ICME-13 Monographs

Lynda Ball · Paul Drijvers
Silke Ladel · Hans-Stefan Siller
Michal Tabach · Colleen Vale *Editors*

Uses of Technology in Primary and Secondary Mathematics Education

Tools, Topics and Trends

ICME 13
Hamburg 2016

Springer

ICME-13 Monographs

Series editor

Gabriele Kaiser, Faculty of Education, Didactics of Mathematics, Universität Hamburg, Hamburg, Germany

Each volume in the series presents state-of-the art research on a particular topic in mathematics education and reflects the international debate as broadly as possible, while also incorporating insights into lesser-known areas of the discussion. Each volume is based on the discussions and presentations during the ICME-13 Congress and includes the best papers from one of the ICME-13 Topical Study Groups or Discussion Groups.

More information about this series at http://www.springer.com/series/15585

Lynda Ball · Paul Drijvers
Silke Ladel · Hans-Stefan Siller
Michal Tabach · Colleen Vale
Editors

Uses of Technology in Primary and Secondary Mathematics Education

Tools, Topics and Trends

Springer

Editors
Lynda Ball
Melbourne Graduate School of Education
The University of Melbourne
Melbourne, VIC
Australia

Paul Drijvers
Freudenthal Institute
Utrecht University
Utrecht
The Netherlands

Silke Ladel
University of Saarland
Saarbrücken
Germany

Hans-Stefan Siller
Institute of Mathematics
University of Würzburg
Würzburg
Germany

Michal Tabach
School of Education
Tel Aviv University
Tel Aviv
Israel

Colleen Vale
Faculty of Education
Monash University
Melbourne, VIC
Australia

ISSN 2520-8322 ISSN 2520-8330 (electronic)
ICME-13 Monographs
ISBN 978-3-319-76574-7 ISBN 978-3-319-76575-4 (eBook)
https://doi.org/10.1007/978-3-319-76575-4

Library of Congress Control Number: 2018933013

Printed on acid-free paper

This Springer imprint is published by the registered company Springer International Publishing AG part of Springer Nature
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Contents

Chapter 1
Introduction

Lynda Ball, Silke Ladel and Hans-Stefan Siller

Abstract The use of technology in mathematics education, which encompasses the use of both classical and digital technologies, has a long and broadly discussed tradition. The potential impact of technology on what and how students learn (e.g. Fey et al. in Computing and mathematics. The impact on secondary school curricula. National Council of Teachers of Mathematics, Reston, VA, 1984) is an issue which has existed for decades and there is now a growing corpus of studies which provide insight into the role of technology in mathematics education (see for example, Blume and Heid in Research on technology and the teaching and learning of mathematics: volume 2 cases and perspectives. IAP, Charlotte, NC, 2008; Drijvers et al. in Uses of technology in lower secondary mathematics education: a concise topical survey. Springer, Cham, 2016; Heid and Blume in Research on technology and the teaching and learning of mathematics: volume 1 research syntheses. IAP, Charlotte, NC, 2008; Hoyles and Lagrange in Mathematics education and technology–rethinking the terrain. Springer, New York/Berlin, 2010; Moyer-Packenham in International perspectives on teaching and learning mathematics with virtual manipulatives. Springer International Publishing, Switzerland, 2016). Consideration of the impact of technology on the teaching and learning of mathematics has been the topic of considerable research and continues to be of interest as researchers investigate the potential of technology-enabled mathematics education. For these reasons, it is not surprising that technology use was the focus of three Topic Study Groups (TSGs 41, 42 and 43) at the 13th International Congress on Mathematical Education (ICME), held in Hamburg in 2016.

L. Ball (✉)
Melbourne Graduate School of Education, The University of Melbourne,
Melbourne, Australia
e-mail: lball@unimelb.edu.au

S. Ladel
University of Saarland, Saarbrücken, Germany
e-mail: ladel@math.uni-sb.de

H.-S. Siller
University of Würzburg, Würzburg, Germany
e-mail: hans-stefan.siller@mathematik.uni-wuerzburg.de

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_1

These three TSGs focused on the uses of technology (i.e. digital tools) in mathematics education across the spectrum of school mathematics from primary to upper secondary. The aim of this book is to provide an overview of some of these studies related to the use of technology across this age spectrum, as well as point towards future directions for the use of technology in school mathematics. As the field of technology use is a very broad one, the three Topic Study Groups had different foci related to working and using digital tools in teaching and learning of mathematics. The next section provides insight into the intended foci of each of the three TSGs as this foregrounded the work of the three groups at ICME 13.

Keywords Teaching mathematics · Learning mathematics · Technology School mathematics · Digital tools

TSG 41 focused on "Uses of technology in primary mathematics education (up to age 10)". This TSG noted that although many types of digital technology and environments have been available for primary education since before the turn of the century, individual drill and practice software and interactive tools for exposition appeared to be prevalent in many primary classrooms where technology is used (Attard & Orlando, 2016; Bate, 2010; Loong, Doig, & Groves, 2011; Zuber & Anderson, 2013). Today, all over the world, young children bring their experience with hand-held and other technology into the classroom and in recent years, these have included tools to communicate in the cloud. In the context of this experience, two questions were raised:

1. Do primary teachers keep up with digital natives?
2. Which types of technology use are emerging to enrich and foster mathematics learning at primary school?

Taking these questions into account, TSG 41 focused on the issues of 'use of technology', 'key success factors' and 'innovations' in the context of primary mathematics education, with children up to 10 years old, namely:

- *How do schools and teachers around the world, and in differently advantaged communities, use technology to enrich mathematics learning at primary level?*
- *Which factors contribute to successful and sustained use of technology in primary settings?*
- *Which innovations in digital technology for education enable primary children to inquire, problem solve and think mathematically and to share their learning?*

TSG 42 focused on "Uses of technology in lower secondary mathematics education (age 10–14)", with this age range bridging primary and secondary schooling in many countries. TSG 42 considered technology-related issues from both a learner and teacher perspective, focusing on four themes. The four themes and associated research questions shown below initiated the work of TSG 42:

- *Evidence for effect—What are the research findings about the benefits for student learning of the integration of digital tools in lower secondary mathematics education?*
- *Mathematics education in 2025—What will lower secondary mathematics education look like in 2025, with respect to the place of digital tools in curricula, teaching and learning? How can teachers integrate physical and virtual experiences to promote deep understanding of mathematics?*
- *Digital assessment—What are features of appropriate online assessment of, for and as learning?*
- *Communication and collaboration—How can digital technology be used to promote communication and collaborative work between students, between teachers, and between students and teachers? What are the potential professional development needs of teachers integrating digital tools into their teaching, and how can technology act as a vehicle for such professional development activities?*

TSG 43 focused on "Uses of technology in upper secondary mathematics education (age 14–19)". The TSG focused on four themes:

- *Theoretical Aspects. New technologies can create new kinds of activities and new forms of interactions between learners and teachers hence the need to examine current theory for developing and analyzing the implementation of new technologies from cognitive and epistemological perspectives.*
- *Role of Emerging Technologies. For example, how tablets, smartphones, Virtual Learning Environments, Augmented Reality environments, and haptic technologies might mediate new forms of access to mathematics.*
- *Interrelations between technology and the mathematics taught at this age level.*
- *Teacher Education. New challenges and opportunities for teachers to reflect on their practices and how they develop with the use of new technologies.*

One key point to be considered in any discussions about technology in mathematics education is that access to technology does not, of itself, result in improved teaching and learning. Therefore, the topic study groups on technology are crucial to highlight findings from a range of international perspectives, as well as look forward to future research directions. Consideration of the themes and research questions across the three topic study groups highlights the evolutionary nature of research into digital technology and the need for future research in this area. The following section outlines the chapters in this book, discussed in three sections which align with the three topic study groups.

Topic Study Group 41 was concerned with primary mathematics education up to age 10 and Chaps. 2–8 focus on the use of technologies and digital tools in this age range. In Chap. 2 Moyer-Packenham et al. present the results of a study that examines changes in the performance and efficiency of young children's learning as they engaged with several touch-screen virtual manipulative mathematics apps. They found that changes in the children's learning could be explained by the content alignment of the apps, as well as having similarity in the structure of the apps used for assessment and for learning. This suggests that technical familiarity could be a consideration when a teacher is choosing an app to develop or assess a student's understanding. Tucker also focused on touch-screen apps in Chap. 3 applying the Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for learning framework to evaluate potentialities of apps.

Larsen et al., in Chap. 4, found that the purposeful use of screencasting apps supported mathematical discourse and has the potential to impact teacher practices. In Chap. 5 Voltolini questions the bonus brought by technology in situations that link digital and material tools. The author shows how the duo of a digital and a material tool supports the learning processes of children with regards to processes of assimilation and adaptation.

Larkin and Milford (Chap. 6) provided an analysis of 53 apps that support geometry learning to promote discussion about the use of apps in primary mathematics education. In Chap. 7 Walter investigated students' use of a physical 'twenty frame' and the 'twenty frame' tablet-app for a group of students with special learning needs. The different approaches used by different children suggested that potential learning gains may not be achieved by all students, using either physical twenty-frames or the given app. The structure of an app was identified as a potential inhibitor to development of understanding in this case. Calder and Murphy (Chap. 8) also reported on the affordances of an app, Math Shake, and the potential for reshaping learning experiences in primary-school mathematics. While their results show the importance of the affordances of mobile technologies for students' learning, they also show that the teacher's pedagogical approach is influential.

These seven papers focus on different aspects related to the use of technology to enhance mathematics teaching and learning in primary education, but each paper shows the great potential that technologies hold—if used in a useful way.

Drijvers et al. (2016) provided a topical survey to stimulate the work of topic study group 42 at ICME 13; this international perspective included a survey of research findings and future directions for lower secondary mathematics (ages 10–14) in the context of a technological age. Evidence for effect, to assess whether technology has been shown to improve student outcomes, was examined. In addition, the role of the teacher, as well as the role of technology in summative and formative assessment, was considered. The potential for communication and collaboration enabled through technology provides two challenges—how to capitalize on technologies to promote this communication and collaboration and the resultant professional development needs for teachers who are teaching in these contexts. Finally, the topical survey attempted to look ahead to mathematics education in 2025, providing a vision for a technology-rich mathematics education. The presence of technology has provided researchers and teachers with opportunities to re-conceptualize mathematics education at lower secondary education, including a rethinking of goals for curriculum, assessment, teaching and learning (Drijvers et al., 2016).

Chapters 9–14 in this book provide insight into the TSG 42 themes focusing specifically on evidence for effect, assessment and communication. Chapters 9 and 10 provide reviews of quantitative and qualitative studies related to technology in lower secondary mathematics. Drijvers (Chap. 9), provides a review of quantitative studies related to technology use and student achievement. Although significant positive effects are reported, with moderate effect sizes, the question is posed about whether quantitative studies provide the detail about how technology can benefit students' learning of mathematics. In Chap. 10, Heid reviews qualitative literature related to mathematics learning, highlighting the important role that these studies play in probing students' mathematical work and in illustrating that technology use has the potential to enrich a student's mathematical experience. Qualitative studies provide detail into both what has been observed in student work and why this might be the case, thus providing reasons for observed changes in mathematical understanding.

Maschietto, in Chap. 11, provides one case study which highlights the interplay between classical and digital technologies. In this study the Pythagorean Theorem is explored in a laboratory setting with access to both classical and digital technologies. The chapter highlights the cognitive processes evident through kinesthetic experiences with the machines, as well as the role of the teacher in orchestrating the classroom to promote these processes. In Chap. 12, Ball and Barzel focus on an overview of communication in the presence of digital technology. The focus on communication follows on from the previous chapter, where use of a laboratory approach involved communication mediated by technology. Communication in the presence of technology has been categorized by Ball and Barzel as communication through, with and of technology and the ways that this communication can promote conceptual, procedural and metacognitive knowledge is elaborated and illustrated. This provides a lens through which to consider how the development of different types of mathematical knowledge can be supported through the affordances of different types of technology.

Chapters 13 and 14 focus on assessment in the presence of digital technology; this fosters consideration of the ways that technology can assist in assessment and how use of assessment information can inform teaching. In Chap. 13 Grugeon and colleagues analyse results from a study on the use of Pépite, an online assessment and teaching tool which provides information for teachers about students' reasoning and thus can support planning for differentiation in the classroom. The focus here is on formative assessment where the technology provides an analysis of students' algebraic reasoning. In Chap. 14, Dick proposes a prototype for an assessment system that utilizes both a computer algebra system (CAS) and a dynamic geometry environment (DGE) with the goal of assessment being carried out automatically within the system. Both assessment focused papers discuss systems where assessment can be carried out within technology and they provide insight into future possibilities for technology-assisted assessment. These chapters highlight the importance of consideration of online assessment systems that provide teachers with information about students' understanding to inform teaching. This formative assessment can assist teachers in targeting teaching to improve students' outcomes. These six papers serve to highlight that there are still many considerations that need to be addressed with regards to technology use in lower secondary mathematics.

The work in TSG 43 was prompted by a topical survey by Hegedus et al. (2016); this publication identified four challenging themes that impact the use of technology in upper-secondary mathematics education:

- Technology in secondary mathematics education: Theory
- The role of new technologies: Changing interactions
- Interrelations between technology and mathematics
- Teacher education with technology: What, how and why?

Chapters 15–25 each address one theme and focus on either the use of DGE or CAS. Across these chapters, DGE was the most used technology, particularly when the research focus was related to process orientated observations such as exploring or modeling. Some chapters also reported DGE studies investigating the teaching of specific mathematical concepts or skills. In contrast, CAS (handheld calculators, as well as software) was used only for research topics concerning proofs and justification.

By looking at the papers of TSG 43 from a meta-perspective one will be able to recognize several research foci related to taxonomies for orchestration of students' work with technology; the ways that *research informs teachers' knowledge and professional development to optimize students' learning with technology; new opportunities for interactions between teachers and students in the presence of technology and the role of teachers in these interactions.*

Using digital tools in education with the aim to experiment can be identified in two ways. On the one hand the main aim could be the promotion of mathematical thinking and design of educational digital resources, such as in Traglová et al. (Chap. 15). Concerning the issue of implementing new technologies (such as a Wii) or technology using sensors, there is potential to explore changes in the ways that students learn. Ng and Sinclair (Chap. 16) investigate the use of innovative approaches, such as a 3D drawing pen for the learning of functions and calculus, where mathematics moves from the traditional 2-D (such as on paper) to 3D. Ferrari and Ferrara (Chap. 17) suggest that these types of resources can only be produced within an innovative socio-technological environment and therefore requires collaboration by a community of mathematics teachers, computer scientists and researchers in mathematics education.

When working with technology one must be aware that representing, documenting and reflecting are key issues in the context of technological learning environments, as discussed in Chaps. 18–24. For example, Beck in Chap. 18, analyzed written notes of students who worked with CAS in upper secondary mathematics and noted the potential to promote discussion about communication of mathematical working. Interactions with digital tools and technological learning environments seems to be advantageous, as outlined by Moreno-Armella and Brady (Chap. 19). Donevska-Todora (Chap. 20) discusses a framework for developing deep understanding of concepts in linear algebra. Greefrath and Siller (Chap. 21) study the extent to which the systematic application of the dynamic geometry software GeoGebra supports "Mathematical Modelling" and Misfeldt and Jankvist, in Chap. 22, investigate the role of CAS in text books. Trgalová and Tabach (Chap. 23) describe existing ICT standards at the international and national levels, arguing that these standards are too general. In Chap. 24 Bowman proposes that graphing calculators as daily tools can enrich the mathematical learning of students.

In Chap. 25 Thurm provides empirically based recommendations for teacher education. A common concern in the TSG 43 chapters was the necessity for more research about teaching with technology to inform teacher professional development and this issue is evident throughout this book.

The issues associated with teaching and learning mathematics with technology are multi-faceted and the chapters in this book have highlighted some current research and theoretical perspectives in primary and secondary mathematics education. With technology evolving at a fast rate there is a need for qualitative, quantitative and theoretical studies to provide analysis of the benefits of current technologies, but also to drive new questions as we look towards the future of mathematics teaching and learning in the presence of existing and new technologies.

References

Attard, C., & Orlando, J. (2016). Digital natives come of age: The reality of today's early career teachers using mobile devices to teach mathematics. *Mathematics Education Research Journal, 28*(1), 107–121.

Bate, F. (2010). A bridge too far? Explaining beginning teachers' use of ICT in Australian schools. *Australasian Journal of Educational Technology, 26*(7). https://doi.org/10.14742/ajet.1033.

Blume, G. W. & Heid, M. K. (Eds.). (2008). *Research on technology and the teaching and learning of mathematics: Vol. 2. Cases and perspectives*. Charlotte, NC: IAP.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education: A concise topical survey*. Springer Open. Cham: Springer.

Fey, J. T., Atchison, W. F., Good, R. A., Heid, M. K., Johnson, J., Kantowski, M. G., et al. (1984). *Computing and mathematics. The impact on secondary school curricula*. Reston, VA: National Council of Teachers of Mathematics.

Hegedus, S., Laborde, C., Brady, C., Dalton, S., Siller, H. S., Tabach, M., & Moreno-Armella, L. (2016). *Uses of technology in upper secondary mathematics education*. ICME-13 Topical Survey Series: Springer.

Heid, M. K., & Blume, G. W. (Eds.). (2008). *Research on technology and the teaching and learning of mathematics: Vol. 1. Research syntheses*. Charlotte, NC: IAP.

Hoyles, C., & Lagrange, J.-B. (Eds.). (2010). *Mathematics education and technology—Rethinking the terrain*. New York, Berlin: Springer.

Loong, E. Y. K., Doig, B., & Groves, S. (2011). How different is it really?—Rural and urban primary students' use of ICT in mathematics. *Mathematics Education Research Journal, 23*(2), 189–211.

Moreno-Armella, L., & Santos-Trigo, M. (2016). The use of digital technology in mathematical practices: Reconciling traditional and emerging approaches. In *Handbook of international research in mathematics education* (pp. 595–616). London: Routledge.

Moyer-Packenham, P. (Ed.). (2016). *International perspectives on teaching and learning mathematics with virtual manipulatives*. Switzerland: Springer International Publishing.

Zuber, E. N., & Anderson, J. (2013). The initial response of secondary mathematics teachers to a one-to-one laptop program. *Mathematics Education Research Journal, 25*(4), 279–298.

Chapter 2
Using Video Analysis to Explain How Virtual Manipulative App Alignment Affects Children's Mathematics Learning

Patricia S. Moyer-Packenham, Kristy Litster, Emma P. Bullock and Jessica F. Shumway

Abstract In this inquiry, researchers sought to understand changes in young children's learning by examining their performance and efficiency while they engaged with a variety of touch-screen virtual manipulative mathematics apps. We were particularly interested in understanding how the alignment of the apps selected for two different learning sequences might contribute to these changes. A total of 100 children, ages 3–8, participated in interviews. Researchers examined the interviews using a frame-by-frame video analysis to interpret children's interactions with six different mathematics apps on iPads in a clinical interview setting. Results revealed improvements in children's mathematics performance and efficiency between the pre and post assessment apps. Apps that were content aligned and structurally aligned, within each of the learning sequences, helped to explain the changes in children's learning.

Keywords Virtual manipulative · Mathematics apps · Touch screen
Video analysis · Content and structural alignment

2.1 Purpose

Mathematics apps, that contain virtual manipulatives, have become a popular tool and an effective way of supporting children's mathematics learning. Originally, virtual manipulatives were designed as mouse-driven apps for the computer. Since the release of the first iPad in 2010, touch-screen devices have become wide spread platforms for personal and educational use. There are now thousands of mathe-

P. S. Moyer-Packenham (✉) · K. Litster · J. F. Shumway
Utah State University, Logan, USA
e-mail: patricia.moyer-packenham@usu.edu

E. P. Bullock
Sam Houston State University, Huntsville, USA

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_2

matics *apps* (i.e., applications for mobile devices with a touch screen; Gröger, Silcher, Westkämper, & Mitschang, 2013) available for download in online stores. Not all apps have the same quality or value as is evident in the evaluations of apps that have appeared in the literature (Boyer-Thurgood, 2017; Schrock, 2011; Walker, 2010).

The purpose of this project was to utilize frame-by-frame video analysis to examine young children's interactions with virtual manipulative mathematics touch-screen apps. Specifically, we were interested in how app alignment contributed to changes in children's learning. In this study, we identified two types of app alignment: *content* alignment and *structural* alignment. We examined how these two aspects of app alignment contributed to changes in children's learning.

2.2 Research Perspective

Virtual manipulatives (first defined in 2002 by Moyer, Bolyard, & Spikell) are defined as: "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (Moyer-Packenham & Bolyard, 2016, p. 13). Today, there are thousands of virtual manipulatives, with representations of mathematical objects, currently available or under development that can be used with a touch-screen interface (e.g., iPads). The current research on virtual manipulative mathematics apps includes a variety of results on learning outcomes.

2.2.1 Mathematics Apps and Learning Outcomes

The use of touch-screen apps can improve students' mathematics performance. Barendregt, Lindström, Rietz-Leppänen, Holgersson, and Ottosson's (2012) study with 87 five-, six-, and seven-year-olds found that using the subitizing iPad app, Fingu, as part of their practice supported an increase in children's computation abilities with addition and subtraction. In another study, Kermani and Aldemir (2016) designed and implemented mathematics interventions for at-risk preschoolers using iPad apps with a focus on properties of number (i.e., counting and subitizing). They found significant differences in learning between the 25 iPad intervention children and the 25 control children in a traditional classroom intervention. Kiger, Herro, and Prunty (2012) looked at the use of iPod Touch devices as supplemental practice tools for children to use at home. They found that the mobile learning interventions led to a statistically significant difference in performance for the intervention group over children who used the standard curriculum materials. Bakker, van den Heuvel-Panhuizen, and Robitzsch (2015) added new insights to the role of home and school in children's learning. They examined the effects of home and school use of virtual manipulatives with 719 second graders. They found that children who used the app at home after an in-school debrief had significant differences in multiplicative reasoning (i.e., skip counting) over children who used the app just at home or just at school.

These studies show that using mathematics apps on mobile devices can have a positive impact on young children's learning; however, they do not explain *why* they have an impact. This is an important point related to the research in this paper, because through video analysis of children's interactions with apps, we hoped to identify possible indicators that explained children's learning.

2.2.2 Defining Two Types of App Alignment

App alignment may play a role in children's mathematical learning. For the purposes of this study, we defined two types of app alignment: *content* alignment and *structural* alignment. We defined content alignment as the degree to which the specific mathematics topics contained in an app were aligned with the specific mathematics topics contained in each of the other apps in the interview sequence. For example, if one app focused on counting 1–10 blocks and another app focused on identifying the numeral that named the number of blocks from 1 to 10, we would say that the apps were closely aligned in terms of content because they both focused on developing the skill of counting a group of objects from 1 to 10. However, if one app focused on counting 1–10 blocks and another app focused on identifying the place value of a digit in a three-digit number, we would say that the apps were *not* closely aligned in terms of content because one app is developing the skill of counting while the other app is developing an understanding of place value.

We defined structural alignment as the degree to which objects and tasks contained in an app were aligned with the objects and tasks contained in each of the other apps in the interview sequence. For example, if one app displayed a group of squares of different sizes and children were asked to order the squares from largest to smallest, and another app displayed a group of rods of different sizes and children were asked to order the rods from longest to shortest, we would say that the apps were closely aligned in terms of structure because they both contained objects of different sizes and the tasks in both apps asked the child to seriate the objects. However, if one app focused on placing a number on a number line and another app focused on creating a numerical representation for a three-digit number given orally, then we would say that the apps were *not* closely aligned in terms of structure, because one app has a number line as the object with a task of placing the number on the line while the other app has place value cards as the object with the task of creating a numeral with the cards.

2.2.3 Potential Learning Benefits of App Alignment

In this study, we hypothesized that the content alignment of the four apps in the interview sequence would be important for children's learning. In prior research, Edwards Johnson, Campet, Gaber, and Zuidema (2012) suggested that teachers should consider alignment between the activity and the target mathematical content. Their research, using clinical interviews with children in Grades 2–5, found that virtual manipulatives with features that were aligned with mathematical content and procedures reinforced target concepts and addressed children's common error patterns. For example, one error pattern they noted was that children thought that 5 tens and 4 ones equaled 9. The virtual base ten blocks supported the development of place value concepts by allowing students to convert ten unit blocks into one unit of ten and emphasized the meaning of digits in the tens and ones place (p. 203). This shows the potential importance of aligning the mathematical content of each of the apps that children use when they are learning a specific mathematical topic if we want to support children's learning of that topic.

We hypothesized that the structural alignment of the four apps in the interview sequence would be less important for children's learning, because of the research that shows that being able to translate among a variety of mathematics representations supports learning (Lesh, Landau, & Hamilton, 1983). Therefore, if the structure of the apps is not aligned, this simply means that the child is exposed to a variety of different representations (i.e., different objects and different tasks) of the same mathematical topic, which should support learning. While there is little research that directly looks at the structural alignment of apps, there are related findings that may provide some insight about structural alignment. For example, Uttal et al. (2013) reported on the alignment of tests for transfer. They conducted three experiments to examine transfer from: (1) written or physical manipulative instructional methods to written tests, (2) written or physical manipulative instructional methods to physical manipulative tests, and (3) standard and distinctive physical manipulative instruction to written tests. They concluded that posttest performance depended on whether the learning method matched the testing method and suggested that relational similarities may help children transfer learning. In related research, Segal (2011) examined the structural congruence of gestures in direct touch and mouse click applications. Her study compared four different digital conditions: (1) direct touch interface with a congruently mapped application, (2) direct touch interface with an incongruently mapped application, (3) mouse-click interface with a congruently mapped application, and (4) mouse-click interface with an incongruently mapped application. Congruence was defined as matching the gesture children would complete when using a physical manipulative (e.g., turning) to the gesture children used with a virtual manipulative (e.g., swiping to turn vs. tapping to turn). Findings suggested that direct touch interfaces with a congruent mapping of gestures increased student efficiency and accuracy. While these two studies did not directly address structure, their results may provide some insights on how structural alignment may be important.

2.2.4 The Complexity and Diversity of App Features and Structures for Learning

Five categories of affordances were identified in a meta-analysis by Moyer-Packenham and Westenskow (2013): "focused constraint, creative variation, simultaneous linking, efficient precision, and motivation" (p. 35). These five categories are common among virtual manipulatives that have been shown to have positive impacts on mathematics learning. In addition, touch-screen devices, such as iPads, have interactive properties that afford learning opportunities. For example, Segal (2011) found significant differences in haptic modality (mouse vs. touch screen) in that iPads encouraged less guessing, better accuracy, and efficiency when compared with the same app on a computer. This means that app features and device modalities may not affect all children in the same way. In fact research has confirmed these differences. For example, Barendregt et al.'s (2012) Fingu app, intended to develop conceptual subitizing skills, helped different children develop different skills in subitizing. Baccalini-Frank and Maracci (2015) examined preschoolers' number sense with multi-touch devices and found that each app had different characteristics which fostered the development of various aspects of number sense. Children's prior achievement levels also seem to impact their learning with mathematics apps. For example, Moyer-Packenham and Suh (2012) found that low achievers accessed the step-by-step procedures features of fraction apps, while high achievers accessed the evident patterns afforded by the apps. Researchers have also reported that different children access app features in different ways. For example, Moyer-Packenham et al. (2015a) reported that children's access to helping and hindering features (or affordances) in mathematics apps influenced the children's progress. The children who accessed the helping affordances were more likely to progress between the pre and post assessments. These studies imply that the complexity of app features and the diversity of app structures affects different children in different ways.

This paper seeks to contribute to an understanding of why some app experiences help children to progress while others do not by using a frame-by-frame video analysis as a way to identify possible features that may explain children's learning in similar content topics (i.e. counting, subitizing, skip counting) and across different content topics (i.e. seriation, quantities, place value). We were specifically interested in understanding how learning apps that were content aligned and structurally aligned explained changes in children's learning.

2.3 Research Question

While the research base on virtual manipulative mathematics apps is growing, there is a need for further investigation into how content- and structurally-aligned apps may play a role in changes in children's learning performance and efficiency.

This study examined the following research question: How do content-aligned and structurally-aligned virtual manipulative mathematics apps contribute to changes in children's learning performance and efficiency? In this study, learning *performance* was defined as a change in accuracy between the pre- and post-assessment tasks that children completed using virtual manipulative touch-screen apps. Learning *efficiency* was defined as changes in the speed with which children completed the pre- and post-assessment tasks, after completing a variety of learning tasks using virtual manipulative touch-screen apps. Based on the findings of Edwards Johnson et al. (2012), our hypothesis was that aligning the pre- and post-assessment apps with the two learning apps, in terms of their mathematical content, would increase the likelihood of positive changes in children's performance and efficiency.

2.4 Methods

2.4.1 Research Design

To answer the research question, we used an explanatory mixed methods design. We collected and analyzed quantitative and qualitative data and then merged the results to answer our mixed methods research question (Creswell & Plano Clark, 2011; Tashakkori & Teddlie, 2010). The rationale for this design was to obtain complementary data on the same topic to better understand the research problem. We collected the video data for this paper in one of our previous research projects (Moyer-Packenham et al., 2015b). We then used these video data in several different analyses focusing on different research questions, such as the research question in this paper.

In this study we coded videos of children's interactions with a pre-app, two learning apps, and a post-app. We quantitized the learning performance and efficiency data from the pre- and post-assessment activities and explored these data using SPSS. We used qualitative methods to analyze how children's interactions with the apps might explain their outcomes for learning performance and efficiency, which allowed a holistic overall interpretation.

2.4.2 Participants

A total of 100 children (Preschool, ages 3–4, N = 35; Kindergarten, ages 5–6, N = 33; Grade 2, ages 7–8, N = 32) participated in this study. They were recruited using informational brochures and letters distributed to local public and charter elementary schools, the university campus lab school, and the university campus preschools. The demographics of the children were: Asian (1%), Caucasian (89%), Hispanic (2%), and Mixed Race (8%). One-third (34%) of children's parents

Fig. 2.1 Preschooler interacting with an iPad app under the direction of an interviewer in the clinical interview room

reported them receiving free- or reduced-lunch services at school (indicating low socio-economic status). The parents of the participating children completed surveys and reported children's prior iPad use and experiences with technology. Parents reported on the use of touch-screen devices in the home with 11% having more than five touch-screen devices, 78% with between one and four, and 8% with none. Thirteen percent of the children had access to their own touch-screen device at home. Parents reported that the children used the touch-screen devices every day (45%), 4–5 days per week (2%), 1–3 days per week (40%), and never (10%). Figure 2.1 shows a preschooler interacting with an iPad.

2.4.3 Data Sources

We used four instruments to collect data during the study: pre- and post-assessments (to document mathematics accuracy and speed), GoPro video recordings of the iPad screen, wall-mounted video recordings of children and the interviewer, and observation protocols.

The pre- and post-assessment apps used in this study focused on two mathematical content topics for each age-level group. The preschool children (ages 3–4) were assessed on seriation and counting content. The kindergarten children (ages 5–6) were assessed on quantities and subitizing content. The Grade 2 children (ages 7–8) were assessed on place value and skip counting content. The same mathematics app was used for the pre- and post-assessments on each mathematical content topic for each age-level group. To determine mathematics performance (i.e., accuracy), we identified the number of tasks the child completed correctly on the pre-assessment and the number of tasks the child completed correctly on the post-assessment. To determine efficiency (i.e., speed), we identified the time it took the child to complete the tasks on the pre-assessment and the time it took the child to complete the same tasks on the post-assessment. Speed of completion can show several things about the child's learning while using a mathematics app: (1) familiarity and confidence with the mathematics content, (2) familiarity and confidence with the features and tools in the app, or (3) a desire to complete the tasks quickly without regard to the content of the app. By viewing the interview videos to understand the child's overall interactions with the app, we could determine why children became faster or slower when they completed the pre- and post-assessment tasks. The mathematics content topics of seriation, subitizing, counting, skip counting, and place value were selected for study with young children because these concepts are critical foundations to later mathematics learning. Learning the count sequence, object counting, learning cardinal ideas, understanding the seriation of numbers, and skip counting are interrelated counting ideas that serve as the gateway to young children's developing counting strategies and understanding patterns that make up the place value number system. Current research indicates the existence of consistent relationships between counting, number relationships and basic operations, and later mathematics achievement (Jordan, Glutting, & Ramineni, 2010).

Two video views were important sources of data for the project: GoPro video recordings and wall-mounted video recordings. Each child was equipped with a wearable GoPro camera that was positioned to capture an up-close view of their interactions on the touch-screen iPad device. This video recording process captured all of the on-screen motions of the mathematics objects and tasks initiated by the children. It also captured audio interactions between the child and the interviewer as well as audio interactions between the child and the iPad. The wall-mounted video recordings captured a broad view of the child, the interviewer, the iPad, and all actions and interactions that occurred during the interviews. The second video source served as a back-up for the data collected by the GoPro camera and as a broader perspective of the child's actions that were outside the GoPro camera view and away from the iPad.

The final data source was an observation protocol. One observer watched the interview from an observation booth and recorded notes on the interview. Schubert (2009) suggests that the development of these protocols be based on current theories related to the phenomenon of interest and the researcher's own experience with observing the phenomenon. In line with that recommendation, we used the mathematics education literature to focus our attention on how the children interacted with features of the mathematics apps.

2.4.4 Procedures and Data Collection

Parents brought their children to a research building on a university campus. Children participated in individual clinical interviews in an early childhood education research building equipped with two-way mirrors, audio observation rooms, and built-in video cameras. The view that observers had from the observation room is pictured in Fig. 2.2. Prior to each interview, researchers collected information from the parents of the participating children, completed the consent form, and answered questions. During interviews, children used interactive mathematics apps on iPads. The research team had experts with experience in conducting mathematics clinical interviews with young children.

Table 2.1 displays the interview order for each of the mathematics apps used with each age-level group in the study. The research team selected three apps to further preschoolers' (ages 3–4) learning of seriation and three apps to further preschoolers' learning of counting. The team selected three apps to further kindergartens' (ages 5–6) learning of combining amounts and three apps to further kindergarteners' learning of building and representing numbers. Finally, the team selected three apps to further second graders' (ages 7–8) learning of base-10 place value and three apps to further second graders' learning of skip-counting. Screen shots of each of the apps are displayed in Tables 2.3, 2.4, and 2.5 by age level.

Fig. 2.2 A view of the clinical interview room showing observers watching an interview from the observation room

Table 2.1 List of mathematics apps and interview order for each age-level group

Interview order	Preschool (age 3–4)	Kindergarten (age 5–6)	Grade 2 (age 7–8)
	Seriation tasks	*Subitizing tasks*	*Skip counting tasks*
App #1 (pre-assessment)	Pink tower—free moving	10-frame	100s chart
App #2 (learning app 1)	Pink tower—tapping	Hungry guppy	Frog number line
App #3 (learning app 2)	Red rods	Fingu	Counting beads
App #1 (post-assessment)	Pink tower—free moving	10-frame	100s chart
	Counting tasks	*Quantities tasks*	*Place value tasks*
App #4 (pre-assessment)	Base-10 blocks	Base-10 blocks	Base-10 blocks
App #5 (learning app 1)	Base-10 blocks: 1–5	Base-10 blocks: 11–20	Zoom number line
App #6 (learning app 2)	Base-10 blocks: numerals	Base-10 blocks: numerals	Place value cards
App #4 (post-assessment)	Base-10 blocks	Base-10 blocks	Base-10 blocks

As seen in Table 2.1, during each interview, children interacted with a pre-assessment app on the iPad, then interacted with two learning apps that contained a series of mathematical tasks, and finally interacted with a post-assessment app that revisited the tasks from the pre-assessment. This procedure was repeated for a second mathematics content topic for each of the age-level groups using different apps and app tasks. Our goal was to select apps so that each series of learning and assessment tasks (i.e., pre-app, learning app 1, learning app 2, post-app) focused on one specific mathematics content topic that was age-appropriate for the children in that age-level group. This ensured that children spent time interacting with multiple apps, and therefore, interacting with multiple representations of the same mathematics content topic, to support concept development of that particular topic. Apps were selected by content alignment and were not selected based upon structural alignment.

During interviews, one researcher served as the interviewer and presented the mathematics tasks on the iPad to the child. A second researcher started the recording equipment and viewed the interview from the observation booth. A real-time video capture on a laptop allowed the second researcher to record observational notes while the interview was occurring. At the end of each interview, researchers downloaded the video data from the wall-mounted camera and the GoPro camera and secured it on an external hard drive device.

2.4.5 Data Analysis

Researchers first coded the video data through frame-by-frame video analysis to interpret children' interactions with the mathematics virtual manipulative apps. Video data were analyzed and coded for learning performance (i.e., children's accuracy in completing the tasks) and efficiency (i.e., changes in the speed with which the children completed the tasks). In the quantitative analysis, we used descriptive statistics to explore the data. Because the data were not normally distributed, we used the Wilcoxon Signed Ranked Test to analyze changes in learning performance and efficiency. This non-parametric statistical test uses the median of related samples (e.g., pre- and post-assessment scores) to compare data sets and is appropriate for skewed data and small samples.

In the qualitative analysis, we analyzed and coded the video data to identify children's actions, interactions, and access to app features for each app using a process of open coding. As themes emerged, we revisited the video data using axial coding to develop major categories. We identified specific examples to summarize patterns of children's observable interactions, to note when these interactions resulted in changes in performance or efficiency, and to note the content and structure of the apps that were being used at that time. Further, researchers identified samples in the videos to highlight trends in the data and that may contribute to the discussion on app alignment.

Our results in this paper focus specifically on children's learning performance and efficiency during the pre- and post-assessment portions of the interviews and on how the alignment of the apps might explain the changes. Other papers, based on the data collected in this large research project, detail children's learning progressions, explore app affordances, and describe strategies children used during interactions with the apps (e.g., Bullock, Moyer-Packenham, Shumway, Watts, MacDonald, 2015; Moyer-Packenham et al., 2014a, 2014b, 2015a, 2015b; Tucker & Moyer-Packenham, 2014; Tucker, Moyer-Packenham, Shumway, & Jordan, 2016; Watts et al., 2016).

2.5 Results and Discussion

The research question in this study focused on how the use of content-aligned and structurally-aligned virtual manipulative mathematics apps contributed to children's mathematics learning. The results presented discuss the quantitative findings, the qualitative frame-by-frame video analysis, and the complementarity of the results to understand how app alignment may explain some of the changes in children's learning. In the first section, we present the statistical results and discuss each of these results by age group. In the second section, we present the apps children used in each age group, along with figures from the video analysis that provide a representative composite panel of the children's interactions with the apps in each part of the interview sequences (i.e., a basic storyboard that shows a view of what children were doing with the mathematics objects within each of the apps). We then merge the quantitative and qualitative data to discuss the role of app alignment.

Learning Performance and Efficiency Results for All Age Groups

A summary of the pre- and post-assessment results for each age group is presented in Table 2.2. This table focuses on the significant results for all age groups for performance and efficiency.

As Table 2.2 shows, preschool children's (age 3–4) learning performance scores on the seriation and counting sequence tasks remained relatively constant, while their efficiency scores significantly improved for seriation and counting. Improved efficiency on both sequences could be the result of improved understanding of the tasks or it could be a function of learning the technology and more comfortably

Table 2.2 Summary table of performance and efficiency outcomes for pre- and post-assessment apps

Measures	N	Mean rank Post[a]	Mean rank Pre[a]	z	*p*
Preschool seriation	35				
Performance measure					NS
Efficiency measure		16.35	16.89	−2.095	.036*
Preschool counting	35				
Performance measure					NS
Efficiency measure 1		18.52	14.00	−4.244	.000**
Efficiency measure 2		18.65	13.07	−3.522	.000**
Kindergarten subitizing	33				
Performance measure		2.67	7.25	−2.228	.026*
Efficiency measure					NS
Kindergarten quantity	33				
Performance measure					NS
Efficiency measure		18.17	12.22	−2.880	.000**
Grade 2 skip counting	32				
Performance measure 1		.00	3.50	−2.214	.27*
Performance measure 2		.00	3.50	−2.214	.27*
Efficiency measure 1		14.98	20.17	−3.539	.000**
Efficiency measure 2		14.20	15.58	−2.495	.013*
Grade 2 place value	32				
Performance measure					NS
Efficiency measure					NS

[a]Negative ranks are shown first; then positive ranks for each paired condition. *Significant at $p < .05$; **significant at $p < .001$; NS indicates that the measures were not significant. This table is a reproduction of the results which were first reported in Moyer-Packenham et al. (2015a)

working with the apps on the post-assessments. While learning performance remained constant, preschoolers seemed to learn the physical mechanics needed to complete the tasks in a more efficient manner resulting in improved overall efficiency for both seriation and counting tasks.

Kindergarteners (age 5–6) showed significant increases in learning performance for subitizing, and improved efficiency for quantity. Kindergarteners seemed to improve in learning performance while also learning to use the technology efficiently. The Kindergarten quantity task included pre- and post-assessment apps and two learning apps that were all variations of the base-10 block virtual manipulative, which may have allowed the children to become familiar with the design of this app and its features. Additionally, kindergarteners' fine motor skills may have become more refined as they interacted with each base-10 block app.

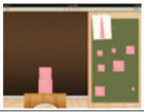
The Grade 2 (age 7–8) results in Table 2.2 showed significant increases in learning performance and efficiency for skip counting, but not for place value. Once again, these results could be due to improved skill in skip counting after working through the learning apps, greater facility with the apps, or a combination of improved mathematical understanding and efficiency with the technology. Results could have also been influenced by the similarity of the skip counting tasks because, in each task for skip counting, children were asked to count by 4s, 6s, and 9s. There seemed to be a ceiling effect on the pre-assessment for place value, with many children mastering the app tasks initially.

2.5.1 App Alignment Results for Preschool

This section presents the six apps used by preschoolers and the composite storyboard panels of typical preschoolers' interviews using video frames taken from the video data. We will use the term *video frame* throughout the paper when we are referring to the still images that were pulled from the video clips as a way to distinguish the static image (video frame) from the dynamic videos (video clip). In the sections that follow the presentation of the preschool data, we also present similar examples for kindergarten participants and Grade 2 participants. A screen shot of the six apps used by the preschool children (age 3–4) is presented in Table 2.3.

The screen shots in the left column of Table 2.3 show the counting task apps. In the Pre and Post App, children build a target number within 9 using base ten blocks. In Learning App 1, children build the sequence of numbers from 1 to 5 using base ten blocks. In Learning App 2, children count a set of base ten blocks within 9. We consider all three apps in the counting sequence to be content aligned because they all asked children to count, and we consider them structurally aligned because they all used the same mathematical objects (base ten blocks) and the same task (counting). All three apps were goal oriented (as opposed to open ended), because there was a correct response for each task.

Table 2.3 Screen shots of preschool apps

Counting tasks		Seriation tasks	
Pre/post app Montessori numbers (1–9)		Pre/post app Pink tower (free moving)	
Learning app 1 Montessori numbers (1–20: 1–5)		Learning app 1 Pink tower (Card #12)	
Learning app 2 Montessori numbers (1–9)		Learning app 2 Intro to math (red rods)	

The right column of Table 2.3 displays the seriation task apps. In the Pre and Post App, children build a tower with different sized free moving blocks from largest to smallest by dragging the blocks. In Learning App 1, children build a tower from largest to smallest with different sized static blocks by tapping the appropriate block. In Learning App 2, children order different sized rods from largest to smallest by dragging the rods. We consider all three apps in the seriation sequence to be content aligned because they all asked children to seriate similar objects from largest to smallest. We consider all three apps to be structurally aligned because they use similar mathematical objects (squares and rectangles) and the same task (seriate from largest to smallest). All three apps were goal oriented.

Preschool children's learning performance remained constant, but they experienced changes in efficiency for counting and seriation; therefore, we reviewed the video data to understand how app alignment may have contributed to changes in efficiency. Figure 2.3 shows a composite storyboard that includes video clips from six different preschool participants on the preschool seriation task. It includes four common participant errors by preschoolers on the pre-assessment, a sample of one participant using the Pink Tower learning app, and a sample of a successful participant on the post-assessment.

The top row of Fig. 2.3 shows four common participant errors made by the preschoolers on the seriation pre-assessment app. These were coded as errors because the expectation was that children would put the blocks in order from largest to smallest, building a pink tower. These four errors illustrate the variety of levels of conceptual understanding that children in the preschool interviews brought with them to the seriation task. Child #1 is an example of the first common error that children made; they stacked blocks directly on top of each other to create a short pile of blocks. Like others who built a pile of blocks, Child #1 did not stack the blocks in seriation order; rather, the blocks were stacked primarily by their proximity to the pile. Child #2 is an example of another common error where children built a misshapen tower. In this example, Child #2 builds a leaning tower with

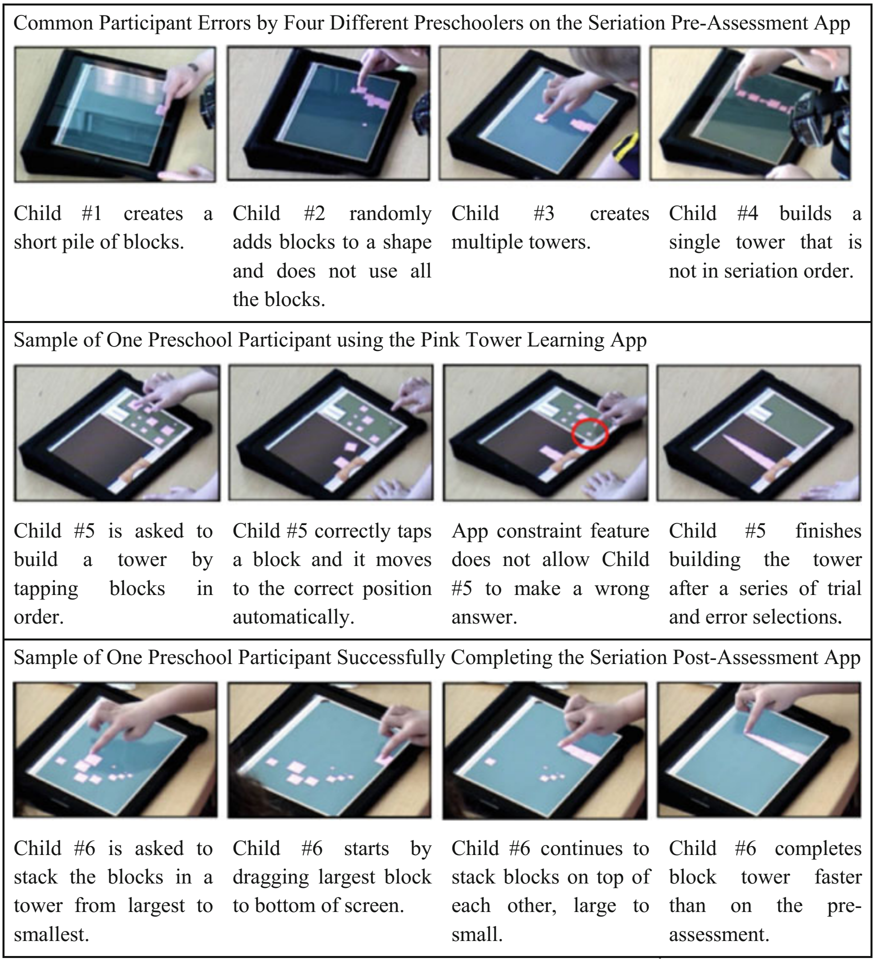


Fig. 2.3 Composite storyboard of Preschool participants' video examples from the seriation learning progression

about half of the blocks and then randomly added blocks to the middle or side of the tower. Sometimes these blocks appeared to be used to fill in gaps or curves in the shape. As seen in this video frame example, the smallest block was often left out of the odd shaped towers completely. Child #3 shows an example of a third common error where children created multiple towers. In this example, Child #3 created a short tower at the bottom of the iPad screen and then created a second tower by stacking blocks in a single pile. Other children stacked their second tower vertically, horizontally, or in a single pile. The fourth common error is shown by Child #4 where the child built a single tower, but not in seriation order from largest to smallest. Other children made similar errors such as having one or two blocks out of order or creating a pattern of alternating small and large blocks as is seen in the example of Child #4.

The middle row in Fig. 2.3 shows one preschooler, Child #5, using the pink tower learning app. The Pink Tower learning app had a close structural alignment with the pink tower pre- and post-assessment apps. As with the Pink Tower pre- and post-assessment apps, in the Pink Tower learning app children are presented with pink blocks of different sizes organized on the screen in a random order. However, in the Pink Tower learning app, when the child selects an incorrect block size, the block does not move. When the child selects the correct block size to put the blocks in seriation order, the blocks move automatically into the tower position. In the first video frame, Child #5 taps the largest block to begin building the pink tower. In the second video frame, after the child taps each block, the app moves the blocks automatically to the appropriate location to build the tower. As the third video frame for Child #5 shows, when the child selects the wrong block the app constraint feature in the Pink Tower learning app does not allow the child to build the tower incorrectly. When an incorrect block is selected, the block shrinks, turns in a circle, and settles back into its original position. The final video frame for Child #5 shows a completed tower after the child has made a series of trials and errors. This completed tower is the same size and structure as the tower presented to children before they interact with the pre- and post-assessment apps. The Pink Tower learning app may have helped increase preschoolers' efficiency on the post-assessment due to its close structural alignment with the pre- and post-assessment apps.

The bottom row of Fig. 2.3 shows one preschooler, Child #6, successfully building the pink tower in correct seriation order on the post-assessment app. In the first video frame, Child #6 is given the pink blocks in random order on the screen. In the second video frame, Child #6 starts building the tower at the bottom of the iPad screen. Although many children started building their tower in the middle of the screen on the pre-assessment app, all children efficiently started building their tower at the bottom of the screen on the post-assessment app. This may be due to children's experiences with the pre-assessment app or it may be due to the fact that the Pink Tower learning app started the tower at the bottom of screen. Overall, the majority of preschool participants were more efficient on the post-assessment app, completing their tower faster than the pre-assessment app. In the third and fourth video frames, Child #6 is seen completing the pink tower by stacking blocks vertically in the correct seriation order. Over half the children accurately stacked the blocks on the post-assessment app. Although this increase in performance was not statistically significant, the qualitative video analysis showed that preschool participants made fewer errors on the post-assessment app. The first two errors (piles as shown by the Child #1 example, and random shapes as shown by the Child #2 example) were virtually eliminated on the post-assessment app. Errors in multiple towers and seriation order were less pronounced on the post-assessment app, with the final towers more closely resembling the Pink Tower learning app in size, order, and orientation.

All preschool apps were chosen for their content alignment with counting and seriation. Repeated practice, due to content alignment, may have played a role in children's learning and efficiency. The close structural alignment of the Pink Tower apps for the seriation task, as well as the similarities in structural alignment of the base ten blocks for the counting task, likely contributed to preschool children's increases in performance and efficiency.

2.5.2 App Alignment Results for Kindergarten

We next present the video results for kindergarten (age 5–6). A screen shot of the six apps selected for kindergarten is presented in Table 2.4.

The left column of Table 2.4 shows that the kindergarten quantities apps all included base-10 blocks. In the Pre and Post App, children build a target number between 10 and 99 using base ten blocks. In Learning App 1, children build the sequence of numbers from 11 to 20 using base ten blocks. In Learning App 2, children count a set of base ten blocks between 10 and 99. We consider all three apps in the quantities sequence to be content aligned because they all asked children to build or identify quantities and we consider them structurally aligned because they all used the same mathematical objects (base ten blocks) and the same tasks (building quantities). All three apps were goal oriented.

The screen shots in the right column of Table 2.4 show the subitizing task apps. In the Pre and Post App, children subitize amounts within 10 and tell "how many more" to build the correct number. In Learning App 1, children subitize amounts within 10, combine amounts to create new quantities, and drag them to the fish. In Learning App 2, children subitize amounts of fruit by using all of their fingers to enter the correct amount on the touch screen. We consider all three apps in the subitizing sequence to be content aligned because they all asked children to subitize

Table 2.4 Screen shots of kindergarten apps

Quantities tasks		Subitizing tasks	
Pre/post app Montessori numbers (10–99)		Pre/post app Friends of ten	
Learning app 1 Montessori numbers (1–20: 11–20)		Learning app 1 Hungry guppy (dots)	
Learning app 2 Montessori numbers (10–99)		Learning app 2 Fingu (level 1)	

and combine quantities. We do not consider them to be structurally aligned because they differ in mathematical objects (ten frame vs. bubbles vs. fruit) and mathematical tasks (build vs. identify). All three apps were goal oriented.

Kindergartener's learning performance and efficiency produced mixed results (i.e., improved performance for subitizing, improved efficiency for quantities); therefore, we reviewed the video data to understand how app alignment may explain these results. Figure 2.4 shows a composite storyboard that includes video frames from six different kindergarten participants on the quantities tasks. It

Common Participant Errors by Four Different Kindergarteners on the Quantities Pre-Assessment App

			
Child #7 randomly places blocks and counts them individually attempting to make 14.	Child #8 counts 1 and 4 with unit cubes for a total of 5 cubes, while attempting to make 14.	Child #9 confuses tens and ones to create 41, while attempting to make 14.	Child #10 counts 9 tens (90) and 5 ones (5) because 9+5=14, while attempting to make 14.

Sample of One Kindergarten Participant using the Montessori Numbers Learning App

			
Child #11 watches as app counts from 11 to 20, highlighting each number as it is spoken.	Child #11 moves a tens rod to model the number 10.	Child #11 moves 5 ones to complete modeling a total of 15 cubes	Child #11 watches as app counts 15 cubes as 10, 11, 12, 13, 14, 15, highlighting each cube.

Sample of One Kindergarten Participant Successfully Completing the Quantities Post-Assessment App

			
Child #12 is asked to show 14 using blocks.	Child #12 places 1 tens rod in the tens column.	Child #12 places 4 unit blocks in the ones column.	Child #12 completes post-assessment faster than pre-assessment.

Fig. 2.4 Composite storyboard of kindergarten participants' video examples from the quantity learning progression

includes four common participant errors by kindergarteners on the pre-assessment app, a sample of one kindergarten participant using the Montessori Numbers learning app and a sample of one kindergarten participant who was successful on the post-assessment app.

The top row of Fig. 2.4 shows four common participant errors made by kindergarteners on the quantities pre-assessment app. These were coded as errors because students were expected to build the target number. These four errors illustrate the variety of levels of conceptual understanding that kindergarteners brought with them to the quantities task. Child #7 is an example of the first common error in which children randomly placed a number of rods in the tens column and unit cubes in the ones column. Child #7 counted each cube individually, counting all 40 cubes in the tens column for the model she created, and ignored the cubes in the ones column. The second common error can be seen in the video frame for Child #8. This child counted one unit cube to represent the digit 1 in the number 14 and then counted four more unit cubes to represent the digit 4 in the number 14 for a total of 5 unit cubes. Other children made similar errors using only ten rods to count out the number 14. The third common error can be seen in the video frame for Child #9 where the tens and ones place values are confused. Child #9 placed one unit cube in the ones column and four tens rods in the tens column for a total of 41 cubes instead of 14. Child #10 shows the fourth common error where children counted a total of 14 rods or unit cubes. Child #10 filled the tens column with 9 rods—the maximum for the tens column. She then continued adding 5 unit cubes to the ones column, counting 10, 11, 12, 13, 14. Other children started with the ones columns and ended in the tens column or switched between tens rods and unit cubes for a total of 14 items. Similar errors were enacted for the other pre-assessment numbers of 31 and 50. All four of these common errors can be categorized as place value errors.

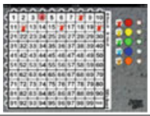
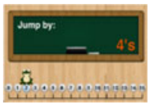
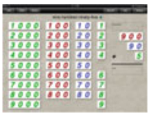
The middle row in Fig. 2.4 shows one kindergartener, Child #11, using the Montessori Numbers learning app. As seen in the first video frame, Child #11 can observe the Montessori Numbers app as the audio portion of the app counts the numbers from 11 to 20. This audio feature allows Child #11 to hear the number names and associate them with the numerals. Children are then prompted to move cubes to build each number, starting with the tens and ending with the ones. The second video frame for Child #11 shows him adding a single tens rod to represent the number 10 in the number 15. The third video frame shows him adding five unit cubes to the tens rod to create a total of 15 (one ten and five ones). The app constraint feature does not allow Child #11 to add more tens rods or unit cubes than needed for each number. If children do not have enough cubes, the app will prompt them to add cubes until the correct number of cubes has been created. The last video frame for Child #11 shows the app audio counting the total number of cubes, starting with the tens rod and saying “10” and continuing to count unit cubes as 11, 12, 13, 14, 15, to the target number. The counting strategy in this learning app focuses children on place value concepts by highlighting tens and ones separately and as a whole. The second learning app also focuses on place value by highlighting the relationship between numerals in the tens or ones place and the number of tens rods or ones unit cubes.

The bottom row of Fig. 2.4 shows one kindergartener, Child #12, who was successful on the post-assessment app by building the number 14 using the appropriate number of tens rods and unit cubes. The first video frame for Child #12 shows him at the starting point with open tens and ones columns, rods and unit cubes at the bottom, and the number 14 at the top of the iPad screen. In the second video frame, Child #12 selects a tens rod and counts “10” out loud. In the next video frame, the child adds four unit cubes to the ones column, counting 11, 12, 13, 14. The final video frame shows Child #12 accurately portraying the number 14 using one tens rod and four unit cubes with the base-ten blocks.

About three-quarters of the children accurately represented the numbers 14, 31, and 50 on the quantities post-assessment app. Alignment of counting strategies in the learning apps that focused on place value may have contributed to a reduction of common place value errors on the post-assessment app for the quantities task. The narrow content alignment of multiple representations of different objects and amounts, which engaged children in app interactions where they repeatedly practiced subitizing amounts, likely contributed to kindergarten children’s increases in performance for the subitizing tasks. The lack of structural alignment between subitizing tasks may have played a role in the lack of efficiency gains for the pre- and post-assessment apps in this mathematical content topic.

The majority of kindergarteners were more efficient on the post-assessment app for the quantities task, building the three numbers faster than on the pre-assessment app. The tens rods and unit cubes in both Montessori learning apps are identical in structure to the pre-assessment app. This structural alignment likely increased kindergarteners’ familiarity with the post-assessment app tasks.

Table 2.5 Screen shots of grade 2 apps

Place value tasks		Skip counting tasks	
Pre/post app Montessori numbers (100–999)		Pre/post app 100s board	
Learning app 1 Math motion zoom (levels 2–4)		Learning app 1 Number lines (skip counting)	
Learning app 2 Place value cards (3-digit)		Learning app 2 Skip counting beads	

2.5.3 App Alignment Results for Grade 2

This section presents the video analysis results for children in Grade 2 (age 7–8). A screen shot of the six apps selected for Grade 2 is presented in Table 2.5.

The left column of Table 2.5 displays the place value tasks. In the place value Pre and Post App, children build target numbers with base-10 blocks. In Learning App 1, children place a target number on a movable number line by swiping the number line left and right. In Learning App 2, children create a target number by dragging place value cards. We consider all three to have a broad content alignment as all three focus on different aspects of place value (e.g., numerical place value, expanded notation place value, and place value on a number line). We do not consider the place value apps to be structurally aligned because they differ in mathematical objects (blocks vs. number line vs. place value cards) and mathematical tasks (build vs. locate). All three apps were goal oriented.

The right column of Table 2.5 displays the skip counting tasks. In the Pre and Post App, children touch numbers on a hundreds board to identify numbers in a skip counting sequence. In Learning App 1, children move a frog along a number line to skip count by a given amount. In Learning App 2, children skip count by grouping beads and matching skip counting numerals to the grouped beads. We consider all three to have close content alignment because all three focus on skip counting. We do not consider them to be structurally aligned because they differ in mathematical objects (hundreds board vs. number line vs. beads) and mathematical tasks (identify vs. build vs. match). All three apps were goal oriented.

Grade 2 learning performance and efficiency improved significantly for skip counting but remained constant for place value tasks; therefore, we reviewed the video data to understand how app alignment may have contributed to student outcomes. Figure 2.5 shows a composite storyboard that includes video frames from six different Grade 2 participants for the apps in the skip counting sequence. It includes four common participant errors by Grade 2 participants on the pre-assessment, a sample of one Grade 2 participant using the Number Lines Learning App 1, and a sample of one Grade 2 participant successfully completing the post-assessment app.

The top row of Fig. 2.5 shows four common errors made by Grade 2 participants on the skip counting pre-assessment app. These were coded as errors because students were expected to choose the correct numbers to count by a given number in the skip counting sequence. These four errors illustrate the variety of levels of conceptual understanding that children brought with them to the skip counting task. Child #13 is an example of the first common error for Grade 2 where children would not attempt to skip count using the hundreds board pre-assessment app. Child #13 told the interviewer: "I do not know how to do nines" and did not complete this portion of the pre-assessment, even though he had previously attempted to skip count by 4 and 6. Child #14 is an example of another common error where children miscounted using their fingers to assist in the counting process. This child counted the first finger as 9 and, after counting several more fingers,

Fig. 2.5 Composite storyboard of Grade 2 participants' video examples from the skip counting learning progression

ended on 16 instead of 18. The third common error, made by Child #15, was children relying on a visual pattern rather than a numerical pattern. Child #15 selected every other number in the same column, starting with 9. Other visual patterns that caused children to make errors included double columns such as counting the number sequence 4, 8, 14, 18, 24, 28. Child #16 is making the fourth common error which was children miscounting or selecting random numbers. Usually, these types of errors did not end on the number requested by the interviewer. Child #16 was asked to skip count by 9s to 36. The child selected 9, 18, 25, 30, and 37. Other miscounting errors included choosing a number smaller than the original number such as the sequence 9, 8, 18.

The middle row of Fig. 2.5 shows one Grade 2 participant, Child #17, interacting with the Number Lines learning app. What is unique about this task, is that the app does not start the skip counting on a multiple of the requested number. For example, in the first picture, Child #17 is asked to skip count by 4s with a starting number of 1. This focuses children on using strategies other than memorization. The app highlights the number in yellow when the frog is correctly placed, as seen in the second picture on the middle row. The app waits 3 s before highlighting a correct answer. This allows children to use a variety of counting strategies. In the third picture, Child #17 uses a strategy of "plus 3 and 1 more" to skip count 4 spaces from 9. He knew that 9 plus 3 was 12 and one more was 13. A similar strategy was used by this child on the post-assessment.

The bottom row of Fig. 2.5 shows one Grade 2 participant, Child #18, successfully skip counting by 9s to 36 on the post-assessment app. In the first picture, Child #18 begins the task with a blank hundreds board. In the second picture, Child #18 selects 9 as the first multiple. In the third picture, he quickly continues the task by selecting multiples of 9. The fourth picture shows Child #18 successfully completing the skip counting task by stopping on 36.

Almost all Grade 2 children increased in accuracy and efficiency for the skip counting sequence. The content alignment of the apps and the tasks played a significant role in the results. Each app used a different representation (i.e., hundreds board, number line, and grouped beads) to visualize skip counting. In addition, the numbers in all three apps were closely content aligned to focus primarily on skip counting by 4s, 6s, and 9s. This close content alignment of the tasks, as well as using multiple representations to complete the tasks, likely explains the significant changes in Grade 2 children's performance on skip counting tasks as well as their increased efficiency with the numbers 4, 6, and 9.

The place value task did not have significant gains in either efficiency or performance. Although the apps appeared to be content aligned, it appeared that the focus of the content covered too broad a range of place value skills (e.g., numeral place values, expanded notation place value, and place value on a number line). In addition, the place value apps did not have a close structural alignment. The lack of a more specific and focused content alignment, as well as the lack of structural alignment, may be one reason that there were no significant changes in children's performance and efficiency on the place value tasks.

2.6 Conclusion

Research on the use of mathematics apps frequently shows that experiences with the apps have a positive influence on young children's learning. However, most studies do not go beyond the performance outcomes to explain *why* the apps have an impact. This study contributes insights that may explain why some mathematics apps may lead to improvements in children's mathematics performance and efficiency. The degree of content alignment and structural alignment may explain why significant results were not reached for performance and efficiency in all of the tasks in this study and in results reported in other studies.

As our results showed, in some cases, changes were related to the mathematical content alignment and the structural alignment of the apps. When the learning apps had a focused mathematical content alignment, as found in the kindergarten subitizing task and the Grade 2 skip counting task, children significantly increased in their performance. Additionally, when the focused content alignment targeted common errors and misconceptions, children's performance increased. As Edwards Johnson et al. (2012) observed, there were learning benefits for a close alignment between tasks and mathematical content topic. In our study, the results support this idea for mathematical topics that were closely aligned (e.g., skip counting in Grade 2) and those that were not as closely aligned (e.g., place value in Grade 2). For the children in this study, content alignment appeared to be beneficial to performance outcomes. However, we have no evidence of long term effects on performance.

When the learning apps were structurally aligned, as found in the preschool tasks and the kindergarten quantities tasks, children demonstrated significant improvements in completing tasks with greater efficiency (and this greater efficiency coincided with greater accuracy, although not statistically significant). Using a variety of apps with the same structure may have reduced technological distance. Technological distance is defined as "the degree of difficulty in understanding how to act up on [something] and interpret its responses" (Sedig & Liang, 2006, p. 184). The opportunities to use structurally similar apps may have reduced some of the technological distance between the app and the child and better allowed the child to focus on the mathematical tasks presented within the app. As Uttal et al. (2013) reported, alignment between structural format of learning and testing method had a positive influence. In our study, we observed a similar phenomenon about the use of the same app for multiple tasks. When the same app and similar apps were used, that is, they had the same structure (i.e., structural alignment) this appeared to be beneficial to efficiency outcomes.

It is important to support young children's conceptual development by designing learning experiences that engage them in the use of multiple representations within the same mathematical content topic. As the research reported in this paper demonstrates, aligning apps for content can contribute to young children's performance; and, aligning apps closely in structure can improve children's efficiency with tasks in the mathematics apps in a short period of time. Further research is needed to explore how the alignment of apps for instruction might influence children's learning over time.

References

Baccalini-Frank, A., & Maracci, M. (2015). Multi-technology and preschoolers' development of number-sense. *Digital Experiences in Mathematics Education,* 1–21. https://doi.org/10.1007/s40751-015-0002-4.

Bakker, M., van den Heuvel-Panhuizen, M., & Robitzsch, A. (2015). Effects of playing mathematics computer games on primary school students' multiplicative reasoning ability. *Contemporary Educational Psychology, 40*, 55–71. https://doi.org/10.1016/j.cedpsych.2014.09.001.

Barendregt, W., Lindström, B., Rietz-Leppänen, E., Holgersson, I., & Ottosson, T. (2012). Development and evaluation of Fingu: A mathematics iPad game using multi-touch interaction. In H. Schelhowe (Ed.), *Proceedings of the 11th International Conference on Interaction Design and Children* (pp. 204–207). New York, NY: ACM. https://doi.org/10.1145/2307096.2307126.

Boyer-Thurgood, J. (2017). *The anatomy of virtual manipulative apps: Using grounded theory to conceptualize and evaluate educational mathematics apps* (Doctoral dissertation). Utah: Utah State University.

Bullock, E. P., Moyer-Packenham, P. S., Shumway, J. F., Watts, C., & MacDonald, B. (2015, March). Effective teaching with technology: Managing affordances in iPad apps to promote young children's mathematics learning. In D. Rutledge & D. Slykhuis (Eds.), *Proceedings of the Society for Information Technology and Teacher Education International Conference* (pp. 2357–2364), Las Vegas, Nevada.

Creswell, J. W., & Plano Clark, V. L. (2011). *Designing and conducting mixed methods research* (2nd ed.). Thousand Oaks, CA: Sage Publications Inc.

Edwards Johnson, P., Campet, M., Gaber, K., & Zuidema, E. (2012). Virtual manipulatives to assess understanding. *Teaching Children Mathematics, 19*(3), 202–206. https://doi.org/10.5951/teacchilmath.19.3.0202.

Gröger, C., Silcher, S., Westkämper, E., & Mitschang, B. (2013). Leveraging apps in manufacturing. A framework for app technology in the enterprise. *Procedia CIRP, 7,* 664–669. https://doi.org/10.1016/j.procir.2013.06.050.

Jordan, N., Glutting, J., & Ramineni, C. (2010). The importance of number sense to mathematics achievement in first and third grades. *Perspectives on Math Difficulty and Disability in Children, 20*(2), 82–88. Downloaded from http://dx.doi.org/10.1016/j.lindif.2009.07.004.

Kermani, H., & Aldemir, J. (2016, March). Using iPads in the classroom to teach young children early math skills. In *Proceedings of the Society for Information Technology & Teacher Education International Conference* (Vol. 1, pp. 1445–1449).

Kiger, D., Herro, D., & Prunty, D. (2012). *The influence of a mobile learning intervention on third-grade math achievement* (Research Brief) (p. 29). Oconomowoc, Wisconsin: Oconomowoc Area School District.

Lesh, R., Landau, M., & Hamilton, E. (1983). Conceptual models in applied mathematical problem solving research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts & processes* (pp. 263–343). NY: Academic Press.

Moyer, P. S., Bolyard, J. J., & Spikell, M. A. (2002). What are virtual manipulatives? *Teaching Children Mathematics, 8*(6), 372–377.

Moyer-Packenham, P. S., Anderson, K. L., Shumway, J. F., Tucker, S., Westenskow, A., Boyer-Thurgood, J., et al. (2014a, January). Developing research tools for young children's interactions with mathematics apps on the iPad. In *Proceedings of the 12th Annual Hawaii International Conference on Education* (*HICE*) (pp. 1685–1694). Honolulu, Hawaii, ISSN# 1541-5880.

Moyer-Packenham, P. S., & Bolyard, J. J. (2016). Revisiting the definition of a virtual manipulative. In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 5–16). New York: Springer.

Moyer-Packenham, P. S., Bullock, E. P., Shumway, J. F., Tucker, S. I., Watts, C., Westenskow, A., et al. (2015a). The role of affordances in children's learning performance and efficiency when using virtual manipulative mathematics touch-screen apps. *Mathematics Education Research Journal,* (Special Issue), 1–27. https://doi.org/10.1007/s13394-015-0161-z.

Moyer-Packenham, P. S., Shumway, J. F., Bullock, E., Tucker, S. I., Anderson-Pence, K. L., Westenskow, A., et al. (2015b). Young children's learning performance and efficiency when using virtual manipulative mathematics iPad apps. *Journal of Computers in Mathematics and Science Teaching, 34*(1), 41–69.

Moyer-Packenham, P. S., & Suh, J. M. (2012). Learning mathematics with technology: The influence of virtual manipulatives on different achievement groups. *Journal of Computers in Mathematics and Science Teaching, 31*(1), 39–59.

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments, 4*(3), 35–50.

Moyer-Packenham, P. S., Westenskow, A., Shumway, J. F., Bullock, E., Tucker, S. I., Anderson-Pence, K. L., et al. (2014b, September). The effects of different virtual manipulatives for second graders' mathematics learning in the touch-screen environment. In *Proceedings of the 12th International Conference of the Mathematics Education into the 21st Century Project* (Vol. 1, pp. 331–336). Herceg Novi, Montenegro.

Schrock, K. (2011). *Critical evaluation of a content-based iPad/iPod app*. Retrieved from http://ipads4teaching.net.

Schubert, C. (2009). Video analysis of practice and the practice of video analysis. In H. Knoblauch, B. Schnettler, J. Raab, & H. G. Soeffner (Eds.), *Video analysis methodology and methods: Qualitative audiovisual data analysis and sociology* (pp. 115–126). New York, NY: Peter Lang.

Sedig, K., & Liang, H. N. (2006). Interactivity of visual mathematical representations: Factors affecting learning and cognitive processes. *Journal of Interactive Learning Research, 17*(2), 179–212.

Segal, A. (2011). *Do gestural interfaces promote thinking? Embodied interaction: Congruent gestures and direct touch promote performance in math* (Unpublished doctoral dissertation). Columbia University, New York, NY.

Tashakkori, A., & Teddlie, C. (Eds.). (2010). *Sage handbook of mixed methods in social & behavioral research* (2nd ed.). Thousand Oaks, CA: Sage.

Tucker, S. I., & Moyer-Packenham, P. S. (2014, July). Virtual manipulatives affordances influence mathematical understanding. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 6, p. 251). Vancouver, Canada: PME.

Tucker, S. I., Moyer-Packenham, P. S., Shumway, J. F., & Jordan, K. E. (2016). Zooming in on students' thinking: How a number line app revealed, concealed, and developed students' number understanding. *Australian Primary Mathematics Classroom, 2*(1), 23–28.

Uttal, D. H., Amaya, M., del Rosario Maita, M., Hand, L. L., Cohen, C. A., O'Doherty, K., et al. (2013). It works both ways: Transfer difficulties between manipulatives and written subtraction solutions. *Child Development Research*, *2013*, 1–13, Article ID 216367. https://doi.org/10.1155/2013/216367.

Walker, H. (2010). *Evaluation rubric of iPod apps*. Retrieved from http://ipads4teaching.net.

Watts, C., Moyer-Packenham, P. S., Tucker, S. I., Bullock, E., Shumway, J. F., Westenskow, A., et al. (2016). An examination of children's learning progression shifts while using touch-screen virtual manipulative mathematics apps. *Computers in Human Behavior, 64,* 814–828.

Chapter 3
Applying the Modification of Attributes, Affordances, Abilities, and Distance for Learning Framework to a Child's Multi-touch Interactions with an Idealized Number Line

Stephen I. Tucker

Abstract Technologies such as touchscreen apps are increasingly popular in mathematics education. Researchers have begun to investigate children's interactions with the apps, outcomes of using apps, and the characteristics that contribute to outcomes. This study applies the Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning Framework to an 11 year-old child's interactions with the mathematics app Motion Math: Zoom to evaluate the outcomes, contributors, and interactions. This framework accounts for relationships among attributes, affordance-ability relationships, and distance involved in interactions. Interacting with Motion Math: Zoom involves using multi-touch gestures to navigate an idealized number line with changeable interval scales. Findings indicate that the framework can contribute to research on the outcomes, contributors, and interactions, as well as linking the three.

Keywords Attributes · Affordances · Distance · Number line · Multi-touch

3.1 Introduction

Technology is becoming ubiquitous in education, including as a support for learning mathematics. In particular, touchscreen mobile devices (e.g., iPads) featuring mathematics apps are becoming popular in schools. An array of research has focused on outcomes of using apps to learn mathematics, finding generally positive results (e.g., Moyer-Packenham et al., 2015; Riconscente, 2013). Researchers have

S. I. Tucker (✉)
Department of Early Childhood and Elementary Education,
College of Education and Human Development, University of Louisville,
1905 South 1st Street, Louisville, KY 40292, USA
e-mail: s.tucker@louisville.edu

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_3

begun to identify characteristics of apps that may contribute to these outcomes (e.g., Larkin, 2015). Other researchers have begun closely examining the children's interactions with multi-touch mathematics apps and their implications for the experience of learning mathematics (e.g., Baccaglini-Frank & Maracci, 2015). Recently, the Modification of Attributes Affordances, Abilities, and Distance (MAAAD) for Learning framework (Tucker, 2015, 2016) emerged, modeling relationships among key constructs that contribute to user-tool interactions. This study applies the MAAAD for Learning framework to a child's interactions with a multi-touch mathematics app involving conceptually congruent gestures to navigate an idealized number line, demonstrating its potential for examining outcomes, characteristics contributing to the outcomes, and user-tool interactions.

3.2 Perspectives on Learning Mathematics with Touchscreen Technology

Research on learning mathematics with technology often focuses on outcomes, characteristics that contribute to outcomes, and the user-tool interactions themselves. In particular, this section focuses on research involving touchscreen mathematics apps ("apps"). Studies often have a main focus among the three areas. Research emphasizing outcomes often found that using apps has positive effects on children's mathematics. In a study of 122 fifth grade children, using the app Motion Math: Fraction Bounce led to statistically significant improvements in performance on fractions assessments and attitude towards fractions relative to instruction without the app (Riconscente, 2013). However, changes in children's performance and efficiency while completing mathematics tasks using apps can vary by age and mathematics content. A study of 100 children completing tasks using apps found that preschool children increased efficiency and maintained performance, Kindergarten children increased performance and maintained efficiency, and second grade children increased performance and efficiency in skip counting but not place value (Moyer-Packenham et al., 2015). Researchers have also documented content-specific progressions in children's learning while using apps, identifying developmental shifts and related behavior patterns (Watts et al., 2016). Outcome-focused research suggests that apps can be effective tools for learning mathematics.

Research has also focused on characteristics of apps that may contribute to outcomes. Influential characteristics include modality (i.e., sensory perception) and gestural congruence (i.e., degree to which the input gesture reflects the mathematics). Research suggests that direct manipulation modalities (e.g., touch object on screen to control) may be more effective for learning mathematics than indirect manipulation modalities (e.g., touch mouse to control object on screen) (Paek, Hoffman, & Black, 2016). Furthermore, using a direct manipulation app with conceptually congruent gestures (e.g., on a static number line, dragging to move an indicator into position rather than only tapping to indicate a location) can have a positive impact on children's mathematical understanding relative to using an otherwise identical mouse-controlled app or an app with non-congruent gestures (Segal, Tversky, & Black, 2014). Affordances related to these characteristics may influence children's outcomes on mathematics tasks. In particular, there may be relationships between helping or hindering affordance access patterns and children's task performance and efficiency (Moyer-Packenham et al., 2016). Many effectively-used apps include virtual manipulatives, which are "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (Moyer-Packenham & Bolyard, 2016, p. 13). Virtual manipulatives offer specific categories of affordances that positively influence learning, including *efficient precision*, which involves allowing efficient, accurate use of precise representations (Moyer-Packenham & Westenskow, 2013, 2016). Researchers also synthesized and applied research on digital technology use and student learning to generate criteria for evaluating apps, generally concluding that few apps support deep learning and meaningful development of mathematical knowledge (e.g., Goodwin & Highfield, 2013; Larkin, 2015). Some evaluation and design frameworks explicitly integrate affordances, such as a set of cognitive guidelines for evaluating and designing apps (Ginsburg, Jamalian, & Creighan, 2013). Research indicates that app characteristics can influence mathematical learning outcomes.

Studies primarily focused on user-tool interactions have found unique occurrences during children's interactions with apps, including multi-touch apps. Researchers investigating preschool children's interactions with three multi-touch number sense apps found patterns in student strategies that provided evidence of components of number sense, leading them to conclude that multi-touch technology may offer unique opportunities for developing number sense and observing development of number sense (Baccaglini-Frank & Maracci, 2015). Other researchers examined groups of 6–8 year-old children's interactions with a multi-touch counting and arithmetic app, TouchCounts, finding that physical and social engagement involving children and the app created rhythms of social interactions that moved from specific to generalizable (Sinclair, Chorney, & Rodney, 2015). However, relatively few apps leverage multi-touch capabilities for mathematically relevant purposes (Byers & Hadley, 2013). Research focusing on the user-tool interactions suggests that using multi-touch technology may have distinctive implications for learning mathematics.

The aims are not mutually exclusive and inform each other, and although many of the aforementioned studies have implications for multiple aims, other research more explicitly addresses these areas. For example, an iterative design-based research study reported on empirical outcomes, app characteristics, and user-tool interactions involved in 5–7 year-old children's use of the early number sense app Fingu (Holgersson et al., 2016). Researchers found that playing Fingu had positive effects on immediate and delayed assessments, that children who answered tasks on advanced levels quickly were more likely to use subitizing affordances on tasks involving non-canonical representations of five than other children, and that children could adapt response patterns to suit their preferences. Research suggests that apps have the potential to support mathematics learning, but that outcomes, contributors, and interactions bear investigation.

3.2.1 Navigating the Number Line

The mathematics in this study focused on navigating a number line. In theory, a number line is infinite in scale and magnitude, but in practice, representations are constrained. Even so, a physical representation of a number line can be an effective tool for developing understandings of magnitude and comparison, concepts children often struggle to comprehend (Rittle-Johnson, Siegler, & Alibali, 2001). Although using number lines may support exploration of otherwise unfamiliar ranges of numbers (Siegler & Booth, 2004), when using a number line, people may process negative numbers more slowly than positive numbers (Fischer & Roitmann, 2005). Many children have difficulty with the concept of density, which involves understanding that intervals always contain infinitely many numbers (Vamvakoussi & Vosniadou, 2010), the foundations of which can be explored on a number line. Thus, despite constraints of commonly used number line representations, they often feature in mathematics instruction.

Representations and interactions used in number line research vary. Research involving number lines frequently includes pictorial representations of number lines in physical space, often drawn on paper (e.g., Geary, Bailey, & Hoard, 2009; Laski & Siegler, 2007; Siegler & Booth, 2004). These representations of number lines are constrained by the space available and the difficulty of depicting density (i.e., inability to fluidly change magnitude and scale). Research involving virtual number lines often resembles those used with a paper-based number line. These number lines range from involving indirect manipulation using non-congruent gestures via a mouse or keyboard (e.g., Fischer & Roitmann, 2005), indirect manipulation using relatively congruent gestures via mouse drag (e.g., Cohen & Blanc-Goldhammer, 2011), direct manipulation using non-congruent gestures via tapping the screen at the intended place (e.g., Schneider, Grabner, & Paetsch, 2009), or direct manipulation using relatively congruent gestures via dragging an indicator across the screen to label the number line (e.g., Dubé & McEwen, 2015; Segal et al., 2014). In each case, participants could not affect the magnitude or scale displayed on the number line.

Multi-touch technology can afford direct manipulation with congruent gestures to navigate a moveable number line. Compared to a physical number line, the “idealized” interactive digital representation available in some virtual manipulative number lines (e.g., in Motion Math: Zoom) is more faithful to a theoretical number line, which is infinite in both scale and magnitude (Kirby, 2013). An interactive idealized number line can include fluid movement and changeable interval scales (Zhang, Trussell, Gallegos, & Asam, 2015) that would not be possible without digital technology (Carpenter, 2013). Combined with conceptually congruent multi-touch gestures, this may offer unique possibilities for exploring a number line and developing understandings of the relevant mathematical concepts. Although studies have included multi-touch apps featuring interactive idealized number lines (e.g., Moyer-Packenham et al., 2015; Tucker, Moyer-Packenham, Shumway, & Jordan, 2016; Zhang et al., 2015), there has been little elaboration on the interactions involving interactive idealized number lines.

3.2.2 The Modification of Attributes, Affordances, Abilities, and Distance for Learning Framework

The MAAAD for Learning framework models relationships among key constructs to describe user-tool interactions, including mathematical practices and changes in these practices (Tucker, 2015, 2016). The framework originated in theories of representation and embodied cognition (see Tucker, 2015 for detailed discussion), which can be set within Activity Theory. *Activity* is an ongoing interaction between subject and object (Leontiev, 1978). Subjects, such as humans, have needs that must be met, including learning (e.g., forming understandings, developing skills). In order to meet these needs, subjects interact with objects, which exist independently of humans and have widely accepted meanings. Through this activity, subjects create an image or understanding of an object. However, the image is a representation of the object and is not necessarily consistent with the widely accepted meaning. Activity can involve learning mathematics, which consists of physically embodied interactions with representations of mathematics, including internalizing external representations (e.g., interpreting graphs, symbols, and pictures) and externalizing internal representations (e.g., writing, speaking, manipulating digital objects) (Goldin & Kaput, 1996). This occurs via perceptuomotor integration during physical interactions with the environment, interrelating perceptual and motor aspects of tool use (Nemirovsky, Kelton, & Rhodehamel, 2013). During this activity, physical engagement in mathematical practices is equivalent to mathematical thinking, and changes in these practices are mathematical learning. In this context, the tools are the artifacts that represent mathematics (i.e., the object), mediating the subject-object interaction (Ladel & Kortenkamp, 2016). Therefore, examining constructs that contribute to physically embodied mathematical practices involved in these user-tool interactions (i.e., activity) can shed light on mathematical learning.

3.3 Building the Framework

Attributes, affordances, abilities, and distance are interrelated in children's interactions with technology tools (see Fig. 3.1). The framework begins with *attributes*, which are characteristics of people or things (Attribute [Def. 5], 2014).

Users and tools have attributes that contribute to the interactions. In this context, an *affordance* relates attributes of a tool to an interactive activity undertaken by a user, whereas an *ability* relates attributes of a user to an interactive activity involving a tool (Gibson, 1986; Greeno, 1994). Therefore, attributes, via affordances and abilities, are involved in the activity involving user and tool. An affordance and its corresponding ability exist only in relation to each other (Greeno, 1994), linked in a continuous system (Chemero, 2003). Affordance-ability relationships are complex, as each attribute may contribute to multiple affordance-ability relationships and attribute changes influence affordance-ability relationships (Tucker, Moyer-Packenham, Westenskow, & Jordan, 2016). For example, a hammer affords driving nails into a wall. The weight, shape, and balance attributes of the hammer contribute to this affordance. To access this affordance, a user draws on an ability that combines attributes of strength, coordination, and perception. These attributes combine differently in the affordance-ability relationship of removing nails from a wall.

Another relevant construct is *distance*, which is the "degree of difficulty in understanding how to act upon [something] and interpret its responses" (Sedig & Liang, 2006, p. 184). In this context, distance is the difference in alignment of related clusters of attributes, which does not remain static (Tucker, 2015, 2016). Modifying attributes may increase or decrease distance. Distance also interacts with affordance-ability relationships. Continuing the example from above, an experienced carpenter may encounter a low degree of distance when using a hammer. Her relevant attributes align with those of the hammer, as her strength is sufficient for moving the hammer, she can coordinate these motions to account for its balance, etc. These attributes also combine to form abilities that permit fluent access to nail driving and nail removal affordances. However, a novice with less strength and coordination may encounter a high degree of distance. Modifying his strength attribute to become stronger, he may be able to swing the hammer, decreasing distance. Nevertheless, he may miss the nail when attempting to access the nail driving affordance and he may not recognize that the claw of the hammer contributes to the nail removal affordance. Throughout the interactions, there is potential for modifying attributes, affordance-ability relationships, and distance, each of which can influence the other constructs.

Fig. 3.1 Modification of attributes, affordances, abilities, and distance for learning framework (Tucker, 2015, p. 117)

3.4 Applying the Framework

The MAAAD for Learning framework can apply to various types of user-tool interactions including children's interactions with mathematics apps (see Fig. 3.2) (Tucker, 2015, 2016). Users and apps have categories of attributes that form the base of these interactions. Both users and apps have *mathematical attributes* related to the mathematics involved in the interactions. For both, subcategories of mathematical attributes include content (e.g., multiplicative identity property) and representations (e.g., pictorial form). Users also have a subcategory of flexibility, which involves connections among content and representations (e.g., one fourth, 0.25, one object out of four). The difference between relevant clusters of mathematical attributes forms *mathematical distance* (e.g., low: strong knowledge of addition as presented by the app). Users and apps also have *technological attributes* related to the physical elements of the interactions. User technological attribute subcategories include motor skills (e.g., coordination) and input familiarity (e.g., do I tap or drag?). App technological attribute subcategories include input range (e.g., pinch recognized unless vertical) and input complexity (e.g., two fingers moving apart). The difference between relevant clusters of technological attributes forms *technological distance* (e.g., high: struggle to coordinate fingers to produce a gesture the app recognizes). Apps have *structural attributes*, which are the elements that support the presentation of mathematical and technological attributes, including scaffolding (e.g., hint option), context (e.g., leveled game), and feedback (e.g., reward depends on objectives met). Users have *personal attributes*, which are personality characteristics that influence the interactions, including affect

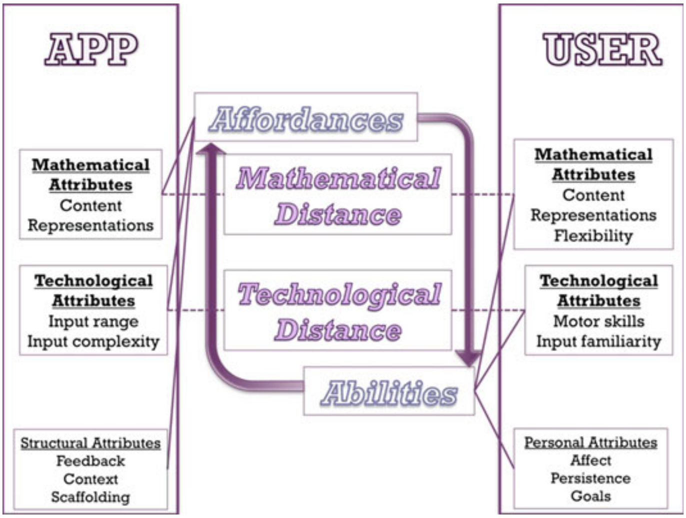


Fig. 3.2 Modification of attributes, affordances, abilities, and distance for learning framework applied to learning mathematics through user-app interactions (Tucker, 2015, p. 119)

(e.g., positive association with the app), persistence (e.g., trying again after mistake), and goals (e.g., finish quickly). Combinations of attributes across categories contribute to affordance-ability relationships (e.g., user coordinates tap gesture to perform additive identity property, app recognizes input and responds by simultaneously linking the gesture to changing representation).

Modifications influence the other elements of the framework (Tucker, 2015, 2016). Users and tools modify attributes by changing attributes or applying different attributes during the interactions. *Reactive attribute modification* involves the app modifying app attributes, the user responding by modifying user attributes, the app responding by modifying app attributes, etc. (e.g., app presents one-step equation, user applies correct property, app presents equation requiring use of same property twice). *Proactive attribute modification* occurs when the app modifies app attributes, the user modifies user attributes, and the user also modifies app attributes (e.g., app presents equation requiring two properties, user struggles to sufficiently modify user attributes to apply second property, user chooses a task that isolates the second property). Reactive attribute modification is relatively common, and although proactive modification can occur at times, children often do not perceive opportunities to proactively modify tool attributes (Tucker & Johnson, 2017). Modifying attributes can modify distance (e.g., user strengthens understanding of required property, decreasing mathematical distance) and modify affordance-ability relationships (e.g., app constrains focus by presenting new property in isolation, then alters constraint by including another property) (Tucker, 2015, 2016). These relationships form the MAAAD for Learning framework, as applied to children's interactions with mathematics apps.

3.5 Methods

This study reports on a single illustrative case that is part of an exploratory qualitative investigation into the focus constructs applying iterative data collection and analysis (Anfara, Brown, & Mangione, 2002). The larger project focused on evidence of and relationships among key constructs in children's interactions with mathematics virtual manipulative iPad apps (Tucker, 2015). As part of the investigation, ten fifth-grade children (aged 10–11 years old) participated in individual semi-structured task-based interviews in interview rooms, during which they interacted with two developmentally appropriate iPad apps selected during piloting: Motion Math: Zoom and DragonBox Algebra 12+. Participants interacted with one app for up to thirty minutes without researcher interference before answering follow-up questions about the interactions. The steps repeated for the second app and the interview closed with summative questions. All follow-up questions were semi-structured, allowing for researchers to accommodate participants' responses and directions while focusing on important and emergent themes (Rossman & Rallis, 2003), supporting the exploratory, iterative nature of the research.

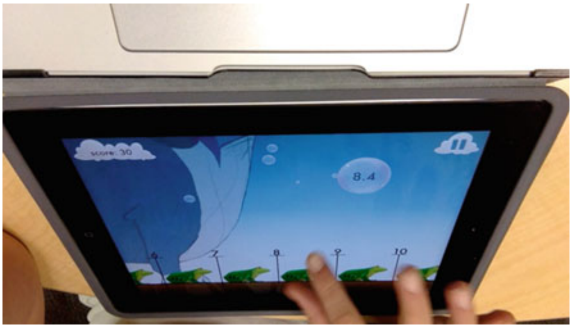


Fig. 3.3 Screenshot from video of hands-on interaction space

Parents were permitted to watch the interview from an adjoining room, through a one-way window. Participants could choose to end the interview at any time.

The researcher collected data during the sessions using video recordings observation field notes. To focus on the physically embodied interactions with the app, which were primarily hand gestures, recordings centered on the hands-on interaction space (see Fig. 3.3), whereas notes focused on occurrences outside the camera views. Interview sessions lasted 55–80 min each, varying by durations of interactions and responses to questions. Data analysis centered on evidence of the focus constructs, relationships among the constructs, and emergent themes. Analysis involved analytic memoing to generate codeable written data to accompany the visual data, and eclectic coding in iterative, interrelated stages of analysis and interpretation (Saldaña, 2013). Due to the exploratory qualitative nature of the study (Stebbins, 2001) and the focus on trustworthiness rather than generalizability or validity (Rolfe, 2006), the researcher collected minimal participant background information beyond eligibility for the study (e.g., 10–11 years old, enrolled in fifth grade, no previous experience with the apps). (For additional detail about piloting and the larger project, see Tucker, 2015.)

3.5.1 Motion Math: Zoom

Motion Math: Zoom is an iPad app with levels presenting tasks involving magnitude, comparison, density, and base-ten relationships on the number line, including positive and negative numbers and intervals from hundredths to thousands. The app includes 24 levels organized by mathematics content, with most levels featuring 10–14 tasks. Each task within a level presents a bubble with a target number in it, requiring the user to navigate the number line to find the space where the target number belongs and pop the bubble (see Fig. 3.4a–d). The app offers an option to use a timer function ("needle") that pops a bubble (i.e., ends a task and level) if the user takes too long to complete an individual task.

The virtual manipulative in Motion Math: Zoom is an interactive idealized number line, allowing changeable intervals and navigation beyond what is

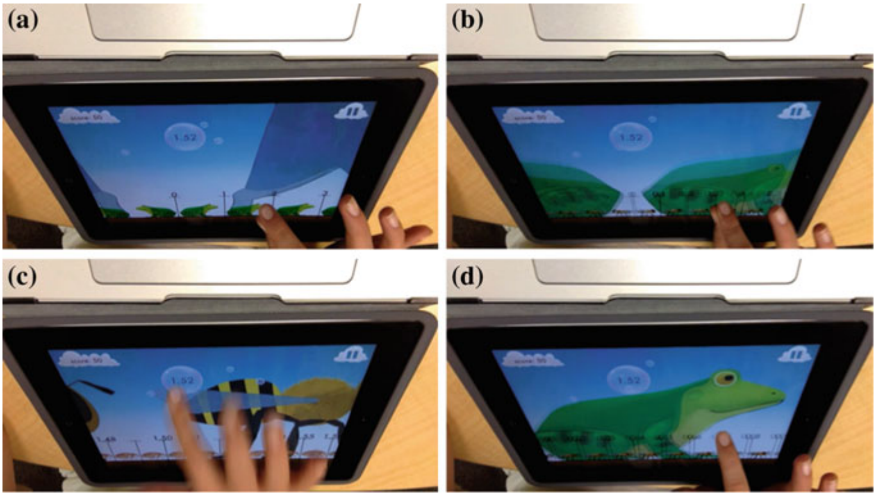


Fig. 3.4 Completing a task in motion math: zoom (clockwise from top left): task presentation (**a**), pinching to change interval scale (**b**), swiping to increase magnitude (**c**), and tapping to indicate answer (**d**)

immediately present on the screen, although navigation is constrained (i.e., not infinite) during tasks. Interacting with the app requires single-touch and multi-touch gestures that are relatively conceptually congruent with widely accepted mathematical meanings. Swiping left or right moves the number line to increase or decrease magnitude, aligning with the continuous nature of the theoretical number line (Fig. 3.4b). Changing interval scale involves pinching, which works most fluidly when performed horizontally instead of vertically or diagonally, aligning with the horizontal orientation of the virtual number line presented in the app (Fig. 3.4c). Pulling fingers apart "zooms in" to smaller base-ten intervals, as if pulling apart the range chosen to find numbers contained within. Pushing fingers together "zooms out" to greater base-ten intervals, as if pushing together the numbers to find ranges that contain them. Although indicating an answer requires tapping the bubble to pop it (Fig. 3.4d) upon reaching the correct placement of the target number, this is conceptually congruent with physics, and the act of finding the placement remains reliant on continuous gestures (e.g., swiping and pinching).

The virtual manipulative in Motion Math: Zoom affords efficient precision, presenting precisely marked ranges on the number line that imply what numbers are within the range. One can use this to guide navigation, such as determining where to change intervals (e.g., to find 24 from intervals of ten, change to ones at 20–30). In the context of this study, "interval" refers to the scale (e.g., tens) and "range" refers to individual spans marked on the number line (e.g., 10–20, 20–30). Efficient navigation refers to the path taken to find the target number. For example, to find 151 from 700 when shown intervals of one hundred, navigating by one hundred to 100–200 and changing to tens, then navigating to 150–160 and changing to ones before popping the bubble at 151. Less efficient navigation would involve taking an alternative path, such as immediately changing to ones at 700 and navigating by ones until reaching 151 and popping the bubble. Although both are mathematically accurate, the former is more efficient than the latter and demonstrates relatively advanced understandings of magnitude, comparison, density, and base-ten relationships on the number line in this context.

3.5.2 Participant: Alex

The entirety of each participant's semi-structured task-based interviews informed the construction of the framework (see Tucker, 2015), but this study focuses on a single case of one child interacting with one app to allow detailed analyses and application of the MAAAD for Learning framework. The researcher selected Alex's experience with Motion Math: Zoom because Alex's experiences included elements commonly seen across most participants (e.g., repeating Level 15; varying navigation efficiency), Alex interacted with the app for an uninterrupted thirty minutes before providing insightful follow-up comments, and the interactions included easily distinguishable phases that support the focus of this study (i.e., outcomes, contributing constructs, interactions, mathematics). During the permission and consent process, 11 year-old Alex enthusiastically proclaimed, "I like math!" Alex remained engaged throughout the interview and requested additional interaction time after the summative questions. During these interactions, Alex attempted 13 of 24 levels of Motion: Math Zoom and did not activate the timer function.

3.6 Findings

Many examples of the constructs and relationships in the MAAAD for Learning framework were present in the data. The findings cover the entirety of Alex's semi-structured task-based interview, highlighting select interrelated examples that evolved throughout the interactions. The findings are organized into five sections: four phases of Alex's interactions with Motion: Math Zoom: (a) integers 0–40, (b) decimals to hundredths, (c) integers −300 to 10,000, and (d) decimals from tenths to hundredths, followed by the semi-structured interview comments. Each phase involved applying and modifying versions of magnitude and comparison on the number line (mathematical attributes), using gestures (technological attributes), differences between Alex's attempts and the app's requirements (distance), and accessing efficient precision (affordance-ability relationship). Figures in each section illustrate relevant applications of the framework.

3.6.1 Phase 1: Integers 0–40

Alex began by requiring four attempts to complete Levels 1–3, which involved positive integers 0–40. Level 1 (introduction to navigation and zooming) posed little challenge (see Fig. 3.5a). Alex repeated Level 2 (range of 0–40 by intervals of one), having first completed it too slowly for the app to prompt a continuation to Level 3 (0–40 by ones, tens allowed but not required). Alex usually accurately identified when to increase or decrease along the number line and correctly placed the target numbers, suggesting alignment of user mathematical attributes with app mathematical attributes (i.e., low degree of mathematical distance) throughout Phase 1. Although Alex swiped within intervals without difficulty (e.g., moving from 3 to 13 by ones), Alex sometimes struggled to zoom. At times, Alex pinched nearly vertically instead of horizontally, which the app did not always recognize as zooming (i.e., some technological distance). Most tasks involved small differences between target numbers, so Alex's choice of range at which to zoom in (e.g., 0–10 to find 13) barely hindered efficiency (i.e., little need to access efficient precision of range contents). By the end of Phase 1 (see Fig. 3.5b), Alex's coordination improved somewhat, leading to consistent use of a diagonal but recognizable, if not smooth pinching gesture (i.e., modifying user attribute leads to modification of distance). After completing Level 3, Alex ignored the prompt to try Level 4 and instead returned to the menu.

3.6.2 Phase 2: Decimals to Hundredths

From the menu, Alex chose the most advanced unlocked option: Level 15 (0–2 by ones, tenths, and hundredths). This proactive user modification of app mathematical attributes immediately increased mathematical distance, as Alex's mathematical attributes no longer aligned with the mathematical attributes required to effectively

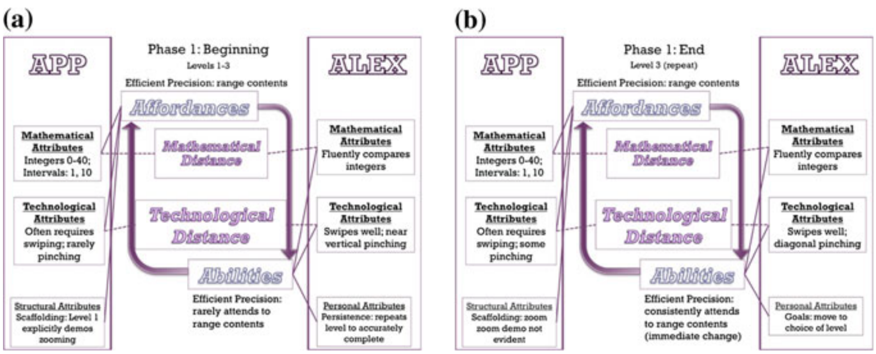


Fig. 3.5 **a**, **b** Applying the MAAAD for learning framework to phase 1

complete the tasks (see Fig. 3.6). Alex struggled to identify correct placement on the number line, repeatedly encountering similar difficulties throughout this phase. The most common mistake involved confusing tenths and hundredths when starting at the opposite interval (e.g., shown intervals of one tenth, trying to place 0.05 at 0.5). Alex frequently traveled using inefficient intervals (e.g., 1.0–1.81 by hundredths), often immediately zooming to the smallest interval in the target before moving toward it (i.e., user mathematical attributes form ability leading to little access to efficient precision of range contents). While rushing through tasks, Alex habitually pinched diagonally when attempting to zoom, which hindered progress (e.g., personal attribute: goal of speed, influences distance). Alex made these mistakes throughout four consecutive unsuccessful attempts to complete Level 15 (i.e., did not modify attributes enough to decrease distance to the point of consistently accurate task completion). Finally, Alex again returned to the menu to choose a different level.

3.6.3 Phase 3: Integers −300 to 10,000

Alex proactively modified app attributes via the menu, choosing to return to Level 3 and play most levels through Level 10, beginning with positive integers and moving to negative integers (see Fig. 3.7a). This decreased mathematical distance, as Alex's mathematical attributes aligned with the app's mathematical attributes at first. However, mathematical distance and technological distance increased when more intervals were accessible. For example, on Level 3, the app permits travel only by

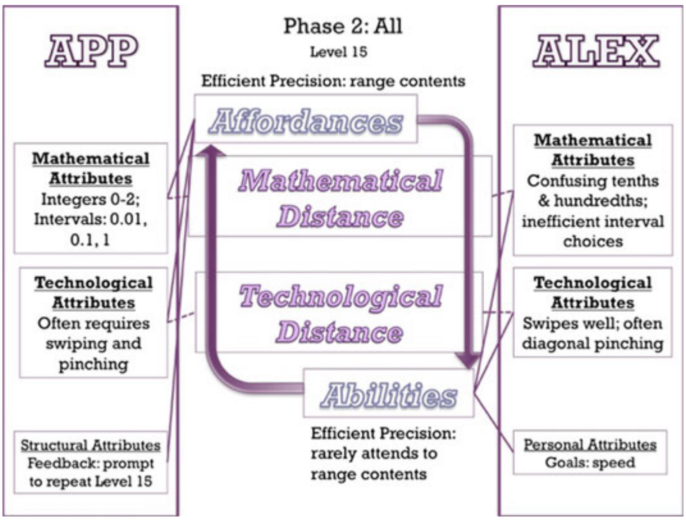


Fig. 3.6 Applying the MAAAD for learning framework to phase 2

ones and tens and no tasks require changing intervals, but by Level 5 (0–1000 by ones, tens, and hundreds), the app also permits travel by more intervals and many tasks require changing intervals. Alex usually navigated in the correct direction (e.g., decreasing to find 33 from 250), but sometimes used a diagonal pinch to zoom and often traveled by inefficient intervals (e.g., when shown intervals of ten to find 33 from 250, zooming in immediately and navigating by ones). When rushing to complete a task, Alex was prone to mixing up zooming in with zooming out and incorrectly answering before the correct placement appeared (i.e., personal attributes influence technological distance and mathematical distance, respectively). Over time, Alex more frequently chose efficient navigation intervals while progressing through integer-only levels, purposefully zooming in when closer to the target (i.e., modified user mathematical attributes, which modified ability involved in accessing efficient precision of range contents and decreased mathematical distance).

Alex fluently completed Level 9 (−25–25 by ones), correctly comparing to place target numbers (i.e., aligned mathematical attributes) without being required to zoom (i.e., modified app technological attributes modifies technological distance) (see Fig. 3.7b). When first attempting Level 10 (−300–300 by ones and tens), Alex rushed to complete each task, often passing the target number (i.e., personal attributes influencing mathematical distance). Upon repeating Level 10, Alex balanced speed and accuracy, used smoother, nearly horizontal pinching gestures, and often chose efficient ranges to change intervals (e.g., zooming in at 0–10 to find 3), successfully completing the level (i.e., modified user personal, technological, and mathematical attributes and accessed efficient precision, decreasing mathematical and technological distance). Next, Alex returned to the menu.

3.6.4 Phase 4: Decimals from Tenths to Hundredths

From the menu, Alex chose the least advanced level to feature decimals, Level 12 (0–3 by ones and tenths), proactively modifying app mathematical attributes

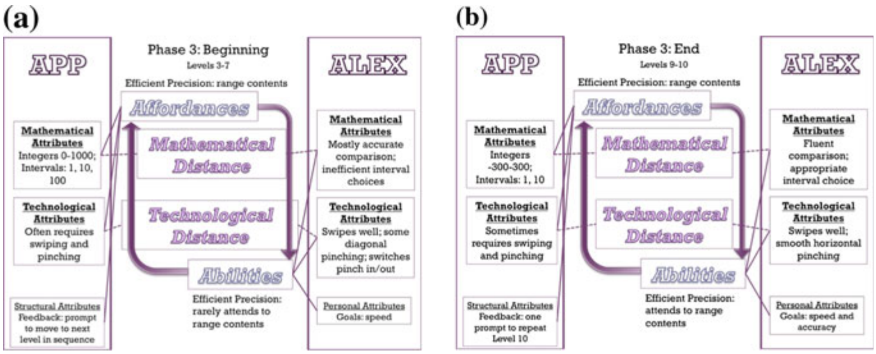


Fig. 3.7 **a**, **b** Applying the MAAAD for learning framework to phase 3

(see Fig. 3.8a). Slowly but accurately completing most tasks, Alex usually chose efficient navigation intervals, traveling in the correct direction but pinching horizontally to zoom on the few tasks that required changing intervals. The only greatly inefficient navigation interval arose when crossing 1.0, as Alex chose to remain at tenths (e.g., shown intervals of one tenth to find 2.5 from 0.2, navigating by tenths instead of zooming out to ones, moving to the midpoint of 2 and 3, then zooming into tenths). While repeating Level 12, Alex further honed the horizontal pinching gesture and zoomed at relatively efficient places (e.g., from previous example, returning to tenths upon reaching 2), quickly and accurately completing the tasks (i.e., modified user technological and mathematical attributes to access efficient precision). Advancing to Level 13 (0–20 by ones and tenths), Alex showed similar performance, navigating the number line with reasonable efficiency and accuracy, with sporadic inefficient navigation choices.

Next, Alex revisited Level 15 (see Fig. 3.8b). Alex made similar errors as on previous attempts at Level 15, but addressed them more often than before, increasing overall accuracy (i.e., modified mathematical attributes decreases mathematical distance). Though continuing to confuse tenths and hundredths when starting at the opposite interval, Alex quickly recognized the error, rather than repeatedly attempting to input the incorrect answer. When beginning to travel by

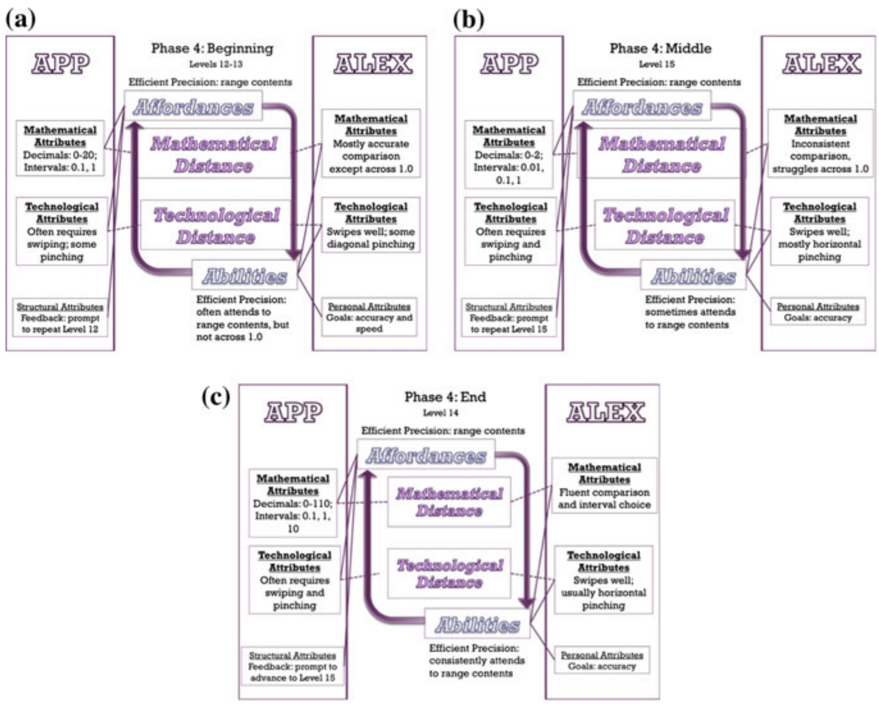


Fig. 3.8 **a–c** Applying the MAAAD for learning framework to phase 4

inefficient intervals, Alex sometimes paused to change intervals. However, especially when crossing 1.0, Alex continued to immediately switch to the interval shown in the target number (e.g., shown intervals of one tenth to find 1.65 from 0.3, zooming into hundredths then increasing toward 1.65). After slowly completing Level 15, Alex chose Level 14 (tens, ones, and tenths, 0–110) (see Fig. 3.8c). Alex usually chose appropriate navigation intervals and zooming ranges throughout this attempt, and consistently used a horizontal pinching gesture to zoom (i.e., relatively low degree of mathematical distance and technological distance). The researcher informed Alex that the interaction time had ended and Alex returned to the menu.

3.6.5 Semi-structured Interview Comments

During the follow-up interview, Alex said that the app was "really fun" and that only Level 15 was challenging. When asked about choosing levels, Alex pointed to Levels 21–24 on the menu, saying "I played the hardest [unlocked] one [but] it wouldn't let me go up levels so I went back and I tried to go back as many as I could [to get there]" (i.e., proactive attribute modification). Regarding navigation, Alex noted, "You could have it stay at the hundredths… but it would be tons harder and long… [but] when you zoom out there's not the numbers" (i.e., varying mathematical distance). Alex also mentioned, "I basically knew where everything was, but you had to go there" (i.e., links between mathematical distance and technological distance).

3.7 Discussion

The MAAAD for Learning framework models interrelated constructs involved in mathematical activity involving user-tool interactions, including a child's interactions with a mathematics app. In this context, the framework supported an investigation of outcomes, contributors to outcomes, and the interactions that occurred. These applications have implications for learning mathematics and using technology, which are relevant for practitioners, technology developers, and researchers.

3.7.1 Linking Outcomes, Contributors, and Interactions

One can use the framework to examine outcomes of user-tool interactions, contributors to the outcomes, and the interactions themselves. For Alex, outcomes included increasing accuracy and efficiency while navigating the number line for integers and decimals, and honing the pinching input gesture. This supports findings from other studies that interactions with iPad apps featuring virtual manipulatives could lead to improvements in children's task performance and efficiency (e.g., Moyer-Packenham et al., 2015). By the end, Alex fluently navigated the number line on integer tasks, but still encountered some difficulties, particularly on two types of decimal tasks. Alex progressively began to use range contents to guide task completion, but remained inconsistent when tasks involved hundredths, often inefficiently navigating by hundredths whenever they were part of the target number, rather than immediately changing to a more efficient interval (e.g., tenths) and returning to hundredths when closer to the target. Follow-up comments reinforced Alex's dilemma of reconciling the easy but inefficient method of navigating by smaller intervals with the option of navigating without smaller intervals visible, which requires stronger understanding of density, comparison, and magnitude in base ten. Alex also accurately completed tasks involving tenths when comparing within the same whole (e.g., 1.1 vs. 1.9) but struggled when tasks involved different wholes, especially when starting from less than 1 (e.g., 0.9 vs. 1.3). This is consistent with findings from other research (e.g., Vamvakoussi & Vosniadou, 2010) which indicate that children develop understandings of density in stages, with recognition of intermediate numbers often beginning between natural numbers (e.g., 0–1 contains 0.1, 0.2, etc.). One can also use the framework to examine what contributed to these outcomes.

In the context of the MAAAD for Learning framework, Alex began with user attributes that aligned with app attributes as required for successful task completion of early levels (e.g., integer comparison, number line, swiping), indicating low degrees of mathematical distance and technological distance. As the app modified app attributes by presenting tasks with different mathematics content (e.g., hundredths) and input gesture requirements (e.g., pinching), both types of distance increased and Alex attempted to modify user attributes to decrease distance (e.g., hone pinching). Interacting with the idealized number line may have supported Alex's efforts to strengthen relevant user mathematical attributes (e.g., density, comparison, and magnitude of integers and decimals) and user technological attributes (e.g., coordination), modifying ability to access the efficient precision of range contents for integers (e.g., 1–10 also contains 2, 3 … 9), with this access in turn influencing further modification of the relevant attributes. Alex also proactively modified app attributes to modify mathematical distance, both by choosing easier content (e.g., Level 15 hundredths to Level 3 integers) and more challenging content (e.g., Level 10 integers to Level 12 tenths). Modifying distance allowed targeted attribute modification, such as strengthening understanding of density as represented in range contents with integers and then attempting to flexibly apply it to decimals. Each of these constructs and relationships may have contributed to the improvements in task completion, aligning with research that applies the framework (Tucker, 2015, 2016) and considers patterns related to specific constructs (e.g., affordances: Moyer-Packenham et al., 2016; attribute modification: Tucker & Johnson, 2017).

In the context of Activity Theory and embodied cognition, the activity consisted of internalizing and externalizing representations via physical engagement in mathematical practices, providing evidence of mathematical thinking and learning

(Nemirovsky et al., 2013). Over time, Alex became adept at the input gestures required to traverse this interactive representation of the idealized number line (i.e., modifying user technological attributes to decrease technological distance). Alex became very accurate and efficient on some tasks (e.g., comparing integers) and began exploring more advanced mathematics content with mixed success (e.g., comparing decimals to hundredths). The interactions may also provide evidence of in-progress transfer across related contexts, as Alex appeared to develop understandings of magnitude, comparison, base-ten, and density for integers that aligned with widely accepted meanings, and attempted to extend these to decimals. The ongoing changes in embodied mathematical practices during the interactions imply that Alex continued learning throughout. This aligns with findings from other research indicating that children's embodied mathematical practices can change during interactions with mathematics apps (e.g., Holgersson et al., 2016; Sinclair et al., 2015; Tucker et al., 2016a). This also suggests that conceptually congruent interactions with an idealized number line representation may facilitate development of mathematics knowledge (i.e., modification of user mathematical attributes). Therefore, the MAAAD for Learning framework may be used to examine outcomes, contributors to outcomes, and interactions that occur when children interact with mathematics apps, including those involving conceptually congruent interactions with idealized representations.

3.8 Implications and Future Directions

The findings and applications of the framework have implications for practitioners, researchers, and technology developers. All three groups may be interested in Alex's changing mathematical practices throughout this activity, as findings suggest that interacting with touchscreen virtual manipulative apps that involve conceptually congruent gestures to navigate an idealized representation of a mathematical concept may have positive effects on mathematical learning. These findings add to research involving non-idealized number lines that suggest using number lines can support development of comparison and magnitude in base ten (e.g., Rittle-Johnson et al., 2001) and facility with unfamiliar numbers (Siegler & Booth, 2004). Notably, these findings also suggest that conceptually congruent interactions with an idealized number line may contribute to developing understandings of density, which research indicates can be difficult even for secondary school students (Vamvakoussi & Vosniadou, 2010). Therefore, it may be beneficial for these stakeholders to consider what constitutes "conceptually congruent" in various contexts, as well as the role of conceptually congruent gestures in the development and evaluation of mathematical practices and mathematics knowledge in these contexts.

Practitioners could use the MAAAD for Learning framework to consider attributes of a child and tool, the distance involved, and affordance-ability relationships that could help bridge this distance. For example, Alex often effectively compared within a given interval (e.g., comparing only by tens). However, the depth of Alex's mathematical understandings involved in the efficient, precise use of the idealized number line was unclear until Alex honed the horizontal pinching gesture to zoom (i.e., mathematical distance influenced by technological distance). Alex encountered a lower degree of mathematical distance on levels featuring only integers than those with decimals, suggesting relatively underdeveloped understandings of decimals. Observations based on these constructs might inform decisions such as which child should use a particular app for a certain learning goal (e.g., Tucker et al., 2016a), which may be especially useful when combined with app evaluations (e.g., Goodwin & Highfield, 2013; Larkin, 2015).

Technology developers can use the MAAAD for Learning framework to inform the design process. For example, some developers consider tool attributes and affordances when designing technology (e.g., Ginsburg et al., 2013; Holgersson et al., 2016). Alex's inconsistent access to efficient precision of range contents supports assertions that affordance access is complex (Tucker et al., 2016b) and can affect learning in various ways (Moyer-Packenham et al., 2016). However, the framework also incorporates distance and affordance-ability relationships that involve the user and tool, encouraging explicit consideration of the interactive links between user and tool. For example, although the app afforded efficient precision of range contents, during the interactions, variations in distance based on differences in user and tool attributes influenced Alex's access to this affordance. Therefore, it may be beneficial to consider interrelationships among the constructs when designing multi-touch technology. Alex's changing mathematical practices related to density based on interactions with the idealized number line support previous research indicating that virtual manipulatives (Moyer-Packenham & Bolyard, 2016; Moyer-Packenham & Westenskow, 2016) and multi-touch technology may offer unique opportunities for mathematics learning (Baccaglini-Frank & Maracci, 2015), while adding to calls for app developers to intentionally utilize multi-touch capabilities (Byers & Hadley, 2013). Using the framework during the design process might also help developers purposefully address this, such as by considering how to minimize technological distance without significantly compromising idealized representations of mathematics or conceptual congruence of gestures.

In addition to studying practical and design applications, researchers may continue using the MAAAD for Learning framework to investigate children's mathematical interactions with technology. The findings support prior research indicating that the framework may be useful for examining activity involving embodied mathematical practices and the relationships that influence the development of these practices (Tucker, 2015, 2016). This may be particularly useful for further exploration of children's mathematical interactions with idealized representations, including those involving conceptually congruent gestures. Furthermore, this study focused on qualitative outcomes, so future studies could examine if quantified outcomes relate to the constructs and relationships in the framework (e.g., specific change in attributes and subsequent affordance access leads to growth from pre- to post-assessment). Thus, the findings in this study related to both the MAAAD for Learning framework and interactions with an idealized number line have applications and implications for researchers, practitioners, and technology developers.

3.9 Conclusion

The Modification of Attributes, Affordances, Abilities, and Distance for Learning framework may be useful for research on activity featuring mathematical interactions with technology, including the outcomes of the interactions, contributors to the outcomes, and interactions themselves. Practitioners may use the framework to evaluate technology tools and learning that occurs during children's interactions with the tools, influencing classroom technology use. Developers of educational technology may use the framework to influence the design process, including consideration of what representations and gestures to incorporate into the app experience. Findings also implied that using conceptually congruent multi-touch gestures to interact with an idealized representation of a number line might support developing understandings of mathematics concepts, including density. Each of these stakeholders and aims are influential for the development and use of educational technology for learning mathematics, and each may benefit from involving the MAAAD for Learning framework.

References

Anfara, V. A., Brown, K. M., & Mangione, T. L. (2002). Qualitative analysis on stage: Making the research process more public. *Educational Researcher, 31*(7), 28–38. https://doi.org/10.3102/0013189X031007028.

Attribute [Def. 5]. (2014). *OED online*. Oxford: Oxford University Press. Retrieved from http://www.oed.com/view/Entry/12931.

Baccaglini-Frank, A., & Maracci, M. (2015). Multi-touch technology and preschoolers' development of number-sense. *Digital Experiences in Mathematics Education, 1*(1), 7–27. https://doi.org/10.1007/s40751-015-0002-4.

Byers, P., & Hadley, J. (2013). Traditional and novel modes of activity in touch screen math apps. In J. P. Hourcade, N. Sawhney, & E. Reardon (Eds.), *Proceedings of the 12th International Conference on Interaction Design and Children*. New York, NY: ACM.

Carpenter, K. K. (2013). *Strategy instruction in early childhood math software: Detecting and teaching single-digit addition strategies* (Doctoral dissertation). Columbia University. Retrieved from http://academiccommons.columbia.edu/catalog/ac:160522.

Chemero, A. (2003). An outline of a theory of affordances. *Ecological Psychology, 15*(2), 181–195. https://doi.org/10.1207/S15326969ECO1502_5.

Cohen, D. J., & Blanc-Goldhammer, D. (2011). Numerical bias in bounded and unbounded number line tasks. *Psychonomic Bulletin & Review, 18*(2), 331–338. https://doi.org/10.3758/s13423-011-0059-z.

Dubé, A. K., & McEwen, R. N. (2015). Do gestures matter? The implications of using touchscreen devices in mathematics instruction. *Learning and Instruction, 40*, 89–98. https://doi.org/10.1016/j.learninstruc.2015.09.002.

Fischer, M. H., & Roitmann, J. (2005). Do negative numbers have a place on the mental number line? *Psychology Science, 47*(1), 22–32.

Geary, D. C., Bailey, D. H., & Hoard, M. K. (2009). Predicting mathematical achievement and mathematical learning disability with a simple screening tool: The Number Sets Test. *Journal of Psychoeducational Assessment, 27*(3), 265–279. https://doi.org/10.1177/0734282908330592.

Gibson, J. J. (1986). *The ecological approach to visual perception*. Hillsdale, NJ: Lawrence Erlbaum.

Ginsburg, H. P., Jamalian, A., & Creighan, S. (2013). Cognitive guidelines for the design and evaluation of early mathematics software: The example of MathemAntics. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 83–120). The Netherlands: Springer. Retrieved from http://link.springer.com/chapter/10.1007/978-94-007-6440-8_6.

Goldin, G. A., & Kaput, J. M. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397–430). Hillsdale, NJ: Erlbaum.

Goodwin, K., & Highfield, K. (2013). A framework for examining technologies and early mathematics learning. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 205–226). The Netherlands: Springer. Retrieved from http://link.springer.com/chapter/10.1007/978-94-007-6440-8_11.

Greeno, J. G. (1994). Gibson's affordances. *Psychological Review*, *101*(2), 336–342. https://dx.doi.org.dist.lib.usu.edu/10.1037/0033-295X.101.2.336.

Holgersson, I., Barendregt, W., Emanuelsson, J., Ottosson, T., Rietz, E., & Lindström, B. (2016). Fingu—A game to support children's development of arithmetic competence: Theory, design and empirical research. In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 123–145). Switzerland: Springer International Publishing. https://doi.org/10.1007/978-3-319-32718-1_6.

Kirby, K. D. (2013). *The development of an idealized number line: Differentiating physical inscription from mathematical object*. Presented at the American Educational Research Association, San Francisco, CA.

Ladel, S., & Kortenkamp, U. (2016). Artifact-centric activity theory—A framework for the analysis of the design and use of virtual manipulatives. In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 25–40). Switzerland: Springer International Publishing. https://doi.org/10.1007/978-3-319-32718-1_2.

Larkin, K. (2015). "An app! An app! My kingdom for an app": An 18-month quest to determine whether apps support mathematical knowledge building. In T. Lowrie & R. J. Zevenbergen (Eds.), *Digital games and mathematics learning* (pp. 251–276). The Netherlands: Springer. https://doi.org/10.1007/978-94-017-9517-3_13.

Laski, E. V., & Siegler, R. S. (2007). Is 27 a big number? Correlational and causal connections among numerical categorization, number line estimation, and numerical magnitude comparison. *Child Development, 78*(6), 1723–1743. https://doi.org/10.1111/j.1467-8624.2007.01087.x.

Leontiev, A. N. (1978). *Activity, consciousness and personality*. Englewood Hills: Prentice Hall.

Moyer-Packenham, P. S., & Bolyard, J. J. (2016). Revisiting the definition of a virtual manipulative. In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 3–23). Switzerland: Springer International Publishing. https://doi.org/10.1007/978-3-319-32718-1_1.

Moyer-Packenham, P. S., Bullock, E. P., Shumway, J. F., Tucker, S. I., Watts, C., Westenskow, A., … Jordan, K. E. (2016). The role of affordances in children's learning performance and efficiency when using virtual manipulatives mathematics iPad apps. *Mathematics Education Research Journal*, *28*(1), 79–105. https://doi.org/10.1007/s13394-015-0161-z.

Moyer-Packenham, P. S., Shumway, J. F., Bullock, E., Tucker, S. I., Anderson-Pence, K. L., Westenskow, A., … Jordan, K. (2015). Young children's learning performance and efficiency when using virtual manipulative mathematics iPad apps. *Journal of Computers in Mathematics and Science Teaching, 34*(1), 41–69.

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments, 4*(3), 35–50.

Moyer-Packenham, P. S., & Westenskow, A. (2016). Revisiting the effects and affordances of virtual manipulatives for mathematics learning. In K. Terry & A. Cheney (Eds.), *Utilizing* *virtual and personal learning environments for optimal learning* (pp. 186–215). Hershey, PA: Information Science Reference. https://doi.org/10.4018/978-1-4666-8847-6.ch009.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372–415. https://doi.org/10.5951/jresematheduc.44.2.0372.

Paek, S., Hoffman, D. L., & Black, J. B. (2016). Perceptual factors and learning in digital environments. *Educational Technology Research and Development, 64*(3), 435–457. https://doi.org/10.1007/s11423-016-9427-8.

Riconscente, M. M. (2013). Results from a controlled study of the iPad fractions game motion math. *Games and Culture, 8*(4), 186–214. https://doi.org/10.1177/1555412013496894.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346–362. https://doi.org/10.1037/0022-0663.93.2.346.

Rolfe, G. (2006). Validity, trustworthiness and rigour: Quality and the idea of qualitative research. *Journal of Advanced Nursing, 53*(3), 304–310. https://doi.org/10.1111/j.1365-2648.2006.03727.x.

Rossman, G. B., & Rallis, S. F. (2003). *Learning in the field: An introduction to qualitative research*. Thousand Oaks, CA: SAGE.

Saldaña, J. (2013). *The coding manual for qualitative researchers* (2nd ed.). Thousand Oaks, CA: SAGE.

Schneider, M., Grabner, R. H., & Paetsch, J. (2009). Mental number line, number line estimation, and mathematical achievement: Their interrelations in grades 5 and 6. *Journal of Educational Psychology, 101*(2), 359–372. https://doi.org/10.1037/a0013840.

Sedig, K., & Liang, H.-N. (2006). Interactivity of visual mathematical representations: Factors affecting learning and cognitive processes. *Journal of Interactive Learning Research, 17*(2), 179–212.

Segal, A., Tversky, B., & Black, J. (2014). Conceptually congruent actions can promote thought. *Journal of Applied Research in Memory and Cognition, 4*(3), 124–130. https://doi.org/10.1016/j.jarmac.2014.06.004.

Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child Development, 75*(2), 428–444. https://doi.org/10.1111/j.1467-8624.2004.00684.x.

Sinclair, N., Chorney, S., & Rodney, S. (2015). Rhythm in number: Exploring the affective, social and mathematical dimensions of using TouchCounts. *Mathematics Education Research Journal, 28*(1), 31–51. https://doi.org/10.1007/s13394-015-0154-y.

Stebbins, R. A. (2001). *Exploratory research in the social sciences*. Thousand Oaks, CA: SAGE.

Tucker, S. I. (2015). *An exploratory study of attributes, affordances, abilities, and distance in children's use of mathematics virtual manipulative iPad apps* (Doctoral dissertation). Retrieved from http://gradworks.umi.com/37/23/3723089.html.

Tucker, S. I. (2016). The modification of attributes, affordances, abilities, and distance for learning framework and its applications to interactions with mathematics virtual manipulatives. In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 41–69). Switzerland: Springer International Publishing. https://doi.org/10.1007/978-3-319-32718-1_3.

Tucker, S. I., & Johnson, T. N. (2017). I thought this was a study on math games: Attribute modification in children's interactions with mathematics apps. *Education Sciences, 7*(2), 50, 1–20. https://doi.org/10.3390/educsci7020050.

Tucker, S. I., Moyer-Packenham, P. S., Shumway, J. F., & Jordan, K. E. (2016a). Zooming in on students' thinking: How a number line app revealed, concealed, and developed students' number understanding. *Australian Primary Mathematics Classroom, 21*(1), 23–28.

Tucker, S. I., Moyer-Packenham, P. S., Westenskow, A., & Jordan, K. E. (2016b). The complexity of the affordance-ability relationship when second-grade children interact with mathematics virtual manipulative apps. *Technology, Knowledge and Learning, 21*(3), 341–360. https://doi.org/10.1007/s10758-016-9276-x.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*(2), 181–209. https://doi.org/10.1080/07370001003676603.

Watts, C. M., Moyer-Packenham, P. S., Tucker, S. I., Bullock, E. P., Shumway, J. F., Westenskow, A., … Jordan, K. (2016). An examination of children's learning progression shifts while using touch screen virtual manipulative mathematics apps. *Computers in Human Behavior*, *64*, 814–828. https://doi.org/10.1016/j.chb.2016.07.029.

Zhang, M., Trussell, R. P., Gallegos, B., & Asam, R. R. (2015). Using math apps for improving student learning: An exploratory study in an inclusive fourth grade classroom. *TechTrends, 59* (2), 32–39. https://doi.org/10.1007/s11528-015-0837-y.

Chapter 4
Using One-to-One Mobile Technology to Support Student Discourse

Shannon Larsen, Kelly McCormick, Josephine Louie and Pamela Buffington

Abstract Education researchers, administrators, and classroom teachers in Auburn, Maine, USA are using a design-based, iterative research approach to examine how screencasting apps can support student discourse in K–2 mathematics classrooms equipped with one-to-one mobile technology (iPads). Preliminary data analysis shows that in addition to enhancing mathematical communication, the purposeful use of screencasting apps supports more equitable opportunities for student participation in mathematics discourse, facilitates effective talk moves such as wait time, involves students in self and peer assessment, and engages students in productive struggle. Early findings also suggest that when teachers utilize this approach in their classroom, their beliefs about student capabilities may increase and their teaching practices may change.

Keywords Screencasting · Mathematical discourse · Formative assessment
Productive struggle · Research-practice partnership

4.1 Introduction

In this paper, we explain how a group of education researchers, higher education faculty, district and building administrators, math coaches, and early elementary classroom teachers worked collaboratively to identify and address persistent learning problems in mathematics within a school district in a small city in Maine. Because early elementary school classrooms (kindergarten–second-grade) in the district were equipped with one-to-one (1–1) mobile technology, specifically iPads,

S. Larsen (✉)
University of Maine at Farmington, Farmington, USA
e-mail: shannon.larsen@maine.edu

K. McCormick
University of Southern Maine, Portland, USA

J. Louie · P. Buffington
Education Development Center, Waltham, USA

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_4

using this technology to support student learning significantly shaped our approach. After a brief review of the literature that influenced the development of the theoretical framework, we identify the overarching research questions and then outline the components of the qualitative, design-based research methodology used in the study. We also describe how ongoing professional learning, tightly coupled with classroom data collection, critically influenced our work. We then share the emerging findings from the first two years of the study, with a focus on the ways in which the use of screencasting apps support students' mathematical discourse and the changing nature of teachers' beliefs as a result.

4.2 Theoretical Framework

Research suggests that interactive digital technologies have the potential to support and enhance the learning of mathematics in the early grades (e.g., Attard & Curry, 2012; Ginsburg, Jamalian, & Creighan, 2013; Goodwin & Highfield, 2013; Soto, 2015; Soto & Ambrose, 2014; Soto & Hargis, 2014). One-to-one, hand-held, mobile technology, such as iPads, provides students with unique learning opportunities. Using recording tools available through apps or the iPad camera, students can document their problem-solving approaches and share their thinking with others (Attard, 2013). Moreover, multisensory recordings allow students to review, reflect on, and critique their own as well as others' written work, representations, and oral explanations. The ease with which students can create and share recordings provides them access to different ways to solve problems and allows them the opportunity to reflect on their own and others' explanations and even discover and correct their own and others' mistakes and misconceptions (Hattie & Timperley, 2007; Soto, 2015; Soto & Hargis, 2014). Producing video recordings of their work engages students and can help them view themselves as creators of their own mathematical ideas (Yelland & Kilderry, 2010). Recordings of students' mathematical thinking can also provide students and teachers with evidence of their learning and be a source of motivation and encouragement (Blair, 2013; Sedig & Liang, 2006; Soto & Ambrose, 2014). Students who use screencastings to explain their mathematical thinking often become aware of and attentive to an audience. Therefore, they adopt a teaching identity through which they describe the process for their mathematical solution and provide a justification for that work (Soto, 2015). Moreover, audio-visual recording capabilities may be particularly beneficial for young students who often are better able to express their thoughts through speaking rather than through writing.

In their 2014 publication, *Principles to Actions: Ensuring Mathematical Success for All*, the National Council of Teachers of Mathematics (NCTM) identifies "facilitat[ing] meaningful mathematical discourse" as a research-based, high-leverage practice that improves the teaching and learning of mathematics. NCTM (2014) indicates that supporting mathematical discourse among students is central to ensuring the meaningful learning of mathematics. Teachers who encourage their students to share their work with one another provide their students with the opportunity to justify and clarify their mathematical ideas, communicate their ideas verbally or in writing using mathematics vocabulary and visual representations, and make sense of other approaches to solving problems (NCTM, 2014). Research suggests that students who have opportunities to engage in mathematical discourse may develop a deeper conceptual understanding of mathematics (Attard, 2013; Moschkovich, 2012; NCTM, 2014). The Common Core State Standards for Mathematics (CCSSI, 2010) includes "construct viable arguments and critique the reasoning of others" as one of the eight Standards for Mathematical Practice (SMP). This student practice standard closely parallels the teaching principle of facilitating meaningful mathematical discourse (NCTM, 2014) and highlights the important nature of discourse in mathematics for all members of the learning community. Supported by this research, and prompted by teachers' observations of students in early elementary grades using screencasting apps in the classroom, the study's co-investigators hypothesized that when students regularly use screencasting apps to record and review their mathematical explanations, their mathematical communication and reasoning skills improve.

4.3 Research Questions

As collaborating partners in the project, the researchers, administrators, and teachers investigated how screencasting apps support students' mathematical learning in the early grades by asking students to use a screencasting app to record their written work and oral explanations as they solve mathematical problems. The following co-developed research questions guided the investigation:

1. In what ways do teachers enact a strategy that encourages students to record and review explanations of their mathematical thinking using iPad-based recording tools?
2. What types of mathematical reasoning and discourse outcomes emerge from use of this strategy?
3. How might use of this strategy be related to teachers' instructional practices and students' mathematical outcomes?

4.4 Design and Methodology

This study is part of the Research + Practice Collaboratory, a project that is funded by the National Science Foundation and is committed to using a partnership approach between researchers and practitioners to develop promising ways to bridge the gap between research and practice in STEM education. The project conjectures that when researchers and educational practitioners work collaboratively to exchange knowledge and to design and develop educational interventions, researchers are more likely to incorporate practitioner knowledge into their research, and educators are more likely to use evidence-based practices in their instruction. Equal positioning of researchers, administrators, and kindergarten—second grade teachers plays a key role in the study's methodological approach. Each partner is considered a co-investigator and plays a critical part in collaboratively identifying important needs to address, designing possible solutions, testing these solutions, and planning for the sustainability and scale of the reform strategies that emerge (Penuel, Fishman, Haugan Cheng, & Sabelli, 2011). The collaborative approach to this study draws from design-based research methodology. Five central principles of design-based research are: (i) the development of theories and learning environments are interconnected, (ii) research and implementation take place in ongoing iterative cycles, (iii) generated theories must be applicable to practitioners and other designers, (iv) research occurs in real environments, and (v) the data collected highlights both enacted work and outcomes (Design-Based Research Collective, 2003).

Our collaborative work means that we have remained deeply committed to providing ongoing professional learning for all participants throughout the duration of the study. Loucks-Horsley et al. (2003) indicate that professional learning for educators needs to support them in acquiring new knowledge, skills, behaviors, attitudes, and depth of content knowledge. A critical feature of effective professional learning is that it provides opportunities for collaboration with colleagues (Loucks-Horsley et al., 2003). Research shows that typical one-day professional development sessions have limited impact on teaching practice as teachers transfer less than 10% of the content into their classroom practices (Showers & Joyce, 1996). Similarly, Attard (2013) found that teachers are less likely to embed effective teaching practices with technology into their teaching without planned and sustained professional dialogue focusing on technology, pedagogy, and content knowledge. Ongoing professional development provides teachers with necessary multiple opportunities to reflect on and re-conceptualize their practice to accommodate the technology and new practices. According to Dorph and Holtz (2000), high-quality professional development meets four conditions. It should be (i) connected to content knowledge, (ii) designed with a clear and focused audience in mind, (iii) sustained over time with a coherent plan, and (iv) structured with opportunities for practitioners to reflect, analyze, and work on their practice. Professional learning communities (PLCs), defined as learning models in which collective inquiry supports changes in attitudes, beliefs and practices (Dufour & Eaker, 1998), can be one tool used to engage in the type of high quality professional learning described by Dorph and Holtz. With this in mind, education researchers, higher education faculty, administrators, math coaches, and classroom teachers participated collaboratively each month in a PLC to shape shared insights into possible research questions, examine research related to mathematics teaching and learning, share tools and strategies, and reflect on the work done in classrooms.

The partnership started in the spring of 2014 when a wide range of stakeholders, including pre-kindergraten through third-grade teachers, specialists, and principals from the six elementary schools in the district as well as the district's curriculum director, assistant superintendent, and superintendent, met to focus on problem identification. This work provided the team with a broad base of input and support, which enabled the work to move forward over two years. It also provided the team with a more clearly defined area of focus. In the summer of 2014, the collaborative work began with eight kindergarten–second-grade teachers, a math coach, and the principals of the three lowest-performing schools. The technology integration specialist, curriculum director, assistant superintendent, and superintendent were also members of the team. Three teachers, the math coach, and another specialist left the project after the first year. In the second year of the project, we scaled up our work, and kindergarten–second-grade teachers and administrators from two additional schools joined the partnership. Practitioner participants in year two included eight administrators, fourteen kindergarten–second-grade teachers (five from year one and nine new teachers) and one new coach.

In the summer of 2014, the co-investigative group met over eight days. During this time, 17 educators from cohort 1 (8 classroom teachers and 9 building administrators) worked collaboratively with a group of education researchers from a non-profit educational research organization and local state universities. We worked together to identify persistent student-learning problems and to study aspects of high quality mathematics learning and teaching. Teachers, administrators and researchers participated in a shared reading of Fosnot and Dolk (2001) book, *Young mathematicians at work: Constructing number sense, addition, and subtraction*. The teachers became particularly interested in two pedagogical ideas highlighted in the reading. The first, "learning landscapes" (Fosnot & Dolk, 2001) presented them with potential learning trajectories that students might follow as they encounter big ideas in early mathematics learning. The second, "math congress" (Fosnot & Dolk, 2001), introduced them to the importance of whole class discourse in early mathematics classes and the purposeful sharing of students' work at the end of a mathematics class. After reading and discussing this work, teachers in the study became more aware of their own students' engagement in mathematical communication and expressed concern over their students' mastery of numeracy.

During the summer months of 2014 and the beginning of the 2014–2015 academic year, we studied research related to the implementation and use of technology, particularly iPads, in early learning classrooms. We examined affordances of various mathematical apps, engaged in mathematics using the iPad apps we explored, and discussed ways in which these tools could support mathematical learning in early mathematics classrooms. In the fall of 2014, we implemented a "toe-in-the-water" induction phase when teachers explored the mathematics learning strategies and mobile technology tools we examined with their own students. Building on our work in the summer of 2014, participating teachers returned to their classrooms that fall with a goal to focus their mathematics instruction on number sense and to pay particular attention to the CCSSI (2010) SMP 3: Construct viable arguments and critique the reasoning of others, SMP 4: Model with mathematics, and SMP 5: Use appropriate tools strategically. All kindergarten–second-grade students in this district have iPads, so we focused our work on how using this mobile technology could support and improve student learning.

From experimenting in their own classes during this trial phase, participating teachers became curious about the ways in which the screencasting app Explain Everything™ (EE) might support student discourse and whole-class discussions in their classrooms. Specifically, the teachers became excited by the level of engagement and mathematical discourse that their students displayed when using screencasting apps to record, explain, and review their thinking when solving mathematics problems. Thus, after seven months of professional work together, we agreed on a group strategy that we would co-investigate. We decided that for the remainder of the school year and the following school year, the teachers would integrate this strategy into their mathematics lessons at least once a month. Employing 30-day plan-do-study-act cycles, we agreed to co-investigate how teachers implemented this strategy, how students responded to this strategy, and for whom and under what conditions this strategy might generate improved mathematics learning outcomes.

Over the 2014–2015 and 2015–2016 academic years, classroom teachers implemented project-related lessons at least once a month during which time students recorded their work using a screencasting app. For each implementation, the teachers completed and submitted a strategy planning and reflection form. Education researchers observed and video-recorded each classroom once a month and then completed an observation log. The team also collected student work done on iPads as evidence of student learning. Finally, teachers and researchers completed surveys and interviews throughout the two years of the project.

Therefore, the project adopted a three-tiered approach that coupled our professional learning and research objectives. The first was ongoing monthly professional learning experiences, facilitated by mathematics education researchers and university faculty, which were driven by the questions that emerged from the teachers' implementation of previous learning or their responses to regular surveys. The second was monthly classroom observations, video recordings, and online logs that teachers completed to record their methods of strategy implementation and observed student outcomes. The third was the sharing and discussing of student work at the monthly meetings. Through these different avenues, we collected data via:

- Reflections on and discussion about student work over one-and-a-half years
- Reflections on and sharing about strategy implementation over one-and-a-half years
- Online logs completed by teachers each month, which addressed strategy implementation in individual classrooms, and observations of potential student impact over one-and-a-half years
- Monthly surveys, conducted during our PLC meetings, in which participants reflected on their strategy implementation, outcomes, and broader themes related to our collaboration

- Student screencasts over two years
- Classroom videos over two years
- Interviews with 17 members of the collaborative team, including teachers, administrators, researchers, teacher educators, and teacher leaders at the end of the second year
- Email correspondence between researchers and educators over two years
- Final written reflections from teachers at the end of the project

Together, the data collected and topics discussed both formally and informally informed our design choices as we cycled through the iterative process.

4.5 A Focus on Ongoing Professional Learning

As previously noted, a critical piece of our work is the fact that the research and ongoing professional learning experiences have been tightly coupled. We knew that without such appropriate professional development, the potential for the iPads to enhance the teaching learning of mathematics may likely be wasted (Attard, 2013). Thus, we highlight below the foci of our ongoing professional learning over the course of the project.

When we first came together in 2014, we quickly identified that the teachers lacked opportunities to engage in learning about best practices related to both teaching and learning mathematics and the use of technology to support students' early mathematics learning in elementary school classrooms. Similarly, they had experienced few chances to reflect deeply on their own classroom practices related to teaching mathematics and using technology to support their students' mathematics learning. In order to identify whether and how the use of technology in the classroom may support students' mathematical discourse, the team realized that we needed ongoing opportunities to examine research, test out ideas, and deeply consider the outcomes of the work. Put simply, we wanted to ensure that we based our study around the use of technology in kindergarten–second-grade classrooms on research-based practices that support high-quality mathematics teaching and learning. Consequently, we needed to work together to identify, understand, and implement high-quality practices.

To achieve this, the team used the data collected to support continued professional learning opportunities. We also considered areas of interest as identified by teachers or needs as identified by both teachers and education researchers. After our initial work together engaging with technological tools and examining research around early mathematics learning trajectories (Clements & Sarama, 2004) and math congress (Fosnot & Dolk, 2001) as an avenue to foster mathematical discourse, we found that teachers needed information regarding additional topics in order to facilitate the type of learning they wished to see in their students. Therefore, in subsequent meetings we learned about and engaged in rich tasks (Stein, Smith, Henningsen, & Silver, 2000; NCTM, 2014) and open questions and parallel tasks (Small, 2012). The introduction of rich tasks into classroom practice led to discussions around productive struggle (Warshaur, 2015) and the use of wait time. Later, we began to examine the district's mathematics textbook and discussed ways to align project work with the text. We also began to explore different models and representations of mathematics problems. To facilitate this learning, we created research and practice briefs, which present short summaries of existing literature, around topics of interest or need. For example, we provided the teachers with and discussed one brief that synthesized the literature on the learning trajectories for counting and cardinality, another about technology in early grades' mathematics classrooms, and a third that detailed research on discourse in mathematics classrooms.

In addition to providing educators the opportunity to deepen their professional knowledge around best practices in mathematics instruction, the monthly PLCs gave the teachers space to share the strategies they used to implement the use of screencasting apps in their classrooms and to observe, analyze, and discuss examples of student screencasts from other teachers' classrooms. Because this time was fundamental for teachers' own learning and efficacy, and because we gained great insight about what was occurring in each classroom, we engaged in these activities monthly.

4.6 Findings

After two years of study, emerging data suggests that the strategic use of screencasting apps in K–2 classrooms can encourage students to communicate, reflect on, and revise their mathematical ideas. Because students record themselves explaining their mathematical thinking, they also listen to their own ideas and the ideas of their classmates through the screencast recording. We see evidence of increased student engagement in self-assessment and peer-assessment, which often prompts them to revise their own work. Similarly, students in our study show evidence of more persistence when working on higher-level math tasks. Participating teachers' teaching practices, meanwhile, have become more closely aligned with the high-leverage practices identified by NCTM in *Principles to Actions: Ensuring Mathematical Success for All* (2014). Specifically, we see evidence of teachers facilitating meaningful mathematical discourse, posing purposeful questions, implementing higher-level tasks that promote reasoning and problem solving, and supporting productive struggle in learning mathematics. Because students in the classrooms now explain their thinking more than before the study began, teachers have become more knowledgeable about what their students know and can do mathematically. In turn, teachers appear to be shifting their beliefs about the learning and teaching of mathematics that are possible in their classrooms.

In this section, we detail qualitative data that suggest the ways in which student discourse in mathematics classes has changed through teachers' participation in this study. We identify pedagogical practices that seem to support this increased discourse including the use of rich tasks, wait time, enabling students to engage in productive struggle, and the use of routines and teacher-created resources. We then highlight how the use of screencasting apps appears to have provided both teachers and students with increased opportunities to participate in meaningful formative assessment work. We also highlight the ways in which these changes in practices align with changes in teachers' beliefs about their students' learning. Namely, teachers are more apt to see their students as capable and competent mathematics learners. We share emerging insights into how the use of screencasting apps in early mathematics classrooms may support equity in student learning by engaging students who are often marginalized in classrooms. Finally, we identify challenges that teachers faced when implementing the research strategy.

4.6.1 Student Discourse

Some research (e.g., Hall, 2015) suggests that using handheld mobile technology in early grades classrooms promotes isolated learning that prohibits social interaction and limits hands-on learning. However, our research indicates that careful and strategic use of screencasting apps (specifically, EE) has quite the opposite effect. When our work began in the district, students most often used iPads individually. They wore headphones and interacted with devices only by tapping their screens. Two years into the study, participating classrooms now look and sound very different. Kindergarten–second-grade students are solving richer mathematical tasks on their own and with their peers. They create their screencasts individually at times and in small groups at other times. Invariably, the students recognize that they will share their work with the teacher, a classmate, in a small group, and/or with the whole class. As students record and share their work, there is generally lively mathematical discourse occurring between students.

Participating teachers noted and are excited about this change as well. They report that students now have conversations with one another about math, sharing their mathematical strategies and improving their ability to communicate about their mathematical reasoning and their use of vocabulary. In one online strategy log entry (12/15/2015), a participating second-grade teacher, Mrs. K, wrote that "EE offered the opportunity for students to practice verbalizing how they subtracted." Mrs. K continued to reflect on her practice over the course of the project. At the end of the second year of the study, she reported in her interview,

> I guess I didn't realize how little I had students talking about math. And I'll find now just sitting on the carpet students will just be engaging in conversation while I'm writing something up on the board and it's about a math problem. And they're arguing with each other but being really reasonable and they're doing those things that we're practicing but without that scaffold of the video. Which I think is really the goal. I feel like the videos are a stepping stone towards just having math conversations with each other.

Mrs. M, another second-grade teacher, indicated in her strategy log (12/9/2015) that "when students talk through what they know and how they know it, they make connections. It also helps their classmates make connections and have 'aha's'. Connections = new learning!" Importantly, teachers identified that the use of screencasting apps in their classrooms changed their overall math instruction. An example of this change comes from Mrs. B, a third second-grade teacher, who noted that without this project "I wouldn't have been doing as much discussion and be really thinking about my instruction to foster discussions among the students" (Online Strategy Log, 10/8/2015). In her interview at the end of the second year, Mrs. B told researchers,

> In the past, my kids have said, 'Well, this is how I did it.' Or, 'I just knew it,' and they don't discuss anymore. This year, they're having discussions and talking about their thinking and responding to others.

Ultimately, survey data indicates that teachers believe that this increased communication, supported by the use of classroom technology, provided their students with more ownership over their learning and improved their mathematical language and justifications.

We see evidence of this increased mathematical discourse not only when students use the screencasting tool itself, but also without the technology. It appears that students and teachers became more comfortable and confident with communication in mathematics class. For example, a researcher in one second-grade classroom observed students sitting around an easel at the beginning of the lesson. After the teacher prompted them to solve an addition problem and share their strategy with the group in a whole class discussion, students began to talk to one another about their approaches using phrases such as, "I'm wondering why you…" or "I started the same way as you, but then I…". In a different second-grade classroom, we observed students frequently engaging in the routine of sharing work with one another before providing each other with a compliment and a question about their work.

4.6.2 Pedagogical Practices that Support Discourse

While we strongly believe that the use of the screencasting apps facilitated students increased mathematical communication both when using mobile devices and in general, we also consider the teachers' use of high-leverage practices and instructional supports, studied and shared during our monthly PLCs, to be instrumental in the improved student discourse. Other researchers, such as Attard (2013), have also noted the need for a more symbiotic connection between professional development that focuses on pedagogical content knowledge and professional development that focuses on technology integration. Attard (2013) reported on a study that highlights the need for appropriate professional development that addresses all aspects of technological and pedagogical content knowledge to ensure successful integration of innovative technologies and to ensure the new teaching practices actually enhance the teaching and learning of mathematics. We found that coupled with such professional development, the nature and affordances of the iPad and screencasting app can help promote the use of high-leverage practices and routines that support productive discourse (e.g. rich tasks, wait time, and productive struggle).

Rich Tasks. When our project first began, like many teachers in the United States, the participants in our study felt pressure to "get through" the district-adopted textbook. Through our examination of research around rich and open tasks, teachers considered how they might open up and/or increase the level of cognitive demand of some of the questions provided in their text and began to create their own rich tasks related to the mathematical content being studied. Teachers also started to identify areas in the textbook that were already ripe for richer mathematical discussion. When describing how she used the screencasting app with her students, Mrs. K reported that the best use of the tool was "definitely the more open-ended math problems." She then added,

> When we come to a more challenging kind of an open response or open-ended math problem that's when we take that time to really explain our thinking. So any opportunity, we have our math curriculum and there's a lot of opportunities within that math curriculum when they say to explain our thinking […] Instead of a few lines or a little space for them to explain how they thought which is hard for a second grader to do. They now get to talk their way around it and figure that out. (Interview, 2/2/2016)

To support their student discourse, teachers realized the importance of giving their students richer, more open tasks. For example, Mrs. B reported,

> I think having those open, rich questions and tasks and having the conversations and having the ability to use all those different apps because those apps let them see things in different ways. I think all that together has really been positive. (Interview, 1/26/2016)

School administrators also noticed that students engaged more frequently with open tasks. In her interview (2/1/2016), Principal Mrs. S said,

> If you have an open enough task, regardless of ability level, a student is going to be [able] to access that task […] How they come up with answers […] might differ, but […] it's been good to sort of facilitate more of those open-ended, multiple entry tasks versus the skill and drill.

When teachers first began to use rich tasks in their classrooms, they primarily drew upon examples provided by researchers at PLC meetings. Later, however, they started to identify areas in their text with richer tasks than they might have first assumed, as is evidenced by Mrs. K's previous statement. Teachers also began to create their own open tasks for students, grounded in contexts related to their own classrooms. For example, we observed this in spring 2015, when the Mrs. S, a second-grade teacher, used the following prompt: "I bought a package of cardstock yesterday. The package contained four colors (red, yellow, blue, and green) and 50 sheets of cardstock. How many of each color might have been in the package?" As the project has progressed, teachers have more deliberately chosen open questions and rich mathematical tasks for their students. These types of problems allow students to identify a variety of possible solutions and use multiple strategies, and they help to promote more prolific mathematical explanations that students can capture on their screencasting apps.

Wait Time and Productive Struggle. Participating teachers indicated that using the screencasting tool promoted their use of wait time. Because students pre-record their solutions and play them from beginning to end, the teachers would not interrupt a student's presentation to prompt his or her thinking as they traditionally would do when the student would orally present the work off of technology. Furthermore, because teachers cannot get to all students as they record their videos, students often talk or think through a problem themselves. The teachers discovered that students often self-correct in the middle of the video when the teacher doesn't interrupt the students' thinking.

One math coach shared that the examination of research around productive struggle was valuable for the primary-level teachers and indicated in her interview at the end of the second year,

> [T]hat's the piece we want to make sure, for some teachers, it's not just regurgitating what you told them to do. It's really letting children think and explore their own thinking and then being able to listen to what the children have said, and identify where they're at, to find out where to move them next. This would be probably a challenge for some teachers, because, especially in the primary level, we're still coddling and motherly types. We don't like to teach in struggle. So needing to really have a clear understanding that productive struggle is where kids learn might be new learning for some.

A school administrator, Mrs. D, echoed this sentiment in her interview (3/11/2016) and told researchers that she sees evidence of this in the classrooms that participate in the study: "[W]hat I've seen is that students are willing to take risks and engage in productive struggle […] And for kids the end result is a better understanding of the learning process itself."

Providing students with wait time and opportunities to experience productive struggle in tandem with the rich, open tasks described above appears to have resulted in students persevering through challenging problems for longer periods of time. When the teacher does not immediately provide the student with the correct answer or an appropriate strategy, and when the student is tasked with creating a video explaining his or her thinking, students invest in sorting through their misconceptions and incorrect answers. Students are also now more comfortable sharing their partial solutions and sharing what they have found challenging with one another. They solicit feedback and ideas from one another when they struggle with a problem. They then use this information to help them record and re-record their work multiple times until they are satisfied with their final product. For example, when second-grade students worked on solving the cardstock problem above, the teacher had the students come together after about 15 min of work time in order to share their ideas. Three students shared their videos even though no one had the correct answer. Before one video played, the student told the class, "I don't have an answer, I only know how many sheets there are." His first attempt showed only the number 50 on his screencast. The students talked about the problem together and, in a conversation facilitated by the teacher, came up with new ideas for approaching the work. The students then continued working on the problem for another 30 min before coming together again to observe three finished videos, all with correct solutions to the problem.

Instructional Routines and Resources. In addition to implementing research-based practices in their classrooms, teachers further adjusted their practice to enable students to use their iPads to communicate their ideas. Teachers adapted, refined, and shared with one another pedagogical practices that support the use of screencasts to improve communication. Kindergarten teacher, Mrs. G, for example, started to incorporate math tools, such as the rekenrek and number frames, into her every day classroom activities. Guided by information in one of the project's research and practice briefs about developmentally appropriate use of technology, Mrs. G provided her students with opportunities to explore both hands-on and technology-based versions of the same tool. One researcher observed a morning routine change in a second-grade classroom so that the teacher now provides students the opportunity to critique their own reasoning and that of their peers. In this routine, each child solves a math problem and records his or her solution on the iPad. The class then observes, discusses, and reflects on some of the videos. Children return to their iPads to continue working on their solutions by re-recording their work if their thinking changed, or to add to their original work if it did not. In another second-grade classroom, the teacher, Mrs. H, introduced the notion of a "Math Guest Teacher of the Day." For this activity, the teacher chooses one student's video to be shared at the beginning of math class. After they share the video, the students in the class discuss what made the video strong and what might make it even stronger. Additionally, they ask one another questions about and discuss the mathematics shared. The teacher tries to incorporate this instructional routine into her lessons at least three times a week.

In their surveys, teachers reported the use of other pedagogical moves such as modeling teacher-created videos of various quality for class discussion and analysis, pausing during a lesson to share "quality" videos in order to keep students on track, ensuring time for students to share their work with partners before coming together as a whole class, and a series of moves that includes student think time, time for partners to record and watch videos together, and sharing and critiquing videos in small groups.

Teachers also identified and researchers observed a variety of tools being created and used by teachers to support their students' communication. These include the use of sentence starters (e.g. I know my work is correct because…), sentence frames (e.g. I started at __ and counted up to 10. That was __ jumps. Then I counted to __. That was __ jumps. Then I added my jumps to get __. So, __- __ = ___), co-constructed checklists of indicators that contribute to a strong video (e.g. I can hear my explanation; I can see a picture that helps to explain my thinking; My picture and writing are clear and easy to read; I say what problem I am solving; My explanation is easy to understand; I explained the math words when needed; My math is correct), discussion guidelines (e.g. It's OK to change your mind, It's OK to feel confused), and anchor charts about quality explanations. Notably, after teachers created these tools, they often shared information about the tools during the monthly PLCs. They shared templates with one another, information about how they use the tools in class, and their reflections on how the tools promote student learning. Some of these teacher-created, shared tools have been adopted by other teachers across the district.

4.6.3 Formative Assessment and Student Discourse

One, unanticipated, promising result of using the EE screencasting app in the early elementary school setting has been a noticeable increase in the use of formative assessment, done by both teachers and students. Research emerged over the last two decades that shows the critical nature of formative assessment, including peer and self-assessment, in fostering growth in student learning and engagement (Black & Wiliam, 1998; Fontana & Fernandes, 1994; Hwang, Hung, & Chen, 2014). Likewise, Wiliam (2000) suggests that "effective learning involves having most of the students thinking most of the time" (pp. 21–22) and that when formative assessment becomes an everyday routine, students think more deeply and reflect on their own academic progress.

Teacher Formative-Assessment. The teachers in our study reported that the recordings provide a valuable source of assessment data. For example, Mrs. M indicated,

> The opportunity to observe students self-correct their thinking as they talk out a problem with a peer gives us lots of information. It gives us a window into their thinking and helps us plan next steps for instruction and explorations for students. (Email message to author, April 11, 2015)

Similarly, Mrs. K reported,

> [W]hereas before it was one teacher and 20 students, so I didn't always get to listen to how every student was listening and sharing their thinking. [Now,] I can look at those students who I know are struggling and take time to review their recording later, and then I can meet with them again. So, it's a nice snapshot of how students are doing with a particular problem at that moment. (Interview, 2/2/2016)

Multiple teachers share that observing their students' screencasts helps them to plan instruction to meet their students where they are in their learning trajectory. We theorize that the fact that teachers now have the opportunity to hear each of their students explain their mathematical ideas allows the teachers to better understand the abilities and misconceptions held by their students. Additionally, because the teachers now have the ability to watch a student work through a problem and listen to their students' explanations via the screencasts, teachers have a better sense of what the children understand than they did when looking at static work, which is typically handed in on paper. In his interview, school Principal Mr. D called the screencasting app a "fantastic" tool for formative assessment and praised its potential to provide teachers with powerful information for their instructional decision making. Teachers also appreciate that the EE videos serve as a "container" that holds student thinking over time and helps show students' learning progress. Some even shared their students' videos with parents during conferences.

Student Peer and Self-Assessment. Teachers not only found more frequent and meaningful ways to assess their students' thinking, but the use of the screencasting app also appears to support students' own ability to engage in self and peer assessment. Black and Wiliam (1998) suggest that this type of formative assessment is "essential to good learning" (p. 6). After students record their solutions on the iPad, they often review their own recording either on their own or with prompting from their teachers. As they hear themselves explain their thinking, they can identify areas in which their explanations are unclear or where they misrepresent their thinking. The students often delete their work and start this process again in order to create a stronger explanation. Soto (2015) identified similar evidence of students engaging in self-assessment when using EE to describe their mathematical work. However, Soto's study examined students' screencasts in a 1–1 environment, in which a researcher and student sat together, while the student recorded his or her video. Our research indicates that Soto's (2015) findings around improved student self-assessment hold true in a classroom environment even when the student and teacher do not interact 1–1.

We found multiple examples of students correcting their work in their screencasting samples. For example, when one second-grade student, Brendan, recorded his solution to the problem "You bought something at the store that costs 72 cents. You paid one dollar. How much change did you receive?," he began by making jumps on an open number line. He started at 72 and jumped to 80. As he did this, he said, "I'm going to skip to 80 and I'll put ten" and then wrote a 10 underneath the jump from 72 to 80. Brendan then made a second jump to 90 and said, "then I'm going to skip to 90, and I'll put ten right there" as he recorded a second 10 under the second jump. He then said, "oh, this isn't ten, I accidentally messed up" at which point the screencast shows him erasing the first ten he recorded while saying, "so, this is actually eight."

Another student in the same class, Brian, recorded a screencasting video for a similar problem. This time, though, the amount of money spent was 63 cents. In his first video, we see the student making 7 jumps of one from 63 to 70 and then a "really big hop" of 30 to 100. When he recorded his final answer, he miscounted the jumps of one and recorded 38 cents as his solution to the problem. Students in this class then shared their work with a peer and used the class' co-constructed "Is My EE Video Complete?" checklist to guide their conversations. The teacher did not expect students to fix their videos during their peer discussions. However, she found that most of her students decided to make new videos. Brian was one of those students. His second video shows a more efficient strategy as he takes a jump of seven and a jump of 30 to reach 100. In his second video, Brian also records the correct answer, 37 cents.

Teachers and researchers also observed students self-correct their work. While the timing, method, and rationale behind the students' self-corrections vary widely, the EE app enables students to decide independently to fix their work in order to make it stronger. In her strategy log (11/18/2015), Mrs. K reported that a math congress, in which a small number of students shared their videos, helped deepen students' understanding of the math content and that "through this conversation, many students were able to revise their thinking and went back to change their work." Mrs. M, also spoke to this during her interview (3/11/2016) when she stated, "I think a really big 'aha' is that kids will self-correct. They self-correct when they make a mistake."

The teachers also indicated that students often report that they "change their minds" about their mathematical work or solution after seeing a partner's video. The teachers suggested that this is because students now observe their peers' work, reflect, and learn from their own errors. Mrs. C, a math coach, told researchers in her end of year interview,

> When they're doing their work and recording, and then they go back and listen, they can either deepen their understanding, or be reflective in the sense of self-correcting. Like, 'Oh, wait a minute. I meant this.' And within that piece, [they] may also clarify a misunderstanding they may have had.

Administrators also noted this change in student learning. In his interview (2/9/2016), Mr. D indicated,

> With the use of that technology as a tool, they're able to go back, and reflect on their work, and revise as needed […] Our biggest learning comes in reflection, and when students can hear their tablets, […] It's so powerful when you can see students self-correct on the spot or, even after the fact, when they're reviewing it […] So, I think students' understanding of math is enhanced by this.

Notably, Mrs. B connected student self-assessment to increased engagement, interest, and confidence in their math work:

> The videos were extremely helpful because we could follow students' thinking as it unfolded. Often students would self-correct as they were making their videos and revise their thinking. Students were engaged and eager to share their thinking. I saw my students gain confidence in solving problems not just to have the correct answer but to be a part of the process of solving the problem and deepen their understanding. (Final Project Reflection, 6/29/16)

Teacher survey data further indicated that students not only correct their own work, but they offer suggestions to help their peers improve their work as well. In one interview (1/26/16) Mrs. M underscored the fact that teachers across the district were noticing and commenting upon improved student assessment:

> I see a difference between my students and what they're learning and how they're talking about their learning and their thinking […] In our groups at our meetings, the other teachers are mentioning that they're seeing the same thing… They're seeing that their students are looking at their work and they're making the corrections and they're pointing out each other's mistakes and they're doing it in a respectful way. And the other teachers, […] they're seeing this growth in their students with math.

Overall, participants reported that students have a greater awareness of their own work, more recognition of multiple strategies for solving problems, and a deeper level of engagement with and reflection about their mathematics due to using the screencasting app.

4.6.4 Changing Teacher Beliefs

Over the course of the study, education researchers observed that participating classroom teachers changed their beliefs about what it means for children to know and do mathematics. Our evidence also suggests that the teachers came to believe that their students are more mathematically capable than they had originally assumed. Teachers' participation in our collaborative study appears to have supported a change in teacher beliefs in two ways.

The first is the way in which they understand what it means for students to be engaged in doing mathematics. For example, during one second-grade class, students struggled to find a correct solution to the problem presented. Students' recorded explanations on their iPads helped to prompt a lively conversation about the problem, and by the end of the day's lesson, the students identified their misconception and how they might fix it. After the observation, the teacher, Mrs. S, indicated to the researcher that in the past she would have considered the lesson a failure because none of the students got the right answer. She reported that she now considers this to be a very successful lesson because her students were able to think deeply, reason mathematically, and determine what their new approach would be. Additionally, Mrs. S noted that the students all remained engaged and excited by the work. A kindergarten teacher, Mrs. G admitted, "In the past, I'd never really thought about how important it is to have [students] explain what they're doing, what they're thinking. So it has been huge for me; it's really been an eye-opener, and a changer, in how I teach my kids" (Interview, 02/25/2016). In her end of project reflection (6/29/2016), Mrs. G added, "My practice has become more thoughtful and reflective in what and how I teach math to my Kindergarten children. I no longer am the teacher in charge, but put that role in the hands of my children." Similarly, Mrs. K reported in an interview (2/2/2016), "I'm finding myself trying to step away and talk less to allow the students to talk more […] Students can really learn a lot from each other and that's valuable." Other teachers echoed this belief in their surveys. They indicated that they now find it important to allow students to struggle and that teachers shouldn't be afraid of this. One teacher stated that she and her students discovered that mistakes help with learning. These findings align with other research in mathematics education, which highlights the importance of productive struggle for learning (NCTM, 2014; Warshaur, 2015).

A second way that teachers' beliefs changed relates to their estimation of their own students' mathematical abilities. Many students in this project come from low-income households, and participating schools have experienced chronic underachievement in mathematics. Kindergarten teacher, Mrs. T, explained in her interview (2/3/2016),

> I have [a student] that just [makes] video after video […] His are just right on and he's picking up and using tools that I haven't even taught yet. And he's using them correctly and it's just a lot of really neat stuff from that kid that I did not expect.

After just one year of work on the project, these changing beliefs were evident. Mrs. S reported,

> "I watched children from other schools explain their thinking and assumed they were just 'smarter' than the kids I work with. Now my students are those 'smart' kids because they can explain and show their thinking" (Email message to author, 05/20/2015).

4.6.5 Equity

Perhaps the most important finding from our work is the way that the use of screencasting apps in mathematics classrooms provided more equitable learning experiences for the children involved in the study. Our findings suggest that using the screencasting tool in mathematics classrooms has the potential to support equitable learning for *all* students in the mathematics class. Only a few students may have the opportunity to present their ideas each day in a typical math class. In classrooms using screencasting apps, every student records and explains his or her work. In essence, each student creates, and often revises, a presentation of his or her mathematical ideas each time the screencasting tool is used, even if that presentation is not shared with another student or with the entire class. In an interview, Mrs. K stated that the screencasting tool "forces everyone to engage in that problem and everyone to talk" (Interview, 02/2/2016). Mrs. K further commented in her log about this fact stating, "creating the videos requires ALL students to think" (01/23/2016) and that during their whole class discussions, students begin to make connections and "light bulbs" go on. Principal Mr. D also shared ways in which using the screencasting app supported equitable learning in the classroom in his interview (2/9/2016),

> We may come up with the same answer, we may all have taken a different route to get there, and I think that being able to capture these things honors that process of we all think differently, and so what students are sharing out in the classroom [is] OK.

Mrs. B an English language learner teacher reported in an interview (01/26/2016),

> Sometimes they don't have the language and the ability to write down what they're thinking. But if they can use this recording tool and an app, they can show it and can talk about what they've done a little bit more easily than if it was pencil/paper.

Moreover, we observed children who receive intervention supports, children with autism, and a student who is selectively mute record their voices using the screencasting tool and share their video-recordings with their classmates. In this manner, students who may commonly be marginalized in a classroom or who may not have opportunities to share their ideas publically participated more fully in mathematics lessons. School administrator, Mrs. D reported in her interview (3/11/2016), "In a classroom where this is happening all of the students are equal participants. And all of the students' ideas are equally valued." This emerging evidence suggests that screencasting tools, such as EE, may provide a platform for a wider range of learners to communicate their ideas and have their voices heard in math class.

4.6.6 Challenges

Unsurprisingly, some challenges occurred with implementing the screencasting tool in classrooms. As expected, both teachers and students experienced a learning curve related to the use of new technology, and this held true with the EE app and its features. In surveys, interviews, and monthly debriefing sessions, teachers and administrators cited overcoming discomfort with the technology itself as a hurdle. Furthermore, the teachers confronted the challenges of poor recording quality, particularly with sound, storing the videos for later reference, and ease of accessing the videos once stored. Teachers also identified difficulties with internet strength, particularly during the state-mandated testing period when students in the upper grades used much of the bandwidth to complete their assessments. Additionally, teachers struggled with finding time to learn the technology themselves, to teach the students how to use it, and to integrate the project strategy into their already busy schedules. With time and practice, many of these challenges dissipated and teachers reported them less frequently. We suggest, however, that these are akin to challenges that might be faced when introducing any new tool in a mathematics classroom, tangible or technological. We know that students and teachers need time to become comfortable and familiar with a new tool and its use.

Beyond the use of technology in the classroom, but closely connected with the work done on the project, teachers and administrators indicated that classroom management (specifically keeping students focused on their task), allowing time for productive struggle, supporting student explanations and critique, and understanding the development of numeracy skills proved to be difficult. We addressed many of these topics during our professional learning time; however, we believe that the teachers might have confronted a disequilibrium between the newly introduced high-leverage, research-based practices and those they currently implemented, which teachers commonly experience when introducing new pedagogical approaches into their teaching, whether or not technology is a part of the work. While we recognize that these challenges were indeed real, we also suggest that they were an important component of improving practice.

4.7 Discussion

After two years of study, our findings suggest that there is not just one way to implement the use of screencasting apps, such as EE, in kindergarten–second-grade mathematics classrooms. Instead, we found that the participating teachers used this tool in a variety of ways, depending on their context and on what was most comfortable for them and their students. Some teachers used screencasting tools more frequently than others. Some teachers had students create videos with a partner, while others had them share their videos with a peer. Some classes used screencasts to respond to problems that came from the district-wide curriculum, and in other classes teachers created problems for students to respond to when using screencasts. In some instances, students shared their work with the whole class, and in other instances, they shared it with small groups. Sometimes, students solved a problem directly on the screencast. Other times, students solved a problem, took a picture of their completed work, and then described it. Regardless of the approach, however, we observed commonalities across classrooms.

In all classrooms, students and teachers began to attend to one another's mathematical ideas. Students first began to explain and later justify their mathematical thinking. Because students in every class shared their screencasts with at least one other person, and because students in every class viewed at least one other video each time the tool was used, the children in this study began to consider how they might present their ideas to someone else in the best way. Our findings suggest that the opportunities to see and hear others' work allow students to assess, edit, and re-do their own work. Additionally, because students viewed one another's work and heard one another's mathematical thinking process, they became more comfortable with the idea that in mathematics there is often more than one appropriate approach and sometimes more than one correct solution to a problem. This allowed students to engage in problem solving and persevere through challenging moments.

We observed that teachers in this study began to change their teaching practices as a result of the increased student discourse. As students began explaining their thinking in more detail, teachers began to recognize the need for richer mathematical tasks. Additionally, because of the pre-recorded nature of the screencasts, teachers were essentially forced to engage in wait time. This allowed students to self-correct as they worked through a problem. As students began to solve more cognitively demanding problems and were provided with sufficient time to think through them, teachers realized that their students were capable of more sophisticated mathematical thinking than they previously thought.

Teachers in this study recognized the need to support their students not only with the task of solving mathematics problems, but also with the task of creating a high-quality explanation. Teachers, therefore began creating tools such as sentence starters, checklists, discussion guidelines, and sentence frames. They also began to take time during math class to discuss not just the mathematics, but also the components of a strong mathematical justification.

The fact that the research (observation, survey, interviews, logs) was tightly coupled with ongoing professional learning was critical to our work on this project. Not only did our summer meetings and monthly PLCs provide teachers with a shared, research-informed, vision of high quality mathematics instruction, it also provided them the opportunity to share tools and implementation strategies with one another. This chance to both reflect on and take ownership over their professional learning and teaching practice is, we believe, strongly linked to the outcomes of the study.

4.8 Limitations

One notable limitation of this study is its sample size. Moreover, school-based partners, particularly participating teachers, self-selected to participate. Participation in this study required an intensive time commitment. In addition to allowing researchers into their classrooms, and to completing monthly strategy logs, participants committed to summer professional development sessions, monthly meetings of the PLC during the school year, and pre- and post-observation meeting times. The time required to participate in this study prohibited some teachers from joining the project and others from returning during the second year. Students in this study each had access to 1–1 iPads in their classrooms, which is atypical in many early elementary school classrooms. The work done on this project would be strengthened with additional research including, but not limited to, studies done in different community settings, with larger numbers of participants, in grades beyond kindergarten–second-grade, and in schools with and without 1–1 mobile technology. While we recognize the significant time challenge for all participants in the study, we also strongly believe and other research supports (Attard, 2013; Attard & Curry, 2012) that the professional learning was a critical feature in the design of the study, and that the outcomes would not be possible if this were removed.

4.9 Conclusion

The project's collaborative team of researchers and practitioners believe that the use of screencasting apps in early elementary mathematics classrooms holds significant potential for supporting mathematical discourse among students. Participating teachers reported and researchers observed improved student communication in mathematics, which in turn allowed for changing teaching practices and beliefs, increased use of formative assessment by both teachers and students, more evidence of student self-correction and perseverance, and a more equitable learning environment for all. As one second-grade teacher, Mrs. M, indicated in her strategy log (10/9/2015), "the EE app is great tool for students to share their thinking. If we start students at an early age talking about problem solving in all curriculum areas, it will be natural for them to share their thinking. We will be able to see learning in action as students use their prior knowledge to make connections to new learning. WOW! This is what learning and school is all about!!"

Acknowledgements Supported by the National Science Foundation (grant DRL-1238253). Opinions expressed in this manuscript are those of the contributors and not necessarily those of the Foundation.

References

Attard, C. (2013). Introducing iPads into primary mathematics pedagogies: An exploration of two teachers' experiences. In V. Steinle, L. Ball, & C. Bardini, (Eds.), *Mathematics Education: Yesterday, Today and Tomorrow—36th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 58–65).

Attard, C., & Curry, C. (2012). Exploring the use of iPads to engage young students with mathematics. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics Education. Expanding Horizons. Proceedings of the 35th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 75–82).

Black, P., & Wiliam, D. (1998). Inside the black box: Raising standards through classroom assessment. *Phi Delta Kappa, 80*(October), 139–144, 146–148.

Blair, K. P. (2013). Learning in critter corral: Evaluating three kinds of feedback in a preschool math app. In *Proceedings of the 12th International Conference on Interaction Design and Children* (pp. 372–375). ACM. Retrieved from http://dl.acm.org/citation.cfm?id=2485814.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics (CCSSM)*. Washington, DC: National Governors Association Center for Best Practices & Council of Chief State School Officers. http://www.corestandards.org/Math/.

Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5–8.

Dorph, G. Z., & Holtz, B. W. (2000). Professional development for teachers: Why doesn't the model change? *Journal of Jewish Education, 66*(1–2), 67–76.

DuFour, R., & Eaker, R. E. (1998). *Professional learning communities at work: Best practices for enhancing student achievement*. Alexandria, VA: ASCD.

Fontana, D., & Fernandes, M. (1994). Improvements in mathematics performance as a consequence of self-assessment in Portuguese primary school pupils. *British Journal of Educational Psychology, 64,* 407–417.

Fosnot, C. T., & Dolk, M. (2001). *Young mathematicians at work: Constructing number sense, addition, and subtraction*. Portsmouth, NH: Heinemann.

Ginsburg, H. P., Jamalian, A., & Creighan, S. (2013). Cognitive guidelines for the design and evaluation of early mathematics software: The example of MathemAntics. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 83–120). Dordrecht: Springer Netherlands. Retrieved from http://link.springer.com/10.1007/978-94-007-6440-8_6.

Goodwin, K., & Highfield, K. (2013). A framework for examining technologies and early mathematics learning. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning*. Dordrecht: Springer Science+Business Media.

Hall, L. (2015, December 2). I gave my students iPads—Then I wished I could take them back. *The Washington Post.* Retrieved from https://www.washingtonpost.com/opinions/i-gave-my-students-ipads–then-wished-i-could-take-them-back/2015/12/02/a1bc8272-818f-11e5-a7ca-6ab6ec20f839_story.html.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research, 77* (1), 81–112. https://doi.org/10.3102/003465430298487.

Hwang, G., Hung, C., & Chen, N. (2014). Improving learning achievements, motivations and problem-solving skills through a peer assessment-based game development approach. *Educational Technology Research and Development, 62,* 129–145.

Loucks-Horsley, S., Love, N., Stiles, K. E., Mundry, S., & Hewson, P. W. (2003). *Designing professional development for teachers of science and mathematics* (2nd ed.). Thousand Oaks, CA: Corwin Press.

Moschkovich, J. N. (2012). How equity concerns lead to attention to mathematical discourse. In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in discourse for mathematics education* (pp. 89–105). Netherlands, Dordrecht: Springer.

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM.

Penuel, W. R., Fishman, B. J., Haugan Cheng, B., & Sabelli, N. (2011). Organizing research and development at the intersection of learning, implementation, and design. *Educational Researcher, 40*(7), 331–337.

Sedig, K., & Liang, H. N. (2006). Interactivity of visual mathematical representations: Factors affecting learning and cognitive processes. *Journal of Interactive Learning Research, 17*(2), 179–212.

Showers, B., & Joyce, B. (1996). The evolution of peer coaching. *Educational Leadership, 53*(6), 12–16.

Small, M. (2012). *Good questions: Great ways to differentiate math instruction* (2nd ed.). New York, NY: Teachers College Press.

Soto, M. (2015). Elementary students' mathematical explanations and attention to audience with screencasts. *Journal of Research on Technology in Education, 47*(4), 242–258.

Soto, M. M., & Ambrose, R. (2014). Making students' mathematical explanations accessible to teachers through the use of digital recorders and iPads. *Learning, Media, and Technology*. https://doi.org/10.1080/17439884.2014.931867.

Soto, M., & Hargis, J. (2014). Students explain everything using iPads. *Learning and Leading with Technology*, 32–33.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards based mathematics instruction: A casebook for professional development*. New York, NY: Teachers College Press.

Warshaur, H. K. (2015). Strategies to support productive struggle. *Mathematics Teaching in the Middle School, 20*(March), 390–393.

Wiliam, D. (2000). Formative assessment in mathematics: Part 3: The learner's role. *Equals: Mathematics and Special Educational Needs, 6*(Spring), 19–22.

Yelland, N., & Kilderry, A. (2010). Becoming numerate with information and communications technologies in the twenty-first century. *International Journal of Early Years Education, 18*(2), 91–106. https://doi.org/10.1080/09669760.2010.494426.

Chapter 5
Duo of Digital and Material Artefacts Dedicated to the Learning of Geometry at Primary School

Anne Voltolini

Abstract Our research project questions the bonus brought by technology and the complementarities of material and digital frameworks in situations which link together digital tools and material tools. We will present here some features to define a duo of digital and material artefacts. We will illustrate our point with a situation to stimulate the use of a pair of compasses as a tool to construct a triangle with given lengths of the sides. We will show that digital technology can bring a didactic bonus, an extra value to a material tool for learning. Digital technology can raise functionalities which refer to a material tool, and vice versa, the material tool can enrich the digital tool. We will show how a duo of artefacts, both material and digital, used in a situation, brings processes of assimilation and adaptation of utilization schemes from one instrument to the other which assist in their instrumental geneses and lead to conceptualisation.

Keywords Digital and material artefacts · Didactic bonus · Duo of artefacts Utilization schemes · Instrumental geneses

5.1 Introduction

The aim of this research work is both the design, and the evaluation, of software allowing mathematics learning at primary school. This software consists of an experimental approach based on playing with representations of mathematical objects on the computer interface, and linking the digital objects with the use of real tools. For this kind of technological framework linked together with the use of concrete material tools we question the effectiveness in stimulating the pupil to uncover mathematical concepts: can the technology bring some extra value to the material tools, helping thus to overcome some difficulties or some epistemological obstacles?

A. Voltolini (✉)
S2HEP, Institute Français de l'Education, ENS Lyon, Lyon, France
e-mail: anne.voltolini@ac-grenoble.fr

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_5

5.2 Digital and Material Artefacts

5.2.1 Literature Review

Moyer-Packenham et al. (2013) studied the effects of physical and virtual manipulatives in mathematics on student achievement during fraction instruction. They randomly assigned participants, from third- and fourth-grade classes, to one of two treatment groups. One treatment group used texts and physical manipulatives in a regular classrooms; the other treatment group used virtual manipulatives in a computer lab. Their results demonstrate that using either physical or virtual manipulatives produces similar student achievement for third- and fourth-grade students learning the fraction concept.

In mathematics classrooms, digital technology is often either seen as a replacement or an adjunct of a text or physical manipulative activity. The examination of the role of the conjunction between digital and physical manipulatives is not often studied. Maschietto and Soury-Lavergne created two situations calling up a duo of artefacts in order to make the pupils learn the number system (2013) and the geometry (2015). In both cases the duo consists in linking a given material artefact with a Cabri Elem[1] e-book which refers to the material artefact. They show that what happens in terms of learning when using the real artefact is different from what happens when using the digital artefact in the e-book.

We take the same basic idea, calling up a duo of material and digital artefacts in order to design and analyze another didactic situation (Brousseau, 1997). Our main idea is not only to substitute a computer framework to the use of a material tool but link a digital artefact to a given material artefact. We claim that on top of the achievement of a task, tools will also become a contribution of mathematics activities of the pupils, yielding the emergence of mathematical concepts. We believe that it is possible to use the potentialities and constraints of a digital technology in order to create an artefact linked to a material artefact in such a way that this linking is an added value for the conceptualization. This program implies to answer the following questions: What is the advantage of such a digital and material duo for the learning process? How can we design the linking together between the physical and digital tools in a complementary way, in order to favour the learning? Is it possible to overpass the technical aspect when calling up a duo? And finally, can the duo contribute to the pupil's individual elaboration of mathematics knowledge?

5.2.2 Instruments Which Benefit from One Another

The use of artefacts, whether they are material or digital, forces the user to build and develop cognitive structures, called schemes (Vergnaud, 2009) to properly use the

[1]The Cabri Elem software developed by the Cabrilog company is used in this project in the framework of the scientific collaboration between Cabrilog and the Frensh Institute for Education.

artefact when accomplishing a task. An artefact's potential uses and restrictions influence the actions and strategies used to solve a given problem. The artefact becomes an instrument through a process of instrumental genesis (Rabardel, 1995) when the user has appropriated it and integrated it in his or her activity. Depending on the type of the given task, while using the artefact, the user creates utilization schemes to properly use it to accomplish the given task. The utilization schemes organise the actions taken, they are developed in situation and bound to one use. An instrument is a dual entity, mixing the artefact as well as the utilization schemes created by the user to accomplish a given task. "*The instrumental genesis is a complex process, needing time, and linked to the artefact characteristics (its potentialities and its constraints) and to the subject's activity, his/her knowledge and former method of working*" (Trouche, 2004).

In a situation which uses a duo of material and digital artefacts, the two artefacts interact with each other as they are used, and thus the two instrumental geneses are mixed. According to Rabardel (ibid), the utilization schemes are bound to their artefacts as well as the objects with which the artefacts interact. "*However, utilization schemes cannot be applied directly. They must be adapted to the specificity of each situation. They are implemented in the form of a procedure relevant to the particularities of the situation*" (ibid, p. 85). When the assimilation of a situation does not allow accomplishment, the scheme is progressively adapted to become a new scheme. "*The implementation of utilization schemes in new but similar situations (assimilation process) leads to the generalization of schemes by extension of the classes of situations, of artefacts and objects they are relevant to. It also leads to their differentiation since most often they have to change to adapt to new and different aspects specific to situations*" (ibid). In a situation which uses a duo of artefacts, both material and digital, we question the scheme's process of assimilation and adaptation from one instrument to the other. We formulated and tested the following research hypothesis: a duo of artefacts, both material and digital, used in a situation, brings processes of assimilation and adaptation of utilization schemes from one instrument to the other which assist in its instrumental geneses and lead to conceptualisation.

5.3 A Duo of Artefacts, Both Material and Digital, to Introduce the Compasses in the Geometric Construction of a Triangle

5.3.1 A Material Artefact: The Material Compasses

We suggest a situation which consists in teaching the geometric construction of a triangle using a ruler and a pair of compasses, knowing the lengths of its three sides. There are two objectives: firstly, to give meaning to the use of the compasses in the construction of a triangle of which we know the lengths of its three sides.

The second objective is to make the pupil's knowledge of triangles grow. Often, the use of the compasses in a geometric construction is linked to a technical ability. The goal of using the compasses is never mentioned. "*There is a confusion between the ability to trace precise lines for a circle with the compasses and the knowledge of reasons why the tool is adequate*"[2] (Artigue & Robinet, 1982). To understand the procedure of constructing a triangle with compasses is one thing, but to understand why the compasses are adequate to accomplish this construction is another. Much like Artigue & Robinet (ibid), we believe that depending on the given task, the conceptions used in the use of compasses are different.

Multiple kinds of difficulties can be shown in the process of using a pair of compasses to construct a triangle of which we know the lengths of its three sides. We identify two of them: the difficulty relative to the dimensional deconstruction of the triangle (Duval, 2005) and the difficulty of the instrumental genesis of the compasses. Indeed, in this construction, the compasses are not the tool which allows the tracing of the outline of the triangle—it's the tool which allows the determination of the triangle's third vertex a 0 dimensional object, hardly apprehended by the pupils.

We discern two types of tools: those which produce the sought-out object, and those who produce an intermediary object which doesn't belong to the sought-out object. The ruler, used to trace a segment, produces a segment, and the compasses, used to trace a circle, produce a circle. But the compasses, used in the construction of a triangle, do not produce one of the triangle's sides. To move from the "compasses to trace a circle", instrument which produces the circle, to the "compasses to determine the triangle's third vertex", instrument which produces circles (circular arcs), an intermediary object, is at the heart of the design of our duo.

We will present later on our methodology regarding the design of a duo and the choices made in a digital environment:

- first of all, to help the pupil to create the compasses instrument to draw a triangle;
- second of all, to put into place the dimensional deconstruction of the triangle without necessarily, first, going to the 0D objects.

5.4 Methodology on the Design of the Duo

"*Tools only having meaning when relating to situations in which they are used*," (see Footnote 2) (Bruillard & Vivet, 1994). A duo of artefacts, material and digital, is attached to a didactical situation (Brousseau, 1997) and makes itself useful in this given situation. Our situation combines both digital and pen-and-paper environments, and allows the articulation between the manipulation of digital tools and the use of a

[2]Translated by the author from the original article in French.

material tool. After having done an epistemological and cognitive analysis of the material artefact, at the same time, we develop a digital artefact articulating with the material one as well as a situation in which the duo is used. In such a situation, our digital and material tools rely on one another, and the technology brings an additional element on a conceptual scale. A duo of digital and material artefacts is not necessarily characterised by a digital artefact which simulates precisely the material artefact. Nevertheless, it is necessary that the articulation between the two artefacts shows the links between the two and presents elements which follow a logical continuity in terms of learning. But some discontinuities are also necessary to promote the evolution of pupil knowledge.

5.4.1 A Situation Which Involves Material Compasses and E-Books

The technology used to design digital environments was the Cabri Elem software. The Cabri Elem technology allows us to create all of the elements with which the pupil will interact: the objects to be manipulated, the possibilities of actions that can be done on these objects as well as the environment's feed-backs. Such a digital environment is organised within an e-book. The user can go through the pages freely, and can do the different suggested tasks (Mackrell, Maschietto, & Soury-Lavergne, 2013). An e-book offers several tasks which bring the user to use appropriate strategies to accomplish each of them. Thus, Cabri Elem technology allows us to design tools and tasks in a digital environment, articulating with material tools. It allows the thought of articulation between the digital tool and the material tool.

Our situation includes two e-books and two pen-and-paper activities. Its orchestration (Trouche, 2004) was put in place with the intention of alternating activities set in a digital environment and in a pen-and-paper environment. Accomplishing the situation consists in successively handling an e-book and a pen-and-paper activity then a second e-book and a second pen-and-paper activity (Fig. 5.1). In the first e-book, the primary objective is to form triangles by manipulating digital segments. It provokes the elaboration of a rotation-dragging instrument to rotate the segments. This rotation-dragging brings the use of compasses in the pen-and-paper environment. The first pen-and-paper activity's objective is to include the material compasses in the triangle's construction. The connection between the first e-book and the first pen-and-paper activity is meant to implement the dimensional deconstruction of the triangle without having, at first, to get down to geometrical points. The second e-book is meant to bring in the circle as a tool in the geometric construction of the triangle. Playing with the tools available through the e-book's pages makes the circle tool a necessary strategy. Finally the objective of the second pen-and-paper activity is to construct triangles with the compasses and ruler. In the design of the duo, a range of didactic variables and

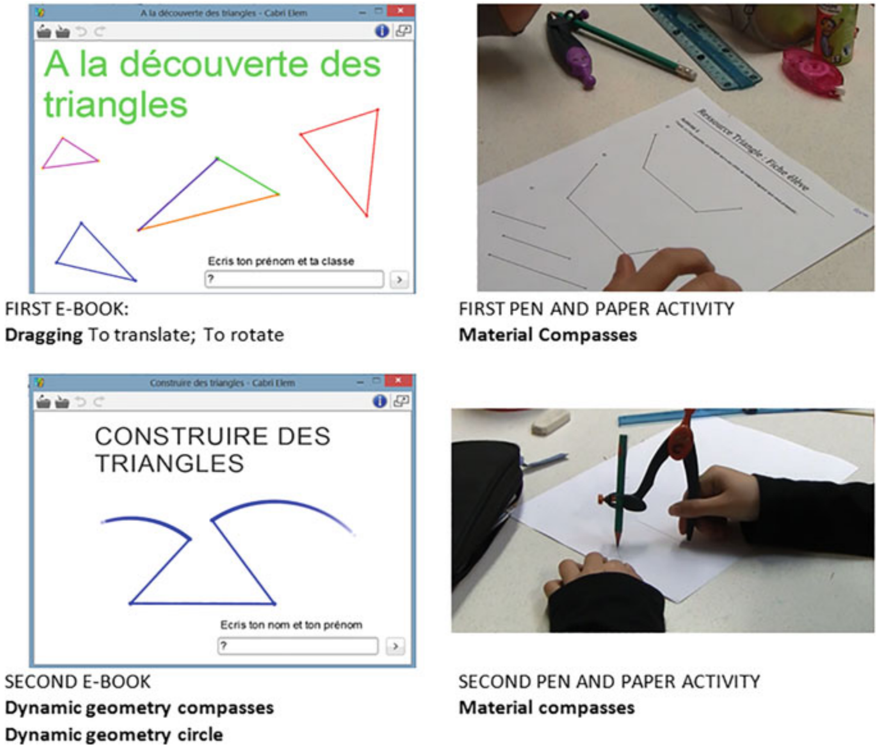


Fig. 5.1 A situation with the material compasses and e-books

mobilized artefacts induce the elaboration and evolution of strategies during the situation and of the connection between the digital artefacts and the material compasses.

5.4.2 A Choice from a Range of Didactic Variables

The first didactic variable is the length of the segments corresponding to the triangle's sides (a, b, c). For the three lengths $0 < a < b < c$, three values are associated to this variable. ($a, b, c = a + b$) The triangle is flat. (a, b, c with $c < a + b$) The lengths verify the triangular inequality and the triangle exists. (a, b, c with $c > a + b$) The triangle does not exist. The three values for this variable intervene in each part of the situation. The segment lengths suggested in the e-books and in the pen-and-paper activities allow the discovery of cases where the triangle does and does not exist. The flat triangle can be seen in the first e-book.

The second didactic variable concerns the possible dragging for the segments of the triangle's sides. Five values are associated to this variable: translation only;

rotation only; simultaneous translation and rotation; dissociated translation and rotation; no movement. The chosen values of this dragging variable at the different steps of the situation influence the implementation of strategies which, as we will later see, are the vehicles of learning.

The third didactic variable concerns the tools of geometry. We choose, in the design of the e-books, to provide the pupils with a toolbox available even when these tools are not useful to resolve the given tasks. In the first e-book all of the dynamic geometry tools are available even if none of them is necessary to resolve the given tasks. In the second e-book however these tools must be used to resolve the given tasks and the chosen values of the tool didactic variable influence the strategies that must be implemented. In the pen-and-paper environment all tools (pencil, ruler, setsquare and compasses) are always available. It is up to the pupil to choose which ones he must use to resolve the problem.

Our intention in designing a situation which makes use of a duo of artefacts, digital and material, in learning about the construction of the triangle with a ruler and the compasses is to characterise a milieu (Brousseau, 1997) that encourages the dimensional deconstruction of the triangle and the instrumental genesis of the compasses through this problem. We want the learning process to be included in the strategies implemented by the subject to resolve the set problems. In the next paragraph we detail the a priori analysis of the situation.

5.5 A Duo: Rotation Dragging and Material Compasses

5.5.1 Asymmetrically Dynamic Segments

The first duo mobilised in the situation consists of the material compasses and a digital artefact included in the first Cabri Elem e-book. The Cabri Elem software suggests the virtual compasses but both of the artefacts instrumented by the subject in this duo are the material compasses and the rotation-dragging of a point in a digital environment. In the e-book, the pupil is lead through the treatment of two tasks: forming triangles by the direct manipulation of segments with fixed and defined lengths, and determining whether three segments can be the three sides of a triangle (Fig. 5.2). The second task, on whether the triangle exists or not, is a mathematical question which problematizes the investigation of a triangle, thus resorting to drag the segments. The displayed segments on this page are represented as asymmetrical on screen and when they are moved.

Two types of movement are possible for a segment: translating the segment by holding the segment or its round end, and rotating the segment around its fixed round end by moving its cruciform end. The graphical distinction between the extremities, round or cruciform, lets the user anticipate the movement before doing it. The asymmetry of the digital segments relative to their movements is an essential part of the milieu constituted by this e-book. In fact, the digital

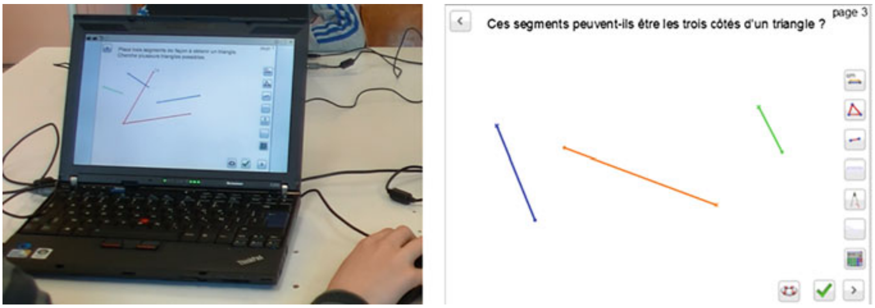


Fig. 5.2 Illustration of manipulations of digital segments in the e-book

environment places limits on the double dragging of the digital segments and forces a dissociation of the rotation and translation motions. When manipulating material objects, these two movements (rotation and translation) are realised simultaneously. Thus the digital environment highlights the rotation necessary to form a triangle from digital segments. Furthermore, the asymmetrical dragging of the segments induces an efficient winning strategy in building a triangle. The fact that both of the segment's ends do not rotate makes difficult an adjustment strategy. The easy adjustments to make are those by rotation, which lets the "broken line strategy" take shape. An efficient winning strategy for building a triangle from the asymmetrical segments given in the digital environment consists in forming, with three segments, a broken line whose extremities are cruciform. The triangle is then formed by rotating the two end-most segments of the broken line (Fig. 5.3). Thus the digital environment creates a milieu which highlights rotation, essential to form a triangle from digital segments, and leads to the implementation of a winning strategy that promotes learning. The broken line is one of the first steps in the dimensional deconstruction of the triangle. The activity in which we form a triangle in a digital environment with three segments by going through the broken line rests on a reconstruction of the two dimensional triangle starting from the one dimensional broken line. This strategy makes apparent that a triangle is a closed broken line.

In this first e-book, two dragging instruments emerge: translation-dragging and rotation-dragging. The rotation-dragging instrument is made to rotate the segments, particularly those segments at the ends of the broken line. It is essential to form a triangle in a digital environment. This instrument is what will spur the use of the compasses in the pen-and-paper environment. Firstly the segment, asymmetrical in its representation on the screen and in its movements, brings to mind the material

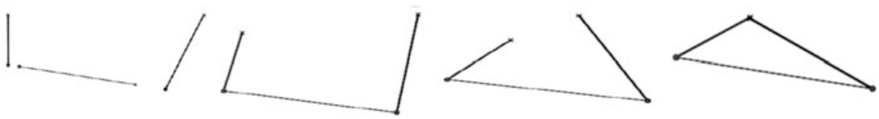


Fig. 5.3 The broken line strategy

Fig. 5.4 Juxtaposed strategies

compasses: one arm remains fixed while another arm turns. Secondly, in the digital environment, the rotation-dragging instrumentation produces utilization schemes which can be adapted to the use of the material compasses.

So as to permit the setup of anticipation strategies, the value of the dragging didactic variable is modified on one of the e-book's pages. Only the translation-dragging is possible on this page. Rotating the segments is no longer possible, and thus forming the triangle is no longer possible. Another strategy must be set up to predict whether the three segments can be the three sides of a triangle. Since the dynamic geometry compasses tool is available on the page, it can be used to set up a strategy. Strategies mobilising the dragging to translate the segments can also be elaborated. The two smaller segments can be juxtaposed over the longest one like a broken line, or in the style of triangular inequality (Fig. 5.4).

5.5.2 A New Function for the Compasses

The e-book is connected to the use of the material compasses in the pen-and-paper environment. This first pen-and-paper activity consists in constructing triangles whose sides are presented as segments drawn on the paper. The given segments are either in broken line formation or parallel to each other (Fig. 5.5). To complete these tasks, the pupil is given geometry tools: a pencil, a ruler, a setsquare and a pair of compasses.

The purpose of the first pen-and-paper activity is to mobilise the compasses in the construction of the triangle. Some of the milieu's elements, generated by the e-book, also appear in the pen-and-paper environment. The segments corresponding to the triangle's sides are already present on the paper, as they were in the digital environment. Furthermore, the broken line has completely integrated into the milieu. The broken line is part of the continuity of the digital and material duo of artefacts. This line whose extremities the pupils must rotate permits the switch from

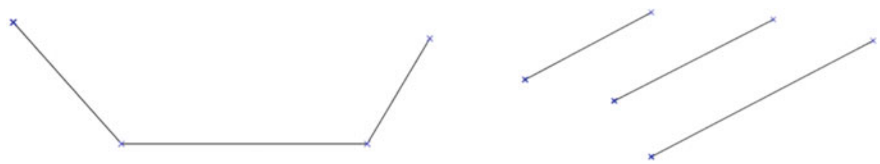


Fig. 5.5 First pen-and-paper activity

the rotation-dragging in the digital environment to the material compasses in the pen-and-paper environment. The material compasses are the artefact that replaces the rotation-dragging. The material compasses allow the subject to rotate a segment. The segment is stuck between the compasses' two arms. A new compasses instrument is created: the compasses to rotate a segment. In the first e-book, during the manipulation of digital segments, the pupil constructs schemes relative to the rotation-dragging to rotate the digital segment. These schemes can be associated to utilization schemes for the material compasses. A utilization schemes for rotating a digital segment can be described as: determine the two ends of the segment then grab the cruciform end and drag to rotate the end point. A utilization scheme for rotating a segment with the help of the compasses can be described as: distinguishing each arm of the pair of compasses, placing the needle on the steady end of the segment, spreading the arms and placing the pen lead on the end to be moved, and at least rotating the compasses whilst keeping the same spread and create a visible trace. Assimilations and adaptations between utilization schemes can be identified from one instrument to another. Whether in a digital or a pen-and-paper environment, in each utilization scheme for rotating a segment, these must be distinguished: the segment's extremities; the compasses arms. In each scheme the segment or the compasses must be rotated. When using the material compasses, there are some necessary adaptations: the compasses produce a visible line, marking the extremity of the rotating segment; since the original does not rotate, a stand-in for the rotated segment must be drawn.

5.6 The Material Compasses and the Dynamic Geometry Circle

5.6.1 The Dynamic Geometry Circle

The goal of the second e-book is to bring the circle as a tool used in the construction of the triangle. It is by playing with the allowed tools that the circle becomes the tool adapted to the situation (Fig. 5.6). At the beginning, the circle is used to verify if a broken line can be the outline of a triangle or not. The circle thus becomes the image of the trajectory of the extremity of a rotating segment.

This second e-book is connected to the use of the material compasses in the first pen-and-paper activity. Some elements of the milieu made in the pen-and-paper environment are kept in the second e-book so as to maintain continuity in the articulation of digital and material artefacts: the material compasses and the dynamic geometry circle. The broken line is still a key element of the milieu in the first pages of the e-book. It is still the element of continuity between the duo and the given situation. The milieu in the e-book is enriched by the dynamic geometry tools, more precisely, by the forbidden dynamic geometry tools.

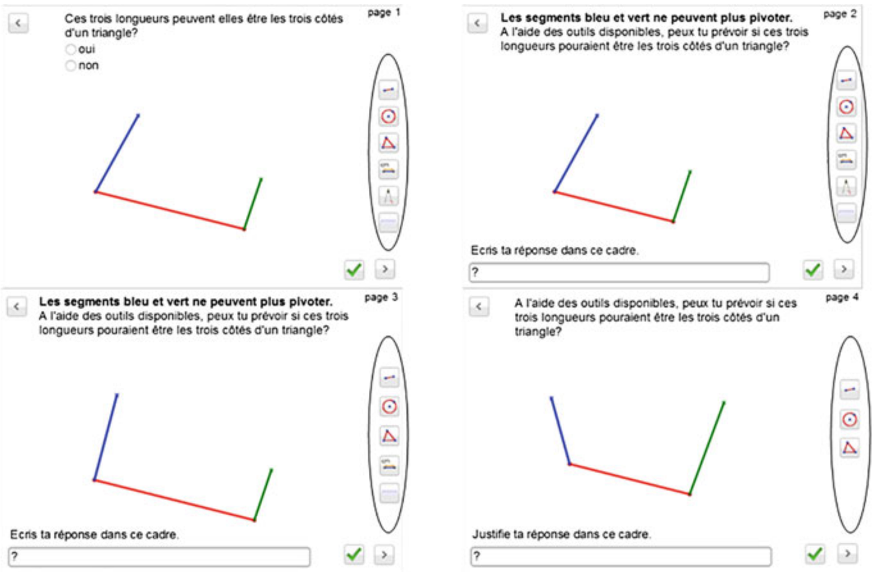


Fig. 5.6 Use of the dynamic geometry tools to involve the circle into the construction of a triangle

A tool initially used to verify then becomes a tool used to create. Afterwards, one must use the circle tool to determine the third vertex of the triangle (Fig. 5.7). The new instrument which is created in this e-book is the circle instrument used to identify a distance.

5.6.2 The Material Compasses to Construct a Triangle

The situation ends with a second pen-and-paper activity using the material compasses to construct a triangle. This second pen-and-paper activity consists, if possible, in constructing triangles of which the lengths of sides are given through numbers. This activity marks the end of the situation and allows the making of a summary of elements taught through the use of digital artefacts used with the material compasses.

5.7 Experiments and Results

Experiments of this situation were done over the course of three consecutive years in French primary school level CM2 classes (ten years' old pupils). These experiments were all done in the same elementary school. Every year, two classes and

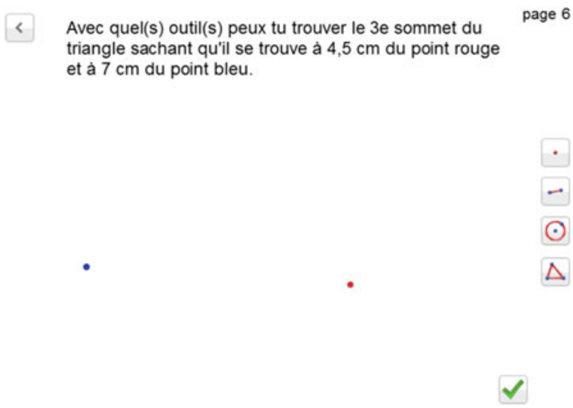


Fig. 5.7 The circle used to determine the third vertex of the triangle

their teachers participated in it. Over 130 pupils tested this situation (38 in 2014, 50 in 2015 and 44 in 2016) over three years. Footage recording the work of each pupil in the e-books were filmed. The pen-and-paper work of three out of four pupils was filmed on video cameras. The reactions of the pupils, as well as the remarks made by the teachers have allowed us to bring modifications to the situation each year. Additionally, these three years of experiments have allowed us to put our research hypotheses to the test.

5.7.1 A Rotation-Dragging Instrument Used to Rotate a Segment

The traces of the work of each pupil in the first e-book allow us to identify two levels of action in the e-book. Firstly, all the pupils interact with the given objects on the first page. We can observe interactions with digital segments in an attempt to move them and interactions with the given dynamic geometry tools. Right from the first page the pupils delve into the manipulation of segments. They grab and drag the segments. We can observe that pupils would like to rotate the segment by one of its ends: the rounded end is clicked upon and the cursor is moved in circular fashion. The two movements, by translation and by rotation around a given point are necessary for pupils. Secondly, strategies have been developed by the pupils to accomplish the given tasks. The double movement of segments is mastered when the pupil understands that only one action on the cruciform end allows the rotating of the segment. If the segment is grabbed by the rounded end or by a point of the segment then it is displaced. If the segment is grabbed by the cruciform end then the segment rotates around the other end which remains still. Finally, throughout the pages of the e-book, the double movement of digital segments is mastered by over 90% of the pupils. Over the course of the manipulation of asymmetrical digital segments in their movement, the pupils have created a utilization scheme to rotate a digital segment: distinguishing each end of the segment, then grabbing the cruciform end and dragging to rotate that end.

After every pupil completes the e-book, the teacher makes a synthesis of what has been done in the e-book with all the pupils. The videos show us that pupils caught three points: they have to form triangle; the rotation-dragging is essential; they can't always form a triangle with three given length.

5.7.2 *The Material Compasses Used to Rotate a Segment*

In the pen-and-paper environment, when a broken line into three segments is given to the pupils, more than 80% of them use a pair of material compasses (Fig. 5.8). These pupils say that the compasses are used here to replace the rotation-dragging used in the digital environment to rotate a segment. Several assimilation and adaptation processes of the utilization schemes used to rotate a digital segment are translated to a utilization scheme of material compasses to rotate a segment in the pen-and-paper environment. In the video we can see the pupils' organisation: distinguishing each arm of the pair of compasses, placing the needle on the steady end of the segment, spreading the arms and placing the pen lead on the end to be moved, and at least rotating the compasses whilst keeping the same spread and create a visible trace. The continuity between the rotation-dragging to move the end of the segment and the movement of the compasses which rotates upon it, give meaning to the compasses used to rotate segments of a broken line in the construction of the triangle. In the pen-and-paper environment, the initial segment does not rotate as it does in the digital environment. The compasses produce the trace of the rotating end—creating thus a circle, or a circular arc. Another trace representing the rotating segment must be drawn. 90% of the pupils who used the compasses to rotate a segment traced the new representations of the rotated segments (Fig. 5.8). These pupils thus drew the triangle obtained by closing the broken line.

To use the pair of compasses to rotate segments of the broken line also allows these pupils to recognize that a broken line may not allow the creation of a triangle (Fig. 5.9).

5.7.3 *The Circle: A Tool in the Construction of the Triangle*

In the second e-book, we observe in the videos that the presence of a broken line on the page leads the pupils to try to rotate its extremities. Even if it is explained that the broken line's segments can no longer rotate (Fig. 5.6), the pupils take one of the extremities and move the cursor in a circular motion. The segments' lack of movement forces them to set up strategies that mobilize the available dynamic geometry tools. When the compasses tool is available on the page, the pupils use it to rotate segments as with the material compasses in the first pen-and-paper activity.

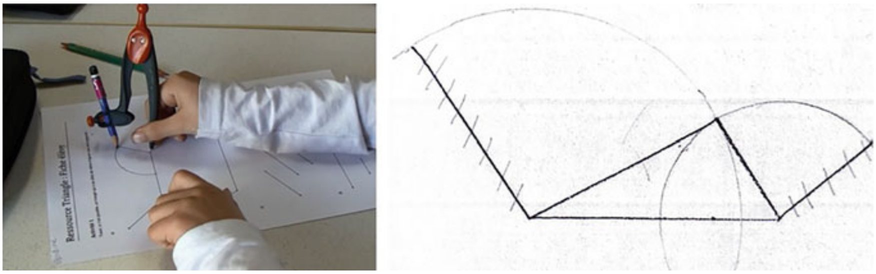
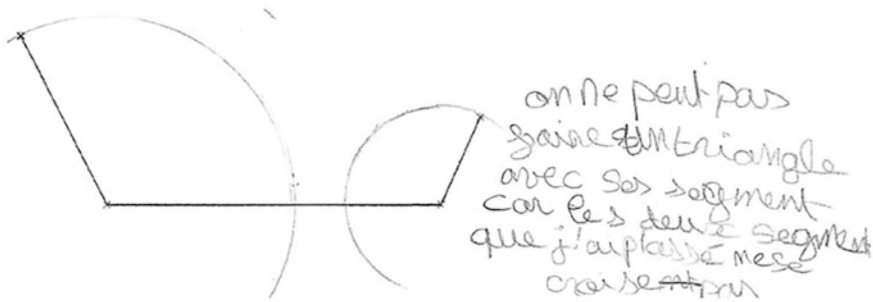


Fig. 5.8 The compasses used to rotate a segment



"One cannot make a triangle with these segments because the two segments that I've moved do not cross each other."

Fig. 5.9 Illustration of a broken line which may not allow the creation of a triangle

The circle tool is mobilized by 70% of pupils to replace the compasses tool. Only 60% of them (only 42% of all pupils) mobilize it correctly. The others activate the circle tool but cannot manage to use it to solve the problem. These pupils' difficulty lies in not knowing where to place the centre of the circle. Even if the utilization schemes for the compasses and dynamic geometry circle tool seem similar, as shown in Table 5.1, we can identify some obstacles in the utilization schemes for the circle tool. On the one hand, the centre of the circle is a 0 dimensional object hardly apprehended by primary school pupils as we already said before. On the other hand, the compasses tool allows a level of visual control that brings to mind the rotating digital segment (Fig. 5.10), which simplifies its use in the task.

So in this situation, the material compasses and the dynamic geometry circle are not a duo of artefacts. There is too wide a gap between the utilization schemes of each artefact to solve the task. Assimilation and adaptation processes of utilization schemes from one instrument to the other cannot be identified.

On the last two pages of the e-book the circle tool must be mobilized to identify a distance, the length of the triangle's side (Fig. 5.7). It is the only available tool to determine the desired distances. Only 15% of pupils manage to use the circle wisely. The perception of a circle as a set of points equidistant from a centre is not yet mastered by pupils at the end of elementary school.

Table 5.1 Utilization schemes for the dynamic geometry

Utilization schemes for the dynamic geometry compasses tool	Utilization schemes for the dynamic geometry circle tool
Click on the compasses tool Click to fix the needle Spread the arms Click to place the lead Pivot the lead-end to draw a trace	Click on the circle tool Click to define the circle's centre Spread the circle Click to define the circle's size and fix its image on the screen

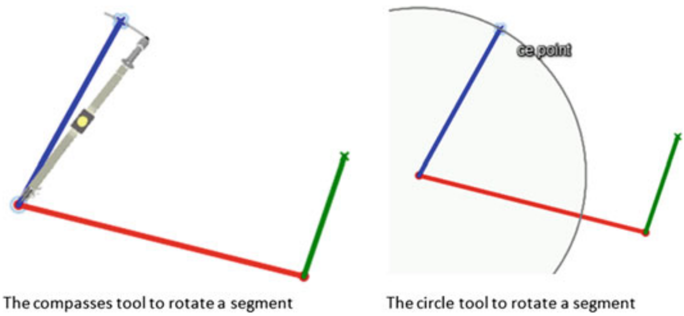


Fig. 5.10 Compasses tool versus circle tool to rotate a segment

5.7.4 A Triangle Is a Broken Line for More Than 75% of Pupils

The second pen-and-paper activity reveals what the pupils have learned in the situation. In the first e-book, the "broken line strategy" comes in as an efficient winning strategy to form a triangle in the digital environment (Fig. 5.3). The videos show us the implementation of a broken line strategy in the pen-and-paper environment. It is transposed into the pen-and-paper environment as a step in the triangle's construction (Fig. 5.8) by more than 75% of pupils. Pupils draw a broken line with the three segments, and then use the compasses to rotate the segments at both ends of the broken line, and finally they draw the triangle if it exists. Video capture allows us to point out certain pupils' remarks. One pupil says: "The broken line helps because before, we didn't know we had to use the compasses to construct a triangle."[3] Only 15% of pupils construct the triangles they are asked for with ruler and compasses without using the broken line. Some of them identify the two solutions for the third vertex. They draw the two triangles. The other 10% of pupils draw the triangle by using only the ruler by successive approximations.

[3]Translation of the original sentence «La ligne brisée ça nous aide parce qu'avant on savait pas qu'il fallait utiliser le compas pour tracer un triangle».

A triangle being a closed broken line is a new conception (Balacheff, 2013) of the triangle resulting from the design of the duo of digital and material artefacts. This conception allows a 1D understanding of the triangle: a triangle is a polygon with three sides. The broken line is a dimensional deconstruction of the 1D triangle, halfway between the 2D three-sided triangle and the triangle determined by its 0D vertices. Furthermore, this conception includes a control structure on the existence or not of the triangle. If the end segments of the broken line meet then the triangle exists. If the end segments of the broken line do not meet then the triangle does not exist.

The rotation-dragging and material compasses' duo for rotating the end segments of a broken line also create a new conception of the circle. A circle is the trajectory of the extremity of a rotating segment. Controls here do not only focus on the compasses' production. A new control is present: pivoting compasses corresponds to pivoting a segment between its arms. This conception highlights the notion of constant distance associated to the circle.

5.8 Conclusion

Our research project questions the added effect brought by digital technology in situations operating on the basis of linking together digital tools and material tools. With an example in geometry, we have illustrated the characteristic elements of a digital and material duo of artefacts, and its incidence on the learning process. The main idea behind a duo is that each artefact improves the other so that the duo encourages the pupils' construction of individual knowledge. Continuity and discontinuity in the two artefacts connection are essential and relay knowledge. In our example we have highlighted how the connection between manipulating in a digital environment and using the material compasses allows the pupils to elaborate a new instrument: the compasses to rotate a segment. The duo of artefacts in situation promotes the assimilation and adaptation of utilization schemes from one instrument to the other. Thus the situation mobilizing a duo of digital and material artefacts participates in the instrumental genesis of the compasses through the triangle-construction task and gives meaning to its usage in this task. Furthermore the duo brings about the elaboration of teaching strategies. The broken line strategy allows the setup of the triangle's dimensional deconstruction 2D/1D. Indeed the duo produces the circumstances that rely on the broken line to implement the triangle's dimensional deconstruction without necessarily going to the point a 0 dimensional object. This broken line also participates in the elaboration of a new conception of the triangle: a triangle is a closed broken line. Thus the digital artefact, or more precisely the duo of artefacts, brings additional value to the material tool which helps in breaking through certain difficulties or epistemological obstacles. The duo and the connection between digital and material artefacts in the situation give meaning to the material artefact's use in solving a task, enrich the system [subject/milieu] by relating problems and reiterating experiences while varying constraints, and are thus favourable to the elaboration of the subject's individual knowledge.

References

Artigue, M., & Robinet, J. (1982). Conceptions du cercle chez des enfants de l'école élémentaire. *Recherche En Didactique Des Mathématiques*, *3*(1).

Balacheff, N. (2013). In M. V. Martinez & A. Castro Superfine (Eds.), *cK¢, a model to reason on learners' conceptions* (pp. 2–15). Chicago IL, USA. Retrieved from https://hal.archives-ouvertes.fr/hal-00853856.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Springer.

Bruillard, E., & Vivet, M. (1994). Concevoir des EIAO pour des situations scolaires. *Recherches En Didactique Des Mathématiques*, *14*(1.2), 275–304.

Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie: développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements. *Annales de didactique et de sciences cognitives, 10,* 5–53.

Mackrell, K., Maschietto, M., & Soury-Lavergne, S. (2013). The interaction between task design and technology design in creating tasks with Cabri Elem. In C. Margolinas (Ed.) (pp. 81–90). Presented at the ICMI Study 22, Oxford, Royaume-Uni.

Maschietto, M., & Soury-Lavergne, S. (2013). Designing a duo of material and digital artifacts: The pascaline and Cabri Elem e-books in primary school mathematics. *ZDM—The International Journal on Mathematics Education*, *45*(7), 959–971. https://doi.org/10.1007/s11858-013-0533-3.

Moyer-Packenham, P. S., Baker, J., Westenskow, A., Anderson, K. L., Schumway, J., Rodzon, K., et al. (2013). A study comparing virtual manipulatives with other instructional treatments in third- and fourth-grade classrooms. *Journal of Education, 193*(2), 25–39.

Rabardel, P. (1995). *Les hommes & les technologies : Approche cognitive des instruments contemporains*. Paris, France: Armand Colin. [Heidi Wood, Trans.]. https://halshs.archives-ouvertes.fr/file/index/docid/1020705/filename/people_and_technology.pdf.

Soury-Lavergne, S., & Maschietto, M. (2015). Articulation of spatial and geometrical knowledge in problem solving with technology at primary school. *ZDM Mathematics Education, 47*(3), 435–449. https://doi.org/10.1007/s11858-015-0694-3.

Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments: Guiding student's command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*(3), 281–307.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52,* 83–94.

Chapter 6
Using Cluster Analysis to Enhance Student Learning When Using Geometry Mathematics Apps

Kevin Larkin and Todd Milford

Abstract Mathematical applications (apps) are becoming commonplace in educational settings. Despite their increasing use, limited quantitative research has been undertaken that might support teachers in making appropriate pedagogical decisions regarding their use, nor how teachers might go about selecting appropriate apps from the multitudes available at iTunes or Google Play. This chapter explores how cluster analysis can be used to identify homogeneity among elements within apps, thus assisting teachers to make decisions regarding which apps might be most appropriate. Based upon selection criteria and rankings generated via a number of scales, the cluster structure of 53 apps to support geometry learning in elementary mathematics classrooms is reported. The chapter concludes by exploring the homogeneity and heterogeneity of these clusters of apps and suggests how to use these apps to enhance student mathematical learning.

Keywords Apps · Cluster analysis · Digital manipulatives · Number
Primary mathematics

6.1 Introduction

The exploration detailed in this chapter is a derivation of a broader ongoing research project. In phase one of the project, Larkin (2013, 2015a) investigated the effectiveness of 142 largely number-oriented mathematics applications (apps). In phase two, a further 53 apps specifically targeted at Geometry, were evaluated using a variety of modified evaluative frameworks (Larkin, 2015b, 2016). Despite a range of worthwhile and meaningful outcomes resulting from those two phases, a number of questions remained unanswered: whether and how various Geometry apps may

K. Larkin (✉)
Griffith University, Southport, Australia
e-mail: k.larkin@griffith.edu.au

T. Milford
University of Victoria, Victoria, Canada

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_6

be used in combination with other apps for a specific teaching purpose; whether and how apps that may have scored poorly overall might still be useful for specific purposes in concert with other apps; and how the data gathered in phase two might be made more useful by additional analysis at a more granular level. This last question is explored here as we employ a targeted methodology to provide more information than just the allocation of an overall cumulative score. To this end we use cluster analysis more closely to examine 53 Geometry apps in order to provide a more fine-tuned selection of apps for subsequent use by teachers. We will argue in this chapter that cluster analysis is a highly versatile and useful methodology that can assist classroom teachers in their initial selection of Geometry apps, as well as providing additional information on pedagogical approaches to support teachers' classroom practices. Because of this versatility and usefulness, it is our intention to use cluster analysis in future research to evaluate more precisely the usefulness of mathematics apps.

6.2 Literature Review

Geometry is broadly defined in curricula documents as a branch of mathematics that deals with shape, size, position and the properties of space (Australian Curriculum and Reporting Authority [ACARA], 2009). It is a core content component of primary mathematics education and is linked to a number of other mathematics strands, as well as informing approaches to learning and teaching in curricula and policy documents. For example, the Australian Curriculum (ACARA) offers Geometry (including sub-strands such as Shape, Transformation and Location, and Geometric Reasoning) as one of the six core strands from Foundation (i.e. Kindergarten) until Year 10 (ACARA, 2009). These sub-strands cover concepts relating to 1D lines, 2D shapes, 3D objects, transformations, co-ordinate geometry, angles and symmetry. Similarly, the National Council of Teachers of Mathematics (NCTM) recognizes the importance of Geometry in establishing productive learning environments by "calling on students to analyse characteristics of geometric shapes and make mathematical arguments about the geometric relationship, as well as to use visualization, spatial reasoning, and geometric modelling to solve problems" (NCTM, ND, 3).

Research suggests that student knowledge and understanding of geometry is vital for a number of reasons. For example, it enables students to understand and interpret their environment; it relates to other mathematics concepts such as arrays or patterning; and it links to other discipline areas such as Science, Geography, Art, Design and Technology (Jones & Mooney, 2003). Consequently, Geometry is now viewed as a core component in promoting higher-level thinking skills in mathematics and beyond (Clements & Sarama, 2011). From this overview, it is therefore clear that curriculum authorities (e.g. ACARA in Australia and the National Research Council (ARC) in the Unites States), recognise the importance of Geometry in their respective national curriculum and policy documents (Moss, Hawes, Naqvi & Caswell, 2015), regarding it as a critical component of student success in school mathematics per se and also in related discipline areas, particularly those in the Science, Technology, Engineering or Mathematics (STEM) domains (Sinclair & Bruce, 2015).

6.2.1 Digital Technologies and Geometry

What is also overtly recognised in numerous international curricula, including ACARA and the NRC, is the important role of digital technologies in the teaching of mathematics. For example, the Australian Curriculum: Mathematics explicitly states that "Digital technologies allow new approaches to explaining and presenting mathematics, as well as assisting in connecting representations and thus deepening understanding" (ACARA, 2009, 12, para 7). As a consequence of this recognition, there has been interest in how best to incorporate digital technologies into teaching. It is therefore not surprising that a range of technologies (Interactive Whiteboards, Laptops, Visualisers, Tablets etc.) are becoming increasingly common in primary mathematics classrooms (Moyer-Packenham et al., 2015), with a more recent trend being the use of mathematical apps on devices such as iPads or other tablets (Ladel & Kortenkamp, 2016; Larkin, 2015a). It is generally accepted in the research literature that both concrete and digital manipulatives support mathematical learning. Therefore, as was the case in previous research (Larkin, 2015a, 2016), it is taken as given in this chapter that manipulatives (concrete and digital) support mathematical learning (e.g. Carbonneau, Marley, & Selig, 2013; Moyer-Packenham et al., 2015; Sarama & Clements, 2009). Although desktop computer virtual manipulatives have been widely researched, there is much less research as to the effectiveness of apps in supporting mathematics learning, despite the rapid expansion of their use in the educational domain in recent years.

We include iPad apps as part of a general class of mathematics objects known as virtual manipulatives (VM). According to Moyer-Packenham and Bolyard (2016), a VM is "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (p. 1). We acknowledge that not all mathematics apps meet this definition i.e. they may be static and merely convey knowledge via text based definitions (e.g. *Basic Geometry App*); however, apps such as this did not meet the initial criteria for further evaluation in this research. Calder (2015) and Larkin (2015a) both acknowledge this lack of current research and note that this has contributed to the largely ad hoc implementation of tablets (iPads and Androids) in many school contexts.

The trend to incorporate digital technologies into mathematics education has significant implications for all strands of mathematics but particularly for Geometry, with the extra requirement of accuracy of external representations (Larkin, 2016; Manches & O'Malley, 2012). Despite the increasing proliferation of iPads in the primary mathematics classroom, how best to select and use many of the more recent technologies—e.g. tablets—are still relatively unexplored (Moyer-Packenham et al., 2015), both conceptually and methodologically. Previous research (Larkin, 2015b) reported that, despite the limited quality of apps available for student learning in Geometry, there are some apps that are very useful for primary school classrooms. Our goal in this chapter is to determine whether some of the lower quality apps (as indicated by raw scores) contain useful content for specific teaching contexts. We use cluster analysis to do so.

6.2.2 *Cluster Analysis*

Cluster analysis is a collection of multivariate (i.e. analysis of more than one statistical outcome variable at a time) techniques that group individuals or objects into clusters so that objects in the same cluster are more similar to one another than they are to objects in other clusters (Hair, Black, Babin, Anderson & Tatham, 2006). Essentially, it seeks to maximize the homogeneity within clusters, while at the same time maximizing the heterogeneity between clusters. In essence, it tries to keep more 'like things' together while keeping 'unlike things' separate. Although similar to factor analysis in assessing structure, cluster analysis groups objects as opposed to grouping variables and is focused more on uncovering the common dimension suggested by natural groupings and proximity than on patterns of variation (correlation). Cluster analysis was selected as a methodological approach for the research outlined here as it was anticipated that the Geometry apps selected for this analysis would show similarities and differences that would allow researchers to identify groupings that could: (a) be linked to relevant curriculum documents (cf. ACARA, NRC) and to theories of geometric learning (cf. van Hiele), and (b) be transferable to teachers to assist their decision-making regarding the use of apps in their classrooms.

Cluster analysis can be found referenced in the literature as supporting a wide variety of educational investigations; however, as Shavelson (1979) indicated, "it is a little used but important technique for examining data in educational research" (p. 1). Some of its uses within educational research include students' learning behaviour during problem-solving activities in an on-line environment (Antonenko, Toy, & Niederhauser, 2012), and student responses to open-ended questions on Algebra (Di Paola, Battaglia, & Fazio, 2016). However, cluster analysis is hardly a silver bullet and is not without its detractors. For example, the method has been criticized for being primarily descriptive and a-theoretical, as clusters always form from data without necessarily indicating meaningful groupings. In addition, results are not generalizable in the parametric sense of the word (Hair et al., 2006).

We use cluster analysis in this research to answer the following research question: How does the cluster structure of apps, based upon three evaluation scales assist teachers make decisions about the quality of apps for use in their mathematics classrooms?

6.3 Study Design—Factor Analysis and Cluster Analysis

6.3.1 Target Population and Criteria for Inclusion

The target population in this study were elementary Geometry apps identified as being appropriate for school children aged 5–11. Evaluation of the apps (initially reported in Larkin, 2015b) began with a targeted search for mathematics apps at the iTunes Appstore. The following search terms were used: *Geometry Elementary Education; Geometry Junior Education; Geometry Primary Education; Symmetry Education* and *Transformations Education*. A variety of criteria were used to immediately critique the apps as worthy of further review: only one app in any series was reviewed; apps categorised as Games, Entertainment or Lifestyle were excluded; apps where mathematics was part of a larger package of reading, writing, and spelling skills were not reviewed; and apps that required access to websites for further (often costly) resources were also excluded. A total of 53 apps met the initial criteria established for this population and these were subsequently used for both the rating according to the previously mentioned scales (Bos, 2009; Dick, 2008; Haugland, 1999), as well as in the cluster analysis. The full list of apps, including a qualitative evaluation of their usefulness or otherwise, can be found at http://tinyurl.com/Geometry-Cluster-Analysis.

6.3.2 Materials and Procedures

The characteristics that formed the data set for later cluster creation were based upon previous work (Larkin, 2015b) which evaluated primary or elementary school Geometry apps, based upon the evaluation criteria of Bos (2009), Dick (2008) and Haugland (1999). For this study, the scales were further modified by the authors to be weighted evenly and targeted specifically at mathematics apps.

Bos' research suggested that the structural format of a virtual digital resource greatly influences the type of mathematical learning that it might support. As noted by Kortenkamp and Ladel (2013), "the actions carried out with a manipulative should support the mathematical design" (p. 188) of an app; therefore, the design format of an app is a critical consideration in determining its future potential to support learning. In brief, the modified Bos scale is based on six types of software: static tools, informationals, quizzes/tests, drill and practice games, virtual manipulatives (VM), and interactive maths objects (IMO). The modifications made by Larkin (2015b), and utilized here, included a more precise matching of evaluative criteria to iPad technology and scoring the apps quantitatively on a scale of 1–10 rather than Bos' descriptive analysis, which only indicated low, medium or high accuracy in terms of mathematics.

The Haugland Software Developmental Scale (Haugland, 1999)—henceforth referred to as the Haugland Scale—is a criterion-based tool used to evaluate the appropriateness of web-based applications and software for use by children. To measure the particular affordances of apps in terms of student use, Larkin (2013) modified the original Haugland Scale which was initially designed to evaluate computer software. In order to more accurately measure mathematics apps, elaborations were added to the initial ten criteria to investigate the quality of the apps in relation to mathematics education. The original scale of (1–10) provided generic information regarding the quality of software in relation to its use by primary students and was not particular to mathematics (e.g. presence of violence or gender stereotypes were indicators of appropriateness). The Haugland scale was further modified for iPad research by clustering the ten dimensions into three sub-dimensions (Child-Centred, Technical Design and Learning Design), and relating each dimension to an aspect of mathematics education. Each sub-dimension contributed to the overall score with child-centred scoring (0–4), technical design (0–3) and learning design (0–3).

The modified fidelity measures were based on the work of Dick (2008) who measured quality according to three types of fidelity: Pedagogical, Mathematical and Cognitive. In brief, Pedagogical fidelity is defined as the degree to which a student can use a tool to further their learning and refers to "the extent to which teachers (as well as students) believe that a tool allows students to act mathematically in ways that correspond to the nature of mathematical learning that underlies a teacher's practice" (Zbiek, Heid, Blume, & Dick, 2007, p. 1187). The second of the three fidelities used to evaluate the apps is Mathematical fidelity—defined as the "faithfulness of the tool in reflecting the mathematical properties, conventions, and behaviors (as would be understood or expected by the mathematical community)" (Zbiek et al., 2007, p. 1173). The final element is Cognitive fidelity—"the faithfulness of the tool in reflecting the learner's thought processes or strategic choices while engaged in mathematical activity" (Zbiek et al., 2007, p. 1173). These notions of fidelity are obviously very important in Geometry apps which are likely to require the use of external representations. The virtual nature of many app objects does allow for high degrees of fidelity; for example, 3D objects can be pulled apart and put back together, and in so doing, can reinforce the link between 3D objects and their 2D representations (i.e. nets). The methodological contribution of the authors was to use each of the three fidelities and score the apps on a (1–10) scoring continuum. Thus, the combination of the scales resulted in five measures overall (Haugland, Bos, and the three separate fidelities) generating a potential range of scores from 5 to 50.

The internal reliability of the evaluation of the apps across each of the Haugland and fidelity scales is presented in Table 6.1 (the Bos scale only had one rating and thus internal reliability could not be calculated). The internal reliability for the three sub-dimensions of the Haugland Scale was calculated at $\alpha = .661$. Although the Haugland Scale's alpha score is slightly less than 0.7, previous research (Larkin, 2015a) using the Haugland Scale reported an alpha score of 0.768. As alpha is sensitive to both number of participants as well as number of items (Tavakol & Dennick, 2011), the decrease noted here is unsurprising. For subsequent analysis, we used a total summed score of all three sub-dimensions (Child-Centred, Technical Design and Learning Design) for the Haugland scale score and treated

Table 6.1 Scales and reliabilities

Scale	Subscale	Reliability
Format (Bos)	–	–
Haugland	Child centred, technical design and learning design	$\alpha = .661$
Fidelity (Dick)	Pedagogical, mathematical and cognitive	$\alpha = .889$

the three fidelity subscales of Pedagogical, Mathematical and Cognitive as separate entities. The internal reliability for these three subscales was calculated at $\alpha = .889$. For subsequent analysis we looked at each of the three scales of the fidelity scale as independent. This information is further detailed in Table 6.1.

Based upon the results of the ratings of each app, as generated by the three scales (i.e. Format, Haugland and Fidelity), a cluster analysis was performed on these 53 apps with SPSS v.22. We initially measured similarities as the squared Euclidian distances between each pair of apps on each of the 5 scale characteristics [i.e. Bos, Haugland and three fidelities (pedagogical, mathematical, and cognitive)]. In this way, smaller distances were viewed as indicating greater similarity. Once the similarity measures were calculated, a hierarchical procedure via Ward's method—which joins cases into clusters such that the variance is minimized—was applied to the clusters. Lastly, the number of clusters was determined, based upon the output, with the objective of generating the simplest structure possible while still representing homogeneous groupings. The number of clusters was determined by both the output and also a decision by the researchers to generate the simplest structure possible while still representing homogeneous groupings. The Dendrogram in Fig. 6.1 provides a graphical portrayal of the clustering process. The vertical axis represents respective apps and the horizontal axis represents the distance used in joining clusters. This horizontal axis is scaled so that closer distances between combinations indicate greater homogeneity (Hair et al., 2006). Based upon the cluster analysis a solution of five clusters was determined. The blue vertical line in Fig. 6.1 represents the final decision.

6.4 Findings

Initial descriptives for the scales used to run the cluster analysis are presented in Table 6.2. All variables are measured on the same 10-point (1 lowest to 10 highest) scales and all meet the assumptions of univariate normality.

A correlational analysis (see Table 6.3) was subsequently run on the five scales to determine if their inclusion in the cluster analysis would be appropriate, or if any overlap (i.e. multicolinearity) might account for double counting. Multicolinearity acts as a weighting process not apparent to the observer but affecting the analysis (Hair et al., 2006).

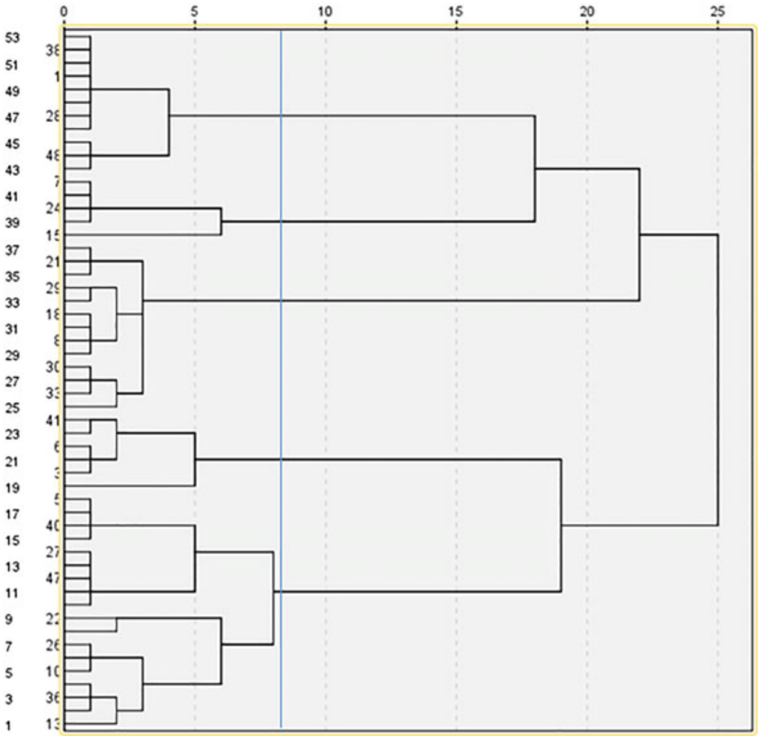


Fig. 6.1 Dendrogram—with blue vertical line indicating point of cluster formation

Table 6.2 Descriptives

Variable	N	Mean (SE)	Median	Skew (SE)	Kurtosis (SE)
Format (Bos)	53	5.69 (.28)	6.00	.33 (.33)	−1.16 (.64)
Haugland	53	5.39 (.28)	5.50	−.12 (.33)	−1.13 (.64)
Fidelity					
Pedagogical	53	4.94 (.31)	4.00	.36 (.33)	−1.12 (.64)
Mathematics	53	4.30 (.26)	4.00	.31 (.33)	−.31 (.64)
Cognitive	53	3.71 (.29)	3.00	.93 (.33)	.28 (.64)

Table 6.3 Correlations of the 5 scales

	Format	Haugland	Pedagogical	Mathematics	Cognitive
Format (Bos)	1.00				
Haugland	.620**	1.00			
Pedagogical	.556**	.878**	1.00		
Mathematics	.622**	.569**	.625**	1.00	
Cognitive	.701**	.745**	.741**	.840**	1.00

**Correlation is significant at the 0.01 level (2-tailed)

Table 6.4 Rotated component matrix

	Component				
	1	2	3	4	5
Format (Bos)	.281	.293	**.902**	.146	.015
Haugland	**.860**	.227	.309	.195	.274
Pedagogical	**.889**	.324	.214	.143	−.198
Mathematics	.284	**.904**	.283	.151	.010
Cognitive	**.448**	**.581**	.355	**.579**	.021

Based upon this analysis, it was determined that the scales were all highly correlated (i.e., between .556 and .878) and a Principal Component Analysis (PCA) was run using Varimax rotation to create orthogonal variables from these scores for the subsequent cluster analysis. The rotated component matrix is provided in Table 6.4.

From the results of the PCA and the rotated component matrix (see Table 6.4), a four-component solution was accepted for use in the subsequent cluster analysis. The four component solution (i.e. components 1 thru 4) was justified as it accounted for over 98% of the variance and none of the scales loaded highly on the 5th component even after the Varimax rotation.

As indicated, a cluster analysis was performed with SPSS using the four component scores of the PCA. Cluster analysis involves three steps: (1) a measure of similarities as the squared Euclidian distances between each pair of apps on each of the four component scores; (2) a hierarchical procedure via Ward's method—which joins cases into clusters such that the variance is minimized; and (3) the number of clusters determined based upon the output and an effort to get the simplest structure possible while still representing homogeneous groupings. The results of this analysis are provided in Figs. 6.1 (Dendrogram) and 6.2 (Scree Diagram) and, based upon the data represented in Figs. 6.1 and 6.2, a solution of five clusters (i.e., 53 − 48 = 5) was determined. Details of the 5 retained clusters and some of their associated apps are provided in Table 6.5.

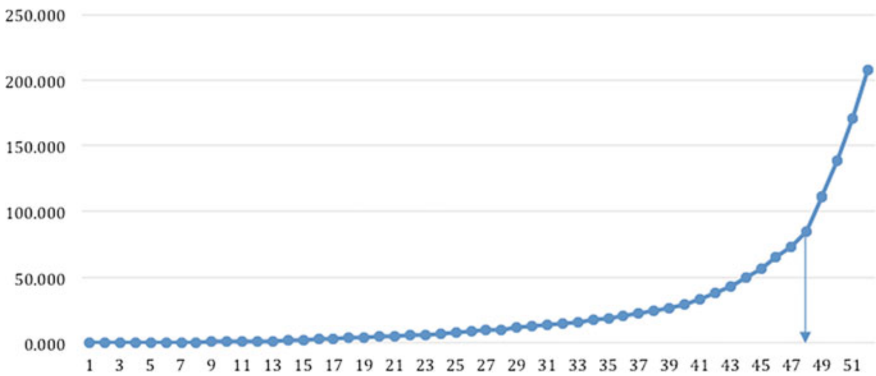


Fig. 6.2 The scree plot

Table 6.5 Cluster labels and example apps

Cluster number	Number of apps	Mean score (SD)	Cluster name	Example apps
1	11	16.7 (5.0)	Visualisation	Geometrie; JustShapes; Koala Math 1-5 Geometry
2	6	39.3 (3.3)	Active learning	Coordinate Geometry (Ventura); Attribute Blocks; Transformations (Investigate)
3	13	17.0 (4.4)	Quizzes	Math Geometry; MathApp - Geometry 1; Geometry Test
4	18	29.3 (5.0)	High fidelity manipulatives	Geometry (Montessori); Pattern Shapes; Isometry Manipulative;
5	5	21.2 (4.6)	Low fidelity manipulatives	Geometry 2D Pad; Hands-on Maths Geoboard;

Because we used an agglomerative method to determine clusters (i.e. each app started out as its own cluster), the Dendrogram detailed in Fig. 6.1 is read from left to right. Starting on the left with each of the 53 apps as its own cluster, using Ward's method of similarity, apps are combined one step at a time, based upon which two are the most similar, and formed into a new cluster. The vertical lines between when clusters are formed are a measure of homogeneity. The longer the vertical line the more dissimilar the clusters are that are merged. Based upon this distance measure, the blue vertical line was placed on the Dendrogram indicating a five cluster solution.

As an additional criterion for the selection of number of clusters, the Scree Plot detailed in Fig. 6.2 is appropriate as it offers more detailed information on the level of dissimilarity of the apps within the cluster. When the "bends" in the scree plot are the most exaggerated, identification of the number of clusters is the most accurate. In the case here, and additionally by examining the Dendrogram in Fig. 6.1, the most exaggerated bend was after five clusters. Thus, based upon both graphical depictions, a five cluster solution was accepted.

6.5 Discussion

The primary purpose of using cluster analysis was to identify homogeneity between apps that may not have been apparent in the earlier work that primarily focused on the cumulative scores. Hence, there was a grounded theory type approach to the research. Once the clusters had been created via the cluster analysis process, the authors began to look at each of the clusters to determine the particular features of each app within each cluster that resulted in their being classed as homogenous. Although not apparent in the earlier research, an examination of the clusters revealed that different clusters of apps appeared to match particular levels of the van Hiele model of geometric thinking (Crowley, 1987). Prior to discussing how and why this is the case, it is useful to quickly provide a refresher on the model which is widely accepted within the mathematics community (see Pegg, 1985; Teppo, 1991) and which forms the basis of the Geometry learning sequence in many mathematics curricula around the world (ACARA, 2015; NGA, 2016). The van Hiele model is generally considered to describe characteristics of student thinking in Geometry and consists of five levels of understanding—Visualisation, Analysis, Informal deduction, Formal deduction, and Rigor (Crowley, 1987). In the presence of appropriate instruction "the learner moves sequentially from the initial, or basic, level (visualisation), where space is simply observed-the properties of figures are not explicitly recognized, through the sequence listed above to the highest level (rigor), which is concerned with formal abstract aspects of deduction" (Crowley, 1987, p. 1). Other authors discern as many as eight levels (Jones, 1998) including, in the view of Clements and Battista (1992), a pre-recognition level evident before the visualisation level. This is not a significant concern as most children are at the visualisation level by the time they commence formal schooling. Additionally, none of the reviewed apps focussed on pre-recognition.

Not all researchers favour the van Hiele model and indicate that it has been subject to some critical discussion (Jones, 1998). For example, the discreteness of the levels and the precise nature of levels 1 and 4 have been queried (Pegg, 1985). A second line of research, questions more broadly, any axiomatic approaches (e.g. van Hiele or Piaget), arguing instead that the generation of meaningful geometrical justifications by students is a more realistic approach (Battista & Clements, 2013). In brief, these approaches favour students working collaboratively to make conjectures and resolving shape/object conflicts by presenting arguments. The teacher's role here is to "involve students in the crucial elements of mathematical discovery and discourse conjecturing" (Battista & Clements, 2013, p. 5). Battista and Clements (2013) also note that "regardless of whether an axiomatic or a justification methodology is favoured—establishing the validity of geometric ideas, and making sense of them mathematically should be the major goal of the Geometry curriculum" (p. 6). Whilst it is apparent that some level of disagreement remains regarding how students develop geometrical understanding, for the purposes of this chapter, the initial van Hiele levels of Recognition, Analysis and Informal Deduction, considered most relevant to the geometric thinking of students aged 5–11, are used. There is little argument that teacher intervention is instrumental in assisting students to move towards deeper levels of geometric thinking (Crowley, 1987; Pegg, 1985). This is confirmed by Teppo (1991) who cites research suggesting that "appropriate instruction can be used to move students successfully from a lower to a higher level of geometric thinking" (p. 214).

A second, largely uncontested, area is the critical role of language in the development of geometric thinking. Each of the initial three van Hiele levels is characterised by a vocabulary that is used to represent the concepts, structures, and networks within that level of geometric understanding. Students at a lower level of thinking are unlikely to understand language presented to them at a higher level of thought. The critical role of language has profound implications for use of mathematical and symbolic language in the Geometry apps. It is the case that some apps, targeted at Foundation or Year One (ages 5–6 years old), are using mathematical or symbolic language likely beyond the level of understanding of young students. Therefore, a key consideration in evaluating the usefulness of the apps relates to linguistics and the potential mismatch between app language and student language. Therefore, in our initial thinking Cluster 5 apps (Low fidelity manipulatives) are more appropriately used in terms of language at the Analysis level, and Cluster 2 (Active Learning) are more useful at the Informal deduction level. Cluster 4 apps (High fidelity manipulatives) are considered appropriate as a transitional device between van Hiele levels 2 and 3. Thus Cluster 4 can operate as a scaffold between Cluster 5 and Cluster 2, supporting language development from Analysis to Informal Deduction. With this emergent learning framework in mind, the process of classifying and later labelling the clusters took shape, based on both the quantitative (Bos, Haugland, Dick) and qualitative data from phase two of the broader project (Larkin, 2015a, b). We analysed how the apps clustered as they did, and in doing so, made pedagogical decisions regarding their usefulness to teachers in terms of a recommended learning framework and in terms of Geometry content. The pattern for the analysis of each cluster is a discussion on the three quantitative measures and then a synthesis of the qualitative data. A full list of the apps in each cluster is available at http://tinyurl.com/Geometry-Cluster-Analysis.

6.5.1 Cluster 1—Visualisation

According to the scores on the summed measures, this was the worst performing cluster, as the average for this cluster was 16.77, significantly lower than the overall average for all apps which was 24.06. Only one app in this cluster, *Symmetry Draw*, scored a pass mark. The cluster is labelled as Visualisation (van Hiele level one) because the apps primarily involved recognition of whole shapes. Despite being adequate in terms of being child-centred, this cluster scored very poorly on the Haugland sub-strand of Maths learning with most apps scoring 0/3. Apps in this cluster therefore do not support deep learning. They may be used in revision activities for testing declarative type knowledge about shapes, objects or basic symmetry. Apps located in this cluster are intuitive with low Mathematical fidelity and low cognitive load. Students at level 1—Visualisation—could use the apps in this cluster. As indicated, the majority of apps in this cluster relate to whole of shape activities. Recommendations for apps to use from this cluster are either *My Geometrie Universe* or *Koala Maths* as they both contain real-world depictions of shapes. A number of symmetry apps are included in this cluster primarily because they relate to early recognition of symmetry in the natural and built environment involving bilateral symmetry. In this cluster there were no opportunities for conjectures or relationships; however, at this stage, this is a positive aspect as students at the Visualisation level are unlikely to be capable of forming conjectures or identifying relationships and will likely become confused if asked to do so.

6.5.2 Cluster 2—Active Learning

This is the best performing of the five clusters: it contains the top three apps overall. All six apps in this cluster are in the top 10 apps overall, and all are evaluated as either VM or IMO. In contrast to the virtual manipulatives in Cluster 5, which are more static, and those in Cluster 4, which are dynamic without necessarily promoting active learning, manipulatives in Cluster 2 are those that promote active learning via engagement. These apps were all exceptional in the Haugland scale across all three sub-strands as well as exceptional across all three fidelities. These apps are therefore considered to be the best of the reviewed apps. The majority of the apps involved transformations and are best used at the later stages of the van Hiele model as they provide linkage, and in some cases capacity, for students to engage in Informal Deductions. These apps, as was the case with Cluster 1, are still highly intuitive. The key difference is that they go beyond ease of use to emphasise a number of mathematical and cognitive elements: namely, active participation; logical transitions from one concept in the app to other; and opportunities for developing patterns and forming conjectures. However, a third of the apps in this cluster are very high in Mathematical fidelity and Cognitive fidelity, but low in Pedagogical fidelity, indicating that students will need significant support from classroom teachers to be able to meaningfully engage with the app. Although this cluster consists primarily of transformations, some 2D Shape apps and 3D Objects apps are included here as the shapes and objects within the apps can be manipulated using rotations and enlargements; thus, the shape work in this app is non-static as opposed to static representations in Cluster 1.

6.5.3 Cluster 3—Quizzes

This was generally a very weak cluster, with the best app, *Math Geometry* only scoring 25/50 overall. The majority of the apps were quizzes and all apps in the cluster are below average on every quantitative measure. This cluster is particularly poor on Maths learning in the Haugland Scale as nine of the 13 apps scored zero in the learning dimension of the scale. Key observations regarding this cluster include the fact that apps are static, contain only standard orientations and prototypes, and lack patterning and connection to real-world mathematics. In effect they are digital worksheets, which are often multiple choice. Ten of the thirteen apps within this cluster primarily related to geometric reasoning, either solely, or in combination with content from another sub-strand such as Shape or Transformations. There is no opportunity to use the apps in this cluster as a transition from level 2 Analysis to level 3 Informal deduction, as patterns and conjecture-forming opportunities are missing. This is despite geometric reasoning being one of the indicators of development from analysis to informal deductions. In addition, they are not appropriate for level 1 Visualisation as mathematics language is utilised.

6.5.4 Cluster 4—High Fidelity Manipulatives

There were a broad range of app formats in this particular cluster, ranging from informational apps through to virtual manipulatives. This is the largest cluster (18 apps), consisting mainly of drill and practice style apps. Manipulative apps in the cluster are closer to Cluster 2 rather than Cluster 5 manipulatives. This means that they are dynamic in nature, demonstrating high fidelity, rather than being static, demonstrating low fidelity. A second difference is that the manipulatives in this cluster are primarily related to shapes and geometrical reasoning rather than transformations. The apps in this cluster were more child-centred, and demonstrated effective design features, but were significantly lower in terms of Maths learning than Cluster 2 apps. These apps all scored above average; however, when compared to Cluster 2 they were similar in Pedagogical fidelity but much lower in Mathematical and Cognitive fidelity scores. This reflects the fact that transformation apps, featuring prominently in Cluster 2, are more likely to support conjecture-forming. The majority of the apps in this cluster were combination apps; combining either shapes and geometric reasoning, or shapes and transformations. The positive of many of the apps in this cluster was that they were accurate mathematically; however, a negative feature was their limited connection to the real world and limited conjecture-forming opportunities which is a clear point of distinction from Cluster 2 apps. Apps in Cluster 4 could be used at level 2 Analysis as the Mathematical fidelity is high and students will be learning accurately about the attributes of shapes, objects and some transformations. However, despite the fact that mathematical accuracy was high, they did not make the transformative step that occurs in Cluster 2 apps where this participation can lead to conjecture forming.

6.5.5 Cluster 5—Low Fidelity Manipulatives

This is a small cluster containing only five apps. Despite the fact that most are either VM or IMO, they generally scored poorly across the three Haugland sub-dimensions and only one app scored in the top half of the 53 apps. It is a useful reminder that interactivity is a necessary, but not sufficient, condition for an app to be considered useful. This cluster can be characterised as manipulatives that are lacking purpose and fidelity. Thus Cluster 5 is similar to Cluster 2, which also contained a small number of manipulative apps. The key difference between the two clusters is that active learning is not encouraged in this cluster. A related difference is that this cluster did not afford opportunities for conjecture-forming or relationship-building. In contrast to Cluster 2, where shape apps included transformations, in this cluster the manipulation of shapes largely involves 2-dimensional concepts or mirror line prototypical flips. Overall in this cluster, the shapes and objects are largely prototypes and presented in vertical or horizontal orientations.

6.6 Limitations

There are a number of limitations to be noted in relation to this novel approach to evaluating apps:

Firstly, and as previously mentioned, cluster analysis has three main limitations (a) that clusters are primarily descriptive and atheoretical; (b) that clusters can form, even if no coherent structure underpins the clusters; and (c) that solutions are not generalizable as they are dependent upon the variables used to differentiate the clusters initially. Our efforts to address these limitations are based upon the careful and transparent efforts we have made to ensure that we had a strong conceptual and theoretical basis for the study that preceded our analysis. To this end, we linked our scales to previous research concerning three types of fidelity; the content area linked to major strands in mathematics curriculum and policy; and we used a well-researched theoretical foundation in the van Hiele model. Our intention is to continue to extend this work, based on the strong conceptual and theoretical position established in this chapter.

Secondly, the cluster analysis is based on scores generated by Larkin (2015a, b) earlier evaluation of the 53 apps. Although a very robust methodology (including qualitative and quantitative aspects), it would be beneficial in future work with cluster analysis if inter-rater reliability on the initial scores could be utilised. Likewise, although the five clusters of apps were generated independently after the cluster analysis process, the synthesis of the clusters was again performed solely by the lead author. When we use the cluster analysis process in future research, a team of mathematics educators will independently examine the clusters generated to identify the themes that emerge from the data. Finally, as noted in Larkin (2013), due to the large and exponentially growing number of "educational" apps, it is very difficult to find, and therefore review, all potentially relevant apps. The problem of the sheer number of apps is compounded by a further limitation of the poorly structured iTunes Appstore user interface. Search results are presented graphically as icons and are not sorted in alphabetical order. They are often labelled inaccurately and are in a state of flux as new apps are added, renamed, upgraded, or deleted. It is therefore possible, and indeed likely, that there are existing, high quality apps available for teachers to use that the lead author did not find and, therefore, did not evaluate.

6.7 Conclusion

In this chapter, we have outlined the use of a multivariate data analysis tool (i.e. cluster analysis) to group Geometry apps grounded in the characteristics they possess. The results presented here suggest that there are five distinct clusters of Geometry apps: Visualization, Active learning, Quizzes, High fidelity manipulatives, and Low fidelity manipulatives. There appears to be a stark division between those apps identified in clusters as Active Learning and High fidelity manipulatives at one end and Visualisations and Quizzes at the other in terms of mean scores. Low fidelity manipulatives appears to occupy the middle ground. This is unsurprising as those apps associated with the lower end of the mean score values are unidimensional in the competencies they seek to develop or assess, while those in the higher end are multidimensional, inquiry-driven, and promote mathematical investigations.

Through these distinctions, the clusters from this study offer teachers further information in terms of suggested classroom usage than that which is provided using the raw scores of the rating scales alone. Thus, cluster analysis "value adds" by providing a much finer-grained analysis of the apps than would be possible to obtain using descriptive analysis alone. Shavelson (1979) has demonstrated the wide application of cluster analysis within educational research by detailing its use both in determining teaching styles for effective pedagogy as well as its use in exploring the nature of mathematics content. In this later example, this author demonstrated how an operational system such as addition could be clustered to correspond very closely with our knowledge of the subject. Similarly, the data generated by our cluster analysis is useful for teachers in identifying either individual apps within a cluster or entire clusters of apps that will be appropriate for meeting the educational requirements of their students.

We argue that the clusters formed from the cluster analysis provide two types of useful information for teachers. Firstly, apps that score highly on the rankings are captured within the clusters so that they can be assured of high quality and pedagogically useful tools for their students. Secondly, the cluster categories offer a more nuanced opportunity for the teacher to align their choice of classroom apps to the curriculum standards for which they are accountable. For example, initial results from the rating scales for individual apps are certainly captured within some of the clusters (e.g. Active learning contains a number of the highest rated apps); however, and we think more usefully, clusters such as Visualisation or Quizzes contain useful apps that can be quickly aligned with curriculum and/or pedagogical needs but which might not otherwise have been selected as they were not individually high-scoring apps on the scale rankings. Despite being very familiar with the apps, the creation of the five clusters, via the cluster analysis process, forced the researchers to look again at why certain apps were homogenous. As a result of the cluster-analysis-guided re-analysis of the Geometry apps, it became clear that the clusters can be utilised by teachers for specific geometry teaching, broadly aligned with the initial three levels of the van Hiele model. This will have educational benefits for students using the apps and also minimises some of the potential negatives of some of the apps e.g. inappropriate mathematics language for younger students or lack of non-prototypical shapes when forming informal deductions. In addition, if teachers use the evaluation system to evaluate new apps, and begin to see patterns in the scoring that correspond to the scoring patterns in the five clusters, this will be a key indication of when and how such new apps should be used by students.

With the proliferation of apps available for selection, and the increased time-pressures on teachers (Larkin, 2015a), robust research tools are required to assist teachers to easily determine the quality of apps. This chapter has indicated, in brief, the usefulness of cluster analysis for researchers investigating the quality of apps. This process was certainly beneficial and we will use cluster analysis again to re-critique the 142 apps initially evaluated in Phase 1 with the intended outcome of determining clusters of apps to promote student learning in other areas of mathematics such as number, measurement and proportional reasoning.

References

Antonenko, P. D., Toy, S., & Niederhauser, D. S. (2012). Using cluster analysis for data mining in educational technology research. *Educational Technology Research and Development, 60*(3), 383–398.

Australian Curriculum and Reporting Authority (ACARA). (2009). *Shape of the Australian Curriculum Mathematics.* Retrieved from http://www.acara.edu.au/_resources/Australian_Curriculum_-_Maths.pdf.

Australian Curriculum Assessment and Reporting Agency (ACARA). (2015). Australian Curriculum: Mathematics. Retrieved from https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/.

Battisa, M., & Clements, D. (2013). *Investigations in number, data and space.* Retrieved from http://investigations.terc.edu/library/bookpapers/geometryand_proof.cfm.

Bos, B. (2009). Technology with cognitive and mathematical fidelity: What it means for the Math classroom. *Computers in the Schools, 26*(2), 107–114.

Calder, N. (2015). Apps: Appropriate, applicable, and appealing? In T. Lowrie & R. Jorgensen (Zevenbergen) (Eds.), *Digital games and mathematics learning. Potential, promises and pitfalls* (Vol. 4, pp. 233–250). Springer Science+Business Media.

Carbonneau, K. J., Marley, S. C., & Selig, J. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology, 105*(2), 380–400.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: Macmillan.

Clements, D. H., & Sarama, J. (2011). Early childhood teacher education: The case of geometry. *Journal of Mathematics Teacher Education, 14,* 133–148.

Crowley, M. (1987). The van Hiele model of the development of geometric thought. In M. M. Lindquist (Ed.), *Learning and teaching geometry, K–12, 1987 Yearbook of the National Council of Teachers of Mathematics* (pp. 1–16). Reston, Va: National Council of Teachers of Mathematics.

Di Paola, B., Battaglia, O. R., & Fazio, C. (2016). Non-hierarchical clustering as a methods to analyse an open-ended questionnaire on algebraic thinking. *South African Journal of Education, 36*(1), 1–14. https://doi.org/10.15700/saje.v36n1a1142.

Dick, T. P. (2008). Fidelity in technological tools for Mathematics education. In G. W. Blume & M. K. Reid (Eds.), *Research on technology and the teaching and learning of Mathematics* (Vol. 2, pp. 333–339). Cases and perspectives Charlotte, N.C.: Information Age Publishing.

Hair, J. F., Black, W. C., Babin, B. J., Anderson, R. E., & Tatham, R. L. (2006). *Multivariate data analysis* (6th ed.). Upper Saddle River, NJ: Pearson.

Haugland, S. (1999). Computers and young children: The newest software that meets the developmental needs of young children. *Early Childhood Education Journal, 26*(4), 245–254.

Jones, K. (1998). Theoretical frameworks for the learning of geometrical reasoning. *Proceedings of the British Society for Research into Learning Mathematics. Geometry Working Group, 18*(1 and 2), 29–34.

Jones, K., & Mooney, C. (2003). Making space for geometry in primary mathematics. In L. Thompson (Ed.), *Enhancing primary mathematics teaching* (pp. 3–15). London: Open University Press.

Kortenkamp, U., & Ladel, S. (2013). Designing a technology based learning environment for place value using artifact-centric activity theory. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 188–192). Kiel, Germany: PME.

Ladel, S., & Kortenkamp, U. (2016). Artifact-centric activity theory—A framework for the analysis of the design and use of virtual manipulatives. In P. Moyer-Packenham (Ed.), *International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives*. Springer Press.

Larkin, K. (2013). Mathematics education. Is there an App for that? In *Proceedings of the 36th Annual Conference of the Mathematics Education Research Group of Australasia*, Melbourne, July 7–11, 2013. Paper presented at the Mathematics Education: Yesterday, Today and Tomorrow.

Larkin, K. (2015a). An App! An App! My Kingdom for An App: An 18-month quest to determine whether apps support mathematical knowledge building. In T. Lowrie & R. Jorgensen (Eds.), *Digital games and mathematics learning* (Vol. 4, pp. 251–276). Netherlands: Springer.

Larkin, K. (2015b). The search for fidelity in geometry apps: An exercise in futility? In *Proceedings of the 38th Annual Conference of the Mathematics Education Research Group of Australasia*, Sunshine Coast, June 30–July 2, 2015. Paper presented at the Mathematics Education in the Margins.

Larkin, K. (2016). Geometry and iPads in primary schools: Does their usefulness extend beyond tracing an Oblong? In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives, mathematics education in the digital era, 7*. Netherlands: Springer.

Manches, A., & O'Malley, C. (2012). Tangibles for learning: A representational analysis of physical manipulation. *Personal and Ubiquitous Computing, 16*(4), 405–419.

Moss, J., Hawes, Z., Naqvi, S., & Caswell, B. (2015). Adapting Japanese lesson study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: A case study. *ZDM Mathematics Education, 47,* 377–390.

Moyer-Packenham, P., & Bolyard, J. (2016). Revisiting the definition of a virtual manipulative. In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives, mathematics education in the digital era*.

Moyer-Packenham, P. S., Shumway, J. F., Bullock, E., Tucker, S. I., Anderson-Pence, K. L., Westenskow, A., … Jordan, K. (2015). Young children's learning performance and efficiency when using virtual manipulative mathematics iPad apps. *Journal of Computers in Mathematics and Science Teaching, 34*(1), 41–69.

National Governors Association (NGA) Center for Best Practices and Council of Chief State School Officers. (2016). *Common core state standards.* Retrieved from http://www.corestandards.org/Math/.

Pegg, J. (1985). How children learn geometry: The van Hiele theory. *Australian Mathematics Teacher, 41*(2), 5–8.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Rutledge.

Shavelson, R. J. (1979). Applications of cluster analysis in educational research: Looking for a needle in a haystack. *British Educational Research Journal, 5*(1), 45–53.

Sinclair, N., & Bruce, C. D. (2015). New opportunities in geometry education at the primary school. *ZDM Mathematics Education, 47,* 319–329.

Tavakol, M., & Dennick, R. (2011). Making sense of Cronbach's alpha. *International Journal of Medical Education, 2,* 53–55.

Teppo, A. (1991). van Hiele levels of geometric thought revisited. *The Mathematics Teacher, 84* (3), 210–221.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169–1207). Charlotte, N.C.: National Council of Teachers of Mathematics.

Chapter 7
How Children Using Counting Strategies Represent Quantities on the Virtual and Physical 'Twenty Frame'

Daniel Walter

Abstract This chapter presents a study that investigated the students' usage of a physical 'twenty frame' and the 'twenty frame' tablet-application. Nineteen students with special learning needs were interviewed, with a focus on those who predominantly solve addition problems through counting strategies. The aim of the research project was to investigate if, and how, students make use of digital media's potential to internalize non-counting strategies. Analysis clarifies that quite a few students make use of these potentials after a short introduction. It was revealed that both the virtual and the physical 'twenty frame' can be detrimental and beneficial. Accordingly, no statements could be made as to which of the tools assisted counting students most in their individual usage preferences. Mathematical-didactical advantages could be identified in specific processing procedures for both materials.

Keywords Representing quantities · Twenty frame · Tablet-app
Special learning needs · Primary mathematics education

7.1 Introduction

With the development of Tablet computers, discussions - some of them highly controversial - have increasingly arisen on questions regarding the expedient application of digital media in lessons across all school levels. This applies in particular to primary schools, for which no common ground can be found either on the question of "whether" nor the "how" in science and practice. The exceptionally wide spectrum of opinions on the use of digital media can amongst other things be reasoned by the fact that, in spite of the unequivocal topicality of the topic, only a few research projects are available which focus in particular on student interactions with digital media and Tablet Apps and thus offer empirical findings on the clari-

D. Walter (✉)
Technische Universität Dortmund, Dortmund, Germany
e-mail: dwalter@math.tu-dortmund.de

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_7

fication of the question (see e.g. Moyer-Packenham et al., 2015). This observation applies in particular for the subject of mathematics. Too little is known on how students with different performance capabilities use software developed for the learning of mathematics. The objective of the research project presented below is to record the methods of use by children of examples of software for Tablet computers in order to contribute towards closure of this research gap.

This chapter is structured as follows: In Sect. 7.2, the theoretical background to the research work is presented. Here the focus is on both current findings on counting arithmetics in mathematics lessons and on indications for the predominant usage of counting strategies. Subsequently, an analysis is conducted on whether the current range of software available appears suitable for the support of children in overcoming counting strategies. Section 7.3.1 contains a description of the research questions and the design of the empirical study. The resulting empirical findings are described in detail in Sect. 7.4. In the closing remarks (Sect. 7.5), the central results are summarised and discussed.

7.2 Theoretical Background

In this section, the theoretical background to the empirical investigation is presented. Initially, research findings on the occurrence of counting strategies in Mathematics lessons are presented in Sect. 7.2.1, after which central difficulties are described which frequently trouble students when learning non-counting strategies. Building on this, Sect. 7.2.2 concerns itself with the question of whether the current range of software offered can, from a theoretical perspective, contribute towards overcoming these difficulties.

7.2.1 Counting Strategies in Simple Arithmetic: Research Findings and Indications

Using predominantly counting strategies to solve simple arithmetic problems can indicate ‘mathematics learning disability’ (e.g. Baroody, 2006; Wartha & Schulz, 2013) and by the end of grade one, students *should* either know addition and subtraction facts till twenty directly or compute them via derived fact strategies (e.g. Gerster, 2009). However, several research projects show that there are many students who do not reach this early goal of mathematics education. For many children, counting remains the main solution strategy when dealing with basic addition tasks over the course of the first school year (e.g. Doschko, 2011; Gaidoschik, 2010; Gray, 1991). The investigation by Benz (2005) shows that children also carry their counting solution strategies into the second school year, and, at least at the beginning of the second school year, use this as the main solution strategy when working through addition and subtraction tasks within a number range up to 100. In addition, the main application of counting strategies has been empirically verified for older children (e.g. Fresemann, 2014; Ostad, 1998) and even at adult age (e.g. Wartha & Schulz, 2013).

Considering these empirical results, there should be more emphasis placed on fostering students' individual pathways to overcome the main usage of counting strategies. Many children do not automatically turn away from preferred counting strategies used in their lessons. They require targeted support methods, which are orientated on the central difficulties for counting children. With regard to this, three empirically-reasoned indications could be identified which counteract the detachment process for children using counting solution strategies:

- *Difficulties in the structured representation and determining of quantities* (e.g. Lorenz, 2011; Lüken, 2012): During the determination of quantities represented through a structured hands-on material (e.g. a 'twenty frame'), it can frequently be observed that many children persist in the slow process of counting up individual elements. Many children also find it hard to structure unstructured quantities so that the number of objects available can be recorded "at a glance". Without these capabilities for structured presentation and recording, the procurement of non-counting strategies may be complicated or even prevented.
- *Sequential understanding of numbers* (e.g. Fuson, 1992; Gaidoschik, 2011): The acquisition of non-counting strategies not only requires the internalisation of sequential but also of cardinal conceptions of numbers. If cardinal number conceptions are not grasped, numbers can only be thought of sequentially when dealing with basic additive tasks. In such cases, the counting strategy remains the only logical consequence.
- *Difficulties when switching between the representative levels* (see e.g. Ladel, 2009; Radatz, 1990): The capacity for translating flexibly between different presentations of a mathematical object is considered of decisive importance for the learning of mathematics in general and also for the acquisition of non-counting solution strategies. At the same time, this capability does not just represent an essential capability for the understanding-based use of derived facts strategies. This is also an obstacle, which many children are unable to overcome.

Based on these three indications, the question arises in the context of this research project as to whether the current range of software available appears suitable for adequate support of students in the overcoming of such difficulties. The following section looks at this question in more detail.

7.2.2 Learning with Digital Media in Primary School Mathematics Lessons

Especially in Germany, research concerning the implementation of digital media has indicated, that so-called 'educational software' is a crucial component of early media use in primary mathematics education (e.g. Institute for Demoscopy Allensbach, 2014). However the observation that the majority of mathematics educational software is mainly based on ‘drill-and-practice’-methods has sufficiently been perceived and criticized as well (e.g. Goodwin & Highfield, 2012; Krauthausen, 2012; Larkin, 2015). There are only a few ICT-based learning arrangements, which can support the development of (and not simply practice on) mathematical conceptual knowledge (e.g. Sinclair & Baccaglini-Frank, 2016; Urff, 2014). Therefore the majority of software currently in existence does *not* appear suitable for overcoming learning difficulties—and therefore for the counting strategy, especially because drill-and-practice software primarily permits the automation of understood knowledge and basic understanding is the prerequisite for adequate use. If children with learning difficulties were to use drill-and-practice software, they would hardly be assisted in overcoming their learning obstacles—on the contrary: their incorrect conceptions would be reinforced as a consequence of the repetitive application of disadvantageous procedures and the subsequent demotivating error feedback.

The comparatively small range of software offers which basically appear suitable for the support of the acquisition of conceptual mathematical knowledge may also explain the scarce amount of research in this field. Above all - but not only - with regard to the research of student interactions with touchscreen Apps, Moyer-Packenham et al. (2015, p. 62), have established that this mathematical-didactical branch of research is “still in its infancy” (see Goodwin & Highfield, 2013; Padberg & Benz, 2011). Accordingly, only individual findings are available which refer, based on research, to the possibilities and opportunities for the use of software developed to build up conceptual knowledge, and to possible learning obstacles when dealing with the same (see Ladel & Kortenkamp, 2014; Sinclair & Heyd-Metzuyanim, 2014).

In view of the described research situation, it is necessary to research the methods of software use by children during the initial phases of a learning process. For the selection of suitable software, it appears to be important not just to use digital translations of physical media already existing. Rather more, the software should take into account the main mathematical-didactical potentials of digital media. The implementation of central potentials is illustrated in Sect. 7.3.2 based on the example of the virtual ‘twenty frame’, which is used in the study described in this article.

7.3 Research questions and the design of the investigation

After the presentation of the theoretical background in the previous section, the next section acts as the link to the empirical results. First the research questions are presented (Sect. 7.3.1). Then the presentation of the functional method and the potentials of the virtual ‘twenty frame’ (Sect. 7.3.2) are conducted before information is given on the investigation process (Sect. 7.3.3) and the sample gets described (Sect. 7.3.4). The Section closes with remarks on the assessment of the collected data (Sect. 7.3.5).

7.3.1 Research Questions

Section 7.2 described how relatively little is known on how children use software—in particular Tablet Apps which are not only intended for the automation of already-understood mathematical concepts, but which also appear suitable for the acquisition of mathematical concepts as yet not understood. This results in the central objective of this research work, which consists of finding out more on how children use this type of software. Over and above this, however, the intention is to work out how the methods of use by children during the operation of a virtual hands-on material differ from those used on a physical hands-on material. The virtual 'twenty frame' (Urff, 2012) and the traditional physical 'twenty frame' are intended for use as examples for research based on the processing procedure for quantities. Appropriately, the following two research questions represent the starting point for the empirical investigation.

- How do students with learning difficulties represent quantities on the virtual 'twenty frame'?
- Which differences result in the methods of use by children during the presentation process of quantities on the virtual 'twenty frame' in comparison to the physical 'twenty frame'?

During research on the methods of use by children, it is of interest whether the use of each 'twenty frames' version supports children in overcoming learning difficulties or rather appears to inhibit them. In addition, investigations are made on to what extent the situations in the 'twenty frames' support the individual user preferences of children during the representation of quantities.

7.3.2 Potentials of the Tablet App 'twenty frame'

The virtual 'twenty frame' (see Fig. 7.1)[1] was developed in order to support the development of mathematical understanding in children *with and without* learning difficulties, and in particular to support them in overcoming counting strategies (see Urff, 2012). This can be reasoned through the fact that the software offers three main features that can be valued as potential to overcome difficulties on the way to

[1]To distinguish the counters in this paper the originally reds are labelled with a R (red) and originally blue counters are labelled with a B (blue).

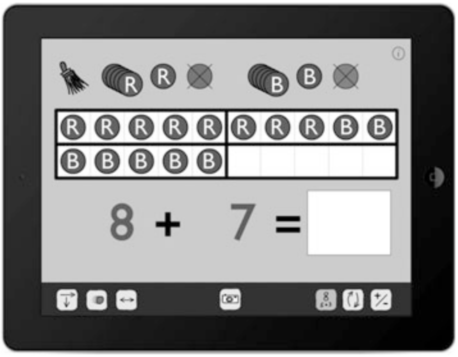


Fig. 7.1 Tablet application, virtual 'twenty frame' (developed by Urff, 2012)

internalize non-counting strategies (Sect. 7.2.1). These *mathematical-didactical* potentials (see Walter 2017) are represented below.

Potential: Add five counters simultaneously As shown in Fig. 7.2, five counters can be represented simultaneously by touching the 'stack of fives'. In this way, the presentation process can be supported by not (only) representing numbers and tasks sequentially using individual counters (e.g. Ladel & Kortenkamp, 2009; Urff 2012). In contrast to the physical 'twenty frame', on the other hand, individual counters can be deleted from five simultaneously placed counters.

The simultaneous representation of five counters provides us with a potential to understand numbers as a compilation of other numbers by taking the sophisticated method of laying out numbers as the subject. For example, the number eight can either be represented as an additive figure using 5 + 1 + 1 + 1 or a subtractive figure using 5 + 5 – 1 − 1 counters. Both methods of representation require the same number of touch inputs, whereby they represent more economical alternatives to the sequential addition of eight individual counters. The comparison of these methods of representation offers us the opportunity to prevent a primarily sequential sense of numbers.

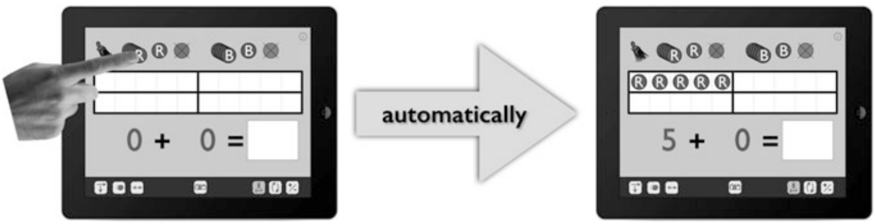


Fig. 7.2 Simultaneous presentation of five counters on the virtual 'twenty frame'

Fig. 7.3 MELRs on the virtual 'twenty frame'

Potential: Multiple External and Linked Representations A further potential of the virtual 'twenty frame' results from the design principle of *multiple external linked representations (MELRs,* see Fig. 7.3*)*. Due to the fact that different representations of a mathematical object are represented and changes to a representative have a direct effect on the other representations, an opportunity arises for understanding the relationship between different representations. Appropriately, this design principle has the potential to support children in overcoming difficulties when switching between different representations (e.g. Ainsworth, 1999; Goldin & Kaput, 1996; Goodwin & Highfield, 2013; Ladel, 2009; Paek, Hoffman, Saravanos, Black, & Kinzer, 2011).

On the virtual 'twenty frame', the MELRs are implemented as follows: a presented task (here: 8 + 6) can be changed virtually-enactively, starting from the nonverbal-symbolic level (here 6 is changed by '+1'). Linked to this, both the appropriate number and the iconic representation change. Alternatively, however, the iconic representation can be changed in order to achieve the same result. To do this, operation of a touch function is required to add a blue counter. Then the appropriate number changes.

Potential: Support in Structuring Different 'supports in structuring' are to be named as the third potential of the virtual 'twenty frame'. For example, represented counters are consistently positioned precisely on the 'twenty frame'. In addition, the counter-representation can be changed flexibly. At the touch of a button, the placement methods 'side by side' and 'line by line' can be changed. Furthermore,

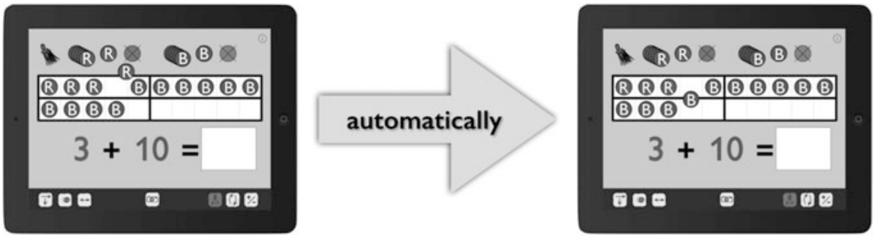


Fig. 7.4 Support in structuring on virtual 'twenty frame'

the represented counter image is always automatically ordered. No iconic representation of the virtual 'twenty frame' can be conducted in an unstructured manner. Therefore, for example, Fig. 7.4 shows how a red counter is deleted, which would produce a gap in the iconic representation. The software prevents this by moving a blue counter from the second row into this position. Originally it was showing four red and ten blue counters.

By *always* presenting the counter image in a structured manner, students can be supported in basically determining quantities simultaneously (e.g. Clements, 1999). The obstacle of first structuring them independently is removed. Children can dedicate themselves directly to the mathematical structures of the 'twenty frame'.

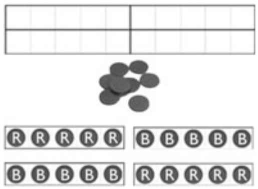
7.3.3 Process of the Investigation and Interview Tasks

In this section, information is given on the investigation process and the interview tasks posited. Table 7.1 first provides an approximate overview over the content of the qualitative interview series, in which both the virtual and the physical 'twenty frame' were used.

As shown in Table 7.1, both the physical and the virtual 'twenty frame' were used. In terms of content, the fields of tasks were orientated on the competence area centrally important for overcoming counting strategies: Calculating with number relations.

After the introduction to work with the physical 'twenty frame' had taken place in Interview 1, the children were requested first to calculate the addition tasks *without* using the hands-on material and to check the results by using the 'twenty frames' to ensure that they were correct. Then they were requested to generate further ways of placement for the tasks until they were of the opinion that they could not generate any more new representations. The same procedure was conducted in Interview 2 based on tasks with the same structure, when working with the virtual 'twenty frame', to be able to determine any differences. In addition, reflection questions are asked, on the basis of which the use of the 'twenty frames' in general and the potentials implemented there in

Table 7.1 Interview series process

(n=19; August/September 2014) Calculating with number relations on the (virtual) 'twenty frame'	
Interview 1	**Interview 2**
	
• Physical 'twenty frame' - Introduction	• Virtual 'twenty frame' - Introduction
• One task (7+6 / 3+8) – many strategies	• One task (8+7 / 4+9) – many strategies

particular were critically observed (e.g. "Which of the 'twenty frames' do you prefer when representing quantities?" or "Can you imagine why it would be good to add five counters at once?"). The interview sessions were conducted on subsequent days in order to limit the possibility of the children having increased their knowledge through lessons.

With the physical 'twenty frame', the children were set the tasks 7 + 6 and 3 + 8 in session 1; on the virtual 'twenty frame', on the other hand, they were set the tasks 8 + 7 and 4 + 9. Precisely those tasks were set which, from the point of view of practised mathematicians, offer the use of derived fact strategies. In this way, the results of the tasks 7 + 6 and 8 + 7 could be derived for example from 7 + 7 using the *'Tie strategy'*. Tasks 3 + 8 and 4 + 9 can also be solved through non-counting methods, for example by *'Bridging through* ten'. Within the scope of this research work, the identical structure of the tasks 7 + 6 and 8 + 7 or 3 + 8 and 4 + 9 is assumed based on the consensual change in the addends and the '*bridging through ten'* occurring in all the tasks. Overall, the use of these tasks of identical structure permitted research to be conducted on the differences in the methods of use by a child when using a virtual medium and its physical counterpart.

The individual interviews conducted within the scope of this study by the author in a separate room of the school were videoed from two perspectives. Whereas one camera provided a frontal view of the child and the interviewer, the second camera recorded the test persons from the side and was focussed on the respective 'twenty frame'. This permitted both the mimicry and gestures of the children to be recorded, as well as their actions.

7.3.4 Information on the Sample

The *initial* selection of the pupils was conducted on the basis of the estimation by the respective specialist Mathematics teacher as to whether the children in the learning group could be classified as 'children with learning difficulties' and whether they primarily used counting strategies when processing basic additive tasks. Then the teacher assessment was inspected within the scope of the qualitative interview session using selected diagnosis tasks (e.g. Peter-Koop, Wollring, Spindeler, & Grüßing, 2007; Wartha & Schulz, 2013). The teacher assessment was considered confirmed if procedural methods could be detected in a child during at least half of the diagnosis tasks relevant for the investigation which appeared detrimental for the overcoming of counting strategies.

Accordingly, children were investigated in this study who were *unable* to achieve the central target of the initial arithmetics lessons of mainly solving the basic additive tasks within a number range up to 20 using derived facts strategies and who were running the risk of remaining stuck in the counting solution strategies they preferred to use.

Whereas, according to the assessments by the participating teachers, 21 children could be described as 'children with learning difficulties', only 19 of these children were estimated as primarily using counting strategies after participating in the qualitative interview. All 19 children took part in the two described qualitative interview sessions. None of the children investigated knew the virtual 'twenty frame' prior to the interview series, whereby the fluency of the functions and the use of the application could be trained exclusively through the introductory sessions.

7.3.5 Data Assessment

The assessment of the data material obtained from the qualitative interview was primarily undertaken based on the transcripts produced, which reproduce both the manner of speaking and the actions of the students. In case uncertainties arise, however—for example in case of ambiguous transcript points—reference was always made to the video recordings of the interviews in order to interpret the respective situation appropriately.

The assessment methods used were qualitative content analysis (Mayring 2015) and comparative analysis (Glaser & Strauss, 2005). Based on the data material, categories were developed for the different methods of use for students in the application of the two 'twenty frames' represented. Accordingly, inductive category formation was undertaken which developed into a structured analysis of the contents as the assessment process continued.

7.4 Results: Representation of Quantities on the 'Twenty Frames'

In this section, the results of the investigation are presented on the methods of use by the children during the representation of quantities on the 'twenty frame'.[2] The focus on representations during the presentation process of quantities is first explained in more detail (Sect. 7.4.1). Then the use of the simultaneous representation of five counters is explained (Sect. 7.4.2), before a closer look is taken at the generation of different iconic representations of the tasks represented previously on the 'twenty frame' (Sect. 7.4.3). Here the focus is initially placed on the methods of use by each child during the work with the virtual 'twenty-frame', whereby in each case the question of whether these procedural methods could also be observed during work with the physical 'twenty frame' is addressed.

7.4.1 Focussing on Representations During the Presentation of Quantities

After the schoolchildren had first processed the additional tasks selected in this investigation (7 + 6 and 3 + 8 on the physical 'twenty frame' or 8 + 7 and 4 + 9 on the virtual 'twenty frame') without using any hands-on material, they were then asked to present the task in the way in which they had calculated it. During the representation of quantities with the 'Twenty frame' three different approaches were detected, which are characterised through different focuses on synchronously presented representations. These three methods of use are described Fig 7.5.

Focussing on Nonverbal-Symbolic Representation If the user of the virtual 'twenty frame' presses one of the buttons symbolised by different counters on the top edge of the screen, this causes individual or several counters to be added. Here the counters 'float' slowly to the respective predetermined position. With regard to the presentation levels, it is important that the nonverbal-symbolic presentation does not change until each counter has taken up position in the 'twenty frame'. However, whilst the counters move across the screen, the number symbols remain unchanged even though more counters are de facto presented than the nonverbal-symbolic representation states (see Fig. 7.6 left).

Resulting from this software characteristic, the initial method of use for presentation of quantities on the virtual 'twenty frame' can be identified, and is described as *focussing on the nonverbal-symbolic representation*. The orientation to

[2]The investigation results shown in this section are taken from Walter (2017).

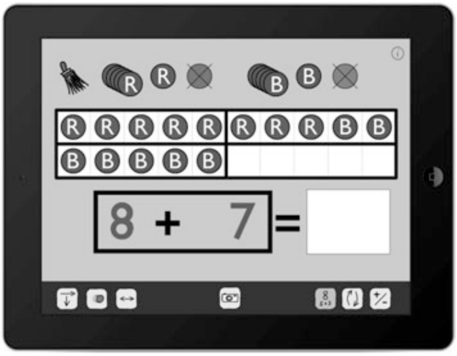


Fig. 7.5 Focussing on the numerals

Addition of counters with regard to the nonverbal-symbolic representation	When the number ‚7' appears on the screen, Lars positions his palms on the table.
	

Fig. 7.6 Focussing of nonverbal-symbolic representation based on Lars as an example

the given symbolic representation can be detected by analysing a student's body tension and attention regarding the tablet's display, as shown below in the example featuring the student Lars.

As Fig. 7.6 shows, Lars keeps his middle finger extended in the direction of the counter selection for addition of counters until the number '7' appears at the bottom edge of the screen. Only then does he lay his hands on the table, thus signalling that he has completed the representation of the task. The suspicion that Lars focusses on the number symbols during the presentation is confirmed in a subsequent dialogue on the procedure he used.

1	I	Tell me, what did you concentrate on when you placed the counters? On the numbers or on the counters?
2	L	On the numbers
3	I	Did you look when an eight and a seven appeared there *(points to number symbols)?*
4	L	Mmh *(positive)*

However, it is not possible to conclude, from the observation that a child focusses on the nonverbal-symbolic representation level, whether this method of use appears to impede or support the overcoming of counting strategies. It occurred both in children who exclusively added counters in sequences, and in those students who also made use of the function by which five counters could be presented at a time, which may be a contribution towards the comprehension of number as a compilation of other numbers. Accordingly, the accompanying mental processes and the related decisions on how a number can be split appear more important than merely focussing on the nonverbal-symbolic representation.

Focussing on the Selection of Counters As a second method of use, *focussing on the selection of counters* could be determined (see the area marked in Fig. 7.7). The procedure by the student David illustrates important properties inherent in this method of use. The transcript excerpt starts at the point after which he had solved the task 8 + 7 without the use of the virtual 'twenty frame' using a counting strategy.

1	I	Would you like to place that the way you calculated it?
2	D	(*places eight individual red counters and seven individual blue counters*)
3	I	What did you just concentrate on when you placed the counters? What did you look at?
4	D	(*points to the upper section of the screen*) Those here
5	I	Not the numbers?
6	D	Hmhm *(negative)*
7	I	Just the pictures?
8	D	Mmh *(positive)*
9	I	And when did you know how to stop with the counters?
10	D	Because I counted every time I pressed

David adds all 15 counters using sequential representation by pressing eight times, one after the other, on the button to add the red counters, and then seven times accordingly to represent the blue counters (line 2). Then he states that he counted each counter as he did so, without focussing on the iconic representation of the counters on the 'twenty frame' or on the number symbols (lines 4–10).

Fig. 7.7 Focussing on the selection of counters

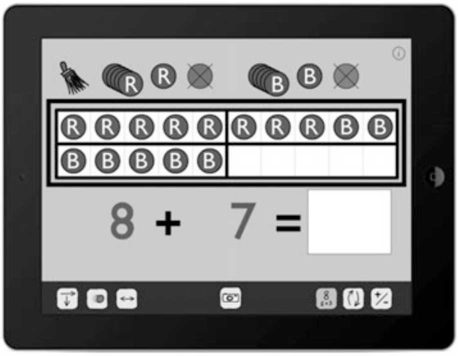


Fig. 7.8 Focussing on the iconic representation

Children who focussed exclusively during the representation of tasks on the selection of counters exclusively added individual counters and did *not* take advantage of the possibility to represent five counters simultaneously. Accordingly, this method of use would appear hardly suitable for overcoming counting solution strategies, as no utilisation is made of the structuring elements offered.

Focussing on the Iconic Representation When *focussing on the iconic representation,* learners only refer during the course of task representations to the resulting counter image located in the central area of the screen (see marked area in Fig. 7.8). Here learners can either use a pure counting strategy and/or make use of the structures. A mixed form is clarified below using the example of the student Valerie. The transcript excerpt begins after she has correctly solved the task 8 + 7, counting on from the initially stated addends without the use of the ‘twenty frame’.

1	I	So now place the task the way you calculated it.
2	V	Fifteen then now?
3	I	Place eight plus seven so that eight plus seven is shown down here (*pointing to nonverbal-symbolic representation*.
4	V	Five (*places five red counters at once*), six (*places one red counter*), seven (*places one red counter*), eight (*states the numbers before they are presented nonverbal-symbolically by the software, and looks at the centre of the screen*). Plus seven.
5	I	Exactly.
6	V	(*places five blue counters and counts the counters one after the other individually on the 'twenty frame'*). Six, seven (*quietly*) (*and places two individual counters*). First I counted the eight and then the seven. 
7	I	What did you concentrate on when you placed the counters? On the numbers (*points to the nonverbal-symbolic representation*) or the counters (*points to the 'twenty frame'*)?
8	V	On the place of the counters
9	I	You didn't look at the numbers at all?
10	V	(*shakes her head*) You can count that yourself.
11	I	Mmh, you can count that yourself. Can you see easily that they are fifteen counters, without having to look at that underneath (*points at the covered-up result*), if you #
12	V	#but I just did that
13	I	If you only look at the counters, can you see that easily #
14	V	#fifteen
15	I	Yes. Can you see that easily somehow?
16	V	Mmh, yes look, here are ten# (*points at the first lines of the 'twenty frame'*)
17	I	#Yes
18	V	And there are five (*points to the second line and then presses the two blue counters in the top half so that these counters turn red and the task 10 + 5 appears*).

Although the interviewer inadvertently orientates Valerie towards the nonverbal-symbolic representation (Line 3), she does *not* refer to this representation during her presentation of the task. This interpretation approach can above all by supported by the fact that she states the numbers she is looking for *before* they appear on the screen (Line 4) and also strictly negates the reference to the nonverbal-symbolic level (Line 10). Also, she doesn't orientate herself on the counter selection presented in the upper screen section, in particular because she frequently looks towards the centre of the screen and counts the counters with her fingers on the 'twenty frame' (Line 6). The fact that Valerie not only records the counters presented in the iconic representation through a counting strategy, but also using a structure, is proven over the course of the interview when she smartly uses the counter-turning function on the virtual 'twenty frame' (Lines 11–18), although we should take into account here that the interviewer had set a targeted impulse for quasi-simultaneous number recording (Line 15).

Focussing of Representations on the Physical 'Twenty Frame' In the previous part of this section, the different focuses on representation in the representation of quantities as observed in the empirical investigation were described for the virtual 'twenty frame'. Below you can find a brief description of whether these methods of use also occurred during the work on the physical 'twenty frame'.

The assessment of the interviews has shown that both the *focussing on the iconic representation* and the *focussing on the selection of counters* could be observed in the same (as described above) or in a mildly different manner during the presentation of quantities on the physical 'twenty frame'. However, focussing on the *nonverbal-symbolic representation* was not pursued by the children on the physical 'twenty frame', and this for a comprehensible reason: the physical 'twenty frame' does not feature nonverbal-symbolic representation on which the children can focus. Accordingly, it is only logical that this method of use cannot exist on the physical 'twenty frame'. In addition, *no* methods of use occurred which were only pursued on the physical but not on the virtual 'twenty frame'.

Final Comments on the Focussing of Representations Overall, it can be recorded here that three different versions of focussing on the presented representations could be observed on the virtual 'twenty frame': (1) focussing on the nonverbal-symbolic representation, (2) focussing on the selection of counters and (3) focussing on the iconic representation. On the physical 'twenty frame', however, only the two latter methods of use occurred. Based on the fact that some children focussed on the nonverbal-symbolic representation level on the virtual 'twenty frame', it can be stated that the additional offer of this representation level caters for the individualism of childrens' access methods to the representation of quantities. However, this does not represent a statement on whether this offer can have positive or negative effects on learning mathematics. This is merely a difference noticed in the design on the two 'twenty frames', which has noticeably been reflected in the methods of use utilised by the children and which therefore has earned attention from mathematical-didactical researchers.

Of the three empirically-reasoned procedural methods for the presentation of quantities on the virtual 'twenty frame', only the *focussing on the selection of counters* can be characterised as impeding the overcoming of numeracy difficulties, in particular as this method of use is distinguished through an external representation of a sequential sense of numbers. On the other hand, the methods of use in which either a *focussing of the nonverbal-symbolic representation* or a *focussing of the iconic representation* is undertaken, the overcoming of counting strategies cannot be classified de facto as detrimental. The quality of the mental processes accompanying the focusses, such as an understanding of numbers as a compilation of other numbers by representing several counters simultaneously, appears to be a primary criteria for whether a child can detach themselves from counting strategies. Whether and how children have used this aspect during the representation of tasks on the 'twenty frames' is shown in the next section.

7.4.2 Simultaneous Representation of Five Counters for the Representation of Quantities

The 'twenty frames' used in this work offer to an equal extent the possibility of presenting five counters simultaneously. Whereas this aspect can be realised on the physical 'twenty frame' by laying strips of five, the virtual 'twenty frame' permits to present a "stack of five" simultaneously simply by pressing a button. A central difference between the two versions is that individual counters can subsequently be deleted from the stacks of five on the virtual 'twenty frame'. No comparable function is available on the physical 'twenty frame', because the 'strips of five' should not be destroyed by cutting them up with scissors or by tearing apart single counters in the investigation. Table 7.2 shows how many children reverted to the simultaneous addition of five counters during the use of the respective 'twenty frame' during the representation of quantities.

The data reveals that the children only rarely used strips of five during the course of the representation of tasks on the physical 'twenty frame'. In the case of the task 7 + 6, only three of the 19 children interviewed used strips of five for the addends. For the representation of the task 3 + 8, again only three children used strips of five, but exclusively for the representation of the second addend. In all other cases, the children *exclusively* reverted to the sequential representation of individual counters.

On the virtual 'twenty frame', however, use was made of the stacks of five far more frequently. During the representation of the task 8 + 7, eight children used stacks of five to represent both addends, and two children for one addend at least. A similar effect could be observed during the representation of the task 4 + 9. Seven children used stacks of five for at least one of the addends, and three children for both addends.

Overall, it can be determined based on the recorded data that the children in the investigation presented here tended to revert to the use of stacks of five on the virtual 'twenty frame' than to the use of strips of five on the physical 'twenty frame' for the representation of quantities with the same structure. The described research results can however merely depict a tendency in the methods of use by the children in view of the low numbers of test persons used. Unambiguous causal interrelationships *cannot* be developed here within the scope of this work. Nevertheless, possible influence factors can be identified which may explain the results of this investigation. Three feasible influence factors are sketched out below.

The first influence factor can be determined in the *compact representation in the counter selection* of the virtual 'twenty frame', which may have led to multiple use of the stacks of five. The counter selection is positioned directly above the iconic representation in the virtual 'twenty frame'. On the other hand, counters and strips

Table 7.2 The use of strips of five and stacks of five for the representation of quantities

Number of children who …	'Strips of five' (physical 'twenty frame')		'Stacks of five' (virtual 'twenty frame')	
	7 + 6	3 + 8	8 + 7	4 + 9
… represented **both** addends through simultaneous addition of five counters	3	0	8	3
… represented **one** addend through simultaneous addition of five counters	0	3	2	7
… represented **no** addends through simultaneous addition of five counters	16	16	9	9

Selection of counters on the virtual 'twenty frame'	Selection of counters on the physical 'twenty frame'
	

Fig. 7.9 Selection of counters on the 'twenty frames'

of five must be removed from a box next to the medium when representing quantities on the physical 'twenty frame', which may involve a comparatively difficult application of motor skills (see Fig. 7.9).

In relation to this, in the reflection phases on their use of both 'twenty frames', almost all children said that they preferred the placement process of the counters on the virtual 'twenty frame', frequently stating as the reason the increased speed of use. In addition, some children *consciously* waived the use of strips of five by frequently removing the strips of five positioned on the counters so that they could remove the individual ones from the box. The children selected this method of use although the advantages of the strips and stacks of five had been mentioned in the course of the introduction phase.

As a second influence factor, the frequent use of stacks of five or the less frequent use of strips of five can be due to the *mathematical-didactical quality of the materials used*. As already mentioned, it is possible on the virtual 'twenty frame' to take individual counters. This aspect is *not* supported through the physical 'twenty frame', as the strips of five would have to be destroyed for the purpose. The representation process of the children may be influenced through this design aspect. Here in particular the role of addends of less than 5 appears to be of particular interest. As shown in Table 7.2, only three children represented a addend with a strip of five during the task 3 + 8 on the physical 'twenty frame'. In all three cases, this concerns the number 8. The number 3 was not represented by any of the children using the strips of five, which isn't necessarily surprising, in particular as this would involve a relatively complex representation process on the physical 'twenty frame'. A strip of five would have to be placed and then swapped directly into five individual counters so that two individual counters could then be removed. The use of three individual counters appears far more economic here. On the virtual 'twenty frame', such a swapping process is not required, as individual counters can be directly removed from the five placed counters. The number 4 was frequently represented by the children in this manner. Five counters were simultaneously represented and one single counter deleted.[3] However, we should also note here that those numbers which were larger than 5 were also represented by the children far more frequently by children using the five simultaneously on the virtual 'twenty frame' than on the physical 'twenty frame'.

The third possible influence factor can be reasoned through *preliminary experiences related to the material*. If children have, in their previous mathematics lessons, (almost) exclusively utilised individual counters when working with the physical 'twenty frame', they may pursue this procedure within the scope of this study, too. It would appear improbable that a brief introductory phase in which the advantages of the use of strips of five on the physical 'twenty frame' were mentioned can decisively change the pattern of action familiar to the children. However, the children had never used the virtual 'twenty frames' before prior to the interview sessions. Therefore the functional method of the software *had* to be processed. In this way, the children may have been inspired to use the stacks of five relatively frequently in the methods of use.

It remains to be noted that the children in this investigation used the stacks of five in the virtual 'twenty frame' far more frequently that the strips of five in the physical 'twenty frame' for the representations of quantities. This result was explained through the description of three possible influence factors: (1) the compact representation of the counter selection on the virtual 'twenty frame', (2) the mathematical-didactical quality of the 'twenty frame' and (3) preliminary experience related to the 'twenty frame'. Linked with this, it is possible to support a hypothesis worthy of further investigation that the virtual 'twenty frame' tends to motivate children to simultaneously represent five counters than is the case for the physical 'twenty frame', whereby with the first hands-on material, with regard to this specific representation process, a larger contribution can be made towards the overcoming of numeracy difficulties.

[3]This data material cannot be used to clarify whether the children would also have proceeded in the same way with the representation of the number 3. However, we hereby note that three pushes of the button are required to represent the number 3 both with and without the use of simultaneous representation. Either three individual counters are added in sequence, or five counters are represented, whereupon two individual counters are then deleted. In this way, the use of stacks of five is in this special case to be viewed theoretically as equally economical as the exclusive use of individual counters.

7.4.3 Generation of Further Iconic Representations of Quantities

Once the children had represented tasks on the ‘twenty frame’, they were requested in this investigation to generate further iconic representations of the respective quantity. In this section, a comparison is drawn between the methods of use for the generation of iconic representations on the ‘twenty frames’.

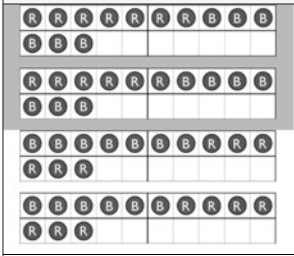
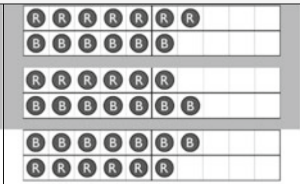
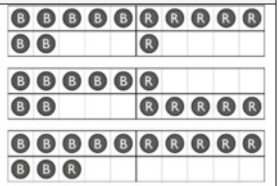
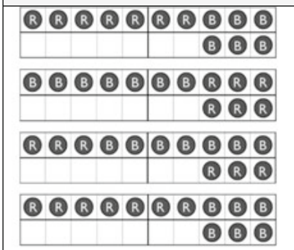
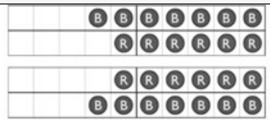
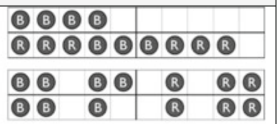
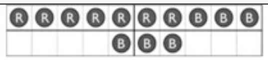
As shown in Sect. 7.3.2, the iconic representation of a task on the virtual ‘twenty frame’ is always presented in structured form. Due to the standard activated automatic sorting function, it can never be the case that an unstructured counter image occurs. Red counters are always represented before blue counters. Gaps can never occur in the iconic representation. These structuring aids have advantages as positive design elements with regard to the support of schoolchildren during the quasi-simultaneous determination of quantities (e.g. Walter 2017). However, it could also be verified that the structuring support on the virtual ‘twenty frame’ in part makes the generation of further representations more difficult—or can even suppress them, as the software *only* permits two different versions of the iconic representation for one task in its standard settings. In this way, for the task 8 + 7 only the iconic representations presented in Fig. 7.10 can be generated.

Based on both of the ‘twenty frames’, the generation of further iconic representations was requested after previous representation of tasks. Whereas on the virtual ‘twenty frame’ the subject concerned the tasks 8 + 7 and 4 + 9, the tasks with the same structure 7 + 6 and 3 + 8 were the subject of the work on the physical ‘twenty frame’. After the representation of the tasks on the respective ‘twenty frame’, the children were asked to generate further counter representations until they signalled that they could derive no further placement methods. Whilst the children were *exclusively* able to generate he placement methods shown in Fig. 7.10 and the appropriate representations of the swapping task 7 + 8 during the course of this process on the virtual ‘twenty frame’, the same children represented numerous methods of placement on the physical ‘twenty frame’ which could not be generated

Line by line	Side by side
	

Fig. 7.10 Iconic representations on the virtual ‘twenty frame’ based on the example 8 + 7

Table 7.3 Observed methods of placement for the representation of the task 7 + 6 on the physical 'twenty frame'

Side by side	Line by line	Further structures
Left-aligned		Power of five
		
Right-aligned		Other
		
Centered		
		

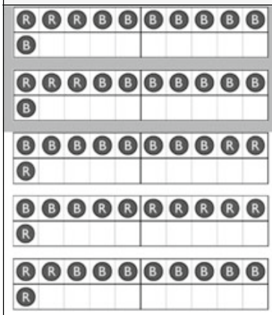
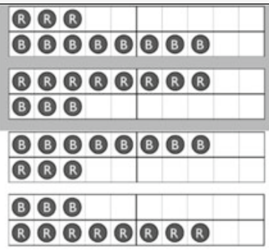
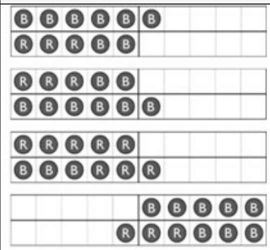
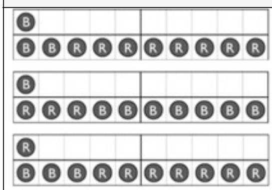
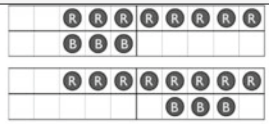
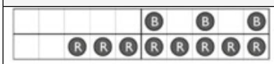
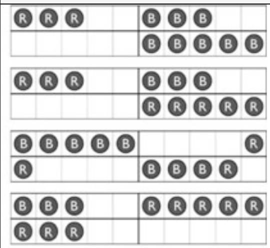
on the virtual 'twenty frame'. Table 7.3 shows an overview of the all the represented counter images for the task 8 + 7. The grey-highlighted representations can be presented on the virtual 'twenty frame', but the others are not possible.

Table 7.3 shows that the children generated far more counter representations during the representation of the task 7 + 6 than the virtual 'twenty frame' can actually permit. This finding applies both for the methods of laying types 'side by side' and 'line by line'. In addition, the children generated diverse counter representations which cannot be assigned to any of the placement methods described above.

Analogue results can also be determined with regard to the representations of counters derived during the course of the representation of the task 3 + 8. The following Table 7.4 shows the iconic representations generated by the children for this task.

The recorded data shows that the children were able to generate far more representations of counters when working with the physical 'twenty frame' than the virtual 'twenty frame' can actually produce. In addition, only a few of these

Table 7.4 Observed laying methods for representation of the task 3 + 8 on the physical 'twenty frame'

Side by side	**Line by line**	**Further structures**
First line filled	Left-aligned	Column-wise
		
Second line filled	Centered	Other
	 With gaps 	

placement methods can transfer onto the structures specified by the 'twenty frame' which permit the quasi-simultaneous ascertainability of counter representations. Accordingly it can be stated that the structuring aids implemented in the virtual 'twenty frame' do not always support the individual - and in part smart - methods of representation used by the children.

Based on these results, however, it is not possible to plead that the structuring aids incorporated into the virtual 'twenty frame' appear to be de facto unsuitable for learning mathematics. Rather more, they serve as an indicator that an excessively rigid framework on computer-supported structures many restrict the individualism of childrens' procedural methods. At the same time, the strengths of the structuring aids in the consistent quasi-simultaneous ascertainability of the counter image can be perceived. This aspect is frequently, but not always given when the children are producing the numbers themselves.

7.5 Closing Remarks

The descriptive analyses of the data have shown that schoolchildren use the virtual 'twenty frame' when representing quantities in very different ways. It became clear that a few students already made use of digital medias' potential (e.g. generating five counters simultaneously by tapping on the 'stack of five'-button) just after getting introduced to the software's features. Hence, the usage of tablet-applications might contribute to overcome central difficulties of students. Nevertheless it should be taken into account that usages with possibly counterproductive effects occurred, too. Several observed approaches might rather facilitate the predominantly application of counting strategies than fostering the development of non-counting approaches. Thus, software features, which are sometimes labelled as auspicious potential of digital media by mathematics experts and software developers does not necessarily fulfil these expectations in its entirety. The fact that software provides a potential does not guarantee that students make use of it in an appropriate way. Likewise, it should be regarded that *how* students use software determines if the mathematical learning processes can be facilitated.

In comparison to the use of the physical 'twenty frame', we were able to determine that *no* conclusions could be made on which of the two 'twenty frames' tended to cater for the individual usage preferences of the children. During the different processing procedures in the course of the representation of quantities, it emerged that the virtual and the physical 'twenty frames' each permit methods of use which cannot be implemented by or can only be implemented with difficulty by the respective other medium. Accordingly the question regarding the use of virtual *or* physical materials appears less relevant than the question of in which specific situations virtual *and* physical materials display mathematical-didactical advantages in order to support children with (and without) numeracy difficulties as effectively as possible in overcoming their learning difficulties.

References

Ainsworth, S. (1999). The functions of multiple representations. *Computer & Education, 33,* 131–152.

Baroody, A. J. (2006). Why children have difficulties mastering the basic number combinations and how to help them. *Teaching Children Mathematics, 13*(1), 22–31.

Benz, C. (2005). *Erfolgsquoten, Rechenmethoden, Lösungswege und Fehler von Schülerinnen und Schülern bei Aufgaben zur Addition und Subtraktion im Zahlenraum bis 100.* [Success rates, calculation methods, solutions and errors by students during addition and subtraction tasks within the number range to 100] Hildesheim: Franzbecker.

Clements, D. H. (1999). Subitizing: What Is It? Why Teach it? *Teaching Children Mathematics*, 400–405.

Doschko, D. (2011). *Lösungshäufigkeiten, Rechenmethoden, Lösungswege und Fehler von Erstklässlern beim Bearbeiten von Aufgaben im Zahlenraum bis Zwanzig.* [Solution frequencies, calculation methods, solutions and errors by first class students when processing tasks within the number range to twenty]. Hamburg: Verlag Dr. Kovač.

Fresemann, O. (2014). *Schwache Rechnerinnen und Rechner fördern: Eine Interventionsstudie an Haupt-, Gesamt- und Förderschulen* [Supporting students with learning difficulties in mathematics: An interventional study in secondary, comprehensive and special needs schools]. Wiesbaden: Springer.

Fuson, K. C. (1992). Research on learning and teaching addition and subtraction of whole numbers. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 53–188). Hillsdale, NJ: Erlbaum.

Gaidoschik, M. (2010). *Die Entwicklung von Lösungsstrategien zu den additiven Grundaufgaben im Laufe des ersten Schuljahres* [The development of solution strategies for additive basic tasks during the first school year]. Dissertation: Universität Wien.

Gaidoschik, M. (2011). *Rechenschwäche—Dyskalkulie. Eine unterrichtspraktische Einführung für LehrerInnen und Eltern* [Learning difficulties in Mathematics—dyscalculia. An introduction to practical lessons for teachers and parents] (Vol. 6). Horneburg: Persen.

Gerster, H.-D. (2009). Schwierigkeiten bei der Entwicklung arithmetischer Konzepte im Zahlenraum bis 100 [Difficulties in the development of arithmetic concepts within a number range up to 100]. In A. Fritz, G. Ricken & S. Schmidt (Eds.), *Handbuch Rechenschwäche* (Vol. 2, pp. 248–268). Weinheim und Basel: Beltz.

Glaser, B. G., & Strauss, A. L. (2005). *Grounded theory. Strategien qualitativer Forschung* [Strategies for qualitative research]. Bern: Huber.

Goldin, G. A., & Kaput, J. J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397–430). Mahwah, NJ: Erlbaum.

Goodwin, K., & Highfield, K. (2012). *iTouch and iLearn—an examination of "educational" apps*. Salt Lake City, Utha: Paper presented at the Early Education and Technology for Children conference.

Goodwin, K., & Highfield, K. (2013). A framework for examining technologies and early mathematics learning. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 205–226). Dordrecht: Springer.

Gray, E. M. (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics, 22,* 551–574.

Institut für Demoskopie Allensbach. (2014). *Digitale Medienbildung in Grundschule und Kindergarten* [Digital media education in primary schools and nursery schools]. Available at www.telekom-stiftung.de.

Krauthausen, G. (2012). *Digitale Medien im Mathematikunterricht der Grundschule [Use of digital media during arithmetics lessons in primary schools]*. Heidelberg: Springer Spektrum.

Ladel, S. (2009). *Multiple externe representationen (MERs) und deren Verknüpfung durch Computereinsatz [Multiple external representations (MERs) and their linkage through the use of computers]*. Hamburg: Verlag Dr. Kovač.

Ladel, S., & Kortenkamp, U. (2009). Virtuell-enaktives Arbeiten mit der "Kraft der Fünf" [Virtually-enactive work using the "power of five"]. *MNU Primar, 1*(3), 91–95.

Ladel, S., & Kortenkamp, U. (2014). Number concepts—processes of internalization and externalization by the use of multi-touch technology. In C. Benz, B. Brandt, U. Kortenkamp, G. Krummheuer, S. Ladel & R. Vogel (Eds.), *Early Mathematics Learning. Selected Papers of the POEM 2012 Conference* (pp. 237–256). New York: Springer.

Larkin, K. (2015). "An App! An App! My Kingdom for An App": An 18-month quest to determine whether apps support mathematical knowledge building. In T. Lowrie & R. Jorgensen (Eds.), *Digital games and mathematics learning: Potential, promises and pitfalls* (Vol. 4, pp. 251–276). Netherlands: Springer.

Lorenz, J. H. (2011). Anschauungsmittel und Zahlenrepräsentation [Representation materials and number representation]. In A. S. Steinweg (Ed.), *Medien und Materialien* (pp. 39–54). Bamberg: University of Bamberg Press.

Lüken, M. (2012). Young children's structure sense. *Journal für Mathematikdidaktik* [Journal for Mathematical Didactics], *33*(2), 263–285.

Mayring, P. (2015). *Qualitative Inhaltsanalyse. Grundlagen und Techniken* [Qualitative content analysis. Basic principles and techniques] (Vol. 12). Weinheim und Basel: Beltz.

Moyer-Packenham, P. S., Shumway, J. F., Bullock, E., Tucker, S. I., Anderson-Pence, K. L., Westenskow, A., et al. (2015). Young children's learning performance and efficiency when using virtual manipulative mathematics ipad apps. *Journal of Computers in Mathematics and Science Teaching, 34*(1), 41–69.

Ostad, S. A. (1998). Developmental differences in solving simple arithmetic word problems and simple number-fact problems: A comparison of mathematically normal and mathematically disabled children. *Mathematical Cognition, 4*(1), 1–19.

Padberg, F., & Benz, C. (2011). *Didaktik der Arithmetik. für Lehrerausbildung und Lehrerfortbildung* [Arithmetic didactics for teacher training and teacher further training] (Vol. 4). Heidelberg: Spektrum Akademischer Verlag.

Paek, S., Hoffman, D., Saravanos, A., Black, J., & Kinzer, C. (2011). *The role of modality in virtual manipulative design.* Paper Presented at the CHI '11 Extended Abstracts on Human Factors in Computing Systems. New York.

Peter-Koop, A., Wollring, B., Spindeler, B., & Grüßing, M. (2007). *Elementar Mathematisches Basis Interview—Zahlen und Operationen [Elementary mathematical basic interview—numbers and operations].* Offenburg: Mildenberger.

Radatz, H. (1990). Was können sich Schüler unter Rechenoperationen vorstellen? [What can students imagine under the term arithmetic operations?]. *Mathemathische Unterrichtspraxis, I. Quartal* (11), 3–8.

Sinclair, N., & Baccaglini-Frank, A. (2016). Digital technologies in the early primary school classroom. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 662–686). New York: Taylor & Francis.

Sinclair, N., & Heyd-Metzuyanim, E. (2014). Developing number sense with TouchCounts. In S. Ladel & C. Schreiber (Eds.), *Von Audiopodcast bis Zahlensinn* (pp. 125–150). Münster: WTM-Verlag.

Urff, C. (2012). Virtuelle Arbeitsmittel im Mathematikunterricht der Primarstufe. Schriften zu Mathematikunterricht und Technologieeinsatz [Virtual working equipment in primary stage arithmetics lessons. Writings on arithmetic lessons and the use of technology]. In S. Ladel & C. Schreiber (Eds.), *Lernen, Lehren und Forschen in der Primarstufe* (pp. 59–82). Hildesheim: Franzbecker.

Urff, C. (2014). *Digitale Lernmedien zur Förderung grundlegender mathematischer Kompetenzen* [Digital learning media to foster basic mathematical competencies]. Berlin: Mensch und Buch Verlag.

Walter, D. (2017). *Nutzungsweisen von Grundschülerinnen und Grundschülern bei der Verwendung von Tablet-Applikationen* [Methods of use by primary school students when using Tablet Apps]. Dissertation: TU Dortmund.

Wartha, S., & Schulz, A. (2013). *Rechenproblemen vorbeugen* [Preventing mahematical learning difficulties] (Vol. 2). Berlin: Cornelsen.

Chapter 8
Reshaping the Learning Experience Through Apps: Affordances

Nigel Calder and Carol Murphy

Abstract This paper reports on how the affordances of the app Math Shake reshaped the learning experience—an aspect of a research project examining the ways mobile technologies are used in primary-school mathematics. Students used different digital tools within the app to solve word problems, while the affordances, including simultaneous linking, focussed constraint, creative variation, dynamic and haptic, made the learning experience different from when using pencil-and-paper technology. However, while the affordances of the mobile technologies are important, the teacher's pedagogical approach was also influential in the learning.

Keywords Digital technologies · Apps · Primary mathematics
Learning · Affordances

8.1 Introduction

There has been a proliferation in the availability and use of mobile technologies (MT) over the last five years, including in educative settings. Their low instrumentation and ease of operation, coupled with the interaction being focused primarily on touch and sight, make using them intuitive for learners. Linked to this increase in MT is the growth in educational apps. Questions have been raised regarding the appropriateness of the content and pedagogical approaches of apps (e.g., Philip & Garcia, 2014) but if MT are an inevitable and relatively enduring element of the evolving digital world, we need to consider their potential for learning. This paper reports on an aspect of a research project examining the ways iPad apps, as an example of MT, are used in primary-school mathematics. The project considers the pedagogy that might best facilitate the learning with students

N. Calder (✉)
University of Waikato, Tauranga, New Zealand
e-mail: nigel.calder@waikato.ac.nz

C. Murphy
University of Tasmania, Launceston, Australia

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_8

(ages 7–10) when engaging in mathematics through MT. In particular, the paper reports on the affordances of the app *Math Shake* and how they might be seen to reshape learning experiences in primary mathematics.

The use of digital technologies has the potential for introducing new ways of engaging with mathematical concepts and processes, and for re-envisaging aspects of mathematical education, along with alternative ways to facilitate understanding (e.g., Borba & Villarreal, 2005; Calder, 2011). Borba and Villarreal (2005) perspective saw understanding emerging from the reconciliation of re-engagements of the collectives of learners, media, and environmental aspects with the mathematical phenomena. Borba and Villarreal contend that each engagement re-organizes the mathematical thinking and initiates a fresh perspective that in turn transforms the nature of each subsequent engagement with the task. This iterative process of re-engaging with the task from each new perspective continues until some form of shared negotiated understanding occurs (Calder, 2011).

Meanwhile, Meyer (2015) suggested that MT offer a socio-material bricolage for learning. Drawing on Fenwick and Edwards' (2012) notion of socio-material approaches to learning, she envisaged complex systems where resources interact with knowledge distributed across people, communities and sites of practice. She used the term socio-material bricolage to describe the "ecological entanglement of material and social aspects of teaching and learning with technology" (Meyer, 2015, p. 28). The notion of bricolage suggests that there is a mutually influential collective of tools and users affecting the dialogue, learning experience, and mathematical thinking, in particular and personalized ways. In an associated viewpoint, de Freitas and Sinclair (2014) identified finger-screen-voice-five assemblages through which a child was involved in a rhythmic engagement when counting to five with the *Touch Counts* app. They contend that these assemblages were not static but fluent, and might manifest in varying constitutions and combinations.

The dynamic nature of these assemblages, with the accompanying relationships between social and technical elements, suggests that the experience will unfold for each individual in a personalized, differentiated way. Hence, the teacher might also structure the learning experience or facilitate a learning environment and class culture so that the students' learning might be both personalized, and differentiated conceptually. The learning experience and associated learning will be contingent on this mutually influential weave of digital features and the accompanying social aspects. Important in this discussion is the symbiotic relationship between the digital media and the user. While the digital medium exerts influences on the student's approach, and hence the understanding that evolves, it is his/her existing knowledge that guides the way the technology is used, and in a sense shapes the technology. The student's engagement is influenced by the medium, but also influences the medium (Hoyles & Noss, 2003). The reciprocity of the relationship between the user and the digital pedagogical medium resonates with the notion of affordances.

8.2 Affordances

In relation to Gibson (1977) notion of affordance as the complementarity of the learner and the environment and to Brown (2005) acknowledgement of the potential relationship between the user and the artifact, the visual and dynamic affordances of MT may be seen to fashion the learning experience in distinctive ways, and so reposition students' engagement with mathematics. This relationship affords opportunities and constraints: Opportunities to envisage and engage with mathematical phenomena in particular ways that might distinctively shape the learning. Some of these affordances might constrain the nature of the engagement or focus the interaction on limited elements of the mathematics or mathematical thinking. We might consider affordances as perceived opportunities and constraints offered through the pedagogical medium in relationship with the propensities and intentions of the user (Calder, 2011).

One affordance frequently associated with digital environments is the notion of multiple representations. The ability to link and simultaneously interact with visual, symbolic, and numerical representations in a dynamic way has been acknowledged extensively in research (e.g., Ainsworth, Bibby, & Wood, 1998; Calder, 2011). Sacristán and Noss (2008) illustrated how the engagement of computational tasks in a carefully designed micro-world might lead to different representational forms (such as visual, symbolic and numeric): a process that they called *representational moderation*. In a similar way, various studies involving dynamic geometry software, report that the dynamic, visual representations enhanced the understanding of functions (e.g., Falcade, Laborde, & Mariotti, 2007). This dynamic affordance, allied with the propensity of MT to give instant feedback to input, transforms the nature of the learning experience compared to pencil-and-paper technology.

Virtual manipulatives (VM) are frequently part of mathematical apps. They are described as interactive, web-based visual representations of dynamic objects (Moyer, Bolyard, & Spikell, 2002) that might afford opportunities for mathematical thinking. In *Math Shake* word problems are generated at various levels, and it provides a range of digital pedagogical tools (e.g., empty number lines, counters, ten frames), that students can select to help with their solutions. Through the use of dynamic, visual representations, VM offer potential to extend the learning experiences beyond those with pencil-and-paper medium (Arcavi & Hadas, 2000).

Moyer-Packenham and Westenskow (2013) identified the affordances of focused constraint, creative variation, simultaneous linking, efficient precision and motivation when students used apps in their mathematical learning. These affordances do not typically manifest in isolation. They interact, and appear to be influential on each other. The *Math Shake* app appears to afford three of them in particular: focused restraint, where the app might focus students' attention on particular mathematical concepts or processes; creative variation, where the app might encourage creativity, hence evoking a range of student approaches and potential solutions; and simultaneous linking, where the app might link representations simultaneously and connect them to student activity (Moyer-Packenham & Westenskow, 2013). In *Math Shake* a word problem is generated, and the students highlight the words associated with the appropriate mathematical process (suggesting focused constraint). They can select tools, such as an empty number line, to help with solving the problem (suggesting focused constraint and creative variation) while these representations can be linked to each other and also to the students' actions in the form of immediate feedback or response to their input (suggesting simultaneous linking). The connections made between pictorial and symbolic representations, mediated through the actions executed on these representations, can emphasize the associated mathematical concepts and processes (Moyer-Packenham & Westenskow, 2013). In a study examining the differences in learning effects of virtual and physical manipulatives, the importance of students' facility with representations and tools was confirmed (Moyer-Packenham et al., 2013). They also indicated that a lack of familiarity with different representations might negatively influence students' understanding.

As well as having affordances similar to other digital technologies, such as the opportunity to engage dynamically, the glass interface of an iPad presents a further affordance through touch. Student interaction is more directly responsive to input, enhancing the relatively high agency of the medium. There is direct interaction with the phenomena, rather than being mediated through a mouse or keyboard, making the iPad more suitable for young children than desktop computers (Sinclair & Heyd-Metzuyanim, 2014). Apps might use this haptic affordance (e.g., with *Multiplier,* where within the task, the student drags out the visual area matrix associated with multiplication facts). This app also evokes multi-touch functionality, enabling students to make sense of individual effects of particular screen touches (Hegedus, 2013), and to create personal explanations of their thinking. This is similar to the simultaneous linking and creative variation that Moyer-Packenham and Westenskow (2013) identified. There is a linking between the various representations of the number fact that the app affords (i.e., symbolic, area, colour and aural) suggesting simultaneous linking, while the student is creating and evaluating their representation, suggesting creative variation.

Others have indicated that affordances of digital technologies, together with the associated dialogue and social interaction, may lead to students exploring powerful ideas in mathematics, learning to pose problems, and create explanations of their own (Sandholtz, Ringstaff, & Dwyer, 1997). They also reported improved high-level reasoning and problem solving linked to learners' investigations in digital environments. iPads and apps also foster experimentation (Calder & Campbell, 2016), allowing space for students to explore. The apps affordances of interactivity and instantaneous feedback foster the learner's willingness to take cognitive risks with their learning (Calder & Campbell, 2016). They allow students to model in a dynamic, reflective way. Others contend that MT can provide new forms of personal ownership (e.g., Meyer, 2015) that in turn supports learners' personal understanding and conceptual frames (Melhuish & Falloon, 2010).

In a similar way to apps such as *Explain Everything, Math Shake* allows students to screencast, a digital recording of the output on the screen, to record individual or group presentations of mathematical processes, strategies and solutions. When students submit this to internal intranets or *Google classroom* spaces, the teacher can attend to them individually and evaluate their strategies and understanding. They can offer feedback and feed forward to better scaffold the learning, with potential to enhance the mathematical understanding. Further to this, *Math Shake* generates word problems at different levels that might be explored through the range of digital pedagogical tools students can select to assist with realizing their solutions. The screencasting feature of the app, and the simplicity with which it is enabled on an iPad, opens up other learning opportunities that would not be possible with approaches that use pencil-and-paper as pedagogical tools exclusively. Such an app introduces a further multi-representational feature that can create an aural representation that students can listen to.

In this paper, we focus on the affordances of simultaneous linking, focused constraint, creative variation (Moyer-Packenham & Westenskow, 2013) as well as the dynamic interaction that is afforded. Simultaneous linking is perceived as the multi-representation affordance used with digital technologies in general, but including the haptic aspect particular to iPads and other mobile devices. With *Math Shake* the different representations used in the screencasting, and with the digital tools also offered opportunity for the simultaneous linking of various representations. We report on teachers' and students' perceptions of the learning opportunities afforded through the use of *Math Shake*, as an example of an app that uses screencasting, the digital recording of the computer screen, along with voice recording.

8.3 Motivation

Much of the discussion and consideration of the ways iPads and apps might influence the learning experience, is centred on the notion of student engagement (e.g., Attard, 2015) with students being actively enthralled and interested, often by the visual and interactive characteristics of the pedagogical medium (Carr, 2012). Meanwhile, motivation is not observed directly, but rather is marked through behaviour and attitudes. Enhanced student engagement would suggest an increase in student motivation. Hannula (2006) described motivation as a preference towards doing some things and avoiding others. Motivation is related to personal interest (Wæge, 2010) and plays a role in student achievement (Pintrich, 2003). The influence of students' positive attitudes on their sense of autonomy, and hence their learning and performance in school has also been reported (e.g., Deci & Ryan, 2000). There appears to be a relationship between motivation and engagement, with peer pressure and classroom culture influencing students' learning opportunities in mathematics (e.g., Sullivan, Tobias, & McDonough, 2006). If students perceive the quality of instruction positively it increases their enjoyment of the learning and lessens feelings of anxiety towards mathematics (Frenzel, Pekrun, & Goetz, 2007). This suggests that such positive perceptions influence students' engagement with mathematics learning.

Research investigating students' perceptions of their learning environments contend that these perceptions affect emotional and social behaviour (e.g., Anderman, 2002). Hannula (2006) contends that there was constrained opportunity to meet students' needs, competence and autonomy in teacher-centred classrooms, implying a need to consider other types of learning environments. Student-centred group learning situations give opportunity for positive dispositions towards mathematics to develop, with enjoyment and enhanced motivation reported in a range of studies (e.g., Hänze & Berger, 2007; Schukajlow et al., 2012).

Learning environments are not just physical attributes, including tools such as mobile technologies and apps, but include social interactions (Frenzel et al., 2007), with classroom culture influential on the nature of the environment (Hunter & Anthony, 2012). Frenzel et al. (2007) also contend that students' perceptions of their environment were related to achievement in mathematics, although they suggested that there were only tentative links between emotions and student achievement.

The literature suggests that there are synergies and inter-relationships between the integration of mobile technologies and apps into mathematics programs; the opportunities and constraints they afford; the assemblage of social and technical elements associated with those affordances; and hence the affective aspects of learning mathematics such as motivation and engagement. When the learning environment, in the broader sense, changes the way that the mathematical phenomena are engaged with, and this in turn evokes more positive dispositions and attitudes in the learner, then we can surmise that the learning might be different and the understanding that emerges might also differ from if the MT was not involved. Hence, the reshaping of the learning experience would be influential in the students' understanding. In the research study undertaken, the data also suggested this.

8.4 Methodology

The research used an interpretive methodology that relates to building knowledge and developing research capability through collaborative analysis and critical reflection of classroom practice and student learning. The research design is aligned with teacher and researcher co-inquiry whereby the university researchers and practicing teachers work as co-inquirers and co-learners (Hennessy, 2014), with an emphasis on collaborative knowledge building. In the first year of the two-year project, three teachers, all experienced with using MT in their programs were involved in the study. One teacher taught a year-4 class using a 'Bring your own Device' (BYOD) approach, while the other two teachers team-taught in a year-5 and 6 class with one-to-one iPad provision. Data, obtained through different sources (focus group interviews, classroom observations, interviews with teachers, and blogs) were analyzed using NVivo via a mainly inductive or grounded method to identify themes. Refinement of the identified themes occurred through joint critical reflection between teacher practitioners and academic researchers in research meetings. The research question that specifically relates to this paper is: How does the use of mathematics apps influence student engagement and learning? In the paper we present extracts of data from the first year of the project: the teacher and student interviews, and the student blogs concerned primarily with the use of the *Math Shake* app. These responses are analyzed in relation to the emerging themes of the project.

8.5 Results

8.5.1 The Teacher Interviews

One teacher commented on the direct interface of the iPad screen, suggesting that the students were interacting more directly with the content of the mathematics. "Like a physical object that they're interacting with." As well as the haptic affordance, this suggests focused restraint as the teacher perceived the app facilitating more direct interaction with mathematical content. The teacher further explained how apps involving screencasting for recording students' strategies were powerful agents in learning as the students were "creating something…explaining their own thinking, creating their own content, their own language." This teacher comment points to the notion of creative variation affordance. The students are creating their own content and language, hence differentiating the experience and learning to some extent.

Another teacher noted how screencasting enabled less confident students to explain their thinking in a "nonthreatening environment" with "no teacher staring at them, no other kids waiting for them to hurry up." "They're in a safe place where they can just record their thinking without any pressure."

The teacher also saw benefits in assessing the students' thinking as the recording provided them with an understanding of "what's going on in the kid's head." Also, in giving feedback to the students on their learning. The non-threatening environment for feedback seemed to resonate with the simultaneous linking affordance. Here the linking was between the student's action and the instantaneous, onscreen feedback.

8.5.2 The Student Interviews

Several students referred to the idea of drawing on the iPad screens or of tapping to select a tool and how this made their work "easier and tidier" (Year 4 student (Y4)). This suggests the haptic affordance, with the students using touch to interact directly with the app. Use of screencasting to record their solution strategies seemed a key feature for the students. The students talked of videoing themselves doing maths, and recording their working out. As one said, “It’s just like making a movie for maths” (Year 6 student (Y6)). The opportunity to record their voices whilst writing and drawing seemed important as it was “hard to explain without writing down. You can write it down as well as explaining it while you’re recording” (Year 5 student (Y5)) This student comment suggests that they were utilising the simultaneous linking affordance to articulate their explanation of the strategy that they were using. It also has elements of simultaneous linking, between the recording, writing and drawing features. The opportunity to pause and edit their recordings also appeared to be significant. “The cool thing is that you can actually pause it and then think about what you’re going to do” (Y5). Students also commented on the assurance that they had a correct solution, and hence had confidence in their strategies to start recording, “when you have your question and you’ve got your equation right” (Y6).

Some students commented how the feedback and opportunity to record their solutions had helped their learning in mathematics: “We can write things down and answer questions to see if we are right or wrong” (Y6). This comment also has elements of the simultaneous linking between the student actions and the digital feedback resulting from the mathematical processes taking place within the digital learning environment. Others referred to the opportunity to use the different tools on the iPad and how these introduced them to new strategies: “I like learning new strategies; using a number line and place value” (Y5). *Math Shake* opened up opportunities for engaging with the mathematics phenomena more easily through the use of digital tools. Some students noted specific instances of learning: “I learnt how to use the reversing strategy on the number line” (Y6). These student comments suggested focused constraint, with the app focusing the student’s attention on particular mathematical concepts or processes. Some could not identify specific learning but noted increased confidence. “I still use the same problems and the way I do them, but for some reason I feel more confident doing my maths” (Y6). Other students indicated emerging learning and confidence when working on a problem involving money with comments such as: “I’m sort of good at counting money now” (Y4).

While it might be argued that these opportunities could be enacted through other pedagogical media, it is the ease of use and the tactile nature of the experience with this app that seemed to facilitate these processes more easily, and make them more student directed when the students used the apps. They are also integrated into one device, one that can be moved seamlessly between learning situations. The students often had choice with where and how they worked and could shift furniture and settings if desired. This enabled a sense of personalisation of the learning space and to some extent helped facilitate a sense of ownership with the learning.

8.5.3 Student Blogs

The student blog data were consistent with the interview data regarding the recording aspect of the screencasting feature of *Math Shake* and the ways that this influenced the learning process through simultaneous linking. The data indicated positive student perceptions about engaging with the mathematics phenomena through the *Math Shake* app. Typically, the blog comments described how the recording of the strategies they used, enabled them to understand the processes that they were using. For example:

> Anna (Y6): I use *Math Shake*—it is a helpful app because you have choices of what you need to work on. You can record your learning and you can see what stage you are working on.

There is also an element of creative variance here as the student chooses and then creates an individual recording and uses that to self-evaluate their learning. Other data also suggested that the ease of recording was conducive to a positive and productive learning experience:

> Josh (Y5): Using math apps helped us so much! Instead of writing stuff in our book we can just record our voices and upload it to a app called *Google Classroom*! It has got us so far by using these apps and no one here wants to go back to writing in our books!

Other student blog data were indicative of the features and affordances of various apps supporting the learning process. For instance, the simultaneous linking and dynamic affordances:

> Matiu (Y6): *Tickle* and *Hopscotch* are a big hit in our class, as you can see your creations move around.

> Jess (Y5): I use these apps to help me with my learning. *Multiplier* helps me because it shows what it looks like, so I know how to do it.

Also, students found the simultaneous linking affordance was facilitated through having various resources embedded in the app.

> Autumn (Y6): The apps were useful for me because it has helped with all the resources in the app *Math Shake*. For example, the number lines and the fraction pieces.

Students were observed moving between these representational resources in a relatively seamless way, discussing and comparing their suitability. *Math Shake* was frequently mentioned in the data in terms of the students' perceptions of the tools that it made available. These tools, in conjunction with other features, appeared to have helped with their mathematical problem solving. The following is typical of the data:

> Alana (Y6): *Math Shake* is a great learning tool because it can help you with your problem solving. So you can choose a level for you, so just say you were genius or easy or confident or even beginner, there are a lot of levels to choose from. And there are also some amazing tools to help you solve your word problem. For instance, number-lines, fractions, counters, and there is also different coloured pencils that you have to earn.

There was also a suggestion of focused constraint as the apps and their features focused the students on particular conceptual or pedagogical elements.

> Trish (Y6): I used apps to help me learn about decimals, different types of triangles, (with *Hopscotch* and *Tickle*) fractions, ratios, proportions, all sorts of very interesting math strategies, lots of things. *Math Shake* is cool as well. It is awesome for showing your learning.

The reflective nature of composing the recording is suggested here. Similarly, with student comments regarding other apps that have a screencasting feature:

> Alice (Y4): Using apps has helped to solve my problems. By using *Explain Everything* you can record and pause and think about what you are saying.

As with the interview data, the student blog entries were relatively cohesive about the affordances making the learning process more engaging and enjoyable. The affordances of the MT motivated some students with their learning. The following are indicative of this motivational aspect.

> Whetu (Y4): I enjoyed using the apps because they make learning much (much!) more fun and intriguing.
>
> Tom (Y5): I like using maths apps because instead of using paper we can explain faster. Using apps makes it easy for me, because they are fun and they are easy to use.
>
> Jay (Y6): Apps like *Explain Everything* and *Math Shake* let you have a creative and fun learning experience.

There is also a suggestion there of the recording of the screencast facilitating a more confident approach in some students. This was an aspect of the recording noted by several students. For instance:

> Mel (Y5): I enjoyed using the apps because I can confidently record my voice and feel OK with others hearing my recordings.
>
> Ella (Y6): *Math Shake*—because it has helped me to be more confident in my math by (me) reading the problems and showing how I work it out.

The benefits for the students' mathematical learning derived from the affordances of the apps reshaping the mathematics learning experience, and motivating students to engage positively with the mathematics phenomena, are difficult to measure. Nonetheless, various mathematics education research in the affective domain suggests the positive influence of student motivation on mathematical learning (e.g., Attard, Ingram, Forgasz, Leder, & Grootenboer, 2016). The student blog entries also indicated this relationship between the affective and cognitive aspects of learning in mathematics. For example:

> Jake (Y5): I enjoyed using the apps because they were exciting to work with. They helped me learn new things. *Math Shake* can help you to challenge yourself.
>
> Jackie (Y6): I enjoyed using the apps because they helped me make my learning better because they showed me different learning skills and strategies.

The data were likewise suggesting that there was an element of student autonomy in the selection and the ways that the apps were used. This is an important aspect, as students being more engaged in the mathematics learning process, and becoming more self-directed, has the potential to facilitate more self-directed learning. Some students were positive about the apps enabling more intrinsic motivation. For example:

Tui (Y6): Some apps challenged me and I really like that.

8.6 Discussion

Teacher and student responses pointed to the acknowledged potential of the iPad use in manipulating objects dynamically onscreen. The teacher spoke of acting directly with the object, in this case the mathematics, and the students related to tapping and drawing on the screen. Students also commented on using the different digital tools to solve their problems, and so engaging with simultaneous linking. In this way, the simultaneous linking and dynamic affordances were acknowledged as part of the emerging theme related to affordances. Some students reflected on examples of new learning through the use of the app, but more clearly the use of screencasting and voice recording had advantages in motivating students and in increasing confidence. As such, the use of the app supports findings regarding simultaneous linking and dynamic affordances and affective aspects such as motivation, enjoyment and confidence.

Furthermore, the screencasting feature of the app was seen to introduce new representations. It seemed that the simultaneous dynamic visual recording (drawing, using images, manipulating digital tools, and writing symbols and words) along with speaking, created a dynamic aural-visual representation. This resonates with Moyer-Packenham and Westenskow (2013) simultaneous linking affordance, as the links between different representations influenced the articulation of mathematical ideas and thinking, and by inference, the students' understanding to some extent.

Not only did the teachers and students note that having the multi-representations being used and linked simultaneously was a way to show the thinking processes in solving a word problem, it appeared that, through pausing and editing, the students took time in preparing and perfecting their recordings. They were able to reflect on what had been said and think about what to say next. The teachers spoke of the students creating their own content and language and of having a safe space to do this. There was a creative element to their work in *Math Shake*. As the students created their own representations and linked these in an individual way when articulating solutions, there was a personal aspect to their screencasts both in the appearance and in the processing of the mathematics (creative variation). This suggested that some differentiating of the learning had taken place as well, with students following more individual learning trajectories.

The student comments suggested the inter-relationships between the opportunities that the apps afforded and the creating of both individual and group discussion, presentations of solutions, and strategies. They used linked multi-representations (simultaneous linking) to present their thinking. The recordings were then available for the individual students to refer back to as a dynamic aural-visual representation of their own thinking, as well as a representation to share with other students for discussion and with their teachers for assessment and feedback.

8.7 Conclusions

Previous research has suggested that MT can offer affordances that might reshape the learning experience. Through a more immediate and explicit interaction, students can manipulate and create dynamic images on the screen to explore mathematical objects. The use of screencasting along with voice recording furthers the simultaneous linking affordance by introducing dynamic aural-visual representations that are created by the students themselves. The process of verbalisation, along with the manipulation of images, drawing and writing in a safe environment that the student controls, would suggest a new learning experience. Further study of this new learning experience is needed to understand how it might be reshaping the learning of mathematics. Insight into the ways screencasting might interact with other forms of technology, the mathematics, the students, and the associated social elements, might enable us to better understand the influence of this learning experience on students' mathematical thinking and understanding.

Math Shake offered three of the affordances that Moyer-Packenham and Westenskow (2013) identified. While simultaneous linking resonates with the multi-representation affordance, it also contains elements of instantaneous feedback, where the students' actions and the ensuing onscreen transformations are directly linked. Their notion of creative variation connects to the emerging project themes of personalization and differentiation of the learning, while the affordances of *Math Shake* constrained the learners' focus on particular virtual learning objects and processes.

Complementing the potential of the affordances of the apps to influence the learning experience, is the pedagogical approach taken by the teacher. The tasks given, and the classroom culture that the teacher develops, are key elements of the learning. The students might have apps available but not necessarily engage with them in ways that optimise the mathematics learning. These aspects were implicit in the interview data rather than being commented on directly. However, coupled with observations in the classroom, the importance of the teacher's pedagogical approach seemed clear. For instance, the task of creating a presentation of their solution and strategies opened opportunity to explore and discuss the mathematics through utilising the affordances of *Math Shake.* The classroom culture including reflecting on processes, exploring in collaborative groupings, and sharing outcomes in a safe learning environment are all conducive to developing mathematical thinking. The seamless engagement with the app, especially in the one-to-one iPad class, was also part of the class culture. The teachers and the students were excited about the learning, and enhanced motivation, cognitive risk taking and confidence were also evident. It appears that it is the assemblage of digital media, learners, mathematics and environmental elements that reshape and hence influence the learning.

The emerging themes of the project and a corresponding framework are being co-constructed with the teacher co-researchers. This process will continue during the second year of the project as the teacher co-researcher group expands into a group of 12 teachers and classes, representing a wide range of year levels, with the teachers having varying levels of experience and expertise in using mobile technologies in their mathematics programmes. This will bring more diverse perspectives as the initial themes and interpretations evolve through iterations of action, reflection and changing perspectives.

We wish to acknowledge the support of the Teacher and Learning Research Initative.

References

Attard, C. (2015). Introducing iPads into primary mathematics classrooms: Teachers' experiences and pedagogies. In M. Meletiou-Mavrotheris, K. Mavrou, Katerina, & E. Paparistodemou (Eds.) *Integrating touch-enabled and mobile devices into contemporary mathematics education* (pp. 193–213*).* Hersley PA, USA: IGI Global.

Attard, C., Ingram, N., Forgasz, H., Leder, G., & Grootenboer, P. (2016). Mathematics education and the affective domain. In K. Makar, S. Dole, J. Visnovska, M. Goos, A. Bennison, & K. Fry (Eds.), *Research in mathematics education in Australasia 2012–2015*. Singapore: Springer.

Ainsworth, S. E., Bibby, P. A., & Wood, D. J. (1998). Analyzing the costs and benefits of multi-representational learning environment. In M. W. van Someren, P. Reimann, H. P. A. Boshuizen, & T. de Jong (Eds.), *Learning with multiple representations* (pp. 120–134). Oxford, U.K.: Elsevier Science.

Anderman, E. (2002). School effects on psychological outcomes during adolescence. *Journal of Educational Psychology, 94*(4), 795–809.

Arcavi, A., & Hadas, N. (2000). Computer mediated learning: An exam-ple of an approach. *International Journal for Computer for Mathematics Learning, 5,* 25–45.

Brown, J. (2005). Affordances of a technology-rich teaching and learning environment. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice*. *Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia*, *Melbourne* (pp. 177–184). Sydney: MERGA.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies*. Modeling, Experimentation and Visualisation, New York, NY: Springer.

Calder, N. S. (2011). *Processing mathematics through digital technologies: The primary years*. Rotterdam, The Netherlands: Sense.

Calder, N. S., & Campbell, A. (2016). Using mathematical apps with reluctant learners. *Digital Experiences in Mathematics Education.* https://doi.org/10.1007/s40751-016-0011-y.

Carr, J. (2012). Does math achievement h'APP'en when iPads and game-based learning are incorporated into fifth-grade mathematics instruction? *Journal of Information Technology Education, 11,* 269–286.

Deci, E., & Ryan, R. (2000). The 'what' and 'why' of goal pursuits: Human needs and the self-determination of behavior. *Psychological Inquiry, 11*(4), 227–268.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge: Cambridge University Press.

Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics, 66*(3), 317–333.

Fenwick, T., & Edwards, R. (Eds.). (2012). *Researching education through actor network theory*. Sussex: Wiley-Blackwell.

Frenzel, A., Pekrun, R., & Goetz, T. (2007). Perceived learning environment and students' emotional experiences: A multilevel analysis of mathematics classrooms. *Learning and Instruction, 17*(5), 478–493.

Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, acting, and knowing: Toward an ecological psychology* (pp. 67–82). Hillsdale, NJ: Lawrence Erlbaum.

Hannula, M. (2006). Motivation in mathematics: Goals reflected in emotions. *Educational Studies in Mathematics, 63*(2), 165–178.

Hänze, M., & Berger, R. (2007). Cooperative learning, motivational effects, and student characteristics: An experimental study comparing cooperative learning and direct instruction in 12th grade physics classes. *Learning and Instruction, 17*(1), 29–41.

Hegedus, S. (2013). Young children investigating advanced mathematical concepts with haptic technologies: Future design perspectives. *The Mathematics Educator, 10*(1&2), 87–107.

Hennessy, S. (2014). *Bridging between research and practice: Supporting professional development through collaborative studies of classroom teaching with technology*. Rotterdam, The Netherlands: Sense.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. Bishop, M. Clements, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Second international handbook of mathematics education* (Vol. 1, pp. 323–349). Dordrecht, NL: Kluwer Academic Publishers.

Hunter, R., & Anthony, G. (2012). Designing opportunities for prospective teachers to facilitate mathematics discussions in classrooms. In J. Dindyal, L. Cheng & S. Ng (Eds.), *Proceedings of the 35th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 354–361). Singapore, SG: MERGA.

Melhuish, K., & Falloon, G. (2010). Looking to the future: M-learning with the iPad. *Computers in New Zealand Schools: Learning, Leading. Technology, 22*(3), 1–15.

Meyer, B. (2015). iPads in inclusive classrooms: Ecologies of learning. In P. Isaias, J. M. Spector, & D. Ifenthaler (Eds.), *E-learning systems, environments and approaches theory and implementation*. Dordrecht: Springer International Publishing.

Moyer, P. S., Bolyard, J. J., & Spikell, M. A. (2002). What are virtual manipulatives? *Teaching Children Mathematics, 8*(6), 372–377.

Moyer-Packenham, P. S., Baker, J., Westenskow, A., Anderson, K., Shumway, J., Rodzon, K., & Jordan, K. (2013). A study comparing virtual manipulatives with other instructional treatments in third- and fourth-grade classrooms. *Journal of Education, 193*(2), 25–39.

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments, 4*(3), 35–50.

Philip, T. M., & Garcia, A. (2014). Schooling mobile phones: Assumptions about proximal benefits, the challenges of shifting meanings, and the politics of teaching. *Educational Policy, 29*(4), 676–707.

Pintrich, P. (2003). A motivational science perspective on the role of student motivation in learning and teaching contexts. *Journal of Educational Psychology, 95*(4), 667–686.

Sacristán, A. I., & Noss, R. (2008). Computational construction as a means to coordinate representations of infinity. *International Journal of Computers for Mathematical Learning*. https://doi.org/10.1007/s10758-008-9127-5.

Sandholtz, J. H., Ringstaff, C., & Dwyer, D. C. (1997). *Teaching with technology: Creating a student centred classroom*. New York: Teachers' College Press.

Schukajlow, S., Leiss, D., Pekrun, R., Blum, W., Müller, M., & Messner, R. (2012). Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. *Educational Studies in Mathematics, 79*(2), 215–237.

Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning number with *TouchCounts:* The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning, 19*(1), 81–99.

Sullivan, P., Tobias, S., & McDonough, A. (2006). Perhaps the decision of some students not to engage in learning mathematics in school is deliberate. *Educational Studies in Mathematics, 62* (1), 81–99.

Wæge, K. (2010). Students' motivation for learning mathematics in terms of needs and goals. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of CERME 6* (pp. 84–93). Lyon, FR: CERME & INRP.

Chapter 9
Empirical Evidence for Benefit? Reviewing Quantitative Research on the Use of Digital Tools in Mathematics Education

Paul Drijvers

Abstract The benefit of using digital tools in education, and in mathematics education in particular, is subject to debate. To investigate this benefit, we focus on effect sizes on student achievement reported in reviews of experimental and quantitative studies. The results show significant positive effects with modest effect sizes. Possible causes for this are discussed and illustrated with one case study. We wonder if the review studies capture the subtlety of integrating digital tools in learning as much as qualitative studies do, and question their potential to address the "how" question. As a conclusion, a plea is made for replication studies and for studies that identify decisive factors through the combination of a methodologically rigorous design and a theoretical foundation in domain-specific theories from mathematics didactics.

Keywords Digital technology · Mathematics education · Effect size
Student achievement

9.1 Introduction

The benefit of using digital tools in education, and in mathematics education in particular, is subject to debate. For example, the header of a September 2015 BBC news item was "Computers 'do not improve' pupil results, says the OECD".[1] A Dutch news site[2] provided an even stronger claim: "Poorer school performance through increased computer use." Both items were based on a report by the

[1] 15 September 2015, http://www.bbc.com/news/business-34174796.
[2] 15 September 2015, http://nos.nl/artikel/2057772-slechtere-schoolprestaties-door-meer-computer gebruik.html.

P. Drijvers (✉)
Freudenthal Institute, Utrecht University, Utrecht, The Netherlands
e-mail: p.drijvers@uu.nl

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_9

Organisation for Economic Co-operation and Development (OECD) on student achievement and the use of computers, that just had been published (OECD, 2015). Indeed, the results of this study included negative correlations between mathematics performance and computer use in mathematics lessons and led to conclude that there is little evidence for a positive effect on student achievement:

> Despite considerable investments in computers, internet connections and software for educational use, there is little solid evidence that greater computer use among students leads to better scores in mathematics and reading. (OECD, 2015, p. 145)

Even if correlations do not imply causality, the "little solid evidence" in the above OECD quote at least challenges the research community. Other voices, however, point out the benefits of using digital technology in education. For the case of mathematics education, the National Council of Teachers of Mathematics claimed that we cannot and should not neglect digital tools: "Technology is an essential tool for learning mathematics in the 21st century" (NCTM, 2008, p. 1). This quote recognizes the potential of digital technology for mathematics teaching and learning, including a possibly changing focus in mathematics curricula towards conceptual understanding and higher order thinking skills. This potential is underpinned by research findings, such as the ones reported by Ronau et al.:

> Over the last four decades, research has led to consistent findings that digital technologies such as calculators and computer software improve student understanding and do no harm to student computational skills. (Ronau et al., 2014, p. 974)

Others (e.g., Hoyles & Lagrange, 2010; Hoyles & Noss, 2003) took a more nuanced stance, claiming that it is the *how* that determines the effect of ICT use on performance in mathematics education: how to design effective ICT environments and how to "exploit" them for student learning?

These different claims and opinions with respect to *if*, *how*, and *how much* to use digital tools in mathematics education raise several questions. What does empirical research really tell us about the effects on student performance of using digital technology in mathematics education? Does the answer depend on student grade, on the mathematical topic, on the type, size, scale and duration of the intervention? Do we see trends in research findings on these questions over the recent decades according to review studies? How can we explain the differences between studies? Is it possible anyway to answer such overarching questions through the review of empirical studies? What are the limitations of this approach? These questions form the core of this chapter, and will lead to considerations on the relationship between qualitative studies, addressed in more detail in Heid's chapter in this volume on the one hand, and quantitative studies and review studies on the other. A reflective stance is taken; as such, this chapter has an essay-like character rather than a traditional research paper format.

In this chapter, we will first revisit and synthesize the results of five important review studies on empirical, quantitative studies on the use of digital technology in mathematics education (Sect. 9.2). This section is central in the chapter. To illustrate the difficulty to find convincing evidence of the potential of digital tools in such (too?) general review studies, Sect. 9.3 describes one empirical study that was grounded in qualitative work and well-focused, but not successful in terms of student performance. Some possible causes are discussed. In the reflective Sect. 9.4, we reflect on the interpretation of effect sizes, the subtlety of using digital tools in mathematics education and some methodological issues. Finally, in the concluding Sect. 9.5 limitations of review studies are addressed, and a plea is made for an appropriate integration of qualitative and quantitative methods, and for methodologically rigorous studies grounded in theories on the learning of mathematics.

9.2 Revisiting Review Studies

9.2.1 Some Relevant Studies Before 2010

Of course, the question of the benefits of integrating digital tools in mathematics education is not new and has been investigated before. In this section, we briefly review early studies in the field, that is, studies that were published before 2010 that try to summarize research findings in the field. In one of the first synthesizing studies, Heid (1997) provided an overview of principles and issues of the integration of digital technology, and sketched the landscape of the different types of tools and their pedagogical potential. On the topic of using handheld graphing technology in particular, Burrill et al. (2002) reported on 43 studies and concluded that these devices can be important in helping students develop a better understanding of mathematical concepts; this conclusion, however, is not quantitatively underpinned. Ellington (2003, 2006) also focused on graphing calculators, which were indeed important in the implementation of digital tools in mathematics education at the end of the 20th century. Her review of 54 studies showed an improvement of students' operational skills and problem-solving skills when calculators are an integral part of testing and instruction. The effect sizes, however, were small—which is not uncommon in educational research. Lagrange, Artigue, Laborde and Trouche (2003) developed a multi-dimensional framework to review a corpus of 662 mostly qualitative research studies on the use of technology in mathematics education and to investigate the evolution of research in the field, to identify trends, without explicitly addressing learning outcomes. Kulik (2003) did address learning outcomes and reported an average effect size of $d = 0.38$ in 16 studies on the effectiveness of integrated learning systems in mathematics.[3]

[3]The effect sizes reported here are means to express the differences between two populations in terms of their pooled standard deviation. The most commonly used methods are Cohen's d and Hedge's g. The difference between the two is important for small sample sizes, but neglected in this paper as we do not want to get into measurement details too much. The d reported here means that the average difference between experimental group and the control group equals 0.38 of their pooled standard deviation, which is considered a weak to medium effect.

Two subsequent large-scale experimental studies by Dynarski et al. (2007) and Campuzano, Dynarski, Agodini and Rall (2009), however, concluded that the effects of the use of digital tools in grade 9 algebra courses was not statistically different from zero. For the use of computer algebra systems, Tokpah (2008) found significant positive effects with an average of $d = 0.38$ over 102 effect sizes.

Altogether, these early studies provided mixed findings on the effect of using digital tools in mathematics education and showed different degrees of quantitative evidence. Also, the dissemination of digital tools and the experience teachers and students had with their use in class were limited by that time. These considerations provide ample reason to look at more recent studies in more detail.

9.2.2 Five More Recent Review Studies

To further investigate more recent findings, we now focus on five review studies that provide information on the effect of using digital technology in mathematics education through reporting effect sizes.[4] The selection of these five studies is not based on a systematic database survey, but on an informal literature and Google Scholar search using terms such as review study, mathematics education, and digital technology. It is interesting to notice that the studies included in each of these review studies are very different and hardly show any overlap, due to different criteria and foci.

The first one is the study by Li and Ma (2010). It reviewed 46 studies on using five different types of computer technology (tutorials, communication media, exploratory environments, tools, and programming languages) on mathematics education in K–12 classrooms, reporting in total 85 effect sizes. The researchers found a statistically significant effect with a weighted average effect size of $d = 0.28$, which led them to report "… a moderate but significant positive effect of computer technology on mathematics achievement" (Li and Ma 2010, p. 232). The reported unweighted average effect size, $d = 0.71$, seems less appropriate as it does not take into account the number of students involved. Additional findings were that higher effect sizes were found in primary education compared to secondary, and in special education compared to general education. Also, effect sizes were bigger in studies that used a constructivist approach to teaching, and in studies that used non-standardized tests. Differences with respect to the five types of technology were not found.

The second review study by Rakes, Valentine, McGatha, and Ronau (2010) focused on algebra in particular. The authors included two studies that were also in the Li and Ma (2010) study, and found 109 effect sizes. The interventions were categorized; here we only report on the categories Technology tools (with calculators, graphing calculators, computer programs, and java applets as categories) and

[4]We addressed three of them in earlier publications (Drijvers, 2014, 2015).

Technology curricula, being computer-based curricula for use in onsite classes, online courses, and tutoring curricula. The average weighted effect sizes for these two categories were $d = 0.151$ and $d = 0.165$, respectively. Over all categories, the authors concluded that interventions focusing on conceptual understanding provide about twice as high effect sizes as the interventions focusing on procedural understanding. Also, they noted that interventions over a small period of time may have significant effect, and that the grain size differences in interventions (whole-school study versus single-teacher interventions) did not make a significant difference.

The third review study by Cheung and Slavin (2013) took into account 74 effect sizes from 45 elementary and 29 secondary studies on K–12 mathematics. The primary studies included one study that was also part of the Rakes et al. review; the secondary studies category included the two studies addressed in the previous paragraphs. The average effect size was $d = 0.16$. The authors' final conclusion refers to a modest difference: "Educational technology is making a modest difference in learning of mathematics. It is a help, but not a breakthrough." (Cheung & Slavin, 2013, p. 102). Some additional findings are worth mentioning. First, the overall effectiveness of educational technology did not improve over time. Second, like Li and Ma (2010), the authors found higher effect sizes in primary than in secondary education. Third, lower effect sizes were found in randomized experiments compared to quasi-experimental studies. Fourth and final, effect sizes in studies with a large number of students were smaller than in small-scale studies.

The fourth review study by Steenbergen-Hu and Cooper (2013) focused on the effectiveness of intelligent tutoring systems (ITS) on K–12 students' mathematical learning. The authors' corpus of studies had four studies in common with the Rakes et al. (2010) study. The 65 effect sizes included in their study ranged from $g = 0.01$ to $g = 0.09$. This led the authors to careful conclusions: "ITS had no negative and perhaps a very small positive effect on K–12 students' mathematical learning relative to regular classroom instruction" (Steenbergen-Hu & Cooper, 2013, p. 982). Additional findings were that the effects of the ICT interventions proved less big in cases of long interventions (more than one school year). Also, the general student population seemed to benefit more from the ITS use than their low achieving peers, which questions the potential of ITS for reducing achievement gaps.

The fifth and final study we address here is a meta-study carried out by Sokolowski, Li, and Willson (2015). The authors particularly investigated the use of exploratory computerized environments (ECEs) for grade 1–8 mathematics. The interventions focused on digital tools for supporting word problem solving and exploration. The average of the 24 effect sizes included was $g = 0.60$, which is a moderate effect size. Additional findings were that the effects were most positive in middle school grades (grades 6–8). Concerning the mathematical domain, the effect sizes tended to be slightly higher for geometry than for algebra. In terms of teaching styles, teacher-based support proved to be more effective than computer-based support, which led the authors to claim that in spite of the positive effects, "this finding does not diminish the importance of good teaching" (Sokolowski, Li, & Willson, 2015, p. 13).

Table 9.1 Effect sizes reported in five review studies

Study	Number of effect sizes	Average effect size	Global conclusion
Li and Ma (2010)	85	d = 0.28 (weighted)	Moderate significant positive effects
Rakes et al. (2010)	109	d range 0.151–0.165	Small but significant positive effects
Cheung and Slavin (2013)	74	d = 0.16	A positive, though modest effect
Steenbergen-Hu and Cooper (2013)	61	g range 0.01–0.09	No negative and perhaps a small positive effect
Sokolowski, Li, and Willson (2015)	24	g = 0.60	A moderate positive effect size

Table 9.1 summarizes the findings of the five review studies with respect to the effect sizes and their global conclusion. The overall image is that the use of digital technology in mathematics education can have a significant positive effect, with effect sizes ranging from small to moderate. The average of these (average!) effect sizes is about 0.2, and we notice quite some variation: comparing the value for g ranging from 0.01 to 0.09 in one study and being 0.6 in another, the results do not really converge. On the one hand, this is somewhat disappointing; on the other, the different studies are based on different sets of research studies with different foci. Meanwhile, we conclude that these studies do not provide an overwhelming evidence for the effectiveness of the use of digital tools in mathematics education.

Of course, this summary of review studies provides a highly (or even too?) aggregated view and neglects detailed differences. Can we learn more about decisive factors that explain these different effects? A first possible factor is *student age and student level*. Sokolowski, Li and Willson (2015) found the effects to be most positive in middle school grades (grades 6–8), whereas Steenbergen-Hu and Cooper found the highest effects in elementary school (grades K–5). Both Li and Ma (2010) and Cheung and Slavin (2013) reported higher effect sizes in primary education compared to secondary. The former also claimed that effects are higher in special education compared to general education. This is in line with the finding by Steenbergen-Hu and Cooper (2013), who concluded that the general student population seemed to benefit more from ITS use than low achieving students. In sum, evidence of benefit is larger in primary and lower secondary education, and it is not self-evident that digital tool use helps to bridge the gap between high and low achieving students. We can conjecture about the reasons for the latter point: if digital environments provide rich learning opportunities, it seems likely that high achieving students manage to better exploit these opportunities. As for grade level, we do not know why digital tools would work better for younger students; maybe other factors such as the availability of the tools and the mathematical sophistication needed play a role here?

This brings us to the second factor: the *mathematical domain*. Steenbergen-Hu and Cooper (2013) found bigger effect sizes for basic math than for algebra. The Rakes et al. (2010) study showed low effect sizes in the domain of algebra, whereas a review study by Chan and Leung (2014) reported a high effect size (d = 1.02) for the use of Dynamic Geometry Systems. These findings are in line with Sokolowski, Li, and Willson (2015), who reported effect sizes to be slightly higher for geometry than for algebra. Again, we wonder why this would be the case. Is using digital tools for geometry more natural, and are geometry tools more intuitively used than algebra tools that may require more syntax? These questions clearly need further investigation.

A third possible factor concerns *learning goals and teaching style*. Li and Ma (2010) found bigger effect sizes in studies that used a constructivist approach to teaching. More or less in line with this, Rakes et al. (2010) reported the largest effect sizes in studies on conceptual understanding rather than on procedural skill acquisition. Sokolowski, Li, and Willson (2015) found high effect sizes in studies explicitly focusing on word problem solving and exploration, and teacher-based support in these studies was more effective than computer-based support. Even if these findings are somewhat eclectic, they suggest that using digital tools can be effective in interventions focusing on higher-order learning goals, such as conceptual insight and problem solving, with a constructivist view on learning and with an important role for the teacher. These findings are interesting as they may challenge the view of digital tools mainly supporting skill acquisition with no important role for the teacher.

Possible external factors that might impact on learning effects are the intervention's *duration and sample size*. Rakes et al. (2010) showed that short interventions may have significant effect, and Steenbergen-Hu and Cooper (2013) claimed that interventions shorter than one school year are more effective than longer ones. It seems that short interventions do not necessarily lead to weaker effects. With respect to sample size, Cheung and Slavin (2013) found that effect sizes in studies with a large number of students were smaller than in small-scale studies. Steenbergen-Hu and Cooper (2013) reported higher effect sizes for studies with less than 200 participants. In contrast to this, Rakes et al. (2010) found that single-teacher interventions were not more effective than whole-school interventions. Apparently, the picture with respect to sample size remains unclear.

As a final factor, we briefly address the *development over time*. Over the last decades, digital tools for mathematics have become more sophisticated, ICT infrastructures have drastically improved both in schools and at home, and both teachers and students have become more familiar with using ICT in education. Therefore, one might expect the benefits for student achievement to increase over time. If we consider these review studies in more detail, however, we agree with Cheung and Slavin (2013) and with Steenbergen-Hu and Cooper (2013) that the effect sizes reported in the different research reports did not significantly increase over time. A possible explanation might be that there indeed is a positive development over time, but that it is compensated by other factors, such as more rigorous study designs and methods, and bigger sample sizes.

One might wonder if *publication bias* might play a role in the review studies addressed above. Would it be possible that actual effect sizes are smaller, due to the fact that studies that did not result in significant effects were not published? Most review studies took this into account. For example, both Steenbergen-Hu and Cooper (2013) and Sokolowski, Li, and Willson (2015) found little evidence that publication bias had impact on their findings.

All in all, the review studies show that the use of digital technology in mathematics education can have a significant positive effect, with effect sizes ranging from small to moderate and with considerable variation in size. Benefits seem to be best for younger students (primary level or early secondary), better for geometry than for algebra, effective in interventions focusing on higher-order learning goals, and already beneficial in short interventions. Over the last decades, effect sizes do not increase and publication bias does not seem to play a role in this picture.

9.3 An Example: The Case of Applets for Algebra

The picture provided by review studies, however, is limited. Different types of interventions, students, mathematical domains and digital tools are merged into one global average effect size. Would this merging of studies with different perspectives explain the modest overall benefits in terms of student performance? In this section, we counterbalance the global picture by briefly presenting one single empirical study that reported no significant results. It illustrates that, in spite of a focus on one mathematical topic and one type of digital tool and the qualitative preparatory study, providing empirical evidence for the benefits of using digital tools is not straightforward. Some tentative explanations will be provided.

In the study (Drijvers, Doorman, Kirschner, Hoogveld, & Boon, 2014), two online algebra modules were used in 8th grade. The modules were designed in the Digital Math Environment, which proved to be successful in improving student achievement in algebra in grade 12 (Bokhove & Drijvers, 2012). Also, teachers had reported success while implementing the online materials in lower grades. Figure 9.1 shows a task from one of the modules.

The study had an experimental design, in which each of the involved teachers taught to two classes in parallel, each randomly assigned to the experimental condition of using the online modules, or to the control condition of regular teaching. Figure 9.2 shows the results of the pretest, the intermediate test (Post_Linear), the posttest (Post_Quadractic) and the two retention tests, all administered with paper-and-pen. In spite of the earlier positive experiences with these types of modules, the results show that the experimental condition did not lead to students outperforming their peers in the control condition. The experimental group did not catch up the small initial (and coincidental) lag; indeed, this gap became significantly larger in the final retention test.

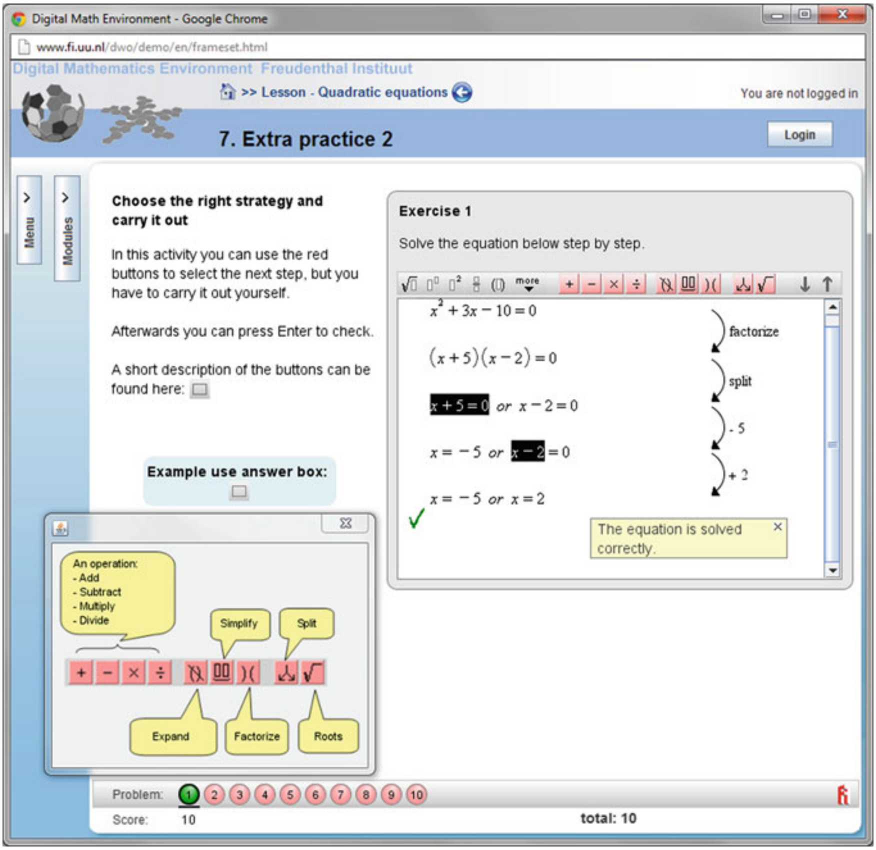


Fig. 9.1 Algebra task in the Digital Math Environment

As a possible explanation for these findings, the authors mention a spill-over effect. All participating teachers taught one control and one experimental class, and they may have picked up pedagogical ideas from the online intervention and used these in the control classes as well. Such a spill-over effect is well-documented in research literature (e.g., see Creemers, Kyriakides, & Sammons, 2010). A second possible explanation is that the work on the online tasks was not an adequate preparation for more complex tasks. Third, the feedback provided in the digital environment might have lacked quality, and, finally, the integration of paper-and-pen skills and digital practice might not have been optimal, so that transfer to the “traditional media” was hindered. In short, in spite of a careful experimental design and an environment that had proven to be useful in other settings, the researchers did not find a positive effect.

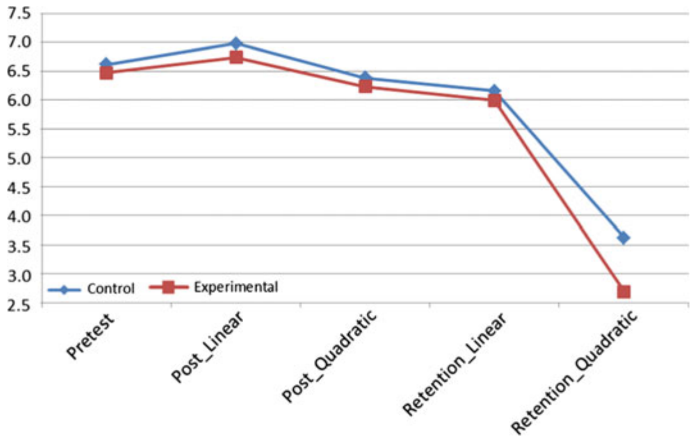


Fig. 9.2 Average grades for control and experimental group (N = 842)

9.4 Reflection

To reflect on the above findings from quantitative studies, we first discuss the interpretation of effect sizes and next address two other factors that may play a role: the too general claims made, which ignore the subtlety of using digital tools for learning, and the methodological weaknesses that some studies suffer from.

9.4.1 Interpreting Effect Sizes

First, let us notice that the results from experimental and quantitative studies are more positive than the correlational findings from the OECD (2015) study cited in this chapter's introduction. However, the effect sizes, with their overall average in the order of $d = 0.2$, are modest. How do we interpret them? Higgins et al. (2012) claimed that technology-based interventions produce just slightly lower effect sizes than other types of educational interventions not involving digital tools, thus suggesting that these results are not that disappointing. Slavin (2016) supported this stance, pointing out that the interpretation of an effect size mainly depends on two factors: the sample size and whether or not the students are assigned randomly to the different conditions. For a number of large scale studies with random assignment on different topics, Slavin found an average effect size of 0.11, suggesting that it is very optimistic to expect more. From this perspective, the reported effect sizes are not that low. In the meanwhile, the interpretation of effect sizes should be done with care and is subject to debate, as is the case for the interpretation of significant *p*-values.[5]

9.4.2 The Too General Claims that Ignore the Subtlety of Using Digital Tools for Learning

Would we not all agree that research findings such as “The use of paper-and-pen has a positive effect on student achievement” would be too general? Why, then, would we try to find evidence for similar claims on the use of ICT? It makes sense to assume that digital technology is not a panacea, and that its effectiveness will largely depend on particular implementations and situations. The following two quotes underline that the effect of ICT in mathematics education is a subtle matter and will depend to an important extent on the specific technological application, the educational setting and the orchestration by the teacher. It is the “how” that counts!

> The range of impact identified in these studies suggests that it is not whether technology is used (or not) which makes the difference, but how well the technology is used to support teaching and learning. There is no doubt that technology engages and motivates young people. However this benefit is only an advantage for learning if the activity is effectively aligned with what is to be learned. It is therefore the pedagogy of the application of technology in the classroom which is important: the how rather than the what. (Higgins, Xiao, & Katsipataki, 2012, p. 3)

> There have been several reviews of the benefits of ICT to student learning in mathematics that suggest positive effects from the use of digital technology. […] However, the type and extent of the gains are a function of how the technology is used in the teaching of mathematics. (Drijvers, Monaghan, Thomas, & Trouche, 2015, p. 15).

If we agree that the learning of mathematics is a complex domain and that we need to know more about the factors that determine the contribution of digital tools to it, it is important that research is grounded in theoretical knowledge from domain-specific mathematics pedagogy and from man-machine interaction. To mention just some possible perspectives, theories on reification (Sfard, 1991), on emergent modeling (Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2012), or on instrumental genesis may offer such a theoretical basis (Drijvers, Kieran, & Mariotti, 2010). Educational research on the use of digital tools for mathematics education that is not based on domain-specific didactical knowledge may miss opportunities to discover decisive factors.

As an aside, we should note that didactical knowledge and practice may also change under the influence of digital technology. In fact, this is what the OECD mentions as a possible explanation for their surprising findings:

[5]For a current debate on *p*-values see http://www.statslife.org.uk/news/2116-academic-journal-bans-p-value-significance-test.

> … we have not become good enough at the kind of pedagogies that make the most of technology. […] Technology can amplify great teaching but great technology cannot replace poor teaching (OECD 2015, pp. 3–4).

In this line of reasoning, an important research question would be "What type of student achievement can be improved through which type of use of which kind of digital tools?" rather than the very general "Does the use of digital tools improve student achievement?"

9.4.3 Methodological Limitations

In this chapter, we limited ourselves to review studies that summarize the results from experimental studies. The body of such experimental studies shows some remarkable methodological characteristics. First, replication studies have hardly ever been carried out. Why is this the case? If replication studies had been done, would we encounter similar replication issues as in the field of cognitive and social psychology?[6] Do we manage to control relevant variables? Second, it is interesting to notice that smaller studies tend to report bigger effect sizes than larger ones and that the reported effect sizes do not seem to increase over time. This suggests that scaling up successful interventions identified in effective small-scale studies may not be so easy. As far as the trend over time is concerned, the criteria for publication and for inclusion in review studies seem to be getting higher, and this is indeed what we should strive for according to Ronau and colleagues, who in a recent study on the quality of 480 mathematics education technology dissertations argued for higher quality in both research reports and reviews:

> The mathematics education technology research community must in turn begin to demand greater quality in its published studies, through both how researchers write about their own studies and how they review the works of others. (Ronau et al. 2014, p. 1002)

A possible cause of the lack of positive trends in reported effect sizes, therefore, might be these higher methodological standards, which might filter out the studies that report high effect sizes. From a methodological point of view, more rigor in research methods to improve the quality of our results is welcomed of course.

[6]See, for example, http://www.theguardian.com/science/2015/aug/27/study-delivers-bleak-verdict-on-validity-of-psychology-experiment-results.

9.5 Conclusion

In the introduction, we raised the question of what empirical research really tells us about the effects on student performance of using digital technology in mathematics education. The literature review revealed mixed results. The OECD correlational study showed little evidence for benefit. Experimental studies, and their review studies in particular, reported significant positive effects, with average effect sizes ranging from small to moderate with considerable variation. Compared to effect sizes reported for other types of innovative interventions, the evidence for benefit is not overwhelming. Also, insight into factors that are decisive for the (lack of) positive benefit of the use of digital tools is limited. Younger students (primary level or early secondary) seem to benefit more, results are better for geometry than for algebra, interventions focusing on higher-order learning goals may be effective, and short interventions may be beneficial. Over the last decades, effect sizes do not increase and publication bias does not seem to play a role in this picture.

Of course, the above conclusion has some important limitations. First, review studies are based on studies that themselves are older, and one might wonder if the picture has changed over, say, the last five years. The fact that effect sizes so far have not been increasing, however, does not favor this argument. Second, we focus on experimental, quantitative studies and neglect qualitative studies and studies that follow a design research paradigm, whereas such studies can contribute to the body of knowledge, and in many cases take an in-depth view on student learning and are firmly grounded in theories from the field of mathematics didactics.[7] The study described in Sect. 9.3 shows that there can be many reasons why the effect of using of ICT in mathematics education may not show up. A third limitation of the type of review studies revisited is that these studies do not differentiate between educational levels, types of technology used, and other educational factors that may be decisive. Rather, they provide an overview without nuances, which may cause us to miss important insights in the phenomenon.

In spite of these limitations, the conclusion is that evidence for the benefit of using technology in mathematics education from experimental studies is modest and that evidence-based insights in factors that affect these benefits are limited. What we need on our research agendas, therefore, are studies (including replication studies) that focus on the identification of decisive factors that determine the eventual benefits in specific cases. Such studies should on the one hand be methodologically well designed according to the standards from educational science, and on the other hand be strongly based in sound theoretical foundations from domain-specific mathematics didactics, as to better address the "how"-question. In many cases, preliminary qualitative studies may show to be indispensable to set up learning arrangements that also will result in positive effects in experimental studies. To combine the best of both worlds is the challenge we are facing.

---

[7]The findings from qualitative studies are addressed in the chapter by Heid in this volume.

References

Bokhove, C., & Drijvers, P. (2012). Effects of a digital intervention on the development of algebraic expertise. *Computers & Education, 58*(1), 197–208.

Burrill, G., Allison, J., Breaux, G., Kastberg, S., Leatham, K., & Sanchez, W. (2002). *Handheld graphing technology at the secondary level: Research findings and implications for classroom practice*. Dallas, TX: Texas Instruments Corp. http://education.ti.com/research.

Campuzano, L., Dynarski, M., Agodini, R., & Rall, K. (2009). *Effectiveness of reading and mathematics software products: Findings from two student cohorts—Executive summary* (NCEE 2009–4042). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.

Chan, K. K., & Leung, S. W. (2014). Dynamic geometry software improves mathematics achievement: Systematic review and meta-analysis. *Journal of Educational Computing Research, 51*(3), 311–325.

Cheung, A. C. K., & Slavin, R. E. (2013). The effectiveness of educational technology applications for enhancing mathematics achievement in K–12 classrooms: A meta-analysis. *Educational Research Review, 9,* 88–113.

Creemers, B. P. M., Kyriakides, L., & Sammons, P. (2010). *Methodological advances in educational effectiveness research*. London/New York: Routledge.

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: From repeated calculations to functional thinking. *International Journal of Science and Mathematics Education, 10*(6), 1243–1267.

Drijvers, P. (2014). Digital technology in mathematics education: A reflective look into the mirror. In J. Roth & J. Ames (Eds.), *Beiträge zum Mathematikunterricht 2014* (pp. 21–28). Münster: WTM-Verlag.

Drijvers, P. (2015). *Denken over wiskunde, onderwijs en ICT*. Inaugurele rede. [Thinking about Mathematics, Education and ICT. Inaugural lecture.] Utrecht: Universiteit Utrecht.

Drijvers, P., Doorman, M., Kirschner, P., Hoogveld, B., & Boon, P. (2014). The effect of online tasks for algebra on student achievement in grade 8. *Technology, Knowledge and Learning, 19,* 1–18.

Drijvers, P., Kieran, C., & Mariotti, M. A. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain* (pp. 89–132). New York/Berlin: Springer.

Drijvers, P., Monaghan, J., Thomas, M., & Trouche, L. (2015). *Use of technology in secondary mathematics. Final report for the international baccalaureate*. http://www.ibo.org/globalassets/publications/ib-research/technologyindpmathematicsfinalreport.pdf.

Dynarski, M., Agodini, R., Heaviside, S., Novak, T., Carey, N., & Campuzano, L. (2007). *Effectiveness of reading and mathematics software products: Findings from the first student cohort. Report to Congress.* Publication NCEE 2007–4005. Washington, DC: U.S. Department of Education.

Ellington, A. J. (2003). A meta-analysis of the effects of calculators on students' achievement and attitude levels in precollege mathematics classes. *Journal for Research in Mathematics Education, 34*(5), 433–463.

Ellington, A. J. (2006). The effects of non-CAS graphing calculators on student achievement and attitude levels in mathematics: A meta-analysis. *School Science and Mathematics, 106*(1), 16–26.

Heid, M. K. (1997). The technological revolution and the reform of school mathematics. *American Journal of Education, 106*(1), 5–61.

Higgins, S., Xiao, Z., & Katsipataki, M. (2012). *The impact of digital technology on learning: A summary for the education endowment foundation.* Report. Durham: Durham University.

Hoyles, C., & Lagrange, J.-B. (Eds.). (2010). Mathematics education and technology-rethinking the Terrain. In *The 17th ICMI Study*. Springer US.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. Leung

(Eds.), *Second international handbook of mathematics education* (Vol. 1, pp. 323–349). Dordrecht: Kluwer Academic Publishers.

Kulik, J. A. (2003). *Effects of using instructional technology in elementary and secondary schools: What controlled evaluation studies say*. Arlington, VA: SRI International. http://www.sri.com/policy/csted/reports/sandt/it/Kulik_ITinK-12_Main_Report.pdf.

Lagrange, J.-B., Artigue, M., Laborde, C., & Trouche, L. (2003). Technology and mathematics education: A multidimensional study of the evolution of research and innovation. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 239–271). Dordrecht, Netherlands: Kluwer Academic Publishers.

Li, Q., & Ma, X. (2010). A meta-analysis of the effects of computer technology on school students' mathematics learning. *Educational Psychology Review, 22,* 215–243.

National Council of Teachers of Mathematics. (2008). *Technology in teaching and learning mathematics. A position of the national council of teachers of mathematics*. http://www.nctm.org/Standards-and-Positions/Position-Statements/Technology-in-Teaching-and-Learning-Mathematics/.

OECD. (2015). *Students, computers and learning. Making the connection*. Paris: OECD Publishing. https://doi.org/10.1787/9789264239555-en. http://www.oecd.org/edu/students-computers-and-learning-9789264239555-en.htm.

Rakes, C. R., Valentine, J. C., McGatha, M. B., & Ronau, R. N. (2010). Methods of instructional improvement in Algebra: A systematic review and meta-analysis. *Review of Educational Research, 80*(3), 372–400.

Ronau, R. N., Rakes, C. R., Bush, S. B., Driskell, S. O., Niess, M. L., & Pugalee, D. K. (2014). A survey of mathematics education technology dissertation scope and quality: 1968–2009. *American Educational Research Journal, 51*(5), 974–1006.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36.

Slavin, R. E. (2016). What is a large effect size? *Huffington Post*. http://www.huffingtonpost.com/robert-e-slavin/what-is-a-large-effect-si_b_9426372.html.

Sokolowski, A., Li, Y., & Willson, V. (2015). The effects of using exploratory computerized environments in grades 1–8 mathematics: A meta-analysis of research. *International Journal of STEM Education, 2*(1), 1–17.

Steenbergen-Hu, S., & Cooper, H. (2013). A meta-analysis of the effectiveness of intelligent tutoring systems on K–12 students' mathematical learning. *Journal of Educational Psychology, 105*(4), 970–987.

Tokpah, C. L. (2008). *The effects of computer algebra systems on students' achievement in mathematics*. Doctoral dissertation. Kent, OH: Kent State University. https://etd.ohiolink.edu/!etd.send_file?accession=kent1215528446&disposition=inline.

Chapter 10
Digital Tools in Lower Secondary School Mathematics Education: A Review of Qualitative Research on Mathematics Learning of Lower Secondary School Students

M. Kathleen Heid

Abstract Mathematics-specific digital technology has an ever-increasing presence in school mathematics learning, and qualitative research has shed light on the potential nature of that learning. Particularly for students at the critical early teen age (ages 10–14, or lower secondary school students), the incorporation of mathematics-specific digital technology in their mathematics instruction can change the representations they see, the mathematical activity in which they engage, and the mathematical content they learn. Research on the impact of mathematics-specific digital technology on lower secondary school students has focused on the mediation of the technology on the relationships between the student and mathematical representation, mathematical activity, and/or mathematical content. To examine the nature of understanding of mathematics learning that can be gleaned from this research, a review of the qualitative research literature on the mathematics learning of lower secondary students in the context of mathematics-specific digital technology was conducted. Fifty-three relevant studies were identified and examined based on a pyramidal model describing the mediation by digital technology of the relationships between the student, and some combination of mathematical activity, mathematical representation, and mathematical content. This chapter uses selected studies from that review to represent ways in which qualitative research probes students' mathematical work. The selected studies are used to illuminate the breadth and depth of students' experience with mathematical activity, mathematical representation, and mathematical content, and the relationships among them.

Keywords Digital technology · Lower secondary school · Mathematical activity Mathematical content · Mathematical representation

M. K. Heid (✉)
The Pennsylvania State University, University Park, PA, USA
e-mail: mkh2@psu.edu

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_10

10.1 Background and Purpose

Options for learning have expanded over the past few decades with the increasingly widespread personal and classroom availability of digital technology. The demands of learning in mathematics and the opportunities to address those demands differ in substantial ways from the demands and opportunities in other disciplines. The nature of mathematics-specific technology, with its frequent involvement of a broad range of reasoning strategies and linked representational registers, further enhances the potential for digital technology to affect learning in mathematics. Since the advent of classroom-accessible graphing and symbolic-manipulation technology more than three decades ago and the explosive growth in new tools and platforms ever since, mathematics-specific digital technology has continually opened new venues for mathematics learning, and mathematics education researchers have engaged in addressing the challenge of understanding the effects of those technologies on mathematics learning.

Mathematics education research on the impact of mathematics-specific digital technology approaches on students' mathematical experience and learning includes both quantitative and qualitative studies, although the results of the studies in these two arenas differ in purpose. Studies that rely on quantitative data and statistical analysis offer a substantively distinctive sort of knowledge–knowledge that offers statistically qualified conclusions about the effects of specific instructional or learning conditions. Such studies are informative to those charged with making decisions about whether to adopt specific curriculum or instructional strategies, but they suffer from the inability to investigate the learning of mathematics more deeply. It is not within the purview of such studies to probe deeply into the nature, the whys, and the affordances of such studies. On the other hand, qualitative studies offer a substantially different set of important opportunities to learn about the learning of mathematics. Much of the qualitative research in this area highlights the mediating effects of digital technology. This set of opportunities is especially helpful in revealing the potential nature of the learning of mathematics with technology.

The purpose of this chapter is to identify and characterize the nature of what can be or has been learned from qualitative research on the learning of mathematics in the context of mathematics-specific digital technology. The goal of this chapter is not to provide a complete synthesis of qualitative research in this area but rather to provide salient representative examples of what qualitative research can contribute to our understanding of the impact of digital tools on students' mathematics experience and learning. Given the widely varying needs and experiences of learners of different age and experience levels, it is useful to focus on a single age group. The mathematics learning of students in lower secondary school is of particular interest, especially with Piaget's characterization of the development of cognitive abilities that begin to crystallize for many during those critical ages (approximately ages 10–14). The research characterized in this chapter describes the mathematics experience and learning of lower secondary school students in the context of mathematics-specific digital technology. Although this chapter focuses on the affordances of qualitative research, it does so in recognition of the limitations of such studies.

10.2 Literature Search

Research studies on the mathematics experience and learning of lower secondary students in the context of digital tools can be parsed in myriad ways. The organization of the research reported in this review was developed to be reflective of the set of studies identified in the search. This review was confined to the mediating effects of mathematics-specific digital technology (e.g., GeoGebra, Desmos, Cabri, Fathom, CAS) on mathematics learning and experience, and did not include the effects of digital tools that are not mathematics-specific on learning in general. The literature search was confined to studies that focused on the mathematics learning of lower secondary school students, and to studies published in or since the year 2000. The search started with an issue-by-issue examination of the following journals over the relevant time period: *Cognition and Instruction, Educational Studies in Mathematics, International Journal for Technology in Mathematics Education, Journal for Research in Mathematics Education, Journal of Mathematical Behavior, Mathematics Education Research Journal,* and *Technology, Knowledge and Learning* (previously published under the title: *International Journal of Computers for Mathematical Learning*). It also included a follow-up Google Scholar search using terms and phrases such as technology, digital tools, mathematics, middle grades, lower secondary school students (or students ages 10–14), and qualitative research. This search for relevant research studies initially yielded 110 sources, not all of which directly addressed the focus of the search (e.g., a number of the studies focused on the learning of high school mathematics).

The list of studies was pared down to the 53 studies (from among the 110 sources located) published in or since 2000 that used qualitative methodologies and empirical data to investigate the impact of mathematics-specific digital tools on the mathematics learning and experience of students in lower secondary school. The relevant 53 studies (and reports) were then reviewed to develop an understanding of the scope of research in the area, so that the smaller set of studies on which the chapter would focus could reasonably represent the relevant body of research.

10.3 A Framework for This Body of Research

Since the body of literature identified for this review focused on students' mathematics learning and experience, it was appropriate to use a framework that included students' relationships to mathematics content and mathematics activity.

Because digital technology provides new mathematical representations, and mathematics-specific digital technology opens the door to new content and to new experiences with content, a framework that included mathematical content, mathematical representation, and mathematical activity seemed reasonable. The framework suggested in Zbiek, Heid, Blume, and Dick (2007) (which, for the purpose of this chapter, will be called a digital mediation framework) includes nodes of Mathematical Content, Mathematical Representation, and Mathematical Activity, and accounts for the impact of digital technology on each of those nodes as well as on the relationships between the students and one or more of these nodes. Using this framework allowed me to categorize each of the studies in the identified body of literature as an investigation of the impact of mathematics-specific digital technology on one of the nodes (student, mathematical content, mathematical representation, and mathematical activity) or on the relationship between student and one or more of the other nodes. Because this digital mediation framework provides an apt structure for categorizing the body of studies under consideration, it was adopted for this review.

Figures 10.1 and 10.2 illustrate the framework. The pyramidal shape depicted in Fig. 10.2 is meant to suggest digital tools as mediators of the relationships between the student and aspects of their mathematics experience and learning categorized as Mathematical Content, Mathematical Activity, or Mathematical Representation. Each section of this chapter highlights a dominant mediation (e.g., how digital technology mediates the relationship between the student and mathematical

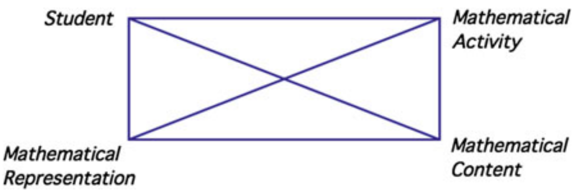


Fig. 10.1 Mathematics learning as the interaction of students with mathematical activity, representation, or content. Printed with permission. Figure reproduced from Zbiek et al. (2007, p. 1172)

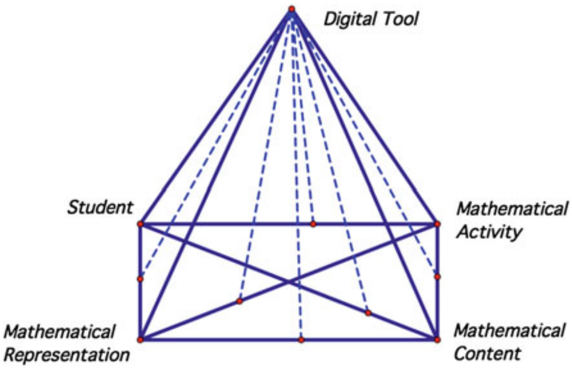


Fig. 10.2 Technology mediating the interaction of students with mathematical activity, mathematical representation, or mathematical content. Printed with permission. Figure reproduced from Zbiek et al. (2007, p. 1172)

representations DT→[S–MR]). This chapter uses versions of the diagram with a specific part in Fig. 10.2 highlighted to represent each dominant mediation.

The purpose of this chapter is not to provide a complete synthesis of research in the targeted area but to characterize what qualitative research can contribute to the field's understanding of the impact of digital tools on mathematics learning and experience. It contains descriptions of some of the ways that studies focusing on students' digital technology experience with Mathematical Representation, Mathematical Content, and Mathematical Activity contribute to that understanding.

10.3.1 Digital Technology as Mediator

Experience with mathematics-specific digital tools can affect the student, the mathematical representations, the mathematical activities, and the mathematical content. As suggested by the pyramidal diagram in Fig. 10.2, digital technology (shown at the top vertex of the pyramid) can also affect the relationships between students and one or more of the other three components (Mathematical Representation, Mathematical Activity, and Mathematical Content). Digital technology can affect the mathematical activity in which students engage, the mathematical representations that students use, and the mathematical content to which students are exposed. Digital tools can also mediate the relationship between the student and the mathematical representation, content, or activity. Many of those mediations have been examined through qualitative research studies, and this chapter is intended to characterize studies representing those mediations. The dashed and solid line segments in Fig. 10.2 suggest the effect of digital technology on a single component or the mediation of digital technology of the relationship between the student and one or more other components. For example, a dashed line between the component of Digital Technology (DT) and the line segment that connect Student (S) and Mathematical Activity (MA) represents an examination of the mediation of digital technology on the ways in which students engage in mathematical activity (signified by DT→[S–MA]). A dashed line between Digital Technology and the triangular region connecting Student, Mathematical Activity, and Mathematical Connection represents an examination of mediation of digital technology in the interaction among these three components (signified by DT→[S–MA–MC]). Following the digital mediation framework suggested in Zbiek et al. (2007), this chapter describes qualitative studies that document ways in which digital technologies can affect these relationships.[1] The following sections illustrate how qualitative research can shed light on different mediation types.

[1]Whereas the chapter in the Lester book uses a digital mediation framework to focus on constructs that research on technology in mathematics education has suggested, this chapter focuses on characterizing the types of findings about students' mathematical experience and learning that qualitative research on different nodes and composition of nodes have and can be generated.

10.3.2 Digital Technology as It Affects Student Characteristics DT→[S]

In addition to affecting the mathematical representations, activity, and content to which students are exposed, digital technology can affect the personal identity of the student who is using the technology. For example, use of digital technology can change a student's preferences or beliefs about mathematics or about technology, or about himself or herself as a doer of mathematics. For example, some qualitative research on digital technology has focused on determining students' preferences, ratings, and beliefs regarding such technology (see Fig. 10.3 for a representation of that mediation). While quantitative research can measure, for example, students' self-reports of their perceptions of the roles of technology in mathematics learning, qualitative research allows triangulation of data and the construction of a more robust, more complex, and more nuanced picture of student perceptions. A study by Bragg (2007), in which the main instructional activity involved learning mathematics through mathematics games, for example, investigated the effects of digital technology on students' attitudes about the use of games. The qualitative approach of the study allowed the researcher not only to document the effects of the technology experience on student beliefs but also to provide potential explanations for those effects. For example, the researcher explained "that for some students the game-playing environment provided the scaffolding needed to bridge constructively their conceptual understandings" (p. 38). In that study, triangulation of survey and interview data yielded contradictory results, generating questions that required further study. Qualitative analysis of the effects of digital technology on the mathematical actions available to students as well as on students' beliefs and preferences allowed a more nuanced analysis of student performance, accounting not only for what they do but also for why they may have done it.

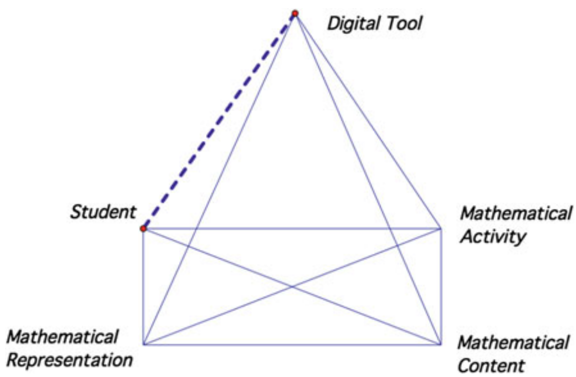


Fig. 10.3 DT→[S]. Digital technology as it affects student characteristics

10.3.3 Digital Technology as It Mediates Students' Relationships to Mathematical Activity. DT→[S–MA]

Digital technology can expose students to new forms of mathematical activity, and qualitative research has provided evidence about the potential impact of that technology on the nature of students' mathematical experience (see Fig. 10.4 for a representation of that mediation). Particularly in the context of digital technology, a qualitative research approach can make the nature of students' mathematical activity more transparent. Some digitally enabled mathematical activity has been shown to foster students' need for mathematical activities such as conjecturing and deductive reasoning. Qualitative research on those efforts have shed light not only on the mathematical processes but also on conditions of the instructional setting that promote activity such as deductive reasoning. In addition, qualitative studies have examined how mathematics-specific digital technologies broaden not only the set of mathematical activities available to students, but also the affordances that mathematics-specific technology allow to students who struggle with mathematics. Finally, the alternative approaches to problem solving fostered within particular digital tools provides a new venue for the study of critical junctures in learning progressions. The following subsections discuss specific studies that address each of these ways that digital technology can mediate students' mathematical activity.

10.3.4 Form and Transparency of Mathematical Activity in the Context of Digital Tools DT→[S–MA]

Mathematics-specific digital tools can engage students in new mathematical activity or in new forms of familiar mathematical activity. Research has shed light on the potentially changing nature of deductive reasoning in digital technology environment, and qualitative research makes the nature of deductive reasoning in digital

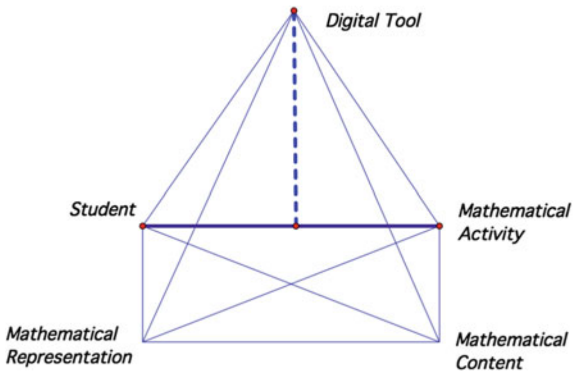


Fig. 10.4 DT→[S–MA]. Digital technology as a mediation factor affecting students' mathematical activity

technology environments more transparent. For example, research focused on environments in which technology such as dynamical geometry environments (DGE) is available has accentuated the concern that students may consider empirical evidence, without further deductive reasoning, as sufficient to convince them of the veracity of a claim. Qualitative research allows the researcher to delve more deeply into this phenomenon noting not only the fact that students are convinced by empirical evidence but also the nature of the empirical evidence that convinces them. Hadas, Hershkowitz, and Schwartz (2000) conducted a study designed to investigate the extent of this behavior in the context of innovative technology-intensive activities designed to cause surprise and uncertainty. Looking for the appearance and frequency of deductive explanations for these surprises, the researchers identified a previously unexpected genre of explanations (which they termed visual/variational) that were either based on the (dynamic) displays or stemmed from students' (presumably DGE-based) imagery. An example of that genre comes from the work of a pair of students on the task of finding the sum of the exterior angles of a polygon as the number of sides increases. These students identified and reasoned from a visual attribute, providing a diagram like the one in Fig. 10.5 and stating: "There is a whole turn around the polygon, therefore the sum is 360°" (Hadas et al., 2000, p. 136). Because the research was designed to examine the explanations students gave rather than only determine the match between students' explanations and those the researchers were expecting, the door was opened to recognizing a possible new norm for mathematical explanations.

Digital tools that make several types of mathematical activity available to students can give researchers the opportunity to characterize and compare the nature of students' cognitive processes in different settings. Research by Parnafes and Disessa (2004) identified connections between type of computational representation and patterns of reasoning students used in their problem solving. The work of eighth-grade and ninth-grade students in the study formed the basis for the researchers' "moment-by-moment" analysis of the students' problem solving. Students had access to a microworld that displayed two informationally equivalent representations of a motion problem: a numerical representation that displayed numerical value for position, velocity, and acceleration for two racing turtles; and a

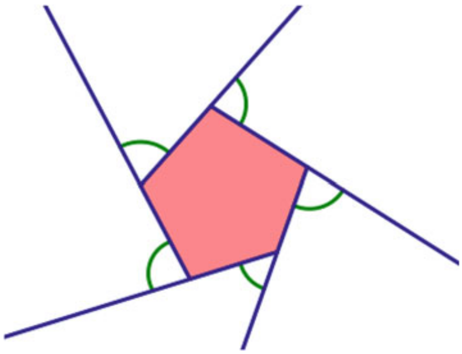


Fig. 10.5 Facsimile of student sketch to explain why the sum of the exterior angles of a pentagon is 360°. Adapted from Hadas et al. (2000, p. 136)

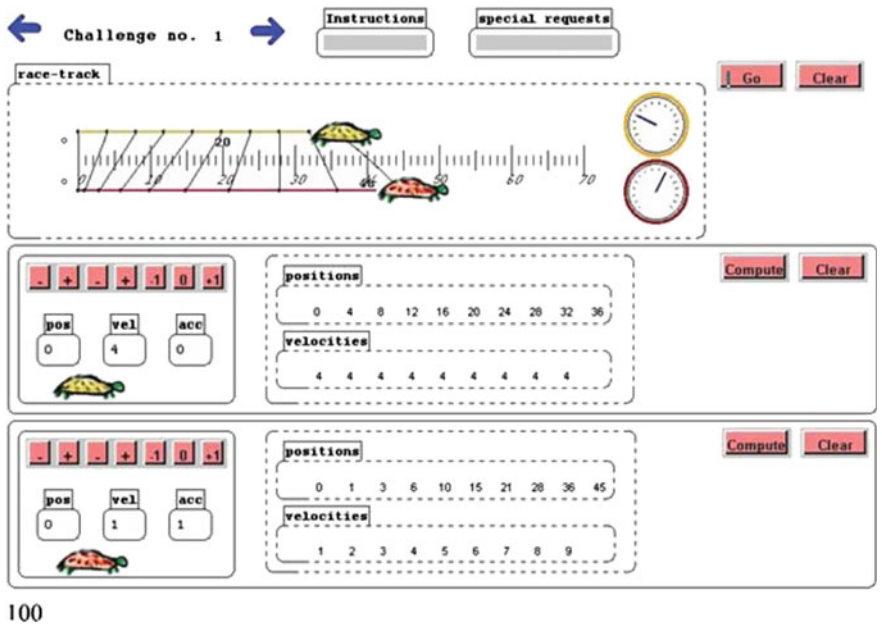


Fig. 10.6 Depiction of two informationally equivalent representations (from p. 254 of Parnafes & Disessa, 2004)

dynamic representation that modeled the race (see Fig. 10.6). The cognitive processes students used were tied to the representations they used, with the numerical representation being associated with what the researchers called “Constraint-based reasoning” (the process of satisfying a given set of constraints by checking whether various sets of values for the variables satisfy a given set of constraints) and the dynamic representation being associated with “Model-based reasoning” (creating and checking a mental model of the motion scenario). Qualitative analysis of the problem-solving process allowed the researchers to delve into the pattern of reasoning and conditions that motivated students to shift their strategies.

Digital tools can allow students access to new forms of mathematical activity, and qualitative research can make the nature of that activity more transparent. With this transparency of mathematical activity across students, researchers can better describe differences and nuances among students’ mathematical engagements. The results are better articulations of students’ mathematical activity and discovery of new patterns of mathematical activity.

10.3.5 Environments that Foster Students' Perceived Need for Deductive Reasoning DT→[S–MA]

Whereas some qualitative studies analyzed how particular digital tools promoted student engagement in particular mathematical activity, others have examined aspects of technology-intensive instruction that may be necessary for the promotion of those activities. To address the goal of understanding how to enhance students' deductive reasoning in the context of dynamical geometry environments, Jones (2000) analyzed data from a longitudinal study that examined how 12-year-old students, who were using dynamical geometry environments, interpreted geometrical objects and relationships. The study examined students' use of hierarchical (an inclusive definitions such as a trapezoid being a quadrilateral with at least one pair of parallel sides) and partitional (an exclusive definition such as a trapezoid having exactly one pair of parallel sides) relationships among different types of quadrilaterals. Jones identified defining features of an instructional setting (e.g., carefully designed tasks, a classroom that fosters conjecturing) that supported students formulating mathematical explanations and coming to terms with inclusive definitions. He conjectured that the technology in settings that did not have these defining features would mediate learning by reducing the students' perceived need for deductive proof. The longitudinal nature of the study and qualitative examination of the data allowed Jones to analyze the mediation of the software on how students interacted with the technology. He found that the dynamical geometry software afforded students a representation of the mathematical idea of functional dependence, and that students developed an understanding of the constraint of robustness of a figure under "dragging" as an important mathematical feature. The rich array of data coupled with the triangulation of the data inherent in qualitative research made it possible to study the complex and nuanced mathematical activity of mathematization in ways that would not have been possible in a strictly quantitative study.

Qualitative studies have led researchers to identify features of technology-intensive instruction that seemed to be necessary to promote student engagement in mathematical activities such as formulating mathematical arguments and mathematization. These features included using carefully designed tasks and developing an environment that encouraged conjecturing and mathematical explanations.

10.3.6 Accommodation of Achievement Levels DT→[S–MA]

Qualitative research has suggested that mathematics instruction facilitated by digital technology allows students of various achievement levels to engage in a range of mathematical activity that was broader than expected. Yerushalmy's (2006) study of "less successful" students in grades 7–9 documented that, when compared with typical behavior of such students, these "slower" students in a functions-based algebra course that had regular access to graphing software adopted a broader view of the contextual problems they were asked to solve than students of similar abilities who did not have such access. They were also more likely to use the technology to confirm their conjectures when compared to the traditional problem solving patterns of students of similar abilities. The range of strategies used by the students in the graphing software/functions approach was greater than expected, although these students delayed the use of symbolic representations and approaches. One might conjecture that this delay was natural because the technology favored numeric and graphical representations rather than symbolic representations and strategies. Left unanswered is the question of whether the delay was related more to a greater conceptual difficulty of symbolic representations or to the nature of the technology used in the study. In this case, qualitative research identified critical areas of needed research that might help explain the mathematical reasoning of students whose mathematical thinking is seldom studied.

10.3.7 Range of Mathematical Activity in Which Students Engage DT→[S–MA]

Qualitative research allows insight into how the robustness of the digital tools available to students can affect the range of solution strategies students use to approach non-routine tasks. For example, one study (Kordaki & Potari, 2002) gave secondary students (14-year olds) access to a computer micro-world designed to provide tools for the measurement of the area of a region by iterating unit shapes. The micro-world was designed to facilitate a variety of approaches to measuring area. The approaches used by students drew on fundamental concepts of area, including the concept of unit of area, the conservation of area, the role of unit iteration, and the inverse relationship between size of covering unit and number of units needed to cover an area. Although their school experience had included area formulas, a majority of the students in the study did not use formulas to measure area but implemented a spatial approach. Because of the open approach of their qualitative analysis, the researchers were able to identify eleven distinct solution strategies that students brought into play. The researchers concluded that “an environment providing the students with the opportunity to select various tools and asking them to produce solutions ‘in any possible way’ can stimulate them to construct a plurality of solution strategies” (p. 65).

The theme of a greater variety of approaches in digital technology environments was corroborated in Papadopoulos and Dagdilesis’s (2008) comparison of the verification processes of fifth and sixth grade students who worked on nonroutine geometry problems that centered on the calculation of area for irregular plane figures. The researchers analyzed students’ paper-and-pencil work and recordings of their computer work, and supplemented this analysis with interviews to clarify parts of the student work that were not clear. In computer environments students’ work evidenced a broader range of verification processes when compared to the processes typically used in a traditional paper-and-pencil environment. Data also allowed researchers to identify and describe a range of students' verification processes.

Given appropriate digital technology environments, students in these qualitative studies used a greater variety of approaches to non-routine problem solving and a greater variety of verification processes. The consequence of this larger menu of options is not yet clear. Having available a greater range of approaches might provide new routes to successful problem solving, and/or it may provide ways to circumvent particularly difficult approaches.

10.3.8 Learning Progressions Seen in Students' Engagement in Mathematical Activity DT→[S–MA]

Not only can qualitative research on use of digital tools document the broad ranges of solution strategies and verification processes open to students of varying achievement levels, but it can test and document learning progressions for students as they overcome roadblocks to their understanding of sophisticated mathematical ideas. One such roadblock is the tendency to students to view sequences only through recursive lenses and to have difficulty progressing to explicitly defined sequences. In the context of a design experiment, Mor, Noss, Hoyles, Kahn, and Simpson (2006) developed a set of mathematical activities in which lower secondary students (ages 10–14) used the ToonTalk programming environment to engage in concrete experiences with sequences through controlling the movement of simulated robots on a screen. Through these experiences, students engaged with concrete manifestations of variables, partial sums, equivalence and rate of change. Through these experiences, they could "make sophisticated arguments regarding the mathematical structures of the sequences without requiring the use of algebra" (p. 65).

10.3.9 Digital Technology and Mediation of Mathematical Activity DT→[S–MA]

Mathematics-specific digital technology has the capacity to change the nature of the mathematical activity in which students engage, and qualitative research can make the nature of that activity more transparent to researchers. Researchers have used widely available software to investigate specific types of mathematical activity. For example, dynamical geometry programs are likely to affect the mathematical activities of conjecturing, deductive reasoning, or proving. On the other hand, researchers have tailored software to specific mathematical goals in order to test hypotheses about the students' mathematical activity in different technological contexts. Researchers have then used their observations of students' work with mathematics-specific digital technology to identify features of instructional settings that support particular types of mathematical activity. Research on the mediation of mathematical activity in the context of digital tools has led to observations about the multiple cognitive styles and achievement levels accommodated and the considerable range of solution strategies and verification processes observed in student work. All told, mathematics-specific digital tools can mediate students' mathematical activity, both by encouraging those activities and by providing the setting in which such activity is natural. A fairly consistent result of the mediation is that the variety of activities and strategies is considerable, the range of personal attributes accommodated is broad, and the capacity of qualitative research to describe the activity in new ways and fine detail is considerable.

10.3.10 Digital Technology as It Mediates Students' Relationships to Mathematical Representation. DT→[S–MR]

Just as digital technology can affect students' experience with mathematical activity, digital technology also mediates students' relationships to mathematical representation, both through its generation of previously unfamiliar representations and through new mathematical activity with familiar representations (see Fig. 10.7 for a representation of that mediation). As discussed previously, Parnafes and Disessa (2004a) identified connections between types of computational representation and patterns of reasoning (constraint-based reasoning and model-based reasoning) students used in their problem solving. The qualitative analysis in this study enabled the researchers to fine-tune observations of patterns in student reasoning.

Students' interpretations of various representations are not unproblematic, however, and students must develop the ability to see particular features in a representation, an ability referenced by Stevens and Hall (1998) as "disciplined

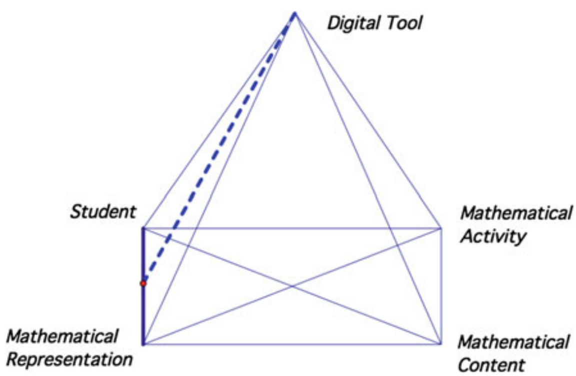


Fig. 10.7 DT→[S–MR]. Digital technology as a mediation factor affecting mathematical representation

perception". Noble, Nemirovsky, Dimattia, and Wright (2004a) studied how, in the context of studying the mathematics of change, sixth grade students learned to see particular features in a representation. In a study of sixth graders' work with representational tools available in their classroom, students worked with software environments that included a simulation of a moving elevator linked to a graph of velocity as a function of time. Using qualitative research methodology, researchers were able to describe the progression of understandings that the sixth graders exhibited as they learned to notice more and more about the representations. Not only did they recognize that students developed an ability to attend to specific features of the representations, but they also documented students' development of the ability to imagine motions associated with given graphs of velocity versus time. Inter-representational connections such as these might be fostered in digital technology arenas in that they have the potential for enriching students' concept images.

The construct of instrumental genesis (Vérillon & Rabardel, 1995), the evolution for individuals of a digital artifact into a digital instrument, is central to understanding students' use of digital tools in their mathematical work. Tabach, Hershkowitz, and Arcavi (2008) studied the instrumental genesis of seventh grade beginning algebra students in the context of a spreadsheet-available, functions approach to algebra. Their qualitative analysis of the work of four to five pairs of students on the same problem enabled them to articulate students' transformation of digital artifacts into instruments that allowed them to make connections between numbers and symbols. Because of the free access to the spreadsheet, students could capitalize on their facility with the spreadsheet's numerical representations to make meaningful symbolic generalizations. The study shed some light on the power of access to different representational registers (e.g., numeric, symbolic, graphical).

Finally, qualitative research allows examination of a broad range of representations, and hence a broad range of ways to understand mathematical ideas. A representation register of increasing interest in mathematics education is that of gestures, and qualitative research opens mathematics education research to the examination of the progression of understanding in the context of a range of representation registers. Specifically, it allows researchers to document the evolution of reification of knowledge in the context of gestures. In a teaching experiment with an eighth grade class using graphing calculators and a motion detector to study motion (still, uniform, and accelerated), Robutti (2006) was able to document a series of semiotic steps in the development of objectified knowledge of motion, including a sequence of physical and iconic gestures, use of metaphor connected to dynamic gestures, and explanation, in terms of a previous classroom experience, of a graph in terms of the relation between variables. Bazzini (2001), in a study of eighth graders' use of an on-line measurement tool and graphing calculators, also used the construct of embodied cognition to identify new roles for grounding metaphors (metaphors that "ground our understanding of mathematical ideas in terms of everyday experience", p. 262) in the development of an understanding of mathematical ideas.

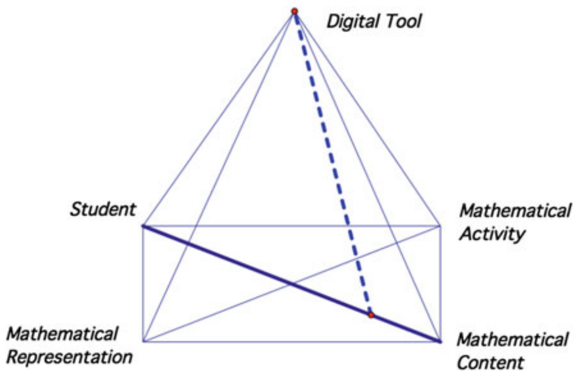
The body of qualitative research that focuses on the mediation of digital tools on students' experience with mathematical representation includes consideration of several different aspects of representation use. Digital tools can support a broader range of representations, they may be tailored to support connections to particular mathematical processes, and they may foster a more fine-tuned understanding of particular representations. Moreover, the effect of digital tools is a function of the relationships of the user to the tool, with progress in the processes of instrumentalization and instrumentation (Artigue, 2002) providing a venue for connections among representation registers.

Digital tools can assist users in focusing on and investigating the meaning of and relationships among specific features of a representation. Just as digital tools can stimulate a wide range of mathematical activity, they can generate a broad range of mathematical representation. The nature of qualitative inquiry and technological advances in capturing and analyzing qualitative data have supported this foray into research on new representation types. Chief among those new representation registers is that of gesture as an indicator of mathematical understanding, and inquiry into the meaning of gestures has been assisted though an embodied cognition lens. With this broader range of representation types in view by researchers, their work has also investigated relationships between representation type and patterns of reasoning, an indicator of the intimate relationship between the type of mathematical representation and the type of mathematical activity in which students engage using that representation. The connections among mathematical activity, mathematical representation, and mathematical content seem to be inescapable.

10.3.11 Digital Technology as It Mediates Students' Relationships to Mathematical Content. DT→[S–MC]

Digital technology can affect students' relationships to mathematical content for several reasons (see Fig. 10.8 for a representation of that mediation). Familiar concepts may be developed more deeply in the context of digital technology. For example, because of the many ways that technology can represent change, digital technology can contribute to students developing a deeper understanding of fundamental concepts such as rate of change. Digital access to representations of rate of change was a major focus of the SimCalc project (Hegedus & Roschelle, 2013), Kaput's technological manifestation of his commitment to democratizing access to calculus. With the ease of digital access to representations of change in tools such as JavaMathWorlds (animated simulation software developed by the SimCalc project), Herbert and Pierce (2008) were able to study the engagement of students (14–15 year olds) in an extensive series of lessons concerning the meaning of rate of change in the context of speed. The researchers noted the importance of the lesson series as "a careful exploration of the mathematical aspects of speed." (p. 248) in developing a deeper understanding of rate of change. The initial "model of" understanding of rate of change in a context of speed held by many of the

Fig. 10.8 DT→[S–MC]. Digital technology as a mediation factor affecting mathematical content



students evolved into a deeper understanding of rate of change as a "model for" that could be applied to understanding of rate of change that did not involve motion (Gravemeijer, 1999). Identification of the subtle difference between understanding a mathematical entity as a "model of" and understanding it as a "model for" was facilitated through the one-on-one interviews conducted in this qualitative research. Similarly, students (13–15 years old) using a dynamical geometry environment in a study by Kordaki and Belamenou (2006) developed a more global and connected view of area than is typical in a paper-and-pencil environment. Qualitative analysis and access to a dynamical geometry tool allowed a more nuanced articulation of students' understanding of area. For example, student understanding advanced to embed the notion of congruent triangles in the broader notion of equivalent triangles (in this case, equivalent triangles are ones with the same area but not necessarily the same perimeter), they used several different measurement representations systems, and, taking advantage of the range of available tools in Cabri-II, they readily linked their prior knowledge to the concept of area in triangles. As a supposed consequence of studying area and perimeter in relation to each other for a variety of figures, what researchers took as evidence of a broader understanding of area included observation of students' construction of equivalence classes of triangles and observation of students' development of the ability to distinguish between area and perimeter.

Of course, not all uses of mathematics-specific digital technology result in enhanced conceptual understanding. Muir (2014), in a study of the use of online mathematical resources accessed by students in grades 5 through 9 (lower secondary school), observed that "The procedural nature, however, of many of the online resources needs to be acknowledged; while they may assist students with procedural understanding or fluency, it is less likely that relational understanding (Skemp 1978) will develop as a result of watching the clips or participating in drill and practice" (p. 835).

10.3.12 Digital Technology as It Mediates Students' Relationships to Mathematical Content and Mathematical Representation. DT→[S–MR–MC]

Digital technology, through its use of new representations, can mediate the relationships among students, mathematical content, and mathematical representation (see Fig. 10.9 for a representation of that mediation). Although the set of research studies discussed in this chapter was selected to focus on the learning of mathematics using mathematics-specific technology, there are times when the use of mathematics-specific technology in those studies entails non-mathematics-specific uses. Some of that research was heavily tied into mathematics-specific technology that facilitated communication and connections across users. Among the more intensively researched technologies that tied mathematics-specific technology to technology for communication and connections is Kaput's aforementioned SimCalc technology (Hegedus & Roschelle, 2013). The body of SimCalc research centered on giving lower secondary students access to fundamental ideas in higher level mathematics (e.g., rate of change) through engaging with multiple dynamic representations. Many of the chapters of the SimCalc book focus on examining the learning of lower secondary students as they engaged in networked activities using simulations and multiple representations of motions (e.g., races and elevator rides) to develop an understanding of important mathematical underpinnings of calculus. Research examined student understanding of the concept of rate of change as well as their use of representations. One qualitative study (Bishop, 2013) focused on student learning and based conclusions on analysis of video footage of the same SimCalc curriculum unit (rates of change through piecewise linear graphs of motion) across 13 seventh grade classrooms. Because the researcher had the opportunity to analyze qualitative data in a large number of similar settings (seventh grade classes using the same curriculum material) she had the unusual opportunity to develop, across a range of settings, a qualitative synthesis of the intellectual work involved in this use of technology. It was through the dynamic representations

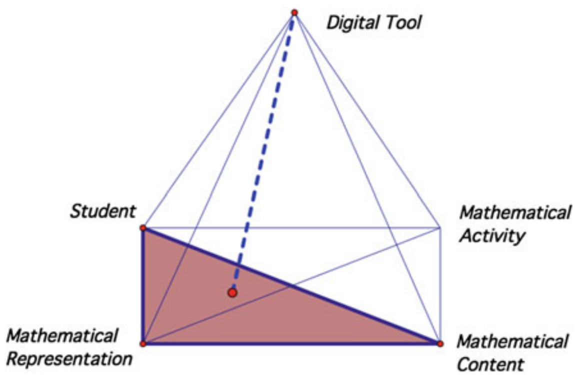


Fig. 10.9 DT→[S–MR–MC]. Digital technology as a mediation factor affecting the relationship among students, mathematical activity, and mathematical representation

inherent in SimCalc that student attention could be drawn to the sophisticated concept of rate of change–evidence of the role of technology in connecting the mathematical content and the representations used to convey that content.

10.3.13 Digital Technology as It Mediates Students' Relationships to Mathematical Activity and Mathematical Representation. DT→[S–MR–MA]

The mathematical activities in which students engage when digital technology is available depend on their interpretation of features of the representations generated by the technology (see Fig. 10.10 for a representation of that mediation), and their interpretation of features can define the mathematical activity in which they engage. Interpretation of the features of dynamic technology comes with its own set of challenges. Central to the challenges inherent in dynamical geometry environments is the dependency relationship between technological "child" and technological "parent". That is, if the construction of one mathematical object builds on an existing object, the existing object is called the "parent" and the object being constructed is called the "child". When Talmon and Yerushalmy (2004) asked ninth grade students in their study to predict the dynamic behavior of points in a construction the students had created, the students erroneously predicted that dragging the child would affect the behavior of the parent. This behavior was consistent with what Jones (2000) described as students thinking of an intersection of two objects as the "glue" that tied together the elements that formed the intersection, regardless of which of the objects was the parent and which was the child. In these cases, students' mathematical activity was a function of their interpretation of the

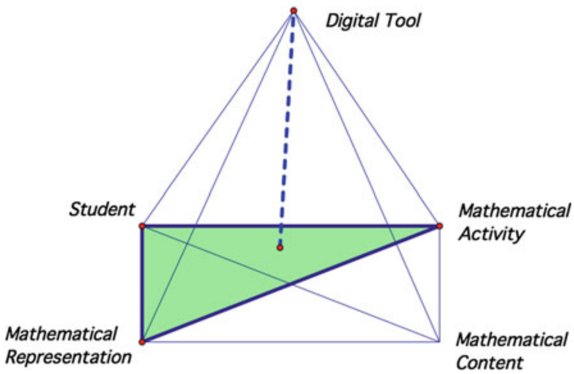


Fig. 10.10 DT→[S–MR–MA]. Digital technology as a mediation factor affecting the relationship among the student, mathematical representation, and mathematical activity

mathematical representations generated by the technology–evidence of the close relationship of mathematical activity to the instrumental genesis of the mathematical representation.

10.3.14 Digital Technology as It Mediates Students' Relationships to Mathematical Content, Mathematical Activity, and Mathematical Representation. DT→[S–MC–MA–MR]

Previous sections of this chapter have examined the effects of digital technology on mathematical content, mathematical activity, or mathematical representation, at times as if these relationships are separable. But throughout these sections was continual evidence of the connectedness of these components. One can easily argue that the components of mathematical content, mathematical activity, and mathematical representation, are inseparable when analyzing the engagement of students in mathematical work using digital technology (see Fig. 10.11 for a representation of that mediation). Digital technology provides access to new forms of representation and new ways to interact with those representations. A result is that the use of this technology can open the door to new mathematical content, although it does not always do so—as reminded by Pea (1987) in his distinction of using technology as a reorganizer and using technology as an amplifier (the reorganizer use suggesting new content and the amplifier use suggesting renewed emphasis on familiar content). One example of a mathematical concept that mathematical digital tools can make more accessible is that of parameter, a focus of Drijvers' extensive study (2004) of the learning of algebra in a computer algebra environment. The computer algebra environment defined not only the mathematical representational registers

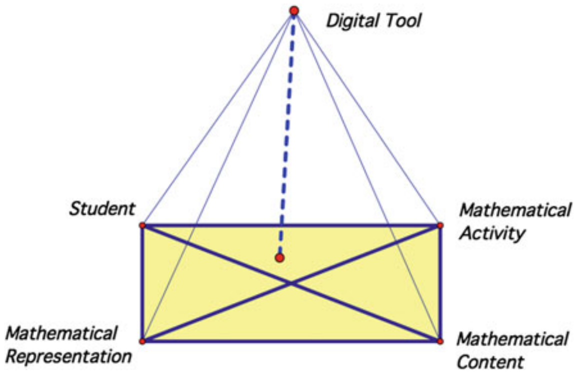


Fig. 10.11 DT→[S–MC–MA–MR]. Digital technology as a mediation factor affecting the relationships among the student, mathematical content, mathematical activity, and mathematical representation

that would be available to students but also the mathematical activities in which they could engage. Drijvers noted the importance of the concept of parameter, pointing to its use in involving students in the mathematical activities of generalizing and abstracting. Computer algebra (CAS) tools gave students fingertip access to multiple representations of variables and of parameters, and CAS tools could ostensibly be used by students to nimbly orchestrate conceptual movement between representations of parameters as fixed values and representations of parameters as varying quantities that defined families of functions. Drijvers' study exemplifies the mediation by digital tools of the relationship among students, mathematical content, mathematical activity, and mathematical representation. Qualitative research enabled the tightly connected study of that relationship. A conceptual analysis of the concept of parameter allowed Drijvers to define growth in student understanding of parameter, and a qualitative approach afforded him the opportunity to document that growth in his participants. His qualitative approach also uncovered the difficulty students experience with instrumentation (i.e., with the ways in which the students' thinking must accommodate the technology) and the impact of that difficulty on the development of conceptual understanding. Because of its extensive and thorough attention to the nature of qualitative data and its sensitivity to the construct of instrumental genesis, the study was able to shed new light on the difficulty of conceptualizing parameters and the ways that sophisticated digital tools may compete for students' cognitive attention. Qualitative analysis afforded Drijvers the opportunity to document the trajectory of students' relationships to the digital tools they were using.

10.4 Affordances of Applying Multiple Theoretical Frameworks in Qualitative Research

A discussion of qualitative research would not be complete without some attention to the construct of theoretical framework. Theory that one uses to frame a study has a substantive and defining impact on the conclusions that might be drawn from the study. Application of multiple frameworks in qualitative research can generate robust and compatible observations. Lagrange and Psycharis (2014) designed a study of the differential effects of using two different theoretical frameworks in the analysis of data from similar studies. The researchers used two very different theoretical traditions to analyze data that focused on computer environments for the teaching and learning of functions and for the conceptualization of functions. The study compared learning in two different national settings (France and Greece) and two different grade levels (middle school and high school) levels. The researchers identified tensions between the goals of the two traditions. One of the traditions, Brousseau's (1997) Theory of Didactic Situations, was concerned with reproducibility and ran into difficulty with the limited predictability of students' interpretation of software feedback while the other tradition sought to develop insight into the learning process and benefited from students' emergent ideas and generalizations. Moreover, the two theoretical traditions required different foci for the data. The constructionist tradition required the analysis of particular students' work while the Theory of Didactic Situations tradition required observation at the level of the classroom. Nevertheless, the researchers concluded that "Overall, the synthesis of the results leads to a more complete view of the potential of digital technologies for the learning of functions from the two perspectives" (Lagrange & Psycharis, 2014, p. 283). The qualitative approach that allowed the collection and analysis of both individual and classroom-level data resulted in a balanced view of learning in the context of mathematical technology. Two theoretical lenses can productively be used to analyze data from qualitative studies, and the analyses provided by the two lenses can complement each other. Drijvers, Godino, Font, and Trouche (2013) demonstrated how, when qualitative data gathered with an Instrumental Genesis theoretical lens was re-analyzed through an Onto-Semiotic lens, a fuller understanding resulted of an excerpt of data describing student CAS-intensive work with a problem centrally focused on parameters. One might hypothesize that multiple theoretical lenses could enrich the field's understanding of mathematics learning, as long as researchers accommodate the need for different foci and different data and as long as the lenses are based on compatible assumptions.

10.5 Conclusion

This chapter categorizes and characterizes qualitative research published in the 21st Century on the effects of mathematics-specific digital technologies on the mathematics learning of lower secondary school students. The lens through which the research is viewed shifts its focus among mediation of the relationship of students to mathematical representation, mathematical activity, or mathematical content, or to relationships between the student and two or more of the other components. Whereas any one of the research studies described may have involved all three of these components, the emphasis on one component or on a relationship among components was chosen to illustrate the nature of the particular type of mediation. The remarkable (and continually expanding) ability of digital technologies to stretch the type of representation to be used to convey a mathematical entity inevitably licenses a broad range of mathematical activity, which, in turn, can lead to consideration of new mathematical content. The omni-presence of mathematical representation, mathematical activity, or mathematical content in the context of research on mathematics-specific digital technology begs for examination of the mediation of the technology on these components—an endeavor that has been well initiated over the past two decades.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Bazzini, L. (2001). From grounding metaphors to technological devices: A call for legitimacy in school mathematics. *Educational Studies in Mathematics, 47*, 259–271.

Bishop, J. P. (2013). Mathematical discourse as a process that mediates learning in SimCalc classrooms. In S. J. Hegedus & J. Roschelle (Eds.), *The SimCalc vision and contributions* (pp. 233–249). Berlin: Springer.

Bragg, L. (2007). Students' conflicting attitudes towards games as a vehicle for learning mathematics: A methodological dilemma. *Mathematics Education Research Journal, 19*(1), 29–44.

Brousseau, G. (1997). *Theory of didactical situations in mathematics* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield, Eds. and Trans.) Dordrecht, NL: Kluwer.

Drijvers, P. (2004). Learning algebra in a computer algebra environment. *International Journal for Technology in Mathematics Education, 11*(3), 77–90.

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses. *Educational Studies in Mathematics, 82*(1), 23–49.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*, 155–177.

Hadas, N., Hershkowitz, R., & Schwarz, B. B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics, 44*(1–2), 127–150.

Hegedus, S. J., & Roschelle, J. (Eds.). (2013). *The SimCalc vision and contributions: Democratizing access to important mathematics*. Berlin: Springer.

Herbert, S., & Pierce, R. (2008). An 'emergent model' for rate of change. *International Journal of Computers for Mathematical Learning, 13*(3), 231–249.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44*(1), 55–85.

Kordaki, M., & Balomenou, A. (2006). Challenging students to view the concept of area in triangles in a broad context: Exploiting the features of Cabri-II. *International Journal of Computers for Mathematical Learning, 11*, 99–135.

Kordaki, M., & Potari, D. (2002). The effect of area measurement tools on student strategies: The role of a computer microworld. *International Journal of Computers for Mathematical Learning, 7*(1), 65–100.

Lagrange, J. B., & Psycharis, G. (2014). Investigating the potential of computer environments for the teaching and learning of functions: A double analysis from two research traditions. *Technology, Knowledge and Learning, 19*(3), 255–286.

Mor, Y., Noss, R., Hoyles, C., Kahn, K., & Simpson, G. (2006). Designing to see and share structure in number sequences. *The International Journal for Technology in Mathematics Education, 13*(2), 65–78.

Muir, T. (2014). Google, Mathletics and Khan Academy: Students' self-initiated use of online mathematical resources. *Mathematics Education Research Journal, 26,* 833–852.

Noble, T., Nemirovsky, R., Dimattia, C., & Wright, T. (2004). Learning to see: Making sense of the mathematics of change in middle school. *International Journal of Computers for Mathematical Learning, 9*, 109–167.

Papadopoulos, I., & Dagdilelis, V. (2008). Students' use of technological tools for verification purposes in geometry problem solving. *Journal of Mathematical Behavior, 27*(4), 311–325.

Parnafes, O., & Disessa, A. (2004). Relations between types of reasoning and computational representations. *International Journal of Computers for Mathematical Learning, 9*(3), 251–280.

Pea, R. D. (1987). Cognitive technologies for mathematics education. *Cognitive Science and Mathematics Education,* 89–122.

Robutti, O. (2006). Motion, technology, gestures in interpreting graphs. *The International Journal for Technology in Mathematics Education, 13*(3), 117–125.

Skemp, R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher, 26*(3), 9–15.

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning*. Cambridge, UK: Cambridge University Press.

Tabach, M., Hershkowitz, R., & Arcavi, A. (2008). Learning beginning algebra with spreadsheets in a computer intensive environment. *Journal of Mathematical Behavior, 27*, 48–63.

Talmon, V., & Yerushalmy, M. (2004). Understanding dynamic behavior: Parent–child relations in dynamic geometry environments. *Educational Studies in Mathematics, 57*, 91–119.

Vérillon, P., & Rabardel, P. (1995). Cognition, and artifact: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology in Education, 9*(3), 1–33.

Yerushalmy, M. (2006). Slower algebra students meet faster tools: Solving algebra word problems with graphing software. *Journal for Research in Mathematics Education*, 356–387.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169–1207). Charlotte, NC: Infoage.

Articles and Chapters in the Body of Reviewed Publications

Ainley, J., Bills, L., & Wilson, K. (2005). Designing spreadsheet-based tasks for purposeful algebra. *International Journal of Computers for Mathematical Learning, 10*(3), 191–215.

Bazzini, L. (2001). From grounding metaphors to technological devices: A call for legitimacy in school mathematics. *Educational Studies in Mathematics, 47*(3), 259–271.

Bishop, J. P. (2013). Mathematical discourse as a process that mediates learning in SimCalc classrooms. In S. J. Hegedus & J. Roschelle (Eds.), *The Simcalc vision and contributions* (pp. 233–249). Berlin: Springer.

Bragg, L. (2007). Students' conflicting attitudes towards games as a vehicle for learning mathematics: A methodological dilemma. *Mathematics Education Research Journal, 19*(1), 29–44.

Calder, N., Brown, T., Hanley, U., & Darby, S. (2006). Forming conjectures within a spreadsheet environment. *Mathematics Education Research Journal, 18*(3), 100–116.

Cedillo, T. E. (2001). Toward an algebra acquisition support system: A study based on using graphic calculators in the classroom. *Mathematical Thinking and Learning, 3*(4), 221–259.

Clark, G., & Redden, T. (2000). Using logo as a scaffolding to develop algebraic thinking in a virtual classroom. *The International Journal for Technology in Mathematics Education, 7*(2), 105.

Drijvers, P. (2004). Learning algebra in a computer algebra environment. *International Journal for Technology in Mathematics Education, 11*(3), 77–90.

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses; A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics, 82*(1), 23–49.

Graham, A. T., & Thomas, M. O. (2000). Building a versatile understanding of algebraic variables with a graphic calculator. *Educational Studies in Mathematics, 41*(3), 265–282.

Gueudet, G. (2008). Learning mathematics with e-exercises: A case study about proportional reasoning. *International Journal of Technology in Mathematics Education, 14*(4), 169–182.

Hadas, N., Hershkowitz, R., & Schwarz, B. B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics, 44*(1–2), 127–150.

Hegedus, S. J., & Penuel, W. R. (2008). Studying new forms of participation and identity in mathematics classrooms with integrated communication and representational infrastructures. *Educational Studies in Mathematics, 68*(2), 171–183.

Hegedus, S. J., & Roschelle, J. (Eds.). (2013). *The SimCalc vision and contributions: Democratizing access to important mathematics*. Berlin: Springer.

Heid, M. K. (2002). How theories about the learning and knowing of mathematics can inform the use of CAS in school mathematics: One perspective. *The International Journal for Technology in Mathematics Education, 9*(2), 95.

Herbert, S., & Pierce, R. (2008). An 'emergent model' for rate of change. *International Journal of Computers for Mathematical Learning, 13*(3), 231–249.

Hoffmann, R., & Klein, R. (2007). Adjusting mathematics teaching and learning to technological changes: Introducing algorithmic problems into elementary school mathematics. *International Journal for Technology in Mathematics Education, 14*(1), 21–30.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44*(1–2), 55–85.

Kieran, C. (2007). Interpreting and assessing the answers given by the CAS expert: A reaction paper. *International Journal for Technology in Mathematics Education, 14*(2), 103.

Kieran, C., & Drijvers, P. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning, 11*(2), 205–263.

Kieran, C., & Saldanha, L. (2007). Designing tasks for the co-development of conceptual and technical knowledge in CAS activity: An example from factoring. In G. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Cases and perspectives* (pp. 393–414). Greenwich, CT: Information Age.

Kordaki, M., & Balomenou, A. (2006). Challenging students to view the concept of area in triangles in a broad context: Exploiting the features of Cabri-II. *International Journal of Computers for Mathematical Learning, 11*(1), 99–135.

Kordaki, M., & Potari, D. (2002). The effect of area measurement tools on student strategies: The role of a computer microworld. *International Journal of Computers for Mathematical Learning, 7*(1), 65–100.

Kuiper, E., & de Pater-Sneep, M. (2014). Student perceptions of drill-and-practice mathematics software in primary education. *Mathematics Education Research Journal, 26*(2), 215–236.

Kynigos, C., & Psycharis, G. (2013). Designing for instrumentalisation: Constructionist perspectives on instrumental theory. *International Journal for Technology in Mathematics Education, 20*(1), 15–20.

Lachmy, R., & Koichu, B. (2014). The interplay of empirical and deductive reasoning in proving "if" and "only if" statements in a dynamic geometry environment. *The Journal of Mathematical Behavior, 36*, 150–165.

Lagrange, J. B., & Psycharis, G. (2014). Investigating the potential of computer environments for the teaching and learning of functions: A double analysis from two research traditions. *Technology, Knowledge and Learning, 19*(3), 255–286.

Lee, H. S., & Hollebrands, K. F. (2006). Students' use of technological features while solving a mathematics problem. *Journal of Mathematical Behavior, 25*(3), 252–266.

Leischner, P. (2002). The collection of interactive solid figures and spatial situations in the Cabri-geometre. *The International Journal for Technology in Mathematics Education, 9*(3), 243.

Leung, A., & Lee, A. M. S. (2013). Students' geometrical perception on a task-based dynamic geometry platform. *Educational Studies in Mathematics, 82*(3), 361–377.

Lin, X. (2001). Reflective adaptation of a technology artifact: A case study of classroom change. *Cognition and Instruction, 19*(4), 395–440.

Miller, D., Glover, D., & Averis, D. (2005, March). Presentation and pedagogy: The effective use of interactive whiteboards in mathematics lessons. In *Proceedings of the Sixth British Congress of Mathematics Education* (Vol. 30, pp. 105–112).

Miyazaki, M., Kimiho, C., Katoh, R., Arai, H., Ogihara, F., Oguchi, Y., et al. (2012). Potentials for spatial geometry curriculum development with three-dimensional dynamic geometry software in lower secondary mathematics. *International Journal for Technology in Mathematics Education, 19*(2), 73–79.

Mor, Y., Noss, R., Hoyles, C., Kahn, K., & Simpson, G. (2006). Designing to see and share structure in number sequences. *International Journal for Technology in Mathematics Education, 13*(2), 65–78.

Muir, T. (2014). Google, Mathletics and Khan Academy: Students' self-initiated use of online mathematical resources. *Mathematics Education Research Journal, 26*(4), 833–852.

Noble, T., Nemirovsky, R., Dimattia, C., & Wright, T. (2004). Learning to see: Making sense of the mathematics of change in middle school. *International Journal of Computers for Mathematical Learning, 9*(2), 109–167.

Norton, S. (2006). Pedagogies for the engagement of girls in the learning of proportional reasoning through technology practice. *Mathematics Education Research Journal, 18*(3), 69–99.

Olive, J. (2011, November). Research on technology in mathematics education: Theoretical frameworks and practical examples. Paper presented at *Korea Society of Educational Studies in Mathematics Conference*, Seoul, South Korea.

Papadopoulos, I., & Dagdilelis, V. (2008). Students' use of technological tools for verification purposes in geometry problem solving. *Journal of Mathematical Behavior, 27*(4), 311–325.

Parnafes, O., & Disessa, A. (2004). Relations between types of reasoning and computational representations. *International Journal of Computers for Mathematical Learning, 9*(3), 251–280.

Pierce, R., & Stacey, K. (2004). Monitoring progress in algebra in a CAS active context: Symbol sense, algebraic insight and algebraic expectation. *International Journal for Technology in Mathematics Education, 11*(1), 3.

Pihlap, S., Sild, S., & Kreutzberg, K. (2013). Notice mathematics around You. *International Journal for Technology in Mathematics Education, 20*(3), 109–114.

Pitta-Pantazi, D., & Christou, C. (2009). Cognitive styles, dynamic geometry and measurement performance. *Educational Studies in Mathematics, 70*(1), 5–26.

Robutti, O. (2006). Motion, technology, gestures in interpreting graphs. *International Journal for Technology in Mathematics Education, 13*(3), 117–125.

Roschelle, J., Tatar, D., Shechtman, N., & Knudsen, J. (2008). The role of scaling up research in designing for and evaluating robustness. *Educational Studies in Mathematics, 68*(2), 149–170.

Soucie, T., Radovic, N., Svedrec, R., & Kokic, I. (2011). Teaching and assessing polygons using technology. *International Journal for Technology in Mathematics Education, 18*(4), 165–170.

Tabach, M., Hershkowitz, R., & Arcavi, A. (2008). Learning beginning algebra with spreadsheets in a computer intensive environment. *Journal of Mathematical Behavior, 27*(1), 48–63.

Talmon, V., & Yerushalmy, M. (2004). Understanding dynamic behavior: Parent–child relations in dynamic geometry environments. *Educational Studies in Mathematics, 57*(1), 91–119.

Turgut, M., & Uygan, C. (2015). Designing spatial visualization tasks for middle school students with 3D modelling software: An instrumental approach. *International Journal for Technology in Mathematics Education, 22*(2), 45–51.

Vale, C. M., & Leder, G. C. (2004). Student views of computer-based mathematics in the middle years: Does gender make a difference? *Educational Studies in Mathematics, 56*(2–3), 287–312.

White, T. (2009). Encrypted objects and decryption processes: Problem-solving with functions in a learning environment based on cryptography. *Educational Studies in Mathematics, 72*(1), 17–37.

Yerushalmy, M. (2006). Slower algebra students meet faster tools: Solving algebra word problems with graphing software. *Journal for Research in Mathematics Education, 37*(5), 356–387.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics, 60*(3), 297–321.

Chapter 11
Classical and Digital Technologies for the Pythagorean Theorem

Michela Maschietto

Abstract This paper aims to discuss the use of material tools, called mathematical machines, and digital tools in approaching the Pythagorean theorem. These mathematical machines are related to different proofs of the theorem. Teaching experiments with 13-year old students were carried out within the laboratory approach developed from the theoretical frameworks of the Theory of Semiotic Mediation and Instrumental approach in mathematics education. Their analysis shows that behind the kinesthetic experience with the machines, there are important cognitive processes such as the identification of invariants, relationships between the components and usage schemes. It also shows the only manipulation of the first machine does not imply the emergence of the mathematical meanings embedded in the materials tools and the crucial role of the teacher with his different instrumental orchestrations in that process.

Keywords Artifacts · Geometry · Laboratory · Lower secondary school education Pythagoras

11.1 Introduction

This paper focuses on the integration of physical and virtual experiences in mathematics teaching and learning through the use of material and digital tools. The use of material tools in mathematics has been witnessed since the Greek geometry and, over the centuries, these tools have accompanied the development of mathematics (Monaghan, Touche, & Borwein, 2016). An example of the relationship

M. Maschietto (✉)
Department of Education and Humanities, University of Modena and Reggio Emilia, Reggio Emilia, Italy
e-mail: michela.maschietto@unimore.it

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_11

between the use of tools and the development of mathematics is represented by projective geometry: the theoretical development has its roots in perspective with its tools for perspective drawings (called perspectographs[1]), that have been built in the XVI and XVII centuries (Bartolini Bussi & Maschietto, 2006). These tools are considered in the collection of mathematical machines and are defined as:

> an artefact designed and built forcing a point, a line segment or a plane figure (supported by a material support that makes them visible and touchable) to move or to be transformed according to a mathematical law that has been determined by the designer. (Maschietto & Bartolini Bussi, 2014, p. 1)[2]

A well-known case is the pair of compasses, in which the lead point is forced to move on a circle. It can be considered the ancestor of many curve-drawing devices. This paper is concerned with the use of some mathematical machines in mathematics education (Maschietto & Bartolini Bussi, 2011).

The reference to tools, in particular material tools, in the discussion about mathematics education started at the end of XIX century. It was often related to the new methodology of mathematics laboratory (Giacardi, 2012). In that context, even the use of squared paper for mathematics was questioned (cf. Maschietto & Trouche, 2010). The interest in the educational use of tools has not decreased over time, but it involves different kinds of tools. Bartolini Bussi and Borba (2010), in WG4 *Resources and technology throughout the history of ICMI*[3] at the ICME Symposium 2008, collected the contributions of different countries to the discussion about the use of tools in mathematics teaching and learning, referring to both classical and new technologies. In particular, for digital technologies, Drijvers (2015) points out a crucial question, which is how to exploit the potential of ICT for learning and teaching mathematics. He recalls three didactical functionalities of digital technology (p. 136):

> (1) the tool function for doing mathematics, which refers to outsourcing work that could also be done by hand, (2) the function of learning environment for practicing skills, and (3) the function of learning environment for fostering the development of conceptual understanding. Even if these three functionalities are neither exhaustive nor mutually exclusive, they may help to position the pedagogical type of use of the technology involved. In general, the third function is the most challenging one to exploit.

The contribution of this paper is between these two strands: it aims to construct and study a learning environment, following Drijvers (2015) in which material and digital tools are available for students' activities and teacher's actions. In our case, it

[1] http://www.macchinematematiche.org/ and http://archiviomacmat.unimore.it/CR/Copertina.html. Accessed: 2 January 2017.

[2] http://www.mathunion.org/fileadmin/ICMI/files/Digital_Library/ICMEs/Bulletin_Maschietto_BartoliniBussi2_01.pdf. Accessed: 2 January 2017.

[3] https://www.unige.ch/math/EnsMath/Rome2008/WG4/WG4.html. Accessed: 2 January 2017.

contains: material tools related to the Pythagorean theorem (described later in Sect. 11.4.1); digital tools as Interactive Whiteboard (IWB) and its software, and Dynamic Geometry Software (DGS). Referring again to Drijvers, the interest is in the third functionality. The teaching experiment analyzed in this paper concerns the Pythagorean theorem at lower secondary school in Italy (Barbieri, Scorcioni & Maschietto, 2014).

The paper is structured in seven sections. Section 11.2 recalls some proofs of the Pythagorean theorem and describes the technologies used in the teaching experiment. Section 11.3 presents the theoretical framework of this research, while the following section contains the analysis of the artifact, the structure of the teaching path and the research questions. Section 11.5 contains the methodology of the experimental research. Then, results are presented and discussed respectively in Sects. 11.6 and 11.7.

11.2 The Pythagorean Theorem

The Pythagorean theorem is a traditional topic not only in the Italian school, but in general in the mathematics curriculum of secondary schools of several countries (Moutsios-Rentzos, Spyrou, & Peteinara, 2014; Sinclair, Pimm, Skelin, & Zbiek, 2012). There exist several proofs of this theorem,[4] some of which are visual and "without word" (Rufus, 1975; Fig. 11.1).

Often, the theorem is first proposed geometrically (e.g. see Fig. 11.1) and it is then soon converted into formulas and related to algebraic calculations. For example, in algebraic proofs of the theorem, the Fig. 11.1 (on the right) is often used: the legs of the right triangles are called a and b, while the hypotenuse is denoted c. If the area of the big square (whose side is equal to $a + b$) is expressed as the sum of the areas of the right triangles and of the square inside, algebraic manipulation leads to the Pythagorean formula.

A result of focusing more on the algebraic relation rather than on the geometrical meaning of the theorem can be observed when visual proof is considered. For instance, Bardelle (2010) proposes a visual proof of this theorem (Fig. 11.2, on the left) to university students. She finds that they tried to look for the algebraic relation among sides starting from the expression "$c^2 = a^2 + b^2$" rather than rearrange the parts of the figures itself, as is characteristic of a visual proof (Fig. 11.2, on the right).

In the treatment of Fig. 11.2, Bardelle finds that the students did not feel the need to prove that the figure that looked like a square was a square. Then, this theorem also represents an opportunity to deal with some related mathematical meanings such as perpendicularity and right triangle (Moutsios-Rentzos et al., 2014) or to focus on its hypothesis of the right triangle (for instance by a DGS, cf. Anabousy & Tabach in Drijvers et al., 2016).

[4]http://www.cut-the-knot.org/pythagoras/index.shtml. Accessed: 2 January 2017.

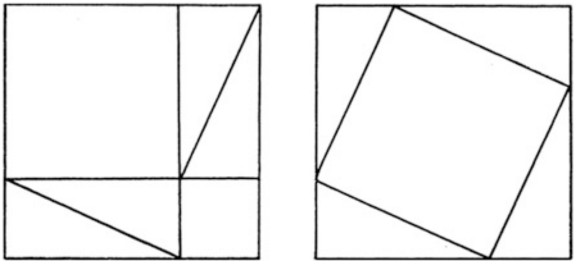


Fig. 11.1 The proof without word by Rufus (1975) (Copyright 1975 Mathematical Association of America. All Rights Reserved)

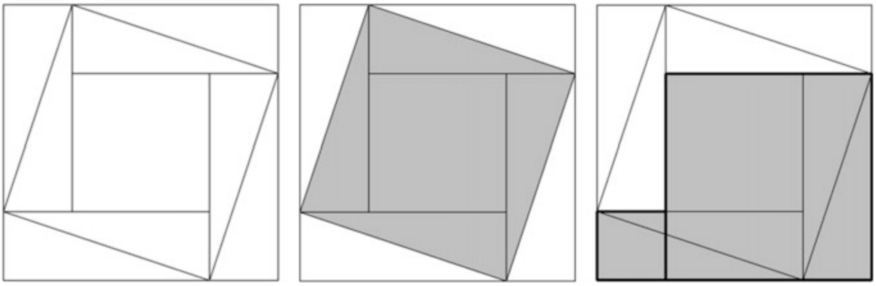


Fig. 11.2 The figure proposed to university students and its rearrangement (*Ibidem*, pp. 253–254)

Moreover, some of these proofs are also available by DGS or java applet,[5] as the proof referring to Fig. 11.1.[6] Concerning applets, Moyer, Bolyard and Spikell (2002) claim that several websites offer interactive experiences for users. In many cases, they contain applets that "demonstrate and verify" (Moyer, Bolyard, & Spikell, 2002, p. 373) this theorem. In particular, the applets allow students to investigate the theorem "by moving pieces of the square that represent a^2 and b^2 and placing them into the area that represents c^2. The various applets use different methods for fitting the pieces into c^2" (Moyer et al., 2002, pp. 373–374).

Visual proofs are often based on the rearrangement of some parts of the figure. Because of this, they can be proposed as material exhibits in mathematical exhibitions (for instance, *Pythagoras and his theorem* by Il Giardino di Archimede[7]) and/or spread as gadgets (Eaves, 1954). In this case, they can be given to the students for manipulation or for evidence of the theorem.

[5]http://www.dynamicgeometry.com/JavaSketchpad/Gallery/Geometry/Pythagoras.html. Accessed: 2 January 2017.

[6]http://nlvm.usu.edu/en/nav/frames_asid_164_g_3_t_3.html?open=instructions&from=category_g_3_t_3.html. Accessed: 2 January 2017.

[7]*Pitagora e il suo teorema*, https://php.math.unifi.it/archimede/archimede/pitagora/immagini/virtuale.php?id=1. Accessed: 2 January 2017.

In the teaching experiment analyzed in this paper, the mathematical machines proposed to classes embed two of those proofs, one of them related to Rufus's proof without words. Even if they can be considered as exhibits to show the theorem, the educational perspective is different here. Following Mariotti (2007) on the relationships between an "intuitive approach to geometry" (Mariotti, 2007, p. 289) and proving processes, the aim is not to visualize the theorem and "convincing pupils of its obviousness" (Mariotti, 2007, p. 289), but to give arguments to justify the theorem. Then, we ask if and how it is possible to approach the Pythagorean theorem starting from some mathematical machines artifacts with 7-grade students in a leaning environment also containing digital tools.

11.3 Theoretical Framework and Research Questions

11.3.1 Theories of Semiotic Mediation and Instrumental Genesis

The use of artifacts is proposed and analyzed within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), grounded in the Vygotskian notion of semiotic mediation and role of artifacts in cognitive development. Within this framework, the teacher chooses artifacts evoking specific mathematical meanings, that are consistent with his/her learning objective for students, and uses them to mediate those meanings. The artifacts are analyzed in terms of their semiotic potential, defined as the double semiotic link established between: the artifact and personal meanings, emerging in students' mind when they use it to accomplish tasks (that have to be accessible for students) on one hand; the artifact and mathematical meanings evoked by that use and recognizable as mathematics by an expert on the other hand. Personal meanings can be related to knowledge that students can recall in solving tasks.

Based on the analysis of the semiotic potential, the teacher plans several activities and tasks; some of them have to be solved with the artifacts. Activities and tasks are organized in terms of didactic cycles with group work, individual work and collective discussions orchestrated by the teacher. The mathematical meanings emerge from the use of the artifacts and from the interactions among peers and between the peers and the teacher, who has the role of an expert guide. In all the activities, students are involved in a semiotic activity (producing gestures, words and/or drawings, which are all called artifact signs, referring to the context of the use of the artifact). The teacher makes evolving artifact signs into mathematical signs (i.e. linked to mathematical content) by the means of pivot signs, acting as bridges between the artifact signs and the mathematical ones. In this sense, the teacher uses the artifact as a tool of semiotic mediation of mathematical meanings. From the students' perspective, there is the passage from a technical instrument to a psychological one.[8]

From the perspective of the instrumental approach (Rabardel & Bourmaud, 2003), the use of an artifact to accomplish specific tasks fosters the emergence of utilization schemes, as a part of a subject's cognitive activity. In particular, two constituents are distinguished: usage schemes, which are related to the management of characteristics and specific properties related to the artifact; and instrument-mediated action schemes (instrumented action schemes in this text), oriented to carry out specific tasks. The instrumental genesis accompanies the constitution of an instrument for the subject, as a cognitive entity composed of utilization schemes and artifact.

In the Theory of Semiotic Mediation, the first didactic cycle usually begins with the exploration of the chosen artifact in small group. The activities are structured following questions such as: "How is the machine made?", "What does the machine make?" or, "How do you use the artifact?" and "Why does it make it?" (Bartolini Bussi, Garuti, Martignone, & Maschietto, 2011, p. 128). In general, the first three questions try to take into account students' processes of instrumental genesis: the first question mainly corresponds to the emergence of artifact components, while the other two correspond to the product of the artifact and to the ways to use it. The fourth question aims to identify the difference between a technical use of the instrument and a psychological one, because it solicits students' processes of formulation of conjectures and argumentation that are very important in mathematical activity and strongly emphasized in a mathematics laboratory. These processes are also supported by the question "What could happen if…?", by which the students are encouraged to vary some parameters of the artifact and to anticipate and interpret the results in the light of what took place before (Bartolini Bussi et al., 2011).

11.3.2 Instrumental Orchestration

The presence and the use of artifacts in a classroom requires specific actions by the teacher. They correspond to the choice of the artifacts and to the construction of tasks, according to the Theory of Semiotic Mediation. The teacher should also take into account students' instrumental geneses. She/he should manage the different artifacts during the lessons, deciding, for instance, when the students can use an artifact or another one, or which artifact is available at a certain moment. Trouche (2004) has proposed the notion of instrumental orchestration, defined by:

[8]Referring to the example of a pair of compasses (Bartolini Bussi & Maschietto, 2008), this can be used as technical tools to produce round shapes. It is externally oriented (Vygotskij, 1978). As a psychological tool it has the potentiality to evoke the peculiar feature of circles (i.e., the constancy of the radius) and to create the link with the geometrical static relational definition of Euclid. It is internally oriented (Vygotskij, 1978).

(1) Didactical configurations, as arrangements of artifacts in the environment;
(2) Exploitation modes of the didactical configurations;

Drijvers, Doorman, Boon, Reed, and Gravemeijer (2010) further developed this notion and added a third component, the didactical performance, characterized as "the ad hoc decisions taken while teaching on how to actually perform the enacted teaching in the chosen didactic configuration and exploitation mode: what question to raise now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals" (Drijvers et al., 2010, p. 215).

The instrumental orchestration has been mainly analyzed in rich technological environments (with graphic calculators, projector, handled devices, and so on; Kratky, 2016). In the case of this paper (as in Maschietto & Soury-Lavergne, 2013), didactical configurations include both material and digital tools.

11.3.3 Research Questions

The research questions concern the didactical exploitation of a learning environment in which material and digital tools are present. In this paper, three questions are central.

1 and 2. If and how the Pythagorean theorem can be approached from using a mathematical machine as M1 (cf. Sect. 11.4.1) with 7-grade students, in a composite environment. In particular, which tasks can be proposed to students for meaning making?

3. Which instrumental orchestrations could the teacher make in such as learning environment?

11.4 Technologies in the Classroom

This section contains the description of the two mathematical machines and the educational context in which the technologies are used in the mathematics classes of Italian schools.

In this work, the use of technologies is rooted in the Italian idea of mathematics laboratory (Maschietto & Trouche, 2010). It is present in the documents of the Commission of the Italian Mathematical Society for Mathematics Instruction (Anichini, Arzarello, Ciarrapico, & Robutti, 2003), in which mathematics laboratory is defined as follows[9]:

[9] http://www.umi-ciim.it/wp-content/uploads/2013/10/Mat2003.zip. Accessed: 2 January 2017.

> A mathematics laboratory is (…) rather a methodology, based on various and structured activities, aimed at the construction of meanings of mathematical objects. (…) We can imagine the laboratory environment as a renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practicing. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other hand, to the interactions between people working together (without distinguishing between teacher and students) (Anichini et al., 2003, PREM_C, p. 23).

In the mathematics laboratory, students' processes of formulation of conjectures and argumentation are strongly motivated (Bartolini Bussi, 2010). The papers on the Italian use of mathematics laboratory mention a number of different tools, including the mathematical machines and DGS.

11.4.1 Material Tools: Mathematical Machines

In this paper, the mathematical rule embedded in the mathematical machines is the Pythagorean theorem according to the proof proposed by Rufus (1975) and the proof attributed to Leonardo da Vinci (Il Giardino di Archimede, 2001).

The first mathematical machine (M1 in Fig. 11.3) is composed of a wooden square base and four wooden right triangles (i.e., triangular prisms) that are all congruent to each other. The square base is surrounded by a frame. For helping the distinction between triangles (figures) and squares ("holes" on the background), a red paper is added into the frame (that corresponds to color the square base and highlights the figure-ground perception, in which one picture can be perceived in two different ways depending on how the students look at it). The triangles are placed within the frame and can shift on it, from one configuration to another one (Fig. 11.3, in the center and at right).

The second mathematical machine (M2) embeds the mathematical proof of the Pythagorean theorem attributed to Leonardo da Vinci. It has two configurations (Fig. 11.4a, b) showing two different 'holes': in Fig. 11.4a there are two right

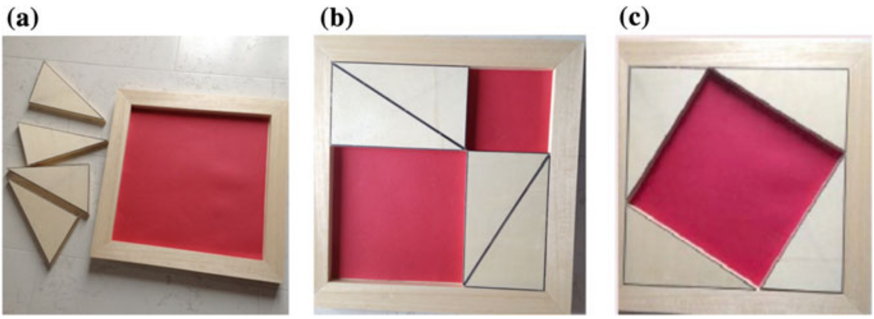


Fig. 11.3 The mathematical machine M1

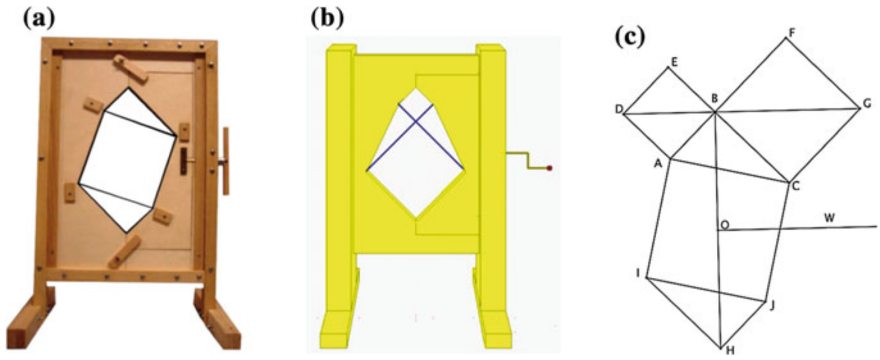


Fig. 11.4 The mathematical machine M2

triangles and a square constructed on their hypotenuse (made by two elastics); in Fig. 11.4b there are two squares on the legs (made by two elastics) of two right triangles. The right triangles are all congruent. The transformation from one configuration to another one is given by the material rotation of a half of M2 by a crank (Fig. 11.4b, on the right side). Referring to Fig. 11.4c, the material rotation corresponds to a reflection with respect to OW (perpendicular to BH); the polygon ABCJHI corresponds to the first configuration (Fig. 11.4a), while the polygon ADEFGC corresponds to the second configuration (Fig. 11.4b). The quadrilaterals ABHI, JHBC, DEFG and ADGC are all equivalent. Let us consider the quadrilaterals ADGC and BCJH. Since the right triangle ABC is a part of the two quadrilaterals, the other parts together have the same area. In other words, the sum of the area of ΔADB and ΔBGC (respectively, half the square constructed on the legs of ΔABC) is equal to the area of half of the square constructed on the hypotenuse AC.

11.4.2 Digital Tools: IWB and DGS

In the classrooms, an IWB was installed, next to a blackboard. Several software were available. Contrary to the mathematical machines, the digital tools were always present in the classrooms. The IWB contains its software for writing and taking notes. With respect to the classical blackboard, this software allows saving of all the texts written on the board, not only to make them available from one lesson to the next, but also to go back to previous steps and, in general, to manage the time stream of the lesson. Furthermore, the teacher can add pictures and use the tool (called the "duplicator") to duplicate chosen figures. Then, these figures can be dragged on the screen. She/he can also easily use colored pens. A DGS was available, and internet too. In such a way, it was possible to show simulations of the machines, above all of M2.[10]

Students and teacher could use the IWB, in the different steps of the teaching experiments. A teacher also used his smartphone and/or video-camera for taking pictures and sharing them through the IWB.

11.5 Methods

The teaching experiment proposes a learning environment in which material and digital artifacts are present. This section contains the analysis of the semiotic potential of the artifacts used in the teaching experiments according to the theoretical framework of Theory of Semiotic Mediation. It is carried out following the components of mathematical content, utilization schemes related to specific tasks and students' personal meaning. Then, the choices and the steps of the teaching path are presented.

11.5.1 Analysis of the Semiotic Potential of the Artifacts

In the mathematical machine M1 (Fig. 11.3) the fundamental relationship between the prisms and the square base (red square in Fig. 11.3) is that the sum of the legs of the right triangles (base of the prism) is equal to the side length of that square. As written above, M1 shows a proof of the theorem (Rufus, 1975). In order to support students' visualization and make evident the interior squares (two squares in Fig. 11.3b and one square in Fig. 11.3c) with respect to the base, we have added a red paper into the frame.

The proof shown in Fig. 11.1 is discussed by Duval (2005) in his analysis of the role of visualization in proving process. He claims that the visualization is not complete if it only considers the two configurations of Fig. 11.1, because the relationship between the big square and the hypotenuse of the right triangles on one hand, and the other two squares and the legs of the same right triangles on the other hand, is expected to be assumed knowledge for the reader. Duval claims that the interpretation of the figures is not obvious. However, if an arrow from left to right, for instance, connects the two representations, the relationship and transformation from one representation to another can be realized. This relationship is based on the analysis of the figures and on a computation (i.e., the difference between the big square and the four triangles). Referring to M1, the transformation of

[10] http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=162&Itemid=243&lang=it. Accessed: 2 January 2017.

representations corresponds to a specific movement of the four right triangles. In contrast to the graphical representations, the ability to view these representations simultaneously is not possible when manipulating M1.

The usage scheme of this mathematical machine is quite simple: shift the prisms-triangles into the frame, without raising them from the base and without overlapping them (this condition is evident because of the height of the prisms).

The mathematical meanings embedded in this artifact are: geometrical figures as the right triangle and square, area and equivalence of area by addition/subtraction of congruent parts. The mathematical machine itself fosters the hypothesis of the theorem since it shows squares and right triangles. We are aware that M1 involves 3D figures, while the statements (and the theorem above) all concern 2D figures. Nevertheless, we make the assumption that students' attention will be paid to 2D shapes because of the movement of the wooden pieces on the red square.

Two tasks are proposed: (1) in the square frame, place the prisms-triangles for obtaining square hole(s) (squares as figures in the background with respect to the triangles); (2) pass to a configuration to the other one (Fig. 11.3b, c). The task (1) involves geometrical figures that should be familiar to lower secondary school, as squares and triangles. Task (2) is based on the previous one. Another task involving the invariance of area can be proposed, for instance looking for several configurations of prisms-triangles on the frame and comparing the area of the holes. But this is not interesting for our purpose. The movement of the prisms is bound by the frame, which ensures the invariance of the sum of the areas of the prisms-triangles and the square holes or, in other words, the invariance of the area of the squares, whatever it is. For students, the movement with the same pieces in different configurations could evoke other manipulations, as Tangram puzzle. In each task, the students can recognize the squares on the legs or on the hypotenuse of the right triangles.

According to the analysis of the semiotic potential, we make the hypothesis that M1 can support a geometrical approach to the theorem and an informal proof leading to a formal one, according to Sinclair et al. (2012).

In the mathematical machine M2 as for in M1, there are two configurations (Fig. 11.4 on the left and in the center). The usage scheme of M2 is turning the crank placed on the middle of a side. When acting, M2 passes from one configuration to another. The two possible tasks are: (1) switching the configurations, (2) comparing the configurations. There are no other manipulations possible for the students, not even changing the lengths of the sides of the triangles.

M2 evokes the mathematical meanings of right triangles, square (above all, squares on legs and hypotenuse), but also perpendicularity, reflection, central symmetry, bisector and the criteria for congruence of triangles. The task (2) requires identifying the figures inside the hexagonal hole that change their sides in the two configurations because of the elastics. For instance, the hypotenuse of the upper triangle in Fig. 11.4a is transformed into a segment whose length is equal to the sum of the length of the legs; or a leg becomes the side of a square. Even if all the triangles appear congruent, this congruence cannot be verified by superposition of components (as in M1), but by measuring sides and angles and referring to the criteria for congruence of triangles.

Based on these analyses, we have chosen to propose to use M1 first in our experiment. We also ask students to reproduce 1:1 M1 on paper (four triangles and base square corresponding to the interior of the frame) and cut the triangles. This choice is due to the fact that there is one wooden model for the classroom and small groups have to be proposed with the same kind of artifact (following the didactic cycle, Sect. 11.3.1). In such a way, the students are supposed to construct a new artifact, that we call M1-paper.

Then, we want to pay attention to the two elements that characterize the semiotic potential of the reproduction of M1: the negligible thickness for all the components of the M1-paper and the lack of the frame. The first element can force the students to transfer constraints of the manipulation of M1, because they have to check that the triangles do not overlap. The material constraint of M1 should become explicit for the corresponding usage scheme for M1-paper. The second element can help to make evident the range of the movement of the right triangles on the square base: they have to remain inside the base square. And this is an usage scheme for M1-paper.

11.5.2 The Design of the Teaching Experiment

According to the theoretical frameworks within the mathematics laboratory methodology, activities for students are organized in didactic cycles (Sect. 11.3.1), consisting of small group work (GW), individual activities (IW), and collective mathematical discussions (CW). They correspond to the three phases A, B and C below.

Phase A concerns the work on M1 and M1-paper until the formulation of the theorem (9 h):

(1) GW: Exploration of the first mathematical machine M1 (Fig. 11.3);
(2) CW: sharing of the description of the M1;
(3) GW: construction of the M1 by paper;
(4) GW: study of the possible configurations of the four triangles of M1;
(5) IW: representation of M1 on workbook;
(6) CW: identification of relationships (invariants) between the components of M1/M1-paper.

Phase B deals with historical aspects and generalization of the theorem (3 h):

(7) History of the Pythagorean theorem and Pythagorean triples;
(8) GW: Generalization of the theorem by different puzzles.

Phase C proposes the exploration of M2 (4 h):

(9) CW: Exploration of the second mathematical machine M2 (Fig. 11.4);
(10) GW: Preparation of posters of the two mathematical machines.

The a priori analysis of these three phases points out some relevant considerations.

The two mathematical machines are proposed in two different phases, or didactic cycles, with two different aims. M1 is used to introduce the Pythagorean theorem to 7-grade students who have not yet met it, while M2 is proposed to the same students at the end of a teaching path when they have already dealt with the theorem.

In the collective discussion at the end of Phase A, the formulation of the relationship between areas is a crucial point. Depending on students' available knowledge, it could be based on (1) writing an algebraic treatment of the areas of the squares on legs and hypotenuse, or (2) other situations of equivalence of areas, as in the Tangram puzzle.

The mathematics laboratory, as it has been presented before, strongly demands students' involvement not only during the group work with the machines, but also during the collective discussion in which the explorations are shared and collective texts are written. In this, argumentation and proving processes are requested. Research on argumentation and proof (Hoyles & Healy, 2007) highlights students' difficulties that are taken into account in the a priori analysis.

The teaching experiments started in 2013, and have involved six Italian classes of 13-years old students (n = 135) and two teachers.[11] In this paper, we consider the data from two classes and a teacher. The analysis is carried out on students' worksheets, videos and photos of the two classes, and IWB files made during classes.

11.6 Results

In this section, we analyze Phase A and Phase C.

11.6.1 Phase A, Steps 1 and 2. Work with the Material Artifact in Small Group and Collective Description

During the first two steps, the students worked in small groups and were given the task of describing the machine M1 and deciding on the elements useful for its

[11]They regularly follow the methodology of mathematics laboratory with their classes and take part of the research team of mathematics education at the Laboratory of Mathematical Machines at the University of Modena and Reggio Emilia.

reproduction with colored paper (for instance, the types of triangles, the length of the sides). After the exploration of M1, the collective discussion enabled students to share their explorations and to agree on a written description of the machine. At the end, the students agreed on a collective description for M1 in which the right triangles were identified as congruent and equivalent, as well as on which kinds of figures can be obtained with the triangles. They also made conjectures about the use of M1; for instance:

> [2*C*] It might serve for the equidecomposability principle, composition of different shapes (rectangle, parallelogram, isosceles trapezoid, rhombus, isosceles triangle, deltoid, some [figures] go out of the base), movement of figures and dynamic geometry. As in Tangram the shapes composed of 4 pieces all have the same area (they are equivalent).

> [2*D*] The machine could be used to build equivalent geometric forms, to calculate the area or the perimeter, to think, and to make calculations.

Students' descriptions of M1 contained their personal meanings that the exploration of the machine evoked. This corresponds to our analysis of the semiotic potential, that is, a link is created between the artifact and students' knowledge. Students' description of M1 also contained relevant elements for approaching the Pythagorean theorem, as Tangram puzzle and the idea of 'same area'. Finally, students' descriptions show their effort to use a geometrical language. We note that it concerns 2D figures, even if the students act on 3D shapes. This confirms the assumption of the a priori analysis at Sect. 11.5.1.

In his instrumental orchestration, the teacher decides two didactical configurations: the first configuration with M1 and paper and pencil, the second configuration with the IWB. In the latter, he used the IWB to collect students' descriptions and write the shared presentation of M1. The IWB, available in the classes, represents an added value with respect to a non-technology classroom under different elements. The first element is that all the descriptions can be written (using a word processor) and shown to all the students. This means that those descriptions, with their artifact signs and mathematical terms, are shared not only by reading them. In such a way, the teacher fosters to compare and/or contrast them. The second element is that the space for writing in the IWB is wider than in the blackboard; the teacher inserts new text within the text already written also using different colors to emphasize correct or wrong sentences.

11.6.2 Phase A, Steps 3 and 4. Construction and Work with M1-Paper

The students easily obtained the reproduction in scale 1:1 by measuring and using tools for drawing (above all, rules and sets square). After this, the students had to fill in a worksheet with the properties of the two figures, square and right triangle, constituting the machine.

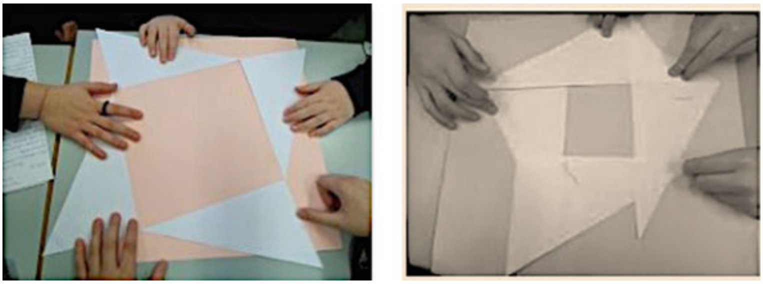


Fig. 11.5 Incorrect configurations by manipulating M1-paper

The manipulation of this new paper machine is guided by the task of looking for "square holes". This task requires applying the usage schemes of M1-paper: the triangles must remain in the big square (Fig. 11.5 on the left) and do not overlap each other (Fig. 11.5 on the right).

During the students' work, the configuration with the two square holes (Fig. 11.6 on the left) often appeared before the configuration with the square, which alone demanded more time (Fig. 11.6 on the right). This could be due to the fact that the sides of the square are not parallel to the side of the square frame, but also to the shared meaning of compositions of figures. In Step 1, the students had composed two prisms-triangles and obtained a rectangle, as in Fig. 11.6 (on the right). In general, they did not consider the prisms-triangle separately inside the frame; only a group wrote about the second configuration in the description of Step 1.

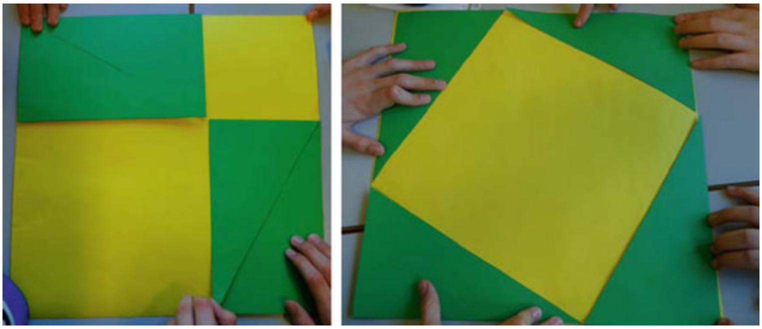


Fig. 11.6 The two configurations by M1-paper

11.6.3 Phase A, Step 4. Individual Work for Representing the Two Configurations with Paper and Pencil

The passage from the manipulation of the artifact (Steps 1–3) to graphic representation on the workbook takes into account other relationships between the elements of the machine. Although the students had correctly described the congruence of the four triangles, in the representation in the workbook several students drew right triangles, not all congruent (Fig. 11.7). In actual fact, the review of all the representations shows that an important invariant (the side of the square base is equal to the sum of the two legs of the machine) is not usually taken into account by the students. When the drawings were not correct, the teacher shared and discussed the wrong representations.

The didactical configuration contains students' drawings, a camera, IWB and M1. The teacher took pictures of some drawings and showed them by the IWB in the collective discussion. In his exploitation mode, he performed an important step: he used students' drawing as pivot signs. The drawings can be considered as artifacts signs, if they are seen from a global and holistic point of view, but the drawings of the two configurations of M1 are required to represent the invariant of the machine. By the IWB, the teacher planned a checklist with the geometrical properties of the components of the machine that had been shared in the previous discussion for comparing the different representations. Considering the drawings by the IWB and comparing them with M1 fosters correct representations.

The analysis shows the exploration of M1 carried out by the students is not enough for the emergence of mathematical meanings embedded in the machine and confirms the role of the teacher as mediator.

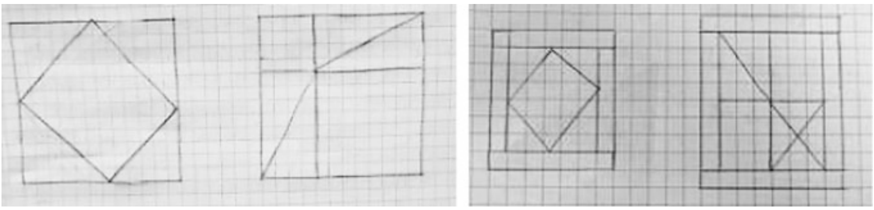


Fig. 11.7 Students' representations of the M1 on their workbooks

11.6.4 Phase A, Step 6. Collective Discussion with IWB

In the collective discussion the teacher took into account the passage from acting on the machine (both wooden and paper) to identify properties and relationships between the two configurations. The machine M1 is represented on the IWB by a photo. This passage is crucial.

Five moments are identified in this discussion. First, the students reproduced M1 on IWB starting from a picture of the configuration with two squares. Secondly, the teacher supported students' argumentations on the geometrical properties of the two holes: they are two squares and they are constructed on the two legs of the right triangle. Then, the students passed from that configuration to the other one and made argumentations on the right angles of the hole. In the fifth and last moment, the two configurations were compared.

In the first moment, after reproducing M1 from a picture (Fig. 11.8a) using "duplicator of figures" (available by the IWB software) for the congruent right triangles, the teacher wrote the known components of M1 below the picture (Fig. 11.8b). The instrumental orchestration with the IWB supported this discussion, enabling a new collective manipulation of the machine.

Then, the students started to answer the question asked by the teacher "Are you sure the two holes are squares?". This is a question pivot between an instrumental and perceptive result and the geometrical property of the holes. The task of Step 3 to look for square holes is based on a perceptive control, which guides the manipulation of M1 and supports the representations of the two configurations. Even if the drawings made by the students in Step 4 confirms the kind of shape of the holes, the justification of those results has not been considered and shared at that moment. For using the expression "the squares on legs and hypotenuse", it needs to be sure that the figures are squares.

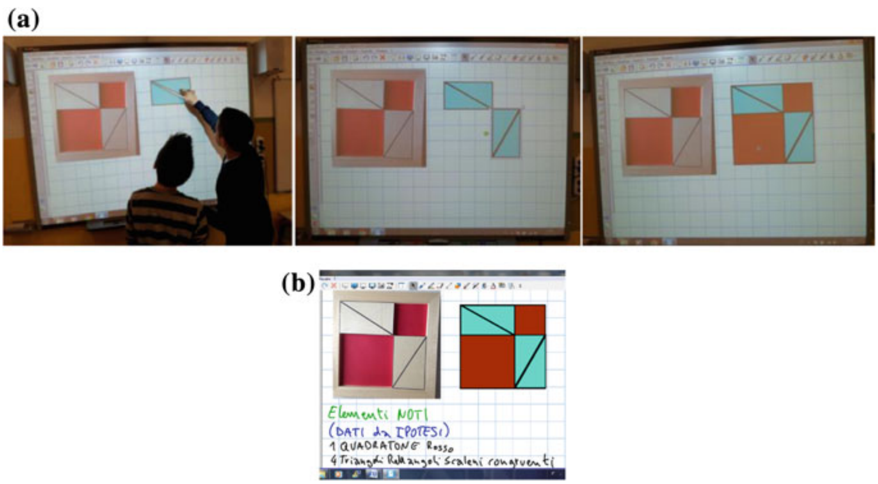


Fig. 11.8 First moment

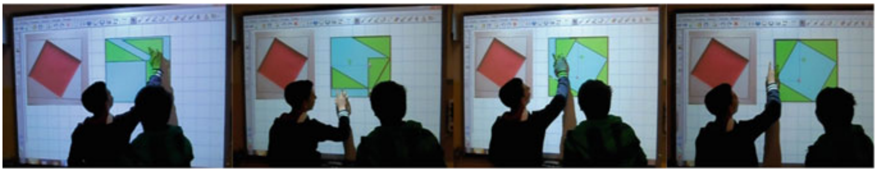


Fig. 11.9 Third moment: the passage from a configuration to the other one

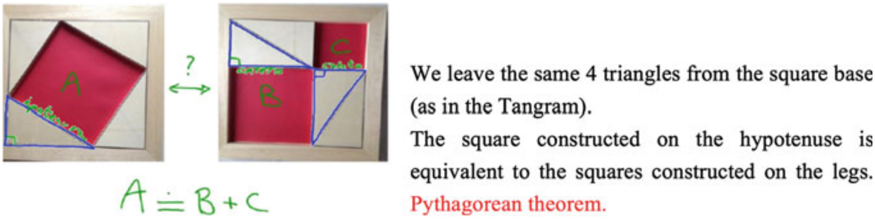


Fig. 11.10 Fifth moment

By the teacher's orchestrations, the IWB became the place for sharing ideas and constructing the justification of what was performed with M1. These orchestrations were based on the teacher's practice and on his use of the IWB as instrument during mathematical lessons, for himself and the students. When this teaching experiment was performed, the students had already started their instrumental genesis concerning the IWB.

In the third moment, the students passed from that configuration to the other one by dragging the right triangles as they made with the material machine (Fig. 11.9).

In the fourth moment, the students were faced with the square on the hypotenuse. This second argumentation process is strongly supported by what happened in the second moment.

In the last moment of Step 6, the teacher put together the two configurations (Fig. 11.10) and supported the comparison of the holes.

The use of IWB allows for the merging of the mathematical machine and its geometrical drawings, paying attention to the sides of the three red squares (Fig. 11.10). The screenshot of the IWB (Fig. 11.10) seems not very different from Fig. 11.1, but in our case it is the final point of the exploring and formulating processes. The standard representation of the Pythagorean theorem was drawn after this moment. The collective use of the digital machine allowed students to link the manipulation of the triangles to the manipulation of Tangram pieces (Fig. 11.10), which had been evoked at the beginning of the activities. In this way, the conservation of the areas of the holes is emphasized. The Pythagorean theorem becomes a particular case of equivalence of areas.

The last element is the numerical interpretation of the relationship between areas, leading to a formulation as "square of the legs and hypotenuse" (Sinclair et al., 2012).

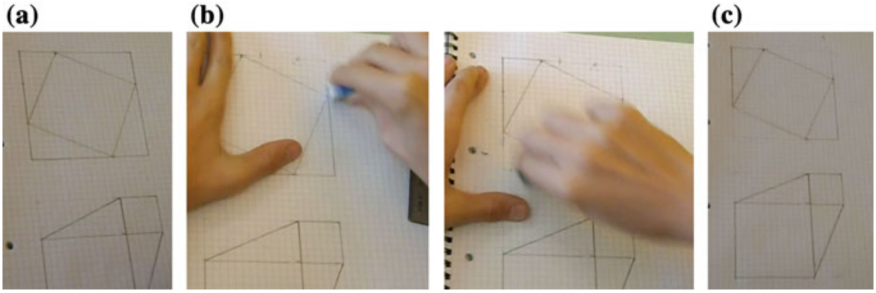


Fig. 11.11 From M1 to M2

11.6.5 Phase C

In this phase, the second mathematical machine was introduced in the classroom. It was explored in a collective session. By the use of ruler and sets square, the students identified the right triangles and the squares in the two configurations. In particular, the triangles are verified congruent by measuring their sides. The students quickly linked this machine to the Pythagorean theorem.

As for M1, the graphical representation of M2 by paper and pencil was not simple at all for the students. With respect to Step 4, the teacher did not revise the representations, keeping them as holistic drawings of the machine. But he supported the argumentation of the equivalence of the two configurations, in which the attention was paid to the hexagonal holes (true holes in this case).

The manipulation of M1, M2 and their graphical representations suggested a graphical treatment allowing two students to show the relationship between the M1 and M2. This aspect was not considered in our a priori analysis. The students drew the two configurations of M1 and deleted four sides in one figure (in the bottom in Fig. 11.11a)

1 S: In this figure too [Fig. 11.11a he indicates the figure on the top], we delete these two triangles (Fig. 11.11b, at the top).

2 S: We obtain the figure (Fig. 11.11c)

The teaching experiment ended with small group work for preparing posters for the two mathematical machines. Those posters were exposed during an exhibition in the school, in which the students presented the machines and the theorem to parents and visitors.

11.7 Conclusion and Discussion

In this teaching experiment, the use of different tools is related to specific steps of the didactic cycles implemented within the methodology of mathematics laboratory.

The first research question about if and how the Pythagorean theorem can be approached by using material and digital tools has been partially addressed.

The first element concerns the link between the planned tasks and students' personal meanings, as required by the definition of semiotic potential. More precisely, students' answers confirm that tasks in Step 1–3 were accessible; for instance, they show recognition of shapes and use geometrical language in the description of the machine. In the same sense, the task of looking for squared holes involves students, and in particular it is based on a global and perceptive apprehension of the machine, fostering a perceptive control of the action. All these elements constitute tasks in which the students know what they have to obtain, but they do not know how to reach the result. I think that this is a strong element in defining the link between artifact and personal meanings. The passage from this perceptive level to a geometrical justification, that is, a link between artifact and mathematical meaning, is taken into account by the teacher in the collective discussion at Step 6. The second relevant link corresponds to the reference to Tangram puzzle and equivalence of areas. It is already present in some students' description at Step 1 and can appear in the work by task 2. But there the students were not asked to look for the invariant of the two configurations. In our analysis that meaning could be less strong that the previous one, or form a different perspective, a meaning for few students. Nevertheless, it becomes relevant to include it as a part of the collective discussion. If that link to Tangram puzzle manipulation does not appear, it becomes a meaning to construct for the teacher, proposing the operation of adding/subtracting areas also through symbolic writing with respect to the area of square base. In any case, this work on areas makes the Pythagorean theorem a particular case of experienced equivalence.

The tasks just discussed are also based on visualization, for the mathematical machines and its representations with paper and pencil and by IWB. Referring to Duval (2005), the representation of a machine results a relevant task for discussing about its geometrical properties and support the passage from holistic drawings to geometrical drawings. In terms of TMS, students' drawings can be used as pivot signs for this passage. In addition, the comparison of drawings contains the germ of generalization for any right triangle.

The analysis shows that behind the kinesthetic experience with the machine, there are important cognitive processes such as the identification of invariants, relationships between the components and usage schemes. The relationship, that is not foreseen in our a priori analysis between M1 and M2, proposed by some students highlights the potential of rearranging pieces for equivalent figures (Sinclair et al., 2012) and confirms the relevance of the choices for the didactical path.

On the second research question about instrumental orchestrations by the teacher, the use of the IWB plays a crucial role during the collective discussions. The IWB is an instrument orchestrated by the teacher, but also used by the students who can perform actions close to the manipulation of the machine. It not only allows the students to manipulate the represented machine, but also allows the teacher to reproduce some components in order to support the argumentation process. In this way, the use of IWB also supports students' visualization and this represents a dynamic dimension with respect to students' personal drawings using paper and pencil. When the machine and its movement are simulated in the IWB, a link between the two technologies is performed for both students and teacher. Following Drijvers, then, we could say this case refers to the third didactical functionality of ICT. But this functionality seems to be based on the fact that IWB is already an instrument for the students, that is, its instrumental genesis has already occurred. Furthermore, it has to be taken into account by the teacher in his teaching practice. The role of the IWB in the collective work should be more studied within the framework of the semiotic mediation. In the classrooms, other instruments were used only by the teacher, such as a videocamera and a camera for taking pictures to show at IWB, but there are also instruments used only by the students such as paper and pencil, M1 and M1-paper.

In this work, the choice of the artifacts, and, in particular, their introduction in specific moments of the teaching path are also part of teacher's instrumental orchestration, corresponding to didactical configurations and exploitation modes, related to the identification of didactical functionalities of the artifacts. The first mathematical machine is used at the beginning, with a strong emphasis on manipulation of its components and its description. The second proposes another proof of the theorem and fosters the identification of its components. In the didactical configuration, we should also consider the other artifacts that could be available in the classroom, as the paper models. The proposition of two different mathematical machines in two different phases of the teaching experiment suggests that it is necessary to distinguish the way in which an artifact can be used by the teacher, introducing the idea of "didactical use". The 'didactical' use of an artifact depends on its semiotic potential (in which students' personal meanings are considered), teacher's goals for students' learning and, finally, by the didactical cycle in which the artifact is introduced. The didactical uses can be: (i) to explore a new property or meanings (in this case by M1), (ii) to assess students' processes and knowledge (in this case by M2). The choice of a didactical use influences the kinds of questions asked to students and expected answers. This is an idea that needs further discussion. We have not analyzed other relationships between the two mathematical machines.

The analysis of this teaching experiment allows to propose the idea of a learning environment composed by different kinds of tools, material and digital, in which students are involved in several tasks and activities with those tools and teacher performs different instrumental orchestrations. We would use the expression of "composite learning environment" for it and deeps the discussion on its features, potentialities and limitations. And this represents an element that needs further research.

This work opens the perspective of further investigation on cognitive processes concerning the use of the mathematical machines and their graphical representations with the geometrical properties embedded in the material machines. The work on graphical representations at Step 4 seems to be consistent with the construction of figural concepts (Mariotti, 2005), in which the visual aspects are dominant at the beginning but then formal constraints become more relevant. It pays attention to what could be implicit in the exploration and manipulation of material tools in mathematics education. In our opinion, this teaching path can also give new insights on the visual proofs of the Pythagorean theorem.

Acknowledgements I wish to sincerely thank the teachers Stefano Barbieri and Francesca Scorcioni, and the anonymous referees for their precious suggestions.

References

Anichini, G., Arzarello, F., Ciarrapico, L., & Robutti, O. (Eds.). (2003). *Matematica 2003. La matematica per il cittadino. Attività didattiche e prove di verifica per un nuovo curricolo di Matematica (Ciclo secondario)*. Lucca: Matteoni stampatore.

Barbieri, S., Scorcioni, F., & Maschietto, M. (2014). Scoperta del Teorema di Pitagora con le macchine matematiche: elementi di discussione di didattica laboratoriale. In B. D'Amore & S. Sbaragli (Eds.), *Parliamo tanto e spesso di didattica della matematica. Incontri con la matematica n. 28* (pp. 155–158). Bologna: Pitagora Editrice.

Bardelle, C. (2010). Visual proofs: An experiment. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of CERME 6* (pp. 251–260). Lyon: INRP.

Bartolini Bussi, M. G. (2010). Historical artefacts, semiotic mediation and teaching proof. In G. Hanna, H. N. Jahnke, & H. Pulte (Eds.), *Explanation and proof in mathematics* (pp. 151–167). New York: Springer.

Bartolini Bussi, M. G., & Borba, M. C. (Eds.). (2010). The role of resources and technology in mathematics education. *ZDM Mathematics Education, 42*(1), 1–4.

Bartolini Bussi, M. G., Garuti, R., Martignone, F., & Maschietto, M. (2011). Tasks for teachers in the MMLAB-ER Project. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 127–130). Ankara, Turkey: PME.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of international research in mathematics education* (II ed., pp. 746–783). New York, NY: Routledge.

Bartolini Bussi, M. G., & Maschietto, M. (2006). *Macchine Matematiche: dalla storia alla scuola. Collana Convergenze*. Milano: Springer.

Bartolini Bussi, M. G., & Maschietto, M. (2008). Machines as tools in teacher education. In D. Tirosh & T. Wood (Eds.), *Tools and processes in mathematics teacher education, The international handbook of mathematics teacher education* (Vol. 2, pp. 183–208). Rotterdam: SensePublishers.

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In S. J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 135–151). Switzerland: Springer.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Use of technology in lower secondary mathematics education*. *ICME 13 Topical Surveys*. Switzerland: Springer.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool; instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234.

Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie: développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements. *Annales de didactique et de sciences cognitives*, *10*, 5–53.

Eaves, J. C. (1954). Pythagoras, his theorem and some gadgets. *Mathematics Magazine, 27*(3), 161–167.

Giacardi, L. (2012). L'emergere dell'idea di laboratorio di matematica agli inizi del Novecento. In O. Robutti & M. Mosca (Eds.), *Atti del Convegno Di.Fi.Ma. 2011* (pp. 55–66). Torino: Kim Williams Books.

Hoyles, C., & Healy, L. (2007). Curriculum change and geometrical reasoning. In P. Boero (Ed.), *Theorems in schools: From history, epistemology and cognition to classroom practice* (pp. 81–115). Rotterdam: Sense Publishers.

II Giardino di Archimede. (2001). *Pitagora e il suo teorema*. Firenze: Edizioni Polistampa.

Kratky, J. L. (2016). *Pedagogical moves as characteristics of one instructor's instrumental orchestrations with tinkerplots and the TI-73 explorer: A case study* (Doctoral dissertations). Retrieved from http://scholarworks.wmich.edu/dissertations/2480 (Order No. 2480).

Mariotti, M. A. (2005). *La geometria in classe. Riflessioni sull'insegnamento e apprendimento della geometria*. Bologna: Pitagora Editrice.

Mariotti, M. A. (2007). Geometrical proof. In P. Boero (Ed.), *Theorems in schools: From history, epistemology and cognition to classroom practice* (pp. 285–304). Rotterdam: Sense Publishers.

Maschietto, M., & Bartolini Bussi, M. G. (2011). Mathematical machines: From history to the mathematics classroom. In P. Sullivan & O. Zavlasky (Eds.), *Constructing knowledge for teaching secondary mathematics: Tasks to enhance prospective and practicing teacher learning* (Vol. 6, pp. 227–245). New York: Springer.

Maschietto, M., & Bartolini Bussi, M. G. (2014). The laboratory of mathematical machines of Modena. *ICMI BULLETIN*. Papers from unpublished issues of the ICMI Bulletin. http://www.mathunion.org/fileadmin/ICMI/files/Digital_Library/ICMEs/Bulletin_Maschietto_BartoliniBussi2_01.pdf. Accessed January 2, 2017.

Maschietto, M., & Soury-Lavergne, S. (2013). Designing a duo of material and digital artifacts: The pascaline and Cabri Elem e-books in primary school mathematics. *ZDM Mathematics Education, 45*(7), 959–971.

Maschietto, M., & Trouche, L. (2010). Mathematics learning and tools from theoretical, historical and practical points of view: The productive notion of mathematics laboratories. *ZDM Mathematics Education, 42,* 33–47.

Monaghan, J., Touche, L., & Borwein, J. M. (2016). *Tools and mathematics. Instruments for learning. Mathematics education library*. Switzerland: Springer.

Moutsios-Rentzos, A., Spyrou, P., & Peteinara, A. (2014). The objectification of the right triangle in the teaching of the Pythagorean theorem: An empirical investigation. *Educational Studies in Mathematics, 85,* 29–51.

Moyer, P. S., Bolyard, J. J., & Spikell, M. A. (2002). What are virtual manipulatives? *Teaching Children Mathematics, 8*(6), 372–377.

Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. *Interacting with Computers, 15*(5), 665–691.

Rufus, I. (1975). Two mathematical papers without words. *Mathematics Magazine, 48*(4), 198. For Fig. 11.1 Copyright 1975 Mathematical Association of America. All Rights Reserved.

Sinclair, N., Pimm, D., Skelin, M., & Zbiek, R. M. (2012). *Developing essential understanding of geometry for teaching mathematics in grades 6-8*. Reston: NCTM.

Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments: Guiding student's command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*(3), 281–307.

Vygotskij, L. S. (1978). *Mind in society. The development of higher psychological processes*. Harvard, MA: Harvard University Press.

Chapter 12
Communication When Learning and Teaching Mathematics with Technology

Lynda Ball and Bärbel Barzel

Abstract In this chapter the role of technology in supporting interactions between students, between students and teachers and between students and technology is investigated. The way that interactions in the presence of technology support the development of different types of mathematical knowledge—conceptual, procedural and metacognitive knowledge—is also considered. These considerations led to our investigation of different types of technology specific to mathematics education and the type of communication supported by these technologies. We developed the distinction between 'communication through technology' (e.g. through use of social networks to work collaboratively on problems), 'communication with technology' (e.g. syntax entry to obtain a result), and 'communication of technology displays' (e.g. when technology displays are used as a stimulus for communication). Opportunities for the development of students' knowledge are discussed from the perspectives of the different types of communication and collaboration enabled through the presence of technology in mathematics education.

Keywords Technology · Mathematics education · Communication with technology
Communication through technology · Communication of technology

12.1 Introduction

There has been considerable research, including research reviews, related to technology in mathematics education over the past few decades (e.g. Barzel, 2012; Blume & Heid, 2008; Drijvers et al., 2016; Heid & Blume, 2008). With increasing

L. Ball (✉)
Melbourne Graduate School of Education, The University of Melbourne, Melbourne, Australia
e-mail: lball@unimelb.edu.au

B. Barzel
University of Duisburg-Essen, Duisburg, Germany

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_12

access to technology, there are increased opportunities for pedagogical use of technology (e.g. Pierce & Stacey, 2010) and these can motivate teachers' reflection on teaching and learning. These increased pedagogical opportunities will require teachers to rethink their personal pedagogical practices, as the process of instrumental genesis in integrating technology in students' individual learning trajectories is manifold and complex (Guin & Trouche, 1999; Trouche & Drijvers, 2014). In this chapter, we use the lens of communication and interaction to improve our understanding of the complexity of teaching and learning with technology, as students' learning of mathematics can be shaped through language and communication (Steinbring, 2015).

Classroom dialogues around use of technology will evolve as pedagogical opportunities become accessible and as teachers become aware of these opportunities through professional development (PD), professional reading and experience. In addition, there will be possibilities for collaboration and communication beyond the classroom as the potential of technology is harnessed, resulting in mathematics education becoming accessible to more students and teachers through distance-education, online interactions and social media. To prepare teachers for the challenges posed by new opportunities it is imperative to understand the influence of communication on development of students' conceptual, procedural and metacognitive knowledge.

12.2 Theoretical Background

In this section, we provide the theoretical background concerning communication and technology to provide insight into the role of technology in communication in the teaching and learning of school mathematics.

12.2.1 Communication in Mathematics Education

Communication is a crucial aspect in students' learning to develop mathematical knowledge (e.g. Mueller, Yankelewitz, & Maher, 2012). Steinbring (2015) noted "Students' learning of mathematics in teaching processes is enclosed in language and communication" (p. 282), indicating that all sorts of communicative actions—speaking, depicting and gesturing—are included in a meaningful exchange. Following the core idea of enactivism and embodied cognition, any intelligent behavior and cognition can only be developed by interaction between the "complete agent" and its environment (Maturana & Varela, 2009; Goodchild, 2014). A big challenge in the frame of mathematical environments is that mathematics is abstract and not directly accessible by senses. Therefore, Duval (2000) points out "The only way gaining access to them [i.e. the mathematical objects] is using signs, words or symbols, expressions or drawings." (p. 61). Signs and symbols do not have a meaning on their own but instead this meaning must be produced by the learners themselves using objects of reference or reference contexts (Steinbring, 2009). The signs refer at the same time to a certain content and mathematical knowledge. The interplay between objects of reference, the concept and the signs are represented in the epistemological triangle (Steinbring, 2009) to model the construction of mathematical knowledge during an interaction—and a sequence of such triangles connected to each other cover the whole process of interactions and learning. This triangle serves as a model for any interaction in the frame of a mathematical environment—either to promote conceptual knowledge by initiating activities like investigating, structuring and categorizing of mathematical objects or to develop procedural knowledge. For both, the observed interplay between signs, concepts and reference contexts helps teachers and researchers to understand how learning occurs.

There has been extensive research related to communication in mathematics education, with one focus on the development of sociomathematical norms in classrooms (e.g. Yackel, 2002). Finding an acceptable justification for the solution to a problem or finding a convincing and correct argumentation for a mathematical relationship are examples of activities which strongly require an exchange of ideas, thoughts and knowledge. These kinds of classroom interactions were observed between primary students, as they discussed their reasoning when solving mathematics problems (Yackel & Cobb, 1996). These interactions clearly contributed to the students' development of an understanding about what was considered to be an acceptable justification for the solution to a mathematical problem. The work of Yackel and colleagues serves to highlight the important role of communication in the mathematics classroom.

Types of mathematical environments with regards to different social settings were a focus of a review of research on interactive learning in mathematics education. Kahveci and Imamoglu (2007) investigated the role of different types of interactions (such as classroom interactions, small group interactions and interactions with technology) on mathematical learning. The studies examined provided examples of interactions that enhanced higher order mathematical skills (such as mathematical reasoning, self-regulation and metacognition). Improvement of such skills requires students to communicate mathematically, hence interaction with peers, teachers or communication through technology.

To use technology for effective collaboration and communication in mathematics classrooms it is necessary to consider the role of technology in the epistemological process developing mathematical knowledge. Given the broad range of technologies available in classrooms currently, there is impetus to consider how technology can be utilized to promote conceptual, procedural, and metacognitive knowledge.

12.2.2 Technology in Mathematics Education

There are two types of technologies to be considered in the development of mathematical knowledge, namely general-purpose technologies and specific-purpose technologies (Drijvers, Barzel, Maschietto, & Trouche, 2006). A general-purpose technology for use in mathematics education has broad applicability across a range of lessons and mathematical topics (i.e. mathematical software such as a computer algebra system, spreadsheet, geometry package or dynamic statistics package). A specific-purpose technology for use in mathematics education, such as an applet, could target scaffolding of a procedure (e.g. a virtual balance model for supporting 'do the same to both sides' when solving linear equations) or support development of a particular concept (e.g. zooming in on a virtual number line to show that between any two numbers you can always find another number—the concept of decimal density). There is a substantial difference in the affordances of a technology that allows integration over time in different topics compared to one focused on one or two specific lessons. There is also an additional consideration and this is concerned with teachers' knowledge of syntax and their familiarity with the technology. For specific purpose technologies, there will be a need for teachers to learn how to use the technology each time a new technology is encountered, even though they are often designed to be intuitive. For a general-purpose technology, a teacher often knows the syntax of the technology, or is at least familiar with aspects of syntax, so the focus can be on how to utilize a familiar technology for a particular pedagogical purpose, rather than learning how to use a new technology. Both types of technology, specific-purpose and general-purpose, have important roles in the mathematics classroom and both can drive communication and collaboration. The role of these types of technologies in fostering communication and collaboration will be discussed later in this chapter.

In addition to specific-purpose and general-purpose technologies for teaching and learning mathematics, there are a range of general online communication technologies (e.g. virtual worlds, audience response systems, social networking software, etc.) and offline communication technologies (e.g. data projector, interactive whiteboards, powerpoint); these communication technologies support interactions between students, and between students and teachers, both inside and outside the classroom.

In many countries, there has been, and continues to be, a focus on the use of offline technologies for teaching and learning mathematics; these can include specific-purpose, general-purpose and communication technologies. Offline technologies are available on a range of devices including handheld (such as graphic or CAS calculators, tablets or laptops), desktop computers or interactive whiteboards. When technologies don't require an internet connection this means that teachers and students can use the technology in a range of classrooms across a year (or longer) and the technology can be available for classwork, homework (if the student has the appropriate handheld device or software or app) and assessment. Given that online networks and internet access are still not stable in all regions of some countries, offline technology is currently still an important option to enable integration of technology in mathematics classrooms. The mathematical software available for offline technologies and online technologies can enable similar pedagogical opportunities for teaching in a mathematics classroom due to the ability for material to be shared. Offline technologies can also support classroom connectivity between a teacher and students or between students by creating a local area network (e.g. Clark-Wilson, 2010a) or simply by displaying a screen (e.g. through use of a data projector or smartboard), thus enabling public display of mathematical work of the teacher or a student, or collection and display of results from different students, or using quick polls with an audience response technology. In addition, there are a range of online options for sharing material and these technologies have the additional affordance that information can be shared outside the physical classroom. For example, Symons and Pierce (2015) studied the type of talk used by year 5 students who participated in online collaborative problem solving. According to the specific orchestration and design of the classroom, technology can foster communication and collaboration when students share screens and observe the work of peers or a teacher (Drijvers, Monaghan, Thomas, & Trouche, 2015; Goos et al., 2009).

Online access becomes crucial in situations where it is necessary to exchange data with the outside world, for example, when downloading a dataset from the web for analysis (for example, when using dynamic statistics), or working collaboratively on problem solving or modeling tasks using synchronous online media together with people who are not collocated. We anticipate that the problem of well-established online access will decrease with time, but currently it provides a need to consider both online and offline technologies.

12.2.3 Communication and Technology

All the above-mentioned types of technology (specific-purpose technologies, general-purpose technologies and general communication technologies) have an influence on communication when learning and teaching mathematics inside and outside the classroom as all of them foster specific cognitive activities in the process of learning and understanding mathematics.

A benefit of using general communication technology was also reported by Roschelle et al. (2010), who found that students in their study using a technology that provided group feedback were more likely to discuss their mathematical work (in this case examples related to fractions) than students in a control group without technology.

Pierce and Stacey (2010) provide information about the range of pedagogical opportunities available in the presence of mathematical analysis software (MAS), a general-purpose technology which integrates CAS features, statistics, graphing and dynamic geometry. The authors highlight the ability to use MAS to promote, among other things, cognitive activities in the classroom that foster communication like discussing, explaining, structuring and classifying. It is possible to use technology displays as a prompt for inquiry learning (Fuglestad, 2009) and class discussions, to use multiple representations (e.g. symbolic, graphical, numerical and dynamic visualisations) to deepen students' understanding, and to scaffold students' learning of procedures through the availability of automatic performance of routine procedures. These approaches can support the development of either conceptual understanding or procedural knowledge and extend the range of task formats possible, for example pattern finding when different graphs or expressions are generated by the technology (e.g. Barzel, 2012). Ainley, Bills, and Wilson (2005) reported use of a spreadsheet to develop students' understanding of the meaning of a variable, thus using technology to support development of conceptual understanding. Vincent, Chick, and McCrae (2005) reported two year 8 students who used pre-made Cabri models to explore virtual representations of real situations and to support argumentation. The ability to collect data from many cases led to cooperative development of conjecture and proof.

Besides these examples with general-purpose technologies you can also find examples of specific-purpose technologies which foster communication as these technologies quite often are constructed to give a certain stimulus to initiate the learning of a certain mathematical idea or they provide a dynamic visualization for exploration. An example is the virtual base-10 blocks available through sites such as National Library of Virtual Manipulatives (http://nlvm.usu.edu). Virtual base-10 blocks is a dynamic visualization to support the development of conceptual knowledge of regrouping, when dealing with whole numbers. The ability to use multiple representations with technology may contribute to students' ability to think flexibly (Reimer & Moyer, 2005), which in turn can provide a stimulus for exploring mathematics to develop understanding.

The purpose of communication is an important consideration when working with technology. For different people, there will be different purposes for communication; teachers may want to provide written examples that give exemplars for solving problems, while using verbal communication to pose questions to students to check their mathematical understanding. Students may work alone, in pairs or in groups, with a technology, to explore mathematical situations and use verbal communication to make sense of technology displays and to find some structure in the results through their explanations. There are a multitude of possibilities for the ways in which communication can occur in the mathematics classroom and the ways that technology can influence the nature of communication.

12.3 Communication Through, With and Of Technology

The interest of this study was to gain deeper understanding of how the different types of technology influence communication in the classroom. For this we investigated the type of cognitive activities which are supported by the different types of technology in the development of mathematical knowledge. The following questions directed our study:

- What role does technology play with respect to communication when learning mathematics, in the interaction between students, between students and teachers and between students and technology?
- How do these interactions in the presence of technology support the development of the different types of mathematical knowledge: conceptual, procedural and metacognitive?

To investigate these questions, we surveyed the literature, and drew on experiences from Australia and Germany in the integration of technology into curriculum.

The result of our investigation is that communication in the presence of technology can be categorized as communication *through* technology, *with* technology or *of* technology displays. These three aspects reflect the different ways that communication occurs in a classroom with access to technology and increasingly between individuals outside the classroom (e.g. in virtual worlds or through social networking).

- Communication *through* technology involves use of technology to support face-to-face communication or communication between students and/or teachers who are not in the same location.
- Communication *with* technology considers the entry of syntax, selection of menu items, programming or any command that drives the technology to produce a display. This communication is, for example, through key strokes, by touching a screen, using gestures to move objects on the screen, or by providing verbal commands.
- Communication *of* technology displays is evident when a technology display is a stimulus for discussion. This discussion could occur in a range of contexts, for example, through two students' consideration of one shared screen or through public display of student work via technology such as an interactive whiteboard or a data projector.

Consideration of these three aspects of technology and associated communication is important to gain insight into potential affordances and constraints of any technology. Although we distinguish between these three roles for technology, we are cognizant of the fact that these do not occur in isolation. It is likely that two or three aspects of communication may occur simultaneously while using technology in mathematics classes. For example, when entering syntax (communicating *with* technology) there can be interpretation of the correctness of the entry (communication *of* technology displays). The inability to separate these two roles in real classrooms is alluded to by Schneider (2002) and Peschek (2007) who suggest that when working with technology (CAS in their case) students will view the CAS as the 'expert'. Peschek (2007) argues that precise communication (i.e. entry of syntax, etc.) is required when viewing technology as an expert (to communicate with the expert), as well as a need to interpret the information provided by the 'expert' (i.e. to understand the display of the technology). These comments serve to highlight the ideas of communication *with* and *of* technology. The notion of a technology as an ‘expert’ suggested an interplay between these two roles in the development of mathematical understanding.

It is important to identify the cognitive activities supported by technology that lead to improved learning for students and use this to guide utilization of technology in classroom interactions. Woo and Reeves (2006) suggest that in order to design activities to facilitate effective interaction within web-based learning environments it is first important to understand the nature of interaction in the presence of technology. The mere presence of technology does not guarantee good teaching and deep learning. To maximize the chance that technology does positively impact student learning it is essential to investigate and understand how the interactions enabled by access to technology can promote mathematics learning.

In the following sections; the influence of communication *through, with* and *of* technology is elaborated and illustrated in terms of the contribution to the development of conceptual, procedural and metacognitive knowledge.

12.3.1 Communication Through Technology

Communication *through* technology should be regarded on three geographical levels. The first level describes the communication through technology inside the classroom enabled through display technologies or linked devices to share screens. This allows the screens to be an object of reference (i.e. mathematical results produced by students or the teacher) used as a stimulus for mathematical discussion between teacher and students or between students. The second level is virtual communication between students of one class taking place outside the classroom. This is realized through technology opening avenues for discussion of mathematics beyond the confines of the physical classroom, for example, when doing homework collaboratively or supporting each other from home via general social networks or specific school networks. In the third geographical level, social networks provide opportunities for cross-cultural communication between mathematics classrooms (e.g. Isoda, McCrae, & Stacey, 2006) from different places in the world showing students that mathematics is learned and used internationally (e.g. Erasmus program of the European Union).

On the first level, inside the classroom, communication through technology occurs whenever a screen is shared to present mathematical objects and discuss them. Guin and Trouche (1999) reported the positive nature of implementing the classroom culture of having a ‘Sherpa student’ who uses the technology under the guidance of the teacher by displaying his or her calculator screen for the whole class (Trouche, 2004). Besides presenting one screen, which could also happen on a data projector or more dynamically on an interactive whiteboard—networking programs allow even more options which can influence the communication in the classroom. For example, Clark-Wilson (2010a) investigated the potential of using a wireless hub as a self-contained system where internet is not required. Calculators are connected to enable a quick collection of classroom data through screen capture.

In this case the technology can provide an aggregate of the class results to support a meaningful mathematical classroom discourse. It also enables formative assessment for current and future lessons by either running a quick poll of individual student answers or by gaining insight into students' understanding during discussions about the technology displays. For example, a teacher could provide students with a photo of a bridge which could be shared using a wireless hub. Students could be given the task of finding a parabola to approximate the shape of the bridge. Screens from multiple devices could be projected simultaneously and a class discussion could follow about approaches used to produce given curves. The focus of the discussion could be about placement of axes on the photo, choice of points, selection of the quadratic form based on given information and how well the given curves fit the shape of the bridge.

The key aspect of technology use here is the ability to display screens and share files, questions and information relatively quickly (e.g. Muir, 2014) enabling more time to be spent discussing the mathematics, prompted by the technological displays. Observation of the work of other students may encourage self-monitoring, when students compare their own mathematics to that being displayed, thus supporting metacognition and reflection.

There is still untapped potential in the harnessing of social networks, virtual environments and other communication technologies to connect students who are learning school mathematics together, but in different locations. When we consider communication through technology, the nature of technology use changes how students do, learn and think about mathematics; it has the potential to shift from being an individual pursuit, where a student uses technology to learn or do mathematics by themselves, to a tool to promote collaboration and communication with the potential to enrich mathematics for students.

12.3.2 Communication With Technology

We regard communication *with* technology as the ability of a student or teacher to drive a technology (e.g. through syntax entry, or programming, or use of a touch screen command). But is this a communication? The German sociologist Luhmann (1994) does not classify an interaction with a machine (i.e. the ability to communicate with technology) as communication as it does not occur in a social system. The underlying philosophy here is that communication must include moments of information, utterance and understanding, so that each participant cannot predict or control what is communicated, which is not the case when entering syntax or driving a technology to perform a given task. But in alignment with Schneider (2002) and Peschek (2007), who investigated students' work with CAS and regarded it as a communication with an "expert", we would classify the commands to make a technology produce a display as communication with technology. Another example is the construction of a point of intersection in a geometrical construction. In contrast to a paper-and pencil-construction it is necessary to specify the point as a "bounded" point based on the two lines rather than a "free" point. These requirements of the technology can trigger reflections on the nature and the dependencies of the different elements of the geometrical construction and by this enhance understanding.

These examples show that even though there are more signs and symbols referring to the specific commands to use the technology in an appropriate way, the use of technology can have the benefit of supporting conceptual and procedural knowledge.

In communicating *with* technology, there is a necessity to know the conventions of the technology, aligned with the necessity for students to communicate with an expert. Choice of appropriate commands and syntax to produce desired results may deepen a student's conceptual knowledge, for example, when working with a graphing package students must think about the roles of letters in functions and identify dependent and independent variables when letters other than x and y appear. It is important to understand the different roles of letters in algebraic expressions, functions or equations when entering commands to obtain results from a computer algebra system. A specific example is the solving of an equation, such as $2x + by = cy + 3$. Students need to correctly enter the equation into CAS; noting that in pen-and-paper mathematics there is implicit multiplication between two letters, but in CAS (refer to the first two lines of Fig. 12.1) cy will be a two-letter variable, with a different meaning to $c \times y$. In Fig. 12.1, the first two lines show that cy is treated differently to $c \times y$ (note that this is displayed as $c{\cdot}y$, rather than show $c \times y$ or cy). In this case the machine convention, where cy is a two-letter variable, is different to pen-and-paper mathematics where two adjacent letters have implicit multiplication unless stated otherwise; this stimulates discussion of the meaning of the letters in an equation. A further consideration in communicating with the machine is that students or teachers will need to indicate whether they are solving for x, y, b or c when solving the equation, as it is possible to solve for any of

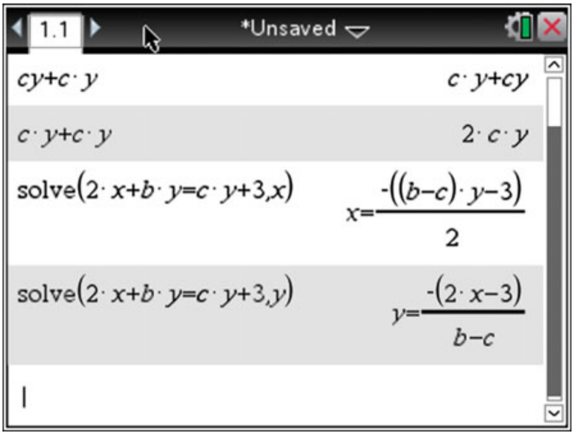


Fig. 12.1 Screen capture showing TI Nspire display for variables

these unknown letters; again, this encourages students to consider the role of the letters. Technology displays can prompt discussion of the different roles of letters in algebraic expressions, functions or equations; this requires communication *of* technology, in other words, discussion of the results of communication *with* technology. These types of communication rarely happen in isolation, as technology users need to monitor their work with technology, through consideration of syntax, conventions and reflection on technology results.

Correct choice of syntax or commands also enables procedures to be carried out correctly, which supports procedural knowledge. In addition, the display produced once commands or syntax is entered provides immediate feedback to students so that they can be in control of their own work and monitor their work. In this way, the communication with the technology includes personal interpretation of the display to enable students to be able to continue to communicate with the technology and enter the next command or syntax. Finally, decisions about appropriate features of technology for a given purpose and choice of syntax or commands can promote reflection and mathematical discussion, thus supporting metacognition.

12.3.3 Communication Of Technology Displays

Hiebert and Carpenter (1992) highlight that it is important to help students build mental models and internal representations of procedures that become part of larger conceptual networks. For example, the use of an area model to support the understanding of fraction multiplication (Lamon, 2012) may help students to develop a mental model to support the learning of an algorithm for multiplication of fractions. It is important to develop mental models and internal representations prior to application of routine procedures to a range of problems. Students should be able to carry out the steps required for given procedures and be able to decide on the appropriateness of a procedure in a given context through awareness of conditions for applicability of a procedure (Barzel, Leuders, Prediger, & Hußmann, 2013).

One means of building rich webs of relationships of mathematical phenomena is through multiple representations of mathematical objects, which can be fostered by using technology (Dick & Edwards, 2008). In many ways this can be considered to be contributing to larger conceptual networks as students explore and make sense of mathematical phenomena through the use of technological representations. Rezat and Sträßer (2012) note the important role that technology takes in mediating learning of mathematics in the classroom and we believe that discussion of technology displays is crucial in this regard. Communication *of* technology displays, where students and students (or the teacher) consider the display of the technology as a prompt for mathematical discussion, can foster students' mathematical knowledge. Conceptual knowledge can be supported by use of a technology display as a stimulus for verbal communication about mathematical concepts. Some general purpose technologies can be used to focus on specific skills or concepts. For example, Fitzallen and Watson (2014) noted secondary students using Tinkerplots

a	2*a*
1	2
2	4
3	6
4	8
5	10
6	12
7	14

a	2*a*
1	2
1.5	3
2	4
2.5	5
3	6
3.5	7
4	8

a	2a
1	2
1.1	2.2
1.2	2.4
1.3	2.6
1.4	2.8
1.5	3
1.6	3.2

Fig. 12.2 Investigating patterns in results to explore the concept of a variable

(https://www.tinkerplots.com/) to explore multiple representations in order to deepen their understanding of statistics. A further example of use of technology to promote conceptual understanding is when students learn about a variable as a generalized number through use of a spreadsheet. As shown in Fig. 12.2, students can use the spreadsheet to explore the option of generating values for a and 2a, starting with whole numbers (1–7) then exploring increments of 0.5, followed by increments of 0.1. The ability to quickly produce many values in a table (only a few shown here) and the ease with which students can carry out their own exploration and produce results to discuss can promote understanding of the meaning of a variable.

In addition, a cell in a spreadsheet operates as a variable (see Fig. 12.3) when the number in the cell (e.g. 2 in this case) is used in the formula in a second cell (=A1 + 3 here), and as the number in cell A1 changes there is a resultant change in cell B1. This ability to explore situations quickly with technology has provided opportunities in the mathematics classroom (and outside the classroom) that were not possible in pen-and-paper only classrooms.

Some specific-purpose technologies can also support the development of students' conceptual understanding. Use of these technologies will require students or teachers to know the specific communication with the technology to obtain required results, as well as the need to interpret the displays and recognize any constraints in using the technology. For example, use of an applet, such as National Library of Virtual Manipulatives (NLVM) base blocks (http://nlvm.usu.edu/en/nav/frames_asid_152_g_1_t_1.html) for making and writing numbers can help students deepen

	A	B
1	2	=A1+3

Fig. 12.3 A cell in a spreadsheet represents a variable

their understanding of the base ten system. NLVM base blocks enables students or teachers to move objects on the screen to represent specific numbers and then combine or separate blocks in line with the base ten system. The students must interpret the display provided, for example if the display shows twelve unit blocks then the number '12' will not be displayed until ten of the unit blocks are combined to form 1 ten to give 1 ten and 2 units. This example highlights the interplay between communication *with* and *of* technology, where interpretation/discussion of a display can prompt syntax or choice of the next step with technology. The provision of immediate feedback to students enables them to be in control of their own work. Advantages in using technology to prompt discussion also occur in use of dynamic geometry or dynamic applets, for example the exploration of dynamic applets for Pythagoras' theorem can provide a stimulus for discussion about links between the theorem and the dynamic representation. The use and subsequent interpretation of virtual manipulatives can strengthen students' understanding of geometric transformation (e.g. Gulkilik, 2016). In a study of third grade students it was found that the use of virtual balances supported development of students' relational thinking through multiple representations (Suh & Moyer, 2007). The students in the study used the technology display showing step-by-step working to support their explanations of their solution processes. Thus, the technology provided a prompt to support verbal reasoning about solutions, which in turn supports procedural knowledge.

More generally, when doing constructions within a dynamic geometry package it is important to follow a clear sequence of steps to take account of dependencies and hierarchies within the program, to ensure that constructions are robust and maintain their underlying structure when points are moved. Making these decisions about commands to produce a given construction can foster the conceptual knowledge behind procedures. This potential is also evident in other technologies.

Comparison of technology results to results obtained through pen-and-paper work or mental strategies can prompt verbal communication of procedural knowledge. This may occur when the technological result is in a different format to that produced using pen-and-paper, for example, when a computer algebra system reorders letters in an expression due to inbuilt conventions, or when an applet produces an answer in a format different to expected; this can prompt discussion of what the technology has done, treating the technology as a third party in the classroom. For example an automatic simplification of an algebraic expression by a CAS can be quite surprising for a student and discussion of why a given input and output belong together can foster "algebraic insight" (Pierce & Stacey, 2004). In this case an unexpected, surprising result given by the technology can foster thinking, reflecting, discussion and maybe further understanding. These "hiccups" (Clark-Wilson, 2010b), in other words, unexpected results, can occur for example when working with a CAS or a Geometry package. Metacognition can also be promoted as individuals work with technology, notice unexpected results and then discuss and potentially rethink concepts/procedures as a result of discussion about unexpected results (Barzel, 2006).

Interactive media, such as programs that enable students to work together, can be used to solve problems collaboratively, with students reflecting on and discussing personal understanding as problems are solved. Through discussion of technology here, metacognition can be promoted as students develop their personal understanding of mathematics through working with other students.

We believe that technology can contribute to development of mathematical knowledge by promoting deeper understanding, scaffolding learning and by providing immediate feedback to students on their mathematical work. In addition to feedback provided to students, with teachers having increased access to formative assessment through technology there is the potential for teachers to target discussion with students to provide impetus for metacognition through reflection.

12.4 Conclusion

Communication and collaboration with technology is multifaceted. A focus on communication and collaboration enables reconsideration of ways to promote development of conceptual, procedural and metacognitive knowledge. Consideration of the communication fostered through, with and of technology is a useful construct for analyzing and understanding the different roles of technology in teaching and learning mathematics. Although we distinguish between the three different roles, in reality technology use involves intertwining types of communication, as a technology user is unlikely to enter syntax without simultaneously watching the technology display and considering the result of the entry. It is important for teachers to consider the role of technology use for different purposes in order to capitalize on the potential affordances for teaching and learning in school mathematics.

Acknowledgements We would like to thank the reviewers for their insightful comments about our chapter.

References

Ainley, J., Bills, L., & Wilson, K. (2005). Designing spreadsheet-based tasks for purposeful algebra. *International Journal of Computers for Mathematical Learning, 10,* 191–215.

Barzel, B. (2006). *Mathematikunterricht zwischen Instruktion und Konstruktion - Evaluation einer Lernwerkstatt im 11.Jahrgang mit integriertem Einsatz von Computeralgebra* (Dissertation). Fakultät für Mathematik, Universität Duisburg-Essen.

Barzel, B. (2012). *Computeralgebra – Mehrwert beim Lernen von Mathematik – aber wann?* Münster: Waxmann Verlag.

Barzel, B., Leuders, T., Prediger, S., & Hußmann, S. (2013). Designing tasks for engaging students in active knowledge organization. In C. Margolinas (Ed.), *Proceedings of ICMI Study Group 22: Task design in Mathematics Education* (Vol. 1, pp. 285–294). Oxford: University of Oxford.

Blume, G. W. & Heid, M. K. (Eds.). (2008). *Research on technology and the teaching and learning of mathematics: Volume 2 cases and perspectives.* Charlotte, NC: IAP.

Clark-Wilson, A. (2010a). Emergent pedagogies and the changing role of the teacher in the TI-Nspire Navigator-networked mathematics classroom. *ZDM—The International Journal on Mathematics Education, 42*, 747–761.

Clark-Wilson, A. (2010b). *How does a multi-representational mathematical ICT tool mediate teachers' mathematical and pedagogical knowledge concerning variance and invariance?* (Ph. D. thesis). London: Institute of Education, University of London.

Dick, T., & Edwards, B. (2008). Multiple representations and local linearity. In G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 2 cases and perspectives* (pp. 255–275). Charlotte, NC: IAP.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education: A concise topical survey*. Springer Open. Cham: Springer.

Drijvers, P., Barzel, B., Maschietto, M., & Trouche, L. (2006). Tools and technologies in mathematical didactics. In M. Bosch (Ed.), *Proceedings of the fourth congress of the European Society for research in mathematics education* (pp. 927–938). Barcelona, Spain: Universitat Ramon Llull.

Drijvers, P., Monaghan, J., Thomas, M., & Trouche, L. (2015). *Use of technology in secondary mathematics: Final report for the international baccalaureate.* The Netherlands: Utrecht University Repository.

Duval, R. (2000). Basic issues for research in mathematics education. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. I, pp. 55–69). Japan: Hiroshima University.

Fitzallen, N., & Watson, J. (2014). Extending the curriculum with TinkerPlots: Opportunities for early development of informal inference. In K. Makar, B. de Sousa, & R. Gould (Eds.). *Proceedings of the 9th International Conference on Teaching Statistics* (pp. 1–6). Voorburg: International Statistical Institute.

Fuglestad, A. B. (2009). ICT for inquiry in mathematics: A developmental research approach. *Journal of Computers in Mathematics and Science Teaching, 28*(2), 191–202.

Goodchild, S. (2014). Mathematics teaching development: learning from developmental research in Norway. *The International Journal on Mathematics Education*, *46*(2), S. 305–316.

Goos, M., Soury-Lavergne, S., Assude, T., Brown, J., Kong, C. M., Glover, D., et al. (2009). Teachers and teaching: Theoretical perspectives and issues concerning classroom implementation. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain* (pp. 311–2328). New York: Springer.

Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3,* 195–227.

Gulkilik, H. (2016). The role of virtual manipulatives in high school students' understanding of geometric transformations. In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* [electronic resource] (pp. 213–243). Cham: Springer International Publishing.

Heid, M. K. & Blume, G. W. (Eds.). (2008). *Research on technology and the teaching and learning of mathematics: Volume 1 research syntheses.* Charlotte, NC: IAP.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan.

Isoda, M., McCrae, B., & Stacey, K. (2006). Cultural awareness arising from internet communication between Japanese and Australian classrooms. In F. K. S. Leung, K.-D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural traditions: A comparative study of east Asia and the west* (*The 13th ICMI Study*) (pp. 397–408). USA: Springer.

Kahveci, M., & Imamoglu, Y. (2007). Interactive learning in mathematics education: Review of recent literature. In R. Carlsen & D. A. Willis (Eds.), *Proceedings of the Annual Conference of the Society for Information Technology & Teacher Education* (pp. 3269–3276). Chesapeake: Association for the Advancement of Computing in Education.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. New York: Routledge.

Luhmann, N. (1994). *Das Wissen der Gesellschaft*. Frankfurt am Main: Suhrkamp.

Maturana, H. R., & Varela, F. J. (2009). Der Baum der Erkenntnis: Die biologischen Wurzeln menschlichen Erkennens. Berlin: Fischer.

Mueller, M., Yankelewitz, D., & Maher, C. (2012). A framework for analyzing the collaborative construction of arguments and its interplay with agency. *Educational Studies in Mathematics, 80*(3), 369–387.

Muir, T. (2014). Google, Mathletics and Khan academy: Students' self-initiated use of online mathematical resources. *Mathematics Education Research Journal, 26,* 833–852.

Peschek, W. (2007). The impact of CAS on our understanding of mathematics education. *International Journal for Technology in Mathematics Education, 14*(2), 95–101.

Pierce, R., & Stacey, K. (2004). Monitoring progress in algebra in a CAS active context: Symbol sense, algebraic insight and algebraic expectation. *The International Journal of Computer Algebra in Mathematics Education, 11*(1), 3–11.

Pierce, R., & Stacey, K. (2010). Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning, 15*(1), 1–20.

Reimer, K., & Moyer, P. S. (2005). Third graders learn about fractions using virtual manipulatives: A classroom study. *Journal of Computers in Mathematics and Science Teaching, 24*(1), 5–25.

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: Artifacts as fundamental constituents of the didactical situation. *ZDM—The International Journal on Mathematics Education, 44*, 641–651.

Roschelle, J., Rafanan, K., Bhanot, R., Estrella, G., Penuel, B., Nussbaum, M., et al. (2010). Scaffolding group explanation and feedback with handheld technology: Impact on students' mathematics learning. *Educational Technology Research and Development, 58*(4), 399–419.

Schneider, E. (2002). Kommunikationsfähigkeit mit Expert/inn/en und Computeralgebrasysteme. In S. Prediger, K. Lengnink, & F. Siebel (Eds.), *Mathematik und Kommunikation. Darmstädter Texte zur Allgemeinen Wissenschaft 3* (pp. 137–150). Mühltal: Verlag Allgemeine Wissenschaft.

Steinbring, H. (2009). *The construction of new mathematical knowledge in classroom interaction – an epistemological perspective, mathematics education library,* (Vol. 38). Berlin, New York: Springer.

Steinbring, H. (2015). Mathematical interaction shaped by communication, epistemological constraints and enactivism. *ZDM—The International Journal on Mathematics Education, 47* (2), 281–293.

Suh, J., & Moyer, P. (2007). Developing students' representational fluency using virtual and physical algebra balances. *Journal of Computers in Mathematics and Science Teaching, 26*(2), 155–173.

Symons, D., & Pierce, R. (2015). Mathematical language development and talk types in computer supported collaborative learning environments. In M. Marshman, V. Geiger, & A. Bennison (Eds.). *Mathematics education in the margins. Proceedings of the 38th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 579–586). Sunshine Coast: MERGA.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9,* 281–307.

Trouche, L., & Drijvers, P. (2014). Webbing and orchestration. Two interrelated views on digital tools in mathematics education. *Teaching Mathematics and Its Applications, 33,* 193–209.

Vincent, J., Chick, H., & McCrae, B. (2005). Argumentation profile charts as tools for analysing students' argumentations. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 281–288). Melbourne: PME.

Woo, Y., & Reeves, T. C. (2006). *Meaningful online learning: Exploring interaction in a Web-based learning environment using authentic tasks* (Unpublished dissertation). University of Georgia, Athens, GA.

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior, 21,* 423–440.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.

Chapter 13
Online Automated Assessment and Student Learning: The *PEPITE* Project in Elementary Algebra

Brigitte Grugeon-Allys, Françoise Chenevotot-Quentin, Julia Pilet and Dominique Prévit

Abstract The automated digital diagnostic assessment presented in this paper concerns the elementary algebra for students of secondary education (12–16 years) in France. The paper addresses the design of tasks for the test "Pépite", to favour students' algebraic thinking. The selection of the tasks and the analysis of students' responses are based on an epistemological reference of the algebraic domain. The information provided by "Pepite" enables identification of students' consistent reasoning and calculation and assists teachers' planning for differentiated instruction for groups of students. The paper reports some results on the integration of the tools in the usual teaching practices and on students' learning, based on trialling with a group of teachers.

Keywords Online automated assessment · Elementary algebra
Epistemological reference · Design tasks · Algebraic reasoning

13.1 Context of the Study

Diagnostic assessments are an important part of instructional decision-making and support strategies of formative assessment for students' learning (Wiliam & Thompson, 2007). Usually, assessment results are generated from standardized and

B. Grugeon-Allys (✉) · J. Pilet
Laboratoire de Didactique André Revuz (EA 4434), UA, UCP, UPD, UPEC, URN, Université Paris Est Créteil, Paris, France
e-mail: brigitte.grugeon-allys@u-pec.fr

J. Pilet
e-mail: julia.pilet@u-pec.fr

F. Chenevotot-Quentin
Laboratoire de Didactique André Revuz (EA 4434), UA, UCP, UPD, UPEC, URN, Université d'Artois, Lille, France
e-mail: chenevotot.francoise@neuf.fr

D. Prévit
Université Pierre et Marie Curie, Paris, France
e-mail: dominique.previt@gmail.com

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_13

psychometric models. Studies highlight the strengths and limitations of such an approach for making instructional decisions (Kettelin-Geller & Yovanoff, 2009). One may rightfully ask oneself how a didactical approach would make it easy to identify features of appropriate digital diagnostic assessment *of, for* and *as* learning. Since the 1990s, our research team has developed multidisciplinary projects (Delozanne, Prévit, Grugeon-Allys, & Chenevotot-Quentin, 2010; Grugeon-Allys, Pilet, Chenevotot-Quentin, & Delozanne, 2012) concerning the design, development and implementation of a digital diagnostic assessment tool, named "*Pépite*", that provides information for teachers to use to plan differentiated instruction for groups of students. *Pépite* is relevant for learning elementary algebra for students in middle/lower secondary grades (12–16 years old) in France. We have disseminated it on platforms[1] largely used by teachers and students.

Two research questions are considered. *How could an epistemological study support the design and the development of an appropriate digital diagnostic assessment for learning elementary algebra in lower/middle secondary grades? How should the results of this diagnostic assessment be used in a formative way to support every day teaching of elementary algebra according to students' learning needs?*

First, we present the theoretical foundations of diagnostic and formative assessment. We specify it with *Pépite* assessment for grade 9th students in France (14–15 years old). We characterize both the didactical model (test design, response analysis, student's profile) and the computer model that automatically generates generic tasks, analyses students' answers and provides descriptions of students' profiles. Secondly, we describe how to use *Pépite* diagnostic assessment to plan differentiated instruction for groups of students. We present both the didactical model and the computer model of differentiated instruction according to learning needs for groups of students and conclude with some results on the integration of our tools in the usual teaching practices and on students' learning. Finally, we discuss the potentialities and limits of features of this digital assessment and answers to the research questions.

13.2 The Theoretical and Methodological Framework

In the educational system, assessment is a complex issue. To identify the features of appropriate diagnostic and formative assessment, we have chosen both a cognitive and epistemological approach and an anthropological approach, whose potentialities are described in Grugeon-Allys, Pilet, Chenevotot-Quentin, and Delozanne (2012).

---

[1]*Pépite* tools are available on LaboMep platform, developed by Sésamath, a French maths' teachers association (http://www.labomep.net/) and on WIMS environment (an educational online learning platform spanning learning from primary school to the university in many disciplines).

13.2.1 Epistemological and Cognitive Approach

In order to assess students' learning processes, we privilege an epistemological point of view. Designing a test requires selecting a set of tasks that enables the assessment to be realized. We agree with Vergnaud (1986) who stated, "Studying learning of an isolated concept, or an isolated technique, has no sense" (p. 28). Furthermore, Vergnaud introduced a strong assumption: dialectics between genesis of a student's knowledge and mathematical knowledge structure. Beyond a quantitative analysis of responses, we have to define a qualitative didactic analysis, based on a collection of students' responses to the tasks, to identify the type of procedures and knowledge used by students in solving the tasks.

To provide descriptions of a student's consistent reasoning related to their conceptions, it is necessary to define a reference modelling the mathematical competence, in a given mathematical field, while within the scope of a given school grade.

13.2.2 Anthropological Approach

The cognitive approach is not sufficient in order to take into account the impact of the institutional context on students' learning. Indeed, the nature of mathematical content depends on the institution with which we are dealing, as the institution influences mathematical activity. According to the process of didactic transposition (Chevallard, 2006), institutions involve different bodies of knowledge that do not coincide: for instance, "scholar" knowledge produced by scholar mathematicians, "knowledge to be taught" produced by the educational system, "taught knowledge" implemented in classrooms by teachers, and "learnt knowledge" by students.

To study the process of didactic transposition, Chevallard (1999) analyses mathematical activities in terms of praxeologies.[2] Praxeology involves two opposite blocks: the "practical block" or know-how and the "theoretical block" or knowledge, used to describe and justify mathematical activities. More precisely, a praxeology is made of four components: type of tasks, techniques, technologies and theories. The "practical block" contains a set of types of tasks and the techniques used to solve these tasks. The "theoretical block" is made of a double-leveled discourse: a discourse on the technique, named "technological discourse" or technology, developed in order to describe, explain what is done, and justify techniques (properties, rules, logical arguments); theoretical discourse or theory that justify "technological discourse". Therefore, designing an assessment requires the identification of praxeologies that are representative of a mathematical field. It is the

[2]Praxeology originates from two Greek words *praxis* and *logos*.

reason why we defined a praxeology of elementary algebra (Garcia, Gascon, Higueras, & Bosch, 2006; Bosch, 2015). We analyze technological discourse used by students to solve types of tasks in order to evaluate their mathematical activity.

13.2.3 Praxeology of Elementary Algebra

Praxeology of elementary algebra is based on results from the didactics of algebra (Artigue, Grugeon, Assude, & Lenfant, 2001; Chevallard, 1989; Kieran, 2007). This praxeology covers all types of tasks in the algebraic domain. In its *tool* dimension (Douady, 1985), there are tasks for *generalizing*, *modelling, putting into equation*, *proving*. In its *object* dimension, there are tasks focused on calculations with algebraic expressions (*calculating*, *substituting* a number for a letter, *expanding)* or equations (*solving*). This praxeology aims to define appropriate conditions for a reasoned and controlled algebraic calculation, based on equivalence of algebraic expressions and dialectic between numeric and algebraic treatment modes.

Indeed equivalence of algebraic expressions and dialectic between numeric and algebraic treatment modes are two substantial epistemological features. Kieran (2007) sets the equivalence of expressions in the core of theoretical elements of algebraic activity. Equivalence of expressions has a fundamental role in theoretical control to ensure that the transformed expression is equivalent to the initial one. This control can theoretically be made in two ways either by reference to the algebraic properties used (proof when the equivalence is true), either by linking with numeric and substituting numerical values to letters (counter-example when the equivalence is false). However students experience great difficulty in identifying properties used when transforming algebraic expressions (Kieran, 2007) and, particularly in France, in linking transformation of expressions with substitution of numerical values (Chevallard, 1984).

In addition, in the anthropological approach, Ruiz-Munzón, Matheron, Bosch, and Gascón (2012) propose an epistemological model of "algebra to be taught" with different stages as a process of 'algebraization'. The first stage focuses on the need to introduce algebraic expressions and calculation rules and leads to the equivalence of calculation programs (CP) and thus the equivalence of algebraic expressions associated.

13.2.4 What Are the Potentialities of Such an Approach?

This methodological approach presents several potentialities:

- Designing an assessment based on a praxeology has great validity with regards to coverage of mathematical domain and representativeness of tasks;

- This approach allows adaptation of assessment from one grade to another, by playing on values of didactical variables associated with the tasks (for instance, nature and complexity of algebraic expressions, identities or equations, register of representation…) and implementation of automated digital assessment;
- A praxeology of elementary algebra allows the researcher to define criteria and associated values for analysing students' responses. This feature of assessment makes it possible to analyse the responses at the level of technological discourse involved in techniques and not only at the level of technique. It also enables determination of students' profiles. This choice makes it possible to do a transversal analysis on several tasks, that is, to code responses to several tasks using the same codes. It allows identification of consistent reasoning and calculation across the set of tasks. This is a major contribution compared to other approaches;
- A previous study supports strategies for differentiation to serve the groups of students' learning needs, taking into account the learning needs often ignored by curricula (Grugeon-Allys et al., 2012). We present such strategies for differentiation in Sect. 13.4 of this text.

13.3 Features of "*Pépite*" Digital Assessment

We now present the features of *Pépite* digital assessment. The didactical model of *Pépite* digital diagnostic assessment is based on the praxeology of elementary algebra presented above, both for designing tasks and analysing the students' responses to the test. We will rely on the 9th/10th grade level test for 14–15 years old students to explain the modalities of *Pépite* test and to describe the responses analysis.

13.3.1 The Didactical Model

Pépite test

The diagnostic test is composed of ten diagnostic tasks (27 individual items) (Table 13.1) covering the algebraic field.

Representative tasks of elementary algebra are divided among three sets of types of tasks:

- Calculation (expanding algebraic expressions, solving equations) (4 items);
- Production of algebraic objects (expressions, formulas, equations) (8 items);
- Recognition of mathematical relationships from a register of representation to another (16 items).

Table 13.1 Organization of the 9th/10th grade level test in terms of types of tasks

Types of tasks	Number of items	Test items
Calculation	4/27	5.1/5.2/5.3/5.4
Producing algebraic expressions	8/27	3.1/6/8.1/8.2/8.3/9/10.2/10.3
Translation or recognition	16/27	1.1/1.2/1.3/1.4/2.1/2.2/2.3/3.2/4.1/4.2/4.3/4.4/4.5/6/7/10.1

The total of the three categories of items is 28 but the table indicates a total of 27 because Item 6 appears in both categories "Producing algebraic expressions" and "Translation or recognition"

To ensure that the diagnostic test includes all types of tasks involved in elementary algebra, we characterize each item of a diagnostic task by one or more of the previous types of tasks. We consider that the diagnostic test covers the algebraic field if all the four sets of types of tasks are involved. As shown in Table 13.1, the ten diagnostic tasks cover the algebraic field.

The tasks may be multiple-choice items (Fig. 13.1a–c) or open-ended items (Fig. 13.2). For example, the goal of the second task "Determining if an algebraic equality is always true" (Fig. 13.1a) is to identify whether a student recognizes the structure of expressions and also which rules of algebraic writing he/she mobilized and how he/she articulates semantic and syntax.

For the first item of the second task, the student selects a reason according to whether the response is "true" (Fig. 13.1b) or "false" (Fig. 13.1c). The chosen arguments indicate the type of reasoning used in this context by the student.

The ninth task "Proof and calculation program" (Fig. 13.2) is a generalization task whose goal is to identify whether a student generalizes and proves a property with arithmetic or with algebraic strategies. It also provides information about the types of connections between a semiotic register to another and the arguments used by a student.

Responses analysis: the multidimensional model of algebraic assessment

The students' responses are not only evaluated as correct/incorrect but also coded in terms of consistency. The coding of responses is determined by a preliminary analysis of the task in order to anticipate solving procedures or strategies, techniques and reasoning. They correspond either to appropriate skills and abilities for the grade level considered or to recurring errors. Students' responses are evaluated according to technological discourse that justifies the techniques they use (refer to Sect. 13.2).

More precisely, the *Pépite* diagnostic assessment includes three analysis levels:

- The local diagnosis (on a single task) analyses each student's answer on several dimensions and not only in terms of correct/incorrect; the diagnostic system also provides a set of codes that characterize this answer according to anticipated answers;

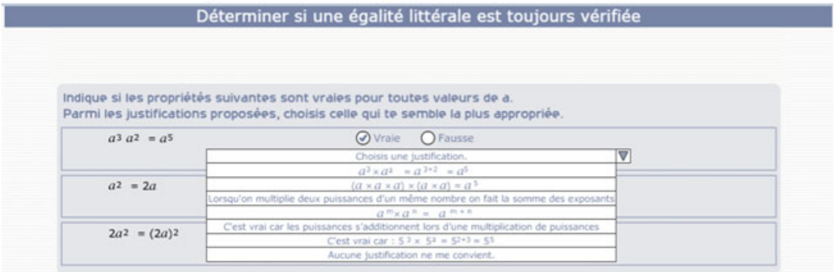
- The individual global diagnosis (on a set of tasks) collects similar codes on different exercises, on the one hand, to build the student's cognitive profile expressed by a level on a three component scale of skills; on the other, success rates and personal features (relative strengths and limitations, false rules and correct rules);

(a)

Determining if an algebraic equality is always true

Indicate whether the following statements are true for all values of *a*.
Among the reasons offered, choose the most appropriate.

(b)



Determining if an algebraic equality is always true

Choose a reason:

- $a^3 \times a^2 = a^{3+2} = a^5$
- $(a \times a \times a) \times (a \times a) = a^5$
- When the square (or the cube) of an integer is multiplied by itself, exponents should be added
- $a^m \times a^n = a^{m+n}$
- It's true because exponents should be added in a multiplication of powers
- It is true because $5^3 \times 5^2 = 5^{2+3} = 5^5$
- None of these

Fig. 13.1 **a** Second task: "determining if an algebraic equality is always true". **b** Second task, first item: arguments suggested if a student selects "true". **c** Second task, first item: arguments suggested if a student selects "false"

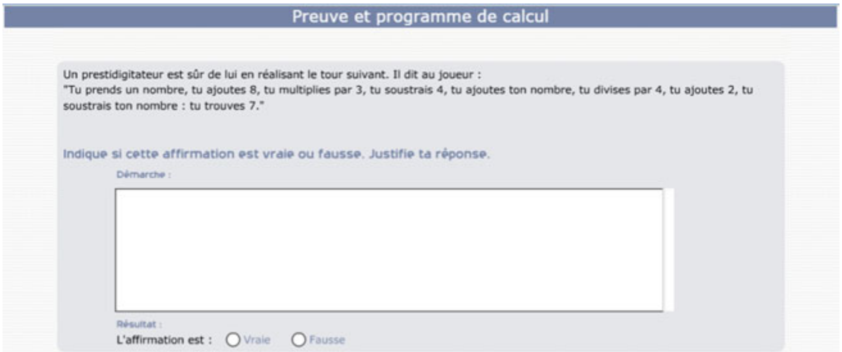
(c)

Determining if an algebraic equality is always true

Choose a reason:

- It's false because 3 × 2 equals 6, not 5
- The property is false because square and cube should be multiplied
- $a^2 \times a^3 = a^{2\times 3} = a^6$
- $a^m \times a^n = a^{m\times n}$
- Exponents shouldn't be added but multiplied
- None of these

Fig. 13.1 (continued)



Proof and calculation program

A magician is certain of the result of the calculation program below:
"Choose a number, add 8, multiply the result by 3, subtract 4, add the initial number, divide by 4, add 2, and subtract the initial number. You will end up with 7."
Indicate whether this statement is true or false. Justify.

Fig. 13.2 Ninth task: "proof and calculation program"

- The collective global diagnosis defines groups of students who have close cognitive profiles.

The local diagnosis (for each task) analyses the students' responses on several assessment dimensions. In addition to the validity of the response (V), student's consistency in algebraic activity is analysed in four dimensions: the use of letters as variables (L), the algebraic writing produced during symbolic transformations (EA), the algebraic rationality (J) and the connections between a semiotic register to another (T) (Grugeon-Allys, 2015). As shown in Table 13.2, students' responses are coded with assessment criteria depending on knowledge and reasoning involved in the techniques.[3]

We illustrate the multidimensional model of algebraic assessment on the task "*Proof and calculation program*" (Fig. 13.2). To solve this task, two strategies are possible: an arithmetic strategy using a number and an algebraic strategy mobilizing a variable. Several incorrect techniques can illustrate an arithmetic strategy (Table 13.3) according to the rules used to translate numerical expressions. Algebraic strategy may be incorrect (J3) if the conversion rules (T3 or T4) or algebraic transformation rules (EA3 or EA4) are inadequate (Table 13.4).

Student's profile

The individual global diagnosis (on a set of tasks) builds the student's cognitive profile, which locates a student on a scale with three components and collects personal features (success rates, strong points/weak points, list of errors/list of success). More precisely, we describe the student's algebraic skills in three components:

- Ability and adaptability in the various uses of Algebraic Calculation (coded CA);
- Use of Algebra for solving problems (coded UA);
- Flexibility in translating a semiotic register to another (geometric figures, graphical representations, natural language, algebraic expressions) (coded TA).

For each of those three components, we identified (Grugeon-Allys, Pilet, Chenevotot-Quentin, & Delozanne, 2012) (Table 13.5) different levels of technological discourse (cf. II). Thus a level describes a student's algebraic skills on CA, a level on UA and a level on TA.

[3]Contrary to usual practices in assessment, we do not attribute a code by technique for each task. This would lead to a multiplicity of codes on various tasks and would be unusable for a cross analysis on all the tasks of the test.

Table 13.2 The multidimensional model of algebraic assessment (partial view)

Assessment dimensions	Assessment criteria
Validity of response (V)	V0: No answer V1: Valid and optimal answer V2: Valid but non optimal answer V3: Invalid answer Vx: Unidentified answer
Use of letters (L)	L1: Correct and optimal use of letters L2: Correct but non optimal use of letters L3: Letters are used with incorrect rules L5: No use of letters Lx: No interpretation
Algebraic writing produced during symbolic transformations (EA)	EA1: Reasoned and controlled algebraic calculation EA2: Correct algebraic calculation but without arguments EA3: Incorrect calculation based on syntactic rules (without taking into account the equivalence of expressions) EA4: Incorrect calculation based on arithmetic rules EAx: No interpretation
Algebraic rationality (J)	J1: Correct algebraic reasoning J2: Arithmetic reasoning J3: Algebraic reasoning but using incorrect rules Jx: No interpretation
Connections between a semiotic register to another (T)	T1: Correct and optimal translation T2: Correct but not optimal translation T3: Incorrect translation taking into account the relationships T4: Incorrect translation without taking into account the relationships Tx: No interpretation

Groups and differentiated instruction strategies

Finally, the collective global diagnosis consists in defining groups of students who have close cognitive profiles. With the aim of proposing viable differentiated instruction in classrooms, students are divided into three predefined groups according to their levels on CA and on UA:

- Group A: use of semantic and syntactic arguments, taking into account structure and equivalence of expressions and expected technology, adapted use of algebra in problem solving (CA1 and UA1-UA2);
- Group B: use of formal syntactic arguments weakly articulated to the numeric, allowing to live the incorrect use of the parenthesis, the use of false rules, for example of type $(a + b)^2 \rightarrow a^2 + b^2$ (CA2),

– unsuitable use of algebra in problem solving (UA3-4, subgroup B−),
– use of algebra in at least one type of problem (UA1-2, subgroup B+);

- Group C: use of arithmetic approaches and inadequate use of symbolism, which could result in errors linked to concatenation rules a + b → ab or duplication errors a^2 → 2a (CA3 and UA3-4).

Table 13.3 Preliminary analysis of arithmetic strategies

Solutions	Reasoning	Coding
For number 3 $((3 + 8) \times 3 - 4 + 3)/4 + 2 - 3 = 7$	Correct arithmetic strategy with global expression that uses parenthesis	V3, L5, EA1, J2, T1
For number 1 $1 + 8 = 9$; $9 \times 3 = 27$; $27 - 4 = 23$; $23 + 1 = 24$; $24/4 = 6$; $6 + 2 = 8$; $8 - 1 = 7$	Correct arithmetic strategy with partial expressions	V3, L5, EA2, J2, T2
For number 36 $36 + 8 \times 3 - 4 + 36/4 + 2 - 36 = 7$	Erroneous arithmetic strategy with global expression that uses no parenthesis	V3, L5, EA3, J2, T3
For number 1 $(1 + 8)3 = 27 - 4 = 23 + 1 = 24/4 = 6 + 2 = 8 - 1 = 7$	Erroneous arithmetic strategy with calculation by step (procedural aspect)	V3, L5, EA3, J2, T4

Table 13.4 Preliminary analysis of algebraic strategies

Solutions	Reasoning	Coding
$((x + 8) \times 3 - 4 + x)/4 + 2 - x$ $= (3x + 24 - 4 + x)/4 + 2 - x$ $= (4x + 20)/4 + 2 - x$ $= x + 5 + 2 - x$ $= 7$	Correct algebraic strategy with global expression that uses parenthesis	V1, L1, EA1, J1, T1
$(x + 8) \times 3 = 3x + 24$; $3x + 24 - 4 = 3x + 20$; $3x + 20 + x = 4x + 20$; $(4x + 20)/4 = x + 5$; $x + 5 + 2 = x + 7$; $x + 7 - x = 7$	Correct algebraic strategy with calculation by step (procedural aspect)	V2, L1, EA1, J1, T1
$x + 8 \times 3 - 4 + x/4 + 2 - x$ $= x + 24 - 4 + x/4 + 2 - x$ $= 2x - x + 24 - 4/4 + 2$ $= 2x + 24 - 1 + 2$ $= 2x + 25$	Erroneous algebraic strategy with global expression that uses no parenthesis	V3, L3, EA32, J3, T3
$(x + 8) \times 3 = 3x + 24 = 27x$; $27x - 4 = 23x$; $23x + x = 24x$; $24x/4 = 6x$; $6x + 2 = 8x$; $8x - x = 7$	Erroneous algebraic strategy with calculation by step (procedural aspect)	V3, L3, EA42, J3, T3

Table 13.5 Description of levels of technological discourse on each component

Component	Level	Description
Algebraic calculation (CA)	CA1	Reasoned and controlled calculation taking into account the equivalence of expressions
	CA2	Calculation based on syntactic rules without taking into account the equivalence of expressions
	CA3	Calculation with arithmetic strategies and without operating priorities
Use of algebra (UA)	UA1	Algebraic tool mastered
	UA2	Algebraic tool adapted in some types of problems
	UA3	Algebraic tool used but without sense for letters
	UA4	Low, because arithmetic reasoning
Algebraic translation (TA)	TA1	Controlled translation
	TA2	Translation without support on the reformulation
	TA3	Translation as to schematise

The definition of groups relies on levels of technological discourse described in Table 13.5. For a learning objective defined by the teacher, *Pépite* generates tasks adapted to students' learning needs (Grugeon-Allys et al., 2012).

13.3.2 The Computer Model

An iterative process between educational researchers, computer scientists and teachers was used to design and test different *Pépite* prototypes to improve the didactical model. Delozanne and Prévit defined the conceptual IT model of classes of tasks, which allows characterizing equivalent tasks on a diagnosis point of view (Delozanne, Prévit, Grugeon, & Chenevotot, 2008). Prévit developed *PépiGen*, a software that automatically generates the tasks and their analysis, at different grade levels. It uses *Pépinière*, a Computer Algebra System that generates anticipated student's correct or incorrect answers, according to a preliminary analysis of the tasks. For example, *Pépinière* deals with similar expressions by referring to the commutative property, correct and incorrect rules, as well as the equivalence of expressions.

Pépite automatically calculates an individual student's profile, as well as profiles for groups of students. Figure 13.3 shows the individual global diagnosis for Colin, a 9th grade student with CA2-UA2-TA2. His personal features enlighten his strong points and weak points.

Pépite diagnostic tasks may be multiple-choice items or open-ended items with multistep reasoning. Of course, the computer programming of the multiple-choice items is easier than the one-line open-ended items. Multiple-choice items are difficult to design but easy to analyse. For one-line open items, the analysis of students' answers is automatic, effective and generic. Howeer, 10–15% answers of the open-ended items (Delozanne et al., 2010) are not analysed due to the complexity of the algebraic reasoning that need specific treatments for each sort of task.[4]

13.4 Differentiated Instruction Adapted to Students' Learning Needs

The information provided by *Pépite* allows the teacher to identify students who have close learning needs in elementary algebra and to plan differentiated instruction.

13.4.1 The Didactical Model

In the French context, the pedagogical differentiation of teaching is advocated by official instruction without the conditions being explained so that this differentiation is profitable for the students (Bolon, 2002; Kahn, 2007). Teachers, who are often destitute, set up pedagogical devices that often take little account of the specificities of the content, for instance by grouping together "good, medium and weak students" but without characterizing the learning needs of students according to the mathematical content. That is why the didactical model of differentiated instruction we have defined considers the mathematical content (Pilet, 2015). For keeping a collective advance of didactical time for the class group, teaching is differentiated in the following way: the learning objective is the same for the whole class—all students work on the same type of tasks—but each task is adapted to individual student's learning needs as identified by the *Pépite* diagnostic assessment. The differentiated instruction supports formative assessment in the sense of Black and Wiliam (1998) since it enables students to understand the gap between what they produce (here in elementary algebra, notably with *Pépite*) and what is expected from them.

The prior identification of learning objectives is based on the praxeology of elementary algebra defined above. It makes it possible to consider both learning needs ignored by the institution and learning needs identified by the diagnostic assessment *Pépite* (Pilet, 2012, 2015). For instance, students in groups B and C, who give a weak meaning to letters, need to visit again the role of algebra in solving problems of generalization and proof; the treatment of this aspect of algebrais weak in French textbooks. Moreover, in order to give meaning to algebraic expressions

[4]Most of the non-coded answers are those in which the student uses natural language together with algebraic expressions, which disrupts the analysis; some are not coded because they are not predicted in the preliminary analysis of the answers. The basis of answers is regularly updated.

and to control algebraic transformations, students need to develop the notion of equivalence of algebraic expressions, which is currently little used in teaching.

A differentiated instruction session is composed of:

- Tasks that relate to a common teaching objective and are adapted to the learning needs of students. A task is characterized by: the involved component(s) (UA, TA, CA), the type of task, the object of algebra involved in the task (expressions, equation, etc.), the nature and complexity of expressions (in relation to the group A, B or C in which the student is assigned), the input and output frames (numeric, algebraic, natural language, geometric, graphic, functional), the complexity of the task.

Colin belongs to the group B, sub-group B+	
Description of the sub-group B+ Use of formal syntactic arguments weakly articulated to the numeric, allowing to live the incorrect use of the parenthesis, the use of false rules, for example of type $(a+b)^2 \rightarrow a^2 + b^2$ Use of algebra in at least one type of problem	
Levels of technological discourse on each component	**Personal features**
Algebraic Calculation **CA2** Calculation based on syntactic rules without taking into account the equivalence of expressions	**Technics** Sucess rate on asked questions: 4/12 **Interpreting algebraic expressions** Success rate on asked questions: 11/23 ✓ **Strong points** **See More** Good mastery of algebraic rules Some good interpretations of algebraic expressions ✓ **Weak points** **See More** Low mastery of algebraic calculation
Use of Algebra **UA2** Algebraic tool adapted in some types of problems	**Mathematical modeling** Success rate on asked questions: 5/9 ✓ **Strong points** **See More** Good mastery of algebra use on some problems ✓ **Weak points** **See More** Justification by school authority
Algebraic Translation **TA2** Translation without support on the reformulation	**Translating situations to algebra** Success rate on asked questions: 12/24 ✓ **Strong points** **See More** Good translation of mathematical relations ✓ **Weak points** **See More** Low mastery of algebraic translation

Fig. 13.3 An overview of Colin's cognitive profile automatically built

- A didactical management articulating an individual work, a collective formulation and validation of a student's procedures and an institutionalization.

We illustrate this model with an example developed by Pilet (2012). This example concerns a differentiated instruction session on equivalence of calculation programs (Ruiz-Munzón, Matheron, Bosch, & Gascón, 2012) and therefore the study of equivalence of algebraic expressions. Given that students in groups B and C take little account of the equivalence of expressions to guide and control the algebraic transformations, the challenge for them is to build the meaning that two expressions can be equal for any value of the letter, even if the expressions presented by the calculation programs have different algebraic writings.

Therefore, the equivalence is first conjectured from numerical substitutions and proved with algebraic reasoning from the distributive property. For this purpose, tasks are differentiated according to groups A, B and C (example in Fig. 13.4). These tasks differ in the choice of values for didactical variables: the nature of the algebraic expressions and the form of the statements (guided task or open task). Help support is also differentiated for each group.

Group B

Are the following three calculation programs equal?

Program 1	**Program 2**	**Program 3**
- Choose a number - Multiply this number by 4 - Add 3 to the product	- Choose a number - Multiply this number by 7	- Choose a number - Multiply this number by 4 - Add the triple of the starting number

1. Choose three numbers and test each program with each of the numbers. You can use a calculator.
2. What programs seem to be equal?
3. Write an algebraic expression for each program.
4. With these three expressions, write an equality that is always true. Justify.
5. Use this equality to check your answer to question 2 and demonstrate which programs are equal.

Group C

Are the following three calculation programs equal?

Program 1	**Program 2**	**Program 3**
- Choose a number - Multiply this number by 2 - Square the result	- Choose a number - Square the result - Multiply by 2	- Choose a number - Multiply this number by 4 - Multiply the result by the starting number

1. Choose three numbers and test each program with each of the numbers. You can use a calculator.
2. What programs seem to be equal?
3. Write an algebraic expression for each program.
4. Demonstrate that programs are equal.

Fig. 13.4 An example of a differentiated task for 9th grade students (15 years old) in France

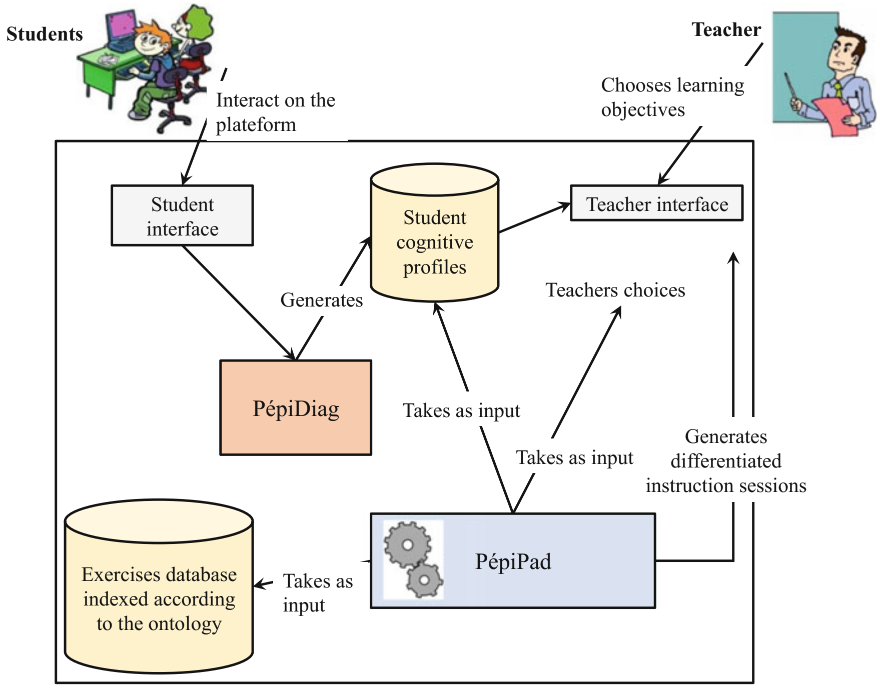


Fig. 13.5 *PépiPad* software

13.4.2 The Computer Model

The automatic generation of differentiated instruction results from collaboration between researchers in mathematics education and computer science researchers (Delozanne et al., 2010). Thanks to an ontology of the algebraic domain, we have indexed tasks involved in the differentiated instruction sessions with the following criteria characterizing the tasks: the type of tasks, the registers of representation given in the task and expected in the student's response, the complexity of the task.

The team of researchers designed and implemented *PépiPad* software (Fig. 13.5). After the students passed *Pépite* assessment, a teacher can choose a teaching objective and then the system automatically selects differentiated instruction sessions for identified groups of students of his class.

13.5 Experiments and Results

To test the relevance of our tools, we conducted experiments in 2011 and 2012 with teachers in six classes of 9th and 10th grades.

13.5.1 A Collaborative Group with Teachers

These teachers and mathematics education researchers collaborated in an IREM (Institute of Research on Mathematics Teaching) group at the Paris-Diderot-University. In this group, teachers and researchers worked on several issues:

- The interpretation and categorization of errors in elementary algebra, in relation to the constitution of groups;
- The types of tasks for working students' learning needs;
- The coherent integration of differentiated instruction with "usual" sequences in order to work learning needs often ignored by curricula (the numeric/algebraic dialectic, the role of equivalence of expressions in an algebraic transformation);
- The statements for the different groups of students;
- The didactical management, especially during the individual work, pooling and institutionalization phases.

The teachers of the IREM group have agreed to set up differentiated instruction in their classes and to collect data (student productions and videos).

13.5.2 Study of a Case: Garance's Class

We now report on an experiment with *Garance* (one of the teachers of the IREM group) and one of her 9th grade classes. Twenty-three students passed the *Pépite* test: thirteen students often calculate with arithmetic strategies and without operating priorities (group C) and seven students calculate expressions without using semantic rules (group B). *Garance* proposed differentiated instruction sessions for motivating the production of algebraic expressions. Students solved problems for generalizing and proving; after their production, they studied their equivalence. For example, the problems proposed in Fig. 13.6 are about calculating the number of square units in patterns. The problems differ by patterns according to the groups of students: the calculation leads either to expressions of the first degree or to expressions of the second degree (Pilet, 2012).

This experiment shows the real potential of both the *Pépite* test and the differentiated instruction sessions but also points to the need of an appropriation of the tools by the teachers. A long period of preparation with teachers is necessary regarding two points: on one hand, they must be aware of the links between students' difficulties in elementary algebra and implicit or ignored students' learning needs and, on the other hand, they must develop their algebraic teaching practices. For this reason, we collaborated with teachers about didactical issues and management in the class.

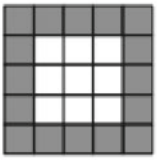
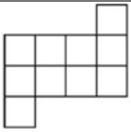
Group B		**Group C**	
Pattern 1	***Expressions***	***Pattern 2***	***Expressions***
	$(a+2)^2 - 2(a+1)$ $a^2 + 2(a+1)$ $a(a+2) + 2$ $a^2 + 2a + 2$		$4x + 4$ $4(x+1)$ $2x + 2(x+2)$ $4(x+2) - 4$

Fig. 13.6 Patterns for students of groups B and C

13.5.3 Results on the Evolution of Students' Cognitive Profile

We also conducted experiments about the evolution of grade 9th students' cognitive profiles.

Students of several grade levels have passed the *Pépite* test. Some passed it at the beginning of the school year, others in the middle or at the end of the school year. The experiment totals 289 passes of the *Pépite* test. Figure 13.7 shows the distribution of the profiles according to the students' grade and the period of the school year. It appears that many students are in groups C and B. This reveals important learning needs on the role of algebra, meaning of letters and equivalence of algebraic expressions. More 9th grade students belong to group C than to the other two groups, whereas the 10th grade students are distributed across groups A, B and C.

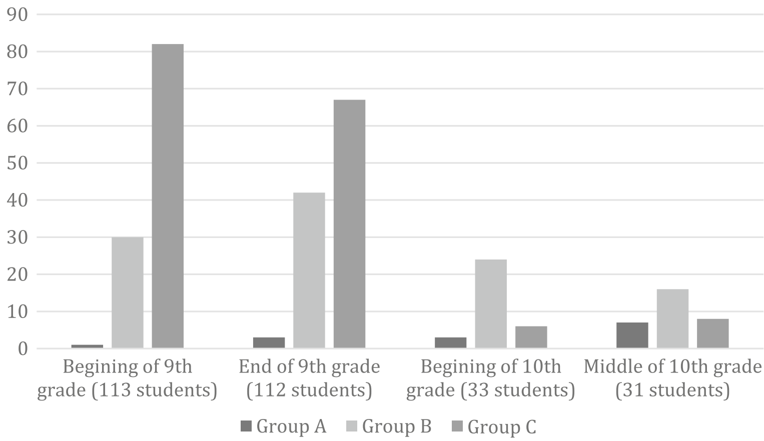


Fig. 13.7 Collective global diagnostic assessment for 9th/10th grade students

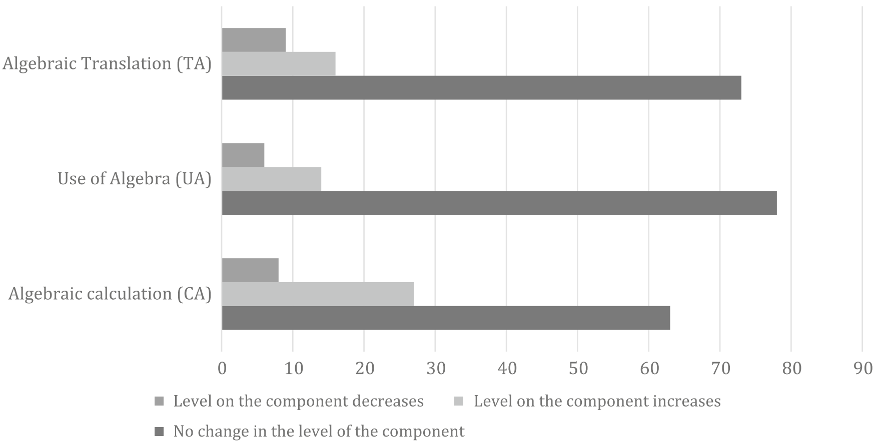


Fig. 13.8 Evolution of levels on the three components (Pilet, 2012) for 98 students

This result highlights the development of algebraic competence from one grade to another.

98 students (15 years old) passed *Pépite* test twice. Between the two assessments, students received differentiated instruction. We followed the evolution of their cognitive profiles between the first and the second assessment. We observed (Fig. 13.8) that the students' cognitive profiles increased, particularly on the CA component (Pilet, Chenevotot, Grugeon, El-Kechaï, and Delozanne, 2013). Students whose level decreased were often students who were not attending classes on a regular basis and were on the margins of the school system.

13.6 Discussion and Perspectives

The research presented here concerns the design of diagnostic assessment tools and differentiated instruction sessions adapted to students' learning needs, implemented on an online platform, and how they can be used in class. This research is part of the *Pépite* project, which is very challenging: helping teachers to manage the heterogeneity of students' knowledge and skills in elementary algebra. We argued the theoretical and methodological framework in order to define the didactical and computer models of diagnostic assessment and differentiated instruction for improving the learning of elementary algebra.

We have shown the potential of an epistemological study to support the design and the development of an appropriate digital diagnostic assessment for learning elementary algebra in middle/lower secondary grades. With such an approach, we defined a praxeology of elementary algebra, which gives us a reference to conceive a valid assessment with regards to coverage of the algebraic field and representativeness of tasks. This praxeology of elementary algebra allows the researcher to define criteria and associated values for analysing students' responses. Thanks to a transversal analysis on several tasks (encoding several tasks with the same codes), we can identify consistent student reasoning and calculation on several tasks that we interpret as level of technological discourse involved in techniques related to the praxeology of elementary algebra. These levels support the definition of groups of students' learning needs and differentiated instruction strategies taking into account the learning needs often ignored by curricula (Grugeon-Allys et al., 2012).

Since 2012, *Pépite* tools have been implemented on *LaboMep and WIMS* platform for 7th/8th grade (13–14 years old), 8th/9th grade (14–15 years old) and 9th/10th grade (15–16 years old) (Chenevotot-Quentin et al., 2016). The diagnostic assessment *Pépite* gives the teacher a very precise cognitive profile, for each student, concerning his or her skills in elementary algebra. The software automatically builds groups of students, identified as having close profiles, and differentiated instruction sessions (refer to Fig. 13.5).

We now return to the conditions for *Pépite* tools in order to support a formative assessment in every day teaching of elementary algebra.

Some teachers, collaborating with researchers, tested them in class. Pilet (2015) analysed the evolution of students' cognitive profiles from 9th to 10th grades where teachers had put in place differentiated teaching practices adapted to the learning needs of the students. We observed that the skills of some students increased (Fig. 13.8), even though the evolution was low.

Bedja (2016) studied the integration of diagnostic assessment tools and differentiated instruction sessions in teaching practices for two teachers involved in collaborative group. They needed a long time to appropriate the new types of algebraic tasks and to develop their algebraic teaching practices.

Those studies concerned only few teachers and students. Furthermore, middle schools and high schools are still not well enough equipped with computers in France, and few teachers regularly use software environments in their teaching. Beyond this qualitative study, a quantitative one would be necessary to confirm the first results, both for the students and the teachers.

This research is continuing in several directions. As differentiated instruction sessions defined by Pilet (2015) concerned only algebraic expressions, Sirejacob (2016) has conducted a similar research on equations based on an epistemological study of equations (Sirejacob, Chenevotot-Quentin, & Grugeon-Allys, to coming). In addition, Grapin has also defined a praxeology of the domain of arithmetic of integers. The transfer of the theoretical and methodological framework is relevant to design the features of an appropriate assessment (Grapin, 2015).

Acknowledgement We warmly thank Elisabeth Delozanne for her valuable contribution to the project.

References

Artigue, M., Grugeon, B., Assude, T. & Lenfant, A. (2001). *Teaching and learning algebra: Approaching complexity trough complementary perspectives.* In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The Future of the Teaching and Learning of Algebra*, *Proceedings of 12 th ICMI Study Conference*, The University of Melbourne, Australia, December 9–14, 2001.

Bedja, S. (2016). *Parcours différenciés d'apprentissage en algèbre élémentaire dans des systèmes de ressources en ligne: De l'usage des ressources par des communautés d'enseignants à leur évolution*. Thèse de doctorat, université Paris Diderot.

Black, P., & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education: Principles, Policy and Practice, 5*(1), 7–73.

Bolon, J. (2002). Pédagogie différenciée en mathématiques: Mission impossible ou défi? *Grand N. Num, 69*, 63–82 (IREM de Grenoble).

Bosch, M. (2015). Doing research within the anthropological theory of the didactic: The case of school algebra. In S. J. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education (ICME12)* (pp. 51–69). Seoul, Korea, July 8–15, 2012.

Chevallard, Y. (1984). Le passage de l'arithmétique à l'algèbre dans l'enseignement des mathématiques au collège. Première partie. L'évolution de la transposition didactique. *Petit x, 5,* 51–94.

Chevallard, Y. (1989). Le passage de l'arithmétique à l'algébrique dans l'enseignement des mathématiques au collège—Deuxième partie: Perspectives curriculaires: La notion de madélisation. *Petit x, 19*, 43–72.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–265.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the IV Congress of the European Society for Research in Mathematics Education* (pp. 21–30). Barcelona, Spain: FUNDEMI-IQS.

Chenevotot-Quentin, F., Grugeon-Allys, B., Pilet, J., Delozanne, E., & Prévit, D. (2016). The diagnostic assessment Pépite and the question of its transfer at different school levels. In *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education CERME9* (pp. 2326–2332). Prague, République Tchèque, 4 au 8 février 2015.

Delozanne, É, Prévit, D., Grugeon, B., & Chenevotot, F. (2008). Automatic multi-criteria assessment of open-ended questions: A case study in school algebra. In *Proceedings of ITS'2008, Montréal* (pp. 101–110). LNCS 5091, Springer, June 2008.

Delozanne, E., Prévit D., Grugeon-Allys, B., & Chenevotot-Quentin, F. (2010), Vers un modèle de diagnostic de compétence. *Revue Techniques et Sciences Informatiques*, *29*, 8–9 (Hermes-Lavoisier, Paris, pp. 899–938).

Douady, R. (1985). The interplay between different settings: Tool-object dialectic in the extension of mathematical ability—Examples from elementary school teaching. In *International Conference of Psychology of Mathematics Education (PME)* (pp. 33–52). Utrecht.

Garcia, F. J., Gascon, J., Higueras, L. R., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM Mathematics Education, 38*(3), 226–246.

Grapin, N. (2015). *Étude de la validité de dispositifs d'évaluation et conception d'un modèle d'analyse multidimensionnelle des connaissances numériques des élèves de fin d'école.* Thèse de doctorat, université Paris Diderot.

Grugeon-Allys, B. (2015). Réguler l'enseignement en algèbre élémentaire: Une approche multidimensionnelle. D. A. C. Mathé & E. Mounier (Eds.), *Actes du séminaire national de Didactique des mathématiques 2015*. Paris, France: IREM Paris 7.

Grugeon-Allys, B., Pilet, J., Chenevotot-Quentin, F., & Delozanne, E. (2012). Diagnostic et parcours différenciés d'enseignement en algèbre élémentaire. In L. Coulange, J. P. Drouhard, J. L. Dorier, & A. Robert (Eds.), *Recherche en Didactique des Mathématiques Enseignement de l'algèbre élémentaire, Bilan et perspectives Hors-série* (pp. 137–162). Grenoble: La pensée sauvage.

Kahn, C. (2007). *Pédagogie différenciée*. Bruxelles: De Boeck.

Ketterlin-Geller, L. R., & Yovanoff, P. (2009). Diagnostic assessment in mathematics to support instructional decision making. *Practical Assessment Research & Education, 14*(16).

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. K. Lester (Eds.), *Second handbook of research on mathematics teaching and learning* (Chapter 16, pp. 707–762).

Pilet, J. (2012). *Parcours d'enseignement différencié appuyés sur un diagnostic en algèbre élémentaire à fin de la scolarité obligatoire: Modélisation, implémentation dans une plateforme en ligne et évaluation* (Thèse de doctorat). Université Paris-Diderot. Available online http://tel.archives-ouvertes.fr/tel-00784039.

Pilet, J. (2015). Réguler l'enseignement en algèbre élémentaire par des parcours d'enseignement différencié. *Recherches en didactique des mathématiques, 35*(3), 273–312.

Pilet, J. Chenevotot, F., Grugeon, B., El-Kechaï, N., & Delozanne, E. (2013). Bridging diagnosis and learning of elementary algebra using technologies. In *Proceedings of the Eighth Congress of the European society for Research in Mathematics Education*. CERME 8. Antalaya, Turquie, 6 au 10 février 2013. 10 p.

Ruiz-Munzón, N., Matheron, Y., Bosch, M., & Gascón, J. (2012). Autour de l'algèbre: les entiers relatifs et la modélisation algébrico-fonctionnelle. In L. Coulange, & J. P. Drouhard (Eds.), *Enseignement de l'algèbre élémentaire. Bilan et perspectives, Recherches en didactique de mathématiques*. Grenoble: La Pensée Sauvage.

Sirejacob, S. (2016). Les organisation de savoirs mathématiques à enseigner: Les équations au college. *Petit x, 102,* 27–55.

Sirejacob, S., Chenevotot-Quentin, F., & Grugeon-Allys, B. (to coming) *Pépite* online automated assessment and student learning: The domain of equations for grade 8th. In *Proceedings of the Tenth Congress of the European society for Research in Mathematics Education* CERME10. Dublin, Irlande, 1er au 5 février 2017.

Vergnaud, G. (1986). Psychologie du développement cognitif et didactique des mathématiques. Un exemple: Les structures additives. *Petit x, 22,* 51–69.

Wiliam, D., & Thompson, M. (2007). Integrating assessment with learning: What will it take to make it work? In C. A. Dwyer (Ed.), *The future of assessment: Shaping teaching and learning* (pp. 53–82). Mahwah, NJ: Erlbaum.

Chapter 14
Using Dynamic CAS and Geometry to Enhance Digital Assessments

Thomas P. Dick

Abstract Summative assessments, and in particular, online digital summative assessments are dominated by items that present a limited number of possible responses for selection (with multiple choice being the most prevalent example). Less constrained constructed responses to a task might be submitted via digital technology, but in many cases require human evaluation. To avoid this, we present a prototype for enhancing digital assessment by using dynamic computer algebra systems linked with dynamic geometry environments. The technology enhanced items created by this system allow students to create mathematical objects required to have stated properties, and the satisfaction of those properties can be evaluated for mathematical correctness automatically by the system.

Keywords Computer algebra systems · Dynamic geometry · Technology enhanced test items · Mathematics assessment

14.1 Background

Our focus in this paper is on describing how linkages between computer algebra systems and dynamic geometry software can be exploited to create summative assessment tasks that (1) require virtually no specialized tool knowledge by the student, (2) require students to construct responses rather than make a selection of presented options, and (3) are entirely machine evaluated for mathematical correctness automatically.

Our premise is that a wider use of such tasks for summative assessment could be influential in encouraging teachers in the use of technology to support student learning in the classroom. We also describe a prototype of an authoring system that would allow teachers or test developers to easily create not only the assessment task but the mechanism for its automatic scoring.

T. P. Dick (✉)
Oregon State University, Corvallis, OR, USA
e-mail: tpdick@math.oregonstate.edu

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_14

In the remainder of this background section we provide a framing for our work by highlighting the distinctions in purpose between summative and formative assessments, the importance of high cognitive level tasks in assessments for promoting student learning, the special challenges of using technology in large scale summative assessments, and our strategy of using linked computer algebra and dynamic geometry to create assessment tasks that admit many solutions but are nevertheless machine scorable. In the second section, we will turn to a detailed description of several exemplars of these assessment tasks and in the third section we describe a prototype of an authoring system for creating them. We conclude the paper with some remarks that include the possible uses of the assessment tasks in the classroom. However, we emphasize that our design work has been motivated by the desire to influence the design of large scale summative assessments, and we have not undertaken any systematic study of the alternative formative uses of the tasks in classroom settings.

14.1.1 Summative and Formative Assessments

Black, Harrison, Lee, Marshall, and Wiliam (2003) have noted the terms *summative* and *formative* as applied to performance tasks indicate a distinction in purpose rather than a descriptor of the content or type of task. In other words, the same performance task could serve either or both of summative and formative purposes. Summative assessment is concerned with documentation of achievement or attainment (whether it be of factual recall, computational proficiency, skill attainment, problem solving, reasoning, or evaluation/justification) as evidenced by the student's successful performance on a task. The designers of summative assessments are accountable to one or more stakeholders (the national or regional government, the school system, future employers or academic programs) to provide a valid and reliable measure that has value for making comparisons. Those comparisons may be of the same student at different times in an instructional sequence, between students in competition for an employment position, or between educational systems or instructional approaches (and hence, could be impactful in making policy decisions).

Black et al. (2003) have focused considerable attention on formative assessments that serve as *assessments for learning*—where the highest priority of purpose is in advancing the learning of students. Such assessments can be quite informal and used during ongoing instruction. The information yielded by a formative assessment may serve as feedback to the student and/or influence the next instructional move of the teacher. The usefulness of a formative assessment task lies in either moving students' thinking forward or in eliciting for the teacher some insight into how students are thinking.

Stein, Smith, Henningsen, and Silver (2000) have described a Mathematical Task Framework to assist teachers in selecting or designing mathematical tasks that promote student learning. The authors contend that the higher the level of cognitive demand of a task, the more opportunities for learning that task affords. The Mathematical Task Framework describes four cognitive demand levels (from lowest to highest):

Memorization (Simple Recall of Facts or Definitions)
Procedures without connections (to understanding, meaning, or concepts)
Procedures with connections (to understanding, meaning, or concepts)
Doing mathematics (explorations of relationships, complex or nonalgorithmic thinking).

The distinction between tasks considered as *procedures without connections* versus *procedures with connections* lies in whether the tasks can be successfully completed by simply applying an algorithm as opposed to also needing to explain how the procedure works, its connections to multiple representations, or its conceptual foundations. The case is made that higher cognitive demand tasks such as *procedures with connections* or *doing mathematics* not only afford more opportunities for student learning, they also are more *discourse worthy* for purposes of facilitating small group or whole class discussions. Smith and Stein (2011) have described five essential practices (anticipation, monitoring, selecting, sequencing, and connecting) for teachers to orchestrate mathematically productive discourse around publicly shared student work on such tasks. Effectively, such discussions can both inform the teacher about student thinking while also moving students' thinking forward.

14.1.2 Technology in Support of Assessment

Viewed through an assessment lens, the literature on technology use in mathematics education has been primarily on the formative side, with the most attention devoted to using technology to promote student learning. Focus in High School Mathematics: Technology to Support Reasoning and Sense Making (Dick and Hollebrands, 2011) includes a wide variety of illustrations of technologies (graphing calculators, computer algebra systems, dynamic geometry environments, probability and statistics applets, and others) used in the mathematics classroom, with special attention to mathematical tasks requiring conceptual understanding and mathematical reasoning. The chapter by Cohen and Hollebrands (2011) in that volume discusses how screen sharing technology can be used in tandem with mathematical software to support the five practices for orchestrating productive mathematical discourse described by Smith and Stein (2011).

In research studies comparing the impact on student learning of instructional approaches employing technology use with approaches that do not, technology use is almost always in an independent variable role. Summative assessment measures play the role of dependent variable and these are usually administered by paper and pencil with no technology tool use allowed, given that the control group did not have experience with the technology.

There has been a long history of work in developing diagnostic assessment items where the technology supports the analysis of student responses. An example of early work in this arena is DEBUGGY, a system that identified common errors in students' arithmetic procedural work (see McFarland & Parker, 1990, for a description of DEBUGGY and other early diagnostic programs). A more recent example is the work on Pépite (Delozanne, Prévit, Grugeon, & Chenevotot, 2008), a technology based diagnostic assessment system that analyzes students' algebraic work entered by typing into response fields for tasks posed by the system. In these examples, the technology supports formative diagnostic assessment and teachers' work in the classroom assisting individual student learners.

Large scale and high stakes summative assessment programs are naturally keenly interested in platforms that allow computer scoring or evaluation of student responses. The savings in time and expense coupled with the accuracy and automaticity of recording student performance are the obvious drawing cards to machine aided assessment. At the same time, the stakeholders accountable to these assessments are naturally deeply concerned with the validity of the psychometric measures they yield—do these measures fairly and adequately reflect the students' proficiencies and knowledge in assessing whether or not the stated standards are met?

There are primarily two types of assessment items, characterized by how students must respond:

(1) Selected response item—a finite collection of possible responses to the question or task are explicitly presented to the student, and the student must choose which of these possible responses is correct or "best."
(2) Constructed response item—the student must present a response requiring creative construction, either within an entirely free format or by making creative changes or additions to a pre-existing structure.

Constructed responses can be very involved, and such items might require an interpretive essay, a complete mathematical proof, the construction of a table or a graph, or the creation of a detailed diagram. However, note the nature of the content of a response is technically independent of the item type (multiple choice or constructed response). A constructed response item might ask for nothing more than a single numerical value with no accompanying supporting work (and many such items exist), while a multiple choice item could present a set of five detailed explanations and ask the student to choose the best one (such items are more rare).

In terms of potential for machine scoring, selected response items are the easiest and most straightforward to implement. The evaluation of student responses is trivial and depends on nothing more than comparison of the student's submitted choices with the coded correct choices (the "answer key"). In terms of cognitive level, selected response assessment items work relatively well for vocabulary (recall knowledge of terminology), application of definitions, recognition tasks, and procedural skill performance.

The most significant limitation of a selected response item is inherent and cannot be remedied—students are only asked to select from presented response choices without needing to present one of their own formulation or construction. The disadvantage of multiple choice items lies in the difficulty in adequately assessing higher level cognitive performance when students are merely selecting from choices and not creating their own responses. Multiple choice items also open up the possibility of students choosing correct responses through either lucky blind guessing or through strategic elimination of the distractor choices (that might display more awareness of "test taking tricks" rather than evidence of actual content knowledge).

For constructed response items to be machine scored, there usually must be some constraints on the form of the student response. For example, suppose a constructed response item called for an answer that has an exact numerical value 2.5, and furthermore, suppose the student is required to express the response in decimal form (i.e., 2½ or 5/2 would be unacceptable forms). This means that the only acceptable response is a completely determined string of three characters—the digit "2", followed by the decimal point, followed by the digit "5". With these constraints, the machine need only perform a sequential symbol by symbol matching check to evaluate the correctness of the response. In a computer testing environment for such an item, the student might be presented with distinct individual response fields for each symbol to aid in the automatic symbol matching process.

In general, constructed response items are widely viewed as more suitable than multiple choice items for assessing higher level cognitive performance. However, if the format of the student response is very strictly constrained to allow machine scoring, then the value for assessing higher level cognition may be compromised. Constructed response items may also provide for partial credit or holistic levels of evaluation, based on the completeness, correctness, and quality of the presented response.

14.1.3 How Can Technology Be Used to Enhance Digital Summative Assessment?

Suppose we consider multiple choice and constructed response assessment items, not as two dichotomous categories, but rather as intervals at the ends of a continuum indicating the level of freedom in student response, and correspondingly, the ease of machine scoring. As shown in Fig. 14.1, the more freedom in form the student is allowed in constructing a response, the more challenging that response will be to evaluate by machine (Drijvers et al., 2016). The territory lying in between these extremes is fertile ground for enhancement through technology.

An overarching goal of technology enhancement is to provide more freedom in student response while maintaining machine evaluation or scoring. The means by which this goal can be achieved is to present an electronic "smart" slate for the student to enter or create their response—an environment that can detect and analyze structure, properties, and qualities of the response that go beyond simply matching against a predetermined answer key. Technology can provide both *surface* and

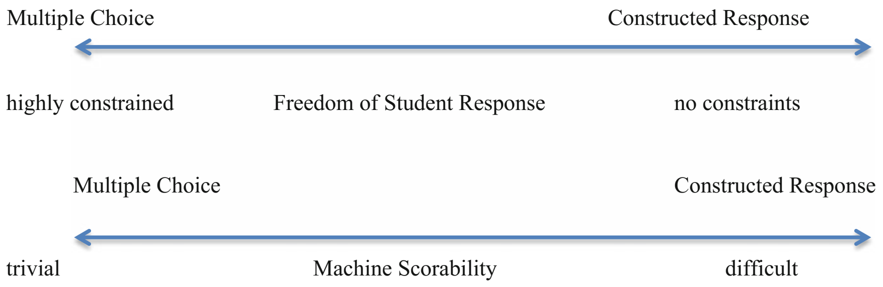


Fig. 14.1 The level of constraints on student response is related to its machine scorability

analytical enhancements in an assessment environment. By surface enhancements are those that provide for ease of construction of a response at the surface of the screen, and so are visible and readily apparent to the student. Ideally, there should be little or no time needed for the student to become familiar with how the surface user interface works to engage in the assessment task. By analytical enhancements are those that provide tools for identifying structure, properties, and qualities of the response, and hence are "under the hood," providing for the machine evaluation of the student response.

For example, if a machine assessment platform was simply capable of performing at least arithmetic calculation of student entered numerical expressions, then more freedom could be allowed in the form of the student response via keyboard entry. If the correct numerical answer to a constructed response question was simply 2.5, an arithmetically "smart" platform would allow students who typed in "5/2" or "1 + 1 + ½" or "3 − 0.5" all to be judged to have submitted the numerical equivalent of 2.5, and these could be evaluated as correct responses. If the machine assessment platform has CAS (Computer Algebra System) capabilities, then evaluated text entry can also be extended to algebraically equivalent expressions.

Unfortunately, much of what are being portrayed as "technology enhanced items" have neither surface enhancements nor analytic enhancements. Rather, they are effectively just multiple choice items where the student selection mechanism has been superficially dressed up or altered. For example, rather than choosing a letter corresponding to one of five presented response options, the student might be directed to drag the chosen response option across the screen to an answer box or location. Selection by clicking on an option or dragging an option to some specific, but arbitrary location, introduces screen level "interaction" that is cosmetic only and essentially devoid of any content implications.

The machine scoring by simply matching the label of a response (A–B–C–D–E) to the keyed response has now been replaced by matching the location of a click or origination/destination of a drag on screen to a predetermined keyed location. Hence, there is no analytical enhancement at play. A truly technology enhanced mathematical assessment item is one where either the surface tools aid the student in constructing a response that has discernable mathematical meaning (examples:

numbers, expressions, graphs, tables, geometric objects, proofs), or some aspects of the mathematical meaning of a constructed response are discernable by the technology.

14.1.4 Harnessing Dynamic CAS and Geometry to Enhance Digital Summative Assessment

There can be a powerful synergy between the surface and analytical enhancements to an assessment task, provided the technology allows

- the linking of the results of student use of surface construction tools to variables;
- the values of those variables being sufficient for the machine to analytically determine whether or not, or to what extent, the requirements of the task have been fulfilled.

The opportunities abound if the surface construction tools themselves lie within a mathematically sensitive environment. That is, suppose the student can create and/or manipulate objects (expressions, parameter values, points, lines, geometric objects, graphs, etc.) such that properties of, and relationships among these objects can effectively be linked "under the hood" to variable values. If these variable values can, in turn, be dynamically linked to logical expressions whose truth values can be evaluated by CAS, then this opens up a tremendously rich constructed response setting that is instantly machine scorable.

If a constructed response task asks for students to come up with a symbolic expression that is completely determined up to algebraic equivalence, then the only analytic role of CAS is to check the student's submitted expression for equivalence with an answer key expression. While this is not an insignificant enhancement, it is only the tip of the iceberg in terms of potential. Many routine mathematical tasks present the student with an object and ask them to find something related to it, such as: find the solutions to this equation, find the area and perimeter of this polygon, find the mean and standard deviation of these data, etc. Such tasks can be "jeopardized" (in the sense of the game show "Jeopardy" where contestants are presented with answers and must come up with the questions), by providing students with a mathematically aware environment and asking them to create a mathematical object with certain properties.

Two design principles that Dick and Burrill (2016) have proposed for dynamic interactive mathematics learning environments are also relevant to the surface design of digital assessment tasks: *mathematical fidelity* and *cognitive fidelity* (Zbiek, Heid, Blume, & Dick, 2007). Mathematical fidelity refers to the faithfulness of the technology-based behavior and properties of objects on screen to the mathematical behaviors they are intended to represent, while cognitive fidelity refers to the match between cognitive perception and the actual mathematical action. For example, if the underlying vertical and horizontal scales in a coordinate plane are not the same, rigid motions of objects might not be perceived as angle preserving by the user, even when they are mathematically correct.

14.2 Examples of Technology Enhanced Assessment Items

In this section we illustrate the potential for enhancing assessment items through the use of dynamic Computer Algebra Systems and Dynamic Geometry technology. The particular platform used to create the exemplars is the TI-Nspire CAS software, but a similar strategy could be used with other systems that provide for dynamic linking of variables in computer algebra and dynamic geometry.

Each assessment task presents the student with a mathematical object in the form of a graph or a geometrically figure, and a set of mathematical conditions or properties that the object should satisfy. The student can readily modify this object by moving one or more points on screen, or by editing an algebraic equation defining the object. The coordinates of movable points are linked to variables in stored logical expressions whose truth-value indicate whether the conditions or properties have been satisfied. Similarly, if an object is defined by an algebraic expression, that expression is linked to the underlying computer algebra system for analysis.

The evaluation of the assessment item is actually dynamic, as the truth value of the logical expressions is continuously updated as the student modifies the object. Of course, this dynamic truth-value is not normally visible to students as they modify the object, for in many cases this would open up a "trial and error" search strategy that is not purposeful mathematically. (Making the dynamic truth-value visible could have some merit in a formative assessment setting if accompanied by reflective questioning requiring student explanation of this feedback.) In the summative assessment setting, a student would indicate completion of the task by way of a "submit" button. In the screenshots shown for the examples discussed below, we have made the dynamic machine evaluation visible, and added the reminder that this would normally not be visible to the student undertaking the assessment task.

The assessment tasks we describe were motivated by mathematical content standards found in the *Common Core State Standards—Mathematics* (http://www.corestandards.org/Math/) for algebra and geometry in middle grades:

- Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. (Grade 7, Geometry)
- Understand congruence and similarity using physical models, transparencies, or geometry software. (Grade 8, Geometry)
- Find the equation of a line parallel or perpendicular to a given line that passes through a given point. (High School Geometry: Expressing Geometric Properties with Equations)
- Translate between the geometric description and the equation for a conic section. (High School Geometry: Expressing Geometric Properties with Equations)

Example 1
The opening screen for the task is shown in Fig. 14.2 (note that the dynamic machine evaluation shown would normally not be visible to the student). Points P and Q are fixed, and the remaining two points R and S can be moved by the student by simple "click and drag" using either a mouse or touchscreen. The placements of the points R and S are restricted to a rectangular lattice (on which P and Q also lie).

While there are no visible coordinate axes, the lattice is coordinatized for the purposes of the machine evaluation of whether or not the segments connecting points P and R and the points Q and S have a common midpoint. The screenshots in Fig. 14.3 show two of the possible successful completions of the task.

Note that many correct solutions to the task are possible, even under the constraint of the lattice. The lattice allows for a more precise placement of the points by the student. Without it, placing the points to achieve the condition on the diagonals would be quite difficult. If used in a classroom setting where student solutions could be compared with each other and discussed, the task could lead to a conjecture that

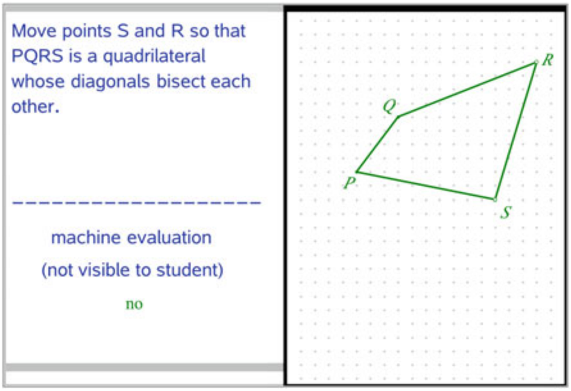


Fig. 14.2 Opening screen for a quadrilateral creation task

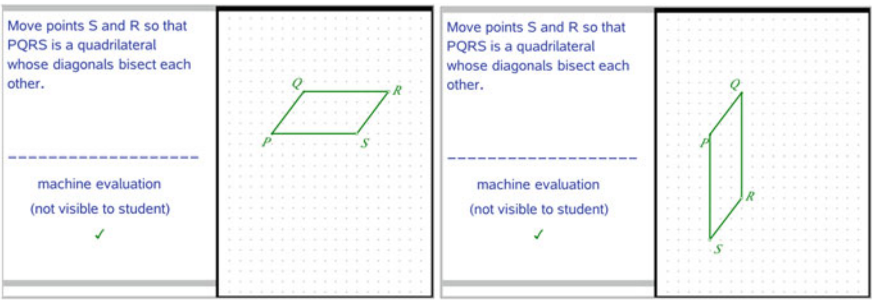


Fig. 14.3 Two successful attempts at creating a quadrilateral with mutually bisected diagonals

the condition forces the quadrilateral to be a parallelogram—a potential theorem to be proven. The next two screenshots in Fig. 14.4 show solutions that share an additional property: the diagonals are also of the same length. This additional property suggests a conjecture that such a quadrilateral is forced to be a rectangle. The screen in Fig. 14.5 shows a figure that exhibits two mutually bisected segments, but this is a self-intersecting figure that is not a quadrilateral.

Example 2

The opening screen for the task is shown in the first screenshot of Fig. 14.6 (again, the dynamic machine evaluation shown would not be visible to the student). While the previous example admitted several correct solutions, for this task there is a unique location on the available screen lattice that satisfies the condition. (Other points would satisfy the condition if an "extended" screen lattice were available.) The task requires identification of the required scale factor (2 or 1/2, depending on direction of scaling) as a critical step. Once the scale factor is determined, the rectangular lattice provides an important means of comparing segment lengths in locating the unique point satisfying the condition. That point is shown in the second screenshot of Fig. 14.6.

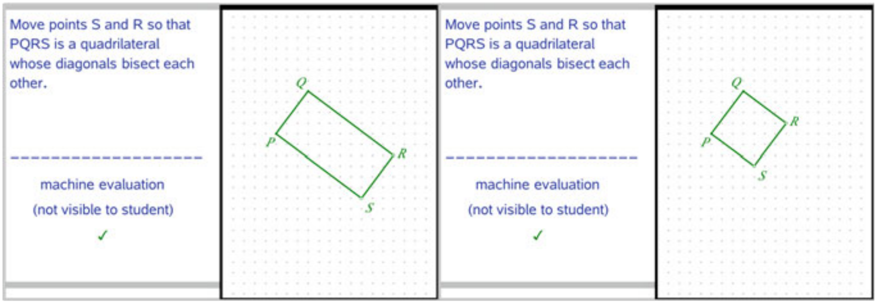
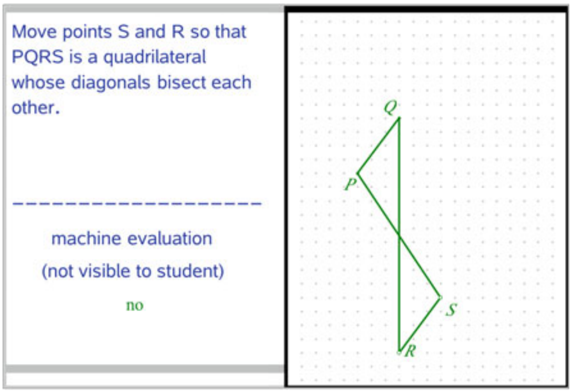


Fig. 14.4 Two solutions to the quadrilateral task that also have diagonals of equal length

Fig. 14.5 An interesting incorrect solution to the quadrilateral task



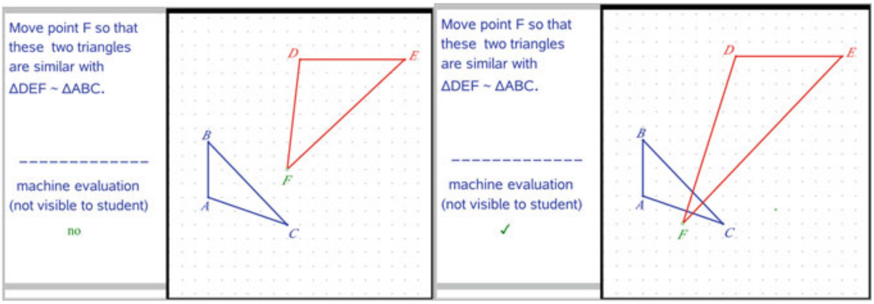


Fig. 14.6 Opening screen and a successful completion of the similar triangle creation task

Example 3

The opening screen for the task is shown in Fig. 14.7, as well as one typical solution. The isosceles trapezoid shown in Fig. 14.7 is not the only possible solution, however. There is a family of "kites" that can be created having the line through *Q* and *S* as a line of symmetry. Two such kites are shown in Fig. 14.8.

The point *S* can be placed in many lattice point positions that make the line through *Q* and *S* a line of symmetry for the figure (and these point positions themselves are collinear), but not all of these locations satisfy the desired condition (Fig. 14.9). Quadrilaterals satisfying the condition need not be convex (Fig. 14.10).

In all of the solutions except the isosceles trapezoid, the line through *Q* and *S* is the line of symmetry. For the isosceles trapezoid, the line of symmetry goes through the two midpoints of opposite of the quadrilateral. Is it possible to create a different isosceles trapezoid that satisfies the reflection symmetry condition? In this case, the constraint of the given rectangular lattice does not allow for the placement of this point (an attempt that comes "close" is shown in Fig. 14.11), but this limitation itself opens up opportunities for discussion: Exactly where would the desired point need to be placed between lattice points?

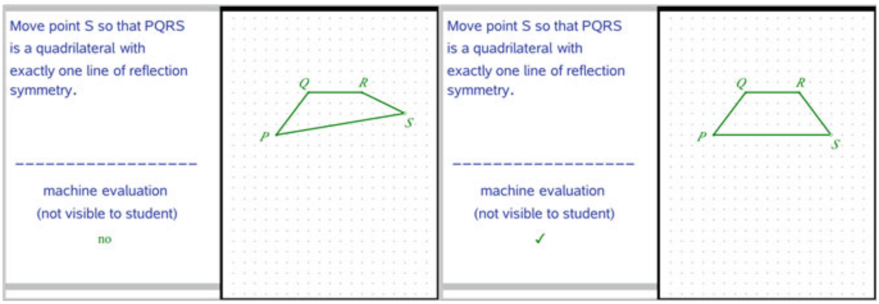


Fig. 14.7 Opening screen and a typical solution to the quadrilateral symmetry task

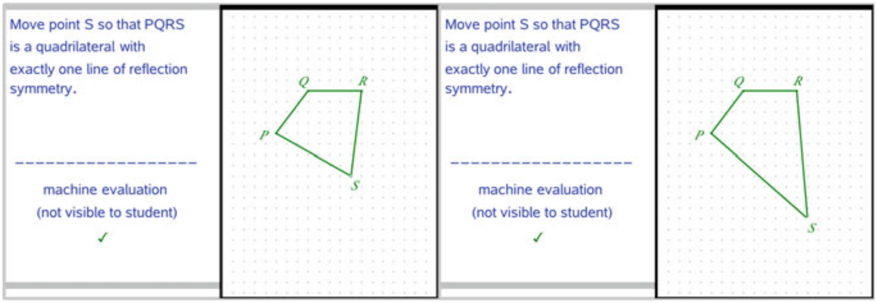


Fig. 14.8 Two "kite" solutions to the quadrilateral symmetry task

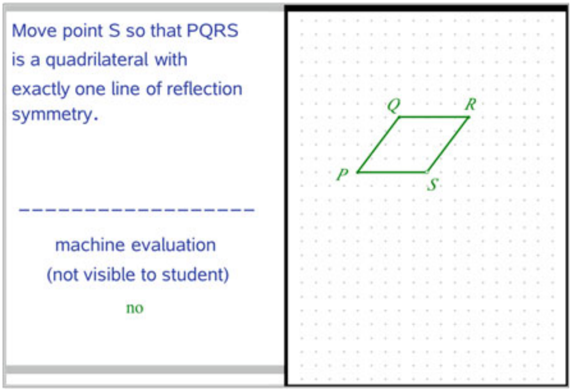


Fig. 14.9 Does this figure have exactly one line of symmetry?

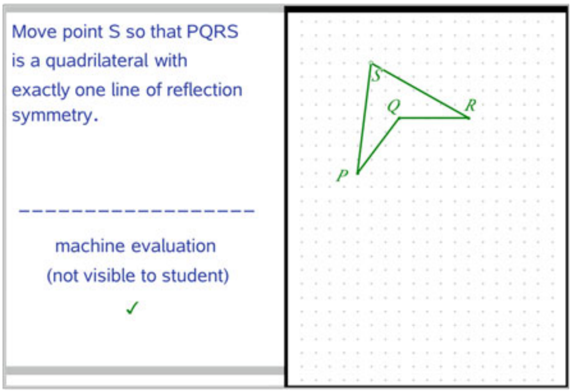


Fig. 14.10 One of the non-convex "dart" quadrilaterals that can be created

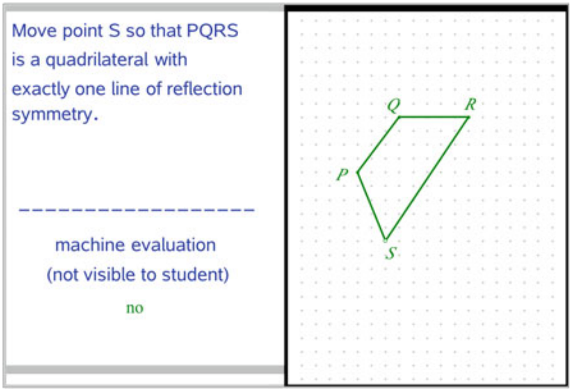


Fig. 14.11 A quadrilateral that is "close" to being an isosceles trapezoid

These first three examples all employ a (coordinatized) rectangular lattice of points to provide an easy means for students to create geometric figures. The dynamic links of the point coordinates to variables allow for the machine evaluation of whether the desired properties are satisfied. No programming was required to create these items, for the machine is simply the evaluating the truth of a logical statement that is the coordinatized translation of the condition(s).

In the following examples involving algebraic functions and their graphs, the coordinate system for the Cartesian plane is now on display to the student. The connection between a function's algebraic expression and its graph is exploited in both directions. In some cases, the task for the student is to create an algebraic expression for a function that satisfies certain graphical properties. In other cases, the student creates a graph that fits algebraic specification.

Example 4

The opening screen for this task is shown in Fig. 14.12. The graph of the prototypical $y = x^2$ is the dotted curve, while the solid parabola is manipulated by moving the two points indicated.

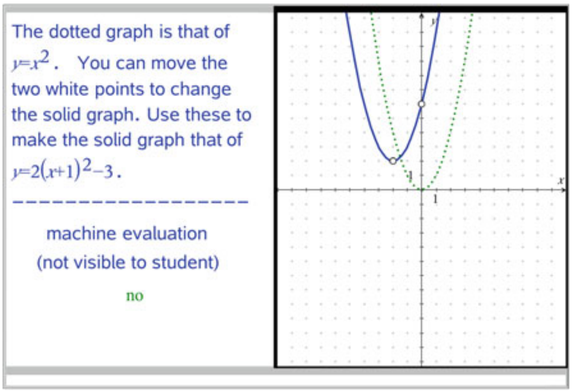


Fig. 14.12 Opening screen of a parabola task

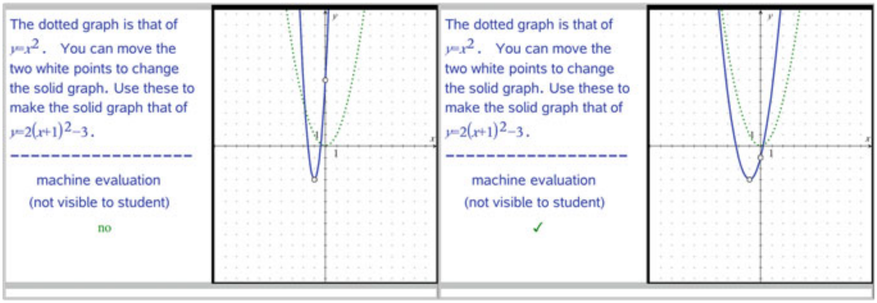


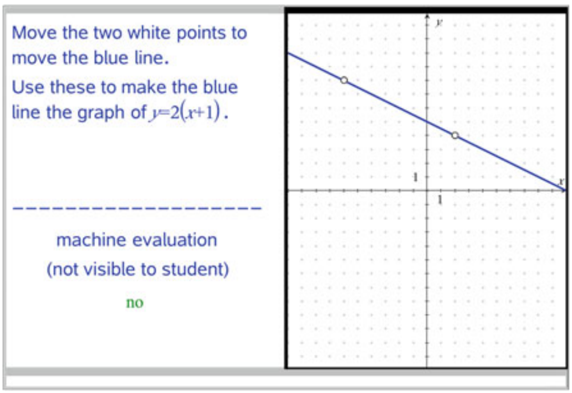
Fig. 14.13 A first move (locating the correct vertex) and a successful parabola creation

A student might make a first step of moving the vertex point to its correct location. Adjusting the second point's location can then successfully complete the task (Fig. 14.13).

Example 5
Students are presented with two distinct control points to determine the line (Fig. 14.14). While only one line satisfies the requirements, there are many choices of locations for the two points determining that line. Figure 14.15 shows screens resulting from first using one point to locate the x-intercept, and then second point to locate the y-intercept.

Example 6
In this example, the representational direction is reversed. The opening screen is shown in Fig. 14.16. Now the student's action is to submit a functional equation for the line. Figure 14.17 shows two typical incorrect choices for the slope. Figure 14.18 shows one of the correct solutions (any line with slope −3).

Fig. 14.14 Opening screen of a line creation task



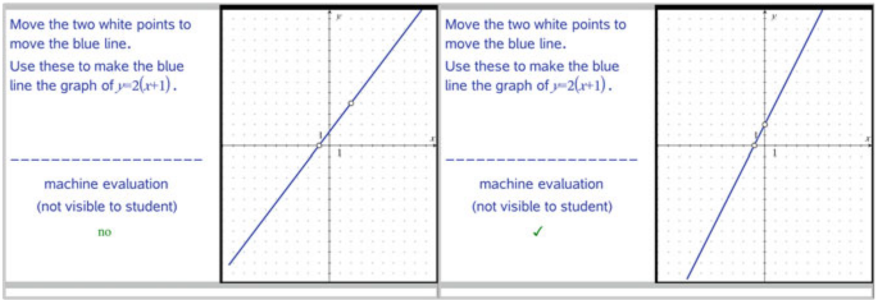


Fig. 14.15 Completing the line creation task by locating the intercepts

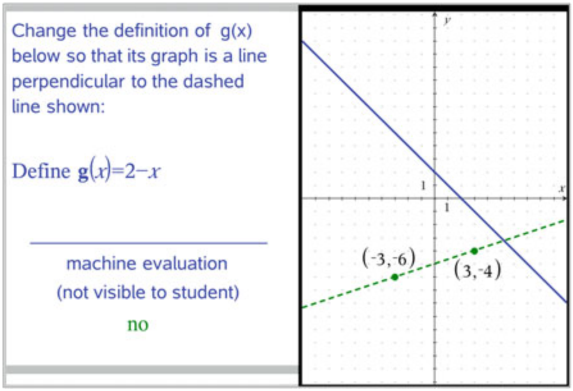


Fig. 14.16 Opening screen of a task asking for an algebraic expression

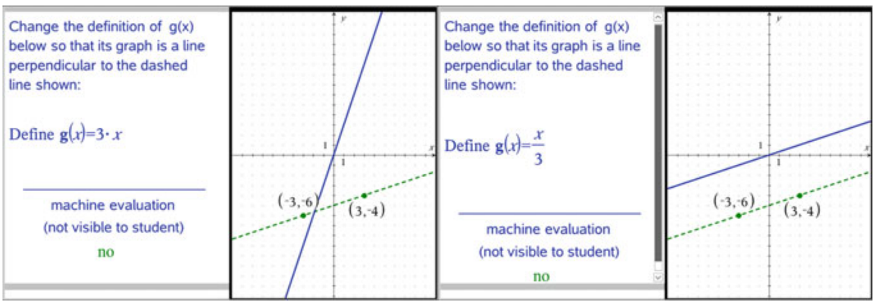
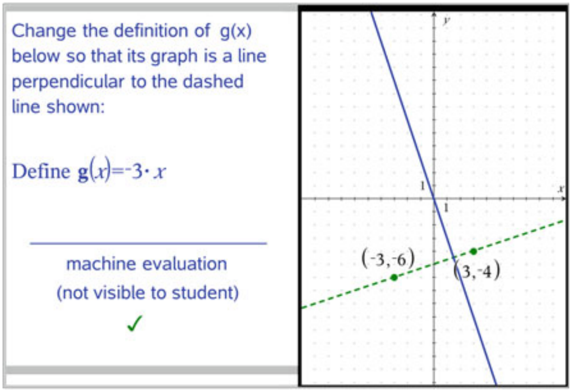


Fig. 14.17 Two incorrect student solutions (reciprocal slope, equal slope)

Fig. 14.18 A correct student solution to the linear function task



Example 7
The opening screen is shown in Fig. 14.19. Two correct solutions are shown in Fig. 14.20. Note that the computer algebra capabilities of the system allow for different but equivalent expressions representing the same parabola.

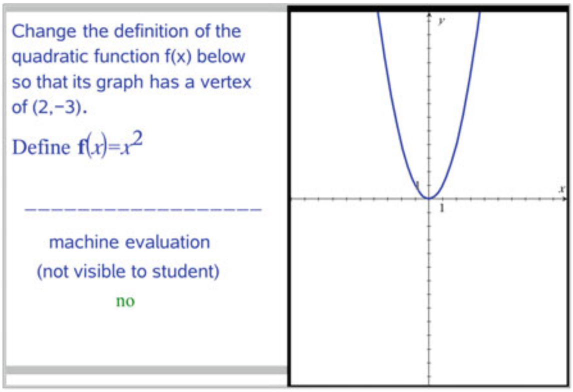


Fig. 14.19 Opening screen of a task asking for a quadratic expression

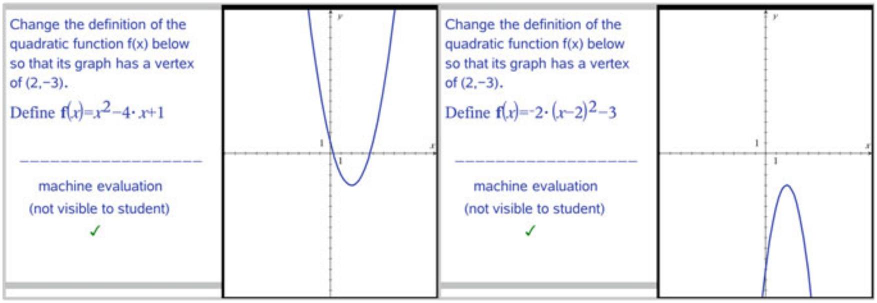


Fig. 14.20 Two correct quadratic expression (one in standard form, one in vertex form)

14.3 Next Steps: Technology Enhanced Item Authoring Tools

We hope that these examples serve to illustrate that there is vast potential for technology-based assessments to move well beyond the constraints of multiple-choice format while still allowing for automatic machine evaluation. While no programming was required for the examples discussed above, the logical statement evaluated by the machine can be complicated. In Example 3 where the student is asked to create a quadrilateral with exactly one line of symmetry, the logical statement is quite involved, for it needs to additionally check for non-collinearity of the points as well as ruling out a second line of symmetry.

We are now investigating the development of an authoring tool that would allow the teacher to easily create such assessment items simply by specifying the mathematical requirements of the object to be created by the student. Based on these specifications, the authoring tool would create both the item task as well as formulating all the required logical checks. We have developed such a prototype item generator for triangle properties as an exemplar. The prototype system allows the teacher to present a student with a dynamic triangle $\triangle ABC$ on a rectangular lattice grid that can be manipulated by moving one or more vertices (zero, one, or two of the vertices A, B, or C could be fixed). The teacher selects from a variety of properties and/or measurements to be satisfied by the student's triangle.

Example 8
Suppose the teacher wished to author a machine scorable assessment item that asks the student to create an acute triangle having area 12. The author's specification screen is shown in Fig. 14.21.

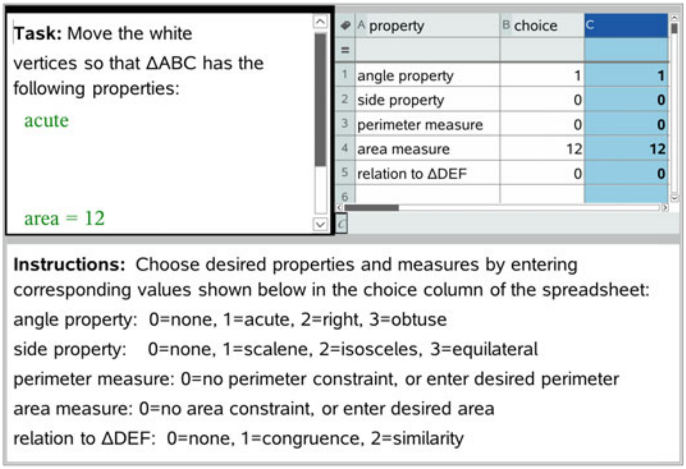


Fig. 14.21 Setting the specifications for a triangle task using an authoring tool

On the properties setup screen, the item author chooses which triangle properties are desired by entering the measurements or category codes in the appropriate line of the properties table. The task statement is then automatically generated here and on the task screen that will be presented to the student. The possible specifications include requiring a particular perimeter and/or area measurement, the category of triangle determined by its angles (acute, right, obtuse) or by its side lengths (equilateral, isosceles, or scalene). The author could also specify a second triangle for which the student must create a congruent or similar triangle.

Solid black points and moveable vertices indicate locked vertices by "open" (white) points. If a second triangle ΔDEF is to be provided, then the item author moves its vertices to their desired positions and then locks all three vertices D, E, and F. (Locking or unlocking any vertex is accomplished on the TI-Nspire by simply using the attributes menu for the vertex points.)

Through the use of computer algebra linked to the geometry environment, all these properties are dynamically checked by the system, and in turn, the system can provide feedback regarding exactly which of the specified properties or measurements were satisfied and which were not.

For this example, if the author wished to make all three vertices movable, the screen presented to the student for this task would appear as shown in Fig. 14.22.

All measurements and properties of the triangle are dynamically recorded on a report page. In addition, whether or not the requirements specified by the author are also checked for satisfaction. For the triangle initially shown in Fig. 14.22, the report page is shown in Fig. 14.23, indicating that this triangle is actually a right triangle having area 25.

Figure 14.24 shows a student's triangle that does successfully satisfy the requirements, as well as the dynamically updated report page.

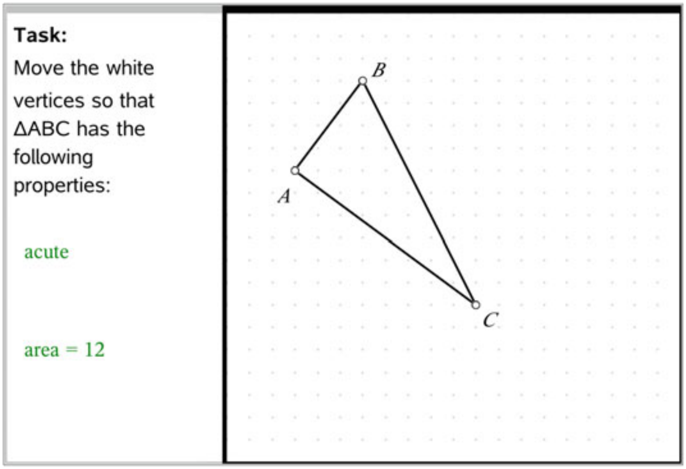


Fig. 14.22 Triangle creation task as presented to student

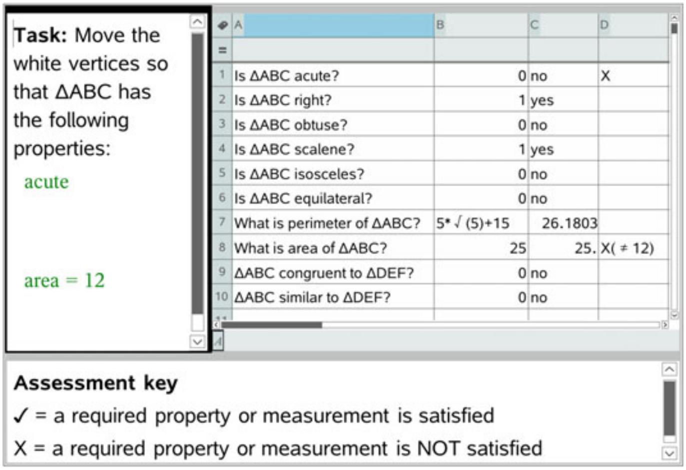


Fig. 14.23 The initial triangle satisfies neither the angle nor area conditions

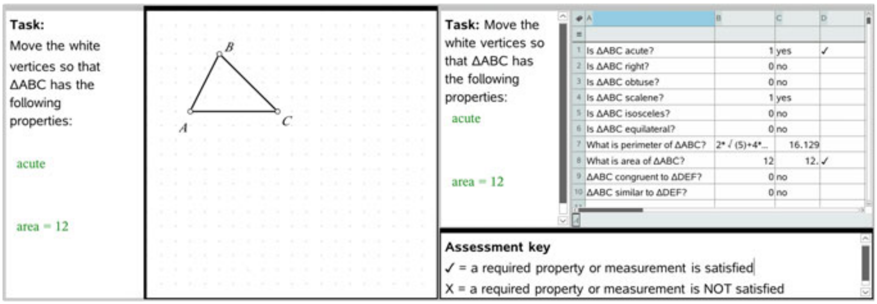


Fig. 14.24 A triangle that satisfies the task requirements

14.4 Concluding Remarks

There is a tension that exists between technology use on summative assessment tasks and technology use in the classroom for teaching and learning. For too many teachers, a restriction on allowable technology on high stakes summative assessments may be interpreted as implying a corresponding restriction on the appropriate use of technology in the classroom. These imagined limitations that perceived accountability to the summative purposes unnecessarily imposes on instruction are unfortunate. An even more serious concern arises, when the measurement (test score) of a learning goal becomes the goal itself. If high performance on the summative assessment becomes the target objective rather than the actual student learning that the same assessment purports to measure, then instructional practice may be warped into “teaching to the test,” that is, instruction is judged by how directly it supports student performance on summative assessment items. For example, if the summative assessment tasks tend to be at the lower cognitive levels as described by the Mathematical Task Framework, then it may become more difficult to convince teachers to engage their students around higher cognitive level tasks in the classroom.

Mathematically “active” software tools such as computer algebra systems and dynamic geometry environments are valued for the impact they can have on mathematics teaching and learning. Our intent in this paper is to illustrate the potential these tools could have to enhance technology-based summative assessment to be more open-ended and of higher cognitive level. The tasks that can be created by such tools are far less constrained than multiple choice, and can have multiple correct solutions, all of which can be evaluated automatically by the machine. The implications are significant for changing the landscape of large scale online digital summative assessments.

While we have focused our attention on the use of mathematically active environments to widen the types of items that could be used for summative assessment, we close by noting that such systems could easily be repurposed as tools for formative assessment for teachers and as a source of feedback for students. Paired with screen sharing capabilities, students’ responses could be also be used by teachers to facilitate productive mathematical discourse in the classroom. A significant advantage to these tasks lies in the absence of specialized tool knowledge needed by the student to engage in the tasks, making these items suitable for administration to students with a very wide variety of previous experience with technology tools in learning settings, including no experience at all. Again, our hope is that imaginative employment of mathematically active environments in summative assessments in turn could encourage more teachers to make use of technology as tools to promote student learning of mathematics.

References

Black, P., Harrison, C., Lee, C., Marshall, B., & Wiliam, D. (2003). *Assessment for learning: Putting it into practice*. Maidenhead, U.K.: Open University Press and McGraw-Hill Education Publishers.

Cohen, J., & Hollebrands, K. (2011). Technology tools to support mathematics teaching. In T. P. Dick & K. F. Hollebrands (Eds.), *Focus in high school mathematics: Technology to support reasoning and sense making* (pp. 105–120). Reston, VA: National Council of Teachers of Mathematics.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education; A concise topical survey*. New York: Springer.

Delozanne, É., Prévit, D., Grugeon, B., & Chenevotot, F. (2008). Automatic multi-criteria assessment of open-ended questions: A case study in school algebra. In B. P. Woolf, E. Aïmeur, R. Nkambou, & S. Lajoie (Eds.), *ITS 2008. LNCS* (Vol. 5091, pp. 101–110). Heidelberg: Springer.

Dick, T., & Burrill, G. (2016). Design and implementation principles for dynamic interactive mathematics technologies. In M. Niess, S. Driskell, & K. Hollebrands (Eds.), *Handbook of research on transforming mathematics teacher education in the digital age* (pp. 23–52). Hershey, PA: IGI Global Publishers.

Dick, T. P., & Hollebrands, K. F. (Eds.). (2011). *Focus in high school mathematics: Technology to support reasoning and sense making*. Reston, VA: National Council of Teachers of Mathematics.

McFarland, T. D., & Parker, R. (1990). *Expert systems in education and training*. Englewood Cliffs, NJ: Educational Testing Service Publications.

National Governors Association Center for Best Practices Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: National Governors Association Center for Best Practices Council of Chief State School Officers. Retrieved from http://www.corestandards.org/Math.

Smith, M. S., & Stein, M. K. (2011). *Five practices for orchestrating productive mathematics discussions*. Reston, VA: National Council of Teachers of Mathematics.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York, NY: Teachers College Press.

Zbiek, R., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In F. Lester (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 1169–1207). Greenwich, CT: Information Age Publishers.

Chapter 15
Design of Digital Resources for Promoting Creative Mathematical Thinking

Jana Trgalová, Mohamed El-Demerdash, Oliver Labs and Jean-François Nicaud

Abstract In this chapter, we present our experience with the design of educational digital resources aiming at promoting creative mathematical thinking, taking place in the MC Squared project, The resources are produced within an innovative socio-technological environment called "C-book technology" (C for creative) by a community gathering together mathematics teachers, computer scientists and researchers in mathematics education. In this chapter, we discuss the choices made in the design of the "Experimental geometry" c-book resource to evidence the affordances of the C-book technology for designing resources promoting creativity in mathematics.

Keywords Digital resource · Design choices · Creative mathematical thinking Geometric locus of points · Dynamic geometry · Dynamic algebra Interoperability

J. Trgalová (✉) · M. El-Demerdash
University of Lyon, Lyon, France
e-mail: jana.trgalova@univ-lyon1.fr

M. El-Demerdash
e-mail: m_eldemerdash70@yahoo.com

M. El-Demerdash
Menoufia University, Shebeen El-Kom, Egypt

O. Labs
MO-Labs, Ingelheim, Germany
e-mail: oliver@MO-Labs.com

J.-F. Nicaud (✉)
Aristod, Palaiseau, France
e-mail: jeanfrancois.nicaud@laposte.net

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_15

15.1 Introduction

Promoting creative mathematical thinking (CMT) is a central aim of the European Union by being connected to personal and social empowerment for future citizens (EC, 2006). It is also considered as a highly valued asset in industry (Noss & Hoyles, 2010) and as a prerequisite for meeting current and future economic challenges. CMT is seen as an individual and collective construction of mathematical meanings, norms and uses in novel and useful ways (Sternberg, 2003). Exploratory and expressive digital media provide users with access to and potential for engagement with creative mathematical thinking in unprecedented ways (Hoyles & Noss, 2003). Yet, new designs are needed to provide new ways of thinking and learning about mathematics and to support learners' engagement with creative mathematical thinking using dynamic digital media.

The MC Squared project, briefly presented in Sect. 15.2, looks for new methodologies that would assist designers of digital educational media to explore, identify and bring to the fore resources stimulating more creative ways of mathematical thinking. The chapter then focuses on the design of one such resource, the "Experimental geometry" c-book, highlighting, in Sect. 15.4, the design choices and the resource affordances to foster creative mathematical thinking (defined in Sect. 15.3) in its users. Concluding remarks summarizing the C-book technology affordances and bringing forward factors stimulating creativity in digital resources collaborative design are proposed in the final Sect. 15.5.

15.2 The MC Squared Project

The MC Squared (MC2) project (mc2-project.eu/) aims at designing and developing an intelligent computational environment, called C-book technology, to support stakeholders from creative industries involved in the production of media content for educational purposes to engage in collective forms of creative design of appropriate digital media. The C-book technology provides an authorable dynamic environment extending e-book technologies allowing meshing text with various dynamic widgets on the same page (Fig. 15.1), an authorable data analytics engine and a tool supporting asynchronous collaborative design of educational resources, called "c-books".

It also comprises a powerful back-end that stores the student's work at any time, so whenever she leaves her c-book and comes back again later, it looks exactly as it has been left. Moreover, the teacher may decide how much of the student's work will be logged to a database and will thus be provided with a large number of analytic tools that will assist her in her teaching. An outstanding feature of the C-book environment is that it does not only come with a large number of existing widgets in the mathematical context, but it also comes with so-called widget factories allowing the teacher to create tailored widgets. Moreover, all these diverse

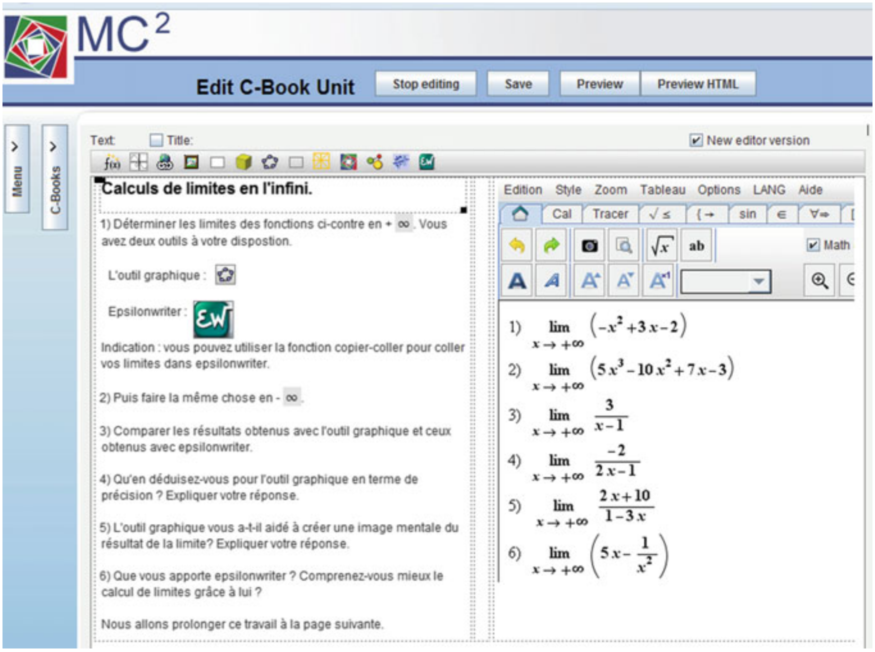


Fig. 15.1 Snapshot of a c-book resource designed with the C-book technology

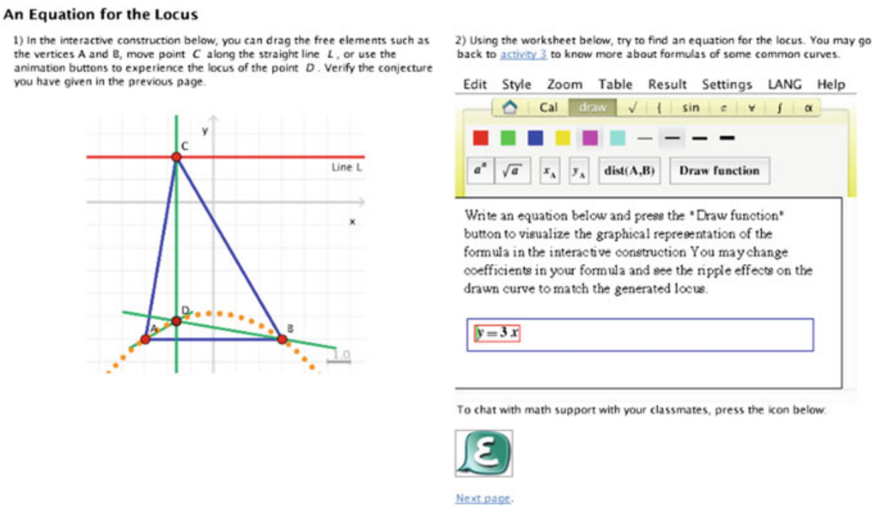


Fig. 15.2 A screenshot of a c-book page (Cinderella widget on the left, EpsilonWriter widget on the top right, and EpsilonChat widget on the bottom right)

widgets work perfectly together in a cross-widget communication. For example, EpsilonWriter is an interesting tool for manipulating formulas and equations via a unique drag and drop interface (right part of Fig. 15.2). But it neither has a built-in function grapher tool nor geometric construction capabilities. These aspects are some of the specialties of the programmable dynamic geometry system Cinderella (upper left part of Fig. 15.2). In the c-book page shown in the screenshot in Fig. 15.2, the mentioned two communication channels between three widgets have been established by the author of the page using drag and drop. When working with the c-book, a student may have produced a reasonable equation of a function with EpsilonWriter, and she can visualize it by using the 'draw' tab. The graph of the function will be shown in the Cinderella construction at the right. As the example illustrates, cross-widget communication is a quite powerful feature that opens the opportunity for the c-book author to make explicit connections between different representations of a mathematical object: a curve represented as a geometric locus, its formula with the ability to modify it dynamically, and a geometric figure combining both the construction as a locus and the visualization of the curve given by the equation. Within the C-book environment, such opportunities exist in other branches of mathematics as well, e.g., via this mechanism statistics and probability widgets may be connected to geometry, algebra, or even to a logo programming widget, to name just a few more cases.

The research reported in this chapter aims at exploring the affordances of the C-book technology for the design of digital resources enhancing creative mathematical thinking that we define in the next section.

15.3 Creative Mathematical Thinking

In this section we elaborate on the concept of creativity and especially mathematical creativity, and present the operational definition of creative mathematical thinking adopted within the MC2 project, which constitutes the theoretical frame of our study.

15.3.1 Creativity

One of the most influential definitions of creativity was proposed by Torrance (1969), seeing it broadly as the process of grasping a problem, searching for possible solutions, drawing hypotheses, testing and evaluating, and communicating the results to others. Most recent definitions can be grouped under what is called 'high' or 'Big-C' creativity and 'ordinary' or 'little-c' creativity. 'Big-C' creativity refers to the achievements and the person characterised by a non-conventional way of thinking and having a substantial contribution to the advancement of our knowledge of the world. On the other hand, 'little-c' creativity assumes that creativity is a quality or a potential all people are capable of displaying, and which can guide choices and route-finding in everyday life (Craft, 2000). These paradigms echo the distinction between *absolute* and *relative* creativity (Lev-Zamir & Leikin, 2011), the former being connected with great historical (mathematical) works and achievements, while the latter refers to discoveries by a certain person in a specific reference group. Applicable to both paradigms are the definitions by Sternberg and Lubart (2000) who see creativity as the ability to predict "non-predictable" conclusions that are useful and applicable, or by Tammadge (cited in Haylock, 1997) who defines creativity as the ability to see new relationships between previously unrelated ideas.

15.3.2 Mathematical Creativity

The problem of defining mathematical creativity is an old and still unresolved one in a sense that no single and widely accepted definition exists. Some conceptualisations of mathematical creativity focus rather on the *process* while others place their emphasis on the *product*. Along the *process* line of thought, Hadamard (1945) refers to the mathematicians' creative process using the four-stage model: preparation—incubation—illumination—verification. Liljedahl (2013) extends Hadamard's model by adding the AHA! experience phenomenon. Ervynck (1991) sees mathematical creativity as the ability to solve problems and/or to develop thinking in structures, taking into account the peculiar logico-deductive nature of the discipline. Liljedahl and Sriraman (2006) refer to it as (1) the process resulting in an unusual (novel) and/or insightful solution to a given problem, and/or (2) the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new perspective. The *product* approach to creativity focuses on the outcomes that result from creative processes. It is based on the assumption that, in order to deem a process or activity as creative, one has to discern the existence of some creative outcome. An example is the suggestion by Chamberlin and Moon (2005) to see creativity as the generation of novel, desired and useful solutions to (simulated or real) problems using mathematical modelling.

Considering creative process that leads to creative products, it is worth raising the question whether there is a considerable input of mathematical knowledge to the development of mathematical creativity. Mann (2006) argues that there is a strong relation between mathematical experience (knowledge and abilities) in a school setting and mathematical creativity. On the contrary, Sriraman (2005), among others, emphasizes that there is not necessarily a relationship between mathematical abilities and creativity, implying thus that mathematical creativity can be developed in students if properly supported. Likewise, Silver (1997) sees creativity as a disposition toward mathematical activity that can be fostered in the school population. This view suggests that teaching toward creativity might be conducive for a broad range of students, and not merely for a few gifted individuals.

15.3.3 Creative Mathematical Thinking in the MC2 Project

In the MC2 project, we have adopted a "little-c" creativity paradigm leading us to assume, in line with Silver (1997), that mathematical creativity can be developed in students through appropriate learning situations. Based on this assumption, we first agreed upon a definition reflecting our vision of creative mathematical thinking (CMT) that defines it as an intellectual activity generating new mathematical ideas or responses in a non-routine mathematical situation. Drawing on Guilford (1950) model of divergent thinking, the generation of new ideas shows the abilities of *fluency* (ability to generate quantities of ideas), *flexibility* (ability to generate different categories of ideas), *originality* (ability to generate new and unique ideas that others are not likely to generate), and *elaboration* (ability to redefine a problem to create others by changing one or more aspects). We then searched for conditions and characteristics of situations likely to foster the development of CMT in students. The following characteristics of situations or problems are deemed as appropriate to engage students in creative mathematical activity:

- Situations based on the interplay between problem-posing and problem-solving (Silver, 1997);
- 'Problematic situations' serving as the organizing centre and context for learning (Torp & Sage, 2002) or open-ended situations that are not solved easily or with a specific formula (El-Demerdash & Kortenkamp, 2009);
- Students seen as active problem solvers and learners; teachers acting as cognitive and metacognitive coaches (Torp & Sage, op.cit.);
- Social interactions in problem-solving processes (Sriraman, 2004).

Our research question is the following: what affordances of the C-book technology can be exploited in the design of resources intended to enhance CMT in students? To bring to the fore such affordances, we present, in the next section, one of the resources designed with the technology, discuss the design choices and highlight the affordances that made them possible.

15.4 The "Experimental Geometry" C-Book

Jareš and Pech (2013) claim that the notion of geometric locus of points is difficult to grasp at all school levels and that technology can be an appropriate media to facilitate its learning. The authors suggest using dynamic geometry software to "*find the searched locus and state a conjecture*" and a computer algebra system to "*identify the locus equation*".

The challenge in designing the "Experimental geometry" c-book was to exploit the C-book technology affordances to propose a comprehensive study of geometric and algebraic characterization of some loci. We decided to create activities aiming at studying loci of special points in a triangle. These loci (e.g., a locus of the orthocentre) are generated by the movement of one vertex of a triangle along a line parallel to the opposite side (see Fig. 15.2). These are classical problems that were solved even before the advent of dynamic geometry (Botsch, 1956). Elschenbroich (2001) revisits the problem of the orthocentre locus with a new media, dynamic geometry software (DGS). El-Demerdash (2010) uses this example to promote CMT in high school mathematically gifted students.

15.4.1 The C-Book Description

The c-book invites students to experiment geometric loci generated by intersection points of special lines of a triangle while one of its vertices moves along a line parallel to the opposite side (see Fig. 15.3a, b). The activity can give rise to a number of various situations, which makes it a rich situation for exploring, conjecturing and proving.

The c-book is organized in three units. The first unit proposes the main activity called "Loci of special points of a triangle". It starts by inviting the students to explore, with the dynamic geometry system (GDS) Cinderella, the locus of the orthocentre D of a triangle while its vertex C moves along a line parallel to the opposite side [AB] (Fig. 15.3a). The students are asked to explore the situation, formulate a conjecture about the locus of D and test the conjecture by visualizing the trace of D (Fig. 15.3b).

The students are then asked to find an algebraic formula of the locus, which is a parabola. The formula is to be written using the EpsilonWriter widget (www.epsilonwriter.com) and the interoperability between this widget and Cinderella allows the students to check whether the provided formula fits the locus or not.

The students are then encouraged to think of, explore, and experiment the geometric loci in other similar situations, such as the locus of the circumcentre (perpendicular bisectors intersection), the incentre (angle bisectors intersection) or the centroid (medians intersection). Other situations can be generated by considering the intersection of two different lines, for example a height and a perpendicular bisector. Twelve such situations can be generated. For each case, one page is devoted offering to the students:

1. a Cinderella widget with a triangle ABC such that the vertex C moves along a line parallel to [AB] and a collection of tools for constructing intersection point, midpoint, line, perpendicular line, angle bisector, locus, and the trace tool;
2. an EpsilonWriter widget enabling a communication with Cinderella;
3. an EpsilonChat widget enabling remote communication among students.

The other two units of the c-book present background knowledge required for completing the main unit of the c-book; students may switch to these units if they need to revise or acquire this knowledge. Unit 2 called "The concept of geometric locus" aims at introducing the concept of locus of points. It starts by leading the students to discover the fact that a circle can be characterized as a locus of points that are at the same distance from a given point. The students first experiment a "soft" locus (Healy, 2000) of a point A placed at the distance 6 cm from a given point M (Fig. 15.4a), and then they verify their conjectures by realizing a "robust" construction of the circle cantered at A with a radius 6 cm (Fig. 15.4b).

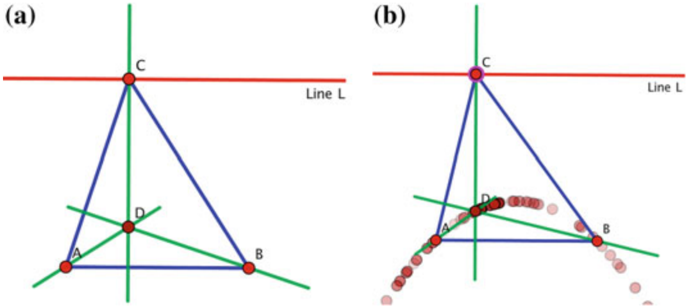


Fig. 15.3 **a** Geometric situation proposed with Cinderella. **b** Visualizing the trace of D while C moves on the line L

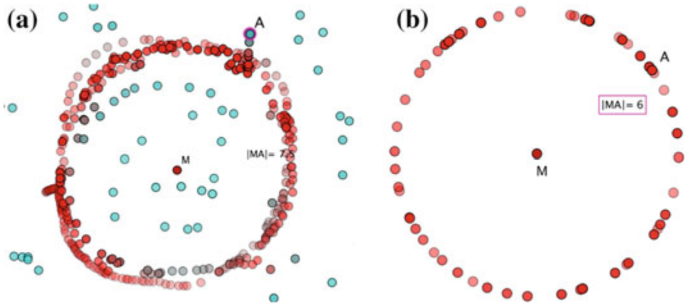


Fig. 15.4 Circle as a locus of points: **a** "soft" locus, and **b** "robust" locus

The next page is constructed in a similar way and allows the students to explore perpendicular bisector as a locus of points that are at the same distance from two given points. Finally, the last page proposes a synthesis of these two activities and provides a definition of the concept of geometric locus of points.

The third, and last, unit, "Algebraic representation of loci", proposes a guided discovery of algebraic characterization of the main curves that can be generated as loci of points: a circle, a perpendicular bisector and a parabola.

15.5 Design Choices and Rationale

Personalized non-linear path. The c-book is designed to allow students to go through it according to their knowledge and interest. They are invited to enter by the main activity in Sect. 15.1 and start exploring a locus of the orthocentre of a triangle. However, the concept of geometric locus is a prerequisite in this activity. In case this knowledge is not acquired yet, or the students need revising it, they can reach Sect. 15.2 via internal hyperlinks from various places of the main activity. Similarly, Sect. 15.3, allowing the students to learn about algebraic characterizations of some common curves, is reachable from the main activity. The students are thus given the opportunity to "read" the c-book in a non-linear personalized way, depending on their knowledge about geometric or algebraic aspects of loci of points. Therefore, the C-book technology affords creating conditions for students to be active problem solvers and learners.

Promoting creative mathematical thinking. The main activity is designed to call for students' *elaboration*: they are invited to modify the initial situation by considering various combinations of special lines in a triangle, whose intersection generates a locus to explore. They thus enter in the interplay of problem posing—problem solving. *Fluency* and *flexibility* are fostered by providing the students with a rich environment in which they can explore geometric situations and related algebraic formulas while benefitting from feedback allowing them to control their actions and to verify their conjectures (see learning analytics below). Specific feedback is implemented toward directing students to produce different and varied situations (*flexibility*) and help them to break down their mind fixation by considering yet different configurations, such as two different kinds of special lines in a triangle passing through the movable vertex (e.g., a height intersecting with an angle bisector), and then the intersection of two different lines that do not pass through the movable vertex. The c-book provides the students not only with digital tools enabling them to explore geometric and algebraic aspects of the studied loci separately, but also with a so-called "cross-widget communication" between Cinderella and EpsilonWriter, dynamic geometry and algebra environments respectively, which makes it possible to experimentally discover the algebraic formula matching the generated locus in a unique way; this feature contributing to the development of original approaches by the students (*originality*). The possibility to provide students with various cross-communicating widgets available within the same working environment enabled us to create a resource intended to support the development of CMT through promoting its components: elaboration; fluency, flexibility and even originality.

Learning analytics and feedback. One of the important aspects of this c-book enabled by the C-book technology affordances for designing appropriate feedback was to decide which of the student's activities should be logged to a database while she is studying the c-book. There are many different types of logs implemented in this c-book. These logs enable the teacher to capture the student's path in studying the c-book, e.g., whether the student starts from the c-book main activity, what pages she goes through and in which order, how far she goes through the additional two activities, whether she goes back and forth through the c-book pages and activities and when, whether she uses the provided hyperlinks to look for further information, how she uses the available hints and how many levels of hints etc.

Moreover, logs are implemented to trace the student's trails or attempts while she is using the provided tools to construct a configuration to elaborate the given problem situation: the time the student spends on each page and each activity as an indicator of motivation; the number of student's trials for each page and each activity of the c-book; the student's use of EpsilonChat as a social aspect of creativity and collaborative work with others whether in pairs or groups. Two types of feedback are provided to students, while they are studying the c-book to guarantee their smooth move from page to page and switch between the c-book activities: mathematical or educational feedback and technical feedback. Mathematical or educational feedback includes hints and comments oriented toward solving the given problem or developing creative mathematical thinking. This type of feedback is in the form of a message sent in a pop-up window, of a hyperlink or of an internal link. Examples of such feedback suggest to the student to explore the two activities complementing the first activity offering the problem to be solved when a lack of prerequisite knowledge is diagnosed (feedback intended to support the student's problem solving), or prompting her to look for another approach to solve the problem (feedback intended to support the development of fluency and/or flexibility). Technical feedback aims at helping students to master the available widgets so that technical issues do not become obstacles to the problem solving processes. This type of feedback is in the form of hints or instructions how to use the provided tools Cinderella or EpsilonWriter, or hints regarding the use of cross-communication between the two widgets.

15.6 Conclusion

The c-book presented in this chapter is the result of a collaborative work of a group of designers coming from various professional backgrounds, as the group comprises researchers in mathematics, mathematics education and computer science, as well as educational software developers. Without the synergy among those group members, a number of design choices would have remained in a hypothetical state, namely the technological advances in terms of cross-widget communication and learning analytics features. The design of the c-book has thus become a driving force in the C-book technology development, and in return, the unique C-book technology features enabled the creation of a resource in which the designers implemented affordances likely to promote creative mathematical thinking.

This experience brings to the fore the C-book technology affordances that the designers can exploit for the design of resources fostering creative mathematical thinking in students. Among these, the availability of numerous dynamic widgets that can be embedded in the same page, the cross-widget communication and the possibility to design appropriate feedback enabling the students to work autonomously are among the most outstanding features making the technology, in our view, a unique authorable environment empowering teachers to envisage more creative ways of mathematics teaching.

Acknowledgements The research leading to these results has received funding from the EU (FP7/2007-2013) under grant agreement n° 610467—project "MC Squared" (mc2-project.eu). The C-book technology is based on the Freudenthal Institute DME portal and is being developed by a consortium of nine partner organizations. This publication reflects only the authors' views and the European Union is not liable for any use that may be made of the information contained therein.

References

Botsch, O. (1956). Reinhard-Zeisberg Band 4b. In *Bewegungsgeometrie*. Frankfurt: Diesterweg-Saite.

Chamberlin, S. A., & Moon, S. M. (2005). Model-eliciting activities as a tool to develop and identify creatively gifted mathematicians. *The Journal of Secondary Gifted Education (JSGE), 17*(1), 37–47.

Craft, A. (2000). *Creativity across the primary curriculum: Framing and developing practice*. London: Routledge/Falmer.

EC. (2006). Recommendation 2006/962/EC of the European Parliament and of the Council of 18 December 2006 on key competences for lifelong learning. *Official Journal of the European Union, L 394*, 10–18. Accessed October 12, 2015 from http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=OJ:L:2006:394:0010:0018:EN:PDF.

El-Demerdash, M. (2010). *The effectiveness of an enrichment program using dynamic geometry software in developing mathematically gifted students' geometric creativity in high schools*. Doctoral dissertation, University of Education, Schwäbisch Gmünd, Germany.

El-Demerdash, M., & Kortenkamp, U. (2009). The effectiveness of an enrichment program using dynamic geometry software in developing mathematically gifted students' geometric creativity. In *Paper Presented at the ICTMT9 Conference*. http://cermat.org/sites/default/files/El-DemerdaKortenkamp-EEPUDGSDMGSGC-2009a.pdf.

Elschenbroich, H. J. (2001). Dem Höhenschnittpunk auf der Spur. In W. Herget et al. (Eds.), *Medien verbreiten Mathematik (Proceedings)* (pp. 86–91). Berlin: Verlag Franzbecker, Hildesheim.

Ervynck, G. (1991). Mathematical creativity. In A. M. Thinking & D. Tall (Eds.), *Advanced mathematical thinking* (pp. 42–53). Dordrecht, the Netherlands: Springer.

Guilford, J. P. (1950). Creativity. *American Psychologist, 5,* 444–454.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. New York: Dover Publications.

Haylock, D. (1997). Recognizing mathematical creativity in schoolchildren. *ZDM Mathematics Education, 27*(2), 68–74.

Healy, L. (2000). Identifying and explaining geometrical relationship: Interactions with robust and soft Cabri constructions. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 103–117). Hiroshima: Hiroshima University.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. J. Bishop et al. (Eds.), *Second international handbook of mathematics education* (pp. 323–340). Dordrecht: Kluwer Academic Publishers.

Jareš, J., & Pech, P. (2013). Exploring Loci of points by DGS and CAS in teaching geometry. *Electronic Journal of Mathematics and Technology, 7*(2), 143–154.

Lev-Zamir, H., & Leikin, R. (2011). Creative mathematics teaching in the eye of the beholder: Focusing on teachers' conceptions. *Research in Mathematics Education, 13*(1), 17–32.

Liljedahl, P. (2013). Illumination: An affective experience? *ZDM—Mathematics Education, 45*(2), 253–265.

Liljedahl, P., & Sriraman, B. (2006). Musing on mathematical creativity. *For the Learning of Mathematics, 26*(1), 20–23.

Mann, E. L. (2006). *Mathematical creativity and school mathematics: Indicators of mathematical creativity in middle school students*. Doctoral dissertation, University of Connecticut.

Noss, R., & Hoyles, C. (2010). Modelling to address techno-mathematical literacies. In R. Work et al. (Eds.), *Modelling students' mathematical modelling competencies*. New York: Springer.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM—Mathematics Education, 3,* 75–80.

Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator, 14*(1), 19–34.

Sriraman, B. (2005). Are giftedness and creativity synonyms in mathematics? *The Journal of Secondary Gifted Education (JSGE), 17*(1), 20–36.

Sternberg, R. J. (2003). The development of creativity as a decision-making process. In R. K. Sawyer et al. (Eds.), *Creativity and development* (pp. 91–138). New York: Oxford University Press.

Sternberg, R. J., & Lubart, T. I. (2000). The concept of creativity: Prospects and paradigms. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 93–115). Cambridge, UK: Cambridge University Press.

Torp, L., & Sage, S. (2002). *Problems as possibilities: Problem-based learning for K–16 education* (2nd ed.). Alexandria, VA: Association for Supervision and Curriculum Development.

Torrance, E. P. (1969). Prediction of adult creative achievement among high school seniors. *Gifted Child Quarterly, 13,* 223–229.

Chapter 16
Drawing in Space: Doing Mathematics with 3D Pens

Oi-Lam Ng and Nathalie Sinclair

Abstract Scholars generally agree that evolutions in technology, such as the printing press, lead to deep changes in thinking, learning and doing mathematics. In this paper, we investigate the potential changes in thinking, learning and doing that may arise from the use of 3D Drawing Pens, which enable mathematics to be done in space, thus shifting a two-millennium old tradition of drawing on 2D sand, paper and screens. We describe our rationale for undertaking this research, theoretical framework, methodology and preliminary findings about the role of 3D drawing in the learning of functions and calculus in a high school mathematics classroom.

Keywords 3D printing · Drawing in space · Inclusive materialism
Gestures · Calculus and functions

16.1 Introduction

Our interest in 3D drawing stems from a long-standing inquiry into the role of gestures and diagrams in mathematics thinking and learning. Specifically, we contend that a mathematical drawing can also be seen as a hand motion that plays both a communicative and an epistemic role. Thus, gestures "in the air" can transform into marks on a page, which can in turn transform into new gestures. According to Châtelet (2000), this interplay between gestures and diagrams is at heart of mathematical invention and crucial in helping shed light on how embodied, material actions can evolve into a formal mathematical discourse. In our context, the three-dimensional nature of gesturing "in the air" is even closer to the potential of drawing "in the air" with a 3D Drawing Pen, thereby further disturbing the boundary between gesture and diagram.

O.-L. Ng (✉)
Chinese University of Hong Kong, Sha Tin, Hong Kong
e-mail: oilamn@cuhk.edu.hk

N. Sinclair
Simon Fraser University, Burnaby, Canada

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_16

The 3D Drawing Pen is a novel handheld 3D printing device that operates in the same manner as a 3D printer. It extrudes small, flattened strings of molten thermoplastic (ABS or PLA) and forms a volume of "ink" as the material hardens immediately after extrusion from the nozzle. As the pen moves along with the hand holding it, a 3D drawing is created at once, either on a surface or in the air (Fig. 16.1a–c). 3D drawing frees the hand—as well as that which the hand makes—from the flat constraints of paper-and-pencil.

Besides the ease of creating and visualising 3D objects generated through the moving hands, 3D drawing enhances the experience of drawing 2D figures. A diagram that would have been drawn with paper-and-pencil, like a triangle, can be re-created and become physical object that can be held, moved and turned. This enables a learner, for example, to interact with 2D figures in ways that were not possible with either paper-and-pencil or the computer screen. Drawings of 3D objects can also be made, as in the cube shown in Fig. 16.1a, without having to rely on the rules of perspective drawing. As such, a 3D drawing has a dual nature: it is both a diagram and a physical manipulative. We hypothesise that this characteristic could be significant in the learning of upper secondary school mathematics topics in which geometrical, diagrammatic and manipulative components are involved, such as: the study of functions, trigonometry and calculus.

3D drawing is dynamic and unregimented; it opens up a new, "3D territory" for mathematising that was unimaginable in the era of paper-and-pencil, and even in the era of the computer screen. The present research aims to explore the potential of 3D drawing in the context of upper secondary school mathematics and to study the effect of its use on mathematical learning, both in terms of the changes in mathematics they might occasion as well as the changes in thinking. Because of its novel nature, our research questions are exploratory, as we investigate:

(1) What are the possible affordances of 3D Drawing pens in the learning of school mathematics?
(2) How might mathematical ideas develop through 3D drawing, and how are they communicated (both verbally and non-verbally)?
(3) How might the interplay between 2D (paper-and-pencil) and 3D drawing support learning?

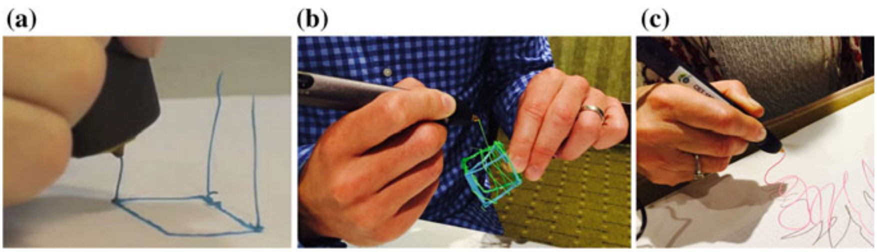


Fig. 16.1 **a** Drawing a cube with a 3D drawing pen. **b** 3D drawing "in the air." **c** A spiral

This exploratory study is informed by previous scholarship on the interaction between technological and mathematical evolution (Hegedus & Moreno-Armella, 2014; Rotman, 2008; Schaffer & Kaput, 1999) and by research on the mathematical impact of new ways of doing and representing mathematics (Cuoco, Goldenberg, & Mark, 1996; Papert, 1980). Finally, our research resonates with the growing interest in the interplay between gestures and new touchscreen technologies that enable learners to interact directly with mathematical objects, and that highlight the material genesis of gesture as hand manipulations having both epistemic and communicative functions (see Sinclair & de Freitas, 2014). Indeed, 'drawing in the air', even while holding a pen, can be seen as a form of gesturing that may function both epistemically and communicatively.

16.2 3D Drawing Tasks

For the present paper, we discuss two learning tasks in which 3D drawing can be incorporated in the learning of functions and calculus. These two tasks were chosen because of our research context, which was a secondary calculus course. One task was used at the beginning of the course related to differential calculus, and the other one near the end of the course related to integral calculus. Our first example illustrates how the three-dimensional nature of 3D drawings may be exploited by utilising them as physical manipulatives in the learning of instantaneous rate of change. Figure 16.2a shows the graph of $y = x^2$ lightly attached on grid paper as it was initially drawn in 3D. It also shows how one can draw a "line" in 3D, place two fingers on it, and push it towards the graph of $y = x^2$ until it "just touches" the parabola at one point, the point of tangency. Note that unlike with physical manipulatives, these graphs were created by the students before being used as manipulatives. We point out that this tangent line can then be moved physically and dynamically along the graph by asserting a force on one of the fingers, and we see this activity as being quite different from, for example, what can be done in DGEs because one can *feel* a force exerted at the point of tangency while the tangent line is physically moved along with the two fingers touching it. Meanwhile, the very movement of the fingers is also the gesture that determines the slope of tangent at different points on the function—which highlights the intricate interaction among gesture, diagram and mathematical thinking that we sought to explore in the study.

Our second example extends our previous work exploring middle school students' (ages 12–14) learning of "area without numbers" (Ng & Sinclair, 2015), where we challenged a measurement- and formula-driven approach of learning area. In this example, the "volume of solids of revolution" (Fig. 16.2b) and the "volume of solids with similar cross sections" (Fig. 16.2c) can be visualised through drawing the mesh of the respective solids in 3D. The actual drawing process is worth describing because of the way different aspects of the concept of volume are highlighted. One might begin, for example, by drawing a parabola, either tracing it from a piece of paper, or making a freehand one. Then one places the 3D Drawing

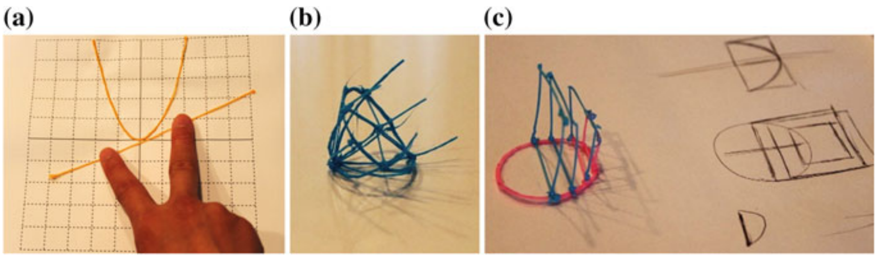


Fig. 16.2 **a** A 3D drawing as both a diagram and a manipulative. **b** Skeleton of a solid of revolution drawn by a 3D drawing pen, and **c** Skeleton of a solid with known cross sections contrasted by a 2D diagram of it

Pen at the tip of one arm of the parabola and rotates the parabola gently while holding the 3D Drawing Pen in place, which results in a curved line in space. When the parabola has been turned by one full rotation, that curved line will have formed a circle. Repeating this drawing process at different points of the parabola would result in a mesh of a paraboloid where its circular cross sections are highlighted, hence supporting students' decomposition of the solid from 3D to its 2D parts. As such, the "3D" nature of the drawing refers to the 'surface' on which the drawing occurs, rather than to the nature of the shape produced.

16.3 Theoretical Considerations

In considering the nature of mathematical concepts and the role of technology in doing mathematics, we adopt de Freitas and Sinclair's (2014) inclusive materialist perspective. This perspective offers a re-conceptualisation of mathematics thinking and learning as the intra-action between mathematical knowledge, teacher, students and material surrounding. This contrasts with the assumptions of other mathematics education theories, which typically conceptualise learners, tools and mathematics as three distinct ontological 'actors' and merely interact one with the other. In these theories, tools are seen as mediating student learning, but as leaving the mathematics more or less independent of these tools. The notion of intra-action highlights the essential intertwinement of humans, concepts and tools.

A significant influence of inclusive materialism is the work of the philosopher Châtelet's (2000), who also advanced a materialist conception of mathematics, seeking to move beyond the dichotomies of concrete/abstract and body/mind that pervade most mathematics education theories. In his study of inventive moments in the history of mathematics, Châtelet highlights the role of diagramming and gesturing in mathematical thinking, showing how the mobility of the human hand (in gestures, which are then "captured" by diagrams) gives rise to new concepts. But these diagrams are not to be seen merely as representations of some static or abstract mathematics. Indeed, the notion of representation reifies the concrete/abstract dichotomy that Châtelet aims to avoid; the diagramming act aims to create a mathematical object, not simply represent an existing, static abstraction. They are the mathematics that gets figured, de-figured and re-figured through repeated drawing and gesturing. This chosen theoretical framework encourages us to focus on the novel gesture-diagram interactions that might arise out of 3D drawing. The ontological assumption made by inclusive materialism enables us to focus less on how the 3D pens might mediate particular, established mathematical ideas and instead explore what new concepts emerge out of the 3D drawing activities.

16.4 Methodology

While we anticipate that 3D drawing could potentially impact a broader range of the K–12 curricula, we have chosen to focus on its role in the learning of upper secondary mathematical topics for two reasons. Firstly, we intend to contest a common perception that the higher level of secondary mathematics, the more "abstract" and "intangible" it becomes. Secondly, we are motivated by the work of researchers such as Gerofsky (2009), who has shown the pedagogical potential of having students gesture functions in the air. If students could also be creating functions as they draw "in the air" they could be productively combining the gestural act with the tracing one, and producing an object that can then be manipulated and shared. 3D drawing may thus be a useful way to introduce curve sketching in calculus and to explore increasing/decreasing functions and concavity, by integrating the gestural forms of thinking as well as the sense of touch into one's graphing experience.

We undertook a classroom-based research, using the two tasks described above, in order to increase the likelihood that the results of research are applicable while also shedding light on how and why certain situations work (Stylianides & Stylianides, 2013). In keeping with this line of research, the teacher-researcher (first author) designed and delivered lessons in a secondary school calculus class with a class set of 3D Drawing Pens. During these 75-min lessons, the teacher invited students to produce 3D drawings as a means to diagram, visualise and explore the target calculus ideas in groups. Then, the teacher led a whole class discussion about the activities and posed problems where students were to provide solutions represented by 3D drawing and on paper. An explicit emphasis was put on the importance of representation with both media. The participants are 25 grade 12 (age 17–18) students enrolled in a culturally diverse high school in Canada with no prior experience with calculus before the study.

Data was collected in the form of videos captured by the students' iPads as well as videos captured by both researchers while observing and interacting with students as they engaged in the 3D drawing tasks during lessons. For example, during the final task of the "derivative functions" lesson, students were asked to draw a cubic function free-hand in 3D and then to graph its derivative function with the aid of the "tangent line" also drawn in 3D, while one student in each group filmed the graphing process. In response to the research questions, we examined the way students communicated about calculus linguistically and with their hands when engaging in the designed learning tasks within a 3D drawing environment. We focused on the specific words, actions and interactions with drawings that seemed to be situated in the task as exploited by the use of 3D Drawing Pens. In doing so, we illustrate how 3D drawings may complement the learning of functions and calculus. In what follows, we present findings upon analyzing the video data collected from both lessons as well as offer some speculations about the affordances of a 3D drawing environment.

16.5 Findings

From the videos gathered during the classroom interventions, we identified four aspects of 3D drawing that were significant to the students' learning during the lessons and that were specifically afforded by the medium of 3D drawings. We discussed these findings in the following sub-sections.

16.5.1 3D Drawing Facilitates the Thinking of Functions as Processes and Objects

At the beginning of the "derivative functions" lesson, the teacher showed a 3D drawing of $y = x^2$ lightly attached on the grid, demonstrated the variance of the tangent slope at different points on the graph by maneuvering the tangent line from one side to another with her fingers, and drew the graph of the corresponding derivative function in between the movement of the tangent line. The students repeated this same activity and were given two follow-up questions immediately after: they were asked to draw the graphs of $y = x^2 - 2$ and then $y = -x^2$ with 3D Drawing Pens along with the graphs of their derivative functions with pencils. We noticed that even though different grids were provided for students to draw the three graphs, all student groups eventually used their previous 3D drawings of $y = x^2$ or $y = x^2 - 2$ as the graph of $y = -x^2$ by detaching the graph of their previous drawings and placing them on the new grid with the appropriate translation and/or reflection. The students did so without being told and explained their strategies in terms of translation ("it is translated 2 units down"), reflections ("it is reflected along the x-axis") and stretches of functions ("the shape of the graph does not change"). These statements indicate that that the students were thinking about functions as objects. These comments were made after they had drawn the graphs temporally and dynamically with the 3D Drawing Pens, which suggests that they thought of the graphs as a process. Hence, 3D drawing gives rise to the creation of functions as processes and objects. We underscore here the ability to pick up a 3D drawing (the graph of a function) and manipulate (translate and reflect) it as an affordance of 3D drawings additional to the ability to touch and feel the function—and not merely a representation of it—physically.

In addition, the students' linguistic communication accompanying their manipulations of the tangent line enabled the emergence of the concept of function as a singular object. Our data shows that the nouns "tangent" and "tangent line" were always used in a singular form. For example, when asked by the researcher to explain his graphing process, one student explained that, "I am using the tangent line to find the slope at each point." We drew attention to the use of nouns in "tangent line" and "slope" and "point" as a singular rather than plural, which were markedly different from what we found in the wording of the students' calculus textbook: "By measuring slopes at points on the sine curve, we get strong visual evidence that the derivative of the sine function is the cosine function" (Stewart, 2008, p. 172). Based on our previous work (Ng, 2016), the use of the singular in all of the utterances found in our data about the "tangent line" provides an indication that the students saw the tangent line as one object moving along the graph *continuously*, whereas the textbook conveys the change of "slopes" in a *discrete* sense.

We observe that the students physically interacted with the functions as "mathematical objects" that were not possible with the paper-and-pencil medium. During the integral calculus part of the course, students were invited to draw the mesh of "solids of revolution" before they learned to solve for their volume using definite integrals. The students employed various 3D drawing strategies which are worth describing because of how the students made use of their own 3D drawings in the process. For example, when asked by the classroom teacher to visualize the solid formed by revolving a curve about the *x*-axis, they invented a strategy that made use of the "*x*-axis" as a manipulative and the action of spinning the axis. Having drawn a curve and the coordinate axes with a 3D Drawing Pen, they picked up the drawing from the piece of paper, held the two ends of the *x*-axis and began rotating it physically and rapidly (Fig. 16.3a–c).

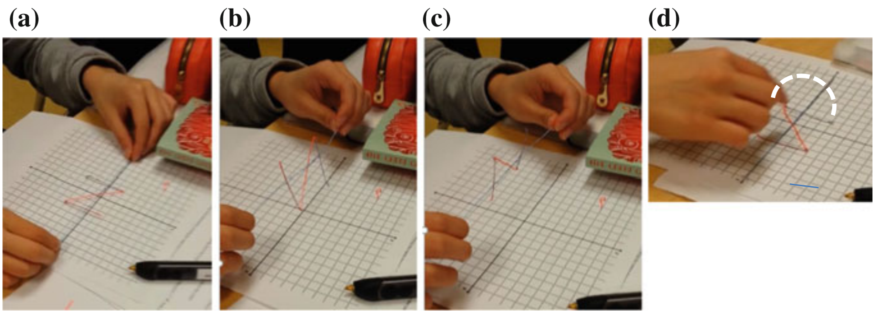


Fig. 16.3 **a**–**c** Picking up the graph drawn and rotating the axis physically to visualize the solid formed. **d** Gesturing a semi-circle above the diagram

16.5.2 3D Drawing Slows Down the Drawing of Functions

While drawing graphs with their 3D Drawing Pens, the students took much time to finish their drawings. From the video data collected, the students spent an average of 14 s to draw the graph of a cubic function free-hand on a regular-sized paper during the final task. Although we did not obtain comparison data about students' drawing of the same cubic function with paper-and-pencil free-hand, we suggest that the time that they took to draw in 3D seemed much longer than what they would take to draw functions with paper-and-pencil. Hence, we point out that 3D drawing slows down the drawing of graphs. Furthermore, the two types of speeds of extruding "ink" in 3D offered by the 3D Drawing Pens, although not that different, seemed to take on a decisive factor of how fast (or slow) and steady one draws in 3D. In contrast, ink from a pen or lead from a pencil is not extruded, and we speculate this difference as having implications on the students' attention to the drawing process as well as their sense of the temporal nature of graphing.

16.5.3 3D Drawing Supports Continuous Constructions

Another interesting observation about drawing graphs in 3D was the continuity of the drawing process. Our data shows that all student groups drew the functions from left to right, continuously without stopping and picking up the 3D Drawing Pens (Fig. 16.4a). By comparison, when students drew graphs (either the function or its derivative) with the paper-and-pencil medium, they drew them in a discontinuous manner, in the sense that they drew different parts of the graph with different domains, sometimes from left to right, and other times from right to left, as if they were piece-wise functions (Fig. 16.4b). Some students also drew over their sketches repeatedly on paper by stroking with the pencils to refine their sketches, which was not observed when they explored the derivative tasks with the 3D Drawing Pens. Rather than refining a 3D drawing, they would discard the drawing as a whole and draw a new one when they were dissatisfied with their 3D drawings.

Like the speed of their drawing process, we had not anticipated that the students would draw graphs in 3D in these significantly different ways. Perhaps, extruding "ink" in 3D is more closely tied to temporality than drawing with a pen or pencil because of the way 3D material is "coming out" or being created over time while drawing. We think that there is room for further exploration of the effect of 3D drawing on temporal and motion-based thinking with respect to both of these findings.

(a) (b)

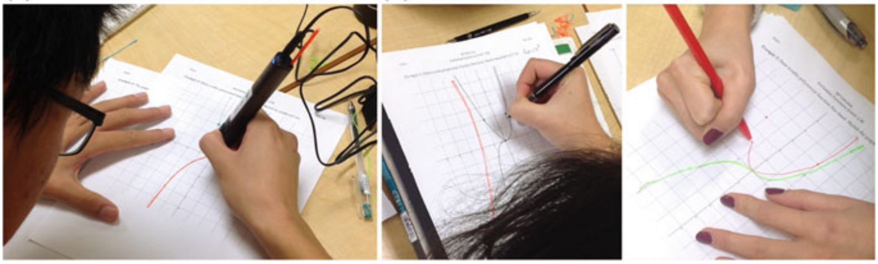


Fig. 16.4 **a** Drawing graphs continuously with a 3D pen and **b** discontinuously with a pen

16.5.4 3D Drawing Offers New Gestural Forms of Thinking

Through the students' hand movements accompanying their interactions with the 3D drawings, we recognise that 3D drawings enabled students to physically feel the mathematical idea of tangent to a curve during the "derivative functions" lesson. This was evident in our observations that students used their index fingers to push their tangent lines against the graph while estimating the slope of tangent during the final task. Physically, when a straight line is pushed onto a curve, the slope of the line is re-oriented to be the same as the one locally straight to the curve at the point of contact. During the final task, one student took the extra effort to adjust the tangent line and make it lean parallel to the curve with her index finger (Fig. 16.5a) after it was initially pushed against the curve by another student with her finger. Similarly, another student used his left and right index fingers to push the tangent line and the graph towards each other (Fig. 16.5b) to re-orient the tangent line more precisely. Besides achieving a more accurate reading of the slope of tangent with their own fingers, these hand movements gave rise to new gestural forms of thinking about tangent to a curve that is afforded by the 3D drawings. Given our theoretical consideration, these new gestures are the very movement that give rise to new mathematical meanings that are both physical and abstract.

(a) (b) (c)

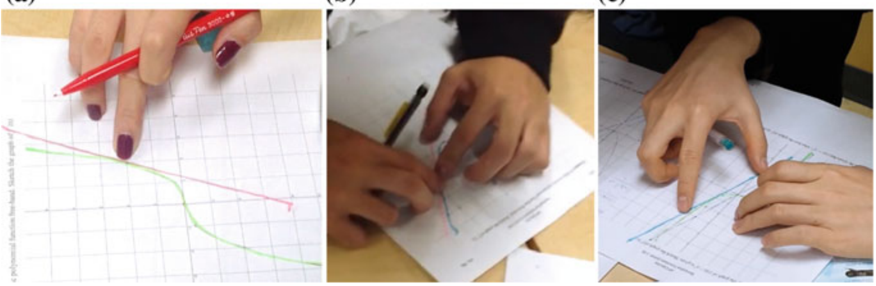


Fig. 16.5 **a** Physically feeling tangent to a curve with one or **b** two fingers. **c** Gesture-diagram interaction while exploring slope of tangent at different points on the function

We also noticed that the students did use their two fingers to act as “anchors” of the tangent line, which we anticipated would constitute a gesture-diagram interaction since the fingers that moved along with tangent line at different points were also the fingers that determine the slope of tangent (Fig. 16.5c). However, we did not observe students talking about the slope of tangent while making this gesture or made reference about this gesture when exploring slope of tangent. Therefore, we did not obtain evidence to claim that the students took advantage of the gesture-diagram interaction in the learning of derivative functions. We aim to study further the impact of gesture-diagram interaction in mathematical thinking and learning.

16.6 Discussion

In response to the first research question, our findings suggest two main features of 3D drawings that distinguish them from drawing on 2D surfaces or pre-made manipulatives. The first is related to its manipulative dimension. The three-dimensional nature of 3D drawings facilitated the emergence of the concept of function at an object level, offered a form of physical interactions with them, and generated new gestural forms of thinking. Even if the task is 2D in nature like the first one, in that students draw parabolas and lines, the third dimension does come into play with the moving and touching of the tangent lines which offered new gesture-diagram interaction and a physical substantiation of tangent to a curve. Secondly, 3D drawings were much more than pre-made physical manipulatives because they are created over time in the process of drawing with the 3D Drawing Pens. In this way, the 3D drawings captured the diagramming process and facilitated thinking about functions as process. Therefore, the act of 3D drawing is a gestural act that give rise to new mathematical concepts. We are intrigued by the future possibilities of incorporating both of these features in more 3D drawing tasks, particularly the “drawing in space” kind, which seem to offer more significantly novel opportunities in terms of the dimensional decomposition that Duval (2005) has argued is central to the learning of geometry. The 3D Drawing pens provide an unusual form of deconstruction in that the shapes that inhabit a 3D space are being created using essentially 2D objects (lines), even though those 2D lines are created in 3D! With respect to the second research question, we saw that a 3D drawing environment influenced the way “change” is conveyed. Based on the students’ linguistic communication, we suggest that they thought of the tangent line as one object (a singular) moving along the graph continuously rather than discretely. In terms of hand movements, we noticed that 3D drawings slowed down the students’ drawing process and made them draw in a continuous manner. We hypothesise that since “ink” in 3D was extruded at a steady speed, this may facilitate temporal and motion-based mathematical thinking. On the other hand, when creating solids of revolution, the students’ hand movements were much more complex; they spun their drawings and gestured over it (Fig. 16.3d) in order to generate the solids *virtually*. In other words, their hands were very much entangled with the solids formed. We encourage similar studies to examine what we call “mathematising in 3D”, where doing mathematics is much more than writing symbols or drawing diagrams on paper but is integrated with the 3D space, the moving hands and the physical experience of interacting with 3D drawings.

Finally, with regards to the third research question, the task we provided for the lesson on derivative functions did invite the students to work with two media: they needed to draw graphs with the 3D Drawing Pens and then the corresponding derivative functions with paper-and-pencil. The students worked with both media back and forth throughout the lesson, during which they drew graphs differently with each medium. We also found evidence of students' gesture-diagram interactions during both tasks and interactions between 2D and 3D drawings in the "volume by revolution" and "volume of solids with similar cross sections" lessons. Typically, in Canadian secondary school classrooms, teaching these topics required students to visualize the solids formed by drawing representations of the 3D solids on paper. In contrast, a 3D drawing environment offered the possibility of drawing a solid in 3D.

Based on the two core features of 3D drawing, we can speculate that 3D drawing may impact teaching and learning mathematical topics in the current curricula in two ways. First, we recognise that our subjects found it quite powerful even to produce 3D drawings that were flat. They were attracted to the tangibility of their creations—the ability to pick up and interact with the drawings physically even if they were 2D in nature. Thus, 3D drawing may be useful to support learning of 2D shapes or functions in ways that were not possible with either paper-and-pencil or the computer screen. For example, it is suggested that the physical movement and tactile interactions of drawing, touching and turning a 2D figure may be helpful for young children to learn about shapes, in particular, overcome difficulties related to non-prototypical shapes.

In the secondary mathematics level, the learning of trigonometry and plane geometry may be exploited with 3D Drawing Pens through drawing, manipulating and comparing angles and line segments without introducing numerical measurements. These geometrical topics are traditionally introduced with measurements with the media of paper-and-pencil or computer screen because it would be otherwise difficult to compare lengths or angles. However, the ability to draw, pick up and manipulate 3D drawings make it possible to compare geometrical objects by superimposing one onto another or by the sense of touch. While these drawings are illustrations of particular examples, they also maintain a sense of generality since they did not rely on numerical measurements.

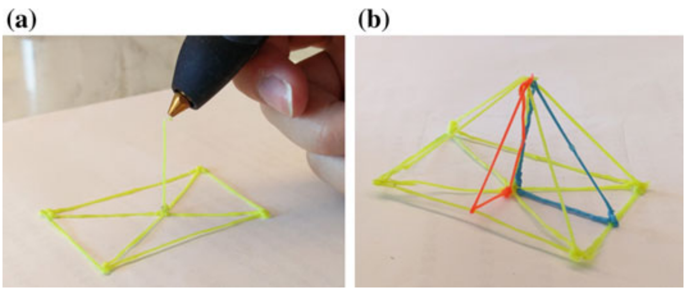


Fig. 16.6 a–b Drawing a rectangular pyramid with a 3D drawing pen

Secondly, when drawing 3D figures, one does not need to rely on the rules of perspective drawing. For this reason, diagramming prisms, cones, pyramids and polyhedrons with a 3D pen can be very significant for one's learning about the geometrical properties of 3D solids. As opposed to drawing flat diagrams, 3D drawing requires reconstructing the 3D solids in space, through which particular features, such as perpendicularity, parallelism, height, and relationships between faces and vertices can be observed. Teachers can exploit the diagrammatic and manipulative dimensions by guiding students to draw, touch and explore certain planes, cross sections and line segments of 3D figures, which can be underscored by drawing them in different colours. Figure 16.6 shows the process of constructing a regular pyramid with a 3D pen. The drawing process makes it possible to observe the relationship between the height and the diagonals of the base of the pyramid (Fig. 16.6a) as well as the three different right triangular plane that are perpendicular to the rectangular base of a pyramid (Fig. 16.6b). Drawing these with a 3D Drawing Pen may help support students who may otherwise have difficulties visualising 3D figures drawn on paper or computer screen.

16.7 Conclusion

In summary, our first encounter with the 3D Drawing Pens provided important insights on the affordances of a 3D drawing environment for mathematics learning. We also illustrated how 3D drawings may enhance the learning of 2D shape recognition and transformation in early grades; shapes and space in the middle school level; as well as functions and calculus in the secondary level. In terms of future research directions, we are interested in what sorts of learning and thinking possibilities arise when there is a material record of one's gestural history. It is this detail that most intrigues us around the pedagogical possibilities of doing mathematics with the 3D Drawing Pens.

References

Châtelet, G. (2000). *Figuring space: Philosophy, mathematics, and physics* (R. Shore & M. Zagha, Trans.). Dordrecht: Kluwer.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15,* 375–402.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge, UK: Cambridge University Press.

Duval, R. (2005). Les conditions cognitives de l'apparentissage de la géométrie: Développement de la visualisation, differenciation des raisonnement et coordination de leurs fonctionnements. *Annales de didactique et sciences cognitives, 10,* 5–53.

Gerofsky, S. (2009). 'Being the graph': Using haptic and kinesthetic interfaces to engage students learning about functions. In C. Bardini & P. Fortin (Eds.), *Proceedings of ICTMT 9*. France: Université de Metz, Metz.

Healy, L., & Sinclair, N. (2007). If this is our mathematics, what are our stories? *International Journal of Computers for Mathematical Learning, 12*(1), 3–21.

Hegedus, S., & Moreno-Armella, L. (2014). Information and communication technology (ICT) affordances in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education SE—78* (pp. 295–299). Netherlands: Springer.

Ng, O. (2016). *Language, gestures, and touchscreen-dragging in school calculus: Bilinguals' linguistic and non-linguistic communication (Unpublished doctoral dissertation)*. Canada: Simon Fraser University.

Ng, O., & Sinclair, N. (2015). 'Area without numbers': Using touchscreen dynamic geometry to reason about shape. *The Canadian Journal of Science, Mathematics, and Technology Education, 15*(1), 84–101.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books.

Rotman, B. (2008). *Becoming beside ourselves: The alphabet, ghosts, and distributed human beings*. Durham: Duke University Press.

Shaffer, D. W., & Kaput, J. (1999). Mathematics and virtual culture: A cognitive evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics, 37* (2), 97–119.

Sinclair, N., & de Freitas, E. (2014). The haptic nature of gesture: Rethinking gesture with new multitouch digital technologies. *Gesture, 14*(3).

Stewart, J. (2008). *Calculus: Early transcendental* (6th ed.). Belmont, CA: Brooks Cole.

Stylianides, A. J., & Stylianides, G. J. (2013). Seeking research-grounded solutions to problems of practice: Classroom-based interventions in mathematics education. *ZDM—The International Journal on Mathematics Education, 45*(3), 333–342.

Chapter 17
Diagrams and Tool Use: Making a Circle with WiiGraph

Giulia Ferrari and Francesca Ferrara

Abstract Using Châtelet's perspective on the gesture/diagram interplay, we aim to contribute to the current discussion on the role of technology showing influences of tool use on some grade 9 students' diagrammatic activity. The students have been engaged in graphing motion experiences in the context of a teaching experiment with WiiGraph, a software application modelling the movement of two controllers of the game console Nintendo Wii. In particular, we focus on the activity of making a circle and the circle emerging as a gradient of speeds and directions out of students' movements. Telling the story from the point of view of the diagram, we focus on the new dimensions and movements that arise from, within and about the working surface, as dynamic sources and sites of mathematical thinking.

Keywords Circle · Diagram · Gesture · Movement · Tool use · WiiGraph

17.1 Introduction

Drawing on lines of flight offered by the recent book "Mathematics and the Body" of de Freitas and Sinclair (2014), we propose to use the work of the philosopher of mathematics Gilles Châtelet to analyse a technology-related task and the diagrams used by some secondary school students to face the task. Châtelet allows us to explore the huge potential of the diagram in mathematics education. He ascribes several functions to diagrams and sees diagrams as technologies intertwined with other technologies of writing. Interestingly, Châtelet discusses the diagram of an interval pointing out the virtual dimensions of its length, which is traditionally seen as originating from a point stuck in the plane. In so doing, Châtelet rethinks the notion of length imagining two extremities spreading out at the same time and the creation of "the diagram of the opening out with its indifference centre" (Châtelet, 1993/2000, p. 151). He points out the way in which tool use is tangled up in this mobilization of the interval. Pincers or compasses might:

G. Ferrari (✉) · F. Ferrara
Dipartimento di Matematica, Università di Torino, Turin, Italy
e-mail: giulia.ferrari@unito.it

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_17

> give a *point of view* to the hand, by associating an angle in which the interval is 'seen' with the grasp. The angle refers to a degree of embracing and allows the lateral to resume its rights. It invites a second dimension, another world where the geometer may virtually propel himself: the hand inhabits the angle and the angle incites the end to open out into two sides. (ibid., p. 151)

Another aspect, which de Freitas and Sinclair discuss in their work, is the pedagogical force of diagrams, a force that could be integrated in the mathematics classroom. In this paper, we propose to join elements of this view with tool use in mathematical thinking.

In particular, our interests are on the way that the technology in use in the mathematical activity can mobilize, reconfigure and expand the diagrams created by the students, unfolding new meanings and mathematical dimensions. Focussing on this aspect, we look at diagrams as "kinematic capturing devices", using the words of de Freitas and Sinclair (2014), instead of looking at them as external representations of mathematical concepts. We also look for diagrammatic and gestural signs, as well as for the gesture/diagram interplay that not simply recalls, but literally brings forth the activity with the tool.

In this chapter, attention is specifically drawn to the activity of making a circle in the context of a classroom-based intervention devoted to studying function through the use of a technology that graphs motion. The activity engages secondary school students in a discussion in which they are asked to relate the diagram of a circle to pairs of one-dimensional motions that combine to make the circle, which the students encountered using the technology. We will see how the initial diagram is mobilized and expanded throughout the discussion and how this movement engenders new and unexpected kinds of mathematical experiences for learners. This helps us better study the relationships between matter and meaning that are implicated in contingent tool use.

17.2 Theoretical Committments: Diagrams and Gestures

The French philosopher of mathematics Châtelet takes episodes in the history of mathematics and physics to explore the interplay gesture/diagram under a perspective that considers the physical in the mathematics, rather than seeing the mathematical and the physical as separated, like in "the Aristotelian division between movable matter and immovable mathematics" (de Freitas & Sinclair, 2014, pp. 63–64). In so doing, he troubles the ontology of the relationship between mathematics and the physical world, as well as the classical vision of what it means to do mathematics. What is peculiar about this relationship is how the concept partakes in the virtual dimensions of the material world.

The virtual is at play when we reconceive concepts less as static abstract entities and more in terms of their power of affecting and being affected, their animating force, their potentiality and mobility, their capacity of giving rise to new configurations, alterations and mutations. So, for example, the circle can be thought of in terms of the virtual motions that it generates instead of being thought of as a static geometrical object. In “L’enchantement du virtuel”, Châtelet (1987) takes the example of the circle when he discusses how points might be considered not as given in the plane, but as being somehow algebraic powers. For him, an “abstract” point of the circle has absolutely no interest. Instead, interesting things about the circle will be really emerging when one will “build functions on the circle or, for example, put some sine [curve]”, when one will “wrap a straight line in a circle with the sine, a constantly dynamic perspective in mathematics”. Researchers have discussed the affordances of technologies that mobilize mathematics implicating gestural and diagrammatic engagement. Nemirovsky and Ferrara (2009), for example, illustrate the case of one girl’s gestures tracing the motion of two laser lights in order to discover a defined triangle shape as the trajectory of the combined motion. Similar work on a circle shape with a drawing machine is instead investigated in Noble, DiMattia, Nemirovsky, and Barros (2006). Others, e.g. Hegedus and Tall (2016), Sinclair, Chorney, and Rodney (2016), focus on how multitouch devices might offer new mathematical experiences in environments mainly designed for the study of geometry and early arithmetic.

Following Châtelet, the relationship between gestures and diagrams can be rethought, through their coupling and looking at gestures as “capturing devices” and diagrams as “physico-mathematical” entities. De Freitas and Sinclair (2014) notice the relevance of his vision with respect to present literature: “In contrast to current work on gestures, on the one hand, and diagrams, on the other hand, Châtelet insists that separating one from the other is both awkward and possibly misleading” (p. 64). For him:

> A diagram can transfix a gesture, bring it to rest, long before it curls up into a sign, which is why modern geometers and cosmologists like diagrams with their peremptory power of evocation. They capture gestures mid-flight; for those capable of attention, they are the moments where being is glimpsed smiling. (1993/2000, p. 10)

Châtelet argues that the diagram is by its very nature never complete, and the gesture is never just the enactment of an intention. Instead, the two participate in each other’s provisional ontology:

> Like the metaphor, they [diagrams] leap out in order to create spaces and reduce gaps: they blossom with dotted lines in order to engulf images that were previously figured in thick lines. But unlike the metaphor, the diagram is never exhausted: if it immobilizes a gesture in order to set down an operation, it does so by sketching a gesture that then cuts out another. (*ibid.*, p.10)

Châtelet insists that gestures and diagrams are both pivotal sources of mathematical meaning and they mutually presuppose each other and share similar mobility and potentiality. De Freitas and Sinclair (2014) underline that “For Châtelet, diagrams ‘lock’ or ‘capture’ gestures. ‘Capturing’ is contrasted to ‘representing’ in that the latter is bound to a regime of signification that curtails our thinking about diagramming and gesturing *as events*” (p. 64). Briefly speaking, if the gestural gives rise to the very possibility of diagramming, so the diagrammatic gives rise to new possibilities for gesturing. Instead of being seen as external representations of existing knowledge, the diagrams are “kinematic capturing devices, mechanisms for direct sampling that cut up space and allude to new dimensions and new structures" (p. 65). The coupling and interplay of diagramming and gesturing are of interest for their being "embodied acts that constitute new relationships between the person doing the mathematics and the material world" (de Freitas & Sinclair, 2012, p. 134).

Rotman (2008) considers as an essential aspect of diagrams in Châtelet's understanding their "after-life", the future alterations that are latent in them and that never get "exhausted", suddenly bringing into life new unexpected gestures and movements. Thus, the diagram is a material surface, which is not an inert static part of the mathematical event but actively and dynamically partakes in mathematics thinking and learning, with its gaps and flaws. Mathematics emerges out of this actual and virtual mobility of the gestural and the diagrammatic, prompted by the continual movement and becoming that shape the activity. Elsewhere (Ferrara & Ferrari, 2017), we have taken a similar perspective to discuss the assembling of learners with mathematical meanings and their material surrounding in pattern generalisation activity that did not involve technological tools. Once gestures and bodily activity are entangled with tool use, the interplay gesture/diagram, and therefore the mathematical activity, is enriched and reconfigured by those actions that have incorporated the tool: new dimensions can be unfolded from the static appearance of the diagram, and this is sustained by new unexpected ways of capturing mathematical relationships.

To investigate the complex processes of becoming that new technologies can create in teaching and learning mathematics, the main idea of this study is to focus on the ways in which past experiences with tool can re-inject movement and time into the mathematics that the students are exploring, rather than focussing on specific tool use. Drawing on an example of students' diagrammatic activity through Châtelet's vision, we look at the mathematical activity telling the story from the point of view of the diagram, so that we can recognise the partaking of the technological devices in the essence of the diagrams as "kinematic capturing devices". We embrace a vision similar to that offered by Roth (2016), who proposes to look, rather than at the "finished and finalized diagrams students make", at "the actual flows of the movements that got their makers to those end points" (p. 4). We are interested in looking at how diagrams are in becoming as lines or traces of students' mathematical thinking, and at the gestures and words around and about these diagrams. In particular, in this study we want to shed light on the dynamic process of "making a circle" as it emerges out from an activity of graphing motion in two dimensions.

17.3 The Study

17.3.1 Participants and Methodology

The data presented here come from a medium-term teaching experiment that involved a class of thirty grade 9 students and their regular mathematics teacher in graphing motion experiences to introduce the concept of function and the mathematics of change. The teaching experiment held in Northern Italy and consisted of 9 two-hour sessions, during the 4-months period from December 2014 to March 2015. The tasks were all designed by three people, the two authors and the teacher. During the sessions, the second author led the activity, while the first author and the teacher were active observers. The students were engaged in collective discussions, group work and individual written tasks, and were filmed by the first author. The students made use of a specific software application for modelling motion called WiiGraph, which works by means of two controllers of the game console Nintendo Wii. The meetings took place in a laboratory room, which was used for the embodied interactions with the technological tools.

In the specific case of this chapter, we will mainly devote attention to the contingent interventions of five students in the activity: Barbara and Lucrezia as volunteers who move the controllers to make a circle; Emanuele, Federico and Tiziana as participants in the following discussion about the mathematics of the circle and its relationship with the movement performed by the two girls. As participant observers involved in the teaching and learning setting, we are materially implicated in the research process and entangled with spaces, resources and technologies, as well as with learners, method, theory and the data—which is our research matter and consists of all the written productions and the video. This speaks directly to the ethical dimension of the research, as discussed in Ferrara and Ferrari (2017) (see also Haraway (2008), for extensive discussion on the space of ethics).

17.3.2 WiiGraph: Line and Versus

WiiGraph has been built for didactic purposes by Ricardo Nemirovsky and colleagues at the Centre of Research in Mathematics and Science Education of San Diego State University. It allows for exploring and creating many types of motion graphs, by capturing over time the positions of two Wiimotes (the controllers of the Wii) with respect to a sensor bar. Here we focus on two specific options: *Line* and *Versus*. *Line* furnishes in real time two space-time graphs that depicts the distance of the controllers over time, being the Wiimotes moved by two users in the interaction space in front of the sensor (Fig. 17.1a). Time and spatial ranges can be set and modified for the Cartesian axes. Labelled with a and b the two distances, the lines of $a(t)$ and $b(t)$ appear coloured in different ways (according to the colours associated to the controllers by default). In our experiment, we projected the graphs on an IWB (Interactive White Board) so that they could be shared and watched by all the students. *Versus* plots at each time t an ordered pair of the two distances: $(b(t), a(t))$, leaving time implicit (on the Cartesian plane ba). One of the most interesting challenges for a *Versus* graph involves the creation of plane shapes, like rectangles and circles (Fig. 17.1b). The circle is the shape of interest in the case of the collective discussion, which is the focus of our study. From the mathematical point of view, the case of the circle is intriguing with respect to $a(t)$ and $b(t)$ being the sinusoidal functions that describe a circular motion.

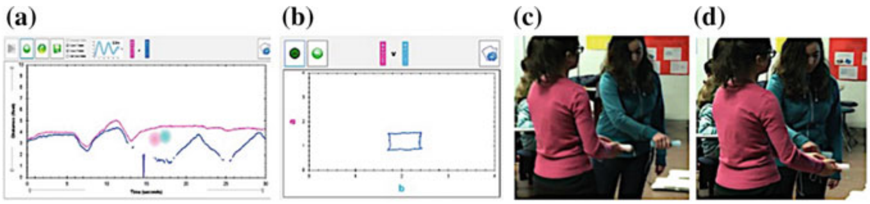


Fig. 17.1 **a** *Line* session; **b** *Versus* session; **c**, **d** Lucrezia and Barbara making the circle

17.4 Making a Circle

During the collective discussion, we aimed to connect the experience of making a circle with *Versus*, with the corresponding *Line* graphs. This explicitly drew attention to the nature of the circle as a planar motion trajectory (a parametric curve) with respect to the nature of the sinusoidal line graphs as the distance-time functions associated to the two-dimensional motion. The students had already worked on this connection in an intuitive way when they had first tried to make the circle using *Versus*, and then, keeping moving the controllers in the same way, had switched to the *Line* option to discover the associated graphs of $a(t)$ and $b(t)$. On that occasion, Lucrezia and Barbara had first moved the controllers in a coordinated way along parallel directions (Fig. 17.1c, d) being able to obtain quite a circular shape more times (Fig. 17.2a). If one thinks of the circle as a gradient of speeds and directions, one realises how challenging the task is in that it affects and is affected by the students' need for becoming coordinated together and with the software. This agreement in movement is a sort of obligation to the task that pushes the students (both those moving and those watching by their seats) to begin perceiving how mutual position, direction and change of speed are implicated in the process of creating a circle.

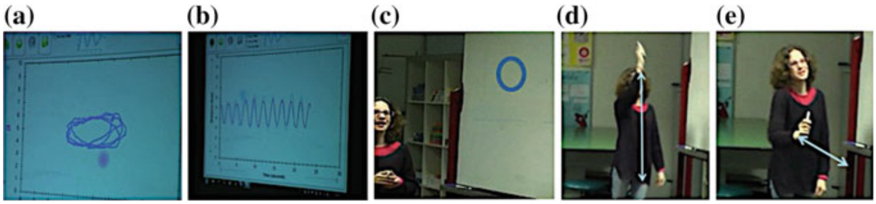


Fig. 17.2 **a** Circular shapes with *Versus*; **b** sinusoidal *Line* graphs; **c** the circle; **d–e** Tiziana's orthogonal movements

In a second moment, while Lucrezia and Barbara were trying to maintain their coordination, we had changed *Versus* to *Line* and all the students had been able to see the graphs shown in Fig. 17.2b (that we might take as sinusoidal curves). In this case, the two graphs appear only little translated in respect to each other, but the students could start perceiving such a geometrical property as part of Lucrezia and Barbara's coordinated movement. Partaking in the experience as observers or movers becomes relevant as soon as the explicit connection between the graphs and the circle, as well as the relation between the two graphs, comes to be discussed. Attention was shifted during the next meeting from making the circle with WiiGraph to talking about the circle, without using WiiGraph, in a collective discussion. In light of the theoretical commitments outlined above, our analysis of the discussion will not specifically focus on the students' speech but on the alterations and changes in the diagrammatic activity, which speak directly to changes in mathematical thinking about the circle.

17.4.1 Moving Hands and Changing Diagrams

Even if the coordinated movement of the movers/controllers requires agreement not only in space but also in time, in the making of a circle with *Versus* time seems to disappear. This is true in the only sense that time as a variable does not belong to the Cartesian plane displayed by *Versus*, where a figure is indeed captured as a set of couples of positions. However, time is still present in the circle as the parameter that fundamentally determines and rules the relation between the elements of each couple. From the mathematical point of view then, its role is crucial to grasp the connection between the circle as a trajectory and the sinusoidal graphs of distance versus time that describe circular motion. Accordingly, the collective discussion mainly focuses on making explicit the role of time in this connection.

The starting point of the episode is the drawing on a whiteboard of two kinds of Cartesian planes: *ba* on the left and *st* on the right, where *s* is the label used for distance (no matter whether corresponding to *a* or to *b*). We draw attention to a brief passage of the discussion, in which we look at the evolution of the diagram changing throughout the movement of the body and the hands of several students. To this aim, we centre on some pictures, extracted from the video of the discussion, that capture changes of the diagram and that we see as the site of the gestural/diagrammatic interplay (Figs. 17.2c–e and 17.3). The first picture is that of a circle added by Tiziana on the *ba* plane (Fig. 17.2c). At this moment, the circle is simply a circular shape, like that obtained at the end of a *Versus* session with WiiGraph. While two orthogonal movements seem to be implicated in the graphical shape

(Tiziana: Fig. 17.2d, e), those movements are still detached from the vision of the circle as a geometrical object, so they do not bring about new elements to the diagram. As soon as a point is drawn on the circle (Federico: Fig. 17.3a), the circle is no longer only a closed line but becomes a motion trajectory that reveals and unfolds a dimension in which the point is movable. The dotted lines spilling out from the point in orthogonal directions add to the diagram another dimension, which freezes the movement of the point in a specific position, capturing two specific distances *a* and *b* of the controllers from the sensor bar. The few hyphens that bring into being the two projections now tie the variable point on the circle to its position in the plane through entanglement with the Cartesian axes. Notation for the position is explicitly attached to the diagram, correspondingly to the point (Federico: Fig. 17.3b). The circle becomes then a set of points, once the first author points to a different new point on the circle (Fig. 17.3c).

The set of points inherently implies a set of couples of dotted lines, each couple linked to one point. While the circle is discretised by the hand jumping to far positions instead of continuously flowing along the line, the diagram starts telling a different story. The new point, which is actualised by the moving hand, recovers the virtual presence of time in the circular motion trajectory, making present on the diagram a new position in time, which corresponds to an eventual new frozen moment reachable by the moving point (and to a new couple of dotted lines). The dimension of time enriches the diagram with the discreteness of specific positions (*b*, *a*) along the circle. In fact, different positions cannot be reached by the moving point at the same time instant, which the dynamic nature of the diagram makes emerge out of the researcher's pointing to distinct points of the circle. Two points are of course distinct as positions in the plane (distinct couples of coordinates), but also as positions on a motion trajectory, along the temporal dimension. At this moment, the students recognise that time indirectly plays a role in the diagram/trajectory. This shifts the focus of the discussion to that which is instead directly linked to time.

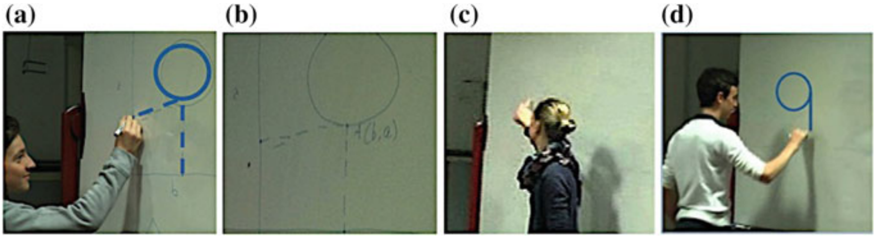


Fig. 17.3 **a** Federico adding point and dotted lines; **b** notation for the point on the circle; **c** new point on the circle; **d** marking room for vertical displacement

The activity changes again when two graphs that unfold the axis of time in relation to the changes of a and b are added to the diagram, in two perpendicular directions (Emanuele: Figs. 17.3d and 17.4a, b). First, the circle implicates the drawing of the horizontal and vertical lines that define room for horizontal and vertical displacement of the position (Fig. 17.3d). Then, the spatial dimension is entangled with the temporal dimension through the new movements that originate the lines of $a(t)$ and $b(t)$ (Fig. 17.4a, b). The new dimension of the functions over time and of their connection with a circular motion appears. This dimension unravels the meaning of the two graphs in terms of the previous physical motion experiences with *Line*. There are significant aspects to take into account here. On the one hand, these lines unexpectedly emerge out of (and really from) the space containing the circle (the quadrant of the Cartesian plane ba): the one on the bottom ($b(t)$), the other on the left ($a(t)$); the one perpendicularly to the axis of b, the other to the axis of a. This perpendicular direction introduces time as that dimension which captures the making of the two graphs in unscripted space. On the other hand, the lines are both constrained to the defined room, no matter which is the direction of their making. Therefore, the new diagram embeds limits in relation to the circle shape (the displacements of position are bounded to the circle's size: see e.g. Fig. 17.4c, d for the vertical displacement). These limits speak directly to the definition of the circle as a geometrical object. But the diagram also embeds freedom(s) about the ways that each line grows as a function of time in the available space (the constraint is given by the borders of the whiteboard).

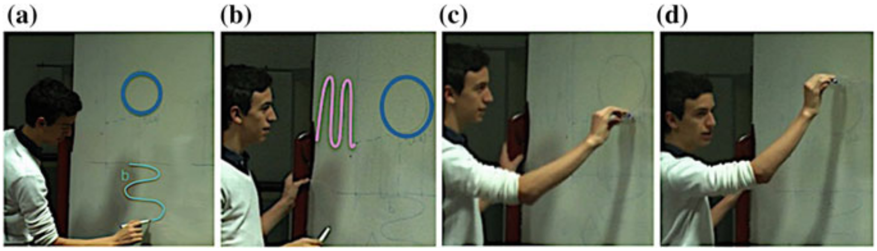


Fig. 17.4 **a**–**b** $b(t)$ and $a(t)$ on the diagram; **c**–**d** vertical bounds

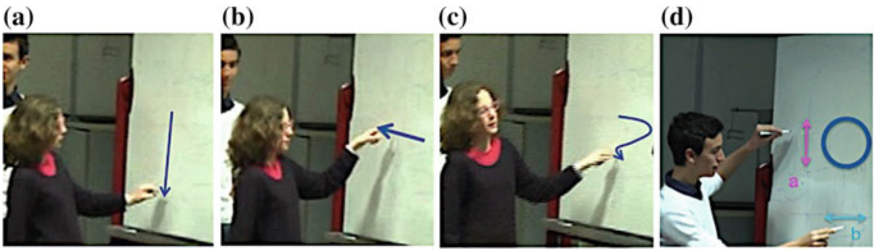


Fig. 17.5 **a**–**c** Tiziana's gestures on the diagram; **d** Emanuele's coordinated movements

At this point, new gestures emerge out of the diagram. Tiziana's left index finger moves close to the diagram to actualise the new Cartesian plane containing the graph of $b(t)$, in particular the axis of time (Fig. 17.5a–c). While this draws attention to the role of time as a variable and to the link with the plane *st*, the following gestures condense the situation by taking time away and, moving thought in reverse, bring back the coordination and the collaboration through which the circle was made. Emanuele's moving hands bring into life the two contemporary movements necessary to have the point moving along the circle (Fig. 17.5d), before unwrapping their developments along the dimension of time. Such new movements recover the essence of the activity with the tool, embedding it in the space constrained to the diameters' projections on the axes, again in perpendicular directions. In the rest of the activity, the students transfer the graphs of the two functions on the plane *st* starting from thinking of the displacement constraints for the vertical and horizontal positions.

Summarizing, the concept of circle emerges from the actualization of constraints that are embedded in the diagrammatic and forged by the gestural, from the breaching and unfolding of dimensions for movement to expand and evolve, and from the exploring of duration and coordination among hands, people and variables. The technology is also implicated in all this emergence in a way that creatively and unexpectedly perturbs both the mathematics of the circle and the entanglement of diagrams, gestures and students in the activity.

17.5 Concluding Remarks

In the very brief episode that we have presented here, it is as if the students included themselves in the drawings of, on and around the circle on the whiteboard, not only projecting themselves into the diagramming activity but also into the past experience with tools. We have seen how learners, tools and diagrams are entangled within the mathematics classroom: boundaries between them always change, mobilizing activity towards/across unexpected and unscripted lines that engender new meanings for the making of the circle. In particular, we have proposed to tell the story from the point of view of the diagram, marking new dimensions and movements within and about the working surface, on which the several students intervened. In this way, we have focused more on all the movements that have originated from and have animated the diagram, rather than on the diagram itself as a finished product of the mathematical activity. This perspective helps us shift attention from seeing diagrams as representations of existing concepts to diagrams as emergent speculative mathematical doings. It allows us to see how mathematical thinking about the circle is throughout developed for the students, and especially how the kind of concept of circle that emerges out of the activity is one of a gradient of speeds and directions, which implicates coordinated movement and timing in/for its making.

This vision also helps us draw attention to how, in the same way as the graphs with WiiGraph arise from and change with movement, the diagram arises from and moves with the students' gestures that incorporate their past experiences with the technology. In this respect, we have shown how the nature of the particular tool used by our students has forged the becoming of the gestural/diagrammatic interplay. On the other hand, de Freitas and Sinclair (2014) point out that the vision of Châtelet "challenges educators to reconsider the power of student diagramming as a disruptive and innovative practice" (p. 84). As mathematics educators, we believe to have taken this challenge in telling the story from the point of view of the diagram. It is this viewpoint that has pushed us to reconsider the power and the affective force of the diagramming as an inventive way of making a circle within the mathematics classroom.

References

Châtelet, G. (1993/2000). *Les enjeux du mobile*. Paris: Seuil (English Transl. by R. Shore & M. Zagha, *Figuring space: Philosophy, mathematics and physics*. Dordrecht, The Netherlands: Kluwer, 2000).

Châtelet, G. (1987). L'enchantement du virtuel. *Chimères, 2* (Online). http://www.revue-chimeres.fr/drupal_chimeres/?q=node/16.

de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics, 80*(1), 133–152. https://doi.org/10.1007/s10649-011-9364-8.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. New York, NY: Cambridge University Press.

Ferrara, F., & Ferrari, G. (2017). Agency and assemblage in pattern generalisation: A materialist approach to learning. *Educational Studies in Mathematics, 94*(1), 21–36. https://doi.org/10.1007/s10649-016-9708-5.

Haraway, D. J. (2008). *When species meet*. Minneapolis: University of Minnesota Press.

Hegedus, S., & Tall, D. (2016). Foundations for the future: The potential of multimodal technologies for learning mathematics. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 543–562). New York, NY: Routledge.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*(2), 159–174. https://doi.org/10.1007/s10649-008-9150-4.

Noble, T., DiMattia, C., Nemirovsky, R., & Barros, A. (2006). Making a circle: Tool use and the spaces where we live. *Cognition and Instruction, 24*(4), 387–437. https://doi.org/10.1207/s1532690xci2404_1.

Roth, W. M. (2016). Growing-making mathematics: A dynamic perspective on people, materials, and movement in classrooms. *Educational Studies in Mathematics, 93*(1), 87–103. https://doi.org/10.1007/s10649-016-9695-6.

Rotman, B. (2008). *Becoming beside ourselves: The alphabet, ghosts, and distributed human beings*. Durham: Duke University Press.

Sinclair, N., Chorney, S., & Rodney, S. (2016). Rhythm in number: Exploring the affective, social and mathematical dimensions of using TouchCounts. *Mathematics Education Research Journal, 28*(1), 31–51. https://doi.org/10.1007/s13394-015-0154-y.

Chapter 18
A Linguistic Approach to Written Solutions in CAS-Allowed Exams

Johannes Beck

Abstract The context of my study is the use of computer-algebra-systems (CAS) as permanently available tools in upper secondary education, where the documentation of CAS-use is a major challenge for both teachers and students. This article focuses on a descriptive model with which students' written solutions can be analysed from a linguistic point of view. Furthermore, this model may help teachers to reflect about the functional aspect of "writing down notes" and thus to allow for a deeper insight into the way the process of communication of mathematics can be shaped.

Keywords Computer-algebra-systems · Exams · Linguistic approach
Documentation · Communication

18.1 Introduction: Solving Tasks with CAS

The use of digital technologies like computer-algebra-systems (CAS) changes various aspects of mathematics education (cf. Barzel, Hußmann, & Leuders, 2005). Thanks to the many available functions of a CAS, students perform typical activities—such as differentiating a function—in a different way than they would have done in a traditional mathematics class. While shortcutting the work of calculating with pen and paper enables classes to have more time for mathematical reasoning, the availability of CAS also forces teachers and students to think about how to document the process of working on a task.

In this study, I focus on mathematics classes that work with CAS as a permanently available tool from 10th to 12th grade. The respective students choose in grade 9 that they want to participate in such a class. The CAS will be allowed in most exams during this three-year-period although this is not obligatory. In the final exam (called Abitur) the students are obligatorily allowed to use the CAS.

J. Beck (✉)
Julius-Maximilians-Universität, Würzburg, Germany
e-mail: johannes.d.beck@uni-wuerzburg.de

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_18

The aim of this study is to analyse students written solutions and identify how they communicate in their final written exam. Furthermore, it is my intention to provide didactical material that helps teachers and students with the challenges and difficulties of documenting their solutions.

18.2 Theoretical Background: Communication in Mathematics

What exactly is the nature of documentation in mathematics? In this article, I distinguish the *solving process*, i.e. the activities, from the *documentation*, which is a written-down “text”. The central aspect of “text” is the fixation of meaning in written form (cf. Beck & Maier, 1994). During the solving process, both mental and non-mental activities—such as operating the computer—are performed by the learner. In traditional CAS-free classes, the performed activities—such as differentiating a function—might need to be written down on paper *in order to perform them* and, hence, the performed activities and their documentation are not separable. In comparison, if a CAS is used many activities can be performed with it automatically and the formerly step-by-step process is hidden within the tool. Consequently, certain CAS-related activities might or might not be documented whereas they would have been written down as part of a traditional, CAS-free solving process.

Any author of a mathematical text can choose different representations to fixate meaning on paper. The extended modelling cycle for modelling with the help of technology distinguishes *three worlds*: the real world, the mathematical world, and the technological world (cf. Greefrath & Siller, 2010). I argue that each world has its own forms of typical representations or, as Moschkovich puts it: “research will need […] to consider the interaction of the three semiotic systems involved in mathematical discourse—natural language, mathematics symbol systems, and visual displays” (Moschkovich, 2010, p. 153). Thus, three different groups of representations can be distinguished: (1) The natural language (e.g. German or English) allows to connect the mathematical content with the real world. Normally, this is the students’ first language. (2) The mathematical world offers the symbolic language of numbers and formulas as well as the mathematical register. Pimm states that “[p]art of learning mathematics is learning to speak like a mathematician, that is, acquiring control over the mathematics register” (Pimm, 1987, p. 76). Furthermore, mathematical content is often represented by a variety of graphical representations such as tables and graphs. (3) The technical world has its own register and symbolic expressions that allow to fixate meaning that is closely connected to the computer, and more prominently the command language with which the digital tool is operated. While all these three worlds have symbols that traditionally have a specific meaning in certain contexts, any author can also create new symbols and other forms of representations. Godfrey and O’Connor speak of the mathematics community as a "community that creates new measures and symbols as they are needed, flexibly applying the means that already exist and adding to those tools when necessary" (Godfrey & O'Connor, 1995, p. 328). In the long term, students have to learn to use the mathematical register and the symbolic language in order to be able to participate in the communication of the community of mathematicians.

Regarding the writing practise, Pimm highlights three different styles of the mathematical writing of students within a continuum of styles: "*verbal*, *mixed* and *symbolic*" (Pimm, 1987, p. 118). Students of each type prefer using representations from the name-giving source. Pimm stresses "that *all* the recording styles […] have the *potential* for accurate and precise written records of mathematics" (Pimm, 1987, p. 118). In the context of CAS-use, a fourth style might occur: a style that relies heavily on writing down computer-commands and results in the representational forms used by the tool.

As mentioned above, the primary object of written solutions in exams is to communicate mathematical knowledge and to make it accessible to the teacher for grading. If Jakobson's model for describing communication situations (cf. Jakobson, 1960) is applied to the examination its six different factors can be clarified as follows: the learner is the ADDRESSER, the teacher is the ADDRESSEE and the written-down solution is the CONTACT (or channel) for the MESSAGE. The CODE in this communication situation consists of the aforementioned representations, which both the addresser and the addressee at least partially must share and understand. The CONTEXT includes the examination tasks as well as the screen of the CAS as its most important factors.

In exams, the written documentation is the only channel by which the addresser sends the message to the addressee. Naturally, in such a situation it is neither possible nor allowed for the corrector to inquire in case he does not understand a part of the solution. Thus, Busse calls all written communication as "reduced communication situation[s]", arguing that only the text itself and the recipient of the text are really participating in the situation (2015, p. 320, translation JB), although the producer of the text is still implicitly present through his text. Thus, the understanding of texts is conceptualized as the allocation of the recipient's knowledge to certain elements of the text. In this regard, Busse claims that the producer of a text can influence the understanding of the text by setting the degree to which this allocation is open or closed (cf. Busse, 2015). In any test, it is in the interest of the addresser to avoid misunderstandings with the addressee. The purpose of documentations in exams is to enable the corrector(s) to understand and judge the way along which the student has found the solution (cf. Ball & Stacey, 2003). Those parts of the written solution, which fail to make the underlying thoughts clear, are commonly marked as "unclear" and not rewarded with the maximum of achievable points. Thus, the addressers have to take care that they *explain* their solving process well enough. Furthermore, most mathematics curricula require students to show a certain degree of formalism, that is, using the mathematics register and the symbolic language appropriately. It has to be part of the preceding lessons that students learn which requirements they have to meet.

Students should also develop the competence to communicate mathematically in different situations and for different purposes, not only for grading. Generally, if somebody writes in mathematics one either writes for oneself or for somebody else (cf. Pimm, 1987). Brenner distinguishes different types of communication in the context of mathematics.

> "Communication About Mathematics entails the need for individuals to describe problem solving processes and their own thoughts about these processes. […] Communication In Mathematics means using the language and symbols of mathematical conventions […] Communication With Mathematics refers to the uses of mathematics which empower students by enabling them to deal with meaningful problems (Brenner, 1994, p. 41)".

At the end of their school life students should be able to communicate competently in all three types. This article focuses mostly on communication in mathematics. How students can develop this competence is a major challenge. Against this theoretical background, different research questions arise. In this article, I focus on the following:

1. How do students write down their solutions in final exams when a CAS is available? Which different forms of documentations do they use? What kinds of problems or difficulties (if any) are connected with these forms?
2. How can students' written solutions be described by means of a category system?

18.3 Methods: A Descriptive Model for Analysing Written Solutions

In order to answer the first and second question, Bavarian teachers of CAS-classes have been asked to send in nine written solutions each from the final exams. Three solutions came from students who have been average, three from students who have been above average, and three from students who have been below average in the preceding semester. Similar data has been collected every year (starting with 2014) for further evaluation and research. Four to five teachers answer this request every year.

The first question is how students document their solving process in exams. So far, in Bavaria (Germany) only little official advice about documentations of solving processes is given. Normally, the Institute for School Quality and Educational Research (ISB) provides such material and official notes in addition to the curriculum. In order to develop such advice, it is a very valuable first step to analyse authentic documentations and to develop a descriptive model with which problems and difficulties can be identified and categorized.

Based on linguistic theories, I developed a preliminary model which will be tested against authentic, students' written solutions. By means of a form-function analysis, which is a typical *pragmatic* approach (cf. Meibauer, 2008), two dimensions can be derived (cf. Fig. 18.1). The following question is the main focus of the analysis: With which forms of representations do students document each step of the solving process? The categories in each dimension have been developed by means of a qualitative content analysis (cf. Mayring, 2010).

The *representational dimension* describes with which forms of representation students document. There might be expressions, which use some kind of formulaic symbols (traditional mathematical, computer-syntax, mixed-forms), verbalisations (both natural language and the special mathematical vernacular) and graphic representations. In the latter category, mixed forms (such as graphs, tables, sketches, etc.) are also counted.

The second, *activity dimension* describes which purpose an element has, that is what actually is documented with it and which step, or activity, in the solving process it is related to. Central categories are:

- *CAS-related notes* make the use of CAS explicit, either by stating the CAS command (input), by writing down the output (e.g. "false", which is odd in a German text), or by unspecifically writing—in short form—that the CAS was used (e.g. "CAS: …").
- According to Wagner and Wörn (2011) *explanations* comprise three different facets: concepts and ideas (what-explanation), algorithms and procedures (how-explanation), argumentations and logical connections (why-explanation). They often focus on:

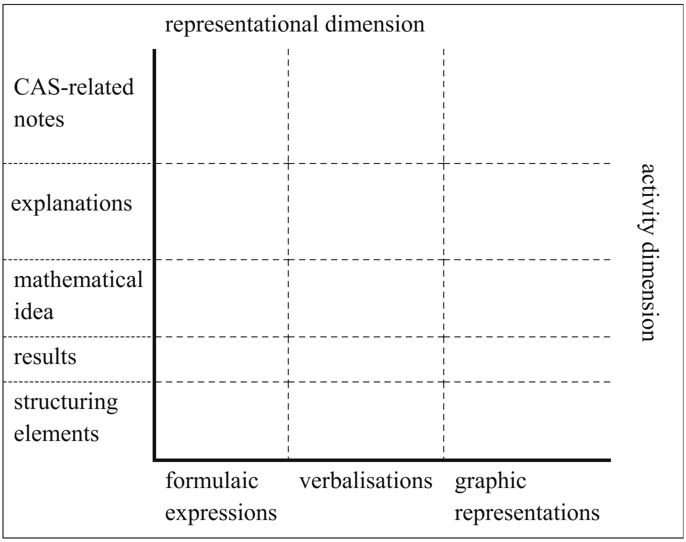


Fig. 18.1 Category system for description of students' solutions

- *mathematizations*, which show that information given in the task-description is translated into mathematical notation or terminology;
- *interpretations*, which are translations of computer-output and the construction of meaning in relation to the task.

The terms mathematizations and interpretations are related to the respective activities in the modelling cycle by Siller and Greefrath (2010).

- Furthermore, there are elements which refer to the underlying *mathematical idea*, e.g. in order to find the maximum of a function f it is necessary to solve the equation $f'(x) = 0$. From this element, the mathematical idea can be reconstructed.
- Results
- *Structuring elements* are used to structure the text on the surface (the layout) and the way the information is presented. They can also be used to set up links between pieces of information such as single steps in the solving process and the chronological order in which they were performed. Thus, one examples in Fig. 18.3 is "then" (the German version has "dann" which means "then" or "next") which adds a chronological order to the text.

18.4 Exemplary Analysis of Some Documentations

In the following part, I present some examples of the analysis of students' written solutions. For the purpose of better readability, I use translations of the authentic material instead of the material itself. The individual characteristics of every documentation are reflected as far as possible, e.g. parts that have been crossed out are crossed out as well. Where the crossed-out text could not be deciphered anymore ~~%~~ ~~&~~ has been used.

18.4.1 Students' Documentation Styles

The model above offers a framework for analysing and comparing students' solutions. Three directions are fruitful and offer some insight into the matter: the analysis of individual students through all tasks of the final exam may show if different styles of documentations occur and are generally solid (cf. above and Pimm, 1987). The comparison of students within one class offers insight into the individual practise of the respective teacher and what he/she deems important in a documentation. The comparison between the documentations of different classes, as the third directions, is a natural by-product of the former analytical direction.

Further distinction: CAS-style (relying heavily on copying CAS-commands from the screen); mathematical style (no CAS-references at all), verbalized style (referring to CAS-use in verbalized form); mixed (different sub-types; any combination of the aforementioned or no preference at all). Here, I present two solutions from one class with different styles and a third solution from a different class (different teacher), where the mathematical style was the only one.

The task is taken from the final exam of 2014 for CAS-classes (CAS-Abitur). All final exams have two different parts: in part A no digital tools are allowed; in part B the students are allowed to use the CAS. Part B is further divided into three thematic blocks—calculus, geometry and data & statistics. For each block one of two different groups of tasks can be chosen by the teacher for his students on the morning of the final exam. The task is taken from the final exam of 2014 for CAS-classes, part B, group of tasks 1. The real-world context is an exit lane of a highway which is described by a polynomic function *s* which was given. In this context different tasks—(a) to (k)—have to be solved by the students. In sub-task (h) the coordinates of the point on the function *s* with the minimal distance to another, fixed point (cell tower in the real-world context) had to be found. The term of the function which describes the distance of a point on *s* was given in sub-task (g). The task was formulated as such: “One point on the southern exit lane is closest to the cell tower. Find this distance on basis of the model” (cf. ISB, translation by JB).

In Fig. 18.2, we see a documentation of a solution to this task in CAS-style. The student starts with writing down the CAS-input with which he defines a function *d* that describes the distance between a point on *s* and the cell tower. In the next two lines, he explicitly but incompletely states CAS-input. The following two lines use the same command as line 3 but do not show them explicitly. Instead, the solve-command is reduced to quotation marks and “<0”, indicating that the same command as in the line above is used for a different inequality. In the last line, the student has already begun to indicate with “(i)” that he starts the next task, yet he completes task (h) without indicating to which task this line actually refers.

Figure 18.3 shows how a student wrote his solution. He documents his use of the CAS (category CAS-related notes) with two different forms of representations. In line 3, he describes his use by stating a CAS command explicitly. In line 1, he verbally describes a mathematical activity that only in relation to line 3 becomes obvious as a reference to the CAS-use. The result of the activity in line 1 is used in

h) ~~d(x):=~~ $d(x){:}\sqrt{(x-6)^2+(s(x)+8)}$

~~ds(x)=~~ $ds(x) := \frac{d}{dx}(d(x)))$

$solve\ ds'(x) = 0 \Rightarrow x = 1.5$~~6~~$4678$

“ > 0 x > “

< 0…..x < ”

⇒ there is a minimum at ~~$(x_2|$~~

(1.54678 | f(1.54678) = S(1.54678 | 5.89293)

i.e. the smallest distance is

i) 20 m 5.89293 ≈ $5.9 \cdot 20\,m = \underline{112\,m}$

Fig. 18.2 Student’s documentation (teacher A) line 3 ("$d_1s(x)$"). Thus, the student uses a mixed style, combining verbalization and CAS-input. In the second half of the documentation, the student uses the CAS as a normal calculator and does not refer to this in any way. The final result is marked by underlining it twice.

The examples (Figs. 18.2 and 18.3) are typical for each student. Both were in the class of teacher A. It can be concluded that teacher A allowed both styles of documentation: the CAS-style of Fig. 18.2 and in Fig. 18.3 the style with the mixture of verbalizations and CAS-commands.

The next example is from a student from a different class (teacher B). It is typical for the documentations from this class that almost no references to the CAS appear. The style is mainly symbolic. Figure 18.4 shows mathematical ideas in the mathematical symbolism (line 2 and 5). Results are given in lines 3 and 5. The explanation in the last line is interesting insofar that it gives mathematical reasoning and that it is not clear if it is part of the solution or not. Normally, if something is crossed out it is clear that it is the student's intention that the teacher does not mark the respective passage. If a student puts something in brackets (outside of the mathematical or CAS notation) then it is not clear if it is a valuable part of the solution or a "meta-comment" on it. Thus, the student does not communicate clearly. The important question (that cannot be answered with my analysis, however) is if the student thinks that verbal explanations should not be part of a mathematical text (conceived of as a series of symbolic expressions) and therefore put it in brackets.

In conclusion of this part, the first result that can be observed is that in regard to the documentation of CAS-commands the style was very homogenous throughout each class. In one of the classes, CAS-commands have been documented (teacher A). In the second class, the CAS-use was indicated by writing "CAS" either over an

h) derive $d_1(x)$

then:

$solve(d_1s(x)| -10 \leq x \leq 3.18 = 0, x)$

$\Rightarrow x = 1.547$

$d_1(1.547) = 5.97$

~~5.~~ $5.97 \cdot 20\,m = \underline{\underline{119.39\,m}}$

Fig. 18.3 Student's documentation (teacher A)

h)

$d'(x) = 0$

$\Rightarrow x_1 \approx 0.54678$ ~~$s(x_1) = -4.14053$~~

$\Rightarrow point\ with\ minmal\ distance\ M(1.54678\,|-4.14053)$

$d(x_1) = 5.89293$ LE $\hat{=}$ $\underline{117.859\,m}$

(there's only one result for $d'(x) = 0$, therefore no test of minimal turning point needed)

Fig. 18.4 Student's documentation (teacher B) equation or at the beginning of a line. In the third and fourth class there were no CAS-commands at all (teachers B and C). This phenomenon can be explained by the normative standards that the respective teacher as the authority in the classroom had set in the preceding year(s). This could happen either explicitly by classroom-standards or conventions about how the solution has to be written down; or an implicit norm is set by the teachers' practice (cf. Ball, 2014, p. 15–16, 145ff). A further result of the analysis of the students' documents is that written solutions *without* verbalised explanations were often harder to understand and that the solving process could not be reconstructed that easily. The following part elaborates on students' explanations.

18.4.2 Explaining and Explanations

In relation to the category system (Fig. 18.1), my expectation that students performing above average explain more than students performing below average, with the average students somewhere in between, was not satisfied. Instead, the quality and style of given explanations differed. Some students from the below-average group formulated their explanations like steps in a recipe or an algorithm at the time they have to be carried out. For example, one student from this group wrote: "Which x does one need so that $f'(x_0) = 0$" (original in German: "Welches x braucht man, damit $f'(x_0) = 0$ wird?", translation JB; note: The original formulation sounds like "Which x does one need for $f'(x_0)$ *to become* 0?") Well-performing students, on the contrary, often tried to give an overview at the beginning or the end of their documentation about how the task can be solved and what the mathematical idea is (Fig. 18.6).

As shown above, elements of written solutions can have different functions. Among them explanations can contribute a lot to make students' documents easily understandable. This is going to be shown in the following example. According to Jörissen and Schmidt-Thieme explanations are characterised as "primarily verbal statements" with the goal that the reader can understand connections (Jörissen & Schmidt-Thieme, 2015, p. 401, translation by JB). The three sub-categories of Wagner and Wörn (2011, see above) appear rudimentarily in the students' solutions in verbalised form although the wording of the task did not explicitly ask for it. The task to the example (Fig. 18.5) is to check whether there is a point at which the exit of a highway—modelled by a polynomial function *s*—runs parallel to another road —the route B299, which is modelled by a linear function.

The student explains the mathematical idea of his solution verbally at the beginning. It is a rudimentary *how-explanation*. The verbal inaccuracy at this point is not that important because the information given in the text is supported by a mathematical formulaic expression, which is the equation. The output ("{ }") follows after an unspecific CAS-use. The student confuses proper mathematical syntax with device-specific CAS-language and mixes both into "an incorrect" expression. As a concluding answer to the task a verbal interpretation of this output is written down.

e) The southern exit route runs parallel to route B299 if the gradient m is equal.

$\Rightarrow s'(x) = m_{B299}$

$s'(x) = -0.5$

$0.03468x^2 + 0.3542x + 0.529 = -0.5$

$\overset{CAS}{\Longrightarrow} x = \{\square\}$

⇒ At no point the exit runs parallel to the B299.

Fig. 18.5 Student's documentation (teacher C)

In terms of the *three worlds* of the extended modelling cycle by Siller and Greefrath (2010) this documentation starts in the real world, then moves on to the mathematical world via the process of mathematization. In line 7, the transition to the computer world and back to the mathematical world is made (the result, as explained above, is written down in the representational form of the computer). The answer at the end returns to the real world. The transitions are not made explicit in verbal form but the student uses arrows whenever such a transition is taking place.

18.5 Discussion

As the examples above show, an analysis of students' written solutions with the descriptive model (Fig. 18.1) can help to understand written solutions better and reveal aspects that need further research. However, the descriptive model cannot avoid the problem that basically no element used in spoken or written language has only one single function. A CAS command such as "solve($f'(x) = 0$,x)" does indicate that this command has been used to solve an equation. But it also shows which mathematical idea originally led the student to use the command. Therefore, the categories of neither the activity nor the representational dimension should be viewed as strictly separate. Instead, the model might help to make the multi-functionality of documentations more apparent and thus help teachers (and learners) to communicate their thoughts more clearly. For the linguistic description of this phenomenon a metaphor is useful: each element is a box and can contain a smaller box in it (and can be part of the content of a bigger box). In the case of "solve($f'(x) = 0$,x)" the CAS-command is the bigger box and contains the mathematical idea "$f'(x) = 0$" in it. Furthermore, I created first drafts of possible best-practice examples of written solutions, using the categories of the model above. It is my intention to raise teachers' awareness for the diversity of approaches to documenting. The following example (Fig. 18.6) shows a documentation which is highly verbalized and makes the transitions between the different worlds clear.

Examples like this (Fig. 18.6) need to be integrated into a model that tries to give ideas about how the broader learning process of students (and teachers) could look.

Category	Exemplary solution
Explanation (mathematisation of the context)	The transition from a right-hand bend to a left-hand bend corresponds to an inflection point of the modelling function *s*.
Mathematical ideas	If $s''(x_W) = 0$ and s'' has a change in the sign, then there is an inflection point.
CAS-related note	Solve $s''(x_W) = 0$ with CAS.
Result	$x_W \approx -5{,}11$
Explanation	s'' has a change in the sign because $s''(x) = 0{,}06939x + 0{,}3542$ is the term of a linear function.
Result (reconnection to context)	In the point $\mathrm{W}(-5{,}11\vert -5{,}01)$ the right-hand bend transforms into a left-hand bend.

Fig. 18.6 Exemplary solution using the categories as a basis

This process has to be integrated into theories of learning mathematics with CAS. Drijvers et al. describe the instrumental approach for example as follows:

> The instrument, then, is the psychological construct of the artefact together with the mental schemes the user develops for specific types of tasks. In such schemes, technical knowledge and domain-specific knowledge (in our case mathematical knowledge) are intertwined. Instrumental genesis, in short, involves the co-emergence of mental schemes and techniques for using the artefact, in which mathematical meanings and understandings are embedded (Drijvers et al., 2010, p. 1349).

In written solutions, students use various signs and symbols to inscribe meaning into a text. How these signs become meaningful to the students and how this process relates to the instrumental genesis has not been explained completely yet.

It is important that students develop the competence to create and judge if a solution is acceptable. This process needs time and may start with students creating their own meaningful symbols for documenting their solutions with CAS. As Godfrey and O'Connor point out: "Through creating and using nonstandard, student-generated symbols and measures, it is hoped that students will more readily deal with traditional mathematical texts and language as real communication" (Godfrey & O'Connor, 1995, p. 328). This of course means that teachers discuss the practise of communicating mathematics in written form in their classes. To provide support for this development is a major challenge for modern mathematics education.

References

Ball, L., & Stacey, K. (2003). What should students record when solving problems with CAS?—Reasons, information, the plan, and some answers. In J. T. Fey et al. (Eds.), *Computer algebra systems in secondary mathematics education* (pp. 289–303). Reston: NCTM.

Ball, L. (2014). *Use of computer algebra systems (CAS) and written solutions in a CAS allowed year 12 mathematics subject: Teachers' beliefs and students' practices* (Ph.D. thesis). Graduate School of Education. The University of Melbourne.

Barzel, B., Hußmann, S., & Leuders, T. (2005). *Computer internet & co. im Mathematikunterricht*. Berlin: Cornelsen Scriptor.

Beck, C., & Maier, H. (1994). Zu Methoden der Textinterpretation in der empirischen mathematikdidaktischen Forschung. In H. Maier & J. Voigt (Eds.), *Arbeiten zur interpretativen Unterrichtsforschung* (pp. 43–76). Köln: Aulis-Verlag Deubner.

Brenner, M. (1994). A communication framework for mathematics: Exemplary instruction in mathematics. In B. McLeod (Ed.), *Language and learning: Educating linguistically diverse students* (pp. 233–244). Albany: SUNY Press.

Busse, D. (2015). *Sprachverstehen und Textinterpretation. Grundzüge einer verstehenstheoretisch reflektierten interpretativen Semantik*. Wiesbaden: Springer.

Drijvers, P., Doorman, M., Boon, P., & van Gisbergen, S. (2010). Instrumental orchestration: Theory and practise. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education. January 28th–February 1st 2009, Lyon*. Lyon: INRP.

Godfrey, L., & O'Connor, M. C. (1995). The vertical handspan: Nonstandard units, expressions, and symbols in the classroom. *The Journal of Mathematical Behavior, 14*(3), 327–345.

ISB. (2014). Abiturprüfung im Fach Mathematik (CAS)—Prüfungsteil B. Retrieved from https://www.isb.bayern.de/schulartspezifisches/leistungserhebungen/abiturpruefung-gymnasium/mathematik/2014/.

Jakobson, R. (1960). Linguistics and poetics. In Th A Sebeok (Ed.), *Style in language* (pp. 350–377). Cambridge, Mass.: MIT-Press.

Jörissen, S., & Schmidt-Thieme, B. (2015). Darstellen, Formalisieren und Kommunizieren. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch der Mathematikdidaktik* (pp. 385–408). Berlin: Springer Spektrum.

Mayring, P. (2010). *Qualitative Inhaltsanalyse—Grundlagen und Techniken*. 11. aktual. u. überarb. Auflage. Weinheim: Beltz.

Meibauer, J. (2008). *Pragmatik—Eine Einführung* (2nd ed.). Tübingen: Stauffenburg Verlag.

Moschkovich, J. N. (2010). *Language and mathematics education. Multiple perspectives and directions for research*. Charlotte, NC: Information Age Publishing.

Pimm, D. (1987). *Speaking mathematically. Communication in mathematics classrooms*. London: Routledge & K. Paul.

Siller, H., & Greefrath, G. (2010). Mathematical modelling in class regarding to technology. In V. Durand-Guerrier, S. Soury-Lavergne, F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (CERME 6). January 28th–February 1st 2009.*

Wagner, A., & Wörn, C. (2011). Erklären lernen—Mathematik verstehen—Ein Praxisbuch mit Lernangeboten. Seelze: Kallmayer–Klett.

Chapter 19
Technological Supports for Mathematical Thinking and Learning: Co-action and Designing to Democratize Access to Powerful Ideas

Luis Moreno-Armella and Corey Brady

Abstract The enterprise of understanding and supporting processes of mathematical cognition is both epistemologically deep and politically urgent. We cannot ignore that new technology-mediated learning environments have the potential to democratize access to powerful ideas. The importance of technology in this respect is bound up with the essential nature of mathematical objects as symbolic entities that can only be expressed and conjured up through the mediation of representations. A key question for the design of technology-enhanced learning environments is whether the cognitive tools—material and digital-symbolic—that have been developed in recent decades might offer learners access to modes of activity with disciplinary structures that have historically been achievable only by 'maestros' of the discipline. In this article we elaborate the construct of "co-action" as a means of describing humans' mathematical interactions with the support of such tools.

Keywords Technological infrastructure · Co-action · Collaboration
Mathematical representations

19.1 Introduction

In this paper, we argue that concrete challenges involved in understanding and supporting processes of mathematical thinking, learning, and activity as they unfold in today's classrooms are both epistemologically deep and politically urgent. Scenes of teaching and learning hold epistemological interest because the behaviors, ideas, and struggles that emerge while students learn to work with mathematical entities can illuminate fundamental structural features of the underlying

L. Moreno-Armella
CINVESTAV-IPN, CDMX, Mexico

C. Brady (✉)
Vanderbilt University, Nashville, USA
e-mail: corey.brady@vanderbilt.edu

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_19

mathematics. Work to support school-based mathematics learning is also politically important because these new technology-mediated learning environments have the potential to democratize access to powerful ideas, enabling learners to experience powerful ways of operating with mathematical symbols and structures.

We argue that the representational peculiarities of mathematics thinking and learning make it an ideal setting for illuminating new modalities of interaction and meaning-making enabled by an array of technologies—technologies that support dynamic, executable representations and technologies that support real-time collaboration among learners. A distinctive form of interaction between learners and learning environments is enabled in such settings, which we call *co-action* (Moreno-Armella & Hegedus, 2009). We articulate our emerging understanding of co-action as a framework for understanding human activity in the context of representation and communication infrastructures through two brief examples. We hope that this construct may help researchers to explore the ways in which technological changes can radically transform thinking and learning processes in the discipline of mathematics.

19.2 The Special Challenges of Mathematical Reference

As we approach mathematical cognition in classroom learning environments, the symbolic dimension of mathematics becomes sharply salient. Mathematical discourse is always social, always culturally situated and always shaped by its institutional context; thus the semiotic dimension is always important. However, in learning settings the nature of mathematical objects is very often in question and not (yet) taken-as-shared, so that efforts to evoke these objects and to communicate clearly about them receive particular attention and social pressure.

As a way of framing the problems involved in the relationships between mathematical representations and objects, consider Magritte's *The Treachery of Images*. This famous painting explores issues of representation, in ways that are relevant to mathematical representations. The artist has written "Ceci n'est pas une pipe" ("This is not a pipe"), in painted script, under the painted image of a pipe. The focus is on the viewer's idea of a pipe: within the painting, there are two explicit "pipes"—the pictorial image *of* a pipe and the painted words "une pipe." The painting puts these two "pipes" in conversation with one another and with the viewer's *Pipe* idea. The image falls short of the idea: it is "not a pipe"—one cannot hold it, fill it with tobacco, or smoke it.

Now suppose, instead of a pipe, Magritte had painted a *circle* with the inscribed legend, "Ceci n'est pas un cercle." A different dynamic would have emerged. Magritte would not, even in theory, have been able to reach into his pocket and produce the geometric *circle* that had served as the model for the painting, and that the painted image is *not*. In fact, one might argue that the legend, "Ceci n'est pas un cercle" would be *false*: at least in the sense that every representation of a circle *does* express circle-ness in some degree, and that, further, nothing *except* a collection of such representations does so.

The essentially symbolic dimension of mathematical thought and discourse highlights a unique epistemological feature. Because a mathematical object cannot be *pointed at* independently of its manifestations within one or more representations, mathematical work and learning must occur in settings that are entirely mediated by representations. This further heightens the importance of *symbolic production* in the learning process, both as learners formulate their thoughts and as they and their teachers exchange symbols and representations in attempting to create shared meanings and understandings. In this vein, Duval (1999) remarks that "the use of systems of semiotic representation for mathematical thinking is essential because, unlike the other fields of [scientific] knowledge (botany, geology, astronomy, physics), there is no other way of gaining access to mathematical objects but to produce some semiotic representations" (p. 4).[1]

19.2.1 The Permanence of Symbolic Beings

The fact that mathematical objects are wholly symbolic beings, which can only be found, expressed, or conjured up through representations, also paradoxically gives them a permanence that cannot be achieved by physical beings or objects. This is why, if we were to read in the newspaper tomorrow morning that the Natural Numbers had been destroyed in a fire, we would smile. We know this is not possible, even though many *instances* of representations of the Natural Numbers do exist in perishable material media.

Part of the reason for the more enduring nature of symbolic entities like the Natural Numbers is the very fact that they do not refer directly to specific objects in the physical or cultural world. Instead, they connect with and express very general features of the human experience of the world. For this reason, the representational and symbolic challenges with which we opened this discussion are also sources of mathematical power. To understand the nature and power of symbolic entities, we can look first at how they emerged in human history and then at how they operate in modern discourse.

[1]We amend Duval's text by adding "scientific" because many forms of knowledge in the arts and the humanities, for example, also face the challenge that the objects of their study are inextricably embedded in semiotic/symbolic representations.

19.2.2 The Emergence of Symbolic Entities

Among the first symbolic entities in human history may have been the records that have been found scratched in bones and dating from about 35 thousand years ago. These marks may have been used to keep track of the number of animals killed in a hunt or the number of days in a lunar cycle. Any external mark or trace that carried and communicated meaning was already a symbolic object: that is, a thing whose purpose was to represent another thing. Moreover, it was perhaps the infeasibility of making an iconographic symbol in available media that led these early humans to produce representations that were loosely coupled to the particular animals or days they described, capturing instead the notion of quantity. The loose coupling of the symbol to its referent made it possible to see relations between two such symbols, even when there was no relation between the objects whose quantities these symbols represented. Thus, the "five-ness" of five sheep, five days, or five pieces of fruit could come to be represented, rather than, and independent of, the "sheep-ness", the "day-ness" or the "fruiti-ness" of the objects. In this way, the number five came to be lifted off of the concrete groups of objects that it described, gaining the status of an independent symbolic entity. In general, symbols can be thought of as crystallized actions—in this case, the action of counting.

As symbolic entities, mathematical objects have a doubly paradoxical relation to the physical world. They exist on a different plane from physical objects, having been decoupled from that world through processes of abstraction and generalization. Moreover, as we have suggested, they cannot be depicted directly or completely. Instead, through representations, certain facets of symbolic entities can be captured, but it is in their nature that they supersede any particular representation. For instance, consider the mathematical symbolic entity of a straight line. In a geometric drawing, we can represent the line as an object in a plane. Applying a coordinate system, we can produce the equation of that line, another representation. Neither of these two representations of the line encompasses the entirety of its mathematical nature, yet each of them captures a facet of that nature. In general, each system of representation highlights or reveals an aspect of the mathematical entities it describes, while concealing or leaving behind other aspects. Thus the choice of a representation is always a consequential choice that constitutes the view and access we have to the mathematical object.

Symbolic entities shared some features with early concrete physical tools, while they also differed from these early tools in other respects. Vygotsky's (1978) famous analysis of this relation was that while tools enabled humans to operate on and exert control over the world, they also enabled humans to exert control over themselves and to regulate their own internal thinking processes, participating actively in these processes. In coming to operate with tools and symbolic entities, human beings gained enormous new powers. Donald (2001) describes this process as the advent of "theoretical culture" and it is the centerpiece of the Baldwinian interpretation of cultural evolution (Baldwin, 1896). With tools, humans encoded processes of labor and craft in physical objects that afforded (Gibson, 2014) the actions that constituted those processes. In this way, tools began to structure human society, so that emerging habits of mind and ways of life were reflected and transmitted in the characteristics sets of tools that supported them. Thus, these extensions to human nature also supported intergenerational development, capturing successful innovations in a transmission medium more flexible and more easily shareable than the biological substrate of DNA. With the symbolic system of written language, communications could be detached from particular interpersonal contacts, enabling new forms of literature, history, science, and philosophy. And with the symbolic system of mathematical discourse, the study of abstract form and structure could take shape and transcend the lives and lifespans of individual thinkers.

19.2.3 Mediated Activity and Co-action

This shift in human history was so significant that many thinkers view human activity as essentially and distinctively mediated activity (e.g., Wertsch, 1991):

> The most central claim I wish to pursue is that human action typically employs mediational means such as tools and language and that these mediational means shape the action in essential ways (p. 12).

Moreover, because tools and human activity are so deeply intertwined, their interaction must be taken account of not only in phylogenetic analyses of development at the level of civilizations and our species, but also (a) in analyses of development at the level of learning communities and of the individuals that compose them, and (b) in micro-genetic analyses of the activity of humans operating with tools. As a broad framework for understanding the dynamic interplay of tools, learning, and activity, Moreno-Armella and Hegedus (2009) have introduced the construct of *co-action*.

Premises of the co-action perspective include: (1) that the activity of tool use involves reflexive adaptation, in which all of the components—human, tool, activity, and context—are continuously transformed; (2) that this interaction is dialogic, not only for learners but also (and perhaps even more radically) for maestros in a discipline; and (3) that analogies in development across the scales suggested above can offer strong clues for both analysis and design. Although there is not space here to articulate the co-action framework fully, in the remainder of this article, these premises will be illuminated in action.

The concept that there are symmetries in the interactions between tools, users, and tasks is encompassed in many socio-cultural perspectives (including Wertsch's and Vygotsky's). For instance, Cole (1996), quoting Alexander Luria, writes that tools, or artifacts, "not only radically change [man's] conditions of existence, they even react on him in that they effect a change in him and his psychic conditions." (Luria, 1928, p. 493, qtd in Cole, 1996, p. 108).

This premise has clear implications for our understanding of maestro performances with tools and artifacts. For instance, consider the relationship between an expert musician and her instrument, as, for example in Jacqueline du Pré's rendering of Elgar's cello concerto. During the performance, the artist and the instrument appear to become one. It is certainly *not* the case that the performance appears effortless; the striking thing about it is that it appears to be co-produced by the musician and the instrument. It seems incorrect to describe the performance as "Du Pré playing on the cello;" instead, it seems appropriate to say, "Du Pré and her cello co-produced the music." Moreover, "her cello" here refers not only to a physical object, but also to the conceptual image of the cello that Du Pré was able to internalize over the course of many years of hard, reflective practice. There is fluidity in this human-artifact integration, making the cello acquire a sound and texture distinctive to the artist (that is, the source of the music is Du Pré and her cello). We use the term co-action (Moreno-Armella & Hegedus, 2009) to describe this generative and creative interplay between humans and tools or symbol systems.

For another example, Gleick's (1993) biography of Richard Feynman records an exchange between Feynman and the historian Charles Weiner. Feynman reacted sharply to Weiner's statement that Feynman's notes offer "a record" of his "day-to-day work."

> "I actually did the work on paper", Feynman said.
>
> "Well," Weiner said, "the work was done in your head, but the record of it is still here."
>
> "No, it's not a record, not really. It's working. You have to work on paper, and this is the paper. Okay?" (Gleick, 1993, p. 409)

The distinction that Feynman makes here shows how he sees his work as intrinsically interconnected with the symbolic system that he is working with. His ideas do not occur separately from their realization in written symbols; rather, they emerge through interaction with that symbol system. It is the same as with Du Pré and her cello, where there is no music without both the artist and the instrument being present.

The Feynman example further clarifies that dialogic interaction with tools and representations is not only a feature of 'finished' performances; it is also a core component in processes of exploration and discovery. For *discovery writing* in literature, this account of exploratory interactions with language is expressed in Forster's (1927) famous question: "How can I tell what I think until I see what I say?".

Beyond maestro performances, co-action can be seen as a theme present within many forms of learning. Indeed, the process of coming to operate fluently and effectively with tools and symbols is common to all learners as they appropriate the practices and "habits of mind" of a discipline. In a way that invites suggestive analogies with phylogenetic processes, the human mind (and indeed the human brain) re-forms itself to accommodate these new discipline-specific ways of operating. For instance, Donald (2001, p. 302) has argued that literacy skills transform the functional architecture of the brain and have a profound impact on how literate people perform their cognitive work. The complex neural components of a literate vocabulary, Donald explains, have to be built into the brain through years of schooling to rewire the functional organization of our thinking. Similar processes take place when we appropriate numbers at school. It is easy to multiply 7 by 8 without representational supports, but if we want to multiply 12,345 by 78,654 then we write the numbers down and follow the specific rules of the multiplication algorithm. It is because we have been able to internalize reading, writing, and the decimal system, that we are able to perform the corresponding operations with an understanding of their meaning.

19.2.4 Democratizing Access to Co-action

Nevertheless, the kind of rich and generative interplay between mind, tool, and symbolic system that we see with Du Pré and Feynman have historically been accessible only to the maestros of a discipline. A key question for the design of technology-enhanced learning environments is whether the cognitive tools that have been developed in the last 30 years might play a role in democratizing access to this generative mode of interacting with disciplinary structures.

If the most sophisticated users of representations and symbolic systems in the past have been able to engage in active and creative interplay with these systems, it is in part because they were able to establish and sustain a dynamic relationship between their thinking and inquiry on one hand and the symbolic system on the other. This is possible because they have internalized the system so thoroughly that they are able to mentally simulate it as a dynamic field of potential, enabling them to engage in "what-if" interactions of an exploratory, conversational nature. In mathematics, this ability is particularly powerful, because of the dependence on representations that we have described above.

We will describe several classes of technology environments that provide dynamic and/or socially-distributed interfaces to representational systems that are fundamental to mathematics. These environments offer the potential for learners (even very young learners) to enter into a relationship with those systems, which we describe as co-action. We argue that the experience of relationships of co-action with mathematical structures can contribute to a transformative educational program. Of course, we do not argue that a technology that opens a possibility for co-action is sufficient in itself to give learners access to mathematical understandings that were the hard-won rewards of a lifetime of study for mathematicians of the past. However, we do suggest that carefully planned educational experiences within such environments can remove barriers to broader participation in a culture of mathematical literacy and fluency.

Extreme care is necessary here, as the long history of teaching and learning with static representations should not be ignored in the work to envision its future successor. Instead, we must proceed by pondering how digital and socially distributed representations of mathematical entities can contribute in new ways to genuine mathematical understanding. We see digital and shared representations as capable of adding dimensions to static representational systems and further improving the cycle of exploration, conjecture, explanation, and justification. Moreover, as educational systems incorporate such environments and experiences, traditional pathways of learning will gradually give way to new cultural and institutional structures that realize the potential of these innovations. In the sections below, we give two brief examples of co-action supported by new technologies: one emphasizing dynamic representations, and the other highlighting socially distributed representations.

19.2.5 Co-action with Dynamic Digital Representations

Consider the family of triangles ABC (see Fig. 19.1) whose side AC contains a given point P in the interior of angle B. The particular triangle in which A and C are chosen so that P is the midpoint of side AC has the least area among all possible triangles.

We explored this situation with teachers, making use of a dynamic geometry environment (in this case, GeoGebra). Beginning from triangle ABC, the teachers built a construction that allowed them to vary a point H along the side BA, thus determining a point D on BC for which triangle HBD included point P. Experimenting with the diagram and watching a readout of the area measure, they began to believe that the proposition about minimum area was true. Nevertheless, significant doubt remained. Following the logic of their construction, the teachers then extended the aspect of their sketch that hinged on the dependency relation between point H and area. They used the length BH as the domain of a function that at each point delivers the area of the corresponding triangle (Fig. 19.1b). Of course we could have—and we did—graph the function using a traditional coordinate system as well. But we show the hybrid Euclidean/Cartesian construction that emerged because we want to emphasize the possibilities that digital media offer learners, in allowing them to manipulate the objects under study. Such environments enable a wide range of interactions that support learners in exploring and building conjectures.

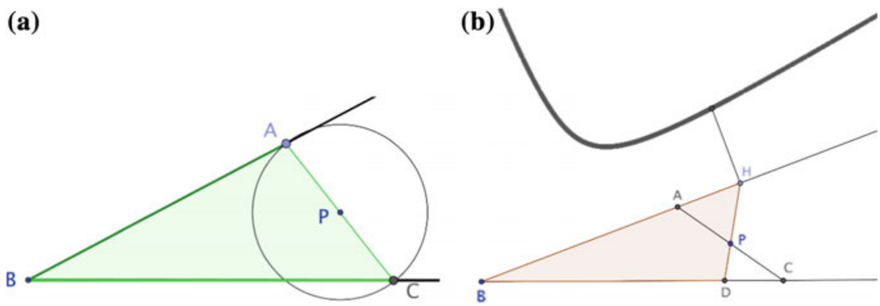


Fig. 19.1 **a** Finding the triangle with the least area. **b** Introducing a graphical representation of the value of the area changing with placements of vertices along rays BA and BC

The interaction between learners and dynamic geometry environments can be theoretically addressed in terms of the complex process Rabardel (1995) studied under the name *instrumental genesis*, which casts light on the mutually defining relationships between a learner and the artifact she is trying to incorporate into her strategies for solving problems. Initially the learner feels the resistance that the artifact opposes, but eventually she can drive it. In the case of GeoGebra, teachers needed to understand, in particular, the syntactical rules inherent to the software in order to use the medium as a mediator of mathematical knowledge. For this to happen, there must be a melody to be played—that is, teachers need an appropriate mathematical task. This task acts as an incentive to integrate in meaningful ways the dynamic power of the symbolic artifact with their own intellectual resources. If this happens, we say with Rabardel, that the artifact has become an *instrument* and the activity for solving problems in partnership with it, becomes an instrumented activity.

In such activities, the mobility of the dynamic digital representation becomes a crucial feature of the represented entity for the learner. Exploring what remains invariant under dragging, for instance, reveals structural aspects of mathematical objects: motion and invariance combine to enable us to see structure. Importantly, too, the motion is induced by the learner, who takes advantage of the executability of the digital representation to reveal structure and meaning. Perceiving structure through motion is a deeply embodied act—similar to how the bird sees the moth as the latter moves about on the bark of the tree. These features, absent from static symbolic representations, help the learner to develop new strategies as she explores mathematical problems. Moreover, they are particularly important for the mathematics taught at upper school levels, supporting a focus on variables and functions. The digital representation here becomes a semiotic mediator—that is, an artifact that supports the creation of meaning within the mathematical system and among its objects. Because the interaction depends on the particular learner's ways of thinking, there is also a strong social dimension to this co-action. The learner makes sense in the context of others, and also through others—co-acting together.

19.2.6 Co-action with Socially-Distributed Representations

Even apparently individual co-action becomes social as learners work together to process the meaning of representations. However, the social dimension can become even more pronounced in learning environments that promote collective work with distributed representations. Our second example of co-action involves students interacting collaboratively with the representation and communication infrastructure (Hegedus & Moreno-Armella, 2009) of a classroom network of graphing calculators.

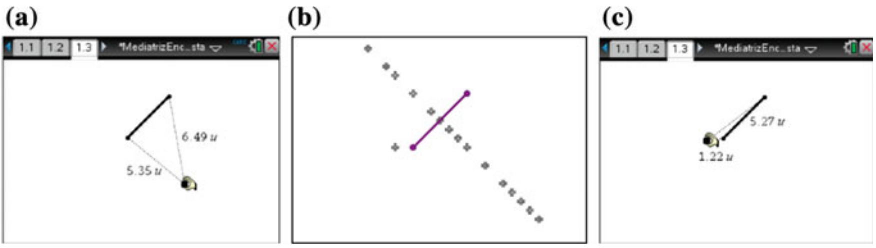


Fig. 19.2 **a**, **c** Students search for points for which the two variable sides of the triangle are of equal length (i.e., which are equidistant from the endpoints of the segment shown in bold). **b** The perpendicular bisector emerges as the locus of such points in the shared space

Within that setting, we can give each student control of a single point in a Cartesian environment, which she can move using the calculator's arrow keys. In real time, the points of all the students in the class are displayed in a shared Cartesian space, which is projected at the front of the classroom. The following activity was created by a teacher to enable his class to encounter the perpendicular bisector of a segment as the locus of points equidistant from the segment's endpoints. As students move their point (point C), they see it represented on their calculator screen as the third vertex of a triangle with the segment AB as its opposite side, where the measures of the variable sides of the triangle are also shown (Fig. 19.2a, c). The teacher asks the class to search for points where the distances from point C to points A and B are the same.[2] As students locate points that satisfy the condition, a pattern emerges in the shared space, indicating the perpendicular bisector of AB as a locus of points, with ever-increasing clarity (Fig. 19.2b).

Of course, a dynamic geometry environment can provide this representation on an individual's screen. However, the socially distributed nature of the locus of points in this activity provides an important experience and tool for thinking for the classroom group. As individuals, they have "felt their way around" the Cartesian space, searching for points that meet the *equidistance* criterion. On finding one, they recognize an isosceles triangle and experience a particular sensation of symmetry. However, based on their own point-based explorations, they can see each of the points in the shared space as solutions to a local problem. This supports a deep and flexible way of thinking about the locus of points and the perpendicular bisector, which has value beyond that which would be gained from the individual experience of a dynamic geometry environment alone.

[2]If the class contains fewer than 25 or so students, this activity can be modified to allow students to mark or stamp their point at two or more locations that satisfy the condition.

19.3 Conclusion: Mathematical Cyborgs

In speaking of mediated action, we have suggested that human cultures are constituted and extended through the creative production of cognitive and symbolic tools. These tools express ways of being in the world, and once internalized, they transform how people perceive and conceive of their worlds. Thus, humans are essentially cyborgs: biological beings who express themselves through tools. In particular, we are already behaving as cyborgs when we engage even in "traditional" mathematical thinking, leveraging the power of Arabic numerals, of the Cartesian system, and so forth.

But in this article we have emphasized the power and importance of dynamic and distributed representations to support new ways of learning how to think and operate with the symbolic entities of mathematics. In the classroom, co-action and the integration of artifact + learner, open the potential to democratize access to the powerful ways of operating with representations that characterize disciplinary ways of knowing. Instrumental genesis, we argue, should be a keystone in the design of new digital curricula that take full advantage of these opportunities. International efforts show ample evidence that this process has already begun. However, school cultures are expressed through institutional forms that have developed over centuries and that are not prepared to adapt nimbly to the rapid changes characteristic of new technologies. Engaging in the mathematics of co-action will require, and produce, a gradual but permanent re-orientation of classroom and school practices, and of the cognitive and epistemological assumptions that underlie them. Our argument here is that as members of a society in which mediated action is deeply entrenched and constitutive, humans are always-already cyborgs. Thus, the question is not *whether* to involve learners in symbiotic relations with technologies, but rather *which* technologies to choose for which purposes, and *how* to integrate them, so as to maximize all students' agency.

References

Baldwin, M. (1896). A new factor of evolution. *The American Naturalist, 30*, 441–451, 536–553.

Cole, M. (1996). *Cultural psychology: A once & future discipline*. Cambridge, MA: Harvard University Press.

Donald, M. (2001). *A mind so rare: The evolution of human consciousness*. New York: Norton.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3–26). Cuernavaca: Mexico.

Forster, E. M. (1927). *Aspects of the novel*. London: Edward Arnold.

Gibson, J. J. (2014). *The ecological approach to visual perception*. New York: Psychology Press.

Gleick, J. (1993). *Genius: Richard Feynman and modern physics*. New York: Pantheon.

Hegedus, S. J., & Moreno-Armella, L. (2009). Intersecting representation and communication infrastructures. *ZDM, 41*(4), 399–412.

Luria, A. R. (1928). The problem of the cultural development of the child. *Genetic Psychology, 35,* 493–506.

Moreno-Armella, L., & Hegedus, S. J. (2009). Co-action with digital technologies. *ZDM, 41*(4), 505–519.

Rabardel, P. (1995). *Les hommes et les technologies: Approche cognitive des instruments contemporains*. Paris: Armand Colin.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Wertsch, J. V. (1991). *Voices of the mind: A sociological approach to mediated action*. Cambridge, MA: Harvard University Press.

Chapter 20
Recursive Exploration Space for Concepts in Linear Algebra

Ana Donevska-Todorova

Abstract The complexity of the role of digital media in facilitation of learning mathematics can be approached by utilizing multiple theoretical frameworks. In this chapter, three theoretical frameworks have been applied with an aim to analyze the contribution of a created Dynamic Geometry Environment in developing deep understanding of concepts in linear algebra. The first one, which is in the main focus, refers to the integration of different description and thinking modes in linear algebra, such as synthetic-geometric, arithmetic and analytic-structural (Hillel in On the teaching of linear algebra. Springer, Netherlands, pp. 191–207, 2000; Sierpinska in On the teaching of linear algebra. Springer, Netherlands, pp. 209–246, 2000). The second one is related to the attributes of Dynamic Technological Environments, such as Recursive Exploration Space (Hegedus et al. in Proceedings of CERME5, WG 9. Tools and technologies in mathematical didactics 1331, pp. 1419–1428, 2007); and the third one is semiotic mediation (Bussi and Mariotti in Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective handbook of international research in mathematics education. New York, pp. 746–783, 2008) of the dragging tool. A landscape of networking strategies for connecting theories (Prediger et al. in ZDM Math Educ 40 (2):165–178, 2008) has been exploited as an attempt to ensure quality of the analysis.

Keywords Linear algebra · Dynamic geometry environment · Recursive exploration space · Thinking modes · Semiotic mediation · Networking strategies · Axiomatic properties

A. Donevska-Todorova (✉)
Humboldt-Universität zu Berlin, Berlin, Germany
e-mail: todorova@math.hu-berlin.de

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_20

20.1 Introduction

This paper represents a continuation of previous work that has been made available to the audience of the 13th International Congress on Mathematical Education in Hamburg (see Donevska-Todorova, 2016). The initial work responded to the Call of the Topic Study Group 43 for investigating cognitive and epistemological perspectives of technological utilization in mathematics education. It focused on mathematical cognition in terms of three kinds of mathematical thoughts, which characterize the learning of linear algebra and students' difficulties regarding such diversity of the thinking modes specific to the mathematical content, in particular to the axiomatic properties. Those elaborations have opened further research questions if and how exactly technologies could contribute to activating and connecting modes of thinking of concepts in linear algebra. To what extent are Dynamic Geometry Systems (DGSs) beneficial towards achieving transparency of the coherence between the thinking modes? In order to approach these questions, this paper discusses the suggested Recursive Exploration Space (RES) about the dot product of vectors further by using an additional theoretical framework about semiotic mediation. Combining the theories, about RES and semiotic mediation allows for an insightful analysis of the benefits of the DGE usage in facilitating integration of the three thinking modes.

20.2 Theoretical Background

Among many cognitive activities in mathematics education, mathematical thinking is a complex phenomenon described by diverse theories. For example, it is viewed as a "kind of thinking in advanced mathematics" (Harel & Sowder, 2005, p. 27) or "an important component of general problem-solving skills" (Bransford, Sherwood, Hasselbring, Kinzer, & Williams, 1990, p. 126) or "thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses" (Edwards, Dubinsky, & McDonald, 2005, p. 15). This paper does not discuss a precise definition of mathematical thinking (e.g., Tall, 1991), rather some of its aspects which are specific for the content of linear algebra. Discussions regarding the ways in which educational technologies can promote mathematical thinking have started almost three decades ago (e.g., Pea, 1987) but are still actively going on. As advanced mathematical thinking supported by technologies is often being brought into context of representational issues, I refer to the theories of Hillel (2000) and Sierpinska (2000) about three different modes of description and thinking in linear algebra.

20.2.1 Three Modes of Thinking at Tertiary and Secondary Level Linear Algebra

Corresponding to the Hillel's (2000) three modes of description and language: abstract, algebraic and geometric, Sierpinska (2000) distinguishes three *modes of thinking in linear algebra*: *analytic-structural, arithmetic and synthetic-geometric*. These three modes, however, do not exist in isolation of each other. Rather, the arithmetic and the synthetic-geometric coexist and could be perceived as nested within the analytic-structural mode (Donevska-Todorova & Steward, 2017, to appear). The modes and their interpretations, distinctions and translations from one into another are already discussed through concepts as vectors, dot product of vectors and determinants at two levels of education from both theoretical (e.g. Donevska-Todorova, 2012, 2014; Filler & Donevska-Todorova, 2012) and empirical perspectives (e.g. Donevska-Todorova, 2015, 2016). Advanced courses in linear algebra, group theory and abstract algebra characterize with the *analytic-structural mode of thinking* of mathematical concepts for the reason that axiomatic definitions of mathematical concepts constitute the formalism of their theories. Unlike the abstract algebras, a typical linear algebra course in school does not use the abstract mode, rather the *geometric* and the *arithmetic* modes of description. At upper-secondary education, these two modes are often kept distant one from another, although they represent the dual nature of a single concept, and this distance is even bigger in lower-secondary mathematics. The geometric mode, further on, can be coordinate-geometric and synthetic-geometric, each with its own strengths and limitations. The abstract mode of description is rarely considered at high school level of education and if so, is usually reduced to axiomatic properties, which are not used for defining mathematical concepts in axiomatic structures, but for their descriptions instead.

20.2.2 Upper High School Students' Difficulties with Axiomatic Properties of Concepts in Linear Algebra

The inverse treatments of the axiomatic properties for defining and applying usage at tertiary and secondary educational level respectively, point out the awareness of possible obstacles for learning. In continuation, I name two affirmed students' difficulties with the axiomatic properties of some concepts in linear algebra.

(1) On the base of the validity of one property, for example *commutativity*, high school students may immediately conclude validity of another property, for example *associativity*, or the other way around. It is not straightforward for a high school student to notice that associativity does not necessarily imply

commutativity of an operation. Indeed, many groups are non-abelian. This may be illustrated by the *cross product of vectors*, an operation that is associative and anti-commutative.

(2) On the basis of the axiomatic properties of one concept students may derive false conclusions for the validity of the same properties for another concept. For example, associativity of the *dot product of two vectors* (or scalar product or inner product in the context of Euclidean space), does not imply associativity of their cross product. Neither holds the opposite statement.

I suspect that both difficulties may further on lead to problematic acquisition of other properties of the same concepts, for example, *distributive properties* of the dot and cross products of vectors (with respect to vector addition), or of other concepts, such as the *scalar triple product of vectors*. Similar warnings are emphasized by Dray and Manogue, who "strongly discourage teaching the dot and cross products at the same time—students tend to get them mixed up!" (Dray & Manogue, 2008, p. 9).

These deficits are a result of various reasons. First, they may originate in the *lower-secondary* mathematics. Due to a longer experience and confidence with operations with numbers in comparison to operations with vectors, students may derive incorrect analogies between the properties of the operations addition and multiplication with real numbers and the properties of the mentioned operations with vectors. This means that they may primary apply the arithmetic mode and fail to link it to the geometric mode of thinking. A second reason may arise from the way that new concepts are introduced in the *upper-secondary* mathematics. Dray and Manogue argue that it should not be done with algebraic formulas, but with geometric definitions that are coordinate independent (Dray & Manogue, 2008, p. 1), although it is unclear if this suggestion refers exclusively to university or also high school students.

20.3 Research Question

The analysis above allows me to pose the following research question:

How can a Dynamic Geometry Environment (DGE) mediate an integration of all three modes of thinking (and therefore facilitate in overcoming some students' difficulties regarding axiomatic properties of concepts in linear algebra) on the basis of its attributes as RES?

The research question is not a trivial one from several aspects. First, it has a content-specific focus on mathematical knowledge, second, it relates the domain of cognition in learning, and third, it refers to the use of technologies in mathematics education. Therefore, in the beginning of the next section, I describe a hypothetical approach regarding the dot product of vectors pointing out the attributes of a DGE and instruments for semiotic mediation. Then, I illustrate it by a concrete example.

20.4 Integration of Thinking Modes in a Dynamic Geometry Environment

When students are being asked to find the dot product of two vectors in a paper-pencil environment and end-up with a resulting scalar, its *geometric* interpretation is the next quite challenging cognitive task. The same problem in a DGE might undergo the following reasoning sequence. Students draw both vectors, but realize that the resulting scalar does not appear on the geometry-window. Then, they open the CAS-window. The vectors appear in their component form, i.e. in *algebraic* mode of description. They try to link the geometric and the algebraic representations by dragging points in the geometry-window. Further on they explore how components of vectors in the CAS-window change and if the resulting scalar will appear on the geometry-window. Since it does not, they may try to zoom in or out. Yet again, the scalar does not appear. They search for the reasons and think what that could mean. Such a sequence, guided by the DGE, exemplifies a *Recursive Exploration Space* (Hegedus, Dalton, & Moreno-Armella, 2007). Namely, students' actions in searching for connections between the geometric and the algebraic modes by the dragging or the zooming tools receive responses from the DGE (constant absence of the scalar on the geometry-window). Consequently, *action-reaction loops* are formed. In this way, the DGE stimulates a distribution of cognition on more than just one thinking mode and challenges further students' engagements with the digital tools. The thinking sequence might proceed in the following way. The students activate the option "Show" from the Main Tool Bar (previously being fixed to "Hide"). Then, a rectangle occurs in the geometry-window (e.g., see Figs. 20.1 and 20.2 in Donevska-Todorova, 2015, pp. 200, 201). In this moment the "*cognitive distance* between the students and the problem diminishes" (Hegedus et al., 2007, p. 1422). Manual dragging and mindful reflecting procedures repeat more frequently and lead to refined and precisely focused examinations. Namely, the students explore the rectangle and the lengths of its sides in connection to projections of vectors. They discover how the area of the rectangle (*geometric mode*) is related to the resulting scalar (*algebraic mode*) of the dot product and conclude that the absolute value of the scalar equals the area of the rectangle. The coherence between the algebraic and the geometric thinking modes becomes transparent now. After students realize and grasp these connections, they may proceed investigating *axiomatic* properties of the dot product, e.g., the *commutative property,* also called *symmetry* at university (related to obstacles (1) and (2) stated above). They enter the vectors in the opposite order and receive feedback from the DGE with the same resulting scalar. Simultaneously, the geometry-window shows a new rectangle. The lengths of its sides are now different, but its area remains unchanged. Finally, students translate this geometric interpretation into algebraic and derive analogies between the commutativity of multiplication of numbers and of the dot product. This means that the students and the DGE exchange their roles in leading the process of integrating *the three thinking modes* of the concept.

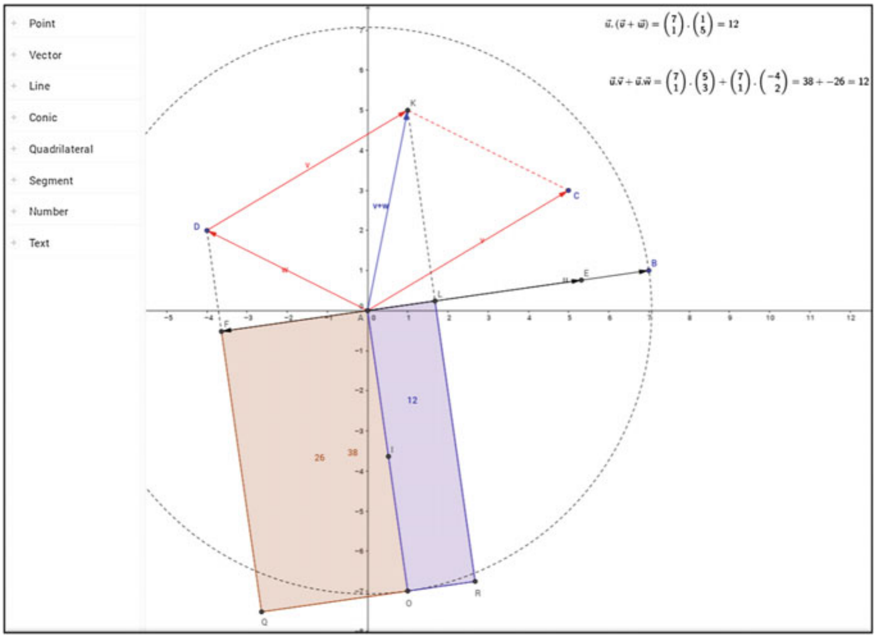


Fig. 20.1 Additive (distributive) property of the dot product

20.5 An Example of a Recursive Exploration Space (RES)

In this section, I aim to exemplify a hypothetical reasoning sequence in a RES, similar to the above, for exploring another property, namely the *additive property of the dot product over vector addition*. The additive and the homogeneity properties form the bi-linearity property that is one of the three axiomatic properties for defining the dot product in the usual university approach (see Donevska-Todorova, 2016). The additive property of the dot product over vector addition is also called a *distributive property of the dot product over vector addition*, which is a designation also used to point out high school students' difficulties in the third section of this article. In order to achieve the aim, I present a dynamic applet, which was created in GeoGebra (Fig. 20.1). The applet shows a simultaneous presence of both the geometric and the algebraic mode of description in the geometry-window, although the algebraic data are also viewable and changeable in the CAS-window. It has three draggable points B, C and D i.e. the terminal points of the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, respectively. Under the dragging modalities described by Arzarello, Olivero, Paola, and Robutti (2002), students may discover invariant properties, for example, that the geometric figures are always rectangles and the area of one of them equals the sum of the areas the other two. Recognition of such invariant properties contributes in "understanding of the underlying abstract mathematical concept" (Leung, 2008, p. 136). Such a created RES has *continuous dynamic* (as much as the software allows it), in comparison to *static inert, static kinesthetic/aesthetic or static computational* features (Moreno-Armella, Hegedus, & Kaput, 2008). They may be changed to *discrete dynamic* features, by implementing integers, instead of decimal numbers, as entries of the components of the vectors or the coordinates of the points. This adaptation depends on the focus group, for example university or high school students, and may vary according to their capabilities or needs.

Support of the geometric mode of thinking in the RES

In the geometric mode of description, the dot product represents an *oriented area* of the rectangle spanned by the vectors $\vec{u}$ and $\vec{v}_{\vec{u}}$—the projection of $\vec{v}$ over $\vec{u}$ (and analogically, $\vec{u}$ and $\vec{w}_{\vec{u}}$—the projection of $\vec{w}$ over $\vec{u}$). With the usage of this applet, students have the possibility to explore the "positive area" when the vectors $\vec{u}$ and $\vec{v}_{\vec{u}}$ have the same direction and orientation (e.g. $\vec{u} \cdot \vec{v} = 38$ in Fig. 20.1) or "negative area" when the vectors $\vec{u}$ and $\vec{w}_{\vec{u}}$ have the same direction but opposite orientation (e.g., $\vec{u} \cdot \vec{w} = -26$ in Fig. 20.1).

By students' *actions* of changing the magnitude, the direction or the orientation of one, or more, of the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ while dragging their terminal points in the geometry-window, the applet simultaneously shows new lengths of the sides and areas of the corresponding rectangles. At the same time, new results of the dot products, which are displaced directly on the corresponding rectangles by matching colors, appear. These responses of the dynamic software in fact represent *reactions* of the RES. Even a single such co-action loop, further on, stimulates students' novel actions immediately followed by reactions of the created applet. As the frequency of these *action-reaction loops* is increasing, the time-intervals between the loops are decreasing. These exchanges proceed until the student ends up with the desired discovery.

Support of the arithmetic mode of thinking in the RES

The expressions in the upper right corner of the applet (Fig. 20.1), where vectors are given with their component form, aim to support the arithmetic mode of thinking of the dot product. Namely, the calculations are embedded in the applet, so that students do not necessarily have to compute. This allows students' activities focusing on the modes rather than distraction of learning processes by time-consuming calculations, in particular with decimals.

An alternative to using this algebraic mode of description of the dot product would be an opening of the field "Point" in the CAS-window (upper left corner in Fig. 20.1) and entering the coordinates of the points $B(7, 1)$, $C(5, 3)$ and $D(-4, 2)$ (terminal points of the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, respectively). Such *action*—a manual entrance of the coordinates of the pints in the CAS-window would also be followed by a *reaction*—a display of the result of the dot product in the upper right corner of the geometry-window. Here, I note that the support of both the geometric and the arithmetic modes of thinking do not necessarily have to be considered by the separation into split windows. Nevertheless, this distinction is made in order to show that this possibility exists.

Support of the analytic-structural mode of thinking in the RES

The dot product of vectors is defined by three axioms: bi-linearity (additivity and homogeneity), symmetry and positivity. The designed applet supports primarily the additive (distributive) property of the dot product over vector addition. In addition to this one, the rest of the axiomatic properties can also be examined with the applet. However, there are similar applets in which they are more transparent (e.g. see Donevska-Todorova, 2015).

What is essential regarding the students' difficulties is that, the distributive property of the *dot product* over vector addition substantially differs, and should not be confused, with the distributive property of the *scalar multiplication* over vector addition. The first operation results in a scalar and the second one in a vector. In my opinion, this differentiation is made viewable by the suggested DGE by connecting the resulting scalar with oriented areas.

20.5.1 Semiotic Potential of the Artifact

The specification of the three modes of thinking above does not aim at classification of different segments of activities rather at an epistemological insight of the concept and easier description of the prospective of the created artifact. The modes of thinking do not appear according to a certain recipe but as an interplay between the modes of description enabled by the artifact. This can be explained further as the following.

Required students' cognitive efforts to accomplish the task of finding the dot product by using the created GeoGebra file on the one hand, and interpreting its geometric meaning by exchanging the modes of description implemented in the artifact on the other hand, clarify *the semiotic potential of the artifact* (Bussi & Mariotti, 2008, p. 754). This twofold semiotic relationship for crystalizing the meaning of the operation dot product result carried out with the use of the RES, guided through direct *activities with the artifact* (e.g. dragging points in the geometry-window), may stimulate *individual* and *collective production of signs* (Bussi & Mariotti, 2008, pp. 754–755), like drawings of the emerging rectangles and assigning corresponding areas. *Artifact signs*, as a category, such as those appearing in the CAS or the geometry-window, initiate occurrence of *mathematical signs*, as another category (Bussi & Mariotti, 2008, pp. 756–757), for expressing the additive (distributive) property of the dot product in upper high school mathematics context. This means that the designed DGE plays a dual role, one as a mean for undertaking a concrete mathematical task and two, as a tool of semiotic mediation to achieve a didactical objective (translating between multiple representations). In such didactical cycle, the teacher acts as a mediator and guides the students towards gaining mathematical knowledge.

20.6 Discussion

As long as we continue to treat the mathematical concepts without their (axiomatic) properties in school, they continue to occur as separate mathematical objects in the students' minds. Technologies may enable us to start treating them as constituents of a single concept, whose existence is impossible if even one property is omitted. If research has been so far treating the contributions of technologies in supporting geometric and algebraic representations it is now the time to more seriously consider their role in supporting the teaching and learning of the third mode of the triple nature of the mathematical concepts, namely the analytic-structural. The RES, in this sense, is more than an infrastructure for embodied actions, more than a technical mediator. It is rather a collaborator in co-actions which allow self-control of the user. The enactive interface of the DGE offers possibilities for rich interactions and activities. The dependency between the *independent*-movable (B, C and D) and the *dependent*-constructed (all other) points allow the students to experience *invariant properties*, e.g. additive (distributive) property of the dot product. The roles of the dragging and the zoom tools in connecting the modes of thinking described above show that these tools might be perceived as instruments of semiotic mediation.

The constrains of this article do not allow for an in-depth analysis of the connections of the created DGE, because of the attributes of being a recursive explorations space with embodied dynamic and executable representations, with the three embodied, symbolic-procedural and formal-axiomatic worlds of Tall (2003, 2004). However, I consider this information worth mentioning for possible further investigations.

The networking of both theoretical frameworks (RES and Semiotic mediation) used in this paper in the sense of Prediger, Bikner-Ahsbahs, and Arzarello (2008), enables understanding of the way DGE may facilitate integration of the thinking modes. They were applied to compare, combine and coordinate information about the epistemic values of variation of the Dragging tool from two perspectives. Both theories have been integrated locally in order to deeply "explain the contribution of the tool mediated action to concept formation" (Falcade, Laborde, & Mariotti, 2007, p. 321), in this case the concept of the dot product. It offers an increased level of explanatory and descriptive data, which may further contribute to development of empirical studies.

20.7 Conclusions

This chapter discusses the growth of mathematical cognition in students at three different levels of education with a focus on thinking modes of linear algebra concepts. The *arithmetical* cognition of the commutative or the distributive property of the dot product of vectors (in lower secondary level) have a major impact on the way other, *geometric* (in upper secondary level) and *abstract* (in tertiary level), cognition proceeds. I find the role of technology in coherence with attributes of a dynamic geometry environment (DGE) being a *Recursive Exploration Space* for *semiotic mediation and integration of three modes of description and thinking*. Executable multiple dynamic characteristics of a technology-rich environment in contrast to single static representations on paper/board mediate and participate in development of mathematical cognition for the reason that students co-act with the environment by exchanging their roles in switching from one into another mode of description and thinking. I summarize with a suggestion for a modification (though not negation) of the paper, as a *frozen* (Hegedus et al., 2007), to the DGE as a *fluid medium* for thinking of mathematical concepts.

References

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri environments. *Zentralblatt für Didaktik der Mathematik, 34*(3), 66–72.

Bransford, J. D., Sherwood, R. D., Hasselbring, T. S., Kinzer, C. K., & Williams, S. M. (1990). Anchored instruction: Why we need it and how technology can help. *Cognition, Education, and Multimedia: Exploring Ideas in High Technology*, 115–141.

Bussi, M. B., & Mariotti, M. A. (2008). *Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective* (pp. 746–783). New York: Handbook of International Research in Mathematics Education.

Donevska-Todorova, A. (2012). Connections between secondary and tertiary curricula for linear algebra with focus on the concept of a determinant—Proposal with technology support. In A. Filler & M. Ludwig (Hrsg.), *Vernetzungen und Anwendungen im Geometrieunterricht. Ziele und Visionen 2020* (Vol. 28, pp. 109–120).

Donevska-Todorova, A. (2014). Three modes of description and thinking of linear algebra concepts at upper secondary education. In J. Roth & J. Ames (Eds.), *Beiträge zum Mathematikunterricht 2014 48*(1), 305–308.

Donevska-Todorova, A. (2015). Conceptual understanding of dot product of vectors in a dynamic geometry environment. *The Electronic Journal of Mathematics and Technology, 9*(3).

Donevska-Todorova, A. (2016). Procedural and conceptual understanding in undergraduate linear algebra. In K. Krainer & N. Vondrova (Eds.), *Proceedings INDRUM 2016* (accepted for publishing).

Donevska-Todorova, A., & Steward, S. (2017, to appear). Teaching and learning linear algebra with digital resources. In S. Steward, C. Andrews-Larson, A. Berman & M. Zandieh (Eds.), *Challenges and strategies in teaching linear algebra*. Springer.

Dray, T., & Manogue, C. A. (2008). The geometry of the dot and cross products. *AMC, 10,* 12.

Edwards, B. S., Dubinsky, E., & McDonald, M. A. (2005). Advanced mathematical thinking. *Mathematical Thinking and Learning, 7*(1), 15–25.

Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics, 66*(3), 317–333.

Filler, A., & Donevska-Todorova, A. (2012). Der Vektorbegriff. Verschiedene Wege zu seiner Einführung. *Mathematik Lehren, 172*(6), 47–51.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning, 7*(1), 27–50.

Hegedus, S., Dalton, S., & Moreno-Armella, L. (2007). Technology that mediates and participates in mathematical cognition. In *Proceedings of CERME5, WG 9. Tools and Technologies in Mathematical Didactics 1331*, 1419–1428.

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In *On the teaching of linear algebra* (pp. 191–207). Netherlands: Springer.

Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning, 13*(2), 135–157.

Moreno-Armella, L., Hegedus, S. J., & Kaput, J. J. (2008). From static to dynamic mathematics: Historical and representational perspectives. *Educational Studies in Mathematics, 68*(2), 99–111.

Pea, R. D. (1987). Cognitive technologies for mathematics education. *Cognitive Science and Mathematics Education*, 89–122.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education, 40*(2), 165–178.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In *On the teaching of linear algebra* (pp. 209–246). Netherlands: Springer.

Tall, D. (1991). *Advanced mathematical thinking* (Vol. 11). Germany: Springer Science & Business Media.

Tall, D. (2003). Using technology to support an embodied approach to learning concepts in mathematics. *Historia e tecnologia no Ensino da Matemática, 1,* 1–28.

Tall, D. (2004). Building theories: The three worlds of mathematics. *For the Learning of Mathematics, 24*(1), 29–32.

Chapter 21
GeoGebra as a Tool in Modelling Processes

Gilbert Greefrath and Hans-Stefan Siller

Abstract Applying digital technology in mathematical teaching is frequently cited as important and fundamental to the understanding-based learning of mathematical content. In this article, we study the extent to which the systematic application of the dynamic geometry software GeoGebra supports the competency "Mathematical Modelling". By giving students an application-oriented modelling problem to solve, modelling processes are analysed, assessed, and represented. By observing students at the 10th grade level with respect to a qualitative study hypotheses are formulated about applying a digital tool at different stages of the modelling cycle.

Keywords Technology · Digital tools · Computer · Qualitative empirical research

21.1 Introduction

Digital tools have much to offer in the context of mathematical teaching. They can be used in such a way so that certain fundamental conditions for technological applications are met (Barzel, 2012). For this purpose, Rögler (2014, p. 983) states: "These conditions include among others utilization within a student-centered and understanding-oriented teaching context, and the stimulation of conceptual knowledge".

In the German national educational standards (KMK, 2015, p. 13), the potential of these tools is emphasised, in particular

- when mathematical relationships are first encountered …;
- in order to facilitate the understanding of mathematical relationships, especially by using multiple possible representations;

G. Greefrath (✉)
University of Münster, Münster, Germany
e-mail: greefrath@wwu.de

H.-S. Siller
University of Würzburg, Würzburg, Germany

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_21

- in order to reduce schematic processes and to make it possible to consider larger datasets;
- to exploit opportunities for self-checking work.

Thus, major topics in mathematical modelling can be identified as within the scope of these tools, and the close relationship between digital tools and the competencies of mathematical modelling is obvious. There are multiple opportunities for integrating digital tools into a modelling cycle (Blum & Leiss, 2007; Siller & Greefrath, 2010) as shown in Fig. 21.1. Geiger (2011, p. 312) also describes an approach that allows technology to be inserted into different stages of the modelling cycle. Such support can be improved by making informed choices about the specific tools to use, including GeoGebra. In addition to specific integrated representations when using digital tools in the context of modelling, the literature also describes digital tools as an independent domain located between mathematical modelling and mathematical results (e.g. see Daher & Shahbari, 2015; Siller & Greefrath, 2010).

Accordingly we focus on a qualitative study which deals with the use of digital tools within the process of modelling. Therefore, we analyse two different categories of action—*using digital tools* and then using them in the various *steps of the modelling cycle.*

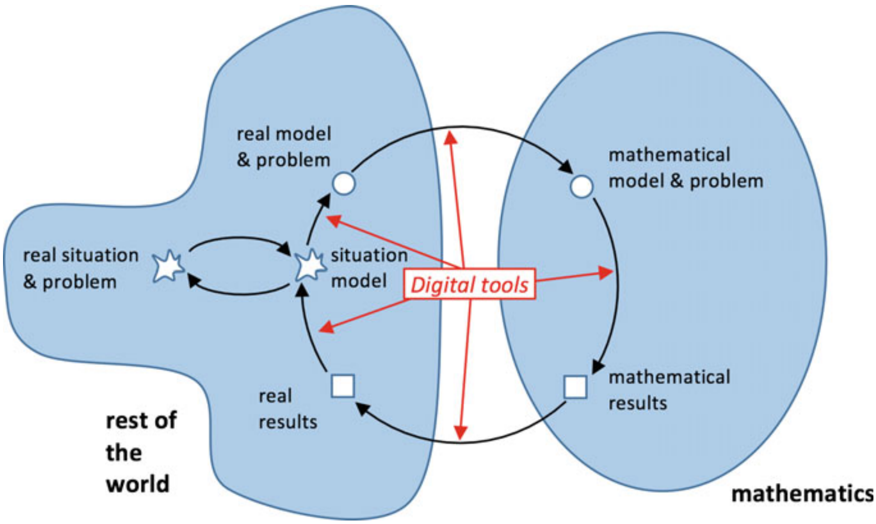


Fig. 21.1 Integration of digital tools into the modelling cycle based on Blum and Leiss (2007)

21.2 Theoretical Background

The concept of mathematical modelling has been discussed comprehensively as Blum (2015) has shown in his review article. In our understanding the concept of mathematical modelling can be understood as a schema, which consists of various elements that have to be addressed—as shown in Fig. 21.1. Starting either in or with a real situation, one is able to create a situation model through cognitive activity, and by idealising this situation model, a real model can be found which is fundamental in the modelling cycle. Here, one starts to translate the given real situation into the world of mathematics. By mathematising, a mathematical model is created which should be solved and re-interpreted, so that the results can be evaluated within the given situation. At first glance this cycle seems easy to handle, but looking at different research results, like Riebel (2010), we can see that especially the process of mathematising and (re-) interpreting, is in fact very difficult for students.

Observing different stages in the modelling cycle enables a sophisticated look at those stages. Because, in mathematics classrooms, the use of digital tools is increasing continuously, it makes sense look at those stages with the help of technology. In this context GeoGebra can take on a wide range of roles.

One of these roles is that of experimenting or exploring (Fahlgren & Brunström, 2014). In this context GeoGebra can serve as a tool for drawing and constructing in the classroom (Hohenwarter, Hohenwarter, Kreis, & Lavicza, 2008). Another application of GeoGebra is that of calculating numerical or algebraic results that would not be feasible for students in the given timeframe, without the help of digital tools (Siller & Greefrath, 2010). Other opportunities for application could include the self-checking of results obtained by other means. Furthermore, GeoGebra can be used for giving presentations.

Such technological tools are considered to promote and improve student understanding of the modelling context and in relation to modelling competencies, so that they exert a decisive impact on mathematical modelling (e.g. see Carreira, Amado, & Canário, 2013; Gallegos & Rivera, 2015).

GeoGebra supports several of the different modelling processes (e.g. Hall & Lingefjärd, 2016). Graphical representations are seen as relevant (Pead, Bill, & Muller, 2007) for understanding the mathematical content. In particular, the diversity of *opportunities for representation* (see also Moreno-Armella, Hegedus, & Kaput, 2008), for example by using the chart window and the spreadsheet window when working with function terms, and the *opportunities for self-checking* (see also Arzarello, Ferrara, & Robutti, 2012), for example by changing a point to control the geometric construction, can support modelling processes at various stages of the modelling cycle—as shown in Fig. 21.1.

Of course, the use of GeoGebra influences the process of mathematical modelling. This is also shown by Galbraith and Stillman (2006) or Confrey and Maloney (2007). In our opinion, this is a crucial point for understanding the connection between mathematical modelling and the use of digital tools, when we investigating by means of a qualitative study in order to raise first hypotheses.

21.3 Empirical Study

21.3.1 Research Questions and Instruments

On the basis of the theoretical background given above, the following research questions were investigated:

- What kinds of action do students undertake in practice when using GeoGebra for modelling?
- Where do these actions fit into the modelling cycle?

To study the modelling processes of students, an open, real-life situation described by Laakmann (2005, p. 86) was used:

> On a foggy November day, a patrol boat sets out from its safe harbor to hunt down pirates. As you can imagine, the conditions are not very favorable, and the visibility of the patrol boat is only 0.5 km. Nonetheless, the captain orders the patrol boat to sail north-east out of the harbor at a speed of 20 km/h. At the same time, a pirate boat is sailing south-east at 15 km/h. As the patrol boat leaves the harbor, the pirate ship is 8 km to the north and 2 km to the east of the harbor (Laakmann 2005, p. 86).

This problem of whether the patrol boat will catch the pirates can be solved in several ways by using the graphical view, the spreadsheet view or the CAS view in GeoGebra. Various mathematical models can be discussed by using those different options, so that different perspectives are raised and can subsequently be taken into consideration (see Siller & Greefrath, 2010). Specifically, we can observe the following differences in the mathematical model, depending on the chosen view of GeoGebra. In the chart window, students have to draw a chart or diagram of the given situation. In the example given above, students can draw two lines with points on them representing the two boats. It is possible to move the points automatically to find the minimum distance of the boats (see Fig. 21.2).

The CAS window allows the students to describe the situation for both ships, by using functions and through functional thinking. One possibility is to use a two-dimensional expression for the position, depending on the time, as used in analytic geometry. The students can then calculate the distance between the two boats. The minimum of the distance may then be determined by using the derivative (see Fig. 21.3). With the help of the spreadsheet window, students compute in a discrete manner with certain data, in order to solve the given problem. In the context of this task, students can calculate the position of each boat, depending on the time (see Fig. 21.4).

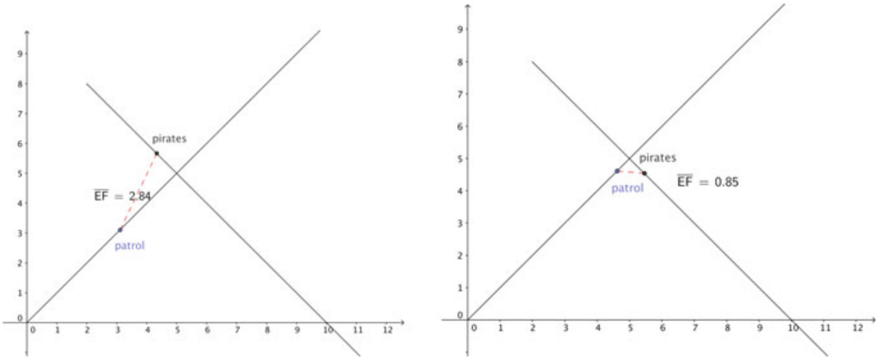


Fig. 21.2 Chart window of GeoGebra (two positions of the boats)

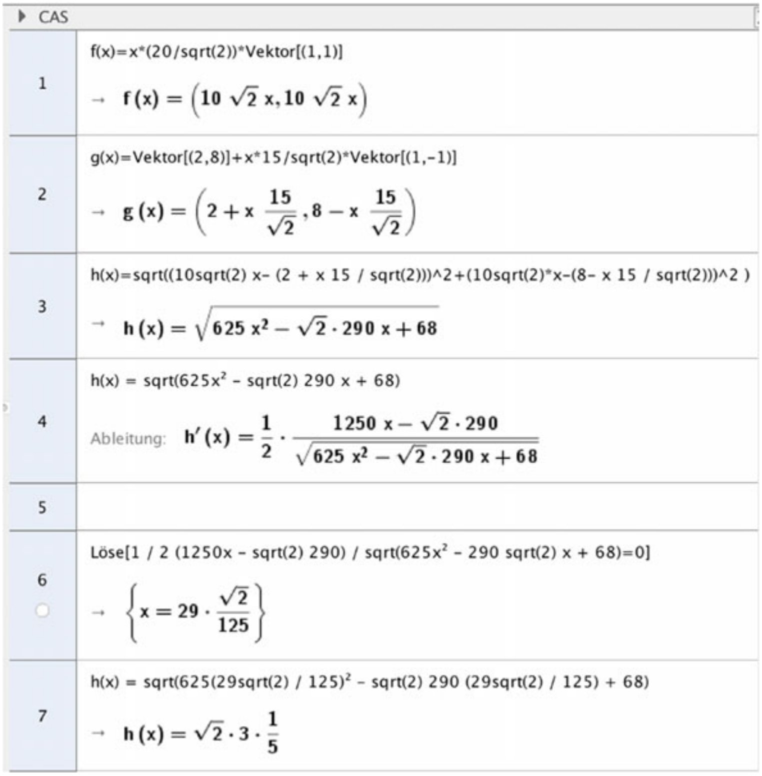


Fig. 21.3 CAS window of GeoGebra

time	x_coord_Pir	y_coord_Pir	x_coord_patrol	y_coord_patrol	distance
0,1	6,939	3,061	1,414	1,414	5,765
0,12	6,727	3,273	1,697	1,697	5,271
0,14	6,515	3,485	1,980	1,980	4,778
0,16	6,303	3,697	2,263	2,263	4,287
0,18	6,091	3,909	2,546	2,546	3,798
0,2	5,879	4,121	2,828	2,828	3,313
0,22	5,667	4,333	3,111	3,111	2,833
0,24	5,454	4,546	3,394	3,394	2,360
0,26	5,242	4,758	3,677	3,677	1,902
0,28	5,030	4,970	3,960	3,960	1,472
0,3	4,818	5,182	4,243	4,243	1,102
0,32	4,606	5,394	4,525	4,525	0,872
0,34	4,394	5,606	4,808	4,808	0,899
0,36	4,182	5,818	5,091	5,091	1,165
0,38	3,969	6,031	5,374	5,374	1,550
0,4	3,757	6,243	5,657	5,657	1,988
0,42	3,545	6,455	5,940	5,940	2,449

Fig. 21.4 Spreadsheet window

21.3.2 Random Sample and Study Implementation

For the presented study, students were chosen who were already experienced in working with the dynamic geometry software package Geogebra in the classroom. This qualitative study was performed in a Gymnasium (academic, university oriented high school) in Muenster (Germany) at 10th grade level. The mathematical background of the students corresponds with the grade. They could solve the selected task using the spreadsheet window or the chart window of GeoGebra. The CAS solution, with the aid of analytical geometry, is not to be expected on the basis of previous knowledge. Four pairs of students were observed as they solved the problem. The task was given to the students with the aim of solving the problem with GeoGebra, but they did not receive any further guidance or advice from a third party. The approach of each students pair was recorded on video.

21.3.3 Data Analysis

In order to assess the observations, the video footage was transcribed in full. The script includes all verbal comments and also the actions in GeoGebra. The script was then marked for assessment in multiple stages. During marking, each of the ideas suggested by the students was given a conceptual description. These descriptions were discussed and modified over the course of multiple revisions of the script (see Strauss & Corbin, 1990). The objective was to describe the students' modelling process independently of the given problem at hand, so that the categories thus obtained could be used as a basis for further study, enabling modelling phases to be compared in terms of these categories. For this purpose, categories were developed which do not depend on the specific task, but describe the processing generally. In a second stage, the categories were corrected and verified.

The following categories were among those identified from the collected material in connection with the application of digital tools (Vehring, 2012):

- Mathematical drawing: Drawing simple geometrical objects within a coordinate system
- Constructing: Drawing more complex geometrical objects and configurations, with the aid of intermediate steps
- Measuring: Determining the distance between points, the length of line segments, the magnitude of angles and the slopes of lines or segments
- Experimenting: Varying the parameters, conditions or assumptions of a sketch and observing the effects
- Calculating: Performing calculations using a physical or software-based calculator.

An example of a situation in which the students measure shows the following detail in the transcript:

C: That's ((points at G)) THE point; that's the ship.
T: We could just have a look if that's right. How about just measuring the length ((looks for a suitable tool, picks the tool "length and distance")). Must be somewhere here ((measures the distance between A and E and the distance between F and G)). From THIS point to THAT one it's 1.41. And from THAT point to that one it should be 1.1 times that. ((looks for his calculator in his backpack)) Wait, Something's wrong.

An example of a situation in which the students experimentation shows the following detail in the transcript:

C: We can just pick another way of calculating this for the time being. Let's have look how long that takes. Just do it. Let's just give it a try. Otherwise we'll never get there.
T: ((selects the tool "circle with centre and radius")) Here, thus with that radius ((points at F))
C: THAT one back to B. No need to draw a circle around here. Around F.

An example of a situation in which the students drawing shows the following detail in the transcript:

T: Northeast direction. So we could theoretically go Northeast ((puts the point C at the position (1.5/1.5) and draws the line AC [direction of the patrol boat]/moves point C and the line AC moves along accordingly/finally C has the coordinates C(4/4).)) So let's just add one more point and make a line now. Yes. Let's move the point now, you see until it Northeast. That should be four–four, theoretically.

Furthermore, the methodology included an analysis of two different categories of action—as mentioned above—*using digital tools* and *steps in the modelling cycle*. To identify the modelling processes within the modelling cycle, the partial competencies of modelling in the cycle described by Blum and Leiss (2007) were used as a basis for categories. Important partial competencies are constructing, simplifying, mathematising, working mathematically, interpreting and validating.

21.4 Results

By using the categories integrated into the modelling cycle, we were able to reconstruct the individual modelling processes of the pairs of students, and are able to formulate hypotheses concerning digital tools when working on modelling problems. In the following Table 21.1 we see a typical part of the modelling process of two students using GeoGebra.

The activities corresponding to each category in the application of digital tools were also established. In this way, the individual modelling path of each student pair, including the way they used GeoGebra, could be reconstructed for each of them. Overall, all of the categories listed above were identified in the modelling processes of the four pairs of students. However, these categories all occurred at different stages of the modelling cycle. Generally, we found that GeoGebra was only applied in the modelling cycle between the stages of real model, mathematical model, and mathematical results. However not all categories were present in each of these steps (see Table 21.2).

Constructing and drawing were identified between the real model and the mathematical model, whereas measuring, calculating and experimenting were identified between the mathematical model and the mathematical results.

Table 21.1 Typical modelling process using GeoGebra

Step in modelling cycle	Using digital tools
Real model—mathematical model	Drawing
Mathematical model—situation model	
Situation model—mathematical model	Drawing
Mathematical model—real model	Measuring
Real model—mathematical model	
Mathematical model—real model	Drawing
Real model—mathematical model	Constructing
Mathematical model—real model	
Real model—mathematical model	Constructing
Mathematical model—mathematical results	Experimenting

Table 21.2 Categories used in the presented study

Step in modelling cycle	Using digital tools
Constructing	
Simplifying	Experimenting
Mathematising	Constructing, drawing
Working mathematically	Calculating, measuring, experimenting
Interpreting	
Validating	

21.5 Discussion

This study shows that the modelling cycle used, as well as the cycle described by Geiger (2011), can meaningfully describe the application of digital tools. Our study suggests that special activities with digital tools can be found, together with special partial competencies in modelling. A combined description of modelling and the use of digital tools as in Fig. 21.5, can therefore be more specific than the general view of technology as in Fig. 21.1. Students make use of the tools in a wide variety of ways, including constructing, drawing, calculating, measuring and experimenting.

This study contributes the debate on how modelling with GeoGebra may be implemented and demonstrated. Our observations show that the digital tool was indeed applied at different stages of the modelling cycle. These results seem to confirm our hypothesis in Siller and Greefrath (2010), where we described the

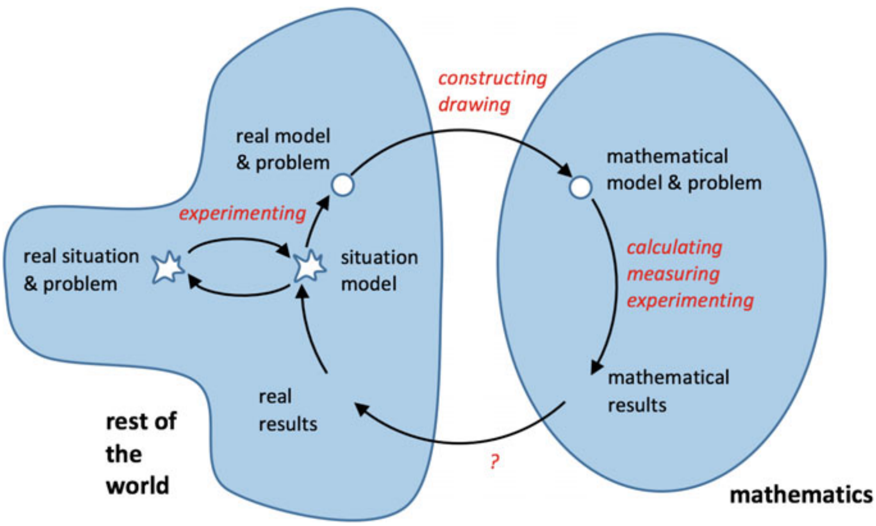


Fig. 21.5 Modelling cycle based on Blum and Leiss (2007) with specific use of digital tools

modelling cycle with technology with respect to three "worlds", namely the real, mathematical and technology worlds. We have stated that these three worlds influence each other. The mathematical model according to a real situation, depends on the situation itself, the available mathematics and the digital tool used. This can be seen here. Moreover, we have evidence supporting the statement of Blum and Niss (1991, p. 58) found in Monaghan, Trouche, and Borwein (2016, p. 166):

> More complex applied problems … relief from tedious routine … Problems can be analysed and understood better by varying parameters … [and] Problems which are inaccessible from a given theoretical basis … may be simulated numerically or graphically.

However, our example revealed that students did not use the tools for the purposes of interpretation or validation but for the other purposes mentioned above. This may be due to the fact that only a small number of student pairs were observed. Also, the chosen problem could not adequately highlight self-checking opportunities.

With the help of this study, we are able to meaningfully describe the utilisation of digital tools in modelling processes. It is evident that students do not focus on different representations or different options in the same program at the same time. Students only focus on a certain way and are unaware of the variety of possibilities shown here. They seem only to follow their individual perceptions and consequently use the accompanying mathematical model to solve the task. Even if the students did not have any analytical geometry skills, the program offered them two very different approaches. It is possible that students are not used to solving tasks in different ways. Therefore, they do not consider this aspect. Yet, the variety of solutions to a task seems to be a crucial point (see Blum, Drueke-Noe, Hartung, & Köller, 2012; Büchter & Leuders, 2005). Obviously, these students are not aware of them, particularly when using digital tools for solving tasks. This could suggest a need to look for more examples like the one shown here. Elementary modifications may help, as shown in Siller and Greefrath (2010). However, we need more examples like this, with relevance to applications and with different approaches. In particular, we need to take a closer look at different student strategies when modelling with a digital tool. Even in teacher education, it would be possible to discuss examples like that of Laakmann (2005), by focussing on different models.

References

Arzarello, F., Ferrara, F., & Robutti, O. (2012). Mathematical modelling with technology: The role of dynamic representations. *Teaching Mathematics and Its Applications, 31*(1), 20–30.

Barzel, B. (2012). *Computeralgebra im Mathematikunterricht: Ein Mehrwert—Aber wann?* Münster: Waxmann.

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do? In: S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education—Intellectual and Attitudinal Challenges* (pp. 73–96). New York: Springer.

Blum, W., Drueke-Noe, C., Hartung, R., & Köller, O. (Eds.). (2012). *Bildungsstandards Mathematik: konkret. Sekundarstufe I: Aufgabenbeispiele, Unterrichtsanregungen, Fortbildungsideen*. Berlin: Cornelsen.

Blum, W., & Leiss, D. (2007). How do students and teachers deal with mathematical modelling problems? The example sugarloaf and the DISUM project. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, engineering and economics* (pp. 222–231). Chichester: Horwood.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects—State, trends and issues in mathematics instruction. *Educational studies in mathematics, 22*(1), 37–68.

Büchter, A., & Leuders, T. (2005). *Mathematikaufgaben selbst entwickeln* (pp. 88–102). Berlin: Cornelsen.

Carreira, S., Amado, N., & Canário, F. (2013). Students' modelling of linear functions: How GeoGebra stimulates a geometrical approach. In B. Ubuz et al. (Eds.), *CERME 8. Proceedings of the 8th Congress of the European Society of Research in Mathematics Education* (pp. 1031–1040). Antalya, Turkey, Ankara: Middle East Technical University.

Confrey, J., & Maloney, A. (2007). A theory of mathematical modelling in technological settings. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 57–68). New York: Springer.

Daher, W., & Shahbari, A. (2015). Pre-service teachers' modelling processes through engagement with model eliciting activities with a technological tool. *International Journal of Science and Mathematics Education, 13*(Suppl 1), 25–46.

Fahlgren, M., & Brunström, M. (2014). A model for task design with focus on exploration, explanation, and generalization in a dynamic geometry environment. *Technology, Knowledge and Learning, 19,* 287–315.

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *Zentralblatt für Didaktik der Mathematik, 38*(2), 143–162.

Gallegos, R. R., & Rivera, S. Q. (2015). Developing modelling competencies through the use of technology. In G. A. Stillman et al. (Eds.), *Mathematical modelling in education research and practice*. Cham/Switzerland: Springer.

Geiger, V. (2011). Factors affecting teachers' adoption of innovative practices with technology and mathematical modelling. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 305–314). Dordrecht: Springer.

Hall, J., & Lingefjärd, T. (2016). *Mathematical modeling—Applications with GeoGebra*. Hoboken, New Jersey: Wiley.

Hohenwarter, M., Hohenwarter, J., Kreis, Y., & Lavicza, Z. (2008). Teaching and learning calculus with free dynamic mathematics software GeoGebra. In *TSG 16: Research and development in the teaching and learning of calculus ICME 11*, Monterrey, Mexico 2008. https://archive.geogebra.org/static/publications/2008-ICME-TSG16-Calculus-GeoGebra-Paper.pdf.

KMK. (2015). *Bildungsstandards im Fach Mathematik für die Allgemeine Hochschulreife*. Bonn und Berlin: Wolters Kluwer.

Laakmann, H. (2005). Die Piratenaufgabe—Verschieden darstellen, verschieden bearbeiten. In B. Barzel, S. Hussmann, & T. Leuders (Eds.), *Computer, Internet & Co. im Mathematikunterricht* (pp. 85–94). Berlin: Cornelsen Scriptor.

Moreno-Armella, L., Hegedus, S., & Kaput, J. (2008). From static to dynamic mathematics: Historical and representational perspectives. *Educational Studies in Mathematics, 68,* 99–111.

Monaghan, J., Trouche, L., & Borwein, J. (2016). *Tools and mathematics: Instruments for learning*. New York: Springer.

Pead, D., Bill, R., & Muller, E. (2007). Uses of technologies in learning mathematics through modelling. In W. Blum et al. (Eds.), *Modelling and applications in mathematics education. The 14th ICMI Study* (pp. 309–318). New York: Springer.

Riebel, J. (2010). *Modellierungskompetenzen beim mathematischen Problemlösen—Inventarisierung von Modellierungsprozessen beim Lösen mathematischer Textaufgaben und Entwicklung eines diagnostischen Instrumentariums*. Dissertation, Universität Koblenz-Landau, Fachbereich Psychologie, Landau.

Rögler, P. (2014). Überzeugungen von Mathematiklehrkräften als Basis zur Entwicklung von Lehrerfortbildung zu Technologien im Unterricht. *Beiträge zum Mathematikunterricht* (pp. 983–986). WTM: Münster.

Siller, H.-St., & Greefrath, G. (2010). Mathematical modelling in class regarding to technology. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the 6th Congress of the European Society for Research in Mathematics Education (CERME 6)*, January 28–February 1, 2009, Lyon (France).

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Thousand Oaks/California: Sage.

Vehring, J. (2012). Mathematische Modellierung mit einer dynamischen Geometriesoftware. Eine empirische Untersuchung anhand der Piratenaufgabe. master thesis, University of Münster.

Chapter 22
Instrumental Genesis and Proof: Understanding the Use of Computer Algebra Systems in Proofs in Textbook

Morten Misfeldt and Uffe Thomas Jankvist

Abstract In this chapter we investigate the role of Computer Algebra Systems (CAS) in textbook proofs. We describe two cases of CAS use in textbook proofs and use the instrumental approach, and in particular the distinction between epistemic and pragmatic mediations, to understand the consequences of the so-called CAS-assisted proofs. We end with a discussion of the experienced shortcomings of the instrumental approach in relation to CAS use in justification of mathematical results, and suggest the inclusion of alternative frameworks for filling the gap.

Keywords Proof · Computer algebra system · Instrumental genesis

22.1 Introduction: CAS and Proofs

Thirteen years ago (as part of a reform), CAS massively entered the upper secondary school mathematics program in Denmark: in the classroom teaching; the written national examinations; and the textbooks. The Danish Ministry of Education provided the guidance that CAS should not only serve the role of a tool for solving problems, etc. but also be seen as an instrument for underpinning conceptual understanding. However, the actual implementation of CAS into the mathematics program was pretty much left up to the schools, the teachers (Jankvist, Misfeldt, & Marcussen, 2016), and not least the textbook authors. As a consequence, textbook authors of more than one textbook system "invented" the notion of "CAS proofs".

A CAS proof, or CAS-assisted proof as we shall prefer to call it, is a textbook proof of a statement or theorem, where one or several steps are outsourced to a CAS

M. Misfeldt (✉)
Aalborg University, Copenhagen, Denmark
e-mail: misfeldt@learning.aau.dk

U. T. Jankvist
Aarhus University, Emdrup, Denmark

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_22

tool. This might be in smaller details of a proof consisting of algebraic manipulations, reductions, etc. But sometimes the textbook authors have gone further and outsourced steps of a proof, which appear to be rather crucial for understanding what is actually going on. And at other times, CAS comes to play the role of an "authority" upon which the correctness of traditional proofs are valued or judged.

The general problem framing our work in this chapter is how to understand such CAS-assisted proofs. We focus on the degree to which the instrumental approach (Artigue, 2002; Trouche, 2005) offers insights to this property. Of course, a number of constructs related to proofs and proving are important in this discussion: e.g. Hanna's (1989) discussion of pedagogical aspects of proof; and Harel's and Sowder's (2007) notion of proof schemes. However, when discussing the role of CAS in relation to proofs, we also need constructs that seriously address the cognitive and pedagogical influences of the advanced tool itself, i.e. CAS. The instrumental approach is broadly adopted in the mathematics education community as a way of looking at students' work with CAS. Hence, it also suggested itself as a natural choice of framework for addressing the problem of CAS in proofs. But as indicated above, we experienced difficulties with this approach. Due to the broad adoption of the instrumental approach, we believe that our experienced problems with the limitations of the framework are of general interest for the mathematics education research community. Our argument consists of showing an example of a CAS-assisted proof from a textbook, introducing the main constructs from the instrumental approach, and describing problems and shortcomings with applying these constructs to the case.

Having looked at how Danish upper secondary school mathematics textbooks use CAS in relation to proofs, we see at least three different approaches: a complete outsourcing of the proof to the CAS tool; an outsourcing of specific technical aspects; and a use of CAS to somehow check the result of an analytic proof. Of course, we do not suggest that these three approaches necessarily make up an exhaustive list of the different ways CAS can be used in proofs and proving, rather we suggest that these three ways of using CAS for proving exists in the Danish upper secondary school, and that the two latter (outsourcing specific technical aspects and checking results) are somewhat common in Danish textbooks. In this chapter we introduce the instrumental approach and describe two cases of CAS-assisted proofs. The first case is a complete outsourcing of the proof to the CAS system, and the second example is a proof where CAS is used to perform a specific technical aspect of a classical algebraic proof, furthermore we briefly describe the third type of CAS use for checking proofs. We discuss how these three cases constitute examples of pragmatic and epistemic uses of CAS for proving and we end with suggesting categories for describing CAS-assisted proofs informed both by the pragmatic/epistemic distinction and by theoretical constructs addressing proof practices among students.

22.2 The Instrumental Approach

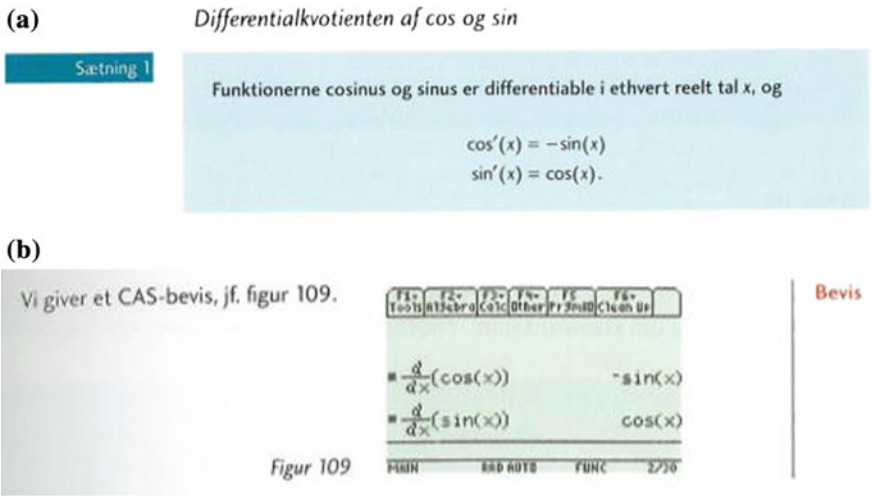
The instrumental approach addresses students' use of technology when learning mathematics from the perspective of appropriating digital tools for solving mathematical tasks, and in that sense it views computational artefacts as mediating between user and goal. The approach suggests that we view a student's goal-directed activity as shaped by the use of tools (this process is often referred to as instrumentation), and simultaneously the goal-directed activity of the student reshapes the tool (this process is often referred to as instrumentalization). In students' work with technology a distinction is made between epistemic mediations and pragmatic mediations. Epistemic mediations relate to goals internal to the user, i.e. affecting his or her conception of, overview of, or knowledge about something —Rabardel and Bourmaud (2003) use the example of a microscope, and Lagrange (2005) refers to experimental uses of computers. Pragmatic mediations relate to goals outside of the user, i.e. making a change in the world. Rabardel and Bourmaud (2003) use the example of a hammer, and Lagrange (2005) refers to the mathematical technique of "pushing buttons". CAS of course serves both pragmatic and epistemic purposes (Artigue, 2002; Trouche, 2005).

22.3 Proof Schemes

Harel and Sowder (2007, p. 809) define "proof scheme" as a combination of ascertaining and persuading. Ascertaining is the process employed to remove one's own doubts about the truth of an assertion, while persuading is the process employed to remove other's doubts. Harel and Sowder (2007) provide a taxonomy consisting of three overall classes of proof schemes: (1) external conviction proof schemes; (2) empirical proof schemes; (3) our usual deductive proof schemes as practiced in mathematics. The external conviction proof schemes may be expressed by an authoritarian proof scheme, e.g. that something is true because the teacher or the textbook says so; a ritual proof scheme, e.g. that a geometry proof must have a two-column format; or a non-referential symbolic proof scheme, e.g. that a proof must contain symbols and symbol manipulations. The empirical proof schemes come into play when using examples to justify the truth of general (universal) statements.

22.4 A Complete Outsourcing to CAS

As an example of the first approach, i.e. complete outsourcing of the mathematical justification, we present an example from a textbook (Clausen, Schomacker, & Tolnø, 2007), where the entire proof is outsourced to CAS.


(a)

Differentialkvotienten af cos og sin

Sætning 1

Funktionerne cosinus og sinus er differentiable i ethvert reelt tal x, og

$$\cos'(x) = -\sin(x)$$
$$\sin'(x) = \cos(x).$$

(b)

Vi giver et CAS-bevis, jf. figur 109.

Bevis

Figur 109

Fig. 22.1 **a** Differential quotient for cos and sin (Clausen et al., 2007, p. 12). **b** We provide a CAS-proof … (Clausen et al., 2007, p. 13)

The example (Fig. 22.1a, b) is taken from a textbook for the third and final year of upper secondary school mathematics ("A level at stx"). The CAS proof, which appears in the very beginning of this textbook, gives a theorem ("sætning")—Theorem 1—stating that "The functions cosine and sine are differentiable in every real number x and $\cos'(x) = -\sin(x)$, $\sin'(x) = \cos(x)$."

As for the proof of this theorem, the authors continue: "We provide a CAS proof, cf. Figure 109" where the box of Figure 109 simply displays the claims of Theorem 1 all over again, only in a screenshot from what appears to be a TI-89. No means for reasoning and explanation whatsoever are provided for the students; the entire act of proving is outsourced to the CAS tool (cf. Fig. 22.1b).

The above example left us somewhat puzzled when attempting an analysis: the proof does not explain anything; and the verification that it does provide is not satisfactory either. So, how can we understand this "proof"?

22.5 Pragmatic Use of CAS in Proof and Proving

If we try to characterize the use of CAS in the textbook case using the instrumental approach and the epistemic-pragmatic distinction, the framework seems to run short. In a sense the "CAS proof" is an example of a very pragmatic use of CAS, since no attempt is made at explaining or visualizing why $\cos'(x) = -\sin(x)$ and $\sin'(x) = \cos(x)$. However, stating that CAS only serves pragmatic purposes in the proof does not really make sense either, since proving is essentially an epistemic activity related to the establishment of truth. Hence, complete outsourcing to CAS in proving activities is difficult to place within either of these two categories. Still, there is a distinction between work process and object of the activity that calls for further clarification. Artigue's distinction between epistemic and pragmatic mediations is developed around a work practice of solving mathematical problems and tasks, and not around practices of working with proofs and proving. As stated, pragmatic mediation serves a purpose outside of the individual. In the case of a proof, pragmatic mediations would mean that the built argument is convincing and trustworthy towards the communicational situation that the proof is presented within. In this respect, the "CAS proof" above is both convincing and trustworthy, but it is neither concerned with questions of meaning nor with explaining the connections between the mathematical interties and ideas constituting the proof. Hence, this CAS-assisted proof is not a proof that explains (Hanna, 1989), but only a proof that may convince. Moreover, the proof convinces by referring to a technological authority—namely CAS. Hence, the educational value of such proofs can surely be discussed, since they place the value of the proof as distinct from the author of the proof. The value is solely in the proof's ability to settle any debate over whether the theorem at stake is true or not. In that sense, the "CAS proof" could give rise to what may be termed a "techno-authoritarian" external conviction proof scheme; techno-authoritarian proof schemes being technical because the proof only makes sense in reference to a specific technology (in this case CAS), and authoritarian because the scheme builds on black boxing (Buchberger, 1990; Jankvist & Misfeldt, 2015; Nabb, 2010), in the sense that the students need to trust the technology in order to believe in the proof, i.e. as if the technology is an authority.

As mentioned, the above proof is rather puzzling in the sense that many of the qualities that we usually assign to a proof in a textbook are missing, e.g. that it has explanatory power, exercises logic, and is concerned with mathematical concepts. These aspects make it likely to believe that the proof is a rare exception, a joke or a mistake. But going through a number of Danish textbooks for upper secondary school we have seen other examples of such heavy outsourcing approaches to the use of CAS in proofs (Jankvist & Misfeldt, in review).

22.6 Outsourcing and Checking of Specific Technical Aspects

Let us now see another example of a CAS-assisted proof. This example displays a different approach to including CAS in proving. The theorem to be proved is again that the derivative of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$. The proof follows a classical approach to calculating the derivative: first setting up the difference quotient $\frac{f(x+h)-f(h)}{h}$, next reduce and/or algebraically manipulate the difference quotient as much as possible—cf. (1) in Fig. 22.2a; and finally calculate the limit of the difference quotient when h approaches 0 (which takes place on Fig. 22.2b). The proof follows this structure and the limit is estimated in the text by using the observation that sin(k) = k, when k is very small. CAS is used to augment the, in other respects classical, proof in two different ways, both supporting the "calculation" that $\lim_{n\to\infty}\frac{\sin k}{k}=1$.

The first use—suggested in the first line of the CAS window of Fig. 22.2b—is to use CAS to calculate $\lim_{n\to\infty}\frac{\sin k}{k}$, when k is a specific but small number (in this example k = 0.05). The second line in the window uses CAS to calculate the limit —verifying the calculation carried out above. These uses represent a different approach to mathematical verification than the naïve outsourcing, as seen in the first example of a CAS-assisted proof. First of all, the core structure of a classical algebraic proof is articulated clearly in the second example; the CAS-based argumentation is merely an augmentation.

However, the CAS use is not redundant, since one of the core points in the proof is only weakly argued for. The argument takes place in the text below Fig. 9 (on Figure 22.2), the unit circle: "On the unit circle we see (Fig. 9) that if k is a number (angle measured in rad) close to 0, is sin k (y-coordinate to k) and k (as measured on the unit circle) almost the same size, hence we have that …" (Carstensen et al., 2007, p. 88).

22.7 Epistemic Use of CAS in Proof and Proving

As described in the above, the first textbook example uses CAS to establish truth in a very efficient, but also very pragmatic way. In the second example, we see a different use of CAS to perform a proof of the same theorem. From a pragmatic perspective this use is much weaker. Whereas the use of CAS in example 1 gets the job done of proving the theorem in question, the second use is merely augmenting the classical argumentation by establishing empirical (by checking with small numbers) and techno-authoritarian (by asking the CAS tool to calculate the limit) reasons to believe the critical step in the proof, namely that $\lim_{n\to\infty}\frac{\sin k}{k}=1$. With concepts from the instrumental approach we may describe the use of CAS for calculation of examples with very small values of k as epistemic, in the sense that the activity of checking these values is meaningfully embedded in testing a specific aspect of the proof, namely the limit when k becomes very small. There is of course no logical necessity related to investigating this, and therefore the activity does not formally play a role in the proof. Hence, the notion of this use of CAS as epistemic is challenged. It would be better to build on the notion of proof schemes and talk about empirical, epistemic use of CAS. The use of CAS to verify the limit calculation may, on the other hand, play a crucial role in the logical argument. As presented here in the text, the CAS-based result is merely an add-on to the algebraic proof, and hence redundant. However, in contrast to example 1, this use of CAS for establishing truth does not deprive but rather highlights a key idea in the proof. For that reason we suggest that this notion is considered epistemic, even though it is verification oriented in the same techno-authoritarian way as described in example 1.

Fig. 22.2 **a** Proof of derivative (Carstensen, Frandsen, & Studsgaard, 2007, p. 87). **b** Proof continued (Carstensen et al., 2007, p. 88)

(a)

SÆTNING 1
Den afledede af $f(x) = \sin x$ er $f'(x) = \cos x$.

Bevis. Differenskvotienten er

$$\frac{\Delta y}{h} = \frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h}.$$

Man kan vise, at der gælder en trigonometrisk formel for en differens mellem to sinus-værdier:

$$\sin x - \sin y = 2 \cdot \cos\frac{x+y}{2} \cdot \sin\frac{x-y}{2},$$

og benytter vi denne formel, fås

$$\frac{\Delta y}{h} = \frac{2 \cdot \cos\frac{2x+h}{2} \cdot \sin\frac{h}{2}}{h} = \cos(x+\tfrac{h}{2}) \cdot \frac{2 \cdot \sin\frac{h}{2}}{h} = \cos(x+\tfrac{h}{2}) \cdot \frac{\sin\frac{h}{2}}{\frac{h}{2}}. \quad (1)$$

Nu skal vi lade h nærme sig 0. Da cos er kontinuert, gælder

$$\cos(x+\tfrac{h}{2}) \to \cos x \;\text{ for }\; h \to 0. \quad (2)$$

(b)

Vi skal derfor bestemme grænseværdien af $\frac{\sin\frac{h}{2}}{\frac{h}{2}}$ når h nærmer sig 0. Da $\frac{h}{2}$ nærmer sig 0, når h gør det, kan vi også sige, at vi skal finde

$$\lim_{k\to 0}\frac{\sin k}{k}.$$

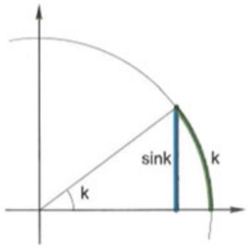
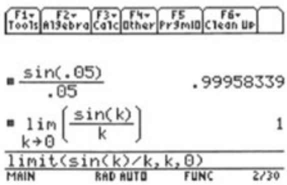


Fig. 9

På enhedscirklen kan vi se (fig. 9), at hvis k er et tal (vinkel i radianer) tæt på 0, er $\sin k$ (y-koordinaten til retningspunktet for k) og k (buelængden for radiantallet k) næsten lige store, så vi har, at

$$\frac{\sin k}{k} \to 1 \;\text{ for }\; k \to 0.$$



Dermed får vi af (1) og (2), at

$$\frac{\Delta y}{h} \to \cos x \cdot 1 = \cos x \;\text{ for }\; h \to 0,$$

og da denne grænseværdi er differentialkvotienten for $f(x) = \sin x$, er det ønskede vist.

22.8 CAS as a Means for Checking Validity

Besides the complete outsourcing and the distributing of specific difficult or technical aspects of a proof, we also see a third approach to the use of CAS in proving, where CAS is used after the formal proof is completed in a traditional manner. Also, this approach to the use of CAS is not unusual in Danish upper secondary school textbooks.

As a simple third example, the classical algebraic proof of the derivative of the tangent function is followed by checking the proof using CAS in a way similar to the CAS proof in the first example—although with the important difference that the algebraic proof has been given just before introducing CAS (Carstensen et al., 2007, p. 90).

It is worth considering both the intentions of the authors as well as the possible outcomes for the students in such a combination of classical and technological proof practice. Of course, it is a good idea to make students aware that there are different routes to the verification of results, and that taking several of these routes is a way to validate mathematical results—and that such approaches to validation is found among professional mathematicians as well (Johansen & Misfeldt, 2016). However, we do know that students in upper secondary school in general have difficulties with developing a sound conception of what a proof is. Hence, it makes sense to ask what idea of proof such CAS-assisted poofs leaves the students with.

The sound practice of checking for human calculation mistakes by using different mathematical routes to the same result might be confused by the students with a process of gathering several instances of evidence that this formula for the derivative of tangent is actually correct. From a proof scheme perspective, such a conception of the activity of checking results will push students in the direction of viewing either mathematical results as something that needs inductive validation (aligning the proof and the CAS check as just two pieces of evidence that the formula is true), or viewing CAS as an authority against which the proof is checked, because “CAS knows best”. The first conception clearly supports an empirical proof scheme, which is known to be a major epistemic obstacle for the transition to work with actual proofs (Education Committee of the EMS, 2011). The second conception supports the development of a techno-authoritarian proof scheme, since students may get the impression that the proof is correct because CAS says so. And hence that “check with CAS” is a necessary process for establishing mathematical truth rather than a practical process of helping us to discover flaws in our own calculations and proofs. This conception is supported by the fact that what is checked is a (flawless) textbook proof, not a (potentially wrong) student calculation. The potentially resulting techno-authoritarian proof scheme may leave students with the impression that mathematics is the output of a digital computer rather than a human endeavour.

22.9 Reflected and Non-reflected Use of CAS in Proving

Looking at the main differences between the three examples, we have phrased the following four questions. We propose that these may be used to structure the discussion of CAS use in relation to proofs and proving.

1. Does the CAS use establish truth? In example 1 it does. In the first line of the CAS window of example 2 it does not, whereas in the second line of the CAS window it does. This is an important aspect of pragmatic use of CAS. When CAS is used for checking proofs that are completed without technological aids, CAS cannot be said to establish truth. However, it does add to the probability that no human calculation error was conducted as part of the proof.
2. Does the CAS use allow interaction and experimentation? In example 1 it does not. In the first line of the CAS window in example 2 it does to some extent, but in the second line of the CAS window this is not the case. If CAS is used to verify classical proofs this can be done in several ways. But in the case considered, there is almost no interaction other than verifying the formula.
3. Is the argumentation inductive, deductive or authoritarian? We see inductive argumentation (first line of the CAS window in example 2), and authoritarian (example 1 and second line of the CAS window in example 2), but it can be argued that the way the limit calculation is part of a larger mathematical argument in example 2 does represent deductive reasoning. When "checking" proofs with CAS in a textbook there is a risk that students experience mathematics as a field, where inductive reasoning is important and closely connected to proof practice. The good practice of checking your work for mistakes and errors by applying different (for example technological) methods to obtain the same result, might in this case end up being misunderstood as a proof practice consisting of gathering a number of more or less contingent pieces of "evidence" suggesting that a mathematical result is true.
4. Does the argument highlight important aspect of the proof or the mathematical relationships? This is the case in example 2, but not in example 1. In example 2 the limit process is outsourced to the CAS tool. This process is two-sided; on the one hand it highlights that taking the limit is a necessary and difficult aspect of the proof, and on the other hand it frees the student from having to work with the limit without the CAS tool. When using CAS for checking a proof, as in the third example, the main idea of the proof will often not be highlighted.

These four questions, we propose, can act as an initial guiding framework for considering the value of CAS in textbook proofs, mediating between the concerns related to ensuring epistemic use of technology, as highlighted in the instrumental approach (Artigue, 2002), and the concerns related to ensuring that students can develop deductive proof schemes (Harel & Sowder, 2007), when working with textbook proofs in upper secondary school—as well as experiencing CAS-based proofs that explain rather than just pragmatically establishing some mathematical truth.

22.10 Conclusion

In conclusion, the shortcoming of the instrumental genesis framework in relation to CAS-assisted proofs may be partially remedied by filling the gap with frameworks from proofs and proving in mathematics education. In particular Hanna's (1989) distinction between proofs that prove and proofs that explain and Harel's and Sowder's (2007) notion of proof schemes appear useful in explaining the educational effects of CAS-assisted proofs (Jankvist & Misfeldt, in review). In relation hereto, it may be remarked that even though verification practices involving CAS can be mathematically healthy and meaningfully included in mathematics textbooks at upper secondary level, the notion of CAS-assisted proofs appears to potentially reinforce the observed educational problems with proofs and proving, e.g. in relation to non-deductive proof schemes, and may thus end up increasing students' difficulties with verification and justification through proof, argumentation, and reasoning.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Buchberger, B. (1990). Should students learn integration rules? *SIGSAM Bulletin, 24*(1), 10–17.

Clausen, F., Schomacker, G., & Tolnø, J. (2007). *Gyldendals Gymnasiematematik. Grundbog A*. København: Gyldendalske Boghandel, Nordisk Forlag A/S.

Carstensen, J., Frandsen, J., & Studsgaard, J. (2007). *Mat B til A stx*. Aarhus: Systime.

Education Committee of the EMS. (2011). Do theorems admit exceptions? Solid findings in mathematics education on empirical proof schemes. *Newsletter of the European Mathematical Society, 82,* 50–53.

Hanna, G. (1989). Proofs that prove and proofs that explain. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the 13th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 45–51). Paris: CNRS.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Charlotte, NC: Information Age Publishing.

Jankvist, U. T., & Misfeldt, M. (2015). CAS-Induced difficulties in learning mathematics? *For the Learning of Mathematics, 35*(1), 15–20.

Jankvist, U. T., & Misfeldt, M. (in review). CAS assisted proofs in upper secondary school mathematics textbooks.

Jankvist, U. T., Misfeldt, M., & Marcussen, A. (2016). The didactical contract surrounding CAS when changing teachers in the classroom. *REDIMAT—Journal of Research in Mathematics Education, 5*(3), 263–286. https://doi.org/10.17583/redimat.2016.2013.

Johansen, M. W., & Misfeldt, M. (2016). An empirical approach to the mathematical values of problem choice and argumentation. In B. Larvor (Ed.), *Mathematical Cultures: The London Meetings 2012–2014* (1st ed., Vol. 1, pp. 259–269). Springer. (Trends in the History of Science).

Lagrange, J. (2005). Using symbolic calculators to study mathematics: The case of tasks and techniques. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 113–135). New York, NY: Springer.

Nabb, K. A. (2010). CAS as a restructuring tool in mathematics education. In P. Bogacki (Ed.), *Electronic Proceedings of the 22nd International Conference on Technology in Collegiate Mathematics* (pp. 247–259).

Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. *Interacting with Computers, 15*(5), 665–691.

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators turning a computational device into a mathematical instrument* (pp. 197–230). New York, USA: Springer.

Chapter 23
In Search of Standards: Teaching Mathematics in a Technological Environment

Jana Trgalová and Michal Tabach

Abstract A number of research studies on teacher professional development (TPD) at the high school level express dissatisfaction regarding the outcomes of TPD programmes. The main reason seems to be a gap between teachers' expectations and TPD programme contents. We assume that this situation may be caused by a lack of standards specifying the knowledge and skills teachers need in order to use technology effectively in their classes. In this chapter, we tackle the issue of standards by describing existing ICT standards at the international and national levels and analysing them through the lenses of the TPACK model and double instrumental genesis. We argue that these standards are too general and do not refer specifically to school level or subject matter. We call on the mathematics education research community to take this issue into consideration.

Keywords Mathematics teachers' knowledge for ICT · Standards for teaching with ICT · TPACK model · Double instrumental genesis

23.1 Introduction

Teacher education was one of the four central themes discussed by the Topic Study Group 43, *Uses of technology in upper secondary mathematics education* (*age* 14–19) at the ICME 13 Congress. In our contribution to this theme (Hegedus et al., 2016), we pointed out that in a number of research papers, authors were disappointed with the outcomes of teacher education programs aimed specifically at high school teachers. The gap between teachers' needs and the teacher education contents was identified as the main reason. This calls attention to the need for teacher educators to better understand what teachers need to know in order to use Information and Communication Technology (ICT) effectively, thus raising the issue of ICT com-

J. Trgalová (✉)
Claude Bernard University, Lyon, France
e-mail: jana.trgalova@univ-lyon1.fr

M. Tabach
Tel-Aviv University, Tel-Aviv, Israel

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_23

petency standards. We therefore searched for an institutional framework regarding teachers' knowledge for teaching mathematics with technology. Surprisingly, we were able to find very few such standards for mathematics teachers or even for teachers in general. Therefore, we recommended that "*Elaboration of ICT standards for mathematics teacher education might become one of the goals of the mathematics education international community*" (ibid., p. 30).

In this chapter we expand on the standards we consider crucial for teacher education: standards aimed at teachers, and in particular mathematics teachers, specifying professional knowledge and skills needed for ICT use in mathematics classes. We discuss both international and national levels.

23.2 Theoretical Perspective

Several researchers have suggested theoretical frameworks for examining and analysing teacher knowledge in general (e.g., Grossmann, 1990; Ball, Thames, & Phelps, 2008). These frameworks draw on Shulman's (1986) construct of pedagogical content knowledge (PCK). Shulman rejected the view of *content knowledge* and *pedagogical knowledge* as two distinct bodies of knowledge and suggested a partial overlap between them. This overlap implies a unique type of knowledge specific for teachers—PCK.

In theorizing about the unique knowledge needed for teaching with digital technology, Mishra and Koehler (2006) introduced the concept of *technological pedagogical content knowledge* (TPCK or TPACK): the knowledge and skills teachers need to meaningfully integrate technology into instruction in specific content areas.

These authors suggested an additional body of knowledge to the PCK model, namely *technological knowledge* (TK), which partially overlaps CK and PK. Figure 23.1 depicts the resulting image of teachers' knowledge, which includes seven bodies of knowledge.

Other models of teachers' professional knowledge have been built on Shulman's PCK. Hill, Schilling, and Ball (2004) introduced the construct of *mathematics knowledge for teaching* (MKT), which they define as "subject-matter knowledge in ways unique to teaching" (p. 122). Their conceptualization of MKT is driven by the search of measurement methods to uncover this knowledge, thus positioning MKT in particular mathematical domains and tasks. Taking technology into account, the MKT construct has given rise to *knowledge for teaching mathematics with technology* (KTMT) (Rocha, 2013).

The TPACK framework offers a theoretical lens that enables us to analyse teachers' professional knowledge at a more general level, as opposed to the KTMT model that requires a specific context to be taken into account. TPACK is used by many researchers and several different interpretations are currently accepted (Voogt, Fisser, Pareja Roblin, Tondeur, & van Braak, 2012): T(PCK) as extended PCK; TPCK as a unique and distinct body of knowledge; and TP(A)CK as the interplay between three domains of knowledge and their intersections. In the current chapter we adopted the TP(A)CK view, as from the perspective of professional development, each knowledge domain and their intersection should be developed.

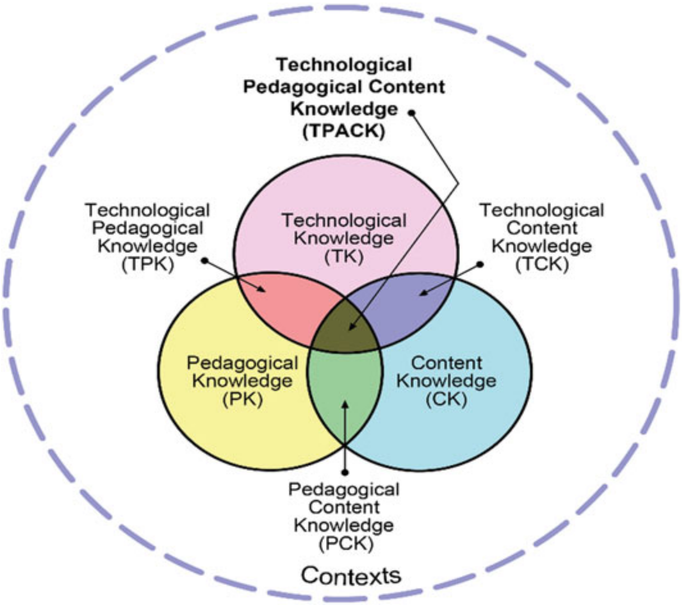


Fig. 23.1 TPACK (with permission from TPACK.org)

The theoretical construct of *double instrumental genesis* (Haspekian, 2011) encompasses both personal and professional instrumental geneses in teachers using ICT. While the personal instrumental genesis is related to the development of a teacher's personal instrument for mathematical activity from a given artefact, the professional instrumental genesis yields a professional instrument for the teacher's didactical activity. These two processes mobilize knowledge of the artefact (TK) and the abilities to solve mathematical problems using it (TCK), to orchestrate ICT-supported learning situations (TPK) and to teach mathematics with ICT (TPACK).

23.3 Methods

In this chapter, we review institutional documents in an attempt to answer the following questions: What competency standards are set for teachers working in technological environments? What are the specificities for mathematics teachers that are unique to this sub-group of teachers?

Two types of data sources were available for us. At the international level, we searched the web for organizations that published documents on the topic. We found the UNESCO ICT Competency Framework for Teachers (2011) and the International Society for Technology in Education (ISTE[1]) Standards-T (2008), both of which relate to teachers in general. At the national level, we considered the NCTM (2011) from the US that is specific for teaching mathematics, yet focused mainly on standards for learning, as well as the recent publication by the Association of Mathematics Teacher Educators (2017). We also considered available documents from France and Israel to obtain a wider national perspective.

While examining each of the data sources, we tried to relate them to one of the four knowledge areas that pertain to technology, as reflected by the TPACK framework.

23.4 Findings

23.4.1 ICT Standards Around the World

UNESCO ICT Competency Framework for Teachers (ICT-CFT) (2011) sets out "*the competencies required to teach effectively with ICT*" (p. 3). The framework stresses that

> it is not enough for teachers to have ICT competencies and be able to teach them to their students. Teachers need to be able to help the students become collaborative, problem solving, creative learners through using ICT so they will be effective citizens and members of the workforce. (ibid.)

The framework is therefore organized according to three different approaches to teaching corresponding to three stages of ICT integration. The first is Technology Literacy "*enabling students to use ICT in order to learn more efficiently*"; the second is Knowledge Deepening "*enabling students to acquire in-depth knowledge of their school subjects and apply it to complex, real-world problems*"; and the third is Knowledge Creation "*enabling students, citizens and the workforce they become, to create the new knowledge required for more harmonious, fulfilling and prosperous societies*" (p. 3). It is interesting to note that these stages are formulated in terms of students' abilities to exploit the ICT potential as a result of the ways teachers use ICT. All aspects of teachers' work, namely understanding ICT in education, curriculum and assessment, pedagogy, ICT, organization and administration, and teacher professional learning, are addressed in all three stages (Fig. 23.2).

[1]International Society for Technology in Education, http://iste.org.

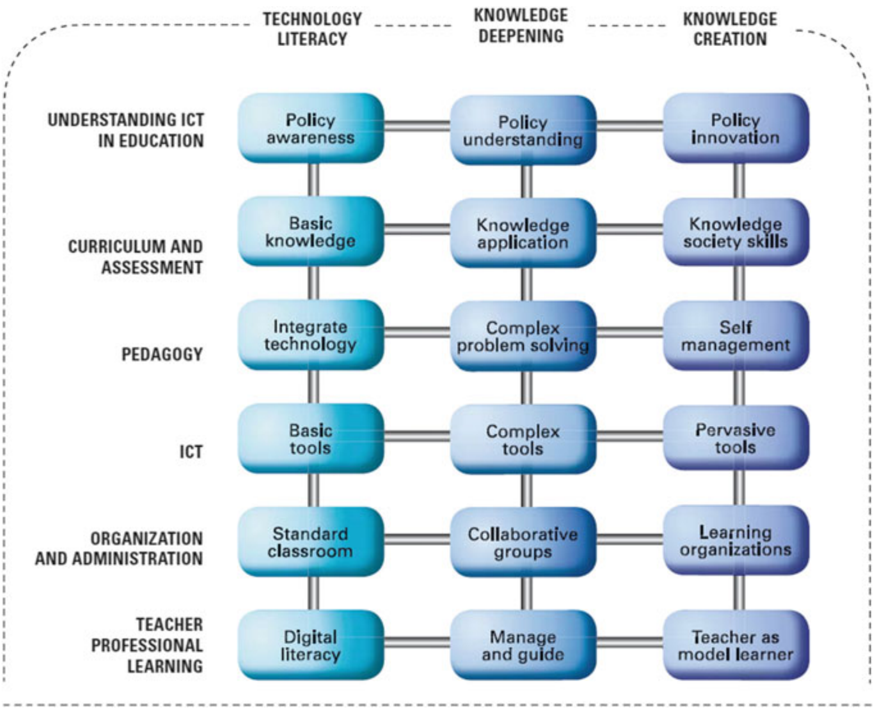


Fig. 23.2 The UNESCO ICT competency framework for teachers (UNESCO, 2011, p. 13)

The authors of the UNESCO framework claim that

> [t]he successful integration of ICT into the classroom will depend on the ability of teachers to structure the learning environment in new ways, to merge new technology with a new pedagogy, to develop socially active classrooms, encouraging co-operative interaction, collaborative learning and group work. This requires a different set of classroom management skills. The teaching skills of the future will include the ability to develop innovative ways of using technology to enhance the learning environment, and to encourage technology literacy, knowledge deepening and knowledge creation. (ibid., p. 8)

The framework specifies competencies teachers need in all aspects of their work. At the level of Technology Literacy,

> teacher competences […] include basic digital literacy skills and digital citizenship, along with the ability to select and use appropriate off–the-shelf educational tutorials, games, drill-and-practice software, and web content in computer laboratories or with limited classroom facilities to complement standard curriculum objectives, assessment approaches, unit plans, and didactic teaching methods. Teachers must also be able to use ICT to manage classroom data and support their own professional learning. (ibid., p. 10)

Referring to the TPACK model, we may consider "basic digital literacy" as part of TK and the ability to select appropriate resources to "complement […] standard didactic teaching methods" as part of TPACK. TPK and TCK are mentioned together with TPACK at the next level, Knowledge Deepening:

> teacher competences […] include the ability to manage information, structure problem tasks, and integrate open-ended software tools and subject-specific applications [TCK] with student-centred teaching methods and collaborative projects in support of students' in-depth understanding of key concepts [TPACK] and their application to complex, real-world problems. To support collaborative projects, teachers should use networked and web-based resources to help students collaborate, access information [TPK], and communicate with external experts to analyze and solve their selected problems. Teachers should also be able to use ICT to create and monitor individual and group student project plans, as well as to access information and experts and collaborate with other teachers to support their own professional learning. (ibid., p. 11)

Finally, at the level of Knowledge Creation, teachers

> will be able to design ICT-based learning resources and environments; use ICT to support the development of knowledge creation and the critical thinking skills of students [TPACK]; support students' continuous, reflective learning [TPK]; and create knowledge communities for students and colleagues. (ibid., p. 14)

The UNESCO document provides examples of syllabi for teacher education that demonstrate ways to operationalize the ICT competency framework. Table 23.1 provides a few examples of tasks suggested in the syllabi at the three levels of teachers' competencies—technology literacy (TL), knowledge deepening (KD) and knowledge creation (KC)—organized according to the TPACK model and the double instrumental genesis concept.

These examples of teachers' competencies show that the UNESCO ICT framework takes into account teachers' personal as well as professional ICT knowledge and skills, although the personal knowledge and skills are only present at the TL and KD levels, while teachers at the KC level are thought to have sufficient personal mastery of technology. All technology-related categories of the TPACK model are present, although the TPACK itself is not specific to subject matter or grade level.

The ISTE Standards-T (2008) define five skills teachers "*need to teach, work and learn in the digital age*":

(1) "*Teachers use their knowledge of subject matter, teaching and learning, and technology to facilitate experiences that advance student learning, creativity, and innovation*", (2) "*Teachers design, develop, and evaluate authentic learning experiences and assessments incorporating contemporary tools and resources*", (3) "*Teachers exhibit knowledge, skills, and work processes representative of an innovative professional*", (4) "*Teachers … exhibit legal and ethical behavior in their professional practices*", and (5) "*Teachers continuously improve their professional practice …, exhibit leadership in their school and professional community by promoting and demonstrating the effective use of digital tools and resources*".

These skills are rather general and refer to various aspects of the teaching profession. They do not relate to TK per se. It seems that in these standards, teachers' TK is taken as a starting point. Moreover, as the standards are not subject specific, they do not relate to TCK. In fact, this set of skills is about TPK.

Table 23.1 Examples of teachers' competencies mentioned in the UNESCO ICT framework

	Personal instrumental genesis	Professional instrumental genesis
Teachers should be able to…	TL—Describe the purpose and basic function of graphics software and use a graphics software package to create a simple graphic display (TK)	TL—Identify the appropriate and inappropriate social arrangements for using various technologies (TPK)
	TL—Use common communication and collaboration technologies, such as text messaging, video conferencing, and web-based collaboration and social environments (TK)	TL—Match specific curriculum standards to particular software packages and computer applications and describe how these standards are supported by these applications (TCK)
	TL—Use ICT resources to support their own acquisition of subject matter and pedagogical knowledge (TCK, TPK)	TL—Incorporate appropriate ICT activities into lesson plans so as to support students' acquisition of school subject matter knowledge (TPACK)
	KD—Identify or design complex, real-world problems and structure them in a way that incorporates key subject matter concepts and serves as the basis for student projects (TCK)	KD—Structure unit plans and classroom activities so that open-ended tools and subject-specific applications will support students in their reasoning with, talking about, and use of key subject matter concepts and processes while they collaborate to solve complex problems (TPACK)
	KD—Operate various open-ended software packages appropriate to their subject matter area, such as visualization, data analysis, role-play simulations, and online references (TCK)	KC—Help students reflect on their own learning (TPK)

Note that the standards encompass various aspects of the teaching profession—designing, teaching, evaluating, leading their peers in school and in their professional community, as well as legal behaviour. An indication that some adaptation to the content taught is needed can be found at the beginning: "*Teachers use their knowledge of subject matter…*". Yet, this does not directly convey that adaptation of these skills to different content areas within K-12, namely focusing on TPACK, may yield different results for different subject matters or grade levels.

To summarize, at the international level the standards are aimed mostly at teachers in general, with no specific adaptation to any school subject. As a result, the documents refer to teachers' TPK, rather than to TCK or TPACK, which are only referred to by evoking "*didactic teaching methods*" or "*support of students' in-depth understanding of key concepts*".

23.4.2 ICT Standards at the National Level

The National Council of Teachers of Mathematics (NCTM) has published documents, statements and position papers aimed at the US, yet these documents are influential beyond the national level. In many countries they serve as a model for national documentation. The NCTM refers specifically to mathematics teachers, as can be viewed from explicit references to mathematics, as well as to digital tools specific to mathematics.

NCTM (2011) claims that

> Programs in teacher education and professional development must continually update practitioners' knowledge of technology and its application to support learning. This work with practitioners should include the development of mathematics lessons that take advantage of technology-rich environments and the integration of digital tools in daily instruction, instilling an appreciation for the power of technology and its potential impact on students' understanding and use of mathematics.

The NCTM position toward technology in mathematics education emphasizes three conditions for an efficient integration of technology that should guide the development of teacher education programs: teachers' awareness of the technology added value in terms of improving students' understanding of mathematics, which refers to TPACK; teachers' continuous upgrading of their knowledge of technology and its use in teaching, which relates both to teachers' TK and to their TPK; and designing teaching resources taking advantage of affordances of digital tools, which refers to TPACK.

In a position statement from 2015 the NCTM further stated that

> Effective teachers optimize the potential of technology to develop students' understanding, stimulate their interest, and increase their proficiency in mathematics. When teachers use technology strategically, they can provide greater access to mathematics for all students.

The document further refers to particular technologies to be used, from mathematical and non-mathematical domains:

> Content-specific mathematics technologies include computer algebra systems; dynamic geometry environments; interactive applets; handheld computation, data collection, and analysis devices; and computer-based applications. Content-neutral technologies include communication and collaboration tools, adaptive technologies, and Web-based digital media.

Teachers are viewed as orchestrators and coaches of strategic use, and their major considerations should stem from the mathematics they are teaching. Technology is used at the service of mathematics. Although not specifically stated, it seems that for the NCTM, TCK, TPK and TK all play central roles in the knowledge teachers must have in order to teach with ICT. This impression is enhanced by the fact that in most of the publications, ICT appears in the background rather than up front.

A recent publication issued by the US Association of Mathematics Teacher Educators (2017) is aimed specifically at mathematics teachers, focusing on preparing teachers to teach quality mathematics. The researchers who compiled the document view the use of mathematical tools and technology as a component of mathematics concepts, practices and curriculum standards. They further detailed this issue for high school mathematics teachers:

> Well-prepared beginning teachers of mathematics at the high school level are proficient with tools and technology designed to support mathematical reasoning and sense making, both in doing mathematics themselves and in supporting student learning of mathematics. In particular, they develop expertise with spreadsheets, computer algebra systems, dynamic geometry software, statistical simulation and analysis software, and other mathematical action technologies as well as other tools, such as physical manipulatives. (p. 117)

In this description we can see references to the double instrumental genesis—both at the level of the teachers' own knowledge of technology use (TK) and in teaching their students (TPK and TPACK). Specific mathematical software packages are mentioned, acknowledging the fact that although general knowledge about communication technology is welcome, high school mathematics teachers also need specific knowledge relevant to mathematics (TCK).

The situation in Israel is quite different in terms of teachers' standards for teaching in an ICT environment in general and for mathematics teachers in particular. At the national level of preservice teacher education, only general reference is made to 21st century skills. In other words, reference is made to TK, which is expected from all citizens and not teachers in particular. At the mathematics education level, again there are no particular standards regarding what mathematics teachers need to know. This is not typical, as the Israel Ministry of Education usually adopts a very centralistic approach.

Until 2014, France was one of the European countries that required a certificate of digital skills known as a "certificate of computer science and Internet" to become a primary or a secondary school teacher. Since 2014, this certification has been integrated into preservice teacher education. This certification was created in 2010 to guarantee professional skills in the pedagogical use of common digital technologies necessary for all teachers and trainers to work in their profession. National standards of competencies related to this certification comprise two main parts: (A) general skills related to the exercise of the profession, and (B) skills needed for ICT integration into teaching practice. The general skills (part A) are organized in three domains: "A1—mastery of professional digital environment" (e.g., select and use the most appropriate tools to communicate with the actors and users of the education system); "A2—development of skills for lifelong learning" (e.g., use online resources or distance learning devices for self-training); and "A3—professional responsibility in the education system" (e.g., take into account the laws and requirements for professional use of ICT). The skills for ICT integration are classified into four domains: "B1—networking with the use of collaborative tools" (e.g., search, produce, index, and share documents, information, resources in a digital environment); "B2—design and preparation of teaching content and learning situations" (e.g., design learning and assessment situations using software that is general or specific to the subject matter, field and school level); "B3—pedagogical enactment" (e.g., manage diverse learning situations by taking advantage of the potential of ICT); and "B4—implementation of assessment techniques" (e.g., use assessment and pedagogical monitoring tools). While the skills from part A refer mostly to TK, those from part B refer to TCK, TPK and TPACK. Numerous intersections can be found between the French national and UNESCO international standards, mainly in considering various aspects of the teaching profession. The standards are not only restricted to teachers' classroom activity, but take into account both personal and professional mastery of ICT. Like the other standards presented above, the French ones are common to all teachers, whatever their school level and the subject matter they teach.

23.5 Conclusion

In this chapter we asked two connected questions: What knowledge standards are set for teachers working in technological environments? What are the specificities for mathematics teachers that are unique to this sub-group of teachers? To answer the two questions, we searched for institutional documents, both at the worldwide level and at the national level. In the findings section we detailed our analysis of the few documents we found, through the lenses of the TPACK framework for teachers' knowledge and the double instrumental genesis concept. We found a document formulated by UNESCO that elaborated ICT standards for teachers in general, regardless of subject matter or grade level. The second document was written by the International Society for Technology in Education, again at the general level. We were surprised to find only these two documents. We would like to point out that these two documents do not address any specific grade level, suggesting that the knowledge and skills for teaching in an ICT environment at any grade level are the same. In addition, the documents do not address any specific subject matter, nor do they suggest that particular adaptations are needed for teaching various school subjects.

At the national level we searched for documentation from three countries: US, France and Israel. There are profound differences between these three countries in terms of national level standards for teaching with ICT, as well as some striking similarities. As is the case at the international level, both in Israel and France reference is made only to teaching in general, with no relation to specific age level or subject domain. Yet, while in Israel some reference is made to 21st century ICT skills needed for any citizen, with emphasis on TK, in France we saw awareness of both personal TK as well as professional knowledge needed for teaching, in line with the double instrumental approach. The findings from the US differ in the sense that the standards are aimed specifically at teaching mathematics and are elaborated per school levels. Indeed, the analysis shows that these standards refer to all types of TPACK knowledge. Nevertheless, they lack specifications.

We are currently at a time of change in terms of teachers' technological knowledge. Newcomers to the profession are expected to be more skilful at the personal level than are veterans. Nonetheless, we think that teachers' mastery of ICT, in terms of both TK and TCK, should not be taken for granted. Rather, this personal level in the double instrumental genesis should be addressed by standards. Moreover, and even after considering the more recent document from the US, we call on the mathematics education research community to consider elaborating sets of standards for teaching with ICT for different age groups and school subjects so as to allow for the promotion of the professional level of instrumental genesis.

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics.* Available online at amte.net/standards.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.

Haspekian, M. (2011). The co-construction of a mathematical and a didactical instrument. In M. Pytlak, E. Swoboda, & T. Rowland (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education. CERME 7*, Rzesvow.

Hegedus, S., Laborde, C., Brady, C., Dalton, S., Siller, H. S., & Tabach, M. (2016). *Uses of technology in upper secondary mathematics education*. In ICME-13 Topical Surveys Series. Springer.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal, 105,* 11–30.

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for integrating technology in teacher knowledge. *Teacher College Records, 108*(6), 1017–1054.

NCTM. (2011). *Technology in teaching and learning mathematics. A position of the national council of teachers of mathematics*. Online http://www.nctm.org/Standards-and-Positions/Position-Statements/Technology-in-Teaching-and-Learning-Mathematics/. Retrieved September 17, 2015.

Rocha, H. (2013). Knowledge for teaching mathematics with technology—A new framework of teacher knowledge. In A. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th PME* (Vol. 4, pp. 105–112). Kiel, Germany: PME.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

UNESCO. (2011). UNESCO ICT competency framework for teachers. Online http://unesdoc.unesco.org/images/0021/002134/213475E.pdf. Retrieved February 20, 2017.

Voogt, J., Fisser, P., Pareja Roblin, N., Tondeur, J., & van Braak, J. (2012). Technological pedagogical content knowledge—A review of the literature. *Journal of Computer Assisted learning, 29*(2), 109–206. https://doi.org/10.1111/j.1365-2729.2012.00487.x.

Chapter 24
Using Graphing Calculators to Graph Quadratics

Elayne Weger Bowman

Abstract Insufficient classroom time to provide mastery to Algebraic content and intense requirements to meet state and national standards create challenges for many secondary (ages 14–17) mathematics educators. Additional test driven impetus to include graphing calculators in the teaching of quadratics and polynomials, along with teacher evaluations tied to student proficiency on mandated exams provide many points for heated discussions among mathematics educators. A culmination of two studies at a large Midwestern United States secondary school combines the findings of graphing calculator use in a secondary course of Algebra while introducing the graphing of quadratics with high stakes testing standards and their combined impact on secondary mathematics teachers. The author believes that using graphing calculators as daily tools can ease the tensions in mathematics education and enrich the mathematical learning in students.

Keywords Graphing calculators · Algebra · Quadratics · High-stakes testing
Common core state standards

24.1 Introduction

"The consequences of a plethora of half-digested theoretical knowledge are deplorable," (Whitehead, 1929, p. 4).

Mathematics education is ever-changing. The challenge of staying current on the diverse directions of mathematics curriculum requires stamina, flexibility, and determination. The decision to become a mathematics instructor should not be taken lightly, as that decision comes with responsibility and commitment to a common vision. Achieving that vision "requires solid mathematics curricula, competent and knowledgeable teachers who can integrate instruction with assessment, education policies that enhance and support learning, classrooms with ready access to

E. W. Bowman (✉)
Oklahoma Christian University, Edmond, Oklahoma, USA
e-mail: elayne.bowman@oc.edu

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_24

technology, and a commitment to both equity and excellence" (NCTM, 2000, p. 3). Not everyone is ready for that challenge. The integration of technology into mathematics continues to be a cause of controversy among mathematics educators and researchers. Many studies exist establishing justification for technology, such as graphing calculators, in mathematics learning, but studies showing how and when to use the technology are difficult to find. This study set forth to fill in the gap in existing literature to examine whether students might benefit from introducing the graphing calculator earlier in the learning sequence when graphing quadratics.

24.2 Testing and Technology

High-stakes testing takes on many forms in different countries. In the United States, continual efforts of mathematics curriculum reform to meet the needs of society's work force have included emphasis on rich technology usage in the mathematics classroom. Classroom teachers continue to disagree on whether the technologies, such as graphing calculators, are necessary for the teaching and learning of Algebraic concepts, such as graphing linear, quadratic, or other polynomial equations. Recent reform efforts, however, have taken the decisions away from the classroom teachers. The inclusion of graphing calculators on the high-stakes exams that students take at the end of their secondary mathematics courses in Algebra is driving the teaching decisions that teachers make inside their own classrooms.

24.2.1 High Stakes Testing

High stakes testing is driving the mathematics curriculum, now more than ever before, and the stakes have changed significantly. The attention-to-standards-mathematics that teachers following National Council of Mathematics recommendations (NCTM, 2000) already practice is now not only expected, but demanded. High stakes testing is often used to evaluate schools and teachers. The past decade has seen an increase in the number of end of instruction exams for secondary core subjects, especially in mathematics. The National Education Association (NEA) cites in a recent article that nearly half of the teachers they surveyed were considering leaving teaching over the standardized testing that "was sucking the oxygen out of the room" (Walker, 2014) and the article reports that over forty percent of the teachers indicated that these same tests were strongly linked to their teacher evaluations.

24.2.2 Mathematics Technology

Along with the mandated testing and extended curriculum, the emphasis on the use of technology in algebra continues to increase. Some teachers instruct their students in step-by-step calculator-button-pressing methods to attain correct answers on the end of instruction exams, without any attention given to meaning. On the other end of the spectrum, other teachers believe that making calculators available in the algebra classroom at all is ill-advised. Studies have shown that the technological beliefs of the instructors implementing the technology use strongly affect the outcome of the students (Martin, 2008; Smith & Shotsberger, 1997). There must be a certain amount of conviction in the technology and certainly, there must be appropriate training for the instructors and the students. The purpose of the technology used must be clear to both instructors and students.

According to the National Council of Teachers of Mathematics, "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (NCTM, 2000, p. 24). However, technology should never be used to replace basic understanding or intuition; rather, its primary use should be to "foster those understandings and intuitions…with the goal of enriching students' learning of mathematics" (p. 25). Technology in algebra is not limited to graphing calculators, but also includes computer spreadsheets, dynamic geometry software, and calculator-based-laboratory (CBL) systems. These tools, used in conjunction with appropriate activities and guidance, help to blur "some of the artificial separations among topics in algebra, geometry, and data analysis by allowing student to use ideas from one area of mathematics to better understand another area of mathematics" (p. 26). Using the recommendations from the National Council of Teachers of Mathematics, numerous researchers have conducted studies and agree that graphing calculators certainly have a place in the algebra classroom (Kastberg & Leatham, 2005; Martin, 2008; Smith & Shotsberger, 1997, e.g.). Nonetheless, the question of how and when to use the graphing calculators in the classroom still remains elusive (Bowman & Conrady, 2014; Kastberg & Leatham, 2005).

24.2.3 Mathematics Teacher Concerns

In the United States, the impact of the Common Core State Standards for Mathematics along with mandated high stakes testing on all offered precollege algebra courses have created additional points of concern for the instructors responsible for teaching the courses. Among these concerns is the use of graphing calculators on the exams. Graphing calculators are currently included in pop-up or drop-down windows in online orchestration of many high-stakes algebra exams, so whether or not mathematics instructors agree with their use is irrelevant. It is essential for instructors to teach their students how to use them appropriately and thoughtfully.

In the United States, “current algebra curriculum has a vast range of discrete information that teachers are asked to cover thoroughly. That vast range of information is quite wide and is growing seemingly wider with each new educational mandate” (Bowman, 2015, p. 91). For example, Algebra 2, an advanced course in precollege algebra that covers quadratics, polynomials, conics, and other similar topics, is the latest of the secondary mathematics courses to come under the auspices of end of instruction testing in the United States. It is a challenging course for most students to fully comprehend, and yet all advanced mathematics requires full understanding of its concepts. The Common Core State Standards for mathematics, adopted in 2010, added significant content standards to be addressed in Algebra 2 bringing the content to the level of course material normally covered in a course of college-level algebra. This course is a minimum requirement for college readiness in many secondary schools and colleges in the United States, so for the majority of students, the impact of changes to the content and rigor in Algebra 2 is powerful.

Innovative teachers are those who engineer and pioneer effective methods of reaching children. Far beyond a classroom lecture and worksheet, familiar in so many mathematics classrooms, innovative teachers’ classrooms are living spaces where students are actively engaged in interactive problem-based learning, known to facilitate greater learning in all students (Reynolds, 2010). The teacher in my study, a male in his mid-50s, had taught mathematics at the secondary level for sixteen years. His students enjoyed his classes and came to my calculus classes well prepared in their mathematical learning. As the standards began to change in our state, this teacher was quite worried over how students’ learning might be affected by including graphing calculators regularly in his algebra class. However, he was also worried that if they did not have practice with the graphing calculators during the learning phases of the course, how that might affect their scores during the high-stakes testing at the end of the course. This teacher was well-versed on current trends in mathematics education and used technology in his classroom in innovative ways. Nonetheless, he was conflicted over being forced to use them, rather than being given the choice to do so when he thought it made sense. I asked him if he was willing to participate in a study to see whether the graphing calculators would make a difference in the students’ test scores in his quadratics equations unit and he agreed.

24.3 The Graphing Calculator Study

24.3.1 Setting

The mixed methods study took place in a large ($n > 2500$) Midwestern United States secondary school (ages 14–17) in four sections of a course of advanced Algebra. The advanced Algebra course covered typical topics of pre–college algebra, such as quadratic equations, polynomial equations, factoring polynomials, conics, and rational equations. The question directing this study was “Does the timing of introducing a graphing calculator matter when teaching students to graph quadratic equations?" The courses were all taught by the same instructor, a mid-50s male who had taught mathematics to this age group for sixteen years. Although over 100 students were enrolled in the four sections of his classes, only 40 signed the paperwork to consent to the study. The results of all 40 students as well as the reflections of the instructor were included in the study.

24.3.2 Methodology

Before the study began, all students received a pretest (see Fig. 24.1) to determine initial knowledge and understanding of the key components of the graphs of quadratic equations. The pretest was a ten question test that asked for key information from quadratic equations or their graphs, such as the vertex, minimum or maximum, and intercepts. Additionally, three questions on the pretest were in the context of a real world problem where the students were to interpret the graph in terms of when and where a projectile would land. Following the pretest, the study was designed in such a way that all students would have the opportunity to learn to graph quadratic equations by two different methods, but the order in which the methods would be presented was reversed. The two methods were a traditional paper and pencil method and a graphing calculator method. In the traditional paper and pencil method, students were led by the teacher to construct t-charts of ordered pairs and then graphed the ordered pairs on coordinate axes. In the graphing calculator method, students were given the graphing calculator as a tool to graph the equations. In both methods, students were to identify the key components of the graphs—intercepts, vertices, direction (opening up or down), and axis of symmetry.

The teacher of the course randomly selected which classes would be taught with each method first. Two sections received the traditional paper and pencil teaching for one week, followed by the graphing calculator teaching. The other two sections received the graphing calculator teaching for one week, followed by the traditional paper and pencil teaching. At the end of the two weeks of instruction, the teacher administered a posttest that tested the same features of quadratic equations and their graphs as the pretest.

24.4 Quantitative Findings from the Study

Table 24.1 shows the results of a paired samples t-test comparing the two groups' pretests to check for variability between the two groups, each group's pretest and posttest, and the two groups' posttests. Group A is the two sections that were first taught traditionally, using paper and pencil only, and then taught the same lesson with use of the graphing calculator. Group B is the two sections that were first taught using the graphing calculators and then taught the same lesson traditionally with paper and pencil only.

Pre-test: Graphing Quadratic Equations

1) Given the quadratic equation $y = (x-3)^2 + 5$, describe the translation of the graph of the parent function $y = x^2$. The graph would be shifted ...
   a) Left 3 units and up 5 units.
   b) Left 3 units and down 5 units.
   c) Right 3 units and up 5 units.
   d) Right 3 units and down 5 units.
2) Given the quadratic equation $y = (x+3)^2 - 5$, describe the translation of the graph of the parent function $y = x^2$. The graph would be shifted ...
   a) Left 3 units and up 5 units.
   b) Left 3 units and down 5 units.
   c) Right 3 units and up 5 units.
   d) Right 3 units and down 5 units.
3) Given the quadratic equation $y = -(x)^2 - 4$, the vertex would be
   a) A minimum at $(0, -4)$.
   b) A maximum at $(0, -4)$.
   c) A minimum at $(-4, 0)$.
   d) A maximum at $(-4, 0)$.
4) Given the quadratic equation $y = -(x+4)^2$, the vertex would be
   e) A minimum at $(0, -4)$.
   f) A maximum at $(0, -4)$.
   g) A minimum at $(-4, 0)$.
   h) A maximum at $(-4, 0)$.
5) The minimum or maximum point of the graph of a quadratic equation is always its ...
   a) Axis of Symmetry.
   b) Apex.
   c) Vertex.
   d) Solution.
6) The x-intercepts of the graph of a quadratic equation are always its ...
   a) Axis of Symmetry.
   b) Apex.
   c) Vertex.
   d) Solution.

The graph of a quadratic equation below models a small model rocket that was launched from the top of the high school gymnasium. Critical points are marked with the letters A, B, C, and D. Pair each letter with the critical point it describes.

7) _____ is the height of the gymnasium.
8) _____ is the maximum height the model rocket reached.
9) _____ is the time when the model rocket reached its maximum height.
10) _____ is the time when the model rocket hit the ground.

Fig. 24.1 Pre test

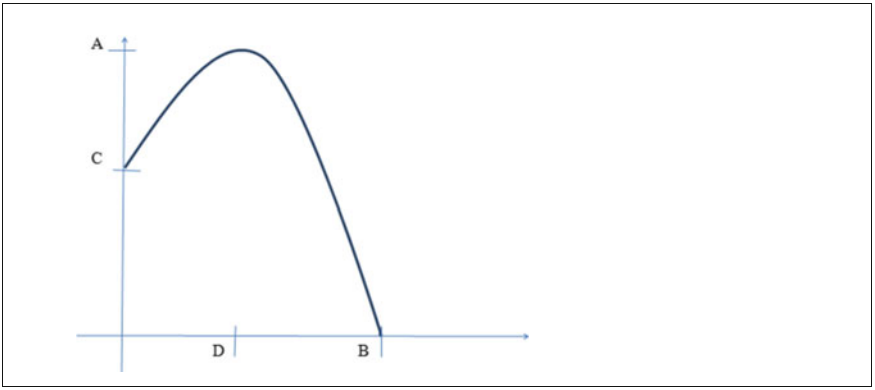


Fig. 24.1 (continued)

Table 24.1 Comparison of Groups A & B

Paired samples	Mean difference	Standard deviation	t-value	*p*-value
Groups A & B pretest	0.15789	2.14121	0.321	.752
Group A posttest/pretest	3.00000	2.44949	5.339	.000
Group B posttest/pretest	4.14286	1.65184	11.493	.000
Group B & A posttest	1.47386	1.89644	3.387	.003

A confidence interval of 95% was used for all tests. As shown in Table 24.1, the two groups of students, those who had the traditional paper and pencil teaching first (Group A) and those who had the graphing calculator teaching first (Group B) did not differ significantly ($p > 0.05$) at the outset of the study.

Both groups A and B had statistically significant ($p < 0.05$) gains in mathematical knowledge over quadratic equations and their graphs between the pretest and posttest. The students in Group B, however, had a statistically significant ($p < 0.05$) gain over the students in Group A.

24.5 Qualitative Findings from the Study

The instructor of the course observed that the levels of student questions during the learning processes differed extensively between the students of Groups A and B. Students in Group A, those who began the study without a graphing calculator, asked critical and higher level questions that teachers would like them to be thinking about. On the other hand, the students in Group B, those who began the study using the graphing calculators, asked questions pertaining only to the use of the technology. The instructor observed during class that his students often relied on the output of the calculator with little thought given to the sense-making of the problem.

24.6 Discussion

Students introduced to the graphing of quadratics with the graphing calculator first engaged with the mathematics at a higher level and retained more knowledge through the unit, even though from a qualitative point of view they seemed to be more concerned about the technology during the initial stages. Graphing calculators allow students to engage in challenging mathematics while still being involved in essential foundation building sense-making. Combining students' need for sense-making and their engagement with technology leads to activities that can utilize both. In our work with teaching students in precollege mathematics, it is essential that we instruct them appropriately with both paper and pencil and with technology, teaching them to think critically and carefully about their answers.

24.7 Seeking Common Ground

Exploring solutions for covering the vast curriculum in the allotted amount of time, integrating graphing calculators, and helping teachers to accomplish both is a worthy goal. In striving for this goal with other teachers, I have observed that when students are simply given the formulas, definitions, and recipes for success, they fail to grasp the concepts; they do not struggle to persevere. However, when introduced to topics in contextual venues they can relate to, their interest as well as their conceptual understanding improves (Bowman, 2015). As a recurring result of the TIMSS (Trends in International Mathematics and Science Study), in which American students are continually being compared to their international counterparts, Gasser (2011) developed Five Ideas for 21st Century Math Classrooms. He proposed that just by focusing on these five ideas, we could better prepare our students for competition against "the best of the best" in the work force. His ideas are to: (1) Incorporate problem-based instruction; (2) Foster student-led solutions; (3) Encourage risk taking; (4) Have fun; and (5) Provide ample collaboration time for both students and teachers (Gasser, 2011). Paul Ernest writes, "The motivation for including 'know how' as well as propositional knowledge as part of mathematical knowledge is that it takes human understanding, activity, and experience to make or justify mathematics—in short, mathematical know-how" (Ernest, 1998, p. 136). Students who are problem-solvers, thinkers, and decision-makers—those with "know how"—will be beyond the touch of button-pressing, machine-replaceable workers.

To reach these goals, curriculum can be reworked into project-sized bits where one project or unit would cover multiple standards and objectives while implementing all eight of the mathematical practices found in the Common Core State Standards for Mathematics: (a) Make sense of problems and persevere in solving them, (b) Reason abstractly and quantitatively, (c) Construct viable arguments and critique the reasoning of others, (d) Model with mathematics, (e) Use appropriate tools strategically, (f) Attend to precision, (g) Look for and make use of structure, and (h) Look for and express regularity in repeated reasoning (NCTM, 2014, p. 8). Projects do not have to be created from scratch. Many such projects have been teacher-created and are available for sharing. Nonetheless, creating the projects is great fun for teachers and students, alike.

Such projects might be like the one below, from Embracing the Common Core State Standards One Project at a Time (Bowman, 2015):

> In order to manufacture widgets, the We Make Do-Dads Company (WMDDC) needs to rent a building, hire employees, pay for utilities, buy materials and equipment, and produce Do-Dads. WMDDC can obtain a start-up loan up to $200,000 at the We Give Loans to Small Companies bank at 5 percent per year. WMDDC has six months after receiving the loan before it must start making monthly payments back to the bank, and it has 10 years to totally repay the loan. As accountants and planners for WMDDC, your responsibility is to propose the amount of the loan needed to start the company. You must determine values for rent, employees, utilities, materials, equipment, and how much to charge for Do-Dads. Your proposal should be accompanied by appropriate calculations, spreadsheets, graphs, or any other supporting mathematics you used to prepare your proposal. (p. 94)

A second project (Bowman, p. 94) explored electrical circuitry, which required a real use for complex numbers. Teachers often complain that they do not have time to add projects to their curriculum. However, such projects are not intended as additions to the teachers' curriculum, but rather replacements. A project such as either of these would address multiple standards and objectives in the Algebra 2 curriculum, require appropriate technology to solve for the variables, and engage the students in a manner not possible with assignments from worksheets or textbooks. The assessment of the unit would be the finished proposals for each group of students.

Such open-ended projects give students a chance to use the technology that they carry with them for educational purposes. It is difficult to explain to students who have computers in their pockets why they need to look up formulas in a textbook, why they needs to memorize the quadratic formula, and why they must learn to factor and solve polynomial equations. Some days, I am not certain that they should. As each project goes along there are opportunities for discussion, techniques, exploration, and even rules of operation; but inside a realistic context so that there is a reason for the tediousness of the algebra. That makes all the difference for students. Being able to cover multiple standards and objectives, engaging students in meaningful activities, and still preparing for the inevitable high-stakes testing, might make teaching fun and perhaps allow teachers to enjoy teaching mathematics and students as much as I do.

References

Bowman, E. (2015). Embracing common core state standards one project at a time. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas, 88*(3), 91–95.

Bowman, E., & Conrady, K. (2014). Graphing calculator quandary: When should graphing calculators be used when graphing quadratics? In *Proceedings of the 113th Annual Convention of the School Science and Mathematics Association* (Vol. 1, pp. 77–84). Jacksonville, FL: SSMA.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. Albany: State University of New York Press.

Gasser, K. W. (2011). Five ideas for 21st century math classrooms. *American Secondary Education, 39*(3), 108–116.

Kastberg, S., & Leatham, K. (2005). Research on graphing calculators at the secondary level: Implications for mathematics teacher education. *Contemporary Issues in Technology and Teacher Education, 5*(1), 25–37.

Martin, A. (2008). Ideas in practice: Graphing calculators in beginning algebra. *Journal of Developmental Education, 31*(3), 20–24.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

NCTM. (2014). *Principles to actions: Ensuring mathematics success for all*. Reston, VA: National Council of Teachers of Mathematics.

Reynolds, A. (2010). *Problem-centered learning in mathematics: Reaching all students*. Saarbrücken, Germany: Lambert Academic Publishing.

Smith, K. B., & Shotsberger, P. G. (1997). Assessing the use of graphing calculators in college algebra: Reflecting on dimensions of teaching and learning. *School Science and Mathematics, 97*(7), 368–376.

Walker, T. (2014). NEA Survey: Nearly half of teachers consider leaving profession due to standardized testing. http://neatoday.org/2014/11/02/nea-survey-nearly-half-of-teachers-consider-leaving-profession-due-to-standardized-testing-2/. Accessed December 6, 2014.

Whitehead, A. N. (1929). *The aims of education*. New York, NY: The Free Press.

Chapter 25
Teacher Beliefs and Practice When Teaching with Technology: A Latent Profile Analysis

Daniel Thurm

Abstract Designing effective teacher education for teaching mathematics with technology requires a profound understanding of teacher beliefs and classroom practice. In this quantitative study with 160 upper secondary in-service teachers from Germany the relation between technology-related beliefs and classroom practice is examined. A latent profile analysis reveals four subgroups of teachers with respect to the relation of beliefs and practice: "positive beliefs—frequent users", "positive beliefs—infrequent users", "negative beliefs—infrequent users" and "negative beliefs—frequent users". Furthermore, beliefs referring to discovery learning and time constraints show the strongest link to frequency of technology use. Based on the results, recommendations for teacher education are given.

Keywords Teacher education · Professional development · Beliefs
Technology · Mathematics

25.1 Introduction

A large amount of research shows that digital technology can facilitate students' conceptual knowledge when teaching complies with specific conditions (e.g. Zbiek, Heid, Blume, & Dick, 2007). However, the use of technology only plays a marginal role in mathematics classrooms (Hoyles & Lagrange, 2010). The introduction of technology into the mathematics classroom is thus not a straight-forward task and it is important for teacher educators to understand the factors associated with technology integration. Research shows that, among other factors, mathematics teachers' beliefs influence the successful implementation (Hennessy, Ruthven, & Brindley, 2005). However, the link between teacher beliefs and technology use has not yet been studied in much detail on a quantitative basis and the lack of research in this area hinders profound development of teacher education programs. In this

D. Thurm (✉)
University of Duisburg-Essen, Essen, Germany
e-mail: daniel.thurm@uni-due.de

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_25

paper we analyze the link between teachers' technology related beliefs and teacher's frequency of technology use in a latent profile analysis. We identify four subgroups of teachers with respect to the relation between beliefs and technology use and discuss consequences for professional development (PD) programs.

25.2 Theoretical Background

Digital technology can play an important role in enhancing the learning of mathematics (e.g. Zbiek et al., 2007). This can be achieved for example by shifting classroom practice from computation and skill development to an emphasis on conceptual understanding, problem solving, modelling and more interpretative tasks (Ellington, 2006; Hoyles & Lagrange, 2010; Simonsen & Dick, 1997). Technology provides the opportunity to easily work with different forms of representations which is crucial for developing an understanding of mathematical concepts (Duval, 2006). In particular, it allows to dynamically link multiple representations and explore relations between different forms of representations (Kaput, 1992). Moreover, technology can promote discovery learning in the classroom (e.g. Barzel & Möller, 2001) and support individual approaches to solving a mathematical problem. If used properly, technology can be a facilitator of change in the classroom, leading to a shift from a teacher centered to a student centered teaching style where students take more responsibility for their learning and where classroom activities comprise more discussion, inquiry and cooperative learning (Penglase & Arnold, 1996). To achieve these benefits, it has been shown, that students need frequent and regular access to technology (Burrill et al., 2002). Only then students develop fluency in operating the technology and the process of instrumental genesis (Drijvers & Trouche, 2008), during which the object or artefact is turned into an instrument can take place. In the following, we use the term 'technology' for all digital tools which afford the described benefits, specifically allowing dynamically linked multiple representations of mathematical concepts (i.e. graphing calculators (GC) or computer algebra systems [CAS]).

However, despite the described benefits, there is a "widely perceived quantitative gap and qualitative gap between the reality of teachers' use of [technology] and the potential for [technology] suggested by research and policy" (Bretscher, 2014, p. 43). The introduction of technology into the mathematics classroom is thus not a straight-forward task. As Kissane (2003, p. 153) pointed out "Availability of technology is not by itself adequate, of course, to effect changes in the mathematics curriculum. A crucial mediating factor is the teacher, and curriculum developers ignore the real needs of teachers at their peril. Mathematics teachers need professional development directly related to graphics calculators if they are to be the main agents of reform, and ultimately directly responsible for whatever happens in the classroom."

Hence it is important for teacher educators to understand the needs of teachers and the factors that influence technology integration. Research shows that, among other factors, mathematics teachers' beliefs have profound implications on their classroom practices as well as on student performance (Staub & Stern, 2002). By the term "belief", we refer to the definition of Philipp (2007, p. 259) who defines beliefs as "Psychologically held understandings, premises, or propositions about the world that are thought to be true". Specifically, it can be shown that teacher's beliefs directly related to teaching with technology have a profound impact on successful technology implementation (e.g. Ertmer, 2005; Hennessy, Ruthven, & Brindley, 2005). And even so "[…] it is not clear whether beliefs precede or follow practice (Guskey, 1986), what is clear is that we cannot expect to change one without considering the other" (Ertmer, 2005, p. 36). Hence if PD programs aim at changing classroom practices and teachers' beliefs, designers of PD programs need to understand the relationship between these two, in order to design effective interventions. In particular, PD programs need to account for individual differences regarding the relation between beliefs and practice among participants. This is especially important since research shows that participant orientation, which means centering on the heterogeneous and individual prerequisites of participants is a crucial design principle for PD programs (Clarke, 1994) and comprises the basis for content-related and methodological design decisions in PD programs.

However despite the need to understand the link between teacher beliefs and technology use when teaching mathematics with technology, this facet has not yet been studied in much detail on a quantitative basis. There are mostly qualitative studies available (e.g. Jost, 1992; Molenje, 2012; Simmt, 1997; Tharp, Fitzsimmons, & Ayers, 1997). These studies indicate consistency between teachers' beliefs and practice. Teachers with beliefs in favor of technology tend to use technology more frequently and in ways more compatible with a constructivist approach to learning. However, studies in other areas than mathematics have also described inconsistencies between teachers' beliefs and their classroom practice (Calderhead, 1996; Chen, 2008; Fang, 1996) and proposed that contextual factors might be responsible for these inconsistencies. Thus it is necessary to further clarify the link between beliefs and practice in the domain of teaching mathematics with technology since the lack of research in this area hinders profound development of teacher education programs.

25.3 Research Questions and Methodology

The study aims at exploring the relation between technology-related beliefs and frequency of technology use in more detail in order to examine aspects which facilitate or inhibit its use and addresses the following questions: Which subgroups of teachers in upper secondary school can be identified with respect to the relation of beliefs and technology use? Which aspects of technology related beliefs show the strongest link to frequency of use?

To answer the question, it is necessary to measure teachers' technology related beliefs as well as teachers' technology use. To measure teachers' technology use, one needs to decide on *how* to measure it. Since research shows that it is difficult for teachers to self-report the *quality* of their teaching we focus on self-reported *frequency* of technology use which has been proven to be validly captured by teacher self-reports (Mayer, 1999). The development of the questionnaire to measure the frequency of technology use started by identifying relevant areas which are influenced by the integration of technology into the classroom. The following areas were identified from literature (e.g. Hoyles & Lagrange, 2010; Penglase & Arnold, 1996; Zbiek et al., 2007):

(*f1*) *use of technology for discovery learning*

Technology can support the exploration and discovery of mathematic relationships. This can be achieved for example by generating examples and exploring patterns, providing students with the opportunity to learn mathematic as a constructive activity.

(*f2*) *use of technology for linking multiple representations*

Technology can be used to link multiple representations. Different forms of representation should be used in the classroom in order to build connections among them in order to facilitate a conceptual understanding of mathematics.

(*f3*) *use of technology when practicing*

Practicing is an important phase in the learning process in order to consolidate and strengthen the knowledge. In this teaching phase, technology can play an important role in supporting the learning process.

(*f4*) *use of technology for individual learning*

Technology can be a catalyst to allow students to pursue individual solution strategies. The teacher may encourage a dual approach to problem solving using graphic or algebraic methods and use tasks which allow multiple solution strategies.

(*f5*) *reflection of technology use*

If technology use is coupled with a reflection of its use, especially on the limitations of technology it may be possible to prevent misconceptions which can arise from technology use.

After these areas were identified, a set of items was developed for each area. In cycles of cognitive interviews (Willis, 2005) with teachers and experts the items were refined until validity of the item set was agreed on by teachers and experts. Response categories for each item were chosen as follows: “almost never”, “once or twice a quarter”, “once or twice a month”, “once a week”, “almost every lesson”.

To measure teacher beliefs we used a questionnaire which consisted of the five following scales (Rögler, Barzel, & Eichler, 2013): (b1) beliefs that technology supports discovery learning, (b2) beliefs that technology support multiple representations, (b3) beliefs that technology is too time consuming (as there is a general concern that there is not enough time to cover the technology and the required curriculum), (b4) beliefs that technology has a negative impact on computational skills (as there is a common concern that pen-paper-skills may be lost in the presence of technology), (b5) beliefs that students must master concepts and procedures prior to technology use. Each scale consisted of 3–5 items. Responses were given on a five point response format ranging from 1 = "strongly disagree" to 5 = "strongly agree". Hence high values on the scales (b1) and (b2) with simultaneously low values on the scales (b3)–(b5) reflect positive beliefs about the use of technology in mathematics education.

Data for the statistical analysis was collected within a larger research study that was carried out in the federal state of North Rhine-Westphalia in Germany (Thurm, Klinger, & Barzel, 2015). In this German federal state, the use of technology (GC or CAS) is compulsory since the schoolyear 2014/15. Both questionnaires were administered to 160 teachers teaching mathematics in grade 10 of upper secondary school. There was a large portion of teachers which were not yet experienced in using technology. Of the 160 teacher, 71 teachers did not have any previous experience and only 50 teachers had more than 4 years' experience in teaching mathematics with technology. Thus, a large portion of the teachers consisted of novices.

We used confirmatory factor analysis to determine whether a five-factor measurement model for the questionnaire assessing frequency of technology use could represent the data well. To assess the goodness of fit we used the chi-square-fit index (χ^2/df), the root mean square error of approximation (RMSEA), the standardized root mean square residual (SRMR) and the comparative fit index (CFI). For identifying subgroups of teachers a latent profile analysis (LPA) was used on the mean values of all subscales. LPA tries to identify subgroups in the data that show similar response patterns and assumes that a categorical latent variable can explain the relationship among indicators (Vermunt & Magidson, 2002). In contrast to a variable-centered approach (e.g. factor analysis,), LPA is person centered approach, where the relation among persons is the main interest. Different than traditional cluster analyses methods (e.g. k-means), LPA is a model-based approach that offer advantages compared to the traditional methods (Goodman, 2002). In particular, different competing models can be assessed through various model fit statistics. However, despite these benefits and the popularity of LPA in other disciplines as psychology, to my knowledge, mixture models have yet not been applied to the study of the relation between beliefs and practice in the domain of teaching mathematics with technology.

25.4 Results: Linking Teacher Beliefs and Classroom Practice

In a first analytical step we validated the five-factor structure of the questionnaire for teacher self-reported frequency of technology use in a confirmatory factor analysis. The analysis yielded a good model fit with RMSEA = 0.069,

Table 25.1 Scales, sample items and Cronbach's alpha for self-reported technology use

Scale (#items)	Sample item	Cronbach's α
(f1) Discovery Learning (5)	How often was the GC/CAS used to explore mathematical relationships?	0.85
(f2) Multiple representation (4)	How often was the GC/CAS used to link graphical and algebraic representation?	0.88
(f3) Practice (4)	How often was the GC/CAS used to practice mathematical content?	0.88
(f4) Individual learning (3)	How often was the GC/CAS to support students to find their own way to solve a task?	0.78
(f5) Reflection (5)	How often did you discuss limitations of GC/CAS?	0.81

Table 25.2 Results of the correlation analysis

	(b1) Discovery learning	(b2) Multiple representations	(b3) Time consuming	(b4) Understanding and skills	(b5) Time point
(f1) Discovery learning	**.319*** **	.107	**−.283** **	**−.193***	**−.211***
(f2) Multiple representations	.78	**.328*** **	−.077	−.018	.030
(f3) Practice	**.241** **	.154	**−.290*** **	−.079	−.164
(f4) Individual learning	**.277** **	.178	**−.266** **	−.159	−.054
(f5) Reflection	−.108	−.011	.013	.036	.013

* (p<0.1)
** (p<0.01)
***(p<0.001)

SRMR = 0.070, CFI = 0.936, and χ^2/df = 1.704 as literature reports the following as sufficient for a good fit: χ^2/df < 3; RMSEA < 0.08; SRMR < 0.11 and CFI > 0.9. Reliability of the scales was high with Cronbach's alpha ranging from 0.78 to 0.88 (Table 25.1).

Subsequently, we examined the correlation between all measured constructs on a latent level (Table 25.2). As expected beliefs in a specific domain correlate highly with technology use in this domain. For example positive beliefs about the use of technology to support multiple representations leads to significant use of technology in this area. Remarkably, the scale (b1) measuring beliefs referring to discovery learning and the scale (b3) measuring beliefs referring to time constraints have significant correlations with frequency of technology use in various areas. Teachers holding the belief that technology can support discovery learning use technology more frequently not only to support discovery learning but also in phases where students practice content and to allow individual learning opportunities. In contrast, the belief that technology integration is too time consuming leads to a significant lower use in exactly these areas.

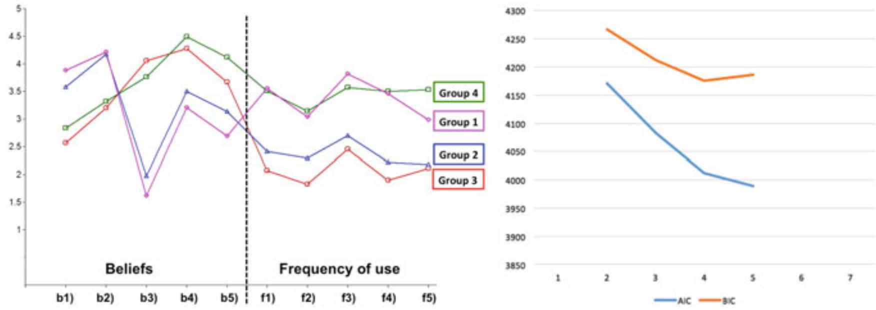


Fig. 25.1 Profiles of the four subgroups (left) and AIC and BIC for the different number of subgroups (right)

Subsequently, using LPA, a four subgroups structure was found where the number of groups was determined by likelihood tests, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). The models were analyzed by using Mplus software 7.31 (Byrne, 2012; Muthen & Muthen, 1998–2015). Figure 25.1 (right) shows the AIC and BIC for different numbers of subgroups. As it can be seen AIC and BIC clearly indicate a four subgroup structure keeping in mind model parsimony.

Figure 25.1 (left) shows the estimated means of the four subgroups for the several scales. When looking at frequency of use it can be seen that teachers can be classified in “frequent users” and “infrequent users” with the first averaging around 2 on the scales (f1)–(f5) and the latter averaging around 3.5 on these scales. Taking teachers beliefs into account as well, the following four subgroups can be identified: Subgroup 1 named “positive beliefs—frequent users” (37.9%) have high values on scales (b1) and (b2) and low values on scales (b3)–(b5), thus holding positive beliefs towards technology and also using technology frequently. In contrast, subgroup 2 named “positive beliefs—infrequent users” (27.6%) has a similar belief structure as subtype 1 but is not using technology frequently. Subgroup 3 named “negative beliefs—infrequent users” (16.8%). Subgroup 4, named “negative beliefs—frequent users” (17.8%), holds negative beliefs about technology use but still uses technology frequently in their classroom. However, the two subgroups holding positive beliefs and the two subgroups holding negative beliefs show almost identical pattern on the belief scales respectively. Thus, it cannot be said, that specific aspects of technology related beliefs are potentially causing the difference in frequency of use. There must be other factors than the beliefs that were included in the study that determine subgroup membership.

25.5 Discussion

The study aimed at connecting teacher beliefs regarding technology and classroom practice when teaching mathematics with technology. Results show that beliefs regarding discovery learning and time constraints are strongly related to frequency of technology use. Furthermore, four subgroups of teachers with respect to the relation between beliefs and classroom usage of technology could be identified.

The subgroups named "positive beliefs—frequent users" and "negative beliefs—infrequent users", can be said to act consistent with regard to their beliefs about technology. The first is holding more positive beliefs about technology integration and hence uses technology frequently. The latter tends to be more negative about technology use and thus integrates technology only from time to time in their classrooms. In contrast, the two subgroups, named "positive beliefs—infrequent users" and "negative beliefs—frequent users" seem to act inconsistent with their beliefs and the question remains what causes this apparent inconsistency. We agree with Philipp (2007) who pointed out "*the inconsistencies exist only in our minds, not within the teachers, and [I] would strive to understand the teachers' perspectives to resolve the inconsistencies. Inconsistencies should still present problems, but for researchers instead of teachers*" (Philipp, 2007, p. 276). It might be, that teachers in the subgroup "positive beliefs—infrequent users" encounter obstacles due to external factors, have low self-efficacy beliefs about technology integration or a limited or inadequate understanding of the promoted concepts that hinders them to use technology more frequently. Furthermore, it might be possible, that teachers other conflicting beliefs (i.e. need to cover a lot content to guide student learning) might hinder technology integration as observed in a study by Chen (2008). The subgroup, "negative beliefs—frequent users" might only use technology because it is compulsory in the curriculum. Another explanation for this subgroup might be, that these teachers do not hold positive beliefs about technology but want to give it a try in order to observe if the learning of students improves through the use of technology. This would be in line with Guskeys model of teacher change (Guskey, 2002) in which "significant change in teachers' attitudes and beliefs occurs primarily after they gain evidence of improvements in student learning" (Guskey, 2002, p. 383).

From the results several recommendations can be derived for professional development programs and teacher education. Teacher educators promoting the use of multi-representation technology in the mathematics classroom should focus specifically on the aspects of how technology can support discovery learning as this factor seems to be linked to frequency of technology use in a strong way. In addition, teacher educators must address the fear of teachers that technology integration is too time consuming. They must give practical approaches showing how the benefits of technology can be utilized while still achieving curricular prescriptions in time.

Finally, teacher educators must be aware of the four different subgroups of teachers identified in this study. This study finds, that the regularly assumed link that positive beliefs about technology use lead to frequent use and negative beliefs about technology use lead to infrequent use does not hold for a large portion of teachers. Consequently, the different subgroups identified in this study probably have different needs that must be addressed in PD programs. Teachers in subgroup "positive beliefs—infrequent users" will most likely be aware of the benefits of technology. A PD program that emphasizes changing teachers' belief might not have much impact on this subgroup. Instead, this subgroup might need to identify, discuss and reflect on the obstacles that hinder them to act according to their beliefs and focus on increasing self-efficacy and on methods to bring teachers beliefs into practice. Teachers in the subgroup "negative beliefs—frequent user" bring a good starting point for changing their beliefs since they are already using technology in their classroom. Hence, this subgroup needs support directly related to their current technology use that enables them to see the benefits so that their beliefs might change subsequently. The subgroup "positive beliefs—frequent users" seems to already have a solid foundation that supports technology integration. Teachers in this subgroup might rather need reflection and support that focus on enhancing the quality of their teaching. The subgroup "negative beliefs—infrequent user" seems to be the most challenging to as they are more technology hostile and not using technology regularly. This subgroup might need a dual approach focusing on reflecting on their beliefs, with direct support aimed at getting started to integrate technology into their practice.

In summary, to our knowledge, this study provides the first quantitative analysis of subgroups of teachers with respect to the relation of beliefs and practice. The study indicates that we must focus on the heterogeneous needs of the different subgroups of teachers if we want to take teachers need seriously and change the degree of technology integration. However, follow up studies are needed to examine the four subgroups in more detail. Qualitative studies should follow to study the different subgroups in more detail and quantitative studies need to investigate if the subgroups can be identified in other populations as well and to what extent covariates like age or teaching experience influence subgroup membership.

References

Barzel, B., & Möller, R. (2001). About the use of the TI-92 for an open learning approach to power functions. *Zentralblatt für Didaktik der Mathematik, 33*(1), 1–5.

Bretscher, N. (2014). Exploring the quantitative and qualitative gap between expectation and implementation: A survey of english mathematics teachers' uses of ICT. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (Vol. 2, pp. 43–70). Dordrecht: Springer.

Burrill, G., Allison, J., Breaux, G., Kastberg, S., Leatham, K., & Sanchez, W. (Eds.). (2002). *Handheld graphing technology in secondary mathematics: Research findings and implications for classroom practice*. Dallas, USA: Texas Instruments.

Byrne, B. M. (2012). *Structural equation modeling with Mplus: Basic concepts, applications, and programming*. New York, NY: Routledge.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 709–725). New York: Macmillan Library Reference.

Chen, C. H. (2008). Why do teachers not practice what they believe regarding technology integration? *The Journal of Educational Research, 102*(1), 65–75.

Clarke, D. M. (1994). Ten key principles from research for the professional development of mathematics teachers. In D. B. Aichele & A. F. Croxford (Eds.), *Professional development for teachers of mathematics* (pp. 37–48). Reston, VA: NCTM.

Drijvers, P., & Trouche, L. (2008). From artifacts to instruments: A theoretical framework behind the orchestra metaphor. In: G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics* (pp. 363–392). Charlotte, NC: Information Age.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131.

Ellington, A. J. (2006). The effects of non-CAS graphing calculators on student achievement and attitude levels in mathematics: A meta-analysis. *School Science and Mathematics, 106*(1), 16–26.

Ertmer, P. (2005). Teacher pedagogical beliefs: The final frontier in our quest for technology integration? *Educational Technology Research and Development, 53*(4), 25–39.

Fang, Z. (1996). A review of research on teacher beliefs and practices. *Educational Research, 38* (1), 47–65.

Goodman, L. A. (2002). Latent class analysis: The empirical study of latent types, latent variables, and latent structures, and some notes on the history of this subject. In J. A. Hagenaars & A. L. McCutcheon (Eds.), *Applied latent class analysis* (pp. 3–55). Cambridge: Cambridge University Press.

Guskey, T. R. (1986). Staff development and the process of teacher change. *Educational Researcher, 15*(5), 5–12.

Guskey, T. R. (2002). Professional development and teacher change. *Teachers and Teaching: Theory and Practice, 8*(3/4), 381–391.

Hennessy, S., Ruthven, K., & Brindley, S. (2005). Teacher perspectives on integrating ICT into subject teaching: Commitment, constraints, caution, and change. *Journal of Curriculum Studies, 37*(2), 155–192.

Hoyles, C., & Lagrange, J.-B. (Eds.). (2010). *Mathematics education and technology: Rethinking the terrain: The 17th ICMI Study*. New York: Springer.

Jost, K. L. (1992). *The implementation of technology in the calculus classroom: An examination of teacher beliefs, practice and curriculum change* (Doctoral dissertation, Syracuse University, 1992). Dissertation Abstracts International, 53/06, 1876.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of teaching and learning mathematics* (pp. 515–556). New York: Macmillan.

Kissane, B. (2003). A model for professional development for graphics calculator use. In A. Rogerson (Ed.), *The mathematics education into the 21st century project* (pp. 153–157). September 19–25, 2003. Brno, Czech Republic.

Mayer, D. P. (1999). Measuring instructional practice: Can policymakers trust survey data? *Educational Evaluation and Policy Analysis, 21*(1), 29–45.

Molenje, L. (2012). *High school teachers' use of graphing calculators when teaching linear and quadratic functions: Professed beliefs and observed practice* (Doctoral thesis). Syracuse University, Syracuse.

Muthen, L. K., & Muthen, B. O. (1998–2015). *Mplus user's guide*. Los Angeles, CA: Muthen & Muthen.

Penglase, M., & Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research. *Mathematics Education Research Journal, 8*(1), 58–90.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 257–315). Charlotte: IAP.

Rögler, P., Barzel, B., & Eichler, A. (2013). Teachers' beliefs referring to teaching with technology. In A. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, p. 154). Kiel: PME.

Simmt, E. (1997). Graphing calculators in high school mathematics. *Journal of Computers in Mathematics and Science Teaching, 16*(2/3), 269–290.

Simonsen, L. M., & Dick, T. P. (1997). Teachers' perceptions of the impact of graphing calculators in the mathematics classroom. *Journal of Computers in Mathematics and Science Teaching, 16*(2/3), 239–268.

Staub, F. C., & Stern, E. (2002). The nature of teachers' pedagogical content beliefs matters for students' achievement gains: Quasi-experimental evidence from elementary mathematics. *Journal of Educational Psychology, 94*(2), 344–355.

Tharp, M. L., Fitzsimmons, J. A., & Ayers, R. L. B. (1997). Negotiating a technological shift: Teacher perception of the implementation of graphing calculators. *Journal of Computers in Mathematics and Science Teaching, 16*(4), 551–575.

Thurm, D., Klinger, M., & Barzel, B. (2015). How to professionalize teachers to use technology in a meaningful way—Design research of a CPD program. In S. Carreira & N. Amado (Eds.), *Proceedings of the 12th International Conference on Technology in Mathematics Teaching*. Faro: University of Algarve.

Vermunt, J. K., & Magidson, J. (2002). Latent class cluster analysis. In J. Hagenaars & A. McCutcheon (Eds.), *Applied latent class models* (pp. 89–106). New York: Cambridge University Press.

Willis, G. (2005). *Cognitive interviewing: A tool for improving questionnaire design*. Thousand Oaks, CA: Sage Publications.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education—A perspective of constructs. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169–1207). Charlotte: Information Age.

Chapter 26
Uses of Technology in K–12 Mathematics Education: Concluding Remarks

Paul Drijvers, Michal Tabach and Colleen Vale

Abstract The aim of this closing chapter is to reflect on the content of this book and on its overall focus on the development of mathematical proficiencies through the design and use of digital technology and of teaching and learning with and through these tools. As such, rather than making an attempt to provide an overview of the field as a whole, or trying to define overarching theoretical approaches, we chose to follow a bottom-up approach in which the chapters in this monograph form the point of departure. To do so, we reflect on the book's content from four different perspectives. First, we describe a taxonomy of the use of digital tools in mathematics education, and set up an inventory of the different book chapters in terms of these types of educational use. Second, we address the learning of mathematics with and through technology. Third, the way in which the assessment of mathematics with and through digital technology is present in this monograph is reflected upon. Fourth, the topic of teachers teaching with technology is briefly addressed. We conclude with some final reflections, including suggestions for a future research agenda.

Keywords Teaching mathematics · Learning mathematics · Digital technology Digital tools · Assessment · Taxonomy

26.1 Taxonomy of Educational Use of Digital Tools

The title of this book mentions tools, topics and trends, so it seems appropriate in this closing chapter to look back at the different types of digital tools that are used throughout the chapters, and the role these tools play in mathematics education.

P. Drijvers (✉)
Freudenthal Institute, Utrecht University, Utrecht, The Netherlands
e-mail: p.drijvers@uu.nl

M. Tabach
Tel-Aviv University, Tel Aviv-Yafo, Israel

C. Vale
Monash University, Melbourne, Australia

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4_26

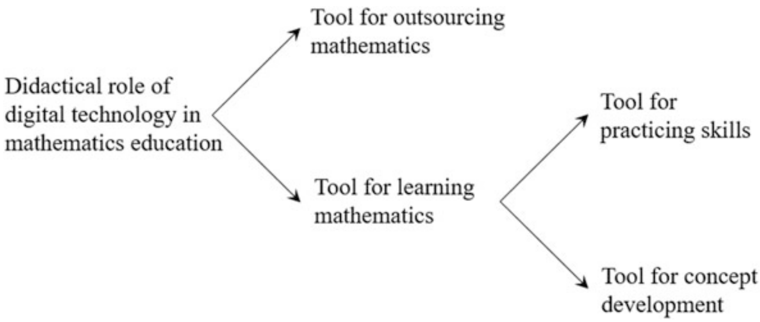


Fig. 26.1 Taxonomy of didactical roles of digital technology for mathematics

To frame this synthesis, a taxonomy might be useful. As a point of departure, we take the model shown in Fig. 26.1, which is adapted from earlier work (Arcavi, Drijvers, & Stacey, 2017; Drijvers, 2015). In this model, three different didactical roles for digital technology in mathematics education are distinguished: as a tool for outsourcing mathematics, as a tool for practicing skills, and as a tool for concept development.

In the first role, the didactical function of the digital tool comes down to outsourcing mathematics. The tool acts as a 'mathematical assistant' who takes over a (often more procedural or low-level) part of the work from the student, so that he can concentrate on the core issues at stake. Even if this may sound easy, the student constantly has to take decisions on what kind of job to outsource to the tool, how to do this, and how to critically interpret the results. As such, there is certainly a learning aspect involved in this type of use and in the process of instrumental genesis (Artigue, 2002).

The second didactical role focuses on practicing mathematical skills. For example, online applets may be used to practice skills such as equation solving, or algebraic expansion and factorization. Students can practice as long as they want, in private, and don't need a teacher. Randomization generates many tasks and automated feedback can be delivered. As is shown in the chapter by Drijvers (Chap. 9), however, online practice does not always lead to better achievement.

The third didactical role of digital tools concerns concept development, and probably is the most subtle one. Through graphical and dynamic representations, digital tools offer expressive power to students. This may lead them to engage in activities that support concept development. In many cases, this type of use includes outsourcing basic work to the technology, so this third role may encompass the first role. Also, we should point out that these different didactical roles do not just rely on the opportunities and constraints of the digital technology, but also on the type of task or activity, and its orchestration in the learning process. As such, we should not assign different roles too tightly to different digital tools.

How do the different chapters in this volume address these didactical roles? The first one, digital technology as a tool for outsourcing mathematics, is apparent in many contributions, even if it is sometimes addresses implicitly. Dick's chapter (Chap. 14) focuses on digital assessment. It is suggested that assessment may be enhanced through the use of digital technology such as dynamic geometry systems. Basic procedural skills, such as graphing lines and parabolas, or changing triangles and quadrilaterals by moving points, are taken care of by the digital environment.

The second didactical role, digital technology as a tool for practicing skills, is addressed only to a limited extent. Moyer-Packenham et al. (Chap. 2) studied the use of apps in a touch screen environment with young children addressing a variety of skills, appropriate to the children's age, like count skipping. Drijvers (Chap. 9) describes an experiment with an applet on practicing equation solving, the results which were slightly disappointing. Some possible causes are discussed.

The third didactical role, digital technology as a tool for concept development, dominates the book, even if it in many cases builds upon the outsourcing role. In Chap. 3, Tucker studied the use of apps to foster the concept of a number line, and its infinite density in particular—a concept known to be challenging for students. Note that the students were only 11 years old. Larsen and colleagues (Chap. 4) focus on discourse and communication that foster higher-order skills such as problem solving. The contributions by Voltolini and the one by Maschietto (Chaps. 5 and 11, respectively) both focus on conceptual understanding in geometry through the use of "duo's of artefacts", consisting of material and digital tools. In the case by Maschietto, this concerns Pythagorean theorem, whereas Voltolini addresses the concept of triangle. Both see the two types of tools as complementary and the articulation of the two as productive for concept development. In Chap. 6, the active learning cluster was the strongest in the cluster analysis carried out by Larkin and Milford. Even if active learning can address different kinds of skills, our impression is that the focus, again, is on conceptual understanding in most cases. In her overview, Heid (Chap. 10) discusses in somewhat more detail a study by Parnafes and Disessa (2004), in which the focus was on reasoning with different computer-based representations of motion. Ball and Barzel's examples in Chap. 12 focus on the concept of variable, as it is manifest in the type of representations and interactions that are provided by computer algebra devices and spreadsheets. In the contribution by Trgalova and colleagues (Chap. 15), the tool draws geometrical elements for the student, and in this sense it may address the first function of outsourcing; yet the whole idea of the c-book is to allow for an expressive way to relate to geometrical concepts so that the students will be creative in experiencing links and general properties. As such, the chapter focuses on concept development. In the chapter by Ng and Sinclair (Chap. 16), the 3D pen is used to conceptualize ideas based on "drawing" a three-dimensional object. The opportunities to create tangible, movable and rotatable drawings is exploited to enhance conceptual learning. In a similar approach, but with different technological tools, Moreno-Armella and Corey Brady (Chap. 19) draw on the dynamic and connected representational power of technological tools to foster conceptual understanding. In Chap. 20 by Donevska-Todorova, the emphasis on outsourcing is more prevalent than in the other chapters mentioned in this paragraph. However, also in this case the main motivation is to use dynamic representations to support the learning of abstract concepts.

To summarize this brief inventory, the first conclusion is that many contributions in this book focus on the didactical opportunities digital technology offers for concept development. Apparently, this is the role that researchers find the most relevant or urgent to investigate. In many cases, the concept development activity is rooted in some functionality that can be outsourced to the digital technology, so from this perspective the outsourcing role is often implicitly present. A second conclusion is that the didactical role of tool for practicing skills is addressed only to a limited extent. Taking into account the popularity of apps, online tutors, exercisers and repositories that focus on skill mastery among students, this is slightly surprising. Is the mathematics education research community not so much interested in skill training? Are there no pressing research issues in this domain? Does this limited attention relate to the preference for qualitative research, addressed in the contribution by Heid? These are interesting questions to further explore.

26.2 Learning With and Through Technology

One of the enduring focuses of research concerning the use of digital technologies in mathematics education has been the design and use of digital tools with regard to their impact on student learning and the ways in which students might engage with mathematics through their use of these tools. Much of this research activity arises because the continual development of digital technology such as tablets, touch screens, virtual manipulatives and screen casting which opens up new possibilities for the use of digital tools and apps for mathematics learning. These developments highlight the need for further research on the interdependence of tools, tasks, pedagogical approaches and learning outcomes (Hoyles & Lagrange, 2010).

Recent reviews of research have reported on the transformation of learning with and through digital technology (Bartolini & Borba, 2010; Hoyles & Lagrange, 2010; Larkin & Calder, 2016; Trouche & Drijvers, 2010). A number of studies included in this monograph focus on the way in which learners interact with digital tools for their learning of mathematics or reported on what students learned when using digital technology. Previous research on learning with and through technology has been dominated by instrumental theories informing studies as the researchers focused on the interplay between the learner and the tool for the learning of mathematics and instrumental genesis theory that is more explicit and focused on the way in which students' thinking is shaped by the tool and shapes their interaction with the tool (Drijvers et al., 2010). However socio-cultural theories of learning have also been evident, for example Borba and Villarreal (2005) *humans-with-media* theory. Beatty and Geiger (2010) reported that as digital technology continued to develop, new possibilities for student investigation and for communication and collaboration between students and between the teacher and students opened up. Studies of tools and learning environments taking advantage of these affordance tend to be informed by Vygotsky's socio-cultural theories of learning. These theories of learning shift the focus from the individual learner and their interaction with the tool to the learner and their communication with and through the tool. Other factors aside from the affordance of the tool such as the classroom environment and the teacher's pedagogical practice become critical to the investigation of student learning in this context.

In this section of the chapter, we consider "learning with" and "learning through" technologies and the learning theories informing these studies as they relate to the purpose of the study or the features of the digital technologies used. We also consider the learning objectives of the mathematical tasks and tools in the research with respect to mathematics proficiencies to consider whether there has been a shift in the learning objectives of research of the impact of digital technologies on students' learning.

26.2.1 Distinguishing the With and the Through

Previous research has noted the different ways in which teachers intend the digital tool(s) to be used and students to interact with and use digital tools for their learning of mathematics. Goos, Galbraith, Renshaw, and Geiger (2003) provided a model with which to frame the purpose and way in which students used or interacted with digital technology. They identified four relationships: *master* where the student become dependent on the digital tool to do the mathematics where their knowledge and skills are limited and thus become more limited; *servant* where the student uses it as an efficient and speedy tool to replace calculating and reasoning by pen and paper; *partner* when the student uses the tool to experience and explore different representations or perspectives or to mediate communication; and *extension of self* when students use the digital technology autonomously to investigate and problem solve. The relationships of servant and partner might be described as concerning learning with digital tools, and depending on the tool, partner along with extension of self could be described as learning through digital tools.

As researchers, we are often concerned at the prevalence of digital tools being the master for some students and the prevalence of pedagogical practices which result in digital tools being a servant for many students. Bowman (Chap. 24) conducted a large study of 14–17 year old secondary students' use of graphics calculators to graph quadratics and their learning in order to address many concerns that teachers have about the use of digital tools as the *master* or a *servant* in upper secondary mathematics. The pre-and post-testing found that students who were introduced to graphic calculators at the beginning of the topic demonstrated higher levels of engagement with mathematical knowledge and retained their acquired knowledge more than students who commenced the topic with traditional transmission teaching and pen and paper exercises. These findings support the notion that students were more inclined to use the graphic calculator as a partner when first being introduced to a topic.

Greefrath and Sillar (Chap. 21) report on 10th grade students' use of *GeoGebra* for mathematical modelling. They report that students used the tool to draw, construct, measure and experiment, and observed that the task and the tool enabled the students to shift their perception and use of the digital tool from *servant* to *partner*. Whilst establishing the use of digital software for a number of steps in the mathematical modelling process they did not relate their work to a particular learning theory and in this chapter did not report on what the students learned. A number of other chapters, Moyer-Packenham, Lister, Bullock, and Shumway (Chap. 2), Tucker (Chap. 3), Walter (Chap. 7) and Calder and Murphy (Chap. 8) also report on studies where the purpose of the study was either to evaluate a tool for its potential to enable to learner to interact with or through the tool as a *partner*, or to use the tool as a mediator for discussion. The teacher in Maschietto (Chap. 11) study used an interactive whiteboard (IWB) to orchestrate whole class discussion of proof of Pythagoras' Theorem using images of proofs using paper-folding. Whilst there may have been opportunity for this study to investigate the use of an IWB to shift students from using digital tools as a servant to that of a partner, the teacher remained in control of the technology and therefore the study did not explore this possibility.

Three research studies investigated students' interactions with touch screens (Moyer-Packenham et al., Chap. 2; Tucker, Chap. 3, and Walter, Chap. 7). Each of these studies investigated the way in which students interacted with these tools. They explored the potential of these tools to represent mathematical concepts in different ways as occurs when learners use tools as a partner. They followed the tradition of instrumentation and instrumental genesis as the theoretical frame informing their studies (Drijvers et al., 2010) though only one of these studies specified a particular learning theory. Tucker (Chap. 3) uses activity theory and case study methods to identify factors that influence students' mathematical thinking when interacting with a dynamic number line tool. According to activity theory learning occurs when students' interaction with an object that is, the mathematical concept represented by the tool or artefact, changes to indicate new understandings of the concept. The study found that the decreased technological distance that is, the touch screen and dynamic number line, enabled the creation of the images that supported students' mathematical thinking and learning.

The other two studies involving touch screens and number sense might have framed their study using instrumental genesis or activity theory. Moyer-Packenham et al. (Chap. 2) argue that students need experiences with multiple representations of concepts to enhance learning. Their study used pre- and post-testing of pre-school to Grade 2 students' use of a number of different touch screen apps each using different representations of number concerning place value, number facts and derived addition strategies. They used pre- and post-testing to measure changes in students' performance and efficiency and found that there were not consistent improvements for both performance and efficiency for each grade level. They conclude that improvement in students' performance and efficiency occurred when there was both mathematical and structural alignment between the tool and the mathematics, and when the mathematical representations of the app accounted for and addressed common student misconceptions. Walter (Chap. 7) study might also have been framed by instrumental genesis or activity theory as he compared the learning that occurred when primary students with mathematic learning needs concerning efficient strategies for adding interacted with physical models and touch screen digital tools. He found that the affordance of the touch screen app provided potential for experiencing alternate representations of number but not all students took advantage of the tools' affordances to interpret the representation of number for learning to occur. So in spite of alternate representations of number that the touch screen tool provided, the tool was not influencing some students' mathematical interpretation of the digital representation and students were not using their mathematical understanding of number to create representations of number that showed evidence of number sense.

Calder and Murphy (Chap. 8) study provides a different perspective on learning with technology as a *partner*. In their study where Grade 4 students used an app to solve worded problems, and present and explain their solution, the digital technology acted as a mediator of classroom discourse. Not surprisingly, they used a social constructivist theory of learning, socio-materialism to frame their study. They cite Meyer (2015) and explain socio-materialism as recognising the complex and dynamic inter-relationships between people, communities such as classrooms, and tools where the dialogue and learning experience is changing and personalised. In their study, students created their own content, using their own language to create a movie to communicate their solution and explain their solution process. Screen casting along with audio recording provided a new mathematical learning experience for these students. Larsen et al. (Chap. 4) also investigated the use of screen casting technology in K–2 grade classrooms over a number of years and report that screen casting technology encourage students to communicate, self-assess and revise their mathematical ideas. Their study illustrated the way in which students in the early years of school interact with both digital tools as partners and with each other for their learning.

Only one study in the manuscript really explored the use of digital technology as an *extension of self*. Ng and Sinclair (Chap. 16) reported on Grade 12 calculus students' use of 3D pens to explore properties of quadratic functions. Their study can be argued to be concerned with extending self as they argued that using 3D pens enabled students to use gesture to explore and develop their understanding when using the 3D pens to 'draw' in 3D space. They framed their study using inclusive-materialism that is, the entwinement of humans, tools and concepts. They observed new gestural forms of thinking to make sense of the curve and tangent to the curve in both a physical and abstract sense.

26.2.2 Developing Mathematical Proficiencies with and Through Digital Technology

While there has been much research that focused on how students learn mathematics with and through digital technology and some studies have focused on mathematics achievement (see section of assessment this chapter), attention with respect to what students learn has mostly been directed at particular mathematics topics and concepts. With the increased emphasis given to the need for mathematics curriculum to include and develop students' problem solving and reasoning, however, it is worth considering which mathematical proficiencies can be supported with and through digital technology and whether or not the research is focused on this question. Mathematical proficiencies are included in the mathematics curriculum of many countries, most being influenced by the *Adding it Up* report (Kilpatrick et al., 2001). This report identified five proficiencies: conceptual understanding; procedural fluency that is, carrying out procedures efficiently, flexibly and accurately; strategic competence that is, formulating and solving problems; adaptive reasoning that is, logical thought, generalising, justifying and proving; and productive disposition that is, seeing mathematics as relevant and useful and makes sense (Kilpatrick, Swafford, & Findell, 2001).

We could argue that teachers in both primary and secondary school have been focused on using digital apps and tools to develop students' fluency (procedural knowledge) for practice and to target teaching and student practise of exercises using digital resources which provides automatic feedback for the student and records for the teacher. This negative perception of the use of digital tools—graphics calculators, was one of the main reasons for Bowman's study (Chap. 24). She was able to show that whilst graphics calculators may serve to assist with procedural fluency, using them at the introduction of the topic along with inquiry type activities promoted deeper engagement with the concepts.

Much research has focused on developing apps and digital tools to develop student understanding, especially in primary school, hence the focus on affordances and constraints and trialing and evaluating of tools and apps with both small and large sample sizes. This is the case with studies reported in this monograph. Moyer-Packenham et al. (Chap. 2), Tucker (Chap. 3), Walter (Chap. 7), Calder and Murphy (Chap. 8), and Ng and Sinclair (Chap. 16) have all focused on students developing understanding with and through digital tools. For younger students this included developing number sense for numbers to 20 and using this sense for adding numbers to 20 (Moyer-Packenham et al., Chap. 11; Walter, Chap. 7) and developing number sense for whole numbers and decimals (Tucker, Chap. 3).

The assumption has been that secondary teachers would use digital tools, software and apps for inquiry to develop new understandings for example by exploring properties of shapes, functions and forming and testing conjectures about these properties. However, the focus of the secondary studies reported here concerned developing understanding. The study by Ng and Sinclair (Chap. 16) focused on students' understanding of tangents to quadratic functions rather than forming and testing conjectures or generalising and Maschietto (Chap. 11), who studied a lesson that used an IWB to explore proofs of Pythagoras' theorem, focused on developing conceptual understanding rather than adaptive reasoning.

Three studies where there was a focus on learning with or through technology were concerned with or reported on developing one or more of the other three proficiencies: strategic competence, adaptive reasoning and productive disposition.

Greefrath and Sillar (Chap. 21) set out to show that *GeoGebra* could be used by secondary students to solve a modelling problem and showed that digital tools and tasks could be used to develop students' strategic competence or problem solving. They did not however report on the learning outcomes in this chapter. In the study by Calder and Murphy (Chap. 8) students solved word problems and reported and explained their solution and solution process using an embedded screen casting technology. They reported that the tool and tasks provided a degree of student autonomy and motivated students to articulate and communicate their mathematical thinking. Likewise, Larsen et al. (Chap. 4) study illuminates the potential for screen casting technology to promote young students' mathematical reasoning. The teachers' learning goal for these mathematics lessons was to construct viable arguments and to critique others' arguments. The way in which these students were able to verbalise mathematical processes and make connections between concepts resulted in teachers improving their attitudes about their students' capabilities.

26.3 Conclusion

The studies reported at the 13th ICME conference and those published in this monograph concerning learners and learning tend to focus on how learning occurs with or through digital technology rather than providing evidence of learning with or through technology. These studies are also predominantly concerned with developing students' understanding of mathematics concepts rather than engaging students in problem solving and adaptive reasoning or developing productive dispositions. Two studies published here, Moyer-Packenham et al. (Chap. 2) and Bowman (Chap. 24), used a large sample, to evaluate the effectiveness of digital tools for learning, however further evidence may be needed to convince teachers that these tools can develop students' proficiencies regarding understanding, problem solving and reasoning in order for them to shift from the dominant practice of using these tools to develop fluency or as servants to do mathematics. One study focused on the learning of students with specific mathematics learning needs, however for the most part these studies do not consider the student cohort, their needs or funds of knowledge. Research is needed concerning particular cohorts of students, especially groups who are disadvantaged or where lower student outcomes have persisted overtime such as girls, low socio-economic and Indigenous students. Students who are differently abled are beginning to receive research attention, especially as digital technology enables other means of interacting with the tool and representations. Instrumental genesis theory and activity theory were the most used frameworks for these studies though others that drew upon socio-cultural theories of learning were evident in the research where students were expected to communicate solutions and problem solving processes or mathematical reasoning. Inevitably, theoretical frameworks will evolve as we deepen our understanding of the role and place of digital technology for learning mathematics.

26.4 Assessment with and Through Technology

Learning, specifically in formal institutions like schools, cannot do without assessment. The more than 20 years old *Assessment Standards for School Mathematics* (NCTM, 1995) are still valid for large-scale and classroom assessments in mathematics education: ensuring that assessments contain high quality mathematics; enhance student learning; reflect and support equitable practices; open and transparent; inferences made from assessments are appropriate to the assessment purpose; and finally the assessment—together with the curriculum and instruction, form a coherent whole. Often “assessments define what counts as valuable learning and assign credit accordingly” (Baird, Hopfenbeck, Newton, Stobart, & Steen-Utheim, 2014, p. 21). It is therefore clear that the integration of technology into mathematics teaching and learning should be coupled by integration of technology into both formative and summative mathematics assessments.

An important distinction between two types of technology-rich assessment, based on the work by Stacey and Wiliam (2013), was made by Drijvers et al. (2016): assessment with technology and assessment through technology. Assessment *with* digital technology concerns paper-and-pen written tests, during which students have access to digital technology such as (graphing or CAS) calculators or computers. In assessment *through* digital technology, the test is delivered and administered through digital means. Think, for example, of online tests, in which all student responses are entered in the digital test player environment. Both types of assessment may relieve students from computation and drawing, hence affects the type of skills assessed, the goals of the assessment, the tasks, and the validity and the reliability of the assessment. Students’ mathematical literacy abilities can be assessed more easily, as well as their conceptual understanding, strategies, and modelling and problem-solving skills.

In their state-of-the-art survey about assessment, Suurtamm et al. (2016) raised several questions for further studies, which concern assessment and technology:

How does the use of technology influence the design of assessment items? What are the affordances of technology? What are the constraints? (p. 12).

What are some of the additional challenges in assessment when hand-held technologies are available (e.g., graphing calculators) or mobile technologies are easily accessible (e.g., smart phones with internet connections)? (p. 19).

Three papers in this volume address these questions and others, and provide initial answers which for the most part reflect three different directions—from theoretical and practical dimensions, at the middle and high school levels: Dick (Chap. 14) focused on geometry; Grugeon-Allys, Chenevotot-Quentin, Pilet, and Prévit (Chap. 13) focused on algebra; and Beck (Chap. 18) focused on written solutions in CAS allowed tests.

In the context of middle school geometry, Dick (Chap. 14) describes the development of an assessment with technology tool that will allow teachers to create their own assessments tasks. To go beyond the trivial, multiple-choice questions, which are checked against a pre-determined given key, he used DGE and

CAS. Students are presented with a geometric figure or graph, and are asked to modify the object so that it fulfils a given set of constrains. The system can check logical conditions or symbolic expressions, hence is able to accept a range of responses as valid. In this way, the system allows design of higher order tasks (Stein et al., 2000). The developers also created an interface by which the teacher can determined the specific tasks that his students will receive, and also retrieve his students' performances. The system was examined with some teachers and their classes, and is seen as a first promising step towards a fuller implementation. Dick's study highlighted both—the possibilities which open up while assessing with technology, as well as some constraints.

A very different example of the opportunities and limitations offered by assessment with technology is provided by Grugeon-Allys et al. (Chap. 13). The system developed by the researchers may be classified as assessing through technology. The aim was to develop a diagnostic tool—that will classify students' performance level using three specific sub-competencies in middle school algebra. The tasks in the diagnostic tool were design by the developers (and not the teachers) based on a comprehensive epistemological and cognitive analysis. This analysis is the basis for the classification of students' answers to three general levels of algebra competencies. It is important to note that the diagnostic tool is able to assess not only technics but also verbal answers by the students. Furthermore, the system also provides the teacher with a lesson or sequence of lessons that includes tasks on their chosen topic directed at learners that where classified to the three levels. First implementations of the system with teachers have already started.

The third study by Beck (Chap. 18) presents yet a different angle stemming from assessment with technology. When technology is used by the students as part of available tools in a test situation, the students may choose to use the technology at hand while answering the test items. Students have to decide what to do with the available technology, and to report on their uses as part of their paper and pencil report on the solution process. Beck used linguistic means to analyze students written reports on their solution process. His study highlighted the importance of communicating in class about how a written solution, which was partially based on open technology like CAS may look like to enable another person to make sense of the process. This is an example of a new challenge that arose as a result of incorporating technology with assessment.

26.5 Teachers Teaching with Technology

The mathematics-education research-community has a specific interest in teachers and technology. This is noted, for example, by the topic working group 15, focused on teaching mathematics with resources and technology during CERME10 (Clark-Wilson, Aldon, Kohanová, & Robutti, in press). The issues under study varied from professional development (PD) aimed at pre-service teachers to that of practicing teachers, possible gains from implementing technologies in teaching and learning mathematics and affective issues concerning implementation of that type. These issues and others are under ongoing study, as reflected also by four authors in the current volume.

An example of possible gains from implementing technology with young learners is provided by Calder and Murphy (Chap. 8). Teachers who worked with young students in class with touch-screen technology commented on its similarity to interacting with physical representation of a mathematical object. Also, students were able to create their own mathematical objects and speak about them using their own terminology in a safe, non-judgmental environment. Such student behavior provided the teachers with access to students' ways of thinking.

A different perspective to understand teachers' frequent use of technology is linking it to teachers' beliefs. Effective aspects have a major influence on human behavior. Specifically, teachers' beliefs about technology are an important factor in the teachers' choices to implement mathematics teaching and learning in a technological environment. Thurm (Chap. 25) studied 160 high school mathematics teachers' responses to questions about the frequency of technology use in class and about teachers' beliefs about the value of such integration. By means of latent profile analysis, he was able to identify four distinct groups of teachers. Two groups acted in a consistent way with regard to technology use and beliefs: "positive beliefs—frequent users" and "negative beliefs—infrequent users". The two other groups acted in inconsistent ways: "positive beliefs—infrequent users" and "negative beliefs—frequent users". More research is needed to understand these teachers' actions. In any case, PD providers need to be aware of these four sub-groups, as they span a range of needs to be taken into account.

Yet another perspective to be considered with respect to teachers' work is suggested by Ball and Barzel (Chap. 12). The researchers pointed at three communicational roles of technology: communication *through* technology (e.g. social networks), communication *with* technology (e.g. syntax entry), and communication *of* technology (e.g. when technology displays are used as a stimulus for communication). From the teacher perspective, they call for professional development aimed first to help teachers to be confident in how to make personal use of the technology. Second, teachers need to learn how to use communication technologies to enable students learning in a technological environment. These two levels of knowledge are captured by the theoretical construct of *double instrumental genesis* (Haspekian, 2011). While the personal instrumental genesis is related to the development of a teacher's personal instrument for mathematical activity from a given artefact, the professional instrumental genesis yields a professional instrument for the teacher's didactical activity.

This same line of thought is further explored by Trgalová and Tabach (Chap. 23). Based on an extensive literature review of PD programs offered and implemented with high-school teachers, it turned out that most PD providers are not satisfied with teachers' knowledge by the end of their program. This dissatisfaction is stressed by the fact that in most cases the researches themselves are highly involved in providing the PD. Interestingly, a search for standards for teachers' knowledge while working in ICT environments at the international level yield only a few and very general recommendations. At the national level, there are variations between countries. The few available documents were analyzed by Trgalová and Tabach to identify if and how they address the double instrumental genesis, and to which of the seven knowledge-area suggested by Mishra and Koehler (2006) they relate.

26.6 Conclusion

To conclude this chapter, we now briefly revisit each of the sections and extrapolate some possible future research topics. In Sect. 26.2, it is remarkable that so many chapters in this monograph seem to focus on the use of digital technology to foster conceptual understanding. Of course, we don't want to argue against the importance of conceptual understanding in mathematics education, but we do recommend further research into the use of digital environments for other didactical purposes, such as practicing skills. It is our impression that online practicing environments are popular among students, so it might be worthwhile to know more about the conditions that make such practice most efficient and fruitful.

Section 26.3 also identified a need for studies that focus on using technology to develop problem solving and reasoning proficiencies. It ends with a plea for more attention to be given to the use of digital technology for students with special needs, which we repeat here. Are there specific characteristics or criteria for digital technology, as to make different types of students all benefit from the opportunities these tools offer? Much remains unknown on this topic.

A core distinction in Sect. 26.4 concerns assessment with and assessment through technology. Even if this distinction seems fruitful in many cases, the question still is how this with-through dimension impacts on different types of assessment, including formative assessment, peer assessment and self-assessment. How to set up these types of assessment in a fruitful way in digital environments?

In Sect. 26.5 on mathematics teachers' professional development with respect to the use of digital technology in their teaching, much is unknown about what exactly the knowledge and skills needed encompass. Also, the different theoretical approaches are not convergent. The TPACK model by Mishra and Koehler (2006), for example, is useful but does not include teachers' beliefs. As teachers' practices are crucial in education, further elaboration of successful models for teachers' professional development is needed.

Overall, the monograph on the one hand provides a wide range of relevant contributions to the knowledge in the field of using digital technology in mathematics education. On the other hand, much work needs to be done to fully exploit its potential in everyday teaching.

References

Arcavi, A., Drijvers, P., & Stacey, K. (2017). *The teaching and learning of algebra: Ideas, insights and activities*. London/New York: Routledge.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7,* 245–274.

Bartolini Bussi, M.G., & Borba, M.C. (Eds.). (2010). The role of resources and technology in mathematics education. *ZDM Mathematics Education, 42*(1).

Beatty, R., & Geiger, V. (2010). Technology, communication and collaboration: Rethinking communities of inquiry, learning and practice. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain. 17th ICMI Study* (pp. 251–286). New York, NY: Springer.

Borba, M., & Villarreal, M. (2005). *Humans-with-media and the reorganization of mathematical thinking*. New York, NY: Springer.

Baird, J., Hopfenbeck, T. N., Newton, P., Stobart, G., & Steen-Utheim, A. T. (2014). *State of the field review: Assessment and learning. Norwegian Knowledge Centre for Education*. Oxford, England: Oxford University Centre for Educational Assessment.

Clark-Wilson, A., Aldon, G., Kohanová, I., & Robutti, O. (in press). *Introduction to the papers of TWG15: Teaching mathematics with resources and technology*.

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In S. J. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 135–151). Switzerland: Springer.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education; A concise topical survey*. New York: Springer.

Drijvers, P., Kieran, C., Mariotti, M., Ainley, J., Andresen, M., Cahn, Y. C., Dana-Picard, T., … Meagher, M. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics Education and Technology—Rethinking the Terrain* (17th ICMI Study, pp. 89–132). New York, NY: Springer.

Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2003). Perspectives on technology mediated learning in secondary school mathematics classrooms. *Journal of Mathematical Behavior, 22,* 73–89.

Haspekian, M. (2011). The co-construction of a mathematical and a didactical instrument. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Congress of the European society for research in mathematics education* (pp. 2298–2307). Rzeszów: University of Rzeszów.

Hoyles, C., & Lagrange, J.-B. (2010). Introduction. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain* (pp. 1–11). New York: Springer.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Larkin, K., & Calder, N. (2016). Mathematics education and mobile technologies. *Mathematics Education Research Journal, 28,* 1–7.

Meyer, B. (2015). iPads in inclusive classrooms: Ecologies of learning. In P. Isaias, J.M. Spector, & D. Ifenthaler. *E-learning systems, environments and approaches theory and implementation*. Dordrecht: Springer International Publishing.

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A new framework for teacher knowledge. *Teachers College Record, 108*(6), 1017–1054.

National Council of Teachers of Mathematics (NCTM). (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

Parnafes, O., & Disessa, A. (2004). Relations between types of reasoning and computational representations. *International Journal of Computers for Mathematical Learning, 9*(3), 251–280.

Stacey, K., & Wiliam, D. (2013). Technology and assessment in mathematics. In M. A. Clements, A. Bishop, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education* (pp. 721–751). New York: Springer.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York, NY: Teachers College Press.

Suurtamm, C., Thompson, D. R., Kim, R. Y., Moreno, L. D., Sayac, N., Schukajlow, S., Silver, E., … Vos, P. (2016). *Assessment in mathematics education.* Springer International Publishing.

Trouche, L., & Drijvers, P. (2010). Handheld technology for mathematics education: Flashback into the future. *ZDM International Journal of Mathematics Education, 42*(7), 667–681.

Index

L. Ball et al. (eds.), *Uses of Technology in Primary and Secondary Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-76575-4