

# Chapter 13

## Continuous Time State Space Modelling with an Application to High-Frequency Road Traffic Data



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### 13.1 Introduction

We present a practical treatment of continuous time state space modelling. The main features of the analysis are highlighted and explored in some generality. We further present and discuss the main results of an empirical study related to road safety analysis. This application of the continuous time methodology in time series analysis shows how it can be used in practice.

A time series is a set of observations which are sequentially ordered over time. In a discrete time state space analysis, the time series observations are assumed to be equally spaced in time. Although missing data may give rise to different time gaps between available observations in discrete time series also, these time gaps are then always multiples of the time unit specific to the time series at hand (e.g. a year for annual data, a month for monthly data, etc.). In this chapter, on the other hand, we concentrate on the continuous time state space model and some of its special cases. In continuous state space models, the time gaps between

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consecutive observations of a series are typically allowed to vary freely from one pair of consecutive observations to the next. The expositions in this chapter rely mostly on the textbooks by Harvey (1989) and by Durbin and Koopman (2012). For an introduction to state space time series analysis, we refer to Commandeur and Koopman (2007).

The literature on continuous time modelling in statistics and econometrics is extensive. It is beyond the scope of this paper to present a full review of this literature. A major and key reference to continuous time models in econometrics is the review of Bergstrom (1984) where various results on parameter estimation for dynamic structural models in continuous time are provided. The many benefits of continuous time modelling are also illustrated. In the statistics literature, there is a considerable focus on smoothing methods that are formulated in continuous time. For example, a standard treatment using the continuous time approach is developed by Green and Silverman (1994). But also the earlier contributions of Wahba (1978) and Silverman (1985) have been of key importance in the development of signal extraction and spline smoothing in continuous time. The connections between spline smoothing and continuous time state space analysis are first established in the work of Wecker and Ansley (1983).

In this review chapter, we provide a detailed account of a continuous time state space approach to time series analysis. The outline of this chapter is as follows. In Sect. 13.2 we formulate the general continuous time state space model and discuss two well-known special cases. Section 13.3 discusses the estimation of the unobserved states together with the unknown model parameters. Finally, in Sect. 13.4 we apply the methodology to an empirical example consisting of road traffic speed data.

## 13.2 A Continuous Time Modelling Framework

Let  $t_\tau$  denote the time point at which observation  $\tau$  in the series was measured,  $\tau = 1, 2, \dots, T$ . Note that  $\tau$  is an integer denoting the *number* of the observation in the time series, while  $t_\tau$  is the *time* at which this observation was made. Thus, unlike  $\tau$ ,  $t_\tau$  can be any non-negative number, for example, 10 years, 200 days, 300.405 ms, etc. The only requirement is that  $t_1 < t_2 < t_3 < \dots < t_T$ . The general linear Gaussian state space model for the  $T$ -dimensional observation sequence  $y_1, \dots, y_T$  is given by

$$y_\tau = Z_\tau \alpha_\tau + \varepsilon_\tau, \quad \varepsilon_\tau \sim \text{NID}(0, H_\tau), \quad (13.1)$$

$$\alpha_{\tau+1} = T_\tau \alpha_\tau + R_\tau \eta_\tau, \quad \eta_\tau \sim \text{NID}(0, Q_\tau), \quad \tau = 1, \dots, T, \quad (13.2)$$

where  $\alpha_\tau$  is the state vector,  $\varepsilon_\tau$  and  $\eta_\tau$  are disturbance vectors and the system matrices  $Z_\tau$ ,  $T_\tau$ ,  $R_\tau$ ,  $H_\tau$  and  $Q_\tau$  are fixed and known. A selection of the elements of the system matrices may depend on an unknown parameter vector. Equation (13.1) is referred as the *observation* or *measurement equation*, while Eq. (13.2) is called the *state* or *transition equation*. The  $p \times 1$  observation vector  $y_\tau$  contains the  $p$  observations at time point  $t_\tau$ , and the  $m \times 1$  state vector  $\alpha_\tau$  is unobserved. The  $p \times 1$  irregular vector  $\varepsilon_\tau$  has zero mean and  $p \times p$  variance matrix  $H_\tau$ .

The  $p \times m$  matrix  $Z_\tau$  links the observation vector  $y_\tau$  with the unobservable state vector  $\alpha_\tau$  and may consist of regression variables. The  $m \times m$  transition matrix  $T_\tau$  in (13.2) determines the dynamic evolution of the state vector. The  $r \times 1$  disturbance vector  $\eta_\tau$  for the state vector update has zero mean and  $r \times r$  variance matrix  $Q_\tau$ . The observation and state disturbances  $\varepsilon_\tau$  and  $\eta_\tau$  are assumed to be serially independent and independent of each other at all time points. In many standard cases,  $r = m$  and matrix  $R_\tau$  is the identity matrix  $I_m$ . In other cases, matrix  $R_\tau$  is a  $m \times r$  selection matrix with  $r < m$ . Although matrix  $R_\tau$  can be specified freely, it is often composed of a selection from the first  $r$  columns of the identity matrix  $I_m$ . It further implies that often we can treat the matrix  $R_\tau$  as a constant matrix that does not vary with  $\tau$ . Similarly, all system matrices are assumed to be (deterministically) varying with  $\tau$ , but in many cases of practical interest, most system matrices are fixed for all  $\tau$ .

The initial state vector  $\alpha_1$  is assumed to be generated as

$$\alpha_1 \sim \text{NID}(a_1, P_1),$$

independently of the observation and state disturbances  $\varepsilon_\tau$  and  $\eta_\tau$ , where initial mean  $a_1$  and initial variance  $P_1$  can be treated as given and known in almost all stationary processes for the state vector. For nonstationary processes and regression effects in the state vector, the associated elements in the initial mean  $a_1$  can be treated as unknown and need to be estimated. For an extensive discussion of *initialisation* in state space analysis, we refer to Durbin and Koopman (2012, Chapter 5).

### 13.2.1 Local Level and Local Linear Trend Models

By appropriate choices of the vectors  $\alpha_\tau$ ,  $\varepsilon_\tau$  and  $\eta_\tau$ , and of the matrices  $Z_\tau$ ,  $T_\tau$ ,  $H_\tau$ ,  $R_\tau$  and  $Q_\tau$ , a wide range of different continuous time state space models can be derived from (13.1) and (13.2). Here we focus on the continuous time equivalents of the discrete local level and local linear trend models. Other model formulations can be considered as well since our state space framework allows for many different linear dynamic specifications that are commonly used in time series analysis. However, the arguments for continuous time formulations are similar, and therefore our treatment below remains relatively general.

Let  $\delta_\tau = t_\tau - t_{\tau-1}$  denote the amount of time elapsed between two consecutive observations  $\tau$  and  $\tau - 1$ . Also defining

$$\alpha_\tau = \mu_\tau, \quad \eta_\tau = \xi_\tau, \quad Z_\tau = T_\tau = R_\tau = 1, \quad H_\tau = \sigma_\varepsilon^2, \quad Q_\tau = \delta_\tau \sigma_\xi^2,$$

(all variables are scalars) for  $\tau = 1, \dots, T$ , model (13.1) and (13.2) reduces to the univariate continuous local level model as given by

$$\begin{aligned} y_\tau &= \mu_\tau + \varepsilon_\tau, & \varepsilon_\tau &\sim \text{NID}(0, \sigma_\varepsilon^2), \\ \mu_{\tau+1} &= \mu_\tau + \xi_\tau, & \xi_\tau &\sim \text{NID}(0, \delta_\tau \sigma_\xi^2), \end{aligned} \quad (13.3)$$

for  $\tau = 1, \dots, T$ . Note that (13.3) reduces to the discrete local level model when the observations are equally spaced, i.e. when  $\delta_\tau = t_\tau - t_{\tau-1} = 1$ , say, for all  $\tau = 1, \dots, T$ .

The local level model can be regarded as the most basic version of a state space model. It is intuitive as it can be interpreted as a model representation for  $y_\tau$  that is generated by the normal distribution with a time-varying mean  $\mu_\tau$  and a fixed variance  $\sigma_\varepsilon^2$ . The continuous time formulation only applies to the dynamic process of the time-varying mean. The local level model also provides a statistical specification for the exponentially weighted moving average (EWMA) forecasting method that is very popular amongst professional practitioners. The forecast function of the local level model is equivalent to the EWMA, but the state space treatment also provides statistical standard errors to the point forecasts; see the discussion below. A full discussion and treatment of the local level model is provided by Harvey (1989) and Durbin and Koopman (2012, Chapter 2).

By defining

$$\begin{aligned} \alpha_\tau &= \begin{pmatrix} \mu_\tau \\ \nu_\tau \end{pmatrix}, \quad \eta_\tau = \begin{pmatrix} \xi_\tau \\ \zeta_\tau \end{pmatrix}, \quad T_\tau = \begin{bmatrix} 1 & \delta_\tau \\ 0 & 1 \end{bmatrix}, \quad Z_\tau = (1 \ 0), \\ H_\tau &= \sigma_\varepsilon^2, \quad \text{Var}(\eta_\tau) = Q_\tau = \delta_\tau \begin{bmatrix} \sigma_\xi^2 + \frac{1}{3}\delta_\tau^2\sigma_\zeta^2 & \frac{1}{2}\delta_\tau\sigma_\zeta^2 \\ \frac{1}{2}\delta_\tau\sigma_\zeta^2 & \sigma_\zeta^2 \end{bmatrix}, \quad \text{and} \quad R_\tau = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

the scalar notation of (13.1) and (13.2) leads to

$$\begin{aligned} y_\tau &= \mu_\tau + \varepsilon_\tau, & \varepsilon_\tau &\sim \text{NID}(0, \sigma_\varepsilon^2), \\ \mu_{\tau+1} &= \mu_\tau + \delta_\tau \nu_\tau + \xi_\tau, \\ \nu_{\tau+1} &= \nu_\tau + \zeta_\tau, \end{aligned} \quad (13.4)$$

for  $\tau = 1, \dots, T$ , and we obtain the univariate continuous local linear trend model. Unlike in the discrete local linear trend model, we see that the disturbances of the level and the slope component are correlated through the off-diagonal elements  $\frac{1}{2}\delta_\tau^2\sigma_\zeta^2$  in matrix  $Q_\tau$  in the continuous local linear trend model. However, as

mentioned by Harvey (1989, p. 487), “this difference is unlikely to be of any great importance”.

The treatment above for the local linear trend model has many connections with the statistical literature on spline smoothing. Reviews of methods related to spline smoothing are given in Silverman (1985), Wahba (1990) and Green and Silverman (1994, Chapter 2). Some of these connections with the approach given above and the more traditional methods are given by Wahba (1990) but are also discussed in Wecker and Ansley (1983). These connections are also highlighted in Durbin and Koopman (2012, Chapter 3).

### 13.2.2 Multivariate Continuous Time State Space Models

The treatments as set out for univariate time series above can be easily extended to multivariate time series. This is one of the advantages of the state space approach since multivariate spline smoothing methods are not widespread.

In case we let  $y_\tau$  denote a  $p \times 1$  vector of observations, a multivariate local linear trend model can be applied to the  $p$  time series simultaneously:

$$\begin{aligned} y_\tau &= \mu_\tau + \varepsilon_\tau, & \varepsilon_\tau &\sim \text{NID}(0, \Sigma_\varepsilon), \\ \mu_{\tau+1} &= \mu_\tau + \eta_\tau, & \eta_\tau &\sim \text{NID}(0, \Sigma_\eta), \end{aligned} \tag{13.5}$$

for  $\tau = 1, \dots, T$ , where  $\mu_\tau$ ,  $\varepsilon_\tau$ , and  $\eta_\tau$  are  $p \times 1$  vectors,  $\Sigma_\varepsilon$  is a  $p \times p$  variance matrix, and

$$\Sigma_\eta = \delta_\tau \begin{bmatrix} \Sigma_\xi + \frac{1}{3}\delta_\tau^2 \Sigma_\zeta & \frac{1}{2}\delta_\tau \Sigma_\zeta \\ \frac{1}{2}\delta_\tau \Sigma_\zeta & \Sigma_\zeta \end{bmatrix}$$

is a  $2p \times 2p$  matrix,  $\Sigma_\xi$  and  $\Sigma_\zeta$  being the  $p \times p$  variance matrices of the level and the slope disturbances, respectively.

### 13.3 State Space Methods for Continuous Time Models

The model formulations as discussed above are all special cases of the general linear Gaussian state space model. We can therefore rely on the associated methods for signal extraction, parameter estimation and forecasting. The most important and well-known method for this class of state space models is the Kalman filter that allows the (predictive and filtered) estimation of the unobserved state vector  $\alpha_\tau$  when the system matrices have given values. It also enables the computation of the log-likelihood function of the model, for a given parameter vector, via the prediction error decomposition. It allows the maximisation of the log-likelihood function with

respect to the parameter vector, in order to obtain its maximum likelihood estimate. On the basis of these parameter estimates, signal extraction and forecasting can take place. We next provide more details of this central part of the state space methodology.

In a similar way as in discrete state space models, in continuous state space models for given values of all system matrices—and for known initial conditions  $a_1$  and  $P_1$ —the state vector can be estimated in three different ways, yielding what are known as the *filtered*, the *predicted* and the *smoothed* state vector. Depending on the types of state estimates required in the analysis, the estimates of the state vector can be obtained by performing one or two passes through the observed time series:

1. A *forward* pass, from  $\tau = 1, \dots, T$ , using a recursive algorithm known as the *Kalman filter* enables the computation of filtered and predicted states and prediction errors, including their variances; from the prediction error and their variances, we can compute the log-likelihood function of the given continuous state space model;
2. A *backward* pass, from  $\tau = T, \dots, 1$ , using all filtered and associated variables from the Kalman filter and using recursive algorithms known as *state and disturbance smoothers* enables the computation of smoothed estimates of states and disturbances; it requires the storage of the Kalman filter variables.

In continuous time state space models, the standard Kalman, state and disturbance smoothing filters can be used; see Durbin and Koopman (2012, Chapter 4) for technical details. A specific difference of substance between discrete and continuous time models is that the variance matrix  $Q_\tau$  in Eq. (13.2) of the state space formulation of the model (containing the variances of the state disturbances) is typically time-invariant in the discrete case while it becomes a time-varying matrix for continuous time state space models.

We have discussed these continuous time state space methods above as if the disturbance variances are given and known. In practice, of course, these parameters are unknown, and they have to be estimated. Just as in the discrete time series situation, the parameter estimates are obtained via maximum likelihood methods which are discussed in Durbin and Koopman (2012, Chapter 7). It requires an optimisation algorithm, and for this purpose quasi-Newton methods are typically used. Each time new parameter values are proposed by the search-for-the-maximum algorithm, the Kalman filter is used to compute the log-likelihood function. In many applications, it is found that the maximum is found quickly and the estimation process does not take much computing time.

## 13.4 An Application in Road Safety and Traffic Control

We consider our continuous time modelling approach to a full day of measurements of the speed of passing motor vehicles on a fixed location in the right lane of a Dutch motorway, starting at midnight and ending at midnight of the following day. For our analysis of this interesting and important time series for road safety studies, we have considered the continuous time models and methods as set out in the previous sections. All computations are implemented in the `OxMetrics` object-oriented programming environment of Doornik (2013) together with the `SsfPack` library of state space routines of Koopman et al. (2008). Initial analyses are carried out by means of the discrete time versions of our models using the `STAMP` software of Koopman et al. (2007).<sup>1</sup>

There is a total of 25,539 passages in this series meaning that we also have 25,539 observations. The time of each passage is measured as the number of milliseconds elapsed since the start of the measurements and the difference between the time of the last and the first observation of the series, i.e.  $t_T - t_1$ , is 86,396 ms which indeed corresponds to a full  $86,396/60^2 = 24$  h. The average time lapse between consecutive observations in the series is 3.383 ms with a minimum of 0.038 ms and a maximum of 1417.4 ms. The variance of the time lapses  $\delta_\tau$  is 230.170.

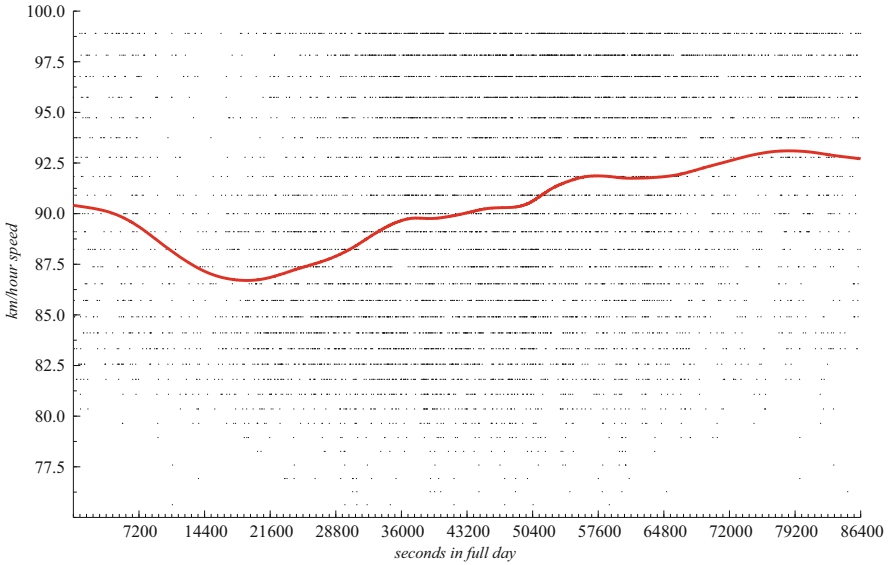
From the perspective of road safety, it is of interest to analyse passages of cars at different speed levels. In our analyses, we consider two groups of speed levels: slow passages with a speed of less than 100 km/h (but faster than 75 km/h as we discard very slow passages which may be due to measurement failings) and fast passages with a speed of higher than 120 km/h. These two different groups constitute a total of 14,435 passages (9010 slow and 5425 fast passages).

The analyses of the two series are based on the continuous time local linear trend model (13.4). To enforce a smoother evolving signal in this highly noisy time series of speed passages, we restrict the variance of the level component to be zero. The remaining variances are estimated by the method of maximum likelihood (ML). We yield the following estimation results. At convergence of the ML process, the parameter estimates for the variance of the slope disturbances and the measurement errors are, respectively, given by  $\sigma_\zeta^2 = 1.061 \times 10^{-6}$  and  $\sigma_\varepsilon^2 = 31.966$  for the slow passages and  $\sigma_\zeta^2 = 5.434 \times 10^{-6}$  and  $\sigma_\varepsilon^2 = 43.295$  for the fast passages.

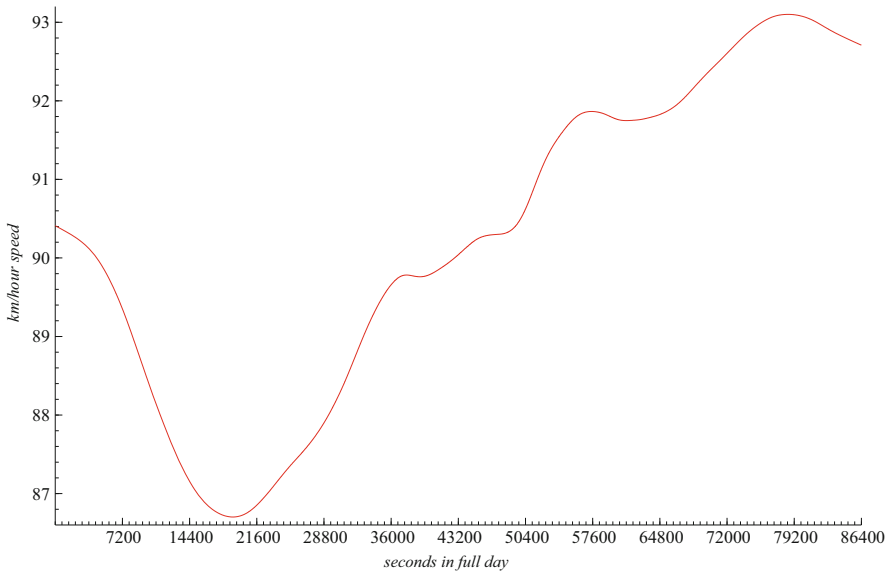
The recorded speed levels of the passages for the slow and the fast groups are presented in Figs. 13.1 and 13.3, respectively, together with their estimated trend components which are also presented separately in Figs. 13.2 and 13.4, respectively. We learn from these graphs that the number of passages of motor vehicles on the motorway diminishes during the night. It is especially observable for the fast-speed passages, between roughly 8000 ms after midnight (i.e. around half past three in the

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<sup>1</sup>The programming code of our analyses is available as supplementary material at the book website <http://www.springer.com/us/book/9783319772189>.

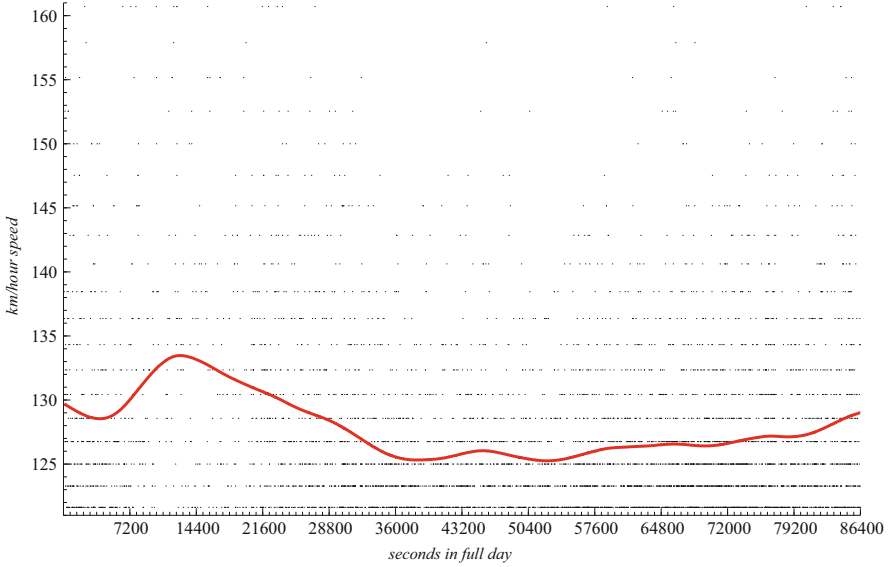


**Fig. 13.1 Slow-speed passages:** speed measures (in km/hour) of slow passages of motor vehicles during a full day on a fixed location in the right lane of a Dutch motorway (in tiny dots) together with the smoothed estimated trend component (solid line) from the continuous time local linear trend model. The horizontal  $x$ -axis represents the time index measured in seconds of a full day starting at midnight 0:00 h

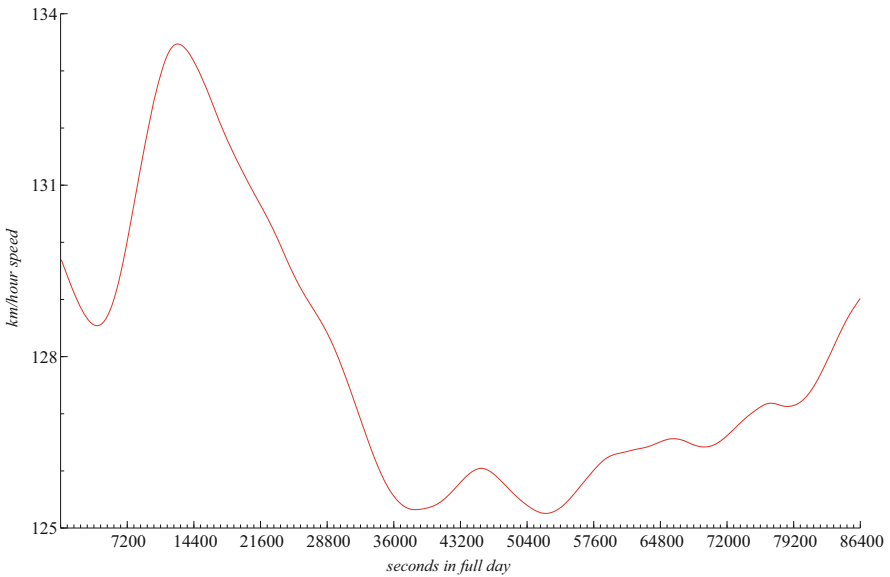


**Fig. 13.2 Slow-speed trend:** the smoothed estimated trend component from the continuous time local linear trend model. The horizontal  $x$ -axis represents the time index measured in seconds of a full day starting at midnight 0:00 h





**Fig. 13.3 Fast-speed passages:** speed measures (in km/hour) of fast passages of motor vehicles during a full day on a fixed location in the right lane of a Dutch motorway (in tiny dots) together with the smoothed estimated trend component (solid line) from the continuous time local linear trend model. The horizontal  $x$ -axis represents the time index measured in seconds of a full day starting at midnight 0:00 h



**Fig. 13.4 Fast-speed trend:** the smoothed estimated trend component from the continuous time local linear trend model. The horizontal  $x$ -axis represents the time index measured in seconds of a full day starting at midnight 0:00 h

morning) and 20,000 ms after midnight (i.e. around six o'clock in the morning), that the number of passages is clearly much smaller. At the same time, we can conclude that the speed of the fast group increases somewhat as it is quiet on the motorway during these night hours. In contrast, somewhat later in the night and up to the early morning hours, the speed of the slow group diminishes to clearly lower speed levels which is possibly due to a relatively intensified presence of more heavy trucks that generally drive slower and on the right lane of the road. This possible explanation can be investigated in more detail since our data set has information on *following distances* between two passing vehicles. In future research we plan to formally test such hypotheses by using statistical procedures based on the continuous time modelling framework developed in this chapter (Figs. 13.1 and 13.3).

We have shown in our current analysis that the unequal time lapses between consecutive vehicles can be handled effectively using our continuous time trend model. In this particular application that is highly relevant for road safety studies, there are several directions in which the results of our analysis can be improved. Diagnostic tests on the one-step ahead prediction errors indicate that neither the assumption of independence nor the assumption of normality of the residuals is satisfied: the Box-Ljung statistic for independence has values for  $Q(10)$  that are too high; their values should be smaller than 16.95 in order to be non-significant at the usual 5% level. Also, the values for the Bowman-Shenton test for normality are too high; their values should be smaller than 5.99 to be non-significant.

Although our reported initial findings are highly interesting, the continuous time trend model appears to be somewhat away from a correct model specification for the analysis of traffic speed data. Further research can be conducted in order to obtain a more satisfactory model that is capable of capturing the remaining autocorrelation and non-normality of the data. However, this research falls outside the scope of our current review on continuous time state space modelling.

## 13.5 Conclusions

We have discussed the basic principles of a model-based continuous time approach using the state space methodology. The methodology is especially designed for the analysis of irregularly spaced data. We have highlighted the potential of this approach in an illustration of high-frequency intra-daily time series of speed measures from vehicles that pass a certain point at a motorway.

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