Chapter 12
Insurance Models Under Incomplete Information

Ekaterina Bulinskaya and Julia Gusak

Abstract The aim of the chapter is optimization of insurance company performance under incomplete information. To this end, we consider the periodic-review model with capital injections and reinsurance studied by the authors in their previous paper for the case of known claim distribution. We investigate the stability of the one-step and multi-step model in terms of the Kantorovich metric. These results are used for obtaining almost optimal policies based on the empirical distributions of underlying processes.

Keywords Incomplete information · Periodic-review insurance model
Reinsurance · Capital injections · Optimization · Stability

12.1 Introduction

The primary goal of any insurer is redistribution of risks and indemnification of policyholders. This explains the popularity of reliability approach in actuarial sciences, that is, thorough analysis of ruin probability. Being a corporation insurance company has a secondary but very important goal, namely dividends payment to the shareholders. So, the alternative cost approach was started by De Finetti in 1957 (see [9]).

Thus, there arose the new research directions in actuarial sciences specific for modern period. They include, along with dividends payments (see, e.g., [1, 2, 11, 15]), reinsurance and investment problems (see, e.g., [4, 8, 13]). Hence, the treatment of complex models (see, e.g., [6]) and consideration of new classes of processes, such as martingales, diffusion, Lévy processes, or generalized renewal ones (see [7]), is needed. It turned out as well that discrete-time models sometimes are more realistic since reinsurance treaties have usually one-year duration, dividends are also paid at the end of financial year (see, e.g., [17]). Several types of objective functions and various methods are used to implement the stochastic models optimization (see, e.g., [19, 22]). It is also important to mention investigation of systems asymptotic

E. Bulinskaya (✉) · J. Gusak
Lomonosov Moscow State University, Moscow, Russia
e-mail: ebulinsk@yandex.ru

J. Pilz et al. (eds.), *Statistics and Simulation*, Springer Proceedings
in Mathematics & Statistics 231, https://doi.org/10.1007/978-3-319-76035-3_12

behavior and their stability with respect to parameters fluctuation and perturbation of underlying processes (see, e.g. [3, 5, 25]). Furthermore, in practice neither the exact values of parameters nor the processes distributions are known. Thus, it is important to study the systems behavior under incomplete information. If there is no a priori information at all it may be useful to employ the empirical processes.

The chapter is organized as follows. In Sect. 12.2, we gather some auxiliary results. The results concerning convergence in distribution in L_1 are transferred to Appendix. Section 12.3 contains a brief description of the model treated in the chapter (Sect. 12.3.1). Further parts of Sect. 12.3 are devoted to stability of the model under consideration. The case of unknown claim distribution is considered in Sect. 12.4. Finally, Sect. 12.5 presents conclusion and further research directions.

12.2 Preliminary Results

To investigate stability of the model, it is necessary to evaluate the difference between the objective functions calculated for two distributions close in some metric. For this purpose, we have chosen Kantorovich or Wasserstein L_1 metric.

12.2.1 Kantorovich or Wasserstein L_1 Metric

We begin by recalling the following definition given, e.g., in [23], see also [20].

Definition 12.1 For random variables (r.v.'s) X and Y defined on some probability space $(\Omega, \mathscr{F}, P)$ and possessing finite expectations, it is possible to define their distance on the base of Kantorovich metric in the following way

$$\kappa(X, Y) = \int_{-\infty}^{+\infty} |F(t) - G(t)| dt,$$

where F and G are the distribution functions (d.f.'s) of X and Y, respectively.

This metric coincides (see, e.g., [12] or [23]) with Wasserstein L_1 metric defined as $d_1(F, G) = \inf E|X - Y|$ where infimum is taken over all jointly distributed X and Y having marginal d.f.'s F and G. It is supposed that both d.f.'s belong to $\mathscr{C}_1$ consisting of all F such that $\int_{-\infty}^{+\infty} |x|\, dF(x) < \infty$.

Lemma 12.1 *The following statements are valid.*
1. Let $F^{-1}(t) = \inf\{x : F(x) \geq t\}$, then $d_1(F, G) = \int_0^1 |F^{-1}(t) - G^{-1}(t)|\, dt$.
2. $(\mathscr{C}_1, d_1)$ is a complete metric space.
3. For a sequence $\{F_n\}_{n\geq 1}$ from $\mathscr{C}_1$ one has $d_1(F_n, F) \to 0$ if and only if $F_n \xrightarrow{d} F$ and $\int_{-\infty}^{+\infty} |x|\, dF_n(x) \to \int_{-\infty}^{+\infty} |x|\, dF(x)$, as $n \to \infty$. Here $\xrightarrow{d}$ denotes, as usual, convergence in distribution.

The proof can be found in [12].

We are going to use also the notion of Lipschitz function.

Definition 12.2 A function f mapping a metric space (S_1, ρ_{S_1}) into a metric space (S_2, ρ_{S_2}) is called Lipschitz if there exists a constant $C \geq 0$ such that $\rho_{S_2}\left(f(s'), f(s'')\right) \leq C\rho_{S_1}(s', s'')$ for any $s', s'' \in S_1$, here ρ_{S_1}, ρ_{S_2} denote metrics in the corresponding spaces.

Now we can formulate

Lemma 12.2 *Let X, Y be nonnegative r.v.'s possessing finite expected values and $\kappa(X, Y) = \rho$. Assume also that $g : R^+ \to R^+$ is a non-decreasing Lipschitz function. Then $\kappa(g(X), g(Y)) \leq C\rho$ where C is the Lipschitz constant.*

Proof The distribution function of the random variable $g(X)$ can be calculated in a following way

$$F_{g(X)}(t) = P\{g(X) \leq t\} = P\{X \leq g^{-1}(t)\} = F_X\left(g^{-1}(t)\right),$$

where $g^{-1}(t)$ is defined as in Lemma 12.1. Similarly, one can write $F_{g(Y)}(t) = F_Y\left(g^{-1}(t)\right)$.

Since g is a non-decreasing Lipschitz function, we get the following sequence of equalities and inequalities

$$\begin{aligned}
\kappa(g(X), g(Y)) &= \int_{R^+} |F_{g(X)}(t) - F_{g(Y)}(t)|dt = \int_{g^{-1}(R^+)} |F_X(s) - F_Y(s)|dg(s) \\
&= \int_{g^{-1}(R^+)} |F_X(s) - F_Y(s)|g'(s)ds \leq C \int_{g^{-1}(R^+)} |F_X(s) - F_Y(s)|ds \\
&\leq C \int_{R^+} |F_X(s) - F_Y(s)|ds = C\rho.
\end{aligned}$$

In the first line, we have used the definition of Kantorovich metric and change of variables $t = g(s)$. As usually, $g^{-1}(R^+)$ is preimage of R^+. Then the properties of Lipschitz functions are employed. □

The next result enables us to estimate the difference between infimums of two functions.

Lemma 12.3 *Let functions $f_1(z)$, $f_2(z)$ be such that $|f_1(z) - f_2(z)| < \delta$ for some $\delta > 0$ and any $z > 0$. Then $|\inf_{z>0} f_1(z) - \inf_{z>0} f_2(z)| < \delta$.*

Proof Put $M_i = \inf_{z>0} f_i(z)$, $i = 1, 2$. According to definition of infimum, for any $\varepsilon > 0$, there exists $z_1(\varepsilon)$ such that $f_1(z_1(\varepsilon)) < M_1 + \varepsilon$. Therefore, $f_2(z_1(\varepsilon)) \leq f_1(z_1(\varepsilon)) + \delta < M_1 + \varepsilon + \delta$ implying $M_2 \leq f_2(z_1(\varepsilon)) < M_1 + \varepsilon + \delta$.

Letting $\varepsilon \to 0$ one gets immediately $M_2 \leq M_1 + \delta$. In a similar way, one establishes $M_1 \leq M_2 + \delta$, thus obtaining the desired result $|M_1 - M_2| < \delta$. □

12.2.2 Distance Between Empirical Functions

First of all we would like to establish that if the difference between two d.f.'s is small in the Kantorovich metric the same is true for the corresponding empirical functions. Thus, let $\{X_i = X_i(\omega)\}_{i=1}^n$ be a sample of size n from the population of r.v.'s with d.f. F. The empirical d.f. is given by $\mathsf{F}_n(\omega, t) = n^{-1}\sum_{i=1}^n I\{X_i \le t\}$, where $\omega \in \Omega$ and $I\{A\}$ is indicator of the set A. Suppose there exists another sample $Y_1, \ldots, Y_n$ from the population of random variables with distribution function, say, G. The empirical distribution function for this sample is denoted by $\mathsf{G}_n(\omega, t)$. Note that we are going to assume further on the samples to consist of independent identically distributed (i.i.d.) r.v.'s.

Proposition 12.1 *Let* $\kappa(F, G) = \rho$, *then* $P(\kappa(\mathsf{F}_n, \mathsf{G}_n) > \rho) \to 0$ *as* $n \to \infty$.

Proof Obviously, $\int_{-\infty}^{+\infty} |\mathsf{F}_n(\omega, t) - \mathsf{G}_n(\omega, t)|dt$ does not exceed

$$\int_{-\infty}^{+\infty} |\mathsf{F}_n(\omega, t) - F(t)|dt + \int_{-\infty}^{+\infty} |\mathsf{G}_n(\omega, t) - G(t)|dt + \int_{-\infty}^{+\infty} |F(t) - G(t)|dt,$$

therefore we get $P\left(\kappa(\mathsf{F}_n, \mathsf{G}_n) > \varepsilon + \rho\right)$ is less than

$$P\left(\kappa(\mathsf{F}_n, F) > \frac{\varepsilon}{2}\right) + P\left(\kappa(\mathsf{G}_n, G)|dt > \frac{\varepsilon}{2}\right) + P\left(\kappa(F, G) > \rho\right). \tag{12.1}$$

The last term in (12.1) is equal to 0, since we assumed $\rho = \int_{-\infty}^{+\infty} |F(t) - G(t)|dt$. Two first terms tend to 0 as $n \to \infty$ for any $\varepsilon > 0$ due to convergence almost surely (a.s.) of empirical function to theoretical one in Kantorovich metric (see, e.g., [10]). Thus, we get the desired result. □

Remark 12.1 Since $\kappa(\mathsf{F}_n, F)$ and $\kappa(\mathsf{G}_n, G)$ tend to zero a.s., as $n \to \infty$, the statement of Proposition 12.1 can be rewritten as follows: $\limsup_{n\to\infty} \kappa(\mathsf{F}_n, \mathsf{G}_n) \le \rho$ if $\kappa(F,\ G) \le \rho$.

12.2.3 Convergence in Distribution for a Fixed t

Suppose for simplicity that two samples are independent. For each fixed $t \in R$ the difference $\mathsf{H}_n(\omega, t) =: \mathsf{F}_n(\omega, t) - \mathsf{G}_n(\omega, t)$ is a real-valued function of the random vector $(X_1, Y_1, \ldots, X_n, Y_n)$ defined on a probability space $(\Omega, \mathscr{F}, P)$, namely

$$\mathsf{H}_n(\omega, t) = \frac{1}{n}\sum_{i=1}^n I\{X_i \le t\} - \frac{1}{n}\sum_{i=1}^n I\{Y_i \le t\} = \frac{1}{n}\sum_{i=1}^n \zeta_i(t), \tag{12.2}$$

where $\zeta_i(t) = I\{X_i \le t\} - I\{Y_i \le t\},\ i = \overline{1, n}$, are i.i.d. r.v.'s. Recall that

$$E\zeta_i(t) = EI\{X_i \le t\} - EI\{Y_i \le t\} = F(t) - G(t),$$
$$\text{Var}\zeta_i(t) = \text{Var}I\{X_i \le t\} + \text{Var}I\{Y_i \le t\} = F(t) + G(t) - (F^2(t) + G^2(t)).$$

Since $\text{Var}\zeta_i(t) < \infty$, the central limit theorem for i.i.d. r.v.'s gives

$$\frac{\sum_{i=1}^{n} \zeta_i(t) - n\,(F(t) - G(t))}{\sqrt{F(t) + G(t) - (F^2(t) + G^2(t))}\sqrt{n}} \xrightarrow{d} N(0, 1),$$

where $N(0, 1)$ is a standard normal variable. In other words,

$$\sqrt{n}\frac{\mathsf{H}_n(\omega, t) - (F(t) - G(t))}{\sqrt{F(t) + G(t) - (F^2(t) + G^2(t))}} \xrightarrow{d} N(0, 1).$$

According to properties of convergence in distribution, we get immediately the following result.

Proposition 12.2 *For any* $t \in R$

$$\sqrt{n}\,|\mathsf{H}_n(\omega, t) - (F(t) - G(t))| \xrightarrow{d} \sqrt{F(t) + G(t) - (F^2(t) + G^2(t))}|N(0, 1)|.$$

12.3 Stability of Insurance Model

We are going to study the stability of the periodic-review model of insurance company performance with capital injections and reinsurance introduced in [8].

12.3.1 Model Description

Let u be the initial surplus of insurance company. It is supposed that the surplus at the beginning of each period has to be maintained above some level $a > 0$. Denote by ξ_n the aggregate claim during the nth period. The sequence $\{\xi_n\}$ is assumed to consist of i.i.d. r.v.'s with a known d.f. F possessing a density and a finite mean γ. The company concludes at the end of each period the stop-loss reinsurance treaty. If the retention level is denoted by $z > 0$, then $c(z) = l\gamma - m\int_z^{+\infty} \overline{F}(t)\,dt$ is the insurer premium (net of reinsurance). Here we supposed that the insurer and reinsurer premiums are calculated on the base of expected value principle, and l and m are the corresponding safety coefficients. As usual $\overline{F}(t) = 1 - F(t)$.

It is necessary to choose the sequence of retention levels minimizing the total discounted injections during n periods.

One-period minimal capital injections are defined as follows

$$h_1(u) := \inf_{z>0} \mathsf{E}J(u, z), \quad \text{where} \quad J(u, z) = (\min(\xi, z) - (u - a) - c(z))^+ .$$

For the n-step model, $n \geq 1$, the company surplus $U(n)$ at time n is given by the relation

$$U(n) = \max(U(n-1) + c(z) - \min(\xi, z), a), \quad U(0) = u.$$

It was also proved in [8] that the minimal expected discounted costs $h_n(u)$ injected in company during n years satisfy the following Bellman equation

$$h_n(u) = \inf_{z>0}(\mathsf{E}J(u,z) + \alpha \mathsf{E}h_{n-1}(\max(u + c(z) - \min(\xi, z), a))), \quad h_0(u) = 0, \tag{12.3}$$

where $0 < \alpha < 1$ is the discount factor.

Put $h_n(u,z) := \mathsf{E}J(u,z) + \alpha \mathsf{E}h_{n-1}(\max(u + c(z) - \min(\xi, z), a))$ for $n \geq 1$. It was established that infimum of the function $h_n(u,z)$ is achieved for some $z > 0$ and function $h_n(u)$ determined by (12.3) is continuous in u.

12.3.2 One-Step Model

We are going to add the label X to all functions depending on ξ if $\xi \sim law(X)$. Putting $\Delta_1 := \sup_{u>a} |h_{1_X}(u) - h_{1_Y}(u)|$ we prove the following result.

Theorem 12.1 *Let X, Y be nonnegative r.v.'s possessing finite expectations, moreover, $\kappa(X,Y) = \rho$. Then*

$$\Delta_1 \leq (1 + l + m)\rho$$

where l and m are the safety loading coefficients of insurer and reinsurer premiums, respectively. Both premiums are calculated according to expected value principle and $1 < l < m$.

Proof Begin by estimating $|\mathsf{E}J_X(u,z) - \mathsf{E}J_Y(u,z)|$. Setting

$$C_X := -(u-a) - l\mathsf{E}X + m\mathsf{E}(X-z)^+, \quad C_Y := -(u-a) - l\mathsf{E}Y + m\mathsf{E}(Y-z)^+,$$

it is possible to write

$$\begin{aligned}
|\mathsf{E}J_X(u,z) - \mathsf{E}J_Y(u,z)| &= |\mathsf{E}\,(\min(X,z) + C_X)^+ - \mathsf{E}\,(\min(Y,z) + C_Y)^+| \\
&\leq \underbrace{|\mathsf{E}\,(\min(X,z) + C_X)^+ - \mathsf{E}\,(\min(X,z) + C_Y)^+|}_{\delta_1(u,z)} \\
&+ \underbrace{|\mathsf{E}\,(\min(X,z) + C_Y)^+ - \mathsf{E}\,(\min(Y,z) + C_Y)^+|}_{\delta_2(u,z)}.
\end{aligned}$$

Now we estimate separately $\delta_1(u,z)$ and $\delta_2(u,z)$.

$$\delta_1(u,z) \le \mathsf{E}|\,(\min(X,z)+C_X)^+ - (\min(X,z)+C_Y)^+\,| \le |C_X - C_Y| \le (l+m)\rho.$$

Applying Lemma 12.2 to r.v.'s X, Y and function $g(x) = (\min(x,z)+C_Y)^+$, we get

$$\delta_2(u,z) = |\mathsf{E}g(X) - \mathsf{E}g(Y)| = \left| \int_{R^+} \overline{F}_{g(X)}(t)dt - \int_{R^+} \overline{F}_{g(Y)}(t)dt \right|$$
$$\le \int_{R^+} |F_{g(X)}(t) - F_{g(Y)}(t)|dt = \kappa(g(X), g(Y)) \le \rho,$$

due to $g'(x) \le 1$. Hence using Lemma 12.3 and just obtained estimates for $\delta_1(u,z)$ and $\delta_2(u,z)$, it is easy to establish the desired result

$$\Delta_1 \le \sup_u |\mathsf{E}J_X(u,z) - \mathsf{E}J_Y(u,z)| \le (1+l+m)\rho.$$

□

12.3.3 Multi-step Model

Now we can prove the following result.

Lemma 12.4 *Function $h_n(u)$ defined by (12.3) is non-increasing in u.*

Proof Since $h_0(u) \equiv 0$ the statement of lemma is valid for $n = 0$. Due to the fact that $\max(u + c(z) - \min(\xi, z), a)$ is non-decreasing in u, we easily see that $h_{n-1}(\max(u + c(z) - \min(\xi, z), a))$ and its expectation are non-increasing in u if we assume $h_{n-1}(u)$ to be non-increasing. Furthermore, $J(u,z) = (\min(\xi, z) - (u - a) - c(z))^+$ does not increase in u; hence, the same is true for $\mathsf{E}J(u,z)$. Summing these results we conclude that $\mathsf{E}J(u,z) + \mathsf{E}h_{n-1}\,(\max(u + c(z) - \min(\xi, z), a))$ is non-increasing in u for any fixed z. It follows immediately that $h_n(u)$ is also non-increasing in u, as infimum in z of previous expression. So, we proved the desired result by means of mathematical induction. □

In the next lemma, we estimate the continuity modulus of function $h_n(u)$.

Lemma 12.5 *For each $n \ge 0$ and any $u \ge a$, $\Delta u \ge 0$ the following inequality is valid*

$$|h_n(u + \Delta u) - h_n(u)| \le C_n \Delta u,$$

where $C_n = (1 - \alpha^n)(1 - \alpha)^{-1}$.

Proof We use the mathematical induction and begin with $n = 0$. Since $h_0(u) \equiv 0$ it is clear that $|h_0(u + \Delta u) - h_0(u)| = 0$. Hence, one has $C_0 = 0 = (1 - \alpha^0)(1 - \alpha)^{-1}$.

Now assume that inequality $|h_{n-1}(u + \Delta u) - h_{n-1}(u)| \le C_{n-1}\Delta u$ is already established. Due to

$$|J(u+\Delta u, z) - J(u,z)|$$
$$= |\,(\min(\xi, z) - (u + \Delta u - a) - c(z))^+ - (\min(\xi, z) - (u-a) - c(z))^+\,| \le \Delta u$$

it follows immediately that $|\mathsf{E}J(u+\Delta u) - \mathsf{E}J(u,z)| \le \Delta u$.

Combining the induction assumption and obvious inequality

$$|\max(u + \Delta u + c(z) - \min(\xi, z), a) - \max(u + c(z) - \min(\xi, z), a)| \le \Delta u,$$

we get

$$|\mathsf{E}h_{n-1}(\max(u + \Delta u + c(z) - \min(\xi, z), a)) - \mathsf{E}h_{n-1}(\max(u + c(z) - \min(\xi, z), a))| \le C_{n-1}\Delta u.$$

Taking into account that $C_{n-1} = (1-\alpha^{n-1})(1-\alpha)^{-1}$ we can write

$$|h_n(u+\Delta u, z) - h_n(u,z)| \le (1+\alpha C_{n-1})\Delta u = C_n \Delta u.$$

Application of Lemma 12.3 with $f_1(z) = h_n(u+\Delta u, z)$ and $f_2(z) = h_n(u,z)$ leads us to the desired result ending the proof. □

Denote by $h_{n_X}(u)$ and $h_{n_Y}(u)$ the minimal injected capital during n years if the claim distribution coincides with $law(X)$ and $law(Y)$, respectively. Our aim is to investigate $|h_{n_X}(u) - h_{n_Y}(u)|$ under assumption $\kappa(X,Y) = \rho$. We put $\Delta_n = \sup_{u>a} |h_{n_X}(u) - h_{n_Y}(u)|$ to formulate the following result.

Theorem 12.2 *Let X, Y be nonnegative random variables having finite means and $\kappa(X,Y) = \rho$. Then*

$$\Delta_n \le \left(\sum_{i=0}^{n-1} \alpha^i C_{n-i}\right)(1+l+m)\rho,$$

here $0 < \alpha < 1$ is the discount factor, $1 < l < m$ are the safety loadings of insurer and reinsurer and C_k, $k \le n$, were defined in Lemma 12.5.

Proof We begin by estimation of $|h_{n_X}(u,z) - h_{n_Y}(u,z)|$. Since

$$(u-a) + l\mathsf{E}X - m\mathsf{E}(X-z)^+ = -C_X, \quad (u-a) + l\mathsf{E}Y - m\mathsf{E}(Y-z)^+ = -C_Y,$$

one can write

$$\max(u + c(z) - \min(X, z), a) = a - (C_X + \min(X,z))^-,$$

$$\max(u + c(z) - \min(Y, z), a) = a - (C_Y + \min(Y,z))^-,$$

where $(C_X + \min(X,z))^- = \min\{0, C_X + \min(X,z)\}$.

Hence, it is possible to get the following expression

$$|h_{n_X}(u,z) - h_{n_Y}(u,z)| \le \underbrace{|\mathsf{E}J_X(u,z) - \mathsf{E}J_Y(u,z)|}_{\delta_{1_n}(u,z)}$$
$$+\alpha \left|\mathsf{E}h_{n-1_X}\left(a-(C_X+\min(X,z))^-\right) - \mathsf{E}h_{n-1_Y}\left(a-(C_Y+\min(Y,z))^-\right)\right|.$$

The first summand in right-hand side of the last inequality is estimated in the one-step model as follows

$$\delta_{1_n}(u,z) \le (1+l+m)\rho.$$

The second summand can be bounded by the sum of three terms.

$$\begin{aligned}
&\left|\mathsf{E}h_{n-1_X}\left(a-(C_X+\min(X,z))^-\right) - \mathsf{E}h_{n-1_Y}\left(a-(C_Y+\min(Y,z))^-\right)\right| \\
&\le \underbrace{\left|\mathsf{E}h_{n-1_X}\left(a-(C_X+\min(X,z))^-\right) - \mathsf{E}h_{n-1_Y}\left(a-(C_X+\min(X,z))^-\right)\right|}_{\delta_{2_n}(u,z)} \\
&+ \underbrace{\left|\mathsf{E}h_{n-1_Y}\left(a-(C_X+\min(X,z))^-\right) - \mathsf{E}h_{n-1_Y}\left(a-(C_X+\min(Y,z))^-\right)\right|}_{\delta_{3_n}(u,z)} \\
&+ \underbrace{\left|\mathsf{E}h_{n-1_Y}\left(a-(C_X+\min(Y,z))^-\right) - \mathsf{E}h_{n-1_Y}\left(a-(C_Y+\min(Y,z))^-\right)\right|}_{\delta_{4_n}(u,z)}
\end{aligned}$$

According to definition of Δ_{n-1} for any $u \ge a$, we have $\left|h_{n-1_X}(u) - h_{n-1_Y}(u)\right| \le \Delta_{n-1}$, therefore

$$\delta_{2_n}(u,z) \le \Delta_{n-1}\int_R dF_X(t) = \Delta_{n-1}.$$

Using Lemma 12.2 for $g(x) = h_{n-1_Y}\left(a-(C_Y+\min(x,z))^-\right)$, one can write

$$\delta_{3_n}(u,z) \le C_{n-1}\rho.$$

To apply Lemma 12.2, it is necessary to verify that $g(x)$ is non-decreasing. This fact clearly follows from Lemma 12.4 due to the form of $g(x)$.

As follows from Lemma 12.5, for any $u \ge a$ one can use inequality $|h_{n-1_Y}(u+\Delta u) - h_{n-1_Y}(u)| \le C_{n-1}\Delta u$ to get

$$\delta_{4_n}(u,z) \le C_{n-1}|C_X - C_Y| \le C_{n-1}(l+m)\rho.$$

Combining the obtained results one gets

$$\begin{aligned}
|h_{n_X}(u,z) - h_{n_Y}(u,z)| &\le (1+l+m)\rho + \alpha\left(\Delta_{n-1} + C_{n-1}(1+l+m)\rho\right) \\
&= \Delta_1 + \alpha C_{n-1}\Delta_1 + \alpha\Delta_{n-1} = C_n\Delta_1 + \alpha\Delta_{n-1},
\end{aligned}$$

whence it follows

$$\Delta_n \le \sup_u |h_{n_X}(u,z) - h_{n_Y}(u,z)| \le C_n\Delta_1 + \alpha\Delta_{n-1}.$$

Since $C_1 = 1$ one gets immediately from the previous formula

$$\Delta_n \le C_n\Delta_1 + \alpha\Delta_{n-1} \le (C_n + \alpha C_{n-1})\Delta_1 + \alpha^2\Delta_{n-2}$$
$$\ldots \le \sum_{i=0}^{n-2}\alpha^i C_{n-i}\Delta_1 + \alpha^{n-1}\Delta_1 = \left(\sum_{i=0}^{n-1}\alpha^i C_{n-i}\right)(1+l+m)\rho.$$

□

Remark 12.2 Letting n tend to infinity it is not difficult to establish that upper bound of Δ_n tends to $(1-\alpha)^{-2}(1+l+m)\rho$. In fact,

$$\sum_{i=0}^{n-1}\alpha^i C_{n-i} = \sum_{i=0}^{n-1}\frac{\alpha^i(1-\alpha^{n-i})}{1-\alpha} = \frac{1}{1-\alpha}\sum_{i=0}^{n-1}\alpha^i - \frac{1}{1-\alpha}n\alpha^n =$$
$$= \frac{1}{1-\alpha}\left(\frac{1-\alpha^{n-1}}{1-\alpha} - n\alpha^n\right) \to \frac{1}{(1-\alpha)^2},$$

as $n \to \infty$ and $0 < \alpha < 1$.

This result shows that the difference between the objective functions diminishes as the distance ρ between the claim distributions decreases. Thus, we have proved the stability of the model under consideration with respect to claim distribution perturbations.

The discount factor α describes the effect of reducing the value of money over time. Hence, it is natural that for α close to 1 the difference is larger than for small values of α.

12.4 Incomplete Information

Up to now, we assumed the claim distribution F per year to be known. In this case, it is possible to find the analytical solution of optimization problem. However in practice, the theoretical d.f. is usually unknown. It is understandable that for calculations the empirical d.f. F_n (n is the sample size) is taken instead of the theoretical one, since $\mathsf{F}_n(t) \to F(t)$ a.s. as $n \to \infty$.

For illustration, we formulate the result from [8] concerning one-step case and show what one can obtain if F is unknown. We need to introduce in addition to $c(z)$ defined in Sect. 12.3.1 the functions $r(z) = \int_z^{+\infty}\overline{F}(x)\,dx$, $k(z) = z + mr(z)$ and $g(z) = k(z) - l\gamma$, that is, $c(z) = l\gamma - mr(z)$ and $g(z) = z - c(z)$. Moreover, we put $z_* = F^{-1}(1-m^{-1})$.

There exist three sets $D_1 = \{m > l > \gamma^{-1}k(z_*)\}$, $D_2 = \{\gamma^{-1}k(z_*) \geq l > \gamma^{-1}z_*\}$ and $D_3 = \{\gamma^{-1}z_* \geq l > 1\}$. It is obvious that $g(z_*) < 0$ in D_1, $g(z_*) \geq 0$ in $D_2 \cup D_3$ and $z_* - c(\infty) \geq 0$ in D_3. Put also $u_* = a + z_* - l\gamma$ and $u_1^* = a + k(z_*) - l\gamma = a + g(z_*)$. Moreover, it was established that inequalities $a > u_1^*$, $u_* < a \leq u_1^*$ and $a \leq u_*$ are equivalent to the relations $(l, m) \in D_1$, $(l, m) \in D_2$ and $(l, m) \in D_3$, respectively.

Recall that the optimal policy depends on system parameters l and m as follows.

Theorem 12.3 ([8]) 1. *If $(l, m) \in D_1$, then $h_1(u) = 0$ for all $u \geq a$. The optimal retention level $z_1(u) = z_*$.*
2. *If $(l, m) \in D_2$, then $h_1(u) = 0$ for $u \geq u_1^*$. The optimal retention level $z_1(u) = z_*$. For $u \in [a, u_1^*)$, the function $z_1(u)$ is the unique solution, for a fixed u, of the equation $u - a + c(z) = z_*$.*
3. *If $(l, m) \in D_3$, then for $u > u_*$ the results coincide with those of part 2, whereas for $u \in [a, u_*]$ it is optimal to use no reinsurance, that is, to take $z_1(u) = \infty$.*

We have reproduced only parts of Theorems 1, 2, and 3 proved in [8] pertaining to our investigation.

Denote by $z_*(n)$, $u_*(n)$, $u_1^*(n)$, and $\gamma(n)$ parameters z_*, u_*, u_1^*, and γ calculated using the empirical d.f. F_n instead of theoretical one.

Corollary 12.1 *For fixed a, l, and m the following relations take place a.s., as $n \to \infty$,*

$$z_*(n) \to z_*, \quad u_*(n) \to u_*, \quad u_1^*(n) \to u_1^*.$$

Proof It is well known that convergence in distribution is equivalent to convergence in quantile. That is, if $F_n \xrightarrow{d} F$, then $F_n^{-1} \xrightarrow{d} F^{-1}$ (quantiles converge in the continuity points of the limit function $F^{-1}(t)$, $0 < t < 1$), see [23]. Moreover, as follows from part 3 of Lemma 12.1, convergence in Kantorovich metric implies convergence in distribution, as well as convergence of expected values, and vice versa. If we take $F_n = \mathsf{F}_n$, then, according to [23], $d_1(\mathsf{F}_n, F) \to 0$ a.s., as $n \to \infty$. Hence, it is clear immediately that $z_*(n) = \mathsf{F}_n^{-1}(1 - m^{-1}) \to F^{-1}(1 - m^{-1}) = z_*$ a.s., as $n \to \infty$.

Since $u_*(n) = a + z_* - l\gamma(n)$, parameters a and l are fixed, whereas $|\gamma(n) - \gamma| \leq d_1(\mathsf{F}_n, F) \to 0$ a.s., as $n \to \infty$, the second statement of corollary is also valid.

Turning to the last statement of corollary, we can write $|r(z_*(n)) - r(z_*)| \leq d_1(\mathsf{F}_n, F) + |z_*(n) - z_*|$. Hence, one easily gets

$$|u_1^*(n) - u_1^*| \leq (m + l)d_1(\mathsf{F}_n, F) + (m + 1)|z_*(n) - z_*| \to 0, \quad \text{a.s.}$$

ending the proof. □

Remark 12.3 Since $z_1(u)$ is equal either to z_* or to $c^{-1}(z_* + a - u)$ for $(l, m) \in D_1 \cup D_2$ it follows immediately from Corollary 12.1 that optimal retention level calculated using empirical d.f. converges a.s. to theoretical one, as the sample size tends to infinity. For the set D_3, there exists also the possibility of no reinsurance.

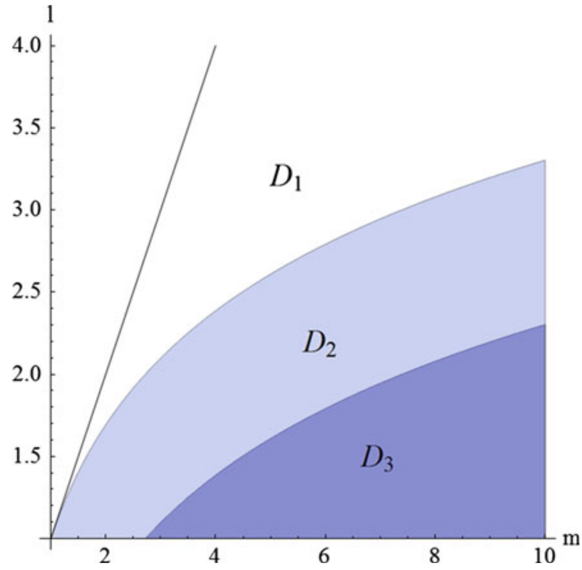

Fig. 12.1 Sets D_i for exponential distribution

The boundary between D_3 and D_2 is given by the curve $l = \varphi_1(m)$ where $\varphi_1(m) = \gamma^{-1}F^{-1}(1 - m^{-1})$, whereas the boundary between D_2 and D_1 is determined by $\varphi_2(m) = \varphi_1(m) + m\gamma^{-1}r(z_*)$. Thus, if we denote by $\varphi_i^{(n)}(m)$, $i = 1, 2$, the corresponding functions calculated on the base of empirical d.f., it is obvious that $\varphi_i^{(n)}(m) \to \varphi_i(m)$ a.s., as $n \to \infty$. So it is possible to specify entirely the "empirical" optimal policy for given parameters l, m, a, and u. Moreover, for a given initial capital u one can choose a providing zero additional costs entailed by capital injection.

The form of the sets D_i, $i = 1, 2, 3$, is depicted by Fig. 12.1 for exponential claim distribution.

It is also interesting to mention that for uniform, as well as, exponential distribution the boundaries $\varphi_i(m)$, $i = 1, 2$, do not depend on distribution parameters.

12.5 Conclusion and Further Research Directions

In this chapter, only the case of no a priori information about the claim distribution was treated for one-step model. The multi-step case is the next step. However to deal with it, we need to prove at first the existence of the so-called asymptotically optimal stationary policy. Then it will be possible to construct empirical asymptotically optimal policy, in other words, to propose a policy based on empirical distribution giving the same long-run injection cost per period as the above-mentioned stationary policy. These results will be published elsewhere. We plan also to carry out the sensitivity analysis as proposed in [18, 21, 24] for finding out the most important scalar parameters. Such analysis was already performed in [5] for two risk models.

12.6 Appendix

Here we prove some results concerning convergence in distribution in L_1 we plan to use in further investigation of two samples case.

Recall that $(X_1, Y_1), (X_2, Y_2), \dots$ is a sequence of independent random vectors defined on a probability space $(\Omega, \mathscr{F}, P)$ taking values in R^2. Moreover, introduced in the previous Sect. 12.2.3 random variables $\zeta_i(t)$ for a fixed t can take values from the set $\{-1, 0, 1\}$. Hence, values of $\mathsf{H}_n(t, \omega)$ for a fixed n belong to the set $\{in^{-1} : i = \overline{-n, n}\}$ for any t, whereas mapping $\mathsf{H}_n(t, .) : \Omega \to R$ is measurable. Thus, the process $H_n(t, \omega)$ for a fixed n is jointly measurable in (t, ω), that is, it is $\mathscr{B}(R) \times \mathscr{F}$-measurable. As usually, $\mathscr{B}(R)$ is the Borel σ-algebra in R.

Theorem 12.4 *Let $X, X_i,\ i \in N$, be i.i.d. r.v.'s with d.f. F and let $Y, Y_i,\ i \in N$, be also i.i.d. r.v.'s but with d.f. G. Put*

$$\eta(t) := (I\{X > t\} - P(X > t)) - (I\{Y > t\} - P(Y > t)), \quad -\infty < t < \infty,$$

whereas $\eta_i,\ i \in N$, are the processes obtained by substitution of X_i instead of X and Y_i instead of Y in the last expression. Then

(a) The processes $\sum_{i=1}^n \eta_i/\sqrt{n} = \sqrt{n}(\mathsf{F}_n - \mathsf{G}_n - (F - G))$ converge in distribution in $L_1(R)$ to the process $B_1(F(t)) + B_2(G(t)),\ t \in R$, where B_1 and B_2 are two independent Brownian bridges, if and only if

$$\int_{-\infty}^{+\infty} \sqrt{F(t)(1 - F(t)) + G(t)(1 - G(t))}\, dt < \infty. \tag{12.4}$$

(b) (1) If the condition (12.4) is valid the sequence

$$\left\| \sum_{i=1}^n \eta_i/\sqrt{n} \right\|_{L_1} = \sqrt{n} \int_{-\infty}^{+\infty} |\mathsf{F}_n(\omega, t) - \mathsf{G}_n(\omega, t) - (F(t) - G(t))|\, dt, \quad n \in N,$$

is stochastically bounded.

(2) If the sequence $\left\|\sum_{i=1}^n \eta_i/\sqrt{n}\right\|_{L_1}$ is stochastically bounded, then

$$\sup_n E \left\| \sum_{i=1}^n (I\{\eta_i > t\} - P(\eta_i > t))/\sqrt{n} \right\|_{L_1} < \infty.$$

Proof According to [14] for any random element $\eta(t)$ from $L_1(R)$ such that $\int ||\eta||_{L_1} dP_\eta < \infty$ and $\int \eta dP_\eta = 0$ condition $\int_{-\infty}^{+\infty} \sqrt{E(\eta(t))^2} dt < \infty$ is equivalent to weak convergence of the measures generated by $\sum_{i=1}^{n} \eta_i(t)/\sqrt{n}$ to a Gaussian measure on $L_1(R)$. First, we show that this condition has the form (12.4) in our case.

Putting $\tilde{X}(t) = I\{X > t\} - P(X > t)$, $\tilde{Y}(t) = I\{Y > t\} - P(Y > t)$, for $s, t \in R$, we have, due to independence of X and Y combined with $E\tilde{X}(t) = E\tilde{Y}(t) = 0$,

$$cov(\eta(s), \eta(t)) = E(\tilde{X}(s) - \tilde{Y}(s))(\tilde{X}(t) - \tilde{Y}(t)) = E\tilde{X}(s)\tilde{X}(t) + E\tilde{Y}(s)\tilde{Y}(t)$$
$$= \min(F(t), F(s)) - F(t)F(s) + \min(G(t), G(s)) - G(t)G(s).$$

Hence, according to the central limit theorem for R^k, $k \in N$, one has

$$(\eta(t_1), \ \ldots, \ \eta(t_k)) \xrightarrow{d} (B_1(F(t_1)) + B_2(G(t_1)), \ \ldots, \ B_1(F(t_k)) + B_2(G(t_k)))$$

for any sequence $t_1, \ldots, t_k$ with $t_i \in R$, $i = \overline{1, k}$. Using the result from [16], we see that processes $\sum_{i=1}^{n} \eta_i(t)/\sqrt{n}$ converge to the process $B_1(F(t)) + B_2(G(t))$ in $L_1(R)$ as $n \to \infty$. Thus paragraph (a) of the theorem and sufficiency of paragraph (b) are established.

Statement of paragraph (b2) is the immediate consequence of the proof of part (b) of Theorem 2.1 in [10]. □

References

1. Albrecher, H., Thonhauser, S.: Optimality results for dividend problems in insurance. Rev. R. Acad. Cien. Serie A. Mat. **103**(2), 295–320 (2009)
2. Avanzi, B.: Strategies for dividend distribution: a review. N. Am. Actuar. J. **13**(2), 217–251 (2009)
3. Bulinskaya, E.: Asymptotic analysis of insurance models with bank loans. In: Bozeman, J.R., Girardin, V., Skiadas, C.H. (eds.) New Perspectives on Stochastic Modeling and Data Analysis, pp. 255–270. ISAST, Athens, Greece (2014)
4. Bulinskaya, E., Gromov, A.: Asymptotic behavior of the processes describing some insurance models. Commun. Stat. Theory Methods **45**, 1778–1793 (2016)
5. Bulinskaya, E., Gusak, J.: Optimal control and sensitivity analysis for two risk models. Commun. Stat. Simul. Comput. **44**, 1–17 (2015)
6. Bulinskaya, E., Muromskaya, A.: Optimization of multi-component insurance system with dividend payments. In: Manca, R., McClean, S., Skiadas, Ch.H. (eds.) New Trends in Stochastic Modeling and Data Analysis. ISAST, Athens, Greece (2015)
7. Bulinskaya, E., Sokolova, A.: Asymptotic behaviour of stochastic storage systems. Mod. Probl. Math. Mech. **10**(3), 37–62 (2015) (in Russian)
8. Bulinskaya, E., Gusak, J., Muromskaya, A.: Discrete-time insurance model with capital injections and reinsurance. Methodol. Comput. Appl. Probab. **17**(4), 899–914 (2015)
9. De Finetti, B.: Su un'impostazione alternativa della teoria collettiva del rischio. Trans. XV-th Int. Congr. Actuar. **2**, 433–443 (1957)
10. Del Barrio, E., Giné, E., Matrán, C.: Central limit theorems for the Wasserstein distance between the empirical and the true distributions. Ann. Probab. **27**(2), 1009–1071 (1999)
11. Dickson, D.C.M., Waters, H.R.: Some optimal dividends problems. ASTIN Bulletin **34**, 49–74 (2004)

12. Dobrushin, R.L.: Prescribing a system of random variables by conditional distributions. Theory Probab. Appl. **15**(3), 458–486 (1970)
13. Eisenberg, J., Schmidli, H.: Optimal control of capital injections by reinsurance in a diffusion approximation. Blätter der DGVFM **30**, 1–13 (2009)
14. Jain, N.C.: Central limit theorems and related questions in Banah space. In: Proceedings of Symposia in Pure Mathematics, vol. 31, pp. 55–65. American Mathematical Society, Providence, RI (1977)
15. Kulenko, N., Schmidli, H.: Optimal dividend strategies in a Cramér-Lundberg model with capital injections. Insur. Math. Econ. **43**, 270–278 (2008)
16. Lawniczak, A.: The Levy-Lindeberg central limit theorem in Orlicz spaces. Proc. Amer. Math. Soc. **89**, 673–679 (1983)
17. Li, S., Lu, Y., Garrido, J.: A review of discrete-time risk models. Rev. R. Acad. Cien. Serie A. Mat. **103**(2), 321–337 (2009)
18. Oakley, J.E., O'Hagan, A.: Probabilistic sensitivity analysis of complex models: a Bayesian approach. J. R. Statist. Soc. B. **66**, Part 3, 751–769 (2004)
19. Peskir, G., Shiryaev, A.: Optimal Stopping and Free-Boundary Problems. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel (2006)
20. Rachev, S.T., Klebanov, L., Stoyanov, S.V., Fabozzi, F.: The Methods of Distances in the Theory of Probability and Statistics. Springer, New York (2013)
21. Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., Saisana, M., Tarantola, S.: Global Sensitivity Analysis. The Primer. Wiley, New York (2008)
22. Schmidli, H.: Stochastic Control in Insurance. Springer, New York (2008)
23. Shorack, G.R., Wellner, J.A.: Empirical Processes with Application to Statistics. Wiley, New York (1986)
24. Sobol, I.M.: Sensitivity analysis for nonlinear mathematical models. Math. Model. Comput. Expt. **1**, 407–414 (1993)
25. Yang, H., Gao, W., Li, J.: Asymptotic ruin probabilities for a discrete-time risk model with dependent insurance and financial risks. Scand. Actuar. J. **1**, 1–17 (2016)