Chapter 9
Coherent Spaces

A. Vourdas

Abstract Coherent spaces spanned by a finite number of coherent states are studied. They have properties analogous to coherent states (resolution of the identity, closure under displacement transformations, closure under time evolution transformations, etc.). The set of all coherent spaces is a distributive lattice and also a Boolean ring (Stone's formalism). The work provides the theoretical foundation, for the description of quantum devices that operate with coherent states and their superpositions.

9.1 Introduction

Coherent states [1–3] play an important role in quantum mechanics, quantum optics and quantum information. In a recent paper [4] we introduced the concept of coherent spaces and the corresponding coherent projectors. They are finite-dimensional subspaces of the Hilbert space, spanned by a finite number of coherent states. They have properties analogous to those of coherent states (compare Propositions 9.1 and 9.2 for coherent states below, with Propositions 9.3 and 9.4 for coherent spaces and the corresponding coherent projectors). The set of all coherent spaces $\mathcal{L}_{\rm coh}$ is a distributive lattice, and is a sublattice of the Birkhoff-von Neumann lattice [5–7] $\mathcal{L}$ of closed subspaces of the harmonic oscillator Hilbert space (or more generally a separable Hilbert space) which is not distributive.

Stone [8–10] has shown that there are deep links between distributive lattices, and some idempotent rings known as Boolean rings [11, 12]. In [4], we used this general result in the present context, to describe the distributive lattice $\mathcal{L}_{\rm coh}$ of coherent spaces, as a Boolean ring. This provides theoretical computer science foundation, for the description of quantum gates operating with coherent states. In particular we studied in detail CNOT gates with coherent states (previous work [13, 14] studied CNOT gates with coherent states which are far from each other, and used the approximation that they are orthogonal).

A. Vourdas (✉)
University of Bradford, Bradford BD7 1DP, UK
e-mail: a.vourdas@bradford.ac.uk

J.-P. Antoine et al. (eds.), *Coherent States and Their Applications*, Springer Proceedings in Physics 205, https://doi.org/10.1007/978-3-319-76732-1_9

In the present work we review this formalism, with different emphasis. In particular:

- We explain that the theory of rings introduces a structure similar to the standard arithmetic for coherent spaces. This can be used in the study of complex circuits with CNOT gates and other devices operating with coherent states.
- We study the statistical properties of density matrices which are coherent projectors divided by their trace (every projector divided by its trace, is a density matrix). This can motivate experimental realization of these density matrices.

In Sect. 9.2 we introduce briefly coherent states, in order to define the notation. In Sect. 9.3 we introduce the Birkhoff-von Neumann lattice $\mathcal{L}$, with emphasis on its non-distributivity. In Sect. 9.4 we define coherent spaces and study their properties. In Sect. 9.5 we study the statistical properties of density matrices related to coherent projectors. In Sect. 9.6 we study the set $\mathbb{C}_{\text{fin}}$ of finite sets of complex numbers, as a distributive lattice, which is isomorphic to the lattice $\mathcal{L}_{\text{coh}}$ of coherent spaces, studied in Sect. 9.7. This means that there is a bijective map g between the lattices $\mathbb{C}_{\text{fin}}$ and $\mathcal{L}_{\text{coh}}$, which preserves the lattice structure (i.e., $g(a) \vee g(b) = g(a \vee b)$ and $g(a) \wedge g(b) = g(a \wedge b)$). We conclude in Sect. 9.8, with a discussion of our results.

9.2 Coherent States

Let $\mathcal{H}$ be the harmonic oscillator Hilbert space. Also, let

$$a = \frac{x + ip}{\sqrt{2}}; \quad a^\dagger = \frac{x - ip}{\sqrt{2}} \tag{9.1}$$

be the annihilation and creation operators, and $D(A)$ the displacement operators

$$D(A) = \exp(Aa^\dagger - A^* a); \qquad A \in \mathbb{C}. \tag{9.2}$$

Coherent states [1–3] are defined as

$$|A\rangle = D(A)|0\rangle = \exp\left(-\frac{|A|^2}{2}\right) \sum_{N=0}^{\infty} \frac{A^N}{\sqrt{N!}} |N\rangle; \qquad a|A\rangle = A|A\rangle \tag{9.3}$$

where $|N\rangle$ are number eigenstates. Let $\varPi(A)$ be the corresponding 'coherent projector':

$$\varPi(A) = |A\rangle\langle A|. \tag{9.4}$$

The following proposition (which is well known and we give without proof), summarizes three important properties of coherent states and coherent projectors:

Proposition 9.1 *(1) Resolution of the identity:*

$$\int_{\mathbb{C}} \frac{d^2A}{\pi} \Pi(A) = \mathbf{1}. \tag{9.5}$$

(2) Closure under displacement transformations:

$$D(z)\Pi(A)[D(z)]^{\dagger} = \Pi(A+z). \tag{9.6}$$

(3) Closure under time evolution:

$$\exp(ita^{\dagger}a)\Pi(A)\exp(-ita^{\dagger}a) = \Pi[A\exp(it)]. \tag{9.7}$$

Let

$$|f\rangle = \sum_N f_N|N\rangle; \quad \sum_N |f_N|^2 = 1 \tag{9.8}$$

be an arbitrary state. Then

$$f(z) = \exp\left(\frac{|z|^2}{2}\right)\langle z^*|f\rangle = \sum_N \frac{f_N z^N}{\sqrt{N!}} \tag{9.9}$$

is an analytic function in the complex plane, called Bargmann function. For example, the coherent state $|A\rangle$ is represented with the function $\exp\left(Az - \frac{1}{2}|A|^2\right)$.

If the state $|f\rangle$ is orthogonal to the coherent state $|A\rangle$, then A^* is a zero of the function $f(z)$ (i.e., $f(A^*) = 0$).

A finite number of coherent states, are linearly independent. To show this, we consider the coherent states $|A_1\rangle$, ..., $|A_n\rangle$ and the relation

$$\lambda_1|A_1\rangle + \cdots + \lambda_n|A_n\rangle = 0. \tag{9.10}$$

From this follows that

$$\sum_{j=1}^{n} \lambda_j \exp\left(-\frac{1}{2}|A_j|^2\right)\frac{A_j^N}{\sqrt{N!}} = 0. \tag{9.11}$$

Here we have an infinite number of equations with a finite number of unknowns, and the only solution is $\lambda_1 = \cdots = \lambda_i = 0$.

Definition 9.1 A set of states $\{|s_i\rangle\}$ is called total, if there is no state in $\mathcal{H}$, which is orthogonal to all $|s_i\rangle$.

Definition 9.2 Let $f(z)$ be an entire function, and $M(R)$ the maximum value of $|f(z)|$ on the circle $|z| = R$. The growth of $f(z)$ is described by the order ρ and the type σ

$$\rho = \limsup_{R\to\infty} \frac{\ln\ln M(R)}{\ln R}; \quad \sigma = \lim_{R\to\infty} \frac{M(R)}{R^\rho}. \tag{9.12}$$

This means that for large R, we get $|f(z)| \approx \exp(\sigma R^\rho)$.

Definition 9.3 Consider a sequence of complex numbers $A_1, A_2, \ldots$ such that

$$\lim_{N\to\infty} |A_N| = \infty \tag{9.13}$$

Let $n(R)$ be the number of terms of this sequence within the circle $|A| < R$. The density of this sequence is described with the numbers

$$\eta = \limsup_{R\to\infty} \frac{\ln n(R)}{\ln R}; \quad \delta = \lim_{R\to\infty} \frac{n(R)}{R^\eta} \tag{9.14}$$

The number of terms of this sequence in a large circle with radius R is $n(R) \approx \delta R^\eta$.

We say that the density (η, δ) of a sequence is greater than (η_1, δ_1) if $\eta > \eta_1$ and also if $\eta = \eta_1$ and $\delta > \delta_1$ (lexicographic order).

The resolution of the identity shows that the set of all coherent states

$$\Sigma_{\text{coh}} = \{|A\rangle \mid A \in \mathbb{C}\}, \tag{9.15}$$

is a total set. But there are many subsets of Σ_{coh} which are also total sets, as shown in the following proposition.

Proposition 9.2 *(1) A subset of Σ_{coh} which is uncountably infinite, is a total set of coherent states.*

(2) Let $A_1, A_2, \ldots$ be a sequence of complex numbers.

- *(a) If the sequence A_n converges to some point A, then the countably infinite set of coherent states $\{|A_1\rangle, |A_2\rangle, \ldots\}$ is a total set.*
- *(b) If the sequence $|A_n|$ diverges, and its density is greater than $(2, 1)$, then the countably infinite set of coherent states $\{|A_1\rangle, |A_2\rangle, \ldots\}$ is a total set.*

Proof The proof of the parts (1) and (2a) of the proposition, is based on the fact that the zeros of analytic functions (in our case of Bargmann functions), are isolated from each other. The proof of the part (2b), is based on the relationship between the growth of Bargmann functions and the density of their zeros. We refer to the literature for the details of these proofs (e.g., [15, 16] and references therein).

There are other well known properties of coherent states (e.g., related to the uncertainty relation). In this paper, the term coherence refers to the properties in the two Propositions 9.1 and 9.2 above, and we will study other more general structures which are coherent in the sense that they obey these properties.

9.3 The Birkhoff-von Neumann Lattice $\mathcal{L}$

We consider the set $\mathcal{L}$ of closed subspaces of the harmonic oscillator Hilbert space $\mathcal{H}$ (or more generally a separable Hilbert space). The zero-dimensional subspace that contains only the zero vector is an element of $\mathcal{L}$ which we denote as $\mathcal{O}$. The space $\mathcal{H}$ is also an element of $\mathcal{L}$.

For any elements h_1, h_2 of $\mathcal{L}$, we define the following operations:

- The conjunction or logical AND operation, denoted with $\wedge$:

$$h_1 \wedge h_2 = h_1 \cap h_2. \tag{9.16}$$

- The disjunction or logical OR operation, denoted with $\vee$:

$$h_1 \vee h_2 = \overline{\text{span}}[h_1 \cup h_2]. \tag{9.17}$$

  This is the closed subspace that contains all superpositions of vectors in the subspaces h_1, h_2. The overline indicates closure.
- The negation or logical NOT operation, denoted with $\neg$:

$$\neg h = h^{\perp}. \tag{9.18}$$

  $h^{\perp}$ is the orthocomplement of h, i.e., an orthogonal space to h such that $h \vee h^{\perp} = \mathcal{H}$.

The set $\mathcal{L}$ with these operations is the Birkhoff-von Neumann orthomodular lattice, that describes the logic of quantum mechanics [5–7].

We note that:

- The partial order in this lattice is 'subspace'. We use the notation $h_1 \prec h_2$ to indicate that h_1 is a subspace of h_2.
- Let $\Pi(h)$ be the projector to the subspace h. Then

$$\Pi(h_1)\Pi(h_2) = 0 \;\rightarrow\; h_1 \wedge h_2 = 0. \tag{9.19}$$

  The converse is not true in general. However, if the $\Pi(h_1), \Pi(h_2)$ commute, then the $h_1 \wedge h_2 = 0$ implies that $\Pi(h_1)\Pi(h_2) = 0$.
- In any lattice the following distributivity inequalities hold:

$$\begin{aligned} &(h_1 \wedge h_2) \vee h_0 \prec (h_1 \vee h_0) \wedge (h_2 \vee h_0) \\ &(h_1 \vee h_2) \wedge h_0 \succ (h_1 \wedge h_0) \vee (h_2 \wedge h_0) \end{aligned} \tag{9.20}$$

  They become equalities in distributive lattices. $\mathcal{L}$ is **not** a distributive lattice. The following projectors can detect deviations from distributivity [17]:

$$\mathfrak{P}_1 = \Pi[(h_1 \vee h_0) \wedge (h_2 \vee h_0)] - \Pi[(h_1 \wedge h_2) \vee h_0]$$
$$\mathfrak{P}_2 = \Pi[(h_1 \vee h_2) \wedge h_0] - \Pi[(h_1 \wedge h_0) \vee (h_2 \wedge h_0)] \tag{9.21}$$

Measurements with these projectors which give a non-zero result, prove the non-distributive nature of the lattice $\mathcal{L}$.

9.4 Coherent Spaces

9.4.1 Coherent Projectors of Rank n

We define coherent subspaces of $\mathcal{H}$, and denote them with upper case H, in order to distinguish them from general closed subspaces, which we denoted with lower case h. $H(A_1)$ is the one dimensional coherent subspace that contains the coherent state $|A_1\rangle$.

Definition 9.4 Let $\mathcal{A} = \{A_1, ..., A_n\}$ be a **finite** set of complex numbers. The coherent subspace of $\mathcal{H}$ denoted as $H(\mathcal{A})$, is the n-dimensional subspace

$$H(\mathcal{A}) = H(A_1, ..., A_n) = H(A_1) \vee ... \vee H(A_n), \tag{9.22}$$

and contains all the superpositions $\lambda_1|A_1\rangle + \cdots + \lambda_n|A_n\rangle$ (which as we explained earlier are linearly independent). If $\mathcal{A} = \emptyset$, the $H(\emptyset) = \mathcal{O}$.

In the Bargmann representation, $H(\mathcal{A})$ contains the functions

$$H(\mathcal{A}) = \left\{ f(z) = \sum \lambda_i \exp(A_i z) | A_i \in \mathcal{A}; \quad \lambda_i \in \mathbb{C} \right\}, \tag{9.23}$$

where the sum is finite. We note that the growth of all these functions has order 1. A function with different order of growth, does not belong in any of the coherent $H(\mathcal{A})$. Examples are, the number states which have order of growth 0, and the squeezed states which have order of growth 2.

We call $\Pi(\mathcal{A}) = \Pi(A_1, ..., A_i)$ the projector to the space $H(\mathcal{A}) = H(A_1, ..., A_i)$, and

$$\Pi^{\perp}(\mathcal{A}) = \Pi^{\perp}(A_1, ..., A_i) = \mathbf{1} - \Pi(A_1, ..., A_i). \tag{9.24}$$

For practical calculations, we can find the $\Pi(A_1, ..., A_i)$ inductively using the Gram-Schmidt orthogonalization algorithm:

$$\Pi(A_1, ..., A_i) = \Pi(A_1, ..., A_{i-1}) + \varpi(A_i|A_1, ..., A_{i-1})$$
$$\varpi(A_i|A_1, ..., A_{i-1}) = \frac{\Pi^{\perp}(A_1, ..., A_{i-1})\Pi(A_i)\Pi^{\perp}(A_1, ..., A_{i-1})}{\mathrm{Tr}[\Pi^{\perp}(A_1, ..., A_{i-1})\Pi(A_i)]} \tag{9.25}$$

The linear independence of a finite number of coherent states ensures that the denominator is different from zero. For example,

$$\Pi(A_1, A_2) = \Pi(A_1) + \frac{\Pi^{\perp}(A_1)\Pi(A_2)\Pi^{\perp}(A_1)}{\mathrm{Tr}[\Pi^{\perp}(A_1)\Pi(A_2)]} \tag{9.26}$$

The following proposition generalizes Proposition 9.1 to coherent projectors:

Proposition 9.3 *(1) Resolution of the identity:*

$$\int_{\mathbb{C}} \frac{d^2A}{n\pi}\Pi(A, A + d_2, ..., A + d_n) = \mathbf{1}. \tag{9.27}$$

Here the $d_2, ..., d_n$ are fixed complex numbers.

(2) Closure under displacement transformations:

$$D(z)\Pi(A_1, ..., A_n)[D(z)]^{\dagger} = \Pi(A_1 + z, ..., A_n + z). \tag{9.28}$$

(3) Closure under time evolution:

$$\exp(ita^{\dagger}a)\Pi(A_1, ..., A_n)\exp(-ita^{\dagger}a) = \Pi[A_1\exp(it), ..., A_n\exp(it)]. \tag{9.29}$$

Proof The proof has been given in [4].

A state $|s\rangle$ is orthogonal to the coherent subspace $H(A_1, ..., A_i)$ (i.e., $\Pi(A_1, ..., A_i)|s\rangle = 0$), if and only if the $A_1^*, ..., A_i^*$ are zeros of its Bargmann function $s(z)$ (i.e., $s(A_j^*) = 0$ for $j = 1, ..., i$).

Definition 9.5 A set of subspaces $\{h_i\}$ is called total, if there is no state in $\mathcal{L}$, which is orthogonal to all h_i.

The following proposition generalizes Proposition 9.2, to coherent projectors:

Proposition 9.4 *(1) A set of coherent subspaces which is uncountably infinite, is a total set.*

(2) Let $\mathcal{A}_i = \{A_{i1}, ..., A_{ik_i}\}$ with $i = 1, 2, ...$, be a countable collection of finite sets of complex numbers. Using a lexicographic order, we relabel the A_{ij} as A_n.

(a) If the sequence A_n converges to some point A, then the countably infinite set of coherent spaces $H(\mathcal{A}_i)$ is a total set.

(b) If the sequence $|A_n|$ diverges and it has density greater than (2, 1) *then the countably infinite set of coherent spaces $H(\mathcal{A}_i)$ is a total set.*

Proof The proof has been given in [4].

Example 9.1 The uncountably infinite set of coherent spaces

$$\Sigma_{\mathrm{coh}}^{(n)} = \{H(A, A + d_2, ..., A + d_n) \mid A \in \mathbb{C}\}, \tag{9.30}$$

is a total set. Here the $d_2, ..., d_n$ are fixed complex numbers.

Proposition 9.5 *Let $\mathcal{A} = \{A_1, ..., A_n\}$ be a finite set of complex numbers, and $g(\mathcal{A})$ the $n \times n$ Hermitian matrix of rank n, with elements*

$$g_{ij}(\mathcal{A}) = \langle A_i|A_j\rangle = \exp\left(A_i^* A_j - \frac{1}{2}|A_i|^2 - \frac{1}{2}|A_j|^2\right). \tag{9.31}$$

Also let $G = g^{-1}$ be its inverse matrix (it exists because the coherent states are linearly independent). Then

$$\Pi(\mathcal{A}) = \sum_{i,j} G_{ij}(\mathcal{A})|A_i\rangle\langle A_j|. \tag{9.32}$$

Proof The proof has been given in [4].

The diagonal elements of g are equal to 1. In the limit $\min_{ki,j}(|A_i - A_j|) \to \infty$ the off-diagonal elements of the matrix g become zero and $g \to \mathbf{1}$ (the set of the n coherent states becomes 'almost orthonormal').

9.5 Physical Applications

9.5.1 The Density Matrix $\frac{1}{n}\Pi(A_1, ..., A_n)$

The operator

$$R(A_1, ..., A_n) = \frac{1}{n}\Pi(A_1, ..., A_n) \tag{9.33}$$

is a density matrix. Using its eigenvectors as a basis, this density matrix becomes

$$R(A_1, ..., A_n) = \frac{1}{n}\begin{pmatrix} \mathbf{1}_{n,n} & 0_{n,\infty} \\ 0_{\infty,n} & 0_{\infty,\infty} \end{pmatrix} \tag{9.34}$$

where the notation is self-explanatory. R represents a mixed state with

$$\mathrm{Tr}\{[R(A_1, ..., A_n)]^2\} = \frac{1}{n}, \tag{9.35}$$

and von Neumann entropy

$$-\mathrm{Tr}[R(A_1, ..., A_n)\log R(A_1, ..., A_n)] = \log n. \tag{9.36}$$

We calculated the average position and momentum

$$\langle x\rangle = \mathrm{Tr}[xR(A_1, ..., A_n)] = \frac{\sqrt{2}}{n}\mathfrak{Re}\left(\sum A_i\right)$$
$$\langle p\rangle = \mathrm{Tr}[pR(A_1, ..., A_n)] = \frac{\sqrt{2}}{n}\mathfrak{Im}\left(\sum A_i\right). \tag{9.37}$$

We also calculated the uncertainties for the special case $R(A_1, A_2)$:

$$\begin{aligned}(\Delta x)^2 &= \mathrm{Tr}[x^2R(A_1, A_2)] - \{\mathrm{Tr}[xR(A_1, A_2)]\}^2\\ &= \frac{1}{2} + \sigma(|A_1 - A_2|) + \frac{1}{2}[\mathfrak{Re}(A_1 - A_2)]^2,\end{aligned} \tag{9.38}$$

and

$$\begin{aligned}(\Delta p)^2 &= \mathrm{Tr}[p^2R(A_1, A_2)] - \{\mathrm{Tr}[pR(A_1, A_2)]\}^2\\ &= \frac{1}{2} + \sigma(|A_1 - A_2|) + \frac{1}{2}[\mathfrak{Im}(A_1 - A_2)]^2,\end{aligned} \tag{9.39}$$

where

$$\sigma(|A|) = \frac{|A|^2}{\exp(|A|^2) - 1} = \frac{1}{1 + \frac{|A|^2}{2!} + \frac{|A|^4}{3!} + ...}. \tag{9.40}$$

Furthermore, we calculated the average number of photons $\mathrm{Tr}(Ra^\dagger a)$ and the second order correlation

$$g_R^{(2)} = \frac{\mathrm{Tr}[(a^\dagger)^2a^2R]}{[\mathrm{Tr}(a^\dagger aR)]^2} \tag{9.41}$$

for the special case $R(A_1, A_2)$. We get:

$$\mathrm{Tr}\left[R(A_1, A_2)a^\dagger a\right] = \frac{1}{2}[|A_1|^2 + |A_2|^2 + \sigma(|A_1 - A_2|)] \tag{9.42}$$

Also

$$g_R^{(2)} = 1 + \frac{(|A_1|^2 - |A_2|^2)^2 - [\sigma(|A_1 - A_2|)]^2 + 2(A_1^*A_2 + A_2^*A_1)\sigma(|A_1 - A_2|)}{[|A_1|^2 + |A_2|^2 + \sigma(|A_1 - A_2|)]^2} \tag{9.43}$$

In the special case $A_2 = -A_1$ it reduces to

$$g_R^{(2)} = 1 - \frac{\sigma(2|A_1|)}{2|A_1|^2 + \sigma(2|A_1|)}. \tag{9.44}$$

It is seen that $g_R^{(2)}$ can take values less than 1, and in these cases we get antibunching.

9.5.2 Generalized Q and P Functions

If ρ is a density matrix, $Q(A_1, ..., A_n|\rho) = \mathrm{Tr}[\rho\Pi(A_1, ..., A_n)]$ is the probability that a measurement with the projector $\Pi(A_1, ..., A_n)$ will give 'yes'. The function $Q(A_1, ..., A_n)$ is a generalized Q-function (or Husimi function). From (9.27) it follows that

$$\int_{\mathbb{C}} \frac{d^2A}{n\pi} Q(A, A + d_2, ..., A + d_n|\rho) = 1. \tag{9.45}$$

The various $\Pi(A_1, ..., A_n)$ do not commute, and $Q(A_1, ..., A_n)$ is not a true probability distribution. It is a quasi-probability distribution of a quantum particle being at the point A_1, OR at the point A_2,..., OR at the point A_n in phase space. We stress that this is the quantum OR in (9.17) that involves superpositions, and it is different from the classical OR in Boolean algebra (which corresponds to union of sets).

It is important for the emerging subject of quantum computation, to involve the OR, AND, NOT logical operations in many practical calculations, and perform related experiments. This 'applied quantum logic' might lead to novel quantum technologies. This is especially true, for the OR operation which involves superpositions, and consequently is very different from its classical counterpart which is union of sets.

The generalized Q-function $Q(A_1, ..., A_n|\rho)$ contains the OR operation, and deserves further study within a generalized phase space formalism, that also defines a generalized P-function:

$$\rho = \int_{\mathbb{C}} \frac{d^2A}{n\pi} P(A, A + d_2, ..., A + d_n|\rho)\Pi(A, A + d_2, ..., A + d_n). \tag{9.46}$$

In the case of coherent states (i.e., projectors of rank 1) it is known that the P-function can be highly singular (e.g., for squeezed states), and it is interesting to study this for projectors of rank n.

9.6 The Set $\mathbb{C}_{\mathrm{fin}}$ of Finite Sets of Complex Numbers

Stone [8–10] has shown that there are deep links between three apparently different areas:

- distributive lattices
- a class of idempotent rings known as Boolean rings
- topological spaces

We use the definition of a ring, which does not require the existence of unity. If the ring has a unity, we indicate this explicitly with the term ring with a unity.

We consider the set $\mathbb{C}_{\text{fin}}$ of all **finite** subsets $\mathcal{A}$ of $\mathbb{C}$ (including the empty set $\emptyset$). $\mathbb{C}_{\text{fin}}$ is a subset of the powerset $2^{\mathbb{C}}$ which contains all (finite and infinite) subsets of $\mathbb{C}$.

Using Stone's formalism, we show that $\mathbb{C}_{\text{fin}}$ can be viewed as distributive lattice, and as a Boolean ring (we do not discuss the topological aspects). Later these results will be extended to the lattice of coherent spaces, which is isomorphic to $\mathbb{C}_{\text{fin}}$.

9.6.1 $\mathbb{C}_{\text{fin}}$ as a Distributive Lattice

In $\mathbb{C}_{\text{fin}}$, we define as disjunction (logical OR) and conjunction (logical AND), the union and intersection correspondingly:

$$\mathcal{A}_1 \vee \mathcal{A}_2 = \mathcal{A}_1 \cup \mathcal{A}_2; \quad \mathcal{A}_1 \wedge \mathcal{A}_2 = \mathcal{A}_1 \cap \mathcal{A}_2 \tag{9.47}$$

These operations are performed only a finite number of times, so that the result is a finite set, i.e., an element of $\mathbb{C}_{\text{fin}}$. With these operations $\mathbb{C}_{\text{fin}}$ is a distributive lattice. $\mathbb{C}_{\text{fin}}$ has least element (denoted as 0) which is the empty set $\emptyset$, but does not have greatest element (denoted as 1). Therefore negation (complementation) is not defined, and $\mathbb{C}_{\text{fin}}$ is not a Boolean algebra. The partial order associated to this lattice is subset (denote as $\prec$ or $\subseteq$). A general property (which actually holds in every lattice) is

$$\begin{aligned} \mathcal{A}_1 \prec \mathcal{A}_2 \;&\rightarrow\; \mathcal{A}_1 \vee \mathcal{B} \prec \mathcal{A}_2 \vee \mathcal{B} \\ \mathcal{A}_1 \prec \mathcal{A}_2 \;&\rightarrow\; \mathcal{A}_1 \wedge \mathcal{B} \prec \mathcal{A}_2 \wedge \mathcal{B}. \end{aligned} \tag{9.48}$$

Let $\mathbb{C}_1$ be the subset of $\mathbb{C}_{\text{fin}}$ that contains all sets $\{A\}$ with cardinality one ($A \in \mathbb{C}$). $\mathbb{C}_1$ is dense in $\mathbb{C}_{\text{fin}}$, in the sense of the following equivalent statements:

- for every $\mathcal{A} \in \mathbb{C}_{\text{fin}}$, there is some $\{A\} \in \mathbb{C}_1$ such that $\{A\} \prec \mathcal{A}$.
- If we consider all $\{A_i\} \in \mathbb{C}_1$ such that $\{A_i\} \prec \mathcal{A}$ (with $i = 1, \ldots, N$), then

$$\mathcal{A} = \{A_1\} \cup \ldots \cup \{A_N\}. \tag{9.49}$$

9.6.2 $\mathbb{C}_{\text{fin}}$ as a Boolean Ring

In the set $\mathbb{C}_{\text{fin}}$ we define multiplication and addition of two elements $\mathcal{A}_1, \mathcal{A}_2$, as the intersection and symmetric difference:

$$\mathcal{A}_1 \cdot \mathcal{A}_2 = \mathcal{A}_1 \cap \mathcal{A}_2; \quad \mathcal{A}_1 + \mathcal{A}_2 = (\mathcal{A}_1 \setminus \mathcal{A}_2) \cup (\mathcal{A}_2 \setminus \mathcal{A}_1) \tag{9.50}$$

In comparison with lattice theory, we replace here $\mathcal{A}_1 \cup \mathcal{A}_2$ (i.e., the logical OR operation), with the $\mathcal{A}_1 + \mathcal{A}_2$ (which is the logical XOR operation). The $\mathcal{A}_1 \cdot \mathcal{A}_2 = \mathcal{A}_1 \cap \mathcal{A}_2$ (i.e., the logical AND operation), is an operation in both lattice formalism and Boolean ring formalism.

With these operations (which are performed a finite number of times), $\mathbb{C}_{\text{fin}}$ is a commutative ring (without unity), with the extra property of idempotent multiplication:

$$\mathcal{A}_1 \cdot \mathcal{A}_1 = \mathcal{A}_1. \tag{9.51}$$

The $\emptyset$ plays the role of additive zero. The additive inverse of $\mathcal{A}$ is $\mathcal{A}$ itself:

$$-\mathcal{A} = \mathcal{A}. \tag{9.52}$$

A ring which has idempotent multiplication is commutative, and it is called a Boolean ring [8, 10].

Boolean rings with a unity are Boolean algebras. $\mathbb{C}_{\text{fin}}$ does not have a unity and as we explained earlier is not a Boolean algebra. The partial order 'subset' obeys the property (9.55)

$$\mathcal{A}_1 \prec \mathcal{A}_2 \;\rightarrow\; \mathcal{A}_1 \cdot \mathcal{B} \prec \mathcal{A}_2 \cdot \mathcal{B}, \tag{9.53}$$

but $\mathcal{A}_1 \prec \mathcal{A}_2$ does **not** imply $\mathcal{A}_1 + \mathcal{B} \prec \mathcal{A}_2 + \mathcal{B}$.

Remark 9.1 If we only consider subsets $\mathcal{A}$ of a finite set Ω of complex numbers, then the Boolean ring has a unity, which is $\Omega = 1$.

9.7 The Set $\mathcal{L}_{\text{coh}} \simeq \mathbb{C}_{\text{fin}}$ of Coherent Subspaces

The Birkhoff-von Neumann lattice $\mathcal{L}$ is not distributive. Here we consider its sublattice $\mathcal{L}_{\text{coh}}$ that contains the coherent subspaces of $\mathcal{L}$, and we show that it is a distributive lattice, isomorphic to $\mathbb{C}_{\text{fin}}$. This can be interpreted as an indication of the semi-classical nature of coherent states (because distributivity is a property of classical logic described with Boolean algebras).

We note that $\mathcal{L}_{\text{coh}}$ contains superpositions of coherent states, which are nonclassical states (e.g., the Wigner function for $|A_1\rangle + |A_2\rangle$ shows quantum interference in phase space). And yet the classical property of distributivity, holds in the lattice $\mathcal{L}_{\text{coh}}$.

Like $\mathbb{C}_{\text{fin}}$, the $\mathcal{L}_{\text{coh}}$ is also a Boolean ring. Many of the results in this section are analogous (isomorphic) to the ones in the previous section. The formalism in this section, provides the theoretical foundation for computation with coherent states, e.g., for quantum gates with coherent states.

9.7.1 $\mathcal{L}_{\text{coh}}$ as a Distributive Lattice

$\mathcal{L}_{\text{coh}}$ is the set of coherent subspaces $H(\mathcal{A})$, where $\mathcal{A}$ is a finite subset of $\mathbb{C}$. The $H(\emptyset) = \mathcal{O}$ is an element of $\mathcal{L}_{\text{coh}}$.

Proposition 9.6 *The disjunction and conjunction of coherent subspaces are given by*

$$H(\mathcal{A}_1) \vee H(\mathcal{A}_2) = H(\mathcal{A}_1 \cup \mathcal{A}_2); \quad H(\mathcal{A}_1) \wedge H(\mathcal{A}_2) = H(\mathcal{A}_1 \cap \mathcal{A}_2). \tag{9.54}$$

Proof We start from the definitions for conjunction and disjunction in (9.16) and (9.17), and we use the fact that the sets $\mathcal{A}_1, \mathcal{A}_2$ are finite, and therefore the corresponding coherent states are linearly independent.

As above, only a finite number of disjunctions and conjunctions are considered. $\mathcal{L}_{\text{coh}}$ has the $H(\emptyset) = \mathcal{O}$ as least element, but does not have a greatest element and it is not a Boolean algebra. The lattices $\mathcal{L}_{\text{coh}}$ and $\mathbb{C}_{\text{fin}}$ are isomorphic to each other. Therefore $\mathcal{L}_{\text{coh}}$ is a distributive lattice, in contrast to the Birkhoff-von Neumann lattice $\mathcal{L}$ which is not distributive.

If $H(\mathcal{A}_1) \wedge H(\mathcal{A}_2) = \mathcal{O}$, the coherent spaces $H(\mathcal{A}_1)$, $H(\mathcal{A}_2)$, have no vectors in common, or equivalently, they have no coherent states in common. The equivalence is based on the fact that a finite number of coherent states are linearly independent.

The partial order associated to this lattice is 'subspace' (denoted as $\prec$). Then

$$\begin{aligned} H(\mathcal{A}_1) \prec H(\mathcal{A}_2) \;&\rightarrow\; H(\mathcal{A}_1) \vee H(\mathcal{B}) \prec H(\mathcal{A}_2) \vee H(\mathcal{B}) \\ H(\mathcal{A}_1) \prec H(\mathcal{A}_2) \;&\rightarrow\; H(\mathcal{A}_1) \wedge H(\mathcal{B}) \prec H(\mathcal{A}_2) \wedge H(\mathcal{B}). \end{aligned} \tag{9.55}$$

Let $\mathcal{L}_1$ be the subset of $\mathcal{L}_{\text{coh}}$ that contains all one-dimensional coherent spaces $H(A)$. $\mathcal{L}_1$ is dense in $\mathcal{L}_{\text{coh}}$, in the sense of the following equivalent statements:

- for every $H(\mathcal{A}) \in \mathcal{L}_{\text{coh}}$, there is some $H(A) \in \mathcal{L}_1$ such that $H(A) \prec H(\mathcal{A})$.
- If we consider all $H(A_i) \in \mathcal{L}_1$ such that $H(A_i) \prec H(\mathcal{A})$ (with $i = 1, ..., N$), then

$$H(\mathcal{A}) = H(A_1) \vee ... \vee H(A_N). \tag{9.56}$$

9.7.2 $\mathcal{L}_{\text{coh}}$ as a Boolean Ring

Proposition 9.7 *The set of vectors $H(\mathcal{A})$ minus the set of vectors $H(\mathcal{B})$, is equal to*

$$H(\mathcal{A}) \setminus H(\mathcal{B}) = H(\mathcal{A} \setminus \mathcal{B}) \tag{9.57}$$

Proof We assume that a vector in $H(\mathcal{A})$

$$|s_A\rangle = \sum \lambda_i |A_i\rangle; \quad A_i \in \mathcal{A}; \quad \lambda_i \neq 0, \tag{9.58}$$

is equal to a vector in $H(\mathcal{B})$

$$|s_B\rangle = \sum \mu_i |B_i\rangle; \quad B_i \in \mathcal{B}; \quad \mu_i \neq 0. \tag{9.59}$$

The fact that a finite number of coherent states are linearly independent, implies that the coherent states $|A_i\rangle = |B_i\rangle$ (and $\lambda_i = \mu_i$). Therefore the subtraction of the vectors from $H(\mathcal{A})$ which also belong to $H(\mathcal{B})$, is equivalent to the subtraction of the coherent states from $H(\mathcal{A})$, which also belong to $H(\mathcal{B})$.

In $\mathcal{L}_{\text{coh}}$ we define addition as

$$H(\mathcal{A}_1) + H(\mathcal{A}_2) = [H(\mathcal{A}_1) \setminus H(\mathcal{A})_2] \cup [H(\mathcal{A}_2) \setminus H(\mathcal{A}_1)] = H(\mathcal{A}_1 + \mathcal{A}_2) \tag{9.60}$$

The proof of the equality follows immediately from Proposition 9.7. We also define multiplication as:

$$H(\mathcal{A}_1) \cdot H(\mathcal{A}_2) = H(\mathcal{A}_1 \cdot \mathcal{A}_2) = H(\mathcal{A}_1) \wedge H(\mathcal{A}_2). \tag{9.61}$$

Only finite sums and finite products, are considered. $\mathcal{L}_{\text{coh}}$ with these operations is a ring with idempotent multiplication:

$$H(\mathcal{A}_1) \cdot H(\mathcal{A}_1) = H(\mathcal{A}_1). \tag{9.62}$$

$H(\emptyset) = \mathcal{O}$ is the zero in this ring. There is no unity element. The additive inverse of $H(\mathcal{A})$ is $H(\mathcal{A})$ itself:

$$-H(\mathcal{A}) = H(\mathcal{A}). \tag{9.63}$$

$\mathcal{L}_{\text{coh}}$ is a Boolean ring, isomorphic to $\mathbb{C}_{\text{fin}}$.

In analogy to (9.53)

$$H(\mathcal{A}_1) \prec H(\mathcal{A}_2) \ \rightarrow \ H(\mathcal{A}_1) \cdot H(\mathcal{B}) \prec H(\mathcal{A}_2) \cdot H\mathcal{B}). \tag{9.64}$$

But $H(\mathcal{A}_1) \prec H(\mathcal{A}_2)$ does not imply $H(\mathcal{A}_1) + H(\mathcal{B}) \prec H(\mathcal{A}_2) + H(\mathcal{B})$.

Example 9.2 Let

$$\mathcal{A}_1 = \{A, B, C\}; \quad \mathcal{A}_2 = \{C, D\}; \quad A, B, C, D \in \mathbb{C}. \tag{9.65}$$

In the Bargmann representation, $H(\mathcal{A}_1) \vee H(\mathcal{A}_2)$ contains the functions

$$f(z) = \lambda_1 \exp(Az) + \lambda_2 \exp(Bz) + \lambda_3 \exp(Cz) + \lambda_4 \exp(Dz); \quad \lambda_i \in \mathbb{C}, \tag{9.66}$$

$H(\mathcal{A}_1) \wedge H(\mathcal{A}_2) = H(\mathcal{A}_1) \cdot H(\mathcal{A}_2)$ contains the functions

$$f(z) = \lambda \exp(Cz); \quad \lambda \in \mathbb{C} \tag{9.67}$$

and $H(\mathcal{A}_1) + H(\mathcal{A}_2)$ contains the functions

$$f(z) = \lambda_1 \exp(Az) + \lambda_2 \exp(Bz) + \lambda_3 \exp(Dz); \quad \lambda_i \in \mathbb{C}. \tag{9.68}$$

Remark 9.2 If we only consider subsets $\mathcal{A}$ of a finite set Ω of complex numbers, then the Boolean ring has a unity, which is $H(\Omega) = 1$.

Remark 9.3 Let U be a unitary transformation and $UH(\mathcal{A})$ the space of all states $U|s\rangle$, where $|s\rangle$ belongs to $H(\mathcal{A})$. We denote as $U\mathcal{L}_{\text{coh}}$ the lattice of all spaces $UH(\mathcal{A})$, with $\mathcal{A} \in \mathbb{C}_{\text{fin}}$. Then $U\mathcal{L}_{\text{coh}}$ is a distributive lattice and a Boolean ring, isomorphic to $\mathcal{L}_{\text{coh}}$.

9.7.3 Applications

Coherent spaces (rather than individual states) could be used as an 'alphabet' for quantum communication purposes. In this case noise which changes an individual state into another individual state within the same coherent space, does not produce an error. This might be useful in areas like quantum error correction and coding.

For this reason, we consider quantum gates or other devices that have as inputs and outputs states in a coherent space (i.e. coherent states or finite superpositions of coherent states). In the Bargmann representation, states in coherent spaces are described with functions that have order of growth 1. An arithmetic-like structure that treats coherent spaces like ordinary numbers, can be very useful in the study of these devices.

Such a device with M inputs and N outputs, can be described as a function from $(\mathcal{L}_{\text{coh}})^M$ to $(\mathcal{L}_{\text{coh}})^N$:

$$f : \begin{pmatrix} H(\mathcal{A}_1) \\ \dots \\ H(\mathcal{A}_M) \end{pmatrix} \rightarrow \begin{pmatrix} H(\mathcal{B}_1) \\ \dots \\ H(\mathcal{B}_N) \end{pmatrix}, \tag{9.69}$$

Here $H(\mathcal{A}_1)$, ..., $H(\mathcal{A}_M)$, $H(\mathcal{B}_1)$, ..., $H(\mathcal{B}_N)$ are elements of the Boolean ring $\mathcal{L}_{\text{coh}}$. For example the CNOT gate, defined in the present context as

$$f : \begin{pmatrix} H(\mathcal{A}_1) \\ H(\mathcal{A}_2) \end{pmatrix} \rightarrow \begin{pmatrix} H(\mathcal{A}_1) \\ H(\mathcal{A}_1) + H(\mathcal{A}_2) \end{pmatrix}. \tag{9.70}$$

Here the two outputs are any states in the coherent spaces $H(\mathcal{A}_1)$, $H(\mathcal{A}_2)$, and the two outputs are any states in the coherent spaces $H(\mathcal{A}_1)$, $H(\mathcal{A}_1) + H(\mathcal{A}_2)$.

As an example, we consider the sets $\mathcal{A}_1$ and $\mathcal{A}_2$ in (9.65). In this case the two inputs are one of the states (in the Bargmann representation)

$$f_1(z) = \lambda_1 \exp(Az) + \lambda_2 \exp(Bz) + \lambda_3 \exp(Cz)$$
$$f_2(z) = \lambda_1 \exp(Cz) + \lambda_2 \exp(Dz); \quad \lambda_i \in \mathbb{C}. \tag{9.71}$$

and the two outputs

$$f_3(z) = \lambda_1 \exp(Az) + \lambda_2 \exp(Bz) + \lambda_3 \exp(Cz)$$
$$f_4(z) = \lambda_1 \exp(Az) + \lambda_2 \exp(Bz) + \lambda_3 \exp(Dz); \quad \lambda_i \in \mathbb{C}. \tag{9.72}$$

As we explained earlier, if all the $\mathcal{A}_1, \mathcal{A}_2$ are subsets of a finite set of complex numbers Ω, then the Boolean ring $\mathcal{L}_{coh}$ has a unity, which is the $H(\Omega) = 1$. In this case we can write (9.70), in a matrix form as

$$f : \begin{pmatrix} H(\mathcal{A}_1) \\ H(\mathcal{A}_2) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \mathcal{O} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} H(\mathcal{A}_1) \\ H(\mathcal{A}_2) \end{pmatrix}. \tag{9.73}$$

We can then use matrices, with elements in the Boolean ring $\mathcal{L}_{coh}$, to describe circuits that involve many CNOT gates. This is an example of the merit of having a ring structure for the coherent subspaces.

9.8 Discussion

Given a finite set of complex numbers $\mathcal{A}$, a coherent space $H(\mathcal{A})$ is a subspace of the Hilbert space, spanned by the $|\mathcal{A}|$ coherent states $|A_i\rangle$ where $A_i \in \mathcal{A}$. A finite number of coherent states are linearly independent, and therefore $H(\mathcal{A})$ is an $|\mathcal{A}|$-dimensional space. The corresponding coherent projectors, have the coherence properties in Propositions 9.3 and 9.4 (resolution of the identity, etc.).

The set of all coherent spaces form the distributive lattice $\mathcal{L}_{coh}$, which is a sublattice of the Birkhoff-von Neumann lattice $\mathcal{L}$ (which is not distributive). Using Stone's formalism that links distributive lattices to Boolean rings, we have studied $\mathcal{L}_{coh}$ as a Boolean ring. This provides an arithmetic-like structure, which treats coherent Hilbert spaces like ordinary numbers. This can be very useful in the study of quantum gates and other devices that operate with coherent states or finite superpositions of coherent states.

The relationship between the non-distributive lattice $\mathcal{L}$, with its distributive sublattice $\mathcal{L}_{coh}$, requires further study. Distributivity is a property that holds in Classical Physics, but does not hold in Quantum Physics. But distributivity holds in sublattices of $\mathcal{L}$ like $\mathcal{L}_{coh}$. This might be interpreted as semi-classical nature of the states in coherent spaces, but we stress that $\mathcal{L}_{coh}$ also contains superpositions of coherent states which are usually viewed as non-classical states.

Coherent states have been studied for a long time, but we believe that the present work presents novel features.

References

1. J.R. Klauder, B.-S. Skagerstam (eds.), *Coherent States* (World Scientific, Singapore, 1985)
2. A. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Heidelberg, 1986)
3. S.T. Ali, J.-P. Antoine, J.-P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, 2nd edn. (Springer, New York, 2014)
4. A. Vourdas, Coherent spaces, Boolean rings and quantum gates. Ann. Phys. **373**, 557 (2016)
5. G. Birkhoff, J. von Neumann, The logic of quantum mechanics. Ann. Math. **37**, 823 (1936)
6. G. Birkhoff, *Lattice Theory* (American Mathematical Society, Rhode Island, 1995)
7. C. Piron, *Foundations of Quantum Physics* (Benjamin, New York, 1976)
8. M. Stone, The theory of representations for Boolean algebras. Trans. Am. Math. Soc. **40**, 37 (1936)
9. M. Stone, Applications of the theory of Boolean rings to general topology. Trans. Am. Math. Soc. **41**, 375 (1937)
10. M. Johnstone, *Stone Spaces* (Cambridge University Press, Cambridge, 1982)
11. P.R. Halmos, *Lectures on Boolean Algebras* (Springer, New York, 1963)
12. R. Sikorski, *Boolean Algebras* (Springer, New York, 1969)
13. T.C. Ralph et al., Quantum computation with optical coherent states. Phys. Rev. **A68**, 042319 (2003)
14. P. Marek, J. Fiurasek, Elementary gates for quantum information with superposed coherent states. Phys. Rev. A **82**, 014304 (2010)
15. A. Vourdas, The growth of Bargmann functions and the completeness of sequences of coherent states. J. Phys. A **30**, 4867 (1997)
16. A. Vourdas, K.A. Penson, G.H.E. Duchamp, A.I. Solomon, Generalized Bargmann functions, their growth and von Neumann lattices. J. Phys. A **45**, 244031 (2012)
17. A. Vourdas, Mobius operators and non-additive quantum probabilities in the Birkhoff-von Neumann lattice. J. Geom. Phys. **101**, 38 (2016)