Chapter 17
Aspects of Spatial Thinking in Problem Solving: Focusing on Viewpoints in Constructing Internal Representations

Mitsue Arai

Abstract What difficulties do seventh grade students have in constructing internal representations and in their mathematizing processes while considering external representations from various viewpoints? Students received a photograph and were asked to mark where on a map they think the photograph was taken. The results reveal seven types of places where students mark a point and six specific perspective cues they use. Different kinds of difficulty students had in each category are found by examining the relational terms, such as *in front of*, or *right side*, used by the students. The study suggests that a possible cause of difficulty in constructing internal representations is a lack of connection between the objects in terms of their position and direction from several perspectives. Finally our data indicates that crating positional relation with information of real world is a significant ability in mathematizing process.

Keywords Internal representation · Mathematizing process · Spatial thinking Viewpoints

17.1 Introduction

Various situations occur in daily life where spatial thinking serves a purpose. Such examples include working with virtual reality like 3D maps on web sites and reading an instruction manual for assembling furniture. Due to the development of information and communication technology, more types of 3D representations like automobile's navigation systems are more prevalent than ever before. This increase

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indicates the importance of spatial thinking. According to the comprehensive report, "*Learning to Think Spatially*" published by the National Research Council Committee on Geography (2006), spatial thinking is a powerful tool, and it is fundamental to problem solving in a variety of contexts in living space, physical space, and intellectual space. In addition to recognition from educational researchers, spatial thinking has been getting attention in school curricula in Japan (Murakoshi, 2012). For example, map reading in geography, understanding solar trajectories in science, and reasoning geometrically in mathematics require students to think spatially. Compared to other subjects, mathematics plays a specific role in fostering students' ability to transform real-world phenomena into mathematical-world problem then solving problems in the mathematical-world.

In the Japanese geometry curriculum, learning goals related to spatial thinking are mainly related to sketching diagrams that include nets and projection views. There has been much research and ideas for practice in this area (e.g., Yamamoto, 2013). However, the majority of such research and ideas for practice deal with abstract objects such as prisms and pyramids. Moreover, results of the national achievement test in Japan report the difficulties students have with mathematizing real world problems (National Institute for Educational Policy Research [NIER], 2014).

Figure 17.1 provides an example of a real world problem. The question is: "there is a cultural festival. A hanging sign needs to be installed on our school building. Decide the lowest position possible for the display so that it is not eclipsed by the tree when someone looks at it from the sidewalk, and explain how to find the position of the sign using words or figures." (ibid., p. 98[1]) The 61.3% of students answered this item correctly but this percentage is lower than achievement on other problems formulated with abstract objects. Therefore, NIER raised the issue that secondary school students have difficulties to simplify phenomena in order to interpret the results mathematically (ibid., p. 102). These mathematical processes are very difficult for students to do in Japan. Therefore, research is needed to understand how students think spatially in real world situations and what difficulties they encounter in their mathematization processes.

In order to examine the role that spatial thinking about real world objects plays in students' ability to mathematize those real world objects, this study explores students' spatial thinking process while they solve problems with planar representations including photographs and maps. A photograph is an "in-between" representation of the actual object and its geometric diagram while a map represents the space with some information from real world. Bishop (1986) considers both photographs and maps as promising avenues in mathematizing space.

[1]Author's translation from the original in Japanese.

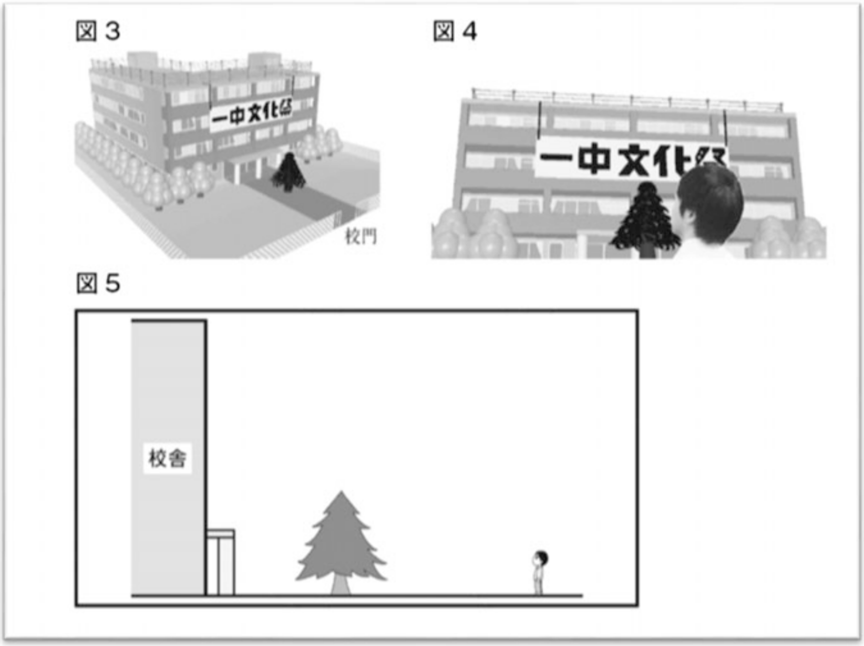


Fig. 17.1 A test item in the Japanese National Assessment of Academic Ability © NIER, used with permission

17.2 Theoretical Background

Research has shown some spatial abilities are present at birth but are slowly realized over years of development (Sarama & Clements, 2009). From a psychological perspective, according to (Krutetskii, 1969), Thurston clarified the structure of human intelligence using factor analysis and showed that the primary mental abilities include a spatial factor. Thurston's notion of primary mental abilities offers a provocative idea that if there is an appropriate combination of primary abilities which constitute mathematical ability, it is possible that mathematical ability could be developed by suitably stimulating those primary abilities besides teaching mathematics (Bishop, 2008). Therefore, spatial ability could be developed through stimulating spatial factor in mathematics education.

From a review of studies on factor analysis regarding spatial abilities, McGee (1979) distinguished two spatial factors, spatial visualization and spatial orientation. Mathematics education also fosters them as competencies. Spatial visualization is the comprehension and performance of imagined movement of objects in 2D and 3D space; spatial orientation is the understanding and operation on the relationship between the objects' positions in space with respect to one's own position (Clements & Battista, 1992). For this paper's focus, spatial thinking is the intellectual exercise of mental operations to create mental spatial images that is supported by intuitive ideas in problem solving situations related to the real or abstract spatial world (Hazama, 2004). From this standpoint, spatial thinking is the activity supported by the competences of spatial visualization and spatial orientation.

The results presented in this paper focus on how students change their viewpoints, which is one of the important intellectual activities related to both spatial visualization and spatial orientation. Saeki (1978) mentioned that changing viewpoints contributes to the reconstruction of internal representations to solve a problem. Also considering an image as a coherent, integrated representation of a scene or object from a particular viewpoint (Eliot, 1987), we believe that looking at viewpoints offers the key to understanding how students create internal representation.

In cognitive psychology, perspective-taking has been discussed since Piaget's "Three Mountain Task." Voluminous literature on the development of perspective-taking provides evidence to support modifying Piaget's theory that young children are spatially egocentric until the age of nine or ten years. Recently, Watanabe and Takamatsu (2014) pointed out that there are processes used to solve a perspective-taking task, one of them being the imagination of body movement from another vantage point in 3D space. Therefore this study takes two types of viewpoint which are considering the part of the object the viewer sees from his or her position (Level 1) and considering the relationship the observer sees among objects as indicated by the cues he or she takes from viewing the objects while solving problems (Level 2) (Flavell, 1974).

With viewpoints thus defined, it is important to refine how spatial descriptions are formed. Spatial descriptions contain statements that locate objects from a reference frame, which includes an origin, a coordinate system, a point of view, terms of reference, and reference objects (Taylor & Tversky, 1996). In order to describe how students construct internal representation, the study focuses on the reference frame. The study's goal is to identify the difficulties students have in solving real world problems by analyzing the terms they use to relate the location of a landmark to a certain origin (i.e. the viewer's position).

17.3 Methodology

The participant sample included 60 seventh graders (33 males and 27 females) in a public school in July 2015. They had not learned how to create nets, map reading, or the topic of similarity. Each student received a questionnaire, which had two components. The first component asked the students if they had seen the objects in a photograph (Fig. 17.2). The second component included two tasks: Task X and Task Y. These tasks were designed based on representational correspondence methods (Liben, 1997). Figure 17.2 shows that the given tasks required students to make a connection between two external representations for one particular place.

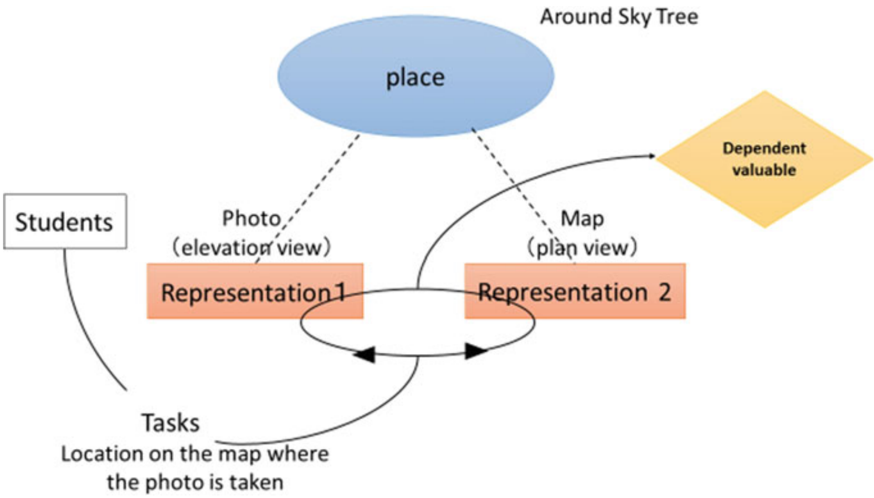


Fig. 17.2 Representational correspondence methods (Task X)

The closed oval line with arrows at the center of Fig. 17.2 represents making a connection in the process of solving the tasks.

In Task X, representation 1 is a photograph, that is a 2D representation, of an elevation view. Representation 2 is a map, that is a 2D representation, representing a view from the top. Students were asked to place a point on the map (Fig. 17.4) to indicate from where the photograph (Fig. 17.3) had been taken and describe the reason for their choice. Both representations show three landmarks: TOKYO SKYTREE^{SM}, East Tower, and a river.

The relational terms *back* and *front* are shown on the photograph. Task X's purpose was to discover what kinds of difficulties seventh graders have in constructing internal representations through focusing on their viewpoints at level 1 and level 2.

In Task Y, Representation 1 included two photographs: One photograph had been taken from an airplane with information about the height and the distance between landmarks; the second photograph gives the appearance of the heights of the two landmarks looking the same from the front (Figs. 17.5 and 17.6). Representation 2 was a map, a 2D representation with view from the top. The two landmarks, TOKYO SKYTREE^{SM} and Mt. Fuji, are well known in Japan. So, every student could have some images of them easily. Task Y asked students to estimate the location in which the photograph was taken (Fig. 17.5) and put a point on the map (Fig. 17.6) or explain it in words. Then, they needed to describe the reason for their location choice with figures and sentences. This task's purpose was to clarify how seventh graders mathematize the given problem and what difficulties exist in their mathematization processes when students analyze the external representations.

Fig. 17.3 The photograph in Task X

The analysis of the two tasks is as follows. In the case of Task X, the cues students described are grouped, and the points students marked are positioned accordingly. Then, specific cues are categorized. The next step is identifying the relationships between the positions and cues using correspondence analysis in order to find strong relations between them. Following the correspondence analysis, the groups are compared based on their descriptions. Finally, our attention shifts to focus on the reference frame expressed in spatial terms. In the case of Task Y, the stages are set based on students' description. Then, cues are selected to solve the problem in each stage. The final part of the analysis of Task Y is examining the relationship between the selected cues in Task Y and Groups A–F in Task X.

17.4 Results and Discussion

All students have had experience seeing TOKYO SKYTREE^{SM} on TV (93%), magazines (60%), from the window (95%), from a distance (55%), from nearby (53%), from the inside of TOKYO SKYTREE^{SM} (35%). All students have seen it in some ways it. Their familiarity with TOKYO SKYTREE^{SM} differs only slightly.

Task X: In this task, there are seven groups of points marked by students, Group A (n = 5), Group B (n = 7), Group C (n = 26), Group D (n = 7), Group E (n = 6), Group F (n = 5), and Group G (n = 4), in the answers (Fig. 17.7). Also identified in the task are six perspective cues: positional relation, distance, direction of stream, curved point, drawing lines, and photograph information (Table 17.1).

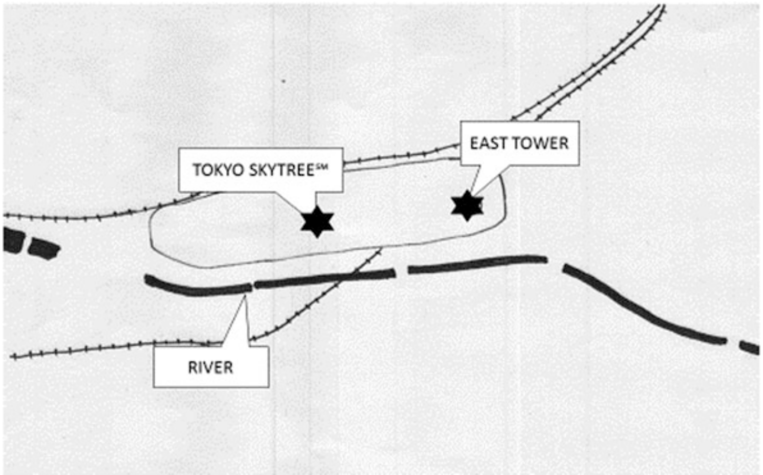


Fig. 17.4 The map in Task X

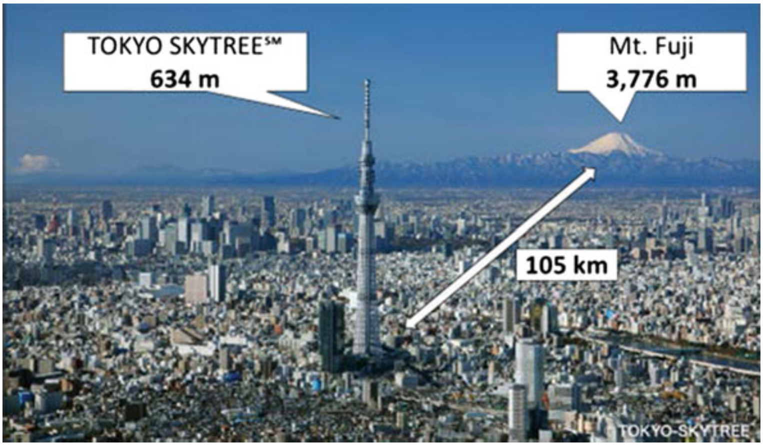


Fig. 17.5 The photograph with information in Task Y © TOKYO-SKYTREE, used with permission

Through correspondence analysis based on the data (Table 17.2), there are three strong relationships between the answers and perspective cues: Groups A & E and Curved Point & Direction of Stream, Group B and Drawing Lines, Groups C & F and Positional Relation (Fig. 17.8). For example in the case of strong relation between Group A and curved Point, the student in Group A describes that "There are three conditions, on the river (bridge), TOKYO SKYTREE℠ should be back and East Tower should be front, the river curved to the right".

Fig. 17.6 The photograph in Task Y Courtesy of Shiroi City Hall, used with permission

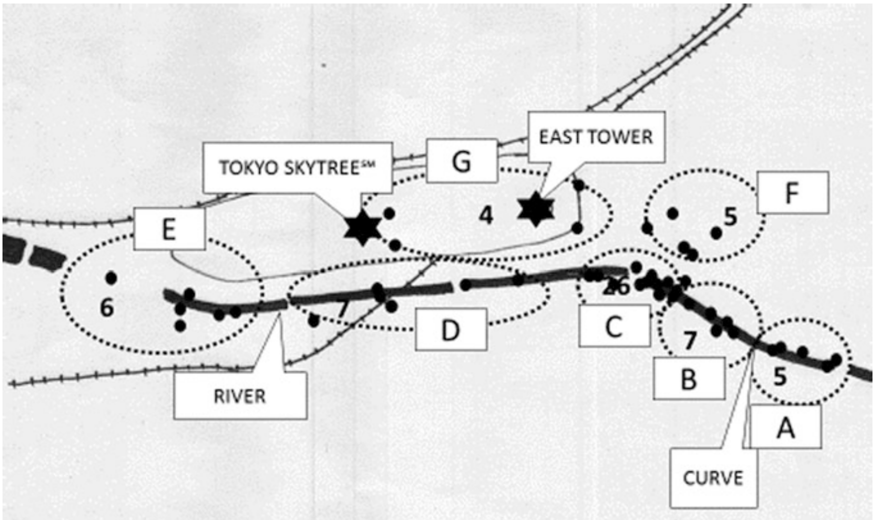


Fig. 17.7 Students' answers (points) in Task Y

Based on the strong relation mentioned above, some groups are compared with the focus on the relational terms, which means terms relating the location of landmark. Comparing Group A and Group E, we observe that students in group A wrote "The river curves *towards* East Tower", "The river curves to the *right*", in contrast students in group E wrote "The river curves to the *side*". Thus, Group A is different from Group E in that using specific terms related to direction. Comparing

Table 17.1 Perspective cues

View point (perspective cues)	Concrete examples
Positional relation	East Tower is in front of TOKYO SKYTREE^{SM}. TOKYO SKYTREE^{SM} is to the left of East Tower
Distance	It looks close
Direction of stream	The river goes to TOKYO SKYTREE^{SM}
Curved point	The river is curved to the right
Drawing line	Drawing the line connecting landmarks on the paper
Photograph information	It might be taken on a bridge

Table 17.2 Ratio of cues in each group

	Positional relation	Distance	Direction of stream	Curved point	Drawing line	Photograph informations
A (n = 5)	80.0 (4)[a]	20.0 (1)	40.0 (2)	80.0 (4)	20.0 (1)	20.0 (1)
B (n = 7)	71.4 (5)	0.0 (0)	14.3 (1)	14.3 (1)	42.9 (3)	0.0 (0)
C (n = 26)	73.1 (19)	7.7 (2)	3.8 (1)	23.1 (6)	15.4 (4)	11.5 (3)
D (n = 7)	14.3 (1)	14.3 (1)	0.0 (0)	0.0 (0)	0.0 (0)	14.3 (1)
E (n = 6)	33.3 (2)	16.7 (1)	33.3 (2)	16.7 (1)	16.7 (1)	0.0 (0)
F (n = 5)	60.0 (3)	20.0 (1)	20.0 (1)	0.0 (0)	20.0 (1)	0.0 (0)
G (n = 4)	50.0 (2)	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)

[a]The figure in parentheses is the number of the students

Group B, C, and Group F, we observed that students in group B drew straight lines connecting buildings and certain point on the river. Thus Group B exploited a mathematical way of drawing lines. On the other hand, 22 students out of 26 in Group C described the river as “The river is curved” and “The photo must have been taken from the bridge.” Group C shows lack of connection between direction of river and position of buildings. Two students out of five in group F described the river’s existence, “The river *is there*.” Group C and F have strong relationship with positional relation yet they only focus on two buildings such as “East Tower is *right*.” The students in group G wrote some words relating to their experience instead of relational terms.

Keeping these conditions of spatial thinking in each group in mind, the study shifts to look at the difficulties in constructing internal representation. Table 17.3 shows the viewpoints in each group. Building, River, and Curve in columns are landmarks students use as viewpoints, showing what they see on the photograph and the map. Positional Relation (Buildings), Positional Relation (River and Buildings), Direction of River, and Direction of Curve are selected as viewpoints, showing how students see or use viewpoints on the photograph and the map. To explain the process, here is an examination of Group C. The viewpoints students in

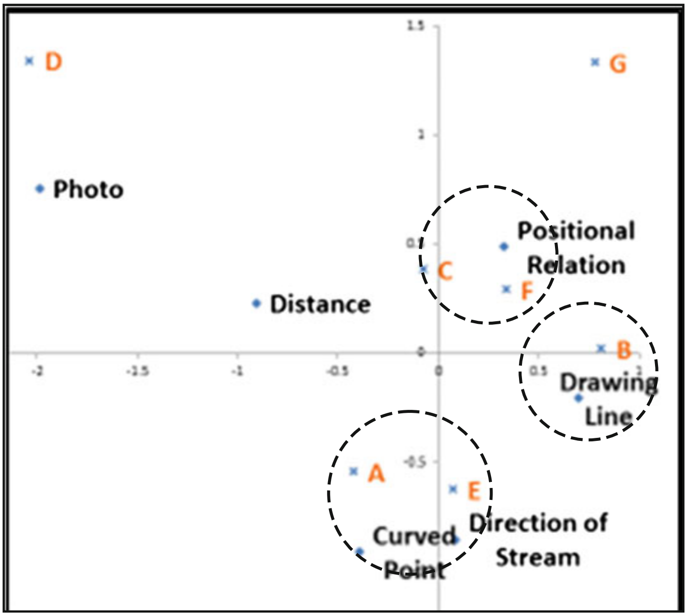


Fig. 17.8 Strong relations

Group C are buildings and the river in the photograph and the map except the curve. When they construct internal representation, the students use these viewpoints and make relationships among them. Some of these relationships are the positional relation of the buildings, right and left, and the front and back from the position on the river, but students do not include the river's direction. These results indicated that students have difficulty in paying attention to the relationships among objects even if they have the information about them. In short, level 2 viewpoints are not sufficient to construct an internal representation under the condition of isolated information.

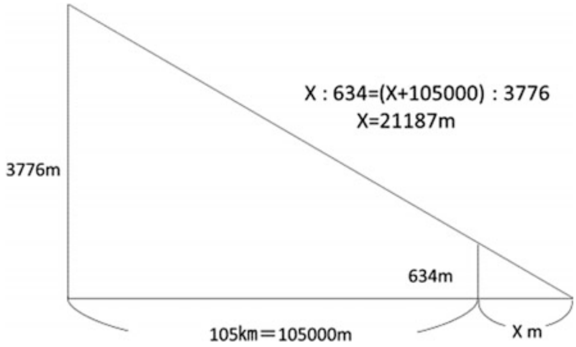
Task Y: This task does not require to find the place the photograph was taken exactly because seventh graders have not learned homothetic ratios. Task Y's purpose is to understand how students construct internal representation in the process of mathematizing through analyzing their descriptions. In order to solve Task Y, students needed to draw figures from the side like Fig. 17.9. Mathematizing process involves making a transformation from the photograph information to the mathematical figures in this task.

Figure 17.10 shows the position of answers on the map. The places students mark are classified in five groups: (1) mark near TOKYO SKYTREE^{SM} (41%), (2) mark far from TOKYO SKYTREE^{SM} (27%), (3) mark vaguely or write "around here" (17%), (4) use words in the answers (12%), (5) wrong answer (3%). Table 17.4 shows ten perspective cues found in the description. The students drawing the line or pictures were divided into three types according to from where

Table 17.3 The viewpoints in each group

	Photo			Map			Construction of internal representation			
	Buildings	River	Curve	Buildings	River	Curve	Positional relation (buildings)	Positional relation (river & buildings)	Direction of river	Direction of curve
A	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
B	✓	✓		✓	✓		✓	✓	✓	
C	✓	✓		✓	✓		✓	✓		
D	✓	✓		✓	✓					
E	✓	✓	✓	✓	✓	✓				
F	✓	✓		✓	✓		✓			
G	✓									

Fig. 17.9 Solution of Task Y



they look at, landmarks are standing on a line from the front of TOKYO SKYTREE[SM] (Straight line (front), Figure (front)), from the sky (Straight line (above), from the side (Figure (side)). They have other cues such as Size, Photograph information, Height of camera. The average of number of cues in each group are that Near (2.0), Far (2.2), Vague (1.2), Words (1.2), Wrong answer (0). It is clear that lack of cues make a decision vaguely (Fig. 17.11).

As previously mentioned, knowledge of homothetic ratio is needed to solve Task Y (Fig. 17.9). Before reaching this stage, students must construct internal and external representations according to the following steps: Step 1 is to recognize that the objects stand on a straight line and estimate the position of the camera should be to the right side of TOKYO SKYTREE[SM] and close to it. Step 2 is to think that the height of the camera should be on the line of sight connecting the top of Mt. Fuji and the top of TOKYO SKYTREE[SM]. Step 3 is to construct internal representations and external representations like figures from the side. Step 4 is to estimate the height of Mt. Fuji as six times as TOKYO SKYTREE[SM] in order to draw a figure

Fig. 17.10 Students' answer

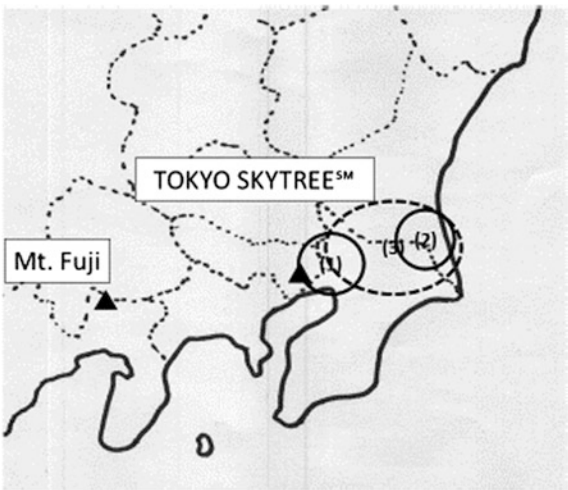


Table 17.4 Ten cues in Task Y

Ten cues	Straight line (front)	Straight line (above)	Size	Height of camera	Figure (front)	Figure (side)	Line of sight	Data	Calculation	Photo informaction
Near (n = 25)	9	11	6	5	2	2	2	4	4	4
Far (n = 16)	3	4	3	5	0	7	3	3	6	1
Vague (n = 10)	3	3	2	1	0	0	0	1	2	0
Words[a] (n = 7)	1	0	0	2	1	2	1	1	0	1
Wrong[b] (n = 2)	0	0	0	0	0	0	0	0	0	0

[a]Students describe the location by words
[b]The answers are not on the straight line

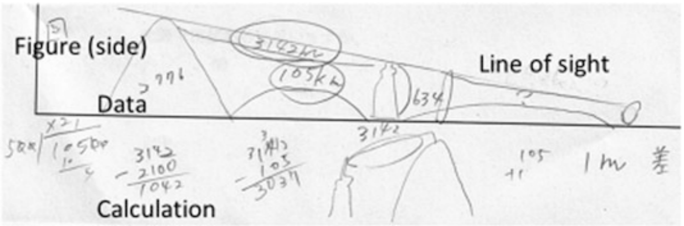


Fig. 17.11 Example of student's description

Table 17.5 Number of students in each step

	A (n = 5)	B (n = 7)	C (n = 26)	D (n = 7)	E (n = 6)	F (n = 5)	G (n = 4)	Total (n = 60)
Step 1	4	7	24	4	5	5	3	52
Step 2	2	3	5	1	0	1	1	13
Step 3	2	1	4	2	1	0	1	11
Step 4	0	0	0	0	0	0	0	0

like Fig. 17.9. In these steps drawing the line of sight is the key point in the mathematical process.

First of all, we would like to describe what kinds of difficulties seventh graders have in these procedures, from Step 1 to Step 4. After that, connecting with the results of Task X, it is shown that the difficulties in each group in Task X are related to the difficulties in the mathematization process in Task Y. Here is Table 17.5, which shows that 87% of seventh graders pass Step 1, however, in Step 2, there is only 22% of seventh graders paid attention to the height of camera with the line of sight. The implication is that realizing the line of sight is the most difficult in the key point of the mathematical process. In Step 3, it clearly appears that drawing a figure from the side is difficult, but the students who understand the positional relations between buildings and river could construct internal and external representation between Mt. Fuji and TOKYO SKYTREE[SM] from above and from the side (see Fig. 17.12). Ten out of thirteen students who described the line of sight belong to Group A, B, and C in Task X. To find the reason why students had difficulties in drawing figures, the focus shifts to the students who tried some cues. The students belonging to Group E had difficulties in drawing figure from the side (Fig. 17.13). They might have been bound to the photograph taken from the front. A student in Group A could build an internal representation among landmarks judging from the description, "the angle of camera is a little bit oblique," however she did not try to draw a figure included a line of sight (Fig. 17.14). Her case indicated that expressing external representations is difficult even if she has an internal representation.

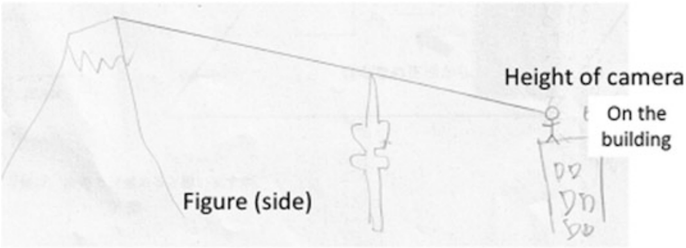


Fig. 17.12 Example of the height of camera (Group C)

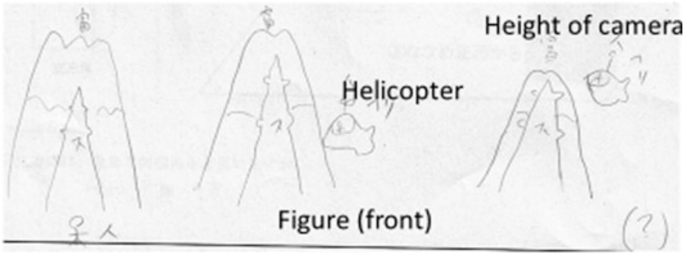


Fig. 17.13 Example of the height of camera (Group E)

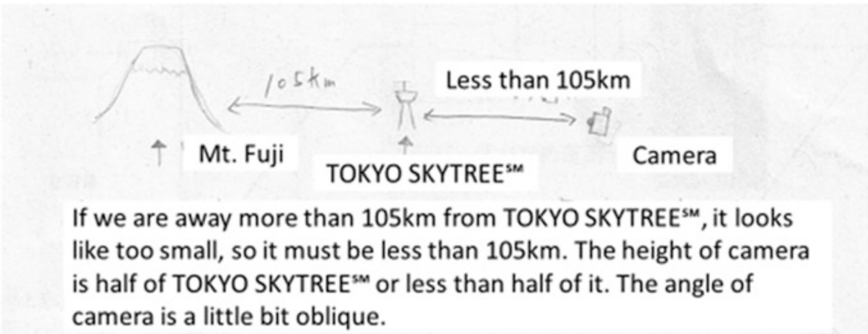


Fig. 17.14 Example of awareness of line of sight (Group A)

In summary, although it is important to draw the figure with the line of sight from a side in the mathematization process, the results of this analysis indicate some obstacles to the next step. The students in Groups D, E, F, and G who could not use the viewpoint of level 2 could not mathematize Task Y. Furthermore, even if the students have the internal representations using the viewpoint of level 2, they have the difficulty to express external representations. Additionally, persistence of the picture may have led to create obstacles in the mathematization process.

17.5 Conclusion

These results lead to the conclusion that there are different types of difficulties. In the case of Task X, the difficulties include the lack of information from the photograph (Group B), making a connection between the direction of the river and the position of buildings (Group C), making a connection among three objects (Groups D, E, F), and few specific cues (Group G). Besides considering the reference frame in the case of Groups D, E and F, there are other difficulties. These difficulties include the lack of relation back and front (Group D), the lack of distance to the buildings (Group E), and the lack of position on the river (Group F). In the actual problem solving situation, the difficulties are to find specific cues, to decide a standing point, and to make a connection among objects relating to their position and direction in the process of structuring the internal representation. Considering these difficulties in each group, it is significant to foster not only the viewpoints of relational position but also utilizing the information about the objects. In the case of Task Y, the difficulties are being aware of line of sight, constructing internal representation that is a figure from the side to include the line of sight, and drawing external representations. However, some of the students in Groups A, B, and C in Task X could recognize the line of sight and draw the figure from a side, enhancing the viewpoint of level 2, which is how objects are seen using cues in real world, the implication is that it is critically important to mathematize real world problems. To foster spatial thinking in mathematics education, two types of viewpoints of level 1 and level 2 need development. In relation to solving real world problems, level 2 viewpoints with utilizing information of real world and expressing internal representation in mathematical way such as drawing line of sight are the key ability in spatial thinking.

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