Asymptotic Results for the Severity and Surplus Before Ruin for a Class of Lévy Insurance Processes

Ekaterina T. Kolkovska and Ehyter M. Martín-González

Abstract We investigate a classical two-sided jumps risk process perturbed by a spectrally negative α-stable process, in which the gain size distribution has a rational Laplace transform. We consider three classes of light- and heavy-tailed claim size distributions. We obtain the asymptotic behaviors of the ruin probability and of the joint tail of the surplus prior to ruin and the severity of ruin, for large values of the initial capital. We also show that our asymptotic results are sharp. This extends our previous work (Kolkovska and Martín-González, Gerber-Shiu functionals for classical risk processes perturbed by an α-stable motion. Insur Math Econ 66:22–28, 2016).

Keywords Two-sided risk process · Stable process · Ruin probability · Severity of ruin · Surplus before ruin · Asymptotic ruin probability

Mathematics Subject Classification 60G51

1 Introduction

For a given risk process $X = \{X(t),\ t \geq 0\}$, the expected discounted penalty function, named also the Gerber-Shiu functional, is defined by

$$\phi(u) = \mathbb{E}\left[e^{-\delta\tau_0} \omega\left(|X(\tau_0)|, X(\tau_0-)\right) 1_{\{\tau_0<\infty\}} \middle| X(0) = u \right],$$

E. T. Kolkovska (✉)
Área de Probabilidad y Estadística, Centro de Investigación en Matemáticas, Guanajuato, Mexico
e-mail: todorova@cimat.mx

E. M. Martín-González
División de Ciencias Naturales y Exactas, Departamento de Matemáticas, Universidad de Guanajuato, Guanajuato, Mexico

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where $\tau_0 = \inf\{t \geq 0 : X(t) < 0\}$ is the ruin time, $\delta \geq 0$ is a constant representing a discounting factor, and $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative penalty function. The random variables $|X(\tau_0)|$ and $X(\tau_0-)$ are known respectively as the severity of ruin and the surplus immediately before ruin. The functional ϕ was introduced in [12] as a generalization of the concept of ruin probability, which is obtained as a particular case when $\delta = 0$ and $\omega \equiv 1$, and has been investigated intensively since then. Many other important risk measures arise as particular cases of the Gerber-Shiu functional, including the distribution of the claim that causes the ruin given that $\tau_0 < \infty$, the joint Laplace transform of the severity of ruin and the surplus prior to ruin, the Laplace transform of the time to ruin and the joint tail distribution of the severity of ruin and the surplus prior to ruin.

The classical two-sided jumps risk process is given by

$$X(t) = u + ct + \sum_{j=1}^{N_1(t)} Y_{j1} - \sum_{j=1}^{N_2(t)} Y_{j2} := u + ct + Z_1(t) - Z_2(t), \tag{1.1}$$

where $u \geq 0$ and $c > 0$ are constants representing, respectively, the initial capital of the insurance company and the prime per unit time that the company receives, and $Z_1 = \{Z_1(t), t \geq 0\}$, $Z_2 = \{Z_2(t), t \geq 0\}$ are two independent compound Poisson processes with respective intensities and jump distributions, λ_i and F_i, $i = 1, 2$, where $\lambda_i \geq 0$ for $i = 1, 2$. Here $Z_1(t)$ and $Z_2(t)$ model respectively the accumulated random gains and random claims at time t. In the case when $\lambda_1 = 0$ the resulting process is called the classical risk process.

In a previous paper [15] we investigated a perturbed two-sided jumps classical risk process $V_\alpha = \{V_\alpha(t), t \geq 0\}$, given by

$$V_\alpha(t) = X(t) - \eta W_\alpha(t), \ \eta > 0, t \geq 0, \tag{1.2}$$

where X is the risk process defined in (1.1) and $\{W_\alpha(t), t \geq 0\}$ is an independent standard α-stable process with index of stability $1 < \alpha < 2$ and skewness parameter $\beta = 1$. Moreover, F_1 possesses a density f_1 whose Laplace transform $\widehat{f_1}$ is a rational function of the form

$$\widehat{f_1}(r) = \frac{Q(r)}{\prod_{i=1}^{N}(q_i + r)^{m_i}}, \quad r \geq 0, \tag{1.3}$$

where $N, m_i \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_N = m$, $0 < q_1 < q_2 < \cdots < q_m$ and Q is a polynomial function of degree at most $m - 1$. The family of distributions satisfying (1.3) is widely used in probability applications. This is a wide class of light-tailed distributions which includes Coxian distributions, combinations of exponential distribution, phase-type distributions, combinations of Erlang distributions and many others. It is dense in the class of general nonnegative distributions (see e.g. [7] and [16, Theorem 8.2.8].) and this property allows for numerical approximations for ϕ in the case of general gain distributions. Under some additional assumptions on the claim size distribution function F_2 and the penalty function ω, in [15] we obtained a formula for the Laplace transform of ϕ and an expression for ϕ as an infinite series of convolutions of given functions. However, such infinite sums of convolutions are hard to work with in practice, and therefore, it is of interest to study the asymptotic behavior of such expressions.

In this paper we investigate the same model as in [15], to which we refer the reader for motivation and explanations about the meaning of the model parameters. Based upon the results obtained in [15], here we obtain an asymptotic formula for the ruin probability $\psi(u) := \mathbb{P}[\tau_0 < \infty | V_\alpha(0) = u]$ as $u \to \infty$, see Theorem 1 below. In Theorem 2 we obtain an asymptotic formula, as $u \to \infty$, of the joint tail distribution

$$\Upsilon_{a,b}(u) := \mathbb{P}[|V_\alpha(\tau_0)| > a, V_\alpha(\tau_0-) > b, \tau_0 < \infty | V(0) = u], \quad a > 0,\ b > 0. \tag{1.4}$$

In Theorem 3 we show that such asymptotic formula holds uniformly in the parameters a and b. These results extend our previous work [14], where we investigated similar behaviors for the classical risk process perturbed by W_α. Other asymptotic results for the ruin probability and the asymptotic distribution of the overshoot of the process about high levels are obtained by Klüppelberg et al. [13] in the case when the Lévy risk process is spectrally positive or spectrally negative. In Doney et al. [6] asymptotic results for the time of ruin, the surplus before the time of ruin and the overshoot at ruin time are obtained for Lévy risk processes under the assumptions that the positive part of the Lévy measure of the process is heavy tailed, and the renewal measure of the descending ladder process is of regular variation. In the case we study here the risk process V_α has two-sided jumps distribution, such that the upward-jump distribution is light-tailed. Therefore, our results complement the investigation in [13] and [6].

We remark that expressions for Gerber-Shiu functionals of a more general class of Lévy risk processes than the one we treat here are given in Biffis and Morales [2] in terms of infinite series of convolutions of integral functions. However, the integrals involved in such convolution formula are not easy to calculate in general, since they are integrals with respect to pure jumps measures and require Laplace transform inversion techniques. In [1] the authors give an expression for a generalized version of the Gerber-Shiu functional for spectrally negative Lévy risk processes in terms of integrals of the associated scale functions of the processes. However, in most cases the scale functions are difficult to obtain explicitly.

The paper is organized as follows: in Sect. 2 we give additional assumptions on the process V_α that we need, as well as several definitions and preliminary results that we use in the sequel. In Sect. 3 we obtain asymptotics for the ruin probability of the process V_α, using Karamata's theorem combined with certain results from [8]. The final Sect. 4 contains our main results, Theorems 2 and 3, and their proofs.

2 Definitions and Preliminary Results

In what follows we consider the process V_α and denote by ψ the corresponding ruin probability $\psi(u) = \mathbb{P}[\tau_0 < \infty | V_\alpha(0) = u]$ starting with an initial capital $u \geq 0$. As above, we write $\Upsilon_{a,b}(u) = \mathbb{P}[|V_\alpha(\tau_0)| > a, V_\alpha(\tau_0-) > b, \tau_0 < \infty | V(0) = u]$, $u \geq 0$, for the joint tail of the severity of ruin and surplus prior to ruin, where a and b are fixed positive numbers. These two functions ψ and $\Upsilon_{a,b}$ are particular cases of ϕ respectively, when $\omega(x, y) = 1$ and when $\omega(x, y) = 1_{\{x>a,y>b\}}$. We recall that the survival probability $\Phi(u) = 1 - \psi(u)$, $u \in \mathbb{R}$, is a distribution function.

We consider the Generalized Lundberg equation

$$L(r) := cr + \eta^\alpha r^\alpha + \lambda_1 \widehat{F_1}(-r) + \lambda_2 \widehat{F_2}(r) - (\lambda_1 + \lambda_2) = 0.$$

In [15, Proposition 3.6] it is proved that L has exactly $m + 1$ roots in the right-half complex plane $\mathbb{C}_+ = \{z \in \mathbb{C} : Re(z) \geq 0\}$, and when $\delta = 0$, 0 is a root of the above equation with multiplicity 1. We denote the roots of L by $\rho_1, \ldots, \rho_{m+1}$, with $\rho_1 = 0$ when $\delta = 0$. We assume that the following conditions hold.

(a) The upward distribution F_1 has a density f_1, whose Laplace transform has the form (1.3).
(b) The Net Profit Condition $\mathbb{E}[V_\alpha(1) - u] = c + \lambda_1\mu_1 - \lambda_2\mu_2 > 0$ holds, where $\mu_j = \mathbb{E}[X_{1j}] < \infty$, $j = 1, 2$.
(c) The roots $\rho_1, \ldots, \rho_{m+1}$, are all different.

Notice that assumption (b) implies that $\lim_{t\to\infty} V_\alpha(t) = +\infty$ with probability 1. For $a > 0$ we denote by $z_{\alpha,a}$ the density of the extremal stable distribution $\zeta_{\alpha,a}$; see e.g. [15, page 376] for the definition of $\zeta_{\alpha,a}$. It is known [11, Lemma 1] that the Laplace transform of $z_{\alpha,a}$ exists for all $r \geq 0$ and is given by $\widehat{z}_{\alpha,a}(r) = \frac{a}{a+r^{\alpha-1}}$. We set $E(\rho_j) = \frac{\prod_{l=1}^N (q_l-\rho_j)^{m_l}}{\prod_{l\neq j}(\rho_l-\rho_j)}$ and denote by T_r the Dickson-Hipp operator introduced in [5], which is defined by $T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y)\, dy$ for any $x \geq 0$, all complex number $r = r_1 + ir_2$ with $r_1 \geq 0$, and all integrable nonnegative functions f. We define the function

$$g_0(x) = \lambda_2 \sum_{j=1}^{m+1} E(\rho_j) T_{\rho_j} f_2(x), \quad x > 0,$$

and for $\alpha < 2$ and $u > 0$ we denote $l_\alpha(u) = \frac{(\alpha-1)u^{-\alpha}}{\Gamma(2-\alpha)}$ and $f_\alpha(u) = \sum_{j=2}^{m+1} E(\rho_j)\rho_j T_{\rho_j} l_\alpha(u)$. It is easily shown that $\widehat{f_\alpha}(r) = \sum_{j=2}^{m+1} E(\rho_j)\rho_j \frac{\rho_j^{\alpha-1}-r^{\alpha-1}}{\rho_j-r}$. From [15, Lemma 5.3] it follows that f_α and g_0 are real valued functions. In the sequel we will assume that these two functions are nonnegative. This assumption holds at least in the case when F_2 is a convex sum of exponential distribution functions with positive coefficients, since in this case it follows similarly as in [4] that the roots ρ_j, $j = 1, \dots, m+1$, of the Lundberg equation $L(r) = 0$ are nonnegative real numbers. This implies, due to the definition of $E(\rho_j)$, that also $E(\rho_j)$ are nonnegative numbers.

Now we define the distribution functions

$$F_\alpha(x) = \frac{1}{\mathcal{C}_F} \int_{0+}^{x} f_\alpha(y)dy, \; G_0(x) = \frac{1}{\mathcal{C}_G} \int_{0+}^{x} g_0(y)dy,$$

$$U_\alpha(x) = \frac{1}{\mathcal{C}_U} \int_{0+}^{x} \nu_\alpha(y)dy, \quad x > 0, \tag{2.1}$$

and $F_{2,I}(x) = \frac{1}{\mu_2} \int_0^x \overline{F}_2(y)\, dy$, $x \geq 0$. Here $\mathcal{C}_F = \int_{0+}^\infty f_\alpha(x)\, dx$, $\mathcal{C}_G = \int_{0+}^\infty g_0(x)\, dx$ and $\mathcal{C}_U = \int_{0+}^\infty \nu_\alpha(x)\, dx$. The functions $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ and $W_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ are defined by their Laplace transforms

$$\widehat{\nu}_\alpha(r) \left(1 + \frac{1}{\theta} \widehat{f}_\alpha(r) \widehat{z}_{\alpha,\theta}(r)\right) = \widehat{z}_{\alpha,\theta}(r), \tag{2.2}$$

where $\theta = c/\eta^\alpha + \kappa$ and $\kappa = \frac{1}{\eta^\alpha} \widehat{g}_0(0) + \widehat{f}_\alpha(0)$, and

$$\widehat{W}_\alpha(r) = \frac{\frac{1}{\eta^\alpha \theta} \widehat{\nu}_\alpha(r)}{1 - \frac{1}{\theta}\left[\kappa \widehat{\nu}_\alpha(r) + \frac{1}{\eta^\alpha} \widehat{g}_0(r) \widehat{\nu}_\alpha(r)\right]}. \tag{2.3}$$

In [15, Proposition 5.6] we give representations of ν and W_α as series of convolutions of given functions.

We recall [15, Proposition 5.4 b)] that the Laplace transform of the ruin probability ψ satisfies the equality

$$\widehat{\psi}(r) = \frac{1}{r} - \frac{(c + \lambda_1 \mu_1 - \lambda_2 \mu_2)}{r} \frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=2}^{m+1} \rho_j} \widehat{W}_\alpha(r), \quad r > 0. \tag{2.4}$$

Notice that the roots of Lundberg's equation appear in conjugate pairs because the equation coefficients are real, hence $\frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=2}^{m+1} \rho_j} > 0$.

We also recall the following definitions: Let F be a distribution function such that $F(0) = 0$ with tail $\overline{F} = 1 - F$. If there exist numbers $c_1, c_2 > 0$ such that $\overline{F}(x) \leq c_1 e^{-c_2 x}$ for all $x > 0$, then F is called light-tailed distribution function. Otherwise

F is a heavy-tailed distribution function and in such a case we write $F \in \mathcal{H}$. In case that $\lim_{x\to\infty} \overline{F^{*2}}(x)/\overline{F}(x) = 2$ we say that F belongs to the class of subexponential distributions and write $F \in \mathcal{S}$. The distribution function F belongs to the class $\mathcal{L}$ if for any $y \geq 0$ there holds $\lim_{x\to\infty} \overline{F}(x-y)/\overline{F}(x) = 1$. Finally, F belongs to the class $\mathcal{R}_c$ for $c \geq 0$ if F has a density f such that $\lim_{x\to\infty} f(x)/\overline{F}(x) = c$. We say that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a regularly varying function of x at ∞, with order $a \in \mathbb{R}$, if $\lim_{x\to\infty} f(xt)/f(x) = t^a$ for $t > 0$, and write $f \in RV_a$. In the particular case when $a = 0$, we say that f is a slowly varying function of x at ∞. If f is regularly varying of order a, then it can be written as $f(x) = x^a L(x)$, where L is a slowly varying function. We define $f \sim g$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. We write $F \in \overline{RV}_a$ if F is such that $\overline{F}(x) \sim x^a L(x)$. The following inclusions hold (see [9]):

$$\overline{RV}_a \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H} \text{ and } \mathcal{R}_0 \subset \mathcal{L}. \tag{2.5}$$

Lemma 1 *Let F_1, F_2 be two distribution functions such that $F_i(0) = 0$, $i = 1, 2$, and let $H = F_1 * F_2$ be their convolution.*

a) If $F_2 \in \mathcal{S}$ and $\overline{F}_1(x) = o(\overline{F}_2(x))$ as $x \to \infty$, then $H \in \mathcal{S}$. Moreover, $\overline{H}(x) \sim \overline{F}_2(x)$.
b) If $\overline{F}_i(x) \sim x^{-\delta} L_i(x)$ for $i = 1, 2$, where L_1 and L_2 are slowly varying functions, then $\overline{H}(x) \sim x^{-\delta}(L_1(x) + L_2(x))$ as $x \to \infty$.
c) If $\overline{F}_2(x) \sim c\overline{F}_1(x)$ for some $c \in (0, \infty)$, then $F_1 \in \mathcal{S}$ if and only if $F_2 \in \mathcal{S}$ and $\overline{H} \sim (1 + c)\overline{F}_2(x)$.
d) If $\beta \in (0, 1)$ and $K(x) = (1 - \beta) \sum_{n=0}^{\infty} \beta^n F_1^{*n}(x)$ then the following three conditions are equivalent:

$$K \in \mathcal{S}, \quad F_1 \in \mathcal{S}, \quad \overline{K}(x) \sim \tfrac{\beta}{1-\beta} \overline{F}_1(x).$$

Proof For a) and d) see, respectively, [8, Proposition 1a) and Theorem 3]. For b) see [10, page 278]. The proof of c) is given in [16, lemmas 2.5.2 and 2.5.4]. ∎

3 Asymptotic Behavior of the Ruin Probability

In what follows we will use the elementary identities

$$\widehat{F}(r) = \frac{\widehat{f}(r)}{r} \text{ and } \widehat{\overline{F}}(r) = \frac{1 - \widehat{f}(r)}{r}, \quad r > 0, \tag{3.1}$$

valid for any distribution function F with $F(0) = 0$ and having density f. First we state the following auxiliary result.

Proposition 1 *The following asymptotics hold:*

a) $\lim_{x\to\infty} \frac{\overline{F}_\alpha(x)}{\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}} = \frac{1}{\mathcal{C}_F}\left(1 - \frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\right)$*, hence* $F_\alpha \in \mathcal{S}$.

b) If $F_2 \in \mathcal{R}_0$*, then* $\lim_{x\to\infty} \frac{\overline{G}_0(x)}{\overline{F}_{2,I}(x)} = \frac{\lambda_2\mu_2}{\mathcal{C}_G}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}$*. If in addition* $F_{2,I} \in \mathcal{S}$*, then* $G_0 \in \mathcal{S}$.

c) If $\overline{F}_2(x) = o(x^{-\alpha})$*, then* $\overline{G}_0(x) = o(x^{1-\alpha})$.

d) $\lim_{x\to\infty} \frac{\overline{U}_\alpha(x)}{\overline{\zeta}_{\alpha,\theta}(x)} = \mathcal{C}_U \frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}$*, hence* $U_\alpha \in \mathcal{S}$.

Proof

a) Let us define $F^*(u) = \int_0^u \overline{F}_\alpha(x)dx$. From (3.1) we obtain $\widehat{F}^*(r) = (1 - \frac{1}{\mathcal{C}_F}\widehat{f}_\alpha(r))/r^2$, hence $\lim_{r\downarrow 0} \frac{r\widehat{F}^*(r)}{r^{\alpha-2}} = \lim_{r\downarrow 0}\frac{1-\frac{1}{\mathcal{C}_F}\widehat{f}_\alpha(r)}{r^{\alpha-1}}$. From the definition of $\mathcal{C}_F$ it follows that $1 - \frac{1}{\mathcal{C}_F}\widehat{f}_\alpha(0) = 0$. Using L'Hospital's rule gives

$$\lim_{r\downarrow 0}\frac{1-\frac{1}{\mathcal{C}_F}\widehat{f}_\alpha(r)}{r^{\alpha-1}} = \lim_{r\downarrow 0}\frac{1+\frac{1}{\mathcal{C}_F}\sum_{j=2}^{m+1}E(\rho_j)\rho_j\frac{\rho_j^{\alpha-1}-r^{\alpha-1}}{\rho_j - r}}{r^{\alpha-1}}$$
$$= \lim_{r\downarrow 0}\frac{\frac{1}{\mathcal{C}_F}\sum_{j=2}^{m+1}E(\rho_j)\rho_j\left(\frac{\rho_j^{\alpha-1}-r^{\alpha-1}}{(\rho_j-r)^2} - \frac{(\alpha-1)r^{\alpha-2}}{\rho_j - r}\right)}{(\alpha-1)r^{\alpha-2}}$$
$$= -\frac{1}{\mathcal{C}_F}\sum_{j=2}^{m+1}E(\rho_j) = \frac{1}{\mathcal{C}_F}\left(1 - \frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\right), \quad (3.2)$$

where the last equality follows by [15, Lemma 5.3]. From [10, Theorem 1, page 443] we obtain the limit in part a), which implies that $\overline{F}_\alpha$ is regularly varying. Using (2.5) we also obtain that $F_\alpha \in \mathcal{S}$.

b) Notice that

$$\lim_{x\to\infty}\left|\frac{\int_x^\infty T_{\rho_j} f_2(y)dy}{\overline{F}_{2,I}(x)}\right|$$
$$\leq \lim_{x\to\infty}\frac{\int_x^\infty\int_y^\infty e^{-Re(\rho_j)(z-y)}f_2(z)dzdy}{\overline{F}_{2,I}(x)}, \quad j = 2,3,\ldots,m+1. \quad (3.3)$$

Taking limits when $x \to \infty$ in the right-hand side of (3.3) yields

$$\lim_{x\to\infty} \frac{\int_x^\infty e^{Re(\rho_j)y} \int_y^\infty e^{-Re(\rho_j)z} f_2(z)dzdy}{\overline{F}_{2,I}(x)} = \lim_{x\to\infty} \frac{\int_x^\infty e^{-Re(\rho_j)z} f_2(z)dzdy}{e^{-Re(\rho_j)x}\overline{F}_2(x)}$$

$$= \lim_{x\to\infty} \frac{e^{-Re(\rho_j)x} f_2(x)}{Re(\rho_j)e^{-Re(\rho_j)x}\overline{F}_2(x) + e^{-Re(\rho_j)x} f_2(x)} = \lim_{x\to\infty} \frac{\frac{f_2(x)}{\overline{F}_2(x)}}{Re(\rho_j) + \frac{f_2(x)}{\overline{F}_2(x)}}, \tag{3.4}$$

where the first and second equalities follow by L'Hospital's rule. Using the assumption that $F_2 \in \mathcal{R}_0$, we obtain from (3.4) and (3.3) that

$$\lim_{x\to\infty} \left| \frac{\int_x^\infty T_{\rho_j} f_2(y)\, dy}{\overline{F}_{2,I}(x)} \right| = 0. \tag{3.5}$$

Since $\int_x^\infty g_0(y)dy = \lambda_2 \frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_j} \mu_2 \overline{F}_{2,I}(x) - \lambda_2 \sum_{j=2}^{m+1} E(\rho_j) \int_x^\infty T_{\rho_j} f_2(y)$, the triangle inequality yields

$$\frac{\lambda_2\mu_2}{\mathcal{C}_G} \frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_j} - \left| \frac{-\frac{\lambda_2}{\mathcal{C}_G} \sum_{j=2}^{m+1} E(\rho_j) \int_x^\infty T_{\rho_j} f_2(y)dy}{\overline{F}_{2,I}(x)} \right| \le \left| \frac{\overline{G}_0(x)}{\overline{F}_{2,I}(x)} \right|, \tag{3.6}$$

and

$$\left| \frac{\overline{G}_0(x)}{\overline{F}_{2,I}(x)} \right| \le \frac{\lambda_2\mu_2}{\mathcal{C}_G} \frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_j} + \left| \frac{-\frac{\lambda_2}{\mathcal{C}_G} \sum_{j=2}^{m+1} E(\rho_j) \int_x^\infty T_{\rho_j} f_2(y)dy}{\overline{F}_{2,I}(x)} \right|. \tag{3.7}$$

The limit in part b) follows from letting $x \to \infty$ in (3.6) and (3.7) and using (3.5). Assuming that $F_{2,I} \in \mathcal{S}$, the relation $G_0 \in \mathcal{S}$ follows from part c) of Lemma 1.

c) Let us assume that $\overline{F}_2(x) = o(x^{-\alpha})$, hence L'Hospital's rule implies that

$$\lim_{x\to\infty} \frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}} = 0. \tag{3.8}$$

This yields

$$\lim_{x\to\infty} \frac{\int_x^\infty \int_y^\infty e^{-Re(\rho_j)(z-y)} f_2(z)dzdy}{x^{1-\alpha}} \le \lim_{x\to\infty} \frac{\int_x^\infty \int_y^\infty f_2(z)dzdy}{x^{1-\alpha}}$$

$$= \mu_2 \lim_{x\to\infty} \frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}} = 0,$$

and from (3.8) we obtain

$$\lim_{x\to\infty}\left|\frac{\int_x^\infty T_{\rho_j} f_2(y)dy}{x^{1-\alpha}}\right| = 0. \tag{3.9}$$

Using (3.6) and (3.7) we obtain the inequalities

$$\frac{\mu_2}{\mathcal{C}_G}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}} - \left|\frac{-\frac{1}{\mathcal{C}_G}\sum_{j=2}^{m+1}E(\rho_j)\int_x^\infty T_{\rho_j} f_2(y)dy}{x^{1-\alpha}}\right| \le \left|\frac{\overline{G}_0(x)}{x^{1-\alpha}}\right| \tag{3.10}$$

and

$$\left|\frac{\overline{G}_0(x)}{x^{1-\alpha}}\right| \le \frac{\mu_2}{\mathcal{C}_G}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}} + \left|\frac{-\frac{1}{\mathcal{C}_G}\sum_{j=2}^{m+1}E(\rho_j)\int_x^\infty T_{\rho_j} f_2(y)dy}{x^{1-\alpha}}\right|. \tag{3.11}$$

The result now follows by letting $x \to \infty$ in (3.10) and (3.11), and using (3.8) and (3.9).

d) Putting $r = 0$ in (2.2) gives $\mathcal{C}_U = (1 + \frac{\mathcal{C}_F}{\theta})^{-1}$. Dividing both sides of (2.2) by $\mathcal{C}_U$ yields $\frac{\widehat{\nu}_\alpha(r)}{\mathcal{C}_U}\left(1 + \frac{1}{\theta}\widehat{f}_\alpha(r)\widehat{z}_{\alpha,\theta}(r)\right) = \frac{\widehat{z}_{\alpha,\theta}(r)}{\mathcal{C}_U}$, hence:

$$\begin{aligned}\left(1 - \frac{\widehat{\nu}_\alpha(r)}{\mathcal{C}_U}\right)\left(1 + \frac{1}{\theta}\widehat{f}_\alpha(r)\widehat{z}_{\alpha,\theta}(r)\right) &= 1 + \frac{1}{\theta}\widehat{f}_\alpha(r)\widehat{z}_{\alpha,\theta}(r) - \frac{\widehat{z}_{\alpha,\theta}(r)}{\mathcal{C}_U}\\ &= 1 + \frac{1}{\theta}\widehat{f}_\alpha(r)\widehat{z}_{\alpha,\theta}(r) - \left(1 + \frac{\mathcal{C}_F}{\theta}\right)\widehat{z}_{\alpha,\theta}(r)\\ &= 1 - \widehat{z}_{\alpha,\theta}(r) - \frac{\mathcal{C}_F}{\theta}\widehat{z}_{\alpha,\theta}(r)\left(1 - \frac{1}{\mathcal{C}_F}\widehat{f}_\alpha(r)\right).\end{aligned} \tag{3.12}$$

We define the function $U_\alpha^*(x) = \int_0^x \overline{U}_\alpha(y)dy$, $x > 0$. From (3.1) we get

$$\widehat{U}_\alpha^*(r) = \frac{\widehat{\overline{U}}_\alpha(r)}{r} = \frac{1 - \frac{\widehat{\nu}_\alpha(r)}{\mathcal{C}_U}}{r^2}. \tag{3.13}$$

It follows from (3.12) that

$$\frac{r\widehat{U}_\alpha^*(r)}{r^{\alpha-2}} = \frac{\frac{1-\widehat{z}_{\alpha,\theta}(r)}{r^{\alpha-1}} - \frac{\mathcal{C}_F}{\theta}\frac{\widehat{z}_{\alpha,\theta}(r)\left(1 - \frac{1}{\mathcal{C}_F}\widehat{f}_\alpha(r)\right)}{r^{\alpha-1}}}{1 + \frac{1}{\theta}\widehat{f}_\alpha(r)\widehat{z}_{\alpha,\theta}(r)}. \tag{3.14}$$

Since $\widehat{z}_{\alpha,\theta}(r) = \frac{\theta}{\theta+r^{\alpha-1}}$ we obtain $\lim_{r\downarrow 0} \dfrac{1-\widehat{z}_{\alpha,\theta}(r)}{r^{\alpha-1}} = \dfrac{1}{\theta}$. Using this equality together with (3.2) and letting $r \downarrow 0$ in (3.14), we obtain

$$\lim_{r\downarrow 0} \frac{r\widehat{U}^*_\alpha(r)}{r^{\alpha-2}} = \frac{\frac{1}{\theta} - \frac{1}{\theta} + \frac{1}{\theta}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}}{1+\frac{\mathcal{C}_F}{\theta}} = \frac{\mathcal{C}_U}{\theta}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}, \tag{3.15}$$

where in the last equality we used that $\mathcal{C}_U = (1+\frac{\mathcal{C}_F}{\theta})^{-1}$. Since U^*_α has the monotone density $\overline{U}_\alpha$, Theorem 1 (page 443) in [10] gives the limit in part d). This implies that the tail of U_α is asymptotically regularly varying with index $1-\alpha$, hence from (2.5) we conclude that $U_\alpha \in \mathcal{S}$.

■

Now we are ready to obtain the main result in this section.

Theorem 1 *Consider the following three cases for the claim size distribution F_2. As $x \to \infty$,*

$$\begin{aligned} &\textit{Case 1}: \overline{F}_2(x) = o(x^{-\alpha}),\\ &\textit{Case 2}: \overline{F}_2(x) \sim \kappa x^{-\alpha} \textit{ for some } \kappa > 0,\\ &\textit{Case 3}: F_{2,I} \in \mathcal{S},\ F_2 \in \mathcal{R}_0 \textit{ and } x^{-\alpha} = o\left(\overline{F}_2(x)\right). \end{aligned} \tag{3.16}$$

Then, as $u \to \infty$, we have:

a) In case 1:

$$\psi(u) \sim \frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\,\Gamma(2-\alpha)} u^{1-\alpha}, \tag{3.17}$$

b) In case 2:

$$\psi(u) \sim \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\kappa}{\alpha-1}\right] u^{1-\alpha}, \tag{3.18}$$

c) In case 3:

$$\psi(u) \sim \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u), \tag{3.19}$$

and in all cases $\Phi \in \mathcal{S}$.

Proof

Case 1. We define the function $G_0^*(x)=\int_0^x \overline{G}_0(y)dy$, $x>0$. Due to (3.1) we have $\widehat{G}_0^*(r)=\frac{1-\widehat{g}_0(r)}{r^2}$. From Proposition 1 c) and the assumption that $\overline{F}_2(x)=o(x^{-\alpha})$ we obtain $\overline{G}_0(x)=o\left(x^{1-\alpha}\right)$, hence Theorem 1 (page 443) in [10] and the equality $\widehat{G}_0^*(r)=\frac{1-\widehat{g}_0(r)}{r^2}$ imply

$$0=\lim_{r\downarrow 0}\frac{r\widehat{G}_\alpha^*(r)}{r^{\alpha-2}}=\lim_{r\downarrow 0}\frac{1-\widehat{g}_0(r)\mathcal{C}_G^{-1}}{r^{\alpha-1}}. \tag{3.20}$$

Using that $0=\psi(\infty)=\lim\limits_{u\to\infty}\psi(u)$, the final value theorem for Laplace transforms $\psi(\infty)=\lim\limits_{r\downarrow 0}r\widehat{\psi}(r)$ and (2.4) we obtain $\widehat{W}_\alpha(0)=\left((c+\lambda_1\mu_1-\lambda_2\mu_2)\frac{\prod_{j=2}^{m+1}\rho_j}{\prod_{i=1}^{N}q_i^{m_i}}\right)^{-1}$. Setting $r=0$ in (2.3) yields

$$\begin{aligned}\frac{1}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\frac{\prod_{j=2}^{m+1}\rho_j}{\prod_{i=1}^{N}q_i^{m_i}}}&=\frac{\frac{1}{\eta^\alpha\theta}\widehat{\nu}_\alpha(0)}{1-\frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(0)+\frac{1}{\eta^\alpha}\widehat{g}_0(0)\widehat{\nu}_\alpha(0)\right]}\\&=\frac{\frac{1}{\eta^\alpha\theta}\mathcal{C}_U}{1-\frac{1}{\theta}\left[\kappa\mathcal{C}_U+\frac{1}{\eta^\alpha}\mathcal{C}_G\mathcal{C}_U\right]},\end{aligned}$$

or equivalently

$$(c+\lambda_1\mu_1-\lambda_2\mu_2)\frac{\prod_{j=2}^{m+1}\rho_j}{\prod_{i=1}^{N}q_i^{m_i}}=\frac{1-\frac{1}{\theta}\left[\kappa\mathcal{C}_U+\frac{1}{\eta^\alpha}\mathcal{C}_G\mathcal{C}_U\right]}{\frac{1}{\eta^\alpha\theta}\mathcal{C}_U}. \tag{3.21}$$

Now we set $\psi^*(u)=\int_0^u\psi(y)dy$. Due to (3.1), (2.4), (2.3) and (3.21) we have

$$\begin{aligned}\widehat{\psi}^*(r)&=\frac{1-\left[\frac{1-\frac{1}{\theta}\left[\kappa\mathcal{C}_U+\frac{1}{\eta^\alpha}\mathcal{C}_G\mathcal{C}_U\right]}{\frac{1}{\eta^\alpha\theta}\mathcal{C}_U}\right]\frac{\frac{1}{\eta^\alpha\theta}\widehat{\nu}_\alpha(r)}{1-\frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(r)+\frac{1}{\eta^\alpha}\widehat{g}_0(r)\widehat{\nu}_\alpha(r)\right]}}{r^2}\\&=\frac{1-\frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(r)+\frac{1}{\eta^\alpha}\widehat{g}_0(r)\widehat{\nu}_\alpha(r)\right]-\left[\frac{1-\frac{1}{\theta}\left[\kappa\mathcal{C}_U+\frac{1}{\eta^\alpha}\mathcal{C}_G\mathcal{C}_U\right]}{\mathcal{C}_U}\right]\widehat{\nu}_\alpha(r)}{r^2\left(1-\frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(r)+\frac{1}{\eta^\alpha}\widehat{g}_0(r)\widehat{\nu}_\alpha(r)\right]\right)}.\end{aligned}$$

It follows that

$$\lim_{r\downarrow 0}\frac{r\widehat{\psi}^*(r)}{r^{\alpha-2}}=\lim_{r\downarrow 0}\frac{1}{r^{\alpha-1}}\left(\frac{1-\dfrac{\widehat{\nu}_\alpha(r)}{\mathcal{C}_U}+\frac{\mathcal{C}_G}{\eta^\alpha\theta}\left[1-\dfrac{\widehat{g}_0(r)}{\mathcal{C}_G}\right]\widehat{\nu}_\alpha(r)}{1-\frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(r)+\eta^{-\alpha}\widehat{g}_0(r)\widehat{\nu}_\alpha(r)\right]}\right)$$

$$=\frac{\dfrac{1}{\theta}\dfrac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\mathcal{C}_U}{1-\frac{1}{\theta}\left[\kappa\mathcal{C}_U+\eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]},\tag{3.22}$$

where the last equality follows from (3.13), (3.15) and (3.20). From (3.21) we obtain

$$\frac{\dfrac{1}{\theta}\dfrac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\mathcal{C}_U}{\dfrac{c+\lambda_1\mu_1-\lambda_2\mu_2}{\eta^\alpha\theta}\dfrac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\mathcal{C}_U}=\frac{\eta^\alpha}{c+\lambda_1\mu_1-\lambda_2},$$

hence from (3.22), $\lim_{r\downarrow 0}\dfrac{r\widehat{\psi}^*(r)}{r^{\alpha-2}}=\dfrac{\eta^\alpha}{c+\lambda_1\mu_1-\lambda_2}$. The asymptotic formula (3.17) now follows from [10, Theorem 1, page 443]. Since (3.17) implies that Φ has a regularly varying tail, from (2.5) we conclude $\Phi\in\mathcal{S}$.

Case 2. We work again with the functions ψ^* and G_0^* defined before. Due to $F_2\in\overline{RV}_{-\alpha}$ and $F_2\in\mathcal{R}_0$, from part b) of Proposition 1 we obtain $\overline{G}_0(x)\sim\dfrac{\lambda_2\mu_2}{\mathcal{C}_G}\dfrac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\overline{F}_{2,I}(x)$. Since $\overline{F}_2(x)\sim\kappa x^{-\alpha}$, an application of L' Hospital's rule to $\lim_{x\to\infty}\frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}}$ yields $\overline{F}_{2,I}(x)\sim\kappa\frac{x^{1-\alpha}}{(\alpha-1)\mu_2}$. Hence $\overline{G}_0(x)\sim\dfrac{\lambda_2\kappa}{\mathcal{C}_G(\alpha-1)}\dfrac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}x^{1-\alpha}$. Applying [10, Theorem 1, page 443] to $G_0^*(x)$ gives

$$\frac{\lambda_2\mu_2\kappa}{\mathcal{C}_G(\alpha-1)}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}=\lim_{r\downarrow 0}\frac{r\widehat{G}_\alpha^*(r)}{r^{\alpha-2}}=\lim_{r\downarrow 0}\frac{1-\widehat{g}_0(r)\mathcal{C}_G^{-1}}{r^{\alpha-1}}.\tag{3.23}$$

From the last equality we see, as in case 1, that the limit (3.22) remains valid also in this case. Therefore

$$\lim_{r\downarrow 0}\frac{r\widehat{\psi}^*(r)}{r^{\alpha-2}}=\lim_{r\downarrow 0}\frac{1}{r^{\alpha-1}}\left(\frac{1-\dfrac{\widehat{\nu}_\alpha(r)}{\mathcal{C}_U}+\frac{\mathcal{C}_G}{\eta^\alpha\theta}\left[1-\dfrac{\widehat{g}_0(r)}{\mathcal{C}_G}\right]\widehat{\nu}_\alpha(r)}{1-\frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(r)+\eta^{-\alpha}\widehat{g}_0(r)\widehat{\nu}_\alpha(r)\right]}\right),$$

hence, substituting (3.13), (3.15) and (3.23) in the above equality gives

$$\lim_{r\downarrow 0}\frac{r\widehat{\psi}^*(r)}{r^{\alpha-2}}=\frac{\dfrac{1}{\theta}\dfrac{\prod_{i=1}^{N}q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\mathcal{C}_U}{1-\frac{1}{\theta}\left[\kappa\mathcal{C}_U+\eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]}\left[1+\frac{\lambda_2\kappa\Gamma(2-\alpha)}{\eta^{\alpha}(\alpha-1)}\right]$$
$$=\frac{\eta^{\alpha}(\alpha-1)+\lambda_2\kappa\Gamma(2-\alpha)}{(c+\lambda_1\mu_1-\lambda_2\mu_2)(\alpha-1)}.$$

The asymptotic formula (3.18) follows from [10, Theorem 1, page 443]. Since the right-hand side of (3.18) is a regularly varying function, it follows that Φ has a regularly varying tail. This finishes the proof of case 2.

Case 3. The equality $W_\alpha(x)=\frac{1}{\eta^{\alpha}\theta}\nu_\alpha*\sum_{n=0}^{\infty}\frac{1}{\theta^n}\left[\kappa\nu_\alpha+\frac{1}{\eta^{\alpha}}g_0*\nu_\alpha\right]^{*n}(x)$ is proved in [15, Proposition 5.6]. From (2.4) we note that $(c+\lambda_1\mu_1-\lambda_2\mu_2)\frac{\prod_{j=2}^{m+1}\rho_j}{\prod_{i=1}^{N}q_i^{m_i}}W_\alpha$ is the density function of the probability of survival Φ, hence using the above equality and the definitions of U_α and G_0 in (2.1), it follows that

$$\Phi(x)=\frac{1}{\eta^{\alpha}\theta}(c+\lambda_1\mu_1-\lambda_2\mu_2)\frac{\prod_{j=2}^{m+1}\rho_j}{\prod_{i=1}^{N}q_i^{m_i}}\mathcal{C}_U U_\alpha$$
$$*\sum_{n=0}^{\infty}\frac{1}{\theta^n}\left(\kappa\mathcal{C}_U U_\alpha+\frac{1}{\eta^{\alpha}}\mathcal{C}_G\mathcal{C}_U G_0*U_\alpha\right)^{*n}(x).$$

Now we define $\beta=\frac{1}{\theta}\left[\kappa\mathcal{C}_U+\eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]$. Using (3.21), we obtain from the last equality that

$$\Phi(x)=(1-\beta)U_\alpha*\sum_{n=0}^{\infty}\frac{\beta^n}{\theta^n}\left[\frac{1}{\beta}\left(\kappa\mathcal{C}_U U_\alpha+\frac{1}{\eta^{\alpha}}\mathcal{C}_G\mathcal{C}_U G_0*U_\alpha\right)\right]^{*n}(x)$$
$$=U_\alpha*K(x), \tag{3.24}$$

where $K(x)=(1-\beta)\sum_{n=0}^{\infty}\beta^n K_0^{*n}(x)$ with $K_0(x)=\left(\frac{1}{\theta\beta}\left[\kappa\mathcal{C}_U U_\alpha+\eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U G_0*U_\alpha\right]\right)(x)$, $x>0$. Since U_α and G_0 are distribution functions, using the definition of β we see that K_0 and K are distribution functions as well, and Φ is the convolution of the distribution functions U_α and K. In view of this, we need to study the asymptotic behaviour of $\overline{K}$.

The assumption that $x^{-\alpha}=o(\overline{F}_2(x))$, together with an application of L'Hospital's rule, imply that $x^{1-\alpha}=o\left(\overline{F}_{2,I}(x)\right)$. Since by assumption $F_2\in\mathcal{R}_0$, part b) of Proposition 1 yields $\overline{G}_0(x)\sim\frac{\lambda_2\mu_2}{\mathcal{C}_G}\frac{\prod_{i=1}^{N}q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\overline{F}_{2,I}(x)$, hence $x^{1-\alpha}=o\left(\overline{G}_0(x)\right)$, and due to part d) of Proposition 1 we get

$\overline{U}_\alpha(x) = o\left(\overline{G}_0(x)\right)$. It follows from the definition of K_0 and Lemma 1 a) that $1 - K_0(x) \sim \frac{\eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U}{\theta\beta}\overline{G}_0(x)$. Since by assumption $F_{2,I} \in \mathcal{S}$, from part b) of Proposition 1 we obtain $G_0 \in \mathcal{S}$. It follows from Lemma 1 d) that

$$\overline{K}(x) \sim \frac{\beta}{1-\beta}\frac{\eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U}{\theta\beta}\overline{G}_0(x)$$
$$\sim \frac{\lambda_2\mu_2}{\frac{c+\lambda_1\mu_1-\lambda_2\mu_2}{\eta^\alpha\theta}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\mathcal{C}_U}\frac{\mathcal{C}_U}{\eta^\alpha\theta}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}\overline{F}_{2,I}(x),$$

which reduces to $\overline{K}(x) \sim \dfrac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(x)$ after simplifying the coefficient in the right-hand side of the asymptotic expression above. From here we obtain (3.19) using (3.24), Lemma 1 a) and the relation $\overline{U}_\alpha(x) = o(\overline{F}_{2,I}(x))$ as $x \to \infty$. Hence $\Phi \in \mathcal{S}$.

■

Corollary 1 *For the three cases in (3.16) the ruin probability $\psi(u)$ admits the asymptotic expression*

$$\psi(u) \sim \frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\,\Gamma(2-\alpha)}u^{1-\alpha} + \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u) \text{ as } u \to \infty. \tag{3.25}$$

In particular, if $\overline{F}_2(u) \sim L_1(u)u^{-\alpha}$ for some slowly varying function L_1 and $\overline{F}_2$ belongs to any of the cases in (3.16), then

$$\psi(u) \sim \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2}{\alpha-1}L_1(u)\right]u^{1-\alpha}. \tag{3.26}$$

Proof The estimate (3.25) follows directly from Theorem 1. To obtain (3.26) we consider the three cases in (3.16).

Case 1. We have $\lim_{u\to\infty}\frac{L_1(u)u^{-\alpha}}{u^{-\alpha}} = \lim_{u\to\infty}\frac{\overline{F}_2(u)}{u^{-\alpha}}\frac{L_1(u)u^{-\alpha}}{\overline{F}_2(u)} = 0$. Hence

$$\lim_{u\to\infty}\frac{\psi(u)}{\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\mu_2}{\alpha-1}L_1(u)\right]u^{1-\alpha}}$$
$$= \lim_{u\to\infty}\frac{\frac{\psi(u)}{u^{1-\alpha}}}{\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\mu_2}{\alpha-1}\frac{L_1(u)u^{1-\alpha}}{u^{1-\alpha}}\right]} = 1,$$

where we used (3.17) to obtain the last equality.

Case 2. We set $C = \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\kappa}{\alpha-1}\right]$.
Using the equality $\lim_{u\to\infty} L_1(u) = \kappa$ and (3.18) we obtain that

$$\lim_{u\to\infty} \frac{\psi(u)}{\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2}{\alpha-1}L_1(u)\right]u^{1-\alpha}}$$

$$= \lim_{u\to\infty} \frac{\frac{\psi(u)}{Cu^{1-\alpha}}}{\frac{1}{C}\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2}{\alpha-1}\frac{L_1(u)u^{1-\alpha}}{u^{1-\alpha}}\right]} = 1.$$

Case 3. Notice that $u^{-\alpha} = o\left(\overline{F}_2(u)\right)$ implies $u^{1-\alpha} = o\left(\overline{F}_{2,I}(u)\right)$. Using now Karamata's theorem (see e.g. [3, Proposition 1.5.10]) we obtain that $\lim_{u\to\infty} \frac{\overline{F}_{2,I}(u)}{L_1(u)u^{1-\alpha}} = \frac{\alpha-1}{\mu_2}$. Hence

$$\lim_{u\to\infty} \frac{\psi(u)}{\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\mu_2}{\alpha-1}L_1(u)\right]u^{1-\alpha}}$$

$$= \lim_{u\to\infty} \frac{\frac{\psi(u)}{\frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u)}}{\frac{\eta^\alpha}{\Gamma(2-\alpha)}\frac{u^{1-\alpha}}{\lambda_2\mu_2\overline{F}_{2,I}(u)} + \frac{1}{\alpha-1}\frac{L_1(u)u^{1-\alpha}}{\overline{F}_{2,I}(u)}} = 1.$$

■

4 Asymptotic Behavior of the Joint Tail of the Severity of Ruin and the Surplus Prior to Ruin

For fixed $\beta > 0$ and $a \geq 0$, we define the function

$$B(x;\beta,a) := \int_x^\infty e^{-\beta(y-x)}\left(\lambda_2\overline{F}_2(y+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right)dy, \quad x \geq 0. \tag{4.1}$$

In order to obtain asymptotic expressions for $\Upsilon_{a,b}(u)$ as $u \to \infty$ in such a way that $\Delta := \max\{u, b\} \to \infty$, we establish some preliminary lemmas.

Lemma 2

a) For all $x \geq 0$ and $\beta > 0$, $B(x;\beta,a) \leq \lambda_2\mu_2 + \frac{\eta^\alpha}{\Gamma(2-\alpha)}a^{1-\alpha}$ for any $a \geq 0$.
b) The asymptotic relation $B(x;\beta,a) = o(\psi(x+a))$ as $x \to \infty$, holds in any of the cases in (3.16).

Proof

a) Since $e^{-\beta(y-x)} \le 1$ when $y \ge x$, and $\overline{F}_2(y+a) \le \overline{F}_2(y)$, we have

$$\begin{aligned} &B(x;\beta,a)\\ &\le \int_x^\infty \left(\lambda_2 \overline{F}_2(y+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right)dy\\ &\le \int_0^\infty \left(\lambda_2 \overline{F}_2(y) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right)dy, \end{aligned}$$

which implies a).

b) Using that $\overline{F}_2(y+a) \le \overline{F}_2(x+a)$ and $(y+a)^{-\alpha} \le (x+a)^{-\alpha}$ for all $y \ge x$, we see that

$$\begin{aligned} B(x;\beta,a) &\le \int_x^\infty e^{-\beta(y-x)}\left(\lambda_2 \overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)dy\\ &= \frac{1}{\beta}\left(\lambda_2 \overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right). \end{aligned} \tag{4.2}$$

For the first two cases in (3.16), the limit $\lim_{x\to\infty} \frac{\overline{F}_2(x)}{x^{-\alpha}}$ exists and is finite, hence in any of these two cases we obtain

$$\begin{aligned} &\lim_{x\to\infty} \frac{\frac{1}{\beta}\left(\lambda_2 \overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{(x+a)^{1-\alpha}}\\ &= \lim_{x\to\infty} \frac{\frac{1}{\beta}\left(\lambda_2 \frac{\overline{F}_2(x+a)}{(x+a)^{-\alpha}} + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}\right)}{x+a} = 0. \end{aligned} \tag{4.3}$$

Due to (3.17) and (3.18) we obtain, again in cases 1 and 2 of (3.16), that $\psi(u) \sim Au^{1-\alpha}$ for some constant $A > 0$. This and (4.3) imply

$$\begin{aligned} &\lim_{x\to\infty} \frac{\frac{1}{\beta}\left(\lambda_2 \overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{\psi(x+a)}\\ &= \lim_{x\to\infty} \frac{\dfrac{\frac{1}{\beta}\left(\lambda_2 \overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{(x+a)^{1-\alpha}}}{\dfrac{\psi(x+a)}{(x+a)^{1-\alpha}}} = 0. \end{aligned}$$

Hence we obtain the result in these two cases by dividing by $\psi(x+a)$ both sides of (4.2) and making $x \to \infty$ afterwards.

In the remaining case 3, the assumption that $F_2 \in \mathcal{R}_0$ and L'Hospital's rule imply that $\overline{F}_{2,I} \in \mathcal{R}_0$. From (3.19) we obtain that $\psi(u) \sim A_2 \overline{F}_{2,I}(u)$ for some constant $A_2 > 0$. Moreover, from the proof of Theorem 1 c) we see that $x^{1-\alpha} = o(\overline{F}_{2,I}(x))$. Using these two results together with $\overline{F}_{2,I} \in \mathcal{R}_0$, it follows that

$$\lim_{x\to\infty} \frac{\frac{1}{\beta}\left(\lambda_2 \overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{\psi(x+a)}$$

$$= \lim_{x\to\infty} \frac{\dfrac{\frac{1}{\beta}\left(\lambda_2 \overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{\overline{F}_{2,I}(x+a)}}{\dfrac{\psi(x+a)}{\overline{F}_{2,I}(x+a)}} = 0.$$

Again, the result follows dividing both sides of (4.2) by $\psi(x+a)$ and making $x \to \infty$.

■

Recall the definition of the joint tail distribution $\Upsilon_{a,b}$ given in (1.4).

Lemma 3 *The joint tail distribution admits the representation*

$$\Upsilon_{a,b}(u) = h_\alpha * W_\alpha(u), \quad u > 0, \tag{4.4}$$

where

$$h_\alpha(u) = \frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_j} \int_u^\infty \left[\lambda_2 \overline{F}_2(a+z) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right] 1_{\{z>b\}} dz + I_{a,b}(u),$$

and $I_{a,b}(x) = \sum_{j=2}^{m+1} E(\rho_j) \int_x^\infty e^{-\rho_j(y-x)} \left(\lambda_2 \overline{F}_2(y+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right) 1_{\{y>b\}} dy$. *Moreover, if F_2 belongs to any of the cases in (3.16), then for fixed $a, b > 0$,*

$$\int_0^u I_{a,b}(u-y)\Phi(dy) = o(\psi(u)) \text{ as } u \to \infty, \tag{4.5}$$

and the following limit holds:

$$\lim_{u\to\infty} \int_0^u I_{a,b}(u-y)\Phi(dy) = 0, \tag{4.6}$$

uniformly on the sets $\{a \geq \xi, b \geq \eta\}$ for all fixed $\xi, \eta > 0$.

Proof Formula (4.4) follows directly from [15, Corollary 5.1]. To prove (4.5) we first note that

$$\left|I_{a,b}(x)\right| \leq \sum_{j=2}^{m+1} |E(\rho_j)| \int_x^\infty e^{-Re(\rho_j)(y-x)} \left(\lambda_2 \overline{F}_2(y+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right) dy,$$

which due to (4.1) is equivalent to

$$\left|I_{a,b}(x)\right| \leq \sum_{j=2}^{m+1} |E(\rho_j)| B\left[x; Re(\rho_j), a\right]. \tag{4.7}$$

Let $\varepsilon > 0$ be given. From Lemma 2 b) there exists $u_0 > 0$ such that $\sum_{j=2}^{m+1} |E(\rho_j)| B\left[u; Re(\rho_j), a\right] < \varepsilon\psi(u)$ for all $u > u_0$. It follows from (4.7) that

$$\begin{aligned}
\left|\frac{\int_0^u I_{a,b}(u-y)\Phi(dy)}{\psi(u)}\right| &< \frac{\varepsilon \int_0^{u-u_0}(1-\Phi(u-y))\Phi(dy)}{\psi(u)} + \frac{\int_{u-u_0}^u |I_{a,b}(u-y)|\Phi(dy)}{\psi(u)} \\
&\leq \frac{\varepsilon \int_0^u (1-\Phi(u-y))\Phi(dy)}{\psi(u)} + \frac{\int_{u-u_0}^u |I_{a,b}(u-y)|\Phi(dy)}{\psi(u)} \\
&\leq \sum_{j=2}^{m+1} |E(\rho_j)| \left(\lambda_2\mu_2 + \frac{\eta^\alpha}{\Gamma(2-\alpha)} a^{1-\alpha}\right) := c_0,
\end{aligned}$$

where in the last equality we used Lemma 2 a). Hence

$$\begin{aligned}
\left|\frac{\int_0^u I_{a,b}(u-y)\Phi(dy)}{\psi(u)}\right| &< \frac{\varepsilon\left(\Phi(u) - \Phi * \Phi(u)\right)}{\psi(u)} + c_0 \frac{\Phi(u) - \Phi(u-u_0)}{\psi(u)} \\
&= \frac{\varepsilon\left(1 - \Phi * \Phi(u)) - \psi(u)\right)}{\psi(u)} + c_0 \frac{\psi(u-u_0) - \psi(u)}{\psi(u)}.
\end{aligned}$$

The estimate in (4.5) follows from the last inequality and the fact that $\Phi \in \mathcal{S}$. Since $I_{a,b}$ is, by its definition, nonincreasing in a and b, it follows that

$$\int_0^u I_{a,b}(u-y)\Phi(dy) \leq \int_0^u I_{\xi,\eta}(u-y)\Phi(dy), \tag{4.8}$$

for all $a \geq \xi$ and $b \geq \eta$.

Since $\lim_{u\to\infty} \psi(u) = 0$, using (4.5) we obtain that $\lim_{u\to\infty} \int_0^u I_{a,b}(u-y)\Phi(dy) = 0$. Hence the result follows from (4.8) by making $u \to \infty$. ∎

We now obtain the main results of this section.

Theorem 2 *Let F_2 belong to any of the three cases given in (3.16). Then, for fixed $a > 0$, the joint tail of the severity of ruin and the surplus prior to ruin, $\Upsilon_{a,b}$, admits the following asymptotic expressions as $u \to \infty$ in such a way that $\Delta = \max\{u, b\} \to \infty$:*

a) in case 1, $\Upsilon_{a,b}(u) \sim \dfrac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\,\Gamma(2-\alpha)}(a+\Delta)^{1-\alpha}$,

b) in case 2, $\Upsilon_{a,b}(u) \sim \dfrac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\dfrac{\eta^\alpha}{\Gamma(2-\alpha)} + \dfrac{\lambda_2\kappa}{\alpha-1}\right](a+\Delta)^{1-\alpha}$,

c) in case 3, $\Upsilon_{a,b}(u) \sim \dfrac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(a+\Delta)$.

Proof From (4.4) and [15, Corollary 5.5] it follows that

$$\begin{aligned}\Upsilon_{a,b}(u) &= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\\ &\quad\times\int_0^u\int_{u-y}^\infty\left[\lambda_2\overline{F}_2(a+z)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right]1_{\{z>b\}}\,dz\,\Phi(dy)\\ &\quad+\frac{\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_j}}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int_0^u I_{a,b}(u-y)\,\Phi(dy). \end{aligned}\tag{4.9}$$

In view of (4.5) we need only to study the asymptotic behavior of

$$\begin{aligned}\Upsilon^*(u,a,b) &:= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\\ &\quad\times\int_0^u\int_{u-y}^\infty\left[\lambda_2\overline{F}_2(a+z)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right]1_{\{z>b\}}\,dz\,\Phi(dy)\end{aligned}$$

as $u \to \infty$ in such a way that $\Delta = \max\{u, b\} \to \infty$. First we suppose that $\Delta = u$ and define

$$\begin{aligned}&\Upsilon_0(u,a)\\ &:= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int_0^u\left[\lambda_2\mu_2\overline{F}_{2,I}(a+u-y)+\frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+u-y)^{1-\alpha}\right]\Phi(dy).\end{aligned}\tag{4.10}$$

Therefore

$$\begin{aligned}\Upsilon^*(u,a,b) &\\ &\leq \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int_0^u\int_{u-y}^{\infty}\left[\lambda_2\overline{F}_2(a+z)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right]dz\,\Phi(dy)\\ &=\Upsilon_0(u,a) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\Upsilon^*(u,a,b) &\\ &\geq \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int_0^u\int_{u}^{\infty}\left[\lambda_2\overline{F}_2(a+z)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right]dz\,\Phi(dy)\\ &=\frac{\lambda_2\mu_2\overline{F}_{2,I}(a+u)+\frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+u)^{1-\alpha}}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Phi(u). \end{aligned} \tag{4.12}$$

The above inequality and Corollary 1 imply that

$$\liminf_{u\to\infty}\frac{\Upsilon^*(u,a,b)}{\psi(u+a)}\geq 1 \tag{4.13}$$

because $\lim_{u\to\infty}\Phi(u)=1$. To finish the proof it suffices to show that

$$\lim_{u\to\infty}\frac{\Upsilon_0(u,a)}{\psi(u+a)}=1 \tag{4.14}$$

for any of the claim size distributions in (3.16). Indeed, the asymptotics in the three cases follow from (4.14) together with (4.11), (4.13), (4.9) and (4.5).

We note that

$$\begin{aligned}\Upsilon_0(u,a) = &\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\\ &\times\left[\lambda_2\mu_2\overline{F}_{2,I}(a)\int_0^u\big(1-F_{a,I}(u-y)\big)\,\Phi(dy)+\frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\int_0^u\big(1-P_{a,\alpha}(u-y)\big)\,\Phi(dy)\right],\end{aligned}$$

where we define for $a > 0$ the functions $F_{a,I}(u) = 1 - \dfrac{\overline{F}_{2,I}(u+a)}{\overline{F}_{2,I}(a)}$ and $P_{a,\alpha}(u) = 1 - \left(\dfrac{au}{a+u}\right)^{\alpha-1} u^{1-\alpha},\ u \geq 0$. Hence,

$$\begin{aligned}
\Upsilon_0(u,a) &= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2} \\
&\quad \times \left[\lambda_2\mu_2\overline{F}_{2,I}(a)\left(\Phi(u) - F_{a,I} * \Phi(u)\right) + \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\left(\Phi(u) - P_{a,\alpha} * \Phi(u)\right)\right] \\
&= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Big(\lambda_2\mu_2\overline{F}_{2,I}(a)\left[\Phi(u) - 1 + 1 - F_{a,I} * \Phi(u)\right] \\
&\quad + \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\left[\Phi(u) - 1 + 1 - P_{a,\alpha} * \Phi(u)\right]\Big) \\
&= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2} \\
&\quad \times \Big(\lambda_2\mu_2\overline{F}_{2,I}(a)\left[1 - F_{a,I} * \Phi(u) - \Psi(u)\right] \\
&\qquad + \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\left[1 - P_{a,\alpha} * \Phi(u) - \Psi(u)\right]\Big).
\end{aligned} \tag{4.15}$$

Case 1. Due to Theorem 1 a) we have $\Phi \in \mathcal{S}$ and $\psi(u) \sim \dfrac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)} u^{1-\alpha}$, hence Lemma 1 c) and the assumption $\overline{F}_{2,I}(u) = o(u^{1-\alpha})$ as $u \to \infty$, imply

$$1 - F_{a,I} * \Phi(u) \sim \frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)} u^{1-\alpha}.$$

This shows that

$$\frac{\lambda_2\mu_2\overline{F}_{2,I}(a)}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[1 - F_{a,I} * \Phi(u) - \psi(u)\right] = o(u^{1-\alpha}) \text{ as } u \to \infty. \tag{4.16}$$

From Lemma 1 b), as $u \to \infty$,

$$1 - P_{a,\alpha} * \Phi(u) \sim \left[\left(\frac{au}{a+u}\right)^{\alpha-1} + \frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)}\right] u^{1-\alpha},$$

which due to (3.17) implies

$$1 - P_{a,\alpha} * \Phi(u) - \psi(u) \sim \left(\frac{a}{a+u}\right)^{\alpha-1}. \tag{4.17}$$

Using the expression for $\Upsilon_0(u,a)$ given in (4.15), together with (4.16) and (4.17) we obtain $\Upsilon_0(u,a) \sim \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Big[\lambda_2\mu_2\overline{F}_{2,I}(a+\Delta) + \frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+\Delta)^{1-\alpha}\Big]$, and (4.14) follows.

Case 2. Since by assumption $\overline{F}_2(u) \sim \kappa u^{1-\alpha}$, L'Hospital's rule gives $\overline{F}_{2,I}(u) \sim \frac{\kappa}{\mu_2(\alpha-1)}u^{1-\alpha}$. Hence $\overline{F}_{a,I}(u) \sim \frac{\kappa}{\mu_2(\alpha-1)\overline{F}_{2,I}(a)}u^{1-\alpha}$. From (3.18) we have $\Psi(u) \sim Cu^{1-\alpha}$, where the constant C is given by $C = \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\kappa}{\alpha-1}\right]$. Using this and Lemma 1 b) gives $1 - F_{a,I} * \Phi(u) \sim \left[C + \frac{\kappa}{\mu_2(\alpha-1)}\right]u^{1-\alpha}$. It follows that

$$1 - F_{a,I} * \Phi(u) - \psi(u) \sim \frac{\kappa}{\mu_2(\alpha-1)}u^{1-\alpha} \sim \frac{\kappa}{\mu_2(\alpha-1)}(a+u)^{1-\alpha}. \tag{4.18}$$

From Lemma 1 b) and (3.18),

$$1 - P_{a,\alpha} * \Phi(u) \sim \left[\left(\frac{au}{a+u}\right)^{\alpha-1} + \frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)} + \frac{\lambda_2\kappa}{c+\lambda_1\mu_1-\lambda_2\mu_2}\right]u^{1-\alpha}.$$

This together with (3.18) yields

$$1 - P_{a,\alpha} * \Phi(u) - \psi(u) \sim \left(\frac{a}{a+u}\right)^{\alpha-1}. \tag{4.19}$$

Now using (4.18) and (4.19), we obtain

$$\Upsilon_0(u,a) \sim \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\kappa}{\alpha-1}\right](a+u)^{1-\alpha}.$$

Case 3. Using the assumption $u^{-\alpha} = o\big(\overline{F}_2(u)\big)$ and L'Hospital's rule we get $u^{1-\alpha} = o\big(\overline{F}_{2,I}(u)\big)$. Since $\overline{P}_{a,\alpha}(u) = \left(\frac{au}{a+u}\right)^{\alpha-1}u^{1-\alpha}$ and $\lim_{u\to\infty}\frac{\left(\frac{auy}{a+uy}\right)^{\alpha-1}}{\left(\frac{au}{a+u}\right)^{\alpha-1}} = 1$ for all $y > 0$, we have $\overline{P}_{a,\alpha}(u) \sim u^{1-\alpha}$. Hence $\overline{P}_{a,\alpha}(u) = o\big(\overline{F}_{2,I}(u)\big)$, and from Corollary 1 and (3.19) we obtain $1 - P_{a,\alpha} * \Phi(u) \sim \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u)$. Using (3.19) again we conclude that $1 - P_{a,\alpha} * \Phi(u) - \psi(u) = o(\overline{F}_{2,I}(u))$. Due to Lemma 1 b),

$$1 - F_{a,I} * \Phi(u) \sim \left(\frac{1}{\overline{F}_{2,I}(a)} + \frac{\lambda_2 \mu_2}{c + \lambda_1 \mu_1 - \lambda_2 \mu_2} \right) \overline{F}_{2,I}(a+u),$$

which implies $1 - F_{a,I} * \Phi(u) - \psi(u) \sim \frac{\overline{F}_{2,I}(u+a)}{\overline{F}_{2,I}(a)}$. In this way we obtain (4.14).

In the case of $\Delta = b$ we have

$$\begin{aligned}
&\Upsilon^*(u, a, b)\\
&= \frac{1}{c + \lambda_1 \mu_1 - \lambda_2 \mu_2} \int_0^u \int_{u-y}^{\infty} \left[\lambda_2 \overline{F}_2(a+z) + \frac{\eta^\alpha (\alpha - 1)}{\Gamma(2-\alpha)} (a+z)^{-\alpha} \right] 1_{\{z>b\}}\, dz\, \Phi(dy)\\
&= \frac{1}{c + \lambda_1 \mu_1 - \lambda_2 \mu_2} \int_0^u \left[\lambda_2 \mu_2 \overline{F}_{2,I}(a+b) + \frac{\eta^\alpha}{\Gamma(2-\alpha)} (a+b)^{1-\alpha} \right] \Phi(dy)\\
&= \frac{\lambda_2 \mu_2 \overline{F}_{2,I}(a+b) + \frac{\eta^\alpha}{\Gamma(2-\alpha)} (a+b)^{1-\alpha}}{c + \lambda_1 \mu_1 - \lambda_2 \mu_2} \Phi(u).
\end{aligned}$$

The asymptotics for $\Upsilon_{a,b}$ follow by dividing $\Upsilon_{a,b}(u)$ by $\dfrac{\lambda_2 \mu_2 \overline{F}_{2,I}(a+b) + \frac{\eta^\alpha}{\Gamma(2-\alpha)} (a+b)^{1-\alpha}}{c + \lambda_1 \mu_1 - \lambda_2 \mu_2}$, letting afterward $u \to \infty$, and proceeding as in the cases 1, 2 and 3 above with u replaced by b. ■

Corollary 2 *For any of the cases in (3.16), the joint tail $\Upsilon_{a,b}$ has the asymptotic expression when $u \to \infty$ and $\Delta = \max\{u, b\} \to \infty$:*

$$\Upsilon_{a,b}(u) \sim \frac{\eta^\alpha}{(c + \lambda_1 \mu_1 - \lambda_2 \mu_2)\, \Gamma(2-\alpha)} (a+\Delta)^{1-\alpha} + \frac{\lambda_2 \mu_2}{c + \lambda_1 \mu_1 - \lambda_2 \mu_2} \overline{F}_{2,I}(a+\Delta).$$

In particular, if $\overline{F}_2(u) \sim L_1(u) u^{-\alpha}$ for some slowly varying function L_1, and $\overline{F}_2$ satisfies any of the cases in (3.16), it follows

$$\Upsilon_{a,b}(u) \sim \frac{1}{c + \lambda_1 \mu_1 - \lambda_2 \mu_2} \left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2}{\alpha - 1} L_1(a+\Delta) \right] (a+\Delta)^{1-\alpha}.$$

We have the following sharper result, which shows that the asymptotics of $\Upsilon_{a,b}$ given in Theorem 2, hold uniformly on the parameters a and b.

Theorem 3 *Let F_2 belong to any of the three cases given in (3.16). The following limits hold, when $u \to \infty$, uniformly on the sets $A_{\xi,\eta} = \{a \geq \xi, b \geq \eta\}$, for fixed $\xi, \eta > 0$.*

1. *In case 1:*

$$\lim_{u\to\infty}\left|\Upsilon_{a,b}(u)-\frac{\eta^{\alpha}}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\,\Gamma(2-\alpha)}(a+u)^{1-\alpha}\right|=0.$$

2. *In case 2:*

$$\lim_{u\to\infty}\left|\Upsilon_{a,b}(u)-\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^{\alpha}}{\Gamma(2-\alpha)}+\frac{\lambda_2\kappa}{\alpha-1}\right](a+u)^{1-\alpha}\right|=0.$$

3. *In case 3:*

$$\lim_{u\to\infty}\left|\Upsilon_{a,b}(u)-\frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(a+u)\right|=0.$$

Proof By (4.9) and (4.6), we only need to study the uniform convergence of $\Upsilon^*(u,a,b) = \dfrac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int\limits_0^u\int\limits_{u-y}^{\infty}\left[\lambda_2\overline{F}_2(a+z)+\frac{\eta^{\alpha}(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right]$ $1_{\{z>b\}}dz\Phi(dy)$ on the sets $A_{\xi,\eta}$.

Using (4.12) we obtain

$$\begin{aligned}\Upsilon^*(u,a,b)-\frac{\eta^{\alpha}(a+u)^{1-\alpha}}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)} &\geq \frac{\eta^{\alpha}(a+u)^{1-\alpha}\left(\Phi(u)-1\right)}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)}\\ &\quad+\frac{\lambda_2\mu_2\overline{F}_{2,I}(a+u)\Phi(u)}{c+\lambda_1\mu_1-\lambda_2\mu_2}\\ &\geq \frac{\eta^{\alpha}(a+u)^{1-\alpha}\left(\Phi(u)-1\right)}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)}.\end{aligned}$$

Since $(a+u)^{1-\alpha}\left(\Phi(u)-1\right)$ is nonincreasing as a function of a, we have

$$(a+u)^{1-\alpha}\left(\Phi(u)-1\right)\leq(\xi+u)^{1-\alpha}\left(\Phi(u)-1\right),$$

and since $\lim\limits_{u\to\infty}(\xi+u)^{1-\alpha}\left(\Phi(u)-1\right)=0$, the convergence

$$\lim_{u\to\infty}(a+u)^{1-\alpha}\left(\Phi(u)-1\right)=0 \tag{4.20}$$

is uniform on $\{a\geq\xi\}$. Hence for all $\varepsilon>0$ and $\xi>0$ there exists $A>0$ such that for $u\geq A$ we have

$$\Upsilon^*(u,a,b)-\frac{\eta^{\alpha}(a+u)^{1-\alpha}}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)}\geq-\varepsilon. \tag{4.21}$$

We will use that, by the definition (4.10) and the equivalent formula (4.15),

$$\overline{F}_{2,I}(a)\left(1 - F_{a,I} * \Phi(u) - \psi(u)\right) = \int_0^u \overline{F}_{2,I}(a+u-y)\Phi(dy)$$

and

$$a^{1-\alpha}\left(1 - P_{a,\alpha} * \Phi(u) - \psi(u)\right) = \int_0^u (a+u-y)^{1-\alpha}\Phi(dy),$$

hence

$$\begin{aligned} &\overline{F}_{2,I}(a)\left(1 - F_{a,I} * \Phi(u) - \psi(u)\right) \text{ and} \\ &\quad a^{1-\alpha}\left(1 - P_{a,\alpha} * \Phi(u) - \psi(u)\right) \text{ are nonincreasing in } a. \end{aligned} \tag{4.22}$$

Case 1: We have, by (4.12):

$$\Upsilon^*(u,a,b) - \frac{\eta^\alpha (a+u)^{1-\alpha}}{(c+\lambda_1\mu_1 - \lambda_2\mu_2)\Gamma(2-\alpha)} \le \Upsilon_0(a,b) - \frac{\eta^\alpha (a+u)^{1-\alpha}}{(c+\lambda_1\mu_1 - \lambda_2\mu_2)\Gamma(2-\alpha)}.$$

Hence, from the definition of $\Upsilon_0(a,b)$ in (4.10) and the equality (4.15) we obtain:

$$\begin{aligned} &\Upsilon^*(u,a,b) - \frac{\eta^\alpha (a+u)^{1-\alpha}}{(c+\lambda_1\mu_1 - \lambda_2\mu_2)\Gamma(2-\alpha)} \\ &\quad \le \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Bigg[\lambda_2\mu_2\overline{F}_{2,I}(a)\left(1 - F_{a,I} * \Phi(u) - \psi(u)\right) \\ &\qquad + \frac{\eta^\alpha a^{1-\alpha}}{\Gamma(2-\alpha)}\left(1 - P_{a,\alpha} * \Phi(u) - \psi(u) - \left(\frac{a+u}{a}\right)^{1-\alpha}\right)\Bigg]. \end{aligned} \tag{4.23}$$

We know from (4.22) that $1 - P_{a,\alpha} * \Phi(u) - \psi(u)$ nonincreasing in a. Since $\left(\frac{a+u}{a}\right)^{1-\alpha} = \left(\frac{1}{1+u/a}\right)^{\alpha-1}$ and $1+u/a$ is decreasing in a, it follows that $-\left(\frac{a+u}{a}\right)^{1-\alpha}$ is decreasing in a. Hence $1 - P_{a,\alpha} * \Phi(u) - \psi(u) - \left(\frac{a+u}{a}\right)^{1-\alpha}$ is decreasing in a.

From this and (4.17) we obtain, similarly as in (4.20), that

$$\lim_{u\to\infty} \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\left|1 - P_{a,\alpha} * \Phi(u) - \psi(u) - \left(\frac{a+u}{a}\right)^{1-\alpha}\right| = 0, \tag{4.24}$$

uniformly on $\{a \ge \xi\}$.

For the remaining term in (4.23), from (4.16) and (4.22) we obtain using the same argument

$$\lim_{u\to\infty} \lambda_2\mu_2\overline{F}_{2,I}(a)\left|1 - F_{a,I} * \Phi(u) - \psi(u)\right| = 0, \tag{4.25}$$

uniformly on $\{a \geq \xi\}$. Due to (4.23), (4.24) and (4.25) it follows that, for all $\varepsilon > 0$, $\xi > 0$ and $\eta > 0$ there exists $A > 0$ such that for all $u > A$ and $a > \xi, b > \eta$ we have

$$\Upsilon^*(u, a, b) - \frac{\eta^\alpha (a+u)^{1-\alpha}}{(c + \lambda_1\mu_1 - \lambda_2\mu_2)\Gamma(2-\alpha)} \leq \varepsilon. \tag{4.26}$$

Hence, the result follows from (4.21) and (4.26).

Case 2: Similarly as in the previous case, we obtain from (4.22) and (4.19)

$$\lim_{u\to\infty} \left| \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)} \left(1 - P_{a,\alpha} * \Phi(u) - \psi(u) - \left(\frac{a+u}{a}\right)^{1-\alpha}\right)\right| = 0, \tag{4.27}$$

uniformly in $\{a \geq \xi\}$. It also follows from (4.18) that

$$\lim_{u\to\infty} \left|\overline{F}_2(a)\left(1 - F_{a,I} * \Phi(u) - \psi(u)\right)\right| = 0, \tag{4.28}$$

uniformly in $\{a \geq \xi\}$. Hence, for all $\varepsilon, \xi, \eta > 0$ there exists an $A > 0$ such that for all $u > A$ and $a > \xi, b > \eta$ we have

$$\Upsilon^*(u, a, b) - \frac{\eta^\alpha (a+u)^{1-\alpha}}{(c + \lambda_1\mu_1 - \lambda_2\mu_2)\Gamma(2-\alpha)} \leq \varepsilon. \tag{4.29}$$

The result follows now from (4.21) and (4.29).

Case 3: By (4.12) it holds

$$\begin{aligned}
\Upsilon^*(u, a, b) &- \frac{\lambda_2\mu_2}{c + \lambda_1\mu_1 - \lambda_2\mu_2}\overline{F}_{2,I}(u+a) \\
&\geq \frac{\lambda_2\mu_2}{c + \lambda_1\mu_1 - \lambda_2\mu_2}\overline{F}_{2,I}(u+a)\left(\Phi(u) - 1\right) \\
&\quad + \frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+u)^{1-\alpha}\Phi(u).
\end{aligned}$$

As in the above cases we obtain that, for all $\varepsilon,\ \xi,\ \eta > 0$ there exists an $A > 0$ such that for all $u > A$ and $a > \xi,\ b > \eta$ we have

$$\Upsilon^*(u, a, b) - \frac{\lambda_2\mu_2}{c + \lambda_1\mu_1 - \lambda_2\mu_2}\overline{F}_{2,I}(u+a) \geq -\varepsilon. \tag{4.30}$$

On the other hand,

$$\Upsilon^*(u,a,b) - \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u+a) \le \Upsilon_0(u,a) - \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_2(a+u)$$
$$= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Big\{\lambda_2\mu_2\left[\overline{F}_{2,I}(a)\left(1-F_{a,I}*\Phi(u)-\psi(u)\right)-\overline{F}_{2,I}(a+u)\right]$$
$$+\frac{a^{1-\alpha}\eta^{\alpha}}{\Gamma(2-\alpha)}\left(1-P_{a,\alpha}*\Phi(u)-\psi(u)\right)\Big\}. \tag{4.31}$$

Since in this case we have $\lim_{u\to\infty}(1-P_{a,\alpha}*\Phi(u)-\psi(u))=0$, using that $1-P_{a,\alpha}*\Phi(u)$ is decreasing in a, it follows that

$$\lim_{u\to\infty}\left|\frac{a^{1-\alpha}\eta^{\alpha}}{\Gamma(2-\alpha)}\left(1-P_{a,\alpha}*\Phi(u)-\psi(u)\right)\right|=0, \tag{4.32}$$

uniformly on $\{a\ge\xi\}$, for any $\xi>0$. For the remaining term in (4.31), there holds

$$\overline{F}_{2,I}(a)\left(1-F_{a,I}*\Phi(u)-\psi(u)\right)$$
$$=\overline{F}_{2,I}(a)\left[\Phi(u)-F_{a,I}*\Phi(u)\right]=\overline{F}_{2,I}(a)\int_0^u\left(1-F_{a,I}(u-y)\right)\Phi(dy)$$
$$=\overline{F}_{2,I}(a)\int_0^u\overline{F}_{a,I}(u-y)\Phi(dy)=\overline{F}_{2,I}(a)\int_0^u\frac{\overline{F}_{2,I}(a+u-y)}{\overline{F}_{2,I}(a)}\Phi(dy)$$
$$=(\overline{F}_{2,I}(a+\cdot)*\Phi)(u). \tag{4.33}$$

Hence $\lim_{u\to\infty}\left(\overline{F}_{2,I}(a+\cdot)*\Phi\right)(u)=0$ uniformly on $\{a\ge\xi\}$ for $\xi>0$. Since

$$\overline{F}_{2,I}(a)\left|1-F_{a,I}*\Phi(u)-\psi(u)-\overline{F}_{2,I}(a+u)\right|$$
$$\le\left[(\overline{F}_{2,I}(a+\cdot)*\Phi)(u)+\overline{F}_{2,I}(a)\overline{F}_{2,I}(a+u)\right],$$

and $\lim_{u\to\infty}\left[(\overline{F}_{2,I}(a+\cdot)*\Phi)(u)+\overline{F}_{2,I}(a)\overline{F}_{2,I}(a+u)\right]=0$ uniformly on $\{a\ge\xi\}$ for $\xi>0$, we obtain that

$$\lim_{u\to\infty}\overline{F}_{2,I}(a)\left|1-F_{a,I}*\Phi(u)-\psi(u)-\overline{F}_{2,I}(a+u)\right|=0, \tag{4.34}$$

uniformly on $\{a\ge\xi\}$, $\xi>0$. Using (4.31), (4.32) and (4.34) we obtain that, for all $\varepsilon,\xi,\eta>0$ there exists $A>0$ such that for all $u>A$ and $a>\xi, b>\eta$, it follows

$$\Upsilon^*(u,a,b)-\frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u+a)\le\varepsilon. \tag{4.35}$$

The result follows now from (4.30) and (4.35). ■

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