Chapter 2
The Principle of Virtual Power (PVP): Application to Complex Media, Extension to Gauge and Scale Invariances, and Fundamental Aspects

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Abstract This work, relative to the principle of virtual power, is composed of three distinct but nevertheless complementary parts. The first part follows the line of thought developed by professor Maugin and his students on complex continuous media subject to the objectivity requirement (translational and rotational invariances). The second part shows that this principle is extensible to other types of invariance such as gauge and scale invariances. Gauge invariance allows to express Maxwell equations, usually derived through a vector approach, by use of a scalar principle having the same formal structure as the principle of virtual power. As to scale invariance, it allows to deal, in a general and unified way whatever the underlying physics, with the passage from a continuous medium to a discontinuous one (singular surfaces, lines or points). The third part concerns the foundations of dynamics where the principle of virtual power appears as a theorem, like other analytical principles, each corresponding to one point of view, deductible from a general intrinsic (viewpoint independent) dynamical framework. The attention will be focused on the origin of the duality notion, at the basis of the principle of virtual power.

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2.1 First Part

2.1.1 Complex Media: Modeling of the Different Continua

If one intends to describe semiconduction effects coupled to ferroelectricity and/or ferromagnetism, one must, on the one hand, distinguish between the various species of carriers by decomposing the total charge and current densities in order to account for generation and recombination phenomena as done in [1]. On the other hand, one must introduce polarization and magnetization gradients to account for electro-magnetic ordering. Phenomenologically, the conduction (or diffusion) currents per unit charge may be considered as the new generalized velocity fields that, by thermodynamic duality, yield generalized internal forces for which constitutive equations will have to be constructed. That is, we increase the complexity of the general scheme of electromechanical interactions of [2] by considering the self-explanatory scheme of Fig. 2.1 where the continua of charge α, correspond to electrons, holes, ions, impurities, etc. Thus the superscript α labels quantities attached to these species of charge and we have the obvious relations and notations.

The volume density of free charges q_f and the total electric current density J correspond to the contribution of each α charge carriers, such that:

$$q_f = \sum_\alpha q_f^\alpha, \quad J = \sum_\alpha J^\alpha \tag{2.1}$$

$$v^\alpha = \frac{J^\alpha}{q_f^\alpha}, \quad u^\alpha = v^\alpha - v = \frac{\mathcal{J}^\alpha}{q_f^\alpha} \tag{2.2}$$
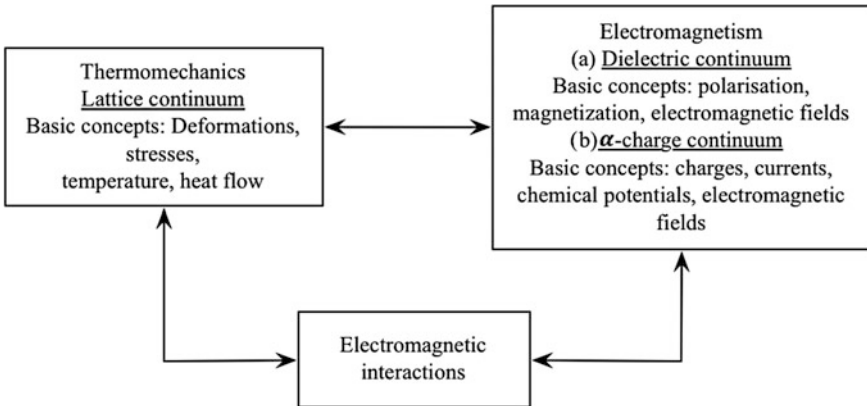

Fig. 2.1 Scheme of interactions in thermo-deformable semiconductors [1]

where q_f^α and J^α denote respectively the volume density and the electric current density of electric charges of type α. The vectors v, v^α and u^α denote respectively the velocity fields describing the global deformable material, the αth continuum (or the velocity of α charges carriers with respect to the frame R_G) and the relative velocity (i.e. with respect to the co-moving frame R_C). $\mathscr{J}^\alpha$ denotes the conduction current density of α charge carriers that is diffused within the material with respect to R_C frame. Similarly to the mass conservation but accounting for the possible recombination and generation [1], we can write for the αth continua of charge the following global balance laws $(\alpha = 1, 2, 3 \ldots)$:

$$\frac{d^\alpha}{dt} \int_{\mathscr{D}} q_f^\alpha dv = \int_{\mathscr{D}} r^\alpha dv \tag{2.3}$$

where $d^\alpha/dt = \partial/\partial t + v^\alpha \cdot \nabla$ denotes the "αth continuum" convective-time derivative and r^α the source terms such that $\sum_\alpha r^\alpha = 0$.

Equation (2.3) yields the local conservation-of-charge equations for the αth type of charge carriers as

$$\frac{\partial q_f^\alpha}{\partial t} + \nabla \cdot J^\alpha = r^\alpha \tag{2.4}$$

By summation over α, the latter equation yields the conservation-of-charge equation for the whole continuum.

In the following we use the convective-time derivative of a vector field A such that

$$\overset{*}{A} = \dot{A} - (A \cdot \nabla)v + A(\nabla \cdot v) \tag{2.5}$$

Maxwell's equations can be written in SI units, like other equations in this communication, in order to be close to what is nowadays done by physicists [3]:

$$\nabla \times \mathscr{E} + \overset{*}{B} = 0, \quad \nabla \cdot B = 0 \tag{2.6}$$

$$\nabla \times \left(\frac{\mathscr{B}}{\mu_0} \right) - \varepsilon_0 \overset{*}{E} = \mathscr{J}^{eff}, \quad \nabla \cdot E = \frac{q^{eff}}{\varepsilon_0} \tag{2.7}$$

where E and B denote the vectors of the electric field and the magnetic induction evaluated in the fixed Galilean frame R_G; $\mathscr{E}$ and $\mathscr{B}$ are the same vector fields as E and B but referred to a co-moving frame R_C in movement with the material velocity v with respect to R_G; ε_0 and μ_0 are respectively the vacuum permittivity and permeability such that $\varepsilon_0 \mu_0 c^2 = 1$ (where c denotes the speed of light in vacuum); q^{eff} and $\mathscr{J}^{eff}$ are the effective charges and currents in R_C defined by:

$$q^{eff} = \sum_\alpha q_f^\alpha - \nabla \cdot \boldsymbol{P} = \sum_\alpha q^{\alpha(eff)}, \qquad q^{\alpha(eff)} = q_f^\alpha \left(1 - \frac{\nabla \cdot \boldsymbol{P}}{\nabla \cdot \boldsymbol{D}} \right) \qquad (2.8)$$

$$\mathscr{J}^{eff} = \mathscr{J} + \overset{*}{\boldsymbol{P}} + \nabla \times \mathscr{M} = \sum_\alpha \mathscr{J}^{\alpha(eff)}, \qquad \mathscr{J}^{\alpha(eff)} = \mathscr{J}^\alpha + q_f^\alpha \left(\overset{*}{\boldsymbol{P}} + \nabla \times \mathscr{M} \right) (\nabla \cdot \boldsymbol{D})^{-1} \quad (2.9)$$

where $\boldsymbol{D}$, $\boldsymbol{P}$ and $\boldsymbol{M}$ denote the vector of the electric displacement, the electric polarization and the magnetization evaluated in R_G; $\mathscr{M}$ and $\mathscr{J} = \sum_\alpha \mathscr{J}^\alpha$ are the same vector fields as $\boldsymbol{M}$ and $\boldsymbol{J}$ but referred to a co-moving frame R_C. When $\nabla \cdot \boldsymbol{D} = 0$, one uses the first equalities of Eqs. (2.8) and (2.9). In the Galilean approximation, we have the following transformation laws between R_G and R_C:

$$\mathscr{E} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \qquad \frac{\mathscr{B}}{\mu_0} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{v} \times (\varepsilon_0 \boldsymbol{E}) \qquad (2.10)$$

$$\mathscr{M} = \boldsymbol{M} + \boldsymbol{v} \times \boldsymbol{P}, \quad \mathscr{J} = \boldsymbol{J} - q_f \boldsymbol{v}. \qquad (2.11)$$

This gives an idea of the effective charge and current densities that must be accounted for the αth species when the material is simultaneously polarized and magnetized.

2.1.2 Thermo-Electro-Magneto-Mechanical Equations

2.1.2.1 General Principles in Global Form

The thermomechanical balance laws of an electromagnetic continuum may be deduced in an elegant manner from three general principles written in global form for the material volume $\mathscr{D}$. These are the principle of virtual power and the first and second principles of thermodynamics [4, 5]. We refer the reader to the review paper [2] and the book [6] for this general approach from which we extract only the required ingredients.

In order to construct the different virtual powers, we construct a space of velocities and velocity gradients $\mathscr{V}$ (see, for instance, [1, 2, 6–8]) from the available "velocities" gathered in $\mathscr{V}^{(0)}$:

$$\mathscr{V}^{(0)} = \left\{ v_i, v_i^\alpha, \dot{\pi}_i, \dot{\mu}_i \right\} \qquad (2.12)$$

where $\pi_i = P_i / \rho$ and $\mu_i = \mathscr{M}_i / \rho$ denote the mass density of polarization and magnetization.

The set of variables is chosen according to the phenomena one is interested in. For simple deformable electro-magneto-mechanical interactions, one needs to account for time rates of polarization $\dot{\pi}_i$ and magnetization $\dot{\mu}_i$, in addition to the

usual mechanical ingredients: velocity v_i and its first gradient $v_{i,j}$ (needed to account for deformation). In the present approach where one deals with complex media including ferroelectricity, ferromagnetism, flexoelectricity, semi-conduction etc., it becomes necessary to add to the time rates of polarization and magnetization first gradients, $(\dot{\pi}_i)_{,j}$ and $(\dot{\mu}_i)_{,j}$, accounting thus for the so-called electromagnetic ordering (ferroelectricity, ferromagnetism …). One also needs to account for a second gradient relative to the velocity $v_{i,jk}$, required to give account of the flexoelectric effect. As to semi-conduction, it is accounted for by introducing new variables reflecting the motion of the different charge carriers (electrons, holes …) i.e. the velocity v_i^α and its first-order gradient $v_{i,j}^\alpha$. In summary, the set of variables $\mathcal{V}^{(0)}$ has been enlarged with these different first and second order gradients to obtain the new set $\mathcal{V}$ such that (see for instance [2, 9]):

$$\mathcal{V} = \left\{ v_i, v_{i,j}, v_{i,jk}, v_i^\alpha, v_{i,j}^\alpha, \dot{\pi}_i, (\dot{\pi}_i)_{,j}, \dot{\mu}_i, (\dot{\mu}_i)_{,j} \right\} \qquad (2.13)$$

In order to pave the way for objective quantities, the velocity gradients may be decomposed into their symmetric and anti-symmetric parts. This may be decomposed as:

$$\mathcal{V} = \left\{ v_i, D_{ij}, \Omega_{ij}, v_{i,jk}, u_i^\alpha, D_{ij}^\alpha, \Omega_{ij}^\alpha, \dot{\pi}_i, (\dot{\pi}_i)_{,j}, \dot{\mu}_i, (\dot{\mu}_i)_{,j} \right\} \qquad (2.14)$$

where $v_{i,j} = v_{(i,j)} + v_{[i,j]} = D_{ij} + \Omega_{ij}$ and $v_{i,j}^\alpha = v_{(i,j)}^\alpha + v_{[i,j]}^\alpha = D_{ij}^\alpha + \Omega_{ij}^\alpha$.

Since the constitutive equations associated with the different continua must be objective, i.e. frame-independent, we construct a subspace $\mathcal{V}_{obj}$ including only objective fields. In order to do that, the Jaumann derivatives, noted D_J, and the specific time derivative tensors are used with the velocity of the deformable continuum and with the velocities of the αth charge continua [1, 3]. Hence, for the polarization, we introduce:

$$\hat{\pi}_i = (D_J \boldsymbol{\pi})_i = \dot{\pi}_i - \Omega_{ij}\pi_j, \qquad \hat{\pi}_i^\alpha = (D_J^\alpha \boldsymbol{\pi})_i = \dot{\pi}_i - \Omega_{ij}^\alpha \pi_j \qquad (2.15)$$

$$\hat{\pi}_{ij} = [D_J(\nabla \boldsymbol{\pi})]_{ij} + D_{kj}\,\pi_{i,k} = (\dot{\pi}_i)_{,j} - \Omega_{ik}\pi_{k,j}, \qquad \hat{\pi}_{ij}^\alpha = (\dot{\pi}_i)_{,j} - \Omega_{ik}^\alpha \pi_{k,j}. \qquad (2.16)$$

For the magnetization, the Jaumann derivatives, $\hat{\mu}_i$ and $\hat{\mu}_i^\alpha$, and the specific time derivative tensors, $\hat{\mu}_{ij}$ and $\hat{\mu}_{ij}^\alpha$, are introduced similarly. Thus, the objective space $\mathcal{V}_{obj}$ is composed of the following set of kinematical objective fields

$$\mathcal{V}_{obj} = \left\{ D_{ij}, v_{i,jk}, u_i^\alpha, D_{ij}^\alpha, \hat{\pi}_i, \hat{\pi}_{ij}, \hat{\mu}_i, \hat{\mu}_{ij}, \hat{\pi}_i^\alpha, \hat{\pi}_{ij}^\alpha, \hat{\mu}_i^\alpha, \hat{\mu}_{ij}^\alpha \right\}$$

As to the set of dynamical objective fields $\mathcal{F}_{obj}$, it is introduced by duality to the set $\mathcal{V}_{obj}$. It is composed of generalized internal forces, such that [2]:

$$\mathscr{F}_{obj} = \left\{ \sigma_{ij}, \mu_{ijk}, q_f^{\alpha}{}^{L}\mathscr{E}_i^{\alpha}, \sigma_{ij}^{\alpha}, \rho{}^{L}E_i, {}^{L}\mathscr{E}_{ij}, \rho{}^{L}B_i, {}^{L}\mathscr{B}_{ij}, \rho{}^{L}E_i^{\alpha}, {}^{L}\mathscr{E}_{ij}^{\alpha}, \rho{}^{L}B_i^{\alpha}, {}^{L}\mathscr{B}_{ij}^{\alpha} \right\}$$

where σ_{ij}^{α} and σ_{ij} are the symmetric first-order stress tensor's components referred to as the intrinsic-stress tensor respectively for the αth charge continua and for the deformable continuum. μ_{ijk} is the intrinsic second-order stress tensor. A dimensional analysis shows that ${}^{L}B_i$ and ${}^{L}B_i^{\alpha}$ are induction fields, and, ${}^{L}\mathscr{E}_i^{\alpha}$, ${}^{L}E_i$ and ${}^{L}E_i^{\alpha}$ are electric fields. ${}^{L}\mathscr{E}_{ij}$, ${}^{L}\mathscr{B}_{ij}$, ${}^{L}\mathscr{E}_{ij}^{\alpha}$ and ${}^{L}\mathscr{B}_{ij}^{\alpha}$ are generalized forces associated to the gradient of the time derivatives of electric polarization and magnetization.

- **Principle of Virtual Power (PVP)**

In a Galilean frame and for a Newtonian chronology, the total virtual power of inertial forces of the system $\mathscr{P}_{(a)}^{*}$ balances the sum of the virtual powers of internal forces $\mathscr{P}_{(i)}^{*}$, of external volume forces $\mathscr{P}_{(v)}^{*}$ and of external contact forces $\mathscr{P}_{(c)}^{*}$ impressed on the system for any virtual velocity field. With the above notation, this reads:

$$\begin{aligned} &\mathscr{P}_{(a)}^{*}(\mathscr{D}, V^{*} \in \mathscr{V}^{(0)*}) \\ &= \mathscr{P}_{(i)}^{*}(\mathscr{D}, V^{*} \in \mathscr{V}_{obj}^{*}) + \mathscr{P}_{(v)}^{*}(\mathscr{D}, V^{*} \in \mathscr{V}^{*}) + \mathscr{P}_{(c)}^{*}(\partial\mathscr{D}, V^{*} \in \mathscr{V}^{(c)*}) \end{aligned} \tag{2.17}$$

- **First Principle of Thermodynamics**

The time rate of change of the total energy contained in the material domain $\mathscr{D}$, considered as a closed system, is equal to the sum of the power developed by "prescribed" forces $\mathscr{P}_{(e)}$, the energy supply by radiation in the volume of $\mathscr{D}$ and the total flux of energy through the boundary $\partial\mathscr{D}$ [2]. Mathematically, this reads:

$$\frac{d}{dt}[K(\mathscr{D}) + E(\mathscr{D}) + U^{em}(\mathscr{D})] = \mathscr{P}_{(e)}(\overline{\mathscr{D}}) + \dot{Q}_h(\overline{\mathscr{D}}) \tag{2.18}$$

where $\overline{\mathscr{D}}$ denotes the outside of domain $\mathscr{D}$ in R^3.

- **Second Principle of Thermodynamics**

For any thermodynamical process the time rate of change of the total entropy of the material domain $\mathscr{D}$ is never less than the sum of the total entropy supply in the volume of $\mathscr{D}$ and the total flux of entropy through its boundary $\partial\mathscr{D}$. Mathematically, this reads:

$$\frac{d}{dt}N(\mathscr{D}) \geq \dot{\mathscr{N}}(\overline{\mathscr{D}}) \tag{2.19}$$

For a general magnetizable, electrically polarized, heat conducting deformable semiconductor, the expressions to be carried in Eqs. (2.17)–(2.19) are given as follows:

$$K(\mathcal{D}) = \int_D \left(\frac{1}{2} \rho \, v^2 + \frac{1}{2} \rho \, d \, \dot{\boldsymbol{\pi}}^2 \right) dv \tag{2.20}$$

$$E(\mathcal{D}) = \int_{\mathcal{D}} \rho \varepsilon dv, \qquad U^{em}(\mathcal{D}) = \int_{\mathcal{D}} \frac{1}{2} \left(\varepsilon_0 \boldsymbol{E}^2 + \frac{\boldsymbol{B}^2}{\mu_0} - 2\mathcal{M} \cdot \mathcal{B} \right) dv \tag{2.21}$$

$$\dot{Q}_h(\overline{\mathcal{D}}) = \int_{\mathcal{D}} \rho h dv - \int_{\partial \mathcal{D}} q \cdot n da \tag{2.22}$$

$$N(\mathcal{D}) = \int_{\mathcal{D}} \rho \eta dv, \qquad \dot{\mathcal{N}}(\overline{\mathcal{D}}) = \int_{\mathcal{D}} \rho \sigma dv - \int_{\partial \mathcal{D}} \boldsymbol{\phi} \cdot n \, da \tag{2.23}$$

where d is the electronic polarization inertia tensor, e is the internal energy per unit mass, η is the entropy per unit mass, h is the radiation heat power source per unit mass, $\boldsymbol{n}$ is the unit exterior normal to the closed surface $\partial \mathcal{D}$ of the material domain $\mathcal{D}$ and $\boldsymbol{q}$ is the total power flux vector, i.e. the sum of the heat power flux vector $\tilde{q}$ and the Poynting's flux vector $\mathcal{S}$ referring to R_C [2]:

$$\boldsymbol{q} = \tilde{\boldsymbol{q}} + \mathcal{S}, \qquad \mathcal{S} = \mathcal{E} \times \mathcal{H}. \tag{2.24}$$

The fields σ and $\boldsymbol{\phi}$ are usually related to h, q and the thermodynamical temperature θ (where $\theta > 0$, $\inf(\theta) = 0$). These relations will be specified later on.

The other expressions to be carried in Eqs. (2.17)–(2.19) are constructed as follows:

Total virtual power of inertial forces $\mathscr{P}^*_{(a)}$

$$\mathscr{P}^*_{(a)}(\mathcal{D}, V^* \in \mathcal{V}^{(0)*}) = \int_{\mathcal{D}} \rho \left[\dot{v}_i v_i^* + d\ddot{\pi}_i \dot{\pi}_i^* + \beta' \gamma^{-1} \dot{\mu}_i \omega_i^* \right] dv \tag{2.25}$$

where γ is the gyromagnetic ratio of electrons. β' is equal to 1 when the spin precession plays an important role (i.e. when the material is ferromagnetic at low temperature) which can be expressed as a constraint on the magnetization velocity $\dot{\boldsymbol{\mu}} = \boldsymbol{\omega} \times \boldsymbol{\mu}$, where $\boldsymbol{\omega}$ is the precession velocity. Otherwise, β' can be set equal to 0 as shown in [1, 2, 6, 8].

Total virtual power of internal forces $\mathscr{P}^*_{(i)}$

The internal forces that reflect the interactions associated with the crystal lattice and polarizable, magnetizable and semi-conducting continua, have to be objective [1, 2, 6, 8]:

$$\mathscr{P}^*_{(i)}(\mathscr{D}, V^* \in \mathscr{V}^*_{obj}) = -\int_{\mathscr{D}} p^*_{(i)} dv \tag{2.26}$$

with

$$
\begin{aligned}
p^*_{(i)} &= \sigma_{ij} D^*_{ij} + \mu_{ijk} v^*_{i,jk} - \rho\,^L E_i \hat{\pi}^*_i - \rho\,^L B_i \hat{\mu}^*_i + {}^L\mathscr{E}_{ij} \hat{\pi}^*_{ij} + \beta\,^L\mathscr{B}_{ij} \hat{\mu}^*_{ij} - \sum_\alpha q^\alpha_f\,{}^L\mathscr{E}^\alpha_i u^{\alpha*}_i \\
&\quad + \sum_\alpha \left(\sigma^\alpha_{ij} D^{\alpha*}_{ij} - \rho\,^L E^\alpha_i \hat{\pi}^{\alpha*}_i - \rho\,^L B^\alpha_i \hat{\mu}^{\alpha*}_i + {}^L\mathscr{E}^\alpha_{ij} \hat{\pi}^{\alpha*}_{ij} + \beta\,^L\mathscr{B}^\alpha_{ij} \hat{\mu}^{\alpha*}_{ij} \right)
\end{aligned}
\tag{2.27}
$$

where, β is equal to 1 when the exchange forces play an important role, via variables $^L\mathscr{B}_{ij}$ and $^L\mathscr{B}^\alpha_{ij}$ (i.e. when the material is ferromagnetic). Otherwise, β can be set equal to 0.

Total virtual power of external volume forces $\mathscr{P}^*_{(v)}$

$$\mathscr{P}^*_{(v)}(\mathscr{D}, V^* \in \mathscr{V}^*) = \int_{\mathscr{D}} \left((f_i + f^{em}_i) v^*_i + \rho \mathscr{E}_i \hat{\pi}^*_i + \rho B_i \hat{\mu}^*_i + \sum_\alpha f^\alpha_i u^{\alpha*}_i \right) dv \tag{2.28}$$

where f and f^α represent volume densities of forces and f^{em} is the volume density of ponderomotive forces. Here, we have assumed for the sake of simplicity that the cofactors of $v_{i,jk}$, $(\dot{\mu}_i)_{,j}$ and $(\dot{\pi}_i)_{,j}$ take the value zero (in fact no physical interpretation of these fields has been found up to now [2]).

Total virtual power of external contact forces $\mathscr{P}^*_{(c)}$

For the external contact power, we obtain the following expression (see [3, 5, 9–13]):

$$
\begin{aligned}
\mathscr{P}^*_{(c)}(\partial\mathscr{D}, V^* \in \mathscr{V}^{(0)*}) &= \int_{\partial\mathscr{D}-\Gamma_{\!\nearrow}} \left((T_i + T^{em}_i) v^*_i + R_i \frac{\partial v^*_i}{\partial n} + \frac{\rho\, Q_i}{\varepsilon_0} \dot{\pi}^*_i \right) da \\
&\quad + \int_{\partial\mathscr{D}-\Gamma_{\!\nearrow}} \left(\beta \mu_0 \rho\, \mathscr{F}_i \dot{\mu}^*_i + \sum_\alpha T^\alpha_i u^{\alpha*}_i \right) da + \int_{\Gamma_{\!\nearrow}} L_i v^*_i\, ds
\end{aligned}
\tag{2.29}
$$

where $\boldsymbol{T}$ and $\boldsymbol{T}^\alpha$ represent surface densities of forces, $\boldsymbol{T}^{em}$ is the electromagnetic surface density of forces, R_i, L_i, Q_i and $\mathscr{F}_i$ denote respectively the normal double traction (per unit length), the reduced linear density of strength along the discontinuous line Γ (i.e. an edge where the unit exterior normal $\boldsymbol{n}$ on the closed surface $\partial\mathscr{D}$ is discontinuous) and the surface distribution of electric and magnetic dipoles.

Total power of "prescribed" forces $\mathscr{P}_{(e)}$

The total power of the "prescribed" forces is obtained by the construction of the principle of virtual power for an actual velocity field [2, 9, 11]. Using the global energetic identity for the electromagnetic fields U^{em} given by Maugin in [6], we obtain the total power of the "prescribed" forces $\mathscr{P}_{(e)}(\overline{\mathscr{D}})$ as

$$
\begin{aligned}
\mathscr{P}_{(e)}(\overline{\mathscr{D}}) = \int_{\mathscr{D}} f_i v_i dv &+ \int_{\partial \mathscr{D} - \Gamma_{\nearrow}} \left(T_i v_i + R_i \frac{\partial v_i}{\partial n} + \frac{\rho Q_i}{\varepsilon_0} \dot{\pi}_i \right) da \\
&+ \int_{\partial \mathscr{D} - \Gamma_{\nearrow}} \left(\beta \mu_0 \rho \mathscr{F}_i \dot{\mu}_i + \sum_\alpha T_i^\alpha u_i^\alpha \right) da + \int_{\Gamma_{\nearrow}} L_i v_i ds
\end{aligned}
\tag{2.30}
$$

In the above-set of Eqs. (2.25)–(2.30), f and T are respectively the volume and surface densities of forces of purely mechanical origin. f^α and T^α are respectively the volume and surface densities of forces associated with the αth charge continuum. The symmetric tensor with the component σ_{ij} is called the intrinsic stress tensor (not to be mistaken for the Cauchy stress tensor to which it is only a symmetric contribution). Constitutive equations will have to be constructed for this tensor. The quantities σ_{ij}^α, $^L E_i$, $^L B_i$ $^L E_i^\alpha$, $^L B_i^\alpha$ and $^L \mathscr{E}_i^\alpha$ all introduced by duality (we need constitutive equations for these) reflect the interactions between, respectively, the neighboring elements of the αth charge continuum; the polarization field and the crystal lattice; the magnetization field and the crystal lattice; the polarization field and the αth charge continuum; the magnetization field and the αth charge continuum; the crystal lattice and the αth charge continuum (this clearly is a "diffusion" process). Finally, the presence of μ_{ijk}, $^L \mathscr{E}_{ij}$, $^L \mathscr{B}_{ij}$, $^L \mathscr{E}_{ij}^\alpha$ and $^L \mathscr{B}_{ij}^\alpha$ are explained by the inclusion of the gradients ($\nabla \nabla v$, $\nabla \mu$ and $\nabla \pi$) and the principle of objectivity. These quantities (we also need constitutive equations for these) reflect, respectively, the second order interaction between the neighboring elements of the deformable continuum, and, the interactions between the polarization gradient field and the crystal lattice; the magnetization gradient field and the crystal lattice; the polarization gradient field and the αth charge continuum; the magnetization gradient field and the αth charge continuum. Finally, f^{em} is the volume density of the ponderomotive force and T^{em} is the corresponding electromagnetic surface density of force.

According to a semi microscopic approach [14] accounting for the effects of only charges and dipoles, the ponderomotive volume density of force is [1, 6, 8, 14–19]:

$$
f^{em} = q^{eff} \mathscr{E} + \left(\mathscr{J}^{eff} - \nabla \times \mathscr{M} \right) \times B + \nabla \cdot (\mathscr{E} \otimes P) + (\nabla B) \cdot \mathscr{M}
\tag{2.31}
$$

where one has set: $[\nabla \cdot (\mathscr{E} \otimes P)]_i = \left(\mathscr{E}_i P_j \right)_{,j}$ and $[(\nabla B) \cdot \mathscr{M}]_i = B_{j,i} \mathscr{M}_j$

This latter expression is equivalent to the following:

$$f^{em} = \nabla \cdot t^{em} - \frac{\partial G}{\partial t} = {}_L f + \nabla \cdot \bar{t}^{em} \qquad (2.32)$$

where we have singularized the "Lorentz force" ${}_L f$

$${}_L f = q^{eff} \mathcal{E} + \mathcal{J}^{eff} \times B = \nabla \cdot t^F - \frac{\partial G}{\partial t} \qquad (2.33)$$

$$t_{ij}^{em} = t_{ij}^F + \bar{t}_{ij}^{em}, \qquad G = \varepsilon_0 E \times B \qquad (2.34)$$

$$t_{ij}^F = \varepsilon_0 E_i E_j + \frac{B_i B_j}{\mu_0} - \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)\delta_{ij}, \qquad \bar{t}_{ij}^{em} = \mathcal{E}_i P_j - \mathcal{M}_i B_j + \mathcal{M} \cdot B\, \delta_{ij}. \quad (2.35)$$

The electromagnetic surface density of forces T^{em} is defined on $\partial \mathcal{D}$ as [1, 6, 8]:

$$T_i^{em} = -(t_{ij}^{em} + G_i v_j)n_j \qquad (2.36)$$

Finally, the ponderomotive couple C^{em} (of electromagnetic origin) is accounted for through the pseudo-vector of the electromagnetic stress tensor C^{em} such that

$$C_{ij}^{em} = -t_{[ij]}^{em} = -\bar{t}_{[ij]}^{em}, \qquad c_k^{em} = \varepsilon_{klm} C_{lm}^{em} = (P \times \mathcal{E} + \mathcal{M} \times B)_k \qquad (2.37)$$

where ε_{ijk} denotes the classical Levi-Civita symbol.

2.1.2.2 Local Electro-Magneto-Mechanical Balance Equations

For any virtual fields v^*, $v^{\alpha *}$, $\dot{\pi}^*$, $\dot{\mu}^*$ and $\partial v^*/\partial n$ and for any element of volume and surface, we obtain the following local field equations from (2.17) that govern the motion and the interactions in a moving magnetized, polarized and semiconducting, material medium (see [1, 2, 4, 9]):

$$\boxed{\forall \tfrac{\partial v^*}{\partial n}} \quad \mu_{ijk} n_j n_k = R_i \qquad\qquad\qquad on\ \partial \mathcal{D} - \Gamma_{\nearrow} \qquad (2.38)$$

$$\boxed{\forall v^*} \quad t_{ij,j} + f_i + f_i^{em} - \sum_\alpha q_f^\alpha \left(\mathcal{E}_i^\alpha + {}^L \mathcal{E}_i^\alpha\right) = \rho\, \dot{v}_i \qquad\qquad in\ \mathcal{D} \qquad (2.39)$$

$$t_{ij} n_j = T_i + T_i^{em} + \left(\nabla_j - n_j\left(\nabla_p n_p\right)\right)\mu_{ijk} n_k - \sum_\alpha T_i^\alpha \qquad on\ \partial \mathcal{D} - \Gamma_{\nearrow} \qquad (2.40)$$

$$L_i = \varepsilon_{jpq}[\![\mu_{ijk} n_k \tau_p n_q]\!] \qquad\qquad\qquad\qquad on\ \Gamma_{\nearrow} \qquad (2.41)$$

where the nonsymmetric Cauchy stress tensor's component t_{ij} is defined by:

$$t_{ij} = \sigma_{ij} - \mu_{ijk,k} + \rho \left({}^L E_{[i}\pi_{j]} + {}^L B_{[i}\mu_{j]} \right) - \left({}^L \mathscr{E}_{[i|k|}\pi_{j],k} + \beta {}^L \mathscr{B}_{[i|k|}\mu_{j],k} \right) \qquad (2.42)$$

and where the symbol $[\![\ldots]\!]$ denotes here the jump across the edge Γ, $\boldsymbol{\tau}$ denotes the unit vector tangent to Γ and oriented in the direct sense about the normal $\boldsymbol{n}$, and, $\hat{\nabla}$ denotes the surface gradient operator.

$$\boxed{\forall \boldsymbol{v}^{\alpha*}} \quad t^\alpha_{ij,j} + q^\alpha_f \left(\mathscr{E}^\alpha_i + {}^L \mathscr{E}^\alpha_i \right) = 0 \qquad\qquad in\ \mathscr{D} \qquad (2.43)$$

$$t^\alpha_{ij}n_j = T^\alpha_i \qquad\qquad on\ \partial\mathscr{D} - \Gamma_{\nearrow} \qquad (2.44)$$

where the nonsymmetric stress tensor's component t^α_{ij} is defined by:

$$t^\alpha_{ij} = \sigma^\alpha_{ij} + \rho \left({}^L E^\alpha_{[i}\pi_{j]} + {}^L B^\alpha_{[i}\mu_{j]} \right) - \left({}^L \mathscr{E}^\alpha_{[i|k|}\pi_{j],k} + \beta {}^L \mathscr{B}^\alpha_{[i|k|}\mu_{j],k} \right) \qquad (2.45)$$

$$\boxed{\forall \dot{\boldsymbol{\pi}}^*} \quad {}^L \mathscr{E}^T_{ij,j} + \rho \mathscr{E}^{eff}_i = \rho\, d\ddot{\pi}_i \qquad\qquad in\ \mathscr{D} \qquad (2.46)$$

$${}^L \mathscr{E}^T_{ij}n_j = \frac{\rho Q_i}{\varepsilon_0} \qquad\qquad on\ \partial\mathscr{D} - \Gamma_{\nearrow} \qquad (2.47)$$

with the effective electric field $\mathscr{E}^{eff}$, and, the local interaction electric fields of the first order ${}^L\boldsymbol{E}^T$ and of the second order with components ${}^L\mathscr{E}^T_{ij}$ defined by:

$$\mathscr{E}^{eff}_i = \mathscr{E}_i + {}^L E^T_i \qquad (2.48)$$

$${}^L E^T_i = {}^L E_i + \sum_\alpha {}^L E^\alpha_i \quad and \quad {}^L \mathscr{E}^T_{ij} = {}^L \mathscr{E}_{ij} + \sum_\alpha {}^L \mathscr{E}^\alpha_{ij} \qquad (2.49)$$

$$\boxed{\forall \dot{\boldsymbol{\mu}}^*} \quad \beta {}^L \mathscr{B}^T_{ij,j} + \rho B^{eff}_i = -\beta' \rho \frac{\omega_i}{\gamma} \ with\ \boldsymbol{\omega}\ such\ that\ \dot{\boldsymbol{\mu}} = \boldsymbol{\omega} \times \boldsymbol{\mu}\ in\ \mathscr{D} \qquad (2.50)$$

$$\beta\varepsilon_{ipq} \left({}^L \mathscr{B}^T_{pj}n_j - \rho\mu_0 \mathscr{F}_p \right) \mu_q = 0 \qquad\qquad on\ \partial\mathscr{D} - \Gamma_{\nearrow} \qquad (2.51)$$

with $\beta' = \beta = 1$, when the material is ferromagnetic at low temperature [13, 15]; $\beta = 1, \beta' = 0$, when the material is ferromagnetic near the Curie temperature; $\beta' = \beta = 0$, otherwise, and, with the effective magnetic induction $\boldsymbol{B}^{eff}$, and, the local interaction magnetic inductions of the first order ${}^L\boldsymbol{B}^T$ and of the second order with components ${}^L\mathscr{B}^T_{ij}$ defined by:

$$B_i^{eff} = B_i + {}^L B_i^T \tag{2.52}$$

$${}^L B_i^T = {}^L B_i + \sum_\alpha {}^L B_i^\alpha \quad \text{and} \quad {}^L \mathscr{B}_{ij}^T = {}^L \mathscr{B}_{ij} + \sum_\alpha {}^L \mathscr{B}_{ij}^\alpha \tag{2.53}$$

2.1.2.3 Local Thermodynamical Equations

Combining the first principle of thermodynamic (2.18) with the principle of virtual power (taken for actual velocities), we obtain the following global statement corresponding to the global form of the energy theorem as:

$$\dot{E}(\mathscr{D}) + \mathscr{P}_{(i)}(\mathscr{D}, \mathcal{V}_{obj}) = \dot{Q}_h(\overline{\mathscr{D}}) + \dot{Q}_{em}(\overline{\mathscr{D}}) \tag{2.54}$$

where we have set

$$\dot{Q}_{em}(\overline{\mathscr{D}}) = \int_\mathscr{D} \dot{q}_{em} dv + \int_{\partial \mathscr{D}} \mathscr{S} \cdot \boldsymbol{n} da \tag{2.55}$$

with

$$\dot{q}_{em} = \sum_\alpha \left(\mathscr{J}^\alpha \cdot \mathscr{E} - \boldsymbol{f}^\alpha \cdot \boldsymbol{u}^\alpha \right) \tag{2.56}$$

Accounting for the generalized transport theorems and balances of mass, from these latter eqns, we deduce the local forms of the first principle of thermodynamics (2.18) (or the local form of the energy theorem) as

$$\rho \dot{\varepsilon} = p_{(i)} + \dot{q}_{em} - \nabla \cdot \tilde{\boldsymbol{q}} + \rho h \tag{2.57}$$

The second principle of thermodynamics remains to be exploited. To that purpose we assume that $\sigma = h/\theta$ and $\phi_i = \tilde{q}_i/\theta$. Only the volume entropy flux differs from the usual ratio of the heat vector to the temperature, which means that non-simple thermodynamic processes are involved (cf. [20], p. 129). The local form of the second principle of thermodynamics (2.19) then reads

$$\rho \theta \frac{d\eta}{dt} \geq \rho h - \nabla \cdot \tilde{\boldsymbol{q}} + \boldsymbol{\phi} \cdot \nabla \theta \tag{2.58}$$

2.1.3 Clausius-Duhem Inequality

The Helmholtz free energy density $\psi = \varepsilon - \eta \, \theta$ is introduced. And we are led to the Clausius-Duhem inequality in the local form:

$$-\rho\left(\frac{d\psi}{dt}+\eta\frac{d\theta}{dt}\right)+p_{(i)}+\dot{q}_{em}-\boldsymbol{\phi}\cdot\nabla\theta\geq 0 \tag{2.59}$$

Introducing the relations between Jaumann derivatives and convective-time ones and a scalar chemical potential, we can evaluate [3]:

$$\begin{aligned}
p_{(i)}+\dot{q}^{em}=&\tilde{t}_{ij}^{T}D_{ij}+\mu_{ijk}v_{i,jk}-{}^{L}E_{i}^{T}\overset{*}{P}_{i}-{}^{L}B_{i}^{T}\overset{*}{\mathscr{M}}_{i}+{}^{L}\mathscr{E}_{ij}^{T}\widehat{\pi}_{ij}+\beta\,{}^{L}\mathscr{B}_{ij}^{T}\widehat{\mu}_{ij}\\
&+\sum_{\alpha}\left[\mathscr{E}^{\alpha(eff)}\cdot\boldsymbol{\mathscr{J}^{\alpha}}-\mu^{\alpha}\nabla\cdot\boldsymbol{\mathscr{J}^{\alpha}}\right]
\end{aligned} \tag{2.60}$$

where we have also introduced the effective electromotive field of the α charge carriers by $\mathscr{E}^{\alpha(eff)}=\mathscr{E}-\nabla\mu^{\alpha}$, where $\tilde{t}^{T}$ is a symmetric tensor such that:

$$\tilde{t}_{ij}^{T}=\sigma_{ij}+\sum_{\alpha}\sigma_{ij}^{\alpha}-{}^{L}E_{(i}P_{j)}-{}^{L}B_{(i}\mathscr{M}_{j)}+{}^{L}\mathscr{E}_{(i|,k|}\pi_{j),k}+\beta\,{}^{L}\mathscr{B}_{(i|,k|}\mu_{j),k} \tag{2.61}$$

and where ${}^{L}E_{i}^{T}$, ${}^{L}B_{i}^{T}$, ${}^{L}\mathscr{E}_{ik}^{T}$ and ${}^{L}\mathscr{B}_{ik}^{T}$ are respectively defined previously.

Recalling that $\boldsymbol{\phi}=\tilde{\mathbf{q}}/\theta$ and accounting for the latter eqns, we can rewrite the Clausius-Duhem inequality (2.59) in the useful form [3]:

$$\begin{aligned}
&-\rho\left(\frac{d\psi}{dt}+\eta\frac{d\theta}{dt}\right)+\tilde{t}_{ij}^{T}D_{ij}+\mu_{ijk}v_{i,jk}-{}^{L}E_{i}^{T}\overset{*}{P}_{i}-{}^{L}B_{i}^{T}\overset{*}{\mathscr{M}}_{i}+{}^{L}\mathscr{E}_{ik}^{T}\widehat{\pi}_{ij}+\beta\,{}^{L}\mathscr{B}_{ik}^{T}\widehat{\mu}_{ij}\\
&+\sum_{\alpha}\mathscr{E}_{i}^{\alpha(eff)}\mathscr{J}_{i}^{\alpha}+\sum_{\alpha}\mu^{\alpha}\left(\rho\frac{dc_{\lambda}^{\alpha}}{dt}-r^{\alpha}\right)+\theta\,\tilde{q}\cdot\nabla\left(\frac{1}{\theta}\right)\geq 0
\end{aligned} \tag{2.62}$$

As is well known, the Clausius-Duhem inequality plays a major role in the building of constitutive relations.

2.2 Second Part

2.2.1 *Extension of the PVP to Gauge and Scale Invariances*

The scalar method known as the principle of virtual power—applied to mechanics with microstructures by Germain [4, 5], then to electro-magneto-mechanics by Maugin and his students [7, 8, 10, 11]—has brought remarkable advances. This method based on the duality notion subject to translational and rotational invariances is extended here to gauge and scale invariances, leading thus to a more unifying principle, apt to account for a wider range of applications as shown in [21–24].

After having extended the principle of virtual power to complex structures including conduction and diffusion effects of various charge carriers (semi-conduction), it seemed advisable to acquire a more unifying, systematic and universal framework, adapting this formal and reliable method to other types of symmetries and invariances thanks to its use of the fruitful concepts of modern geometry.

This was done by borrowing concepts from theoretical physics, particularly gauge theories (serious candidates for the unification of the four forces of interaction of fundamental physics). Thus, scale and gauge invariances were introduced into the physics of continuous media. These two types of invariances add to the well-known translational and rotational invariances (objectivity requirement) already dealt with in electro-magneto-mechanics. Gauge invariance allows to account for Maxwell's electromagnetism analogously to rotational invariance for deformable bodies. As to scale invariance, it allows to deal with the various forms of discontinuities and interfacial properties that occur at singular surfaces, lines and/or points.

These aspects have been presented succinctly in congresses [22–24] and in a synthetic paper [21] more than twenty years ago, both in Newtonian and Einsteinian chronologies, but this theoretical theme, considered too remote from the immediate concerns of the laboratory, had not been pursued further at that time.

In the last decade, some works together with Hirsinger and Devel showed the need for such a general and systematic methodology (see for instance [3]) that will be succinctly recalled here and developed in future works.

2.2.2   Extended form of d'Alembert's Principle

Statical continuum mechanics and magnetism are the simplest examples where the basic ideas are brought out clearly. In addition, since a boundary may be regarded as a particular case of a moving singular surface, one may omit its expression in the present derivation. Only the essential elements are kept here. The attention is focused on the three different invariance principles that govern discontinuities as well as Maxwell electromagnetism and deformable mechanics.

2.2.3   Unified Global Statement

The basic postulate may be expressed in the form of an orthogonality relation as follows:

$$\delta W^* = \langle \boldsymbol{D}, \delta \overset{*}{G} \rangle = 0 \tag{2.63}$$

where δG is an infinitesimal variation of the geometrical parameter $\boldsymbol{G}$ and $\boldsymbol{D}$ is the dynamical contribution introduced by duality. A star * on a field denotes its virtual character. In the present framework, it is convenient to distinguish between three types of energies as follows:

$$\delta W^* = \delta W^*_{GIV} + \delta W^*_{GRI} + \delta W^*_{SCI} = 0 \tag{2.64}$$

In a first-order gradient framework associated with volume and surface physical contributions, one may write:

$$\delta W^*_{GIV} = \int\limits_{\mathcal{D}-\Sigma} \left(K_i \delta R^*_i + K_{ij} \nabla_j \, \delta R^*_i\right) dv + \int\limits_{\Sigma} \left(\hat{K}_i \hat{\delta R}^*_i + \hat{K}_{ij} \nabla_j \, \hat{\delta R}^*_i\right) da \tag{2.65}$$

$$\delta W^*_{GRI} = \int\limits_{\mathcal{D}-\Sigma} \left(A_i \delta R^*_i + A_{ij} \nabla_j \, \delta R^*_i\right) dv + \int\limits_{\Sigma} \left(\hat{A}_i \hat{\delta R}^*_i + \hat{A}_{ij} \hat{\nabla}_j \, \hat{\delta R}^*_i\right) da \tag{2.66}$$

$$\delta W^*_{SCI} = \int\limits_{\Sigma} \left(Z^+_i \delta^+ R^{*+}_i - Z^-_i \delta^- R^{*-}_i + \hat{Z}_i \hat{\delta R}^*_i\right) da \tag{2.67}$$

where dv is the volume element of the bulk medium $\mathcal{D}-\Sigma$, da is the surface element of the interface Σ and $\hat{\nabla}$ denotes the surface gradient, $+$ and $-$ denote quantities on either side of the singular surface and $\wedge$ the quantity at the singular surface.

The expressions of δW^*_{GIV} and δW^*_{GRI} (GIV for given fields, GRI for gauge and rotational invariance) are introduced in a systematic manner. $\left(K_i, \, K_{ij}\right)$ and their surface counterparts $\left(\hat{K}_i, \, \hat{K}_{ij}\right)$ correspond to given fields. A quantity for which no physical support is available can be dropped from Eq. (2.65). As to the expression of δW^*_{GRI}, it should be specified through a physical invariance principle. More precisely, gauge invariance (Electromagnetism) and rotational invariance (Mechanics) will impose restrictions on the form of δW^*_{GRI}. As shown below, the dual field A_{ij} will be skew-symmetrical (in Electromagnetism) and symmetrical (in Mechanics). And the dual fields A_i and $\hat{A}_i$ will vanish in both cases since they violate the invariance requirements. Physically, these invariance principles will give the correct form of the field-field interaction energy (magnetic energy) and of the matter-matter interaction energy (deformation energy). Thus, a net distinction is made between given fields and those deduced from a physical invariance principle.

Across the interface, one loses differentiability, thus, a general form of the interaction energy between the bulk and the interface is given by Eq. (2.67). Its construction is performed by taking all the energies that one may construct at the interface and its surrounding. This leads to the introduction of three vector fields, to be coupled together as well as with the fields present in Eqs. (2.65) and (2.66). A full determination is obtained in two steps. First, the scale invariance principle

relates $\hat{Z}$ to $Z^{\pm}$, then the use of the virtual character through the application of the principle for any $\delta R^{\pm}$ will lead to the determination of $Z^{\pm}$. The scale invariance principle asserts that Eq. (2.67) must remain invariant under the addition of any continuous infinitesimal vector field. This requirement governs the passage from a continuous to a discontinuous medium. Before dealing with the three invariance principles, let us recall that the present formulation may be regarded as a generalization of the well-known Lagrangian approach, (recovered for integrable systems, $A_i \equiv \partial L/\partial R_i$). In the present formulation, no hypothesis of integrability is imposed. This offers richer possibilities, particularly in the framework of dissipative phenomena and irreversible processes.

2.2.4 Derivation of Scale, Gauge and Rotational Invariances

The attention is focused here on the formal unifying structure. The physical details are provided in Refs. [21–24]. One way to deal with invariance principles consists in requiring that the energy remains invariant under the addition of a certain infinitesimal field. Mathematically, one writes

$$\overset{\alpha}{\delta} \overset{\alpha'}{R} = \overset{\alpha}{\delta} \overset{\alpha}{R} + \delta r \quad \alpha = \{+, -, \wedge\} \tag{2.68}$$

A—Scale invariance corresponds to $r = a$, where a is any continuous vector field (i.e. $[\![a]\!] = 0$)

This requirement transforms Eq. (2.67) into

$$\delta W_{SCI}^* = \int [\![Z_i \left(\delta R_i^* - \delta \hat{R}_i^* \right)]\!] \, da \tag{2.69}$$

where $[\![A]\!] \equiv A^+ - A^-$ denotes the jump from the + to the − side of the interface.

B—Gauge invariance consists in taking $r = \nabla \psi$, where ψ is any scalar field. As to δR, it coincides here with an infinitesimal variation of a vector potential δA. The consequence of this invariance on Eq. (2.66) leads to

$$A_i = 0, \qquad \hat{A}_i = 0, \qquad H_{ij} \equiv A_{ij} \tag{2.70}$$

such that

$$H_{ij} + H_{ji} = 0$$

and

$$\overset{S}{\hat{H}}_{ij} \equiv \hat{A}_{ij} \tag{2.71}$$

such that

$$\overset{S}{\hat{H}}_{ij} + \overset{S}{\hat{H}}_{ji} = \overset{S}{\hat{H}}_{ik}\, n_k\, n_j + \overset{S}{\hat{H}}_{jk}\, n_k\, n_i$$

where H_{ij} and $\overset{S}{\hat{H}}_{ij}$ are respectively volume and surface magnetic fields which are pseudo-vectors expressed here in tensorial form.

C—Rotational invariance (or objectivity requirement) is expressed through a rigid body motion transformation $r = X + \boldsymbol{\omega} \times x$ (X: translations, $\boldsymbol{\omega}$: Rotations). This leads to

$$A_i = 0, \qquad \hat{A}_i = 0, \qquad \sigma_{ij} \equiv -A_{ij} \tag{2.72}$$

such that

$$\sigma_{ij} - \sigma_{ji} = 0$$

and

$$\overset{S}{\hat{\sigma}}_{ij} \equiv -\hat{A}_{ij} \tag{2.73}$$

such that

$$\overset{S}{\hat{\sigma}}_{ij} - \overset{S}{\hat{\sigma}}_{ji} = \overset{S}{\hat{\sigma}}_{ik}\, n_k\, n_j - \overset{S}{\hat{\sigma}}_{jk}\, n_k\, n_i$$

In this case δR coincides with an infinitesimal displacement and σ_{ij} is none other than the mechanical stress tensor.

2.2.5 Local Equations

On assuming that Eq. (2.64) holds good for all virtual fields and any element of volume and surface, one obtains the following local equations after using the volume and surface divergence theorems:

$$K_i = \nabla_i A_{ij}, \quad \hat{K}_i = \tilde{\nabla}_i \hat{A}_{ij} + [\![A_{ij}]\!] n_j \tag{2.74}$$

$$\tilde{\nabla} = \hat{\nabla} + 2\overset{m}{\Omega}\,\boldsymbol{n}, \qquad 2\overset{m}{\Omega} = -\hat{\nabla}\cdot\boldsymbol{n} \tag{2.75}$$

($\overset{m}{\Omega}$: mean curvature, $\hat{\nabla}$: surface gradient).

Notice that a tensorial framework offers interesting similarities between mechanical and electromagnetic energies. Here $K_{ij}\,\delta A_{\{i,j\}}$ is none other than deformation energy (mechanics) or magnetic energy (magneto-statics).

2.2.6 Relativistic Framework

Another important feature in such a derivation is its natural generalization to a relativistic framework. Indeed, the basic postulates (2.63)–(2.64) and the invariance requirements (2.68)–(2.73) still hold. The only difference is that one needs to express the fields in a Lorentzian 4-dimensional space. Thus, Eq. (2.74) is to be replaced by

$$K^{\alpha} = \partial_{\beta}\,A^{\alpha\beta}, \quad \hat{K}^{\alpha} = \partial_{\beta}\,\hat{A}^{\alpha\beta} + [\![A^{\alpha\beta}]\!]N_{\beta} \tag{2.76}$$

$$\alpha,\,\beta = \{1,\ldots 4\}$$

where ∂_{β}, ∂_{β} and N_{β} are the 4 dimensional analogues of ∇_{j}, ∇_{j} and n_{j}. For lack of space, we only recall the relation between N and $\boldsymbol{n}$

$$N_{i} = n_{i}/\sqrt{1-\hat{v}_{n}^{2}}, \qquad N_{4} = -\hat{v}_{n}/\sqrt{1-\hat{v}_{n}^{2}} \tag{2.77}$$

when $\hat{v}_{n}^{2} \ll 1$, $\partial_{i} \to \nabla_{i}$ and $\partial_{4} \to \partial/\partial t = \partial/\partial t + \hat{v}_{n}\left(n\cdot\nabla - 2\overset{m}{\Omega}\right)$.

It is important to note here, that dealing with interfaces in a relativistic framework does not only yield more general solutions but also leads to simple covariant expressions. The simplicity criterion is essential here to verify the coherence of the theory. Indeed, the lack of symmetry between space and time in a Galileen framework leads to complicated expressions. When applied to electromagnetism, Eq. (2.76) may be explicitly written as:

$$J^{\alpha} = \partial_{\beta}\,H^{\alpha\beta}, \qquad \hat{J}^{\alpha} = P_{\beta}^{\gamma}\,\partial_{\gamma}\,\hat{H}^{\alpha\beta} + \hat{H}^{\beta\gamma}\Gamma_{\gamma\beta}^{\alpha} + [\![H^{\alpha\beta}]\!]N_{\beta} \tag{2.78}$$

$$P_{\beta}^{\gamma} = \delta_{\beta}^{\gamma} - N_{\beta}N^{\gamma}, \quad \Gamma_{\gamma\beta}^{\alpha} = N_{\gamma}\left\{P_{\beta}^{\theta}\,\partial_{\theta}\,N^{\alpha} - \partial\cdot N\,\delta_{\beta}^{\alpha}\right\}, \quad \overset{s}{J}^{\alpha} = P_{\beta}^{\alpha}\,\hat{J}^{\beta} \tag{2.79}$$

Let us recall that the passage from 4 to (3 + 1) dimensions transforms $J^{\alpha} = \partial_{\beta}\,H^{\alpha\beta}$ such that $H^{\alpha\beta} + H^{\beta\alpha} = 0$ into $q_{f} = \nabla\cdot\boldsymbol{D}$ and $\boldsymbol{J} = -\partial\boldsymbol{D}/\partial t + \nabla\times H$.

The interfacial expression is written in such a manner that one may distinguish between three contributions: (i) the spatio-temporal variation of $\hat{H}^{\alpha\beta}$ (counterpart of the volume expression), (ii) its coupling with the surface curvature $\Gamma^{\alpha}_{\gamma\beta}$ and (iii) the jump relation.

This energy formulation may also account for singular lines by analogy to singular surfaces as explicitly shown in [21–23].

In conclusion, let us recall that the second part of this work extends the ideas expressed in the works of professors Germain and Maugin who developed d'Alembert's principle in different contexts by exploiting the invariance under a rigid body motion.

2.3 Third Part

2.3.1 Foundation of the Principle of Virtual Power (PVP)

The principle of virtual power is a scalar (geometrical) approach, based on the duality notion that corresponds to one point of view among others. We shall go back to the source of this notion thanks to an intrinsic (viewpoint independent) dynamical framework conceptualized by Leibniz and formalized recently in Refs. [25–28]. This framework clearly distinguishes between worlds and points of view. A dynamical world is formally expressed through a relation that links directly the two conserved entities (energy and impulse): $E = F(p)$ or more generally $R(E, p) = C$ (constant). As to a point of view attached to a specific world, it consists in expressing impulse and energy in terms of a motion parameter x: $p = g(x)$, $E = f(x)$. Obviously, if this point of view is relative to the above world $E = F(p)$, then the three functions F, g and f cannot be independent anymore. They must satisfy: $E = F(p) = F(g(x)) = f(x)$.

Unlike usual physics, limited to one world (Newtonian, Einsteinian, Finslerian …) dealt with through one point of view (variational, geometrical, group theoretical …), Leibniz's conception accounts for all physically admissible worlds (i.e. compatible with the relativity and conservation requirements) independently of any a priori imposed point of view whatsoever. Such a conception is characterized by its intrinsic (viewpoint independent) nature where the different dynamical worlds are deduced before the determination, by self-organization, of the appropriate points of view attached to each world.

The principle of virtual power, like other analytical principles such as the principle of least action, appears henceforth as a theorem. These turn out to be deductible from a weaker principle, using qualitative mathematics, from which different quantitative dynamical structures—each constituting one point of view—are derived. Among these dynamical structures, one recognizes the ones that correspond to the well-identified physical principles developed in the history of

dynamics. Here, the attention will be focused on the source of the duality notion which is at the basis of the principle of virtual power.

2.3.2 Main Points of the Leibnizian Dynamical Framework

The Leibnizian formulation takes its origin from a dynamical procedure, due to Huygens, based on the relativity and conservation principles, in $(1 + 1)$ dimensions, recalled in Eqs. (9.11)–(9.14) of Ref. [29], by the physicist and historian of science Barbour.

Elevated to the rank of a principle and expressed in the Leibnizian language of infinitesimal calculus, Huygens dynamics is formally expressed by: $M = d^2E/dw^2$, with $M = m$, $p = dE/dw$ and the limit conditions $w = 0$, $p = 0$, $E = E_0$, where the motion parameter w satisfies an additive composition law $\left(w' = w + W\right)$. Its integration leads to: $p = m\,w$ and $E = 1/2\,m\,w^2 + E_0$. This method (recently justified by a theorem borrowed from group theory) was revived by many authors [30–34] and applied to Einstein's dynamics where the constant $(M = m)$ is replaced by the linear relation $(M = E/c^2)$.

In order to account for all physically admissible worlds and associated points of view, we have extended Huygens procedure according to Leibniz's conceptualization, characterized by the simultaneous presence of an infinity of points of view on each dynamical world [25–28]. Such a conceptualization is called architectonical by opposition to the usual analytical conceptualization, limited to one point of view a priori imposed from the start.

As a consequence, instead of the above differential equation $M = d^2E/dw^2$, relative to Huygens conception, that accounts for one world $(M = m)$, corresponding to: $E = p^2/2\,m + E_0$, dealt with through one point of view $(p = dE/dw)$, expressed by the motion parameter w attached to the operator d/dw, one is led, as shown explicitly in [25, 26], to an infinity of differential equations $M = d_\mu^2E/dv_\mu^2$, corresponding to Leibniz's conception, that account for all dynamically admissible worlds $\left(M = \lambda E + \gamma d_\mu E/dv_\mu + \eta\right)$, each one dealt with through an infinity of points of view $\left(p = d_\mu E/dv_\mu\right)$, expressed by the motion parameters v_μ attached to the infinitely multiple μ-operator: $d_\mu/dv_\mu = I_\mu d/dv_\mu$ where the functions I_μ that depend on v_μ are yet indeterminate. The functions I_μ reflect the non-additive composition laws $\left(v'_\mu \neq v_\mu + V_\mu\right)$ that accompany the additive one $\left(v'_a = v_a + V_a\right)$ for which I_a reduces to unity $(I_a = 1)$.

In brief, the passage from the Huygensian analytical conception to the Leibnizian architectonical one, amounts to replace: $M = m = d^2E/dw^2$ with $p = dE/dw$ by the following under-determinate structure: $M = \lambda E + \gamma d_\mu E/dv_\mu + \eta = d_\mu^2E/dv_\mu^2$ with $p = d_\mu E/dv_\mu$ expressed explicitly by:

$$
\begin{aligned}
M = \lambda E + \gamma\, I_\mu dE/dv_\mu + \eta &= \left[I_\mu d/dv_\mu \right] \left[I_\mu d/dv_\mu \right] E \\
&= I_\mu^2 d^2 E/dv_\mu^2 + \left[I_\mu dI_\mu/dv_\mu \right]\, dE/dv_\mu
\end{aligned}
\tag{2.80}
$$

At first sight, it seems contradictory to associate the term "world" with the above expression of M because of its viewpoint dependence. But this apparent contradiction vanishes by showing, as done in [25], that Eq. (2.80) transform into an intrinsic (viewpoint independent) framework, expressed uniquely in terms of the conserved entities E and p as follows: $M = \lambda E + \gamma p + \eta = [pd/dE][pd/dE]E = [p^2 d^2/dE^2 + (pd/dE)d/dE]E = pdp/dE$. When integrated, this differential equation becomes formally expressed through a relation that links together the two conserved entities (energy and impulse): $R(E,\ p) = C$. One recovers thus what is called above a dynamical world. This procedure, called in [25–28] a "filtering procedure", characterizes the Leibnizian intrinsic approach where the determination of the worlds precedes and contributes to the specification of the points of view.

2.3.3 Determination of the Yet Under-Determinate Framework

The attention will be focused here on the Newtonian (parabolic) and Einsteinian (hyperbolic) worlds that correspond respectively to: $(\lambda,\ \gamma,\ \eta) = (0,\ 0,\ m)$ and $(\lambda, \gamma, \eta) = (E/c^2, 0, 0)$, getting thus: $M = m$ and $M = E/c^2$. These two dynamical worlds can be expressed in a unified differential form by: $M = m(E/m\,c^2)^k = p\,dp/dE$ with $k = 0$ for Newton and $k = 1$ for Einstein.

Its integration will provide valuable information that will actively contribute to the determination of the infinity of the yet indeterminate points of view as shown in [25]. Thus, one is led to the multiple scale law:

$$
I_\mu = (M/m)^{2-\mu} = \left[\left(E/m\ c^2 \right)^k \right]^{2-\mu}
\tag{2.81}
$$

Having specified the functions I_μ, the under-determinate structure (2.80) becomes well determinate: it includes an infinity of quantitative equations, each value of μ corresponding to a particular point of view. Among the infinity of points of view, the formal structure singles out four basic (singular, remarkable and operational) points of view, the others corresponding to more or less complicated combinations of the four basic ones. The three well-identified points of view relative to the three different principles (Lagrangian formulation, d'Alembert's principle and Huygens procedure) expressed in mathematical terms by the calculus of variations, modern geometry and group theory, turn out to be deductible from the points of view of orders $\mu = \{4,\ 1\ and\ 2\}$ respectively as shown in [25–28]. Since we are mainly concerned here with d'Alembert's (or virtual power) principle, the

attention will be focused on the procedure that allows deriving it from the present general Leibnizian approach.

2.3.4 Deduction of the PVP Based on Duality

As shown in [25–28] for the point of view of order one $(\mu = 1)$, we get: $I_1 = M/m = (E/m\,c^2)^k$ and $p = I_1\,dE/dv_1 = m\,v_1$ from which one deduces: $p\,dv_1 = v_1\,dp$ (since $dp = m\,dv_1$), at the basis of the duality notion. Its combination with $p = I_1 dE/dv_1$ leads to: $I_1\,dE - p\,dv_1 = I_1\,dE - v_1\,dp = 0$ so that one is finally left with: $M\,c = m\,c\,I_1$, $p = m\,v_1$ and $I_1\,dE - v_1\,dp = 0$.

With the well-known compact notation: $(M\,c, p) = \boldsymbol{P} = \{P^i\}$, $(E/c, p) = \boldsymbol{p} = \{p^i\}$ and $(c\,I_1, v_1) = \boldsymbol{u} = \{u^i\}$ with $i = 0, 1$, one gets: $\boldsymbol{P} = m\,\boldsymbol{u}$ and $\boldsymbol{u} \cdot d\boldsymbol{p} = 0$ where the scalar product: $\boldsymbol{u} \cdot d\boldsymbol{p} = 0$ is associated with Minkowski's signature $\eta = (1, -1)$.

In order to replace the infinitesimal form: $\boldsymbol{u} \cdot d\boldsymbol{p} = 0$ by a finite one: $\boldsymbol{u} \cdot \boldsymbol{f} = 0$, leading thus to the concept of force, we set: $\boldsymbol{f} = d\boldsymbol{p}/d\tau$, then analogously: $\boldsymbol{F} = d\boldsymbol{P}/d\tau$ and $\boldsymbol{a} = d\boldsymbol{u}/d\tau$. This allows writing: $\boldsymbol{F} = m\,\boldsymbol{a}$ and $\boldsymbol{u} \cdot \boldsymbol{f} = 0$. These two vector and scalar expressions can be unified into a unique scalar formalism: $(\boldsymbol{F} - m\,\boldsymbol{a}) \cdot \boldsymbol{u}^* = 0$ and $\boldsymbol{u} \cdot \boldsymbol{f} = 0$, provided one accounts for a virtual motion $\boldsymbol{u}^*$. This formulation that goes back to d'Alembert corresponds to the principle of virtual power.

2.3.5 Derivation of Einstein's Dynamics

For $k = 1$ (Einstein's world), $\boldsymbol{f}$ reduces to $\boldsymbol{F}$ because $\boldsymbol{p} = \boldsymbol{P}$ since $E/c = M\,c$, getting thus: $(\boldsymbol{F} - m\,\boldsymbol{a}) \cdot \boldsymbol{u}^* = 0$ and $\boldsymbol{u} \cdot \boldsymbol{F} = 0$.

Let us firstly show that this general dynamical approach, will naturally lead to space-time thanks to the duality property. Indeed, by combining: $\boldsymbol{u} \cdot d\boldsymbol{p} = \boldsymbol{u} \cdot \boldsymbol{f}\,d\tau$ with: $\boldsymbol{u} \cdot \boldsymbol{f} = \boldsymbol{f} \cdot \boldsymbol{u}$, one gets the following expressions: $\boldsymbol{u} \cdot d\boldsymbol{p} = \boldsymbol{u} \cdot \boldsymbol{f}\,d\tau = \boldsymbol{f} \cdot \boldsymbol{u}\,d\tau = \boldsymbol{f} \cdot d\boldsymbol{x}$, where we have set: $d\boldsymbol{x} = \boldsymbol{u}\,d\tau$. In the same way as $\boldsymbol{u}$ is the dual of $d\boldsymbol{p}$, $\boldsymbol{f}$ appears as the dual of $d\boldsymbol{x}$. As shown below, when $\boldsymbol{f} = \boldsymbol{F}$, $d\boldsymbol{x}$ corresponds to space-time variation.

On assuming that the relation $(\boldsymbol{F} - m\,\boldsymbol{a}) \cdot \boldsymbol{u}^* = 0$ subject to: $\boldsymbol{u} \cdot \boldsymbol{F} = 0$ holds true for any virtual motion $\boldsymbol{u}^*$, one derives: $\boldsymbol{F} = m\,\boldsymbol{a}$ and $\boldsymbol{u} \cdot \boldsymbol{a} = 0$. Their integration leads to: $\boldsymbol{p} = m\,\boldsymbol{u}$ and $\boldsymbol{u} \cdot \boldsymbol{u} = C$ where C is a constant of integration. On setting $C = c^2$, with c having the dimension of a velocity and accounting for $d\boldsymbol{x} = \boldsymbol{u}\,d\tau$, one is left with: $\boldsymbol{p} = m\,d\boldsymbol{x}/d\tau$ and $d\boldsymbol{x} \cdot d\boldsymbol{x} = c^2 d\tau^2$ where one recognizes Einstein's dynamics with its metrical structure.

Final remark: In order to establish a direct link with the present approach, let us note that the metrical structure may be explicitly written as: $\Gamma^2 - u^2/c^2 = 1$, with $\Gamma = dt/d\tau$ and $u = dx/d\tau$. The couple (Γ, u) that reflects the relativistic factor and

the celerity respectively corresponds to: (I_1, v_1) subject to: $I_1^2 - v_1^2/c^2 = 1$. It is easily deduced from the general relation between I_μ and v_μ:

$$I_\mu^{2/(2-\mu)} - \frac{1}{c^2}\left(\int I_\mu^{(\mu-1)/(2-\mu)} dv_\mu\right)^2 = 1 \qquad (2.82)$$

derived from the Leibnizian architectonical approach. Indeed, for $\mu = 1$, this expression greatly simplifies getting: $I_1^2 - v_1^2/c^2 = 1$ which is formally similar to: $\Gamma^2 - u^2/c^2 = 1$ but with a different interpretation.

According to the architectonical approach where dynamics precedes kinematics and determines it, the principle of virtual power, based on the duality notion between kinematical and dynamical entities (u and f) is not postulated anymore: it corresponds to the point of view of order one $(\mu = 1)$ deduced from a higher intrinsic principle apt to include various singular, remarkable and operational points of view including those developed in the history of science as shown in [26, 27].

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