\epsilon$. Let

$$\begin{aligned} u_t^1(\epsilon, \alpha) &:= \int_0^{t\wedge(\tau+\epsilon)} m_\epsilon^1 1_{[\tau,\tau+\epsilon]}(s)ds \\ u_t^2(\epsilon, \alpha) &:= \int_0^{t\wedge(\tau+\epsilon)} m_{\epsilon,\alpha}^2 1_{[\tau+\epsilon',\tau+\epsilon]}(s)ds. \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} m_\epsilon^1 &:= \frac{\mathsf{c}_{\tau+\epsilon}^1 - \mathsf{c}_{\tau-}^1}{\epsilon} = \frac{1}{\epsilon} \\ m_{\epsilon,\alpha}^2 &:= \frac{\mathsf{c}_{\tau+\epsilon}^2 - \mathsf{c}_{\tau-}^2}{\epsilon - \epsilon'} = \frac{1}{(1-\alpha)\epsilon}. \end{aligned}$$

The functions u_1 and u_2 are illustrated in Fig. 1.

Proposition 3.1 *The functions* $u^1(\epsilon, \alpha)$ *and* $u^2(\epsilon, \alpha)$ *defined in* (3.2) *converge weakly as* $\epsilon \searrow 0$ *to* c^1 *and* c^2 *respectively, and*

$$\mathsf{J}(u^1(\epsilon, \alpha), u^2(\epsilon, \alpha)) = \frac{1+\alpha}{2}.$$

Proof Note that $u^1(\epsilon, \alpha), u^2(\epsilon, \alpha)$ converge pointwise as $\epsilon \searrow 0$ in $[0, \mathbb{T}]/\{\tau\}$ to $\mathsf{c}^1, \mathsf{c}^2$, respectively, and therefore converge weakly. For the second part of the

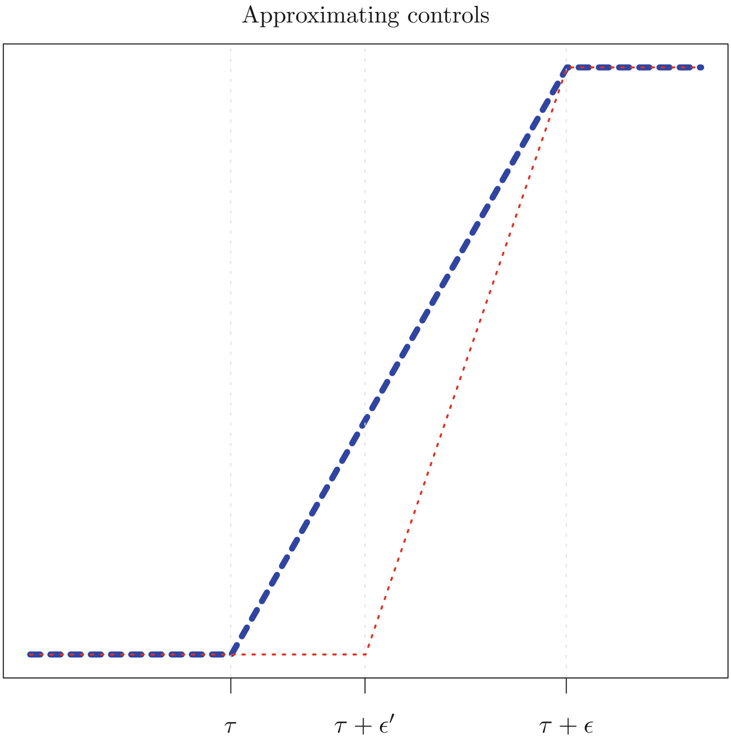

Fig. 1 The two functions u^1 and u^2, defined in Eq. (3.2)

proposition, we have

$$\begin{aligned}\int_0^{\mathbb{T}} du_s^2(\epsilon,\alpha)\int_0^s du_z^1(\epsilon,\alpha) &= \int_{\tau+\epsilon'}^{\tau+\epsilon} m_{\epsilon,\alpha}^2 1_{[\tau+\epsilon',\tau+\epsilon]}(s)ds \int_0^s m_\epsilon^1 1_{[\tau,\tau+\epsilon]}(z)dz \\ &= m_\epsilon^1 m_{\epsilon,\alpha}^2 \int_{\tau+\epsilon'}^{\tau+\epsilon} ds \int_\tau^s dz \\ &= m_\epsilon^1 m_{\epsilon,\alpha}^2 \int_{\tau+\epsilon'}^{\tau+\epsilon} (s-\tau)ds \\ &= m_\epsilon^1 m_{\epsilon,\alpha}^2 \frac{1}{2}\epsilon^2(1-\alpha^2) \\ &= \frac{1+\alpha}{2}.\end{aligned}$$

□

Remark 3.2 Note that it is possible to select a sequence $\{\alpha_m\}_{m\in\mathbf{N}}$ in such a way that the sequence $\left\{\mathsf{J}(u^1(\epsilon,\alpha_m), u^2(\epsilon,\alpha_m))\right\}_{m\in\mathbf{N}}$ generates a dense subset of the interval $[\frac{1}{2}, 1]$, due to Proposition 3.1.

4 A Separation Principle of Sequences

We start this section with Skorokhod's representation of weak convergence in the following form. Let $\{\mu_n\}_{n\in\mathbf{N}}$ be a sequence of probability measures in the interval $[0,\mathbb{T}]$ converging weakly to the measure μ. Then, there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$, a random variable Y and a sequence of random variables $\{Y_n\}_{n\in\mathbf{N}}$ defined in this common space, such that Y_n has distribution μ_n and Y has distribution μ and the sequence converges to Y everywhere in Ω. See e.g., Billingsley [1, Theorem 25.6] for the proof.

Lemma 4.1 *Let $\{F_m\}_{m\in\mathbf{N}}$ be a sequence of elements of C^0. Assume the sequence converges weakly to an element F of C. Thus, the sequence converges pointwise to F except, possibly, for the points $\{\tau_k\}_{k=0}^{\infty}$ where F jumps.*

Then, for $k_0\in\mathbf{N}$ fixed, there exist sequences of non-negative, non-decreasing continuous functions $\{G_m\}_{m\in\mathbf{N}}$ and $\{H_m\}_{m\in\mathbf{N}}$ such that

1. *$F_m = G_m + H_m$ for $m\in\mathbf{N}$.*
2. *The sequence $\{H_m\}_{m\in\mathbf{N}}$ converges pointwise to the function*

$$H(t) := \sum_{k=k_0+1}^{\infty} \Delta F(\tau_k) 1_{\{\tau_k\le t\}}, \tag{4.1}$$

   for $t\in[0,\mathbb{T}]$.
3. *The sequence $\{G_m\}_{m\in\mathbf{N}}$ converges pointwise to the function*

$$G(t) := F(t) - H(t), \tag{4.2}$$

   for $t\in[0,\mathbb{T}]$.

Proof We will do the proof only in the case that $F_m(\mathbb{T}) = F(\mathbb{T}) = 1$, the general case following by normalization.

There exist a probability space $(\Omega,\mathcal{F},P)$ and a sequence of random variables $\{X_n\}_{n\in\mathbf{N}}$ converging to a random variable X, with $X_n \sim F_n$ and $X\sim F$, due to Skorokhod's representation theorem; see e.g., Billingsley [1, Theorem 25.6].

Let

$$A := X^{-1}([0, \mathbb{T}]/\{\tau_{k_0+1}, \tau_{k_0+2} \ldots\}),$$
$$B := X^{-1}(\tau_{k_0+1}, \tau_{k_0+2} \ldots).$$

Let us verify that the function G satisfies

$$G(t) = P\left[\{X \le t\} \cap A\right]. \tag{4.3}$$

Note that

$$\begin{aligned} F(t) - P\left[\{X \le t\} \cap A\right] &= P\left[\{X \le t\}\right] - P\left[\{X \le t\} \cap A\right] \\ &= P\left[\{X \le t\} \cap B\right] \\ &= \sum_{k=k_0+1}^{\infty} P\left[\{X \le t\} \cap X^{-1}(\tau_k)\right] \\ &= \sum_{k=k_0+1}^{\infty} \Delta F(\tau_k) 1_{\{\tau_k \le t\}} \\ &= H(t), \end{aligned}$$

and the equality (4.3) follows. Let

$$G_m(t) := P\left[\{X_m \le t\} \cap A\right], \text{ for } t \in [0, \mathbb{T}] \text{ and } m \in \mathbf{N}.$$

The function G_m has the following properties:

1. The function is clearly non-negative and non-decreasing.
2. G_m is a continuous function. Suppose by way of contradiction that G_m has a jump in $t_0 \in [0, \mathbb{T}]$. Take $\epsilon > 0$ smaller than the size of the jump

   $$0 < \epsilon \le \Delta G_m(t_0).$$

   Then

   $$\epsilon \le P\left[\{X_m = t_0\} \cap A\right] \le P\left[\{X_m = t_0\}\right],$$

   a contradiction with the fact that the function F_m is continuous. Thus, it was false to assume that G_m has a jump.
3. For $t \in [0, \mathbb{T}]$ we claim

   $$\lim_{m \to \infty} G_m(t) = G(t). \tag{4.4}$$

Indeed, we have

$$\lim_{m\to\infty} P\left[\{X_m \le t\} \cap A\right] = \lim_{m\to\infty} E_P\left[1_{(-\infty,t]}(X_m)1_A\right]$$
$$= E_P\left[1_{(-\infty,t]}(X)1_A\right]$$
$$= P\left[\{X \le t\} \cap A\right]$$
$$= G(t),$$

where the second equality holds true due to Lebesgue dominated convergence and the last equality is just (4.3).

Let

$$H_m := P\left[\{X_m \le t\} \cap B\right].$$

Analogously to the sequence $\{G_m\}_{m\in\mathbf{N}}$ we can prove that H_m

1. is a non-decreasing non-negative function,
2. is a continuous function
3. and $\lim_{m\to\infty} H_m(t) = H(t)$, for $t \in [0, \mathbb{T}]$.

The proof concludes with the equalities

$$G_m(t) + H_m(t) = P\left[\{X_m \le t\} \cap A\right] + P\left[\{X_m \le t\} \cap B\right]$$
$$= P\left[\{X_m \le t\}\right]$$
$$= F_m(t).$$

□

5 The Γ-Limit Under Monotonicity

Theorem 5.1 *Let* $\mathsf{J} : \mathsf{C}^0 \times \mathsf{C}^0 \to \mathbf{R}$ *be a monotone functional. Assume* $\mathsf{J}^* : \mathsf{C} \times \mathsf{C} \to \mathbf{R}$ *is the* Γ*-limit of* J *for elements in* $\mathsf{C}_{finite} \times \mathsf{C}_{finite}$ *of distributions with a finite number of jumps. Then, the* Γ*-limit of* J *in* $\mathsf{C} \times \mathsf{C}$ *is given as follows. For a pair* $(\mathsf{c}^1, \mathsf{c}^2) \in \mathsf{C} \times \mathsf{C}$ *with a countable number of jumps* $\{\tau_0, \tau_1, \ldots\}$ *we have*

$$\mathsf{J}^*(\mathsf{c}^1, \mathsf{c}^2) = \lim_{k\to\infty} \mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)),$$

where

$$\widetilde{\mathsf{c}}^i_t(k) := \widehat{\mathsf{c}}^i_t + \sum_{j=0}^{k} \Delta \mathsf{c}^i_{\tau_j} 1_{\{\tau_j \le t\}}$$

and $\widehat{\mathsf{c}}^i$ *is the continuous part of* c^i*, for* $i = 1, 2$.

Proof

1. Let $\{(\mathsf{s}^1(m), \mathsf{s}^2(m))\}_{m\in\mathbf{N}} \subset \mathsf{C}^0 \times \mathsf{C}^0$ be a sequence componentwise weakly-converging to $(\mathsf{c}^1, \mathsf{c}^2) \in \mathsf{C} \times \mathsf{C}$. We first prove that

$$\liminf_{m\to\infty} \mathsf{J}(\mathsf{s}^1(m), \mathsf{s}^2(m)) \geq \lim_{k\to\infty} \mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)). \tag{5.1}$$

For $k \in \mathbf{N}$ fixed and arbitrary $m \in \mathbf{N}$, take a decomposition $\mathsf{s}^i(m) = G^i(m) + H^i(m)$ as in Lemma 4.1 with $G^i(m)$ converging to $\widetilde{\mathsf{c}}^i(k)$ as $m \to \infty$. The functional J is monotone and therefore

$$\mathsf{J}(\mathsf{s}^1(m), \mathsf{s}^2(m)) \geq \mathsf{J}(G^1(m), G^2(m)).$$

As a consequence

$$\liminf_{m\to\infty} \mathsf{J}(\mathsf{s}^1(m), \mathsf{s}^2(m)) \geq \liminf_{m\to\infty} \mathsf{J}(G^1(m), G^2(m)) \geq \mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)),$$

where the last inequality holds true since $G^i(m)$ weakly converges to $\widetilde{\mathsf{c}}^i(k)$. The sequence $\{\mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k))\}_{k\in\mathbf{N}}$ is non decreasing and we obtain the inequality (5.1).

2. Now we construct a sequence where the inequality (5.1) is satisfied with equality.

Let $\{\mathsf{k}^i(k, j)\}_{j\in\mathbf{N}}$ be a sequence of continuous functions weakly converging to $\widetilde{\mathsf{c}}^i(k)$ for $i = 1, 2$ and

$$\mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)) = \lim_{j\to\infty} \mathsf{J}(\mathsf{k}^1(k, j), \mathsf{k}^2(k, j)),$$

such a sequence exists since J^* is the Γ-limit of J in $\mathsf{C}_{finite} \times \mathsf{C}_{finite}$. Let ρ denote the Prokhorov metric on the space of probability measures defined on the interval $[0, \mathbb{T}]$. Next, identify distributions with probability measures. For $k \in \mathbf{N}$ let $j_k \in \mathbf{N}$ be such that $j_k > j_{k-1}$ and for $j \geq j_k$ and $i = 1, 2$

$$\rho(\widetilde{\mathsf{c}}^i(k), \mathsf{k}^i(k, j)) < \frac{1}{2k}$$

$$\rho(\widetilde{\mathsf{c}}^i(k), \mathsf{c}^i) < \frac{1}{2k}$$

$$\left| \mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)) - \mathsf{J}(\mathsf{k}^1(k, j), \mathsf{k}^2(k, j)) \right| < \frac{1}{k}.$$

Then, the sequence $\{(\mathsf{k}^1(k, j_k), \mathsf{k}^2(k, j_k))\}_{k\in\mathbf{N}}$ satisfies (5.1) with equality, since it has the properties

$$\rho(\mathsf{c}^i, \mathsf{k}^i(k, j_k)) < \frac{1}{k},$$
$$\left|\mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)) - \mathsf{J}(\mathsf{k}^1(k, j_k), \mathsf{k}^2(k, j_k))\right| < \frac{1}{k}.$$

□

Let us give an application of Theorem 5.1. To this end, take a non-negative Radon measure η with support in the interval $[0, \mathbb{T}]$. Consider a functional of the form

$$\mathsf{J}(\mathsf{c}^1, \mathsf{c}^2) = \int_{[0,\mathbb{T}]} f(t, \mathsf{c}^1_t, \mathsf{c}^2_t) d\eta_t, \text{ for } (\mathsf{c}^1, \mathsf{c}^2) \in \mathsf{C}^0 \times \mathsf{C}^0,$$

where f is a normal integrand. That is, the correspondence

$$t \in [0, \mathbb{T}] \to \{(c^1, c^2, \alpha) \in \mathbf{R}^2_+ \times \mathbf{R} \mid f(t, c^1, c^2) \leq \alpha\},$$

is closed-valued and measurable. Recall that a set valued mapping (or correspondence) $S : \Xi \mapsto \mathbf{R} \cup \{\infty\}$ defined in a measurable space (Ξ, σ) is measurable if the inverse image $S^{-1}(O) := \{\xi \in \Xi \mid S(\xi) \cap O \neq \emptyset\}$ of every open set O is measurable. We will assume that $f(t, \cdot, \cdot)$ is a continuous non decreasing function for each $t \in [0, \mathbb{T}]$ and it is dominated by an η-integrable function. It is clear that J is a monotone functional. The Γ-limit of J is given in the next result.

Proposition 5.2 *For* $(\mathsf{c}^1, \mathsf{c}^2) \in \mathsf{C} \times \mathsf{C}$ *let* $\mathcal{D}$ *be the set of points where* c^1 *or* c^2 *jumps and let* A *be the set of atoms of the Radon measure* η. *Let* $(A \cap \mathcal{D})^c$ *be the complement of* $A \cap \mathcal{D}$ *in the interval* $[0, \mathbb{T}]$. *The* Γ*-limit of* J *in* $(\mathsf{c}^1, \mathsf{c}^2)$ *is given by*

$$\mathsf{J}^*(\mathsf{c}^1, \mathsf{c}^2) = \int_{(A\cap\mathcal{D})^c} f(t, \mathsf{c}^1_t, \mathsf{c}^2_t) d\eta_t + \sum_{t\in A\cap\mathcal{D}} \eta(\{t\}) f(t, \mathsf{c}^1_{t-}, \mathsf{c}^2_{t-}). \tag{5.2}$$

Proof Take $(\mathsf{c}^1, \mathsf{c}^2) \in \mathsf{C}_{finite} \times \mathsf{C}_{finite}$. For $i = 1, 2$, take a sequence $\{w^i(n)\}_{n\in\mathbf{N}} \subset \mathsf{C}^0$ converging weakly to c^i. We clearly have that

$$\liminf_{n\to\infty} \int_{[0,\mathbb{T}]} f(t, w^1_t(n), w^2_t(n)) d\eta_t$$
$$= \int_{(A\cap\mathcal{D})^c} f(t, \mathsf{c}^1_t, \mathsf{c}^2_t) d\eta_t + \liminf_{n\to\infty} \int_{A\cap\mathcal{D}} f(t, w^1_t(n), w^2_t(n)) d\eta_t,$$

due to the weak convergence, since f is a continuous function.

Take $t \in A \cap \mathcal{D}$. We will do the proof for $t \in (0, \mathbb{T})$, the other cases being more simple. For $\epsilon > 0$ and $\delta > 0$ with $t - \delta, t + \delta \in (0, \mathbb{T})/A \cup \mathcal{D}$ let $N \in \mathbf{N}$ be such that $\left|w^i_{t-\delta}(n) - \mathsf{c}^i_{t-\delta}\right| \leq \epsilon$ and $\left|w^i_{t+\delta}(n) - \mathsf{c}^i_{t+\delta}\right| \leq \epsilon$, for $n \geq N$. Then

$$-\epsilon + \mathsf{c}^i_{t-\delta} \leq w^i_t(n) \leq \epsilon + \mathsf{c}^i_{t+\delta}.$$

As a consequence

$$\mathsf{c}^i_{t-} \leq \liminf_{n\to\infty} w^i_t(n) \leq \limsup_{n\to\infty} w^i_t(n) \leq \mathsf{c}^i_t.$$

The monotonicity and continuity of f implies now that

$$\liminf_{n\to\infty} \int_{A\cap\mathcal{D}} f(t, w^1_t(n), w^2_t(n)) d\eta_t \geq \int_{A\cap\mathcal{D}} f(t, \mathsf{c}^1_{t-}, \mathsf{c}^2_{t-}) d\eta_t.$$

Thus, we have proved that $\mathsf{J}^*(\mathsf{c}^1, \mathsf{c}^2) \leq \liminf_{n\to\infty} \mathsf{J}(w^1(n), w^1(n))$.

Now we are going to construct a sequence $\{(v^1(n), v^2(n))\}_{n\in\mathbf{N}}$ converging weakly to $(\mathsf{c}^1, \mathsf{c}^2)$ with $\mathsf{J}^*(\mathsf{c}^1, \mathsf{c}^2) = \lim_{n\to\infty} \mathsf{J}(v^1(n), v^2(n))$. For $t \in \mathcal{D} \cap (0, \mathbb{T})$ let $B_t(\delta) := (t, t + \delta]$ where $\delta > 0$ is small enough so that $t + \delta \in (0, \mathbb{T})/(A \cup \mathcal{D})$ and the sets $B_t(\delta)$ are pairwise disjoint. For $i = 1, 2$, let l^i_t be the linear function defined by

$$l^i_t(z) = (z - t)\frac{\mathsf{c}^i(t + \delta) - \mathsf{c}^i(t-)}{\delta} + \mathsf{c}^i(t-).$$

We define

$$v^i_z(\delta) := \begin{cases} \mathsf{c}^i_{z-} & \text{for } \; z \notin \bigcup_{t\in\mathcal{D}\cap(0,\mathbb{T})} B_t(\delta), \\ l^i_t(z) & \text{for } \; z \in B_t(\delta). \end{cases}$$

Let $\{\delta_n\}_{n\in\mathbf{N}}$ be a sequence with $\delta_n \leq \frac{1}{n}$ and satisfying the requirements that $t + \delta_n \in (0, \mathbb{T})/(A \cup \mathcal{D})$ and the sets $B_t(\delta_n)$ are pairwise disjoint. It is clear that the sequence $\{(v^1(\delta_n), v^2(\delta_n))\}_{n\in\mathbf{N}}$ converges weakly to $(\mathsf{c}^1, \mathsf{c}^2)$. Indeed, $(v^1(\delta_n), v^2(\delta_n)) = (\mathsf{c}^1, \mathsf{c}^2)$ outside the set $\bigcup_{t\in\mathcal{D}\cap(0,\mathbb{T})} B_t(\delta_n)$. Moreover

$$\int_{A\cap\mathcal{D}} f(t, v^1_t(\delta_n), v^2_t(\delta_n)) d\eta_t = \int_{A\cap\mathcal{D}} f(t, \mathsf{c}^1_{t-}, \mathsf{c}^2_{t-}) d\eta_t,$$

due to the definition of $(v^1_t(\delta_n), v^2_t(\delta_n))$.

We have proved that J^* as defined in (5.2), is the Γ-limit of J for elements in $\mathsf{C}_{finite} \times \mathsf{C}_{finite}$ of distributions with a finite number of jumps. Then, after Theorem 5.1, the Γ-limit of J in $\mathsf{C} \times \mathsf{C}$ is given as follows. For a pair $(\mathsf{c}^1, \mathsf{c}^2) \in$

$\mathsf{C} \times \mathsf{C}$ with a countable number of jumps $\{\tau_0, \tau_1, \ldots\}$ we have

$$\mathsf{J}^*(\mathsf{c}^1, \mathsf{c}^2) = \lim_{k\to\infty} \mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)),$$

with the notation of Theorem 5.1. Note that

$$\mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)) = \int_{[0,\mathbb{T}]} f(t, \widetilde{\mathsf{c}}^1_{t-}(k), \widetilde{\mathsf{c}}^2_{t-}(k)) d\eta_t.$$

Moreover, $\lim_{k\to\infty} \widetilde{\mathsf{c}}^i_{t-}(k) = \mathsf{c}^i_{t-}$ uniformly in $t \in [0, \mathbb{T}]$ and $i = 1, 2$. As a consequence

$$\lim_{k\to\infty} \mathsf{J}^*(\widetilde{\mathsf{c}}^1(k), \widetilde{\mathsf{c}}^2(k)) = \int_{[0,\mathbb{T}]} f(t, \mathsf{c}^1_{t-}, \mathsf{c}^2_{t-}) d\eta_t,$$

due to the continuity of the function f. The right-hand side of the last equation coincides with the right-hand side of (5.2). This proves the proposition. □

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