Limiting Problems for a Nonstandard Viscous Cahn–Hilliard System with Dynamic Boundary Conditions

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Abstract This note is concerned with a nonlinear diffusion problem of phase-field type, consisting of a parabolic system of two partial differential equations, complemented by boundary and initial conditions. The system arises from a model of two-species phase segregation on an atomic lattice and was introduced by Podio-Guidugli in Ric. Mat. **55** (2006), pp. 105–118. The two unknowns are the phase parameter and the chemical potential. In contrast to previous investigations about this PDE system, we consider here a dynamic boundary condition for the phase variable that involves the Laplace-Beltrami operator and models an additional nonconserving phase transition occurring on the surface of the domain. We are interested in some asymptotic analysis and first discuss the asymptotic limit of the system as the viscosity coefficient of the order parameter equation tends to 0: the convergence of solutions to the corresponding solutions for the limit problem is proven. Then, we study the long-time behavior of the system for both problems, with positive or zero viscosity coefficient, and characterize the omega-limit set in both cases.

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1 Introduction

A recent line of research originated from the following evolutionary system of partial differential equations:

$$2\rho\, \partial_t \mu + \mu\, \partial_t \rho - \Delta\mu = 0 \quad \text{and} \quad \mu \geq 0 \tag{1.1}$$

$$-\Delta\rho + F'(\rho) = \mu \tag{1.2}$$

in $Q_\infty := \Omega \times (0, +\infty)$, where $\Omega \subset \mathbb{R}^3$ is a bounded and smooth domain with boundary Γ. The system (1.1)–(1.2) comes out from a model for phase segregation through atom rearrangement on a lattice that has been proposed by Podio-Guidugli [48]. This model (see also [12] for a detailed derivation) is a modification of the Fried–Gurtin approach to phase segregation processes (cf. [34, 41]). The order parameter ρ, which in many cases represents the (normalized) density of one of the phases, and the chemical potential μ are the unknowns of the system. Moreover, F' represents the derivative of a double-well potential F. Besides everywhere defined potentials, a typical and important example of F is the so–called *logarithmic double-well potential* given by

$$F_{log}(r) := (1+r)\ln(1+r) + (1-r)\ln(1-r) + \alpha_1(1-r^2) + \alpha_2 r, \qquad r \in (-1, 1), \tag{1.3}$$

for some real coefficients α_1, α_2. Note that, if α_2 is taken null and $\alpha_1 > 1$, it turns out that F actually exhibits two wells, with a local maximum at $r = 0$. In the case when $\alpha_2 \neq 0$, then one of the two minima of F is preferred, in the sense that there is a global minimum point (positive if $\alpha_2 < 0$, negative if $\alpha_2 > 0$) of the function. As a particular feature of (1.3), observe that the derivative of the logarithmic potential becomes singular at ± 1.

About equations (1.1) and (1.2), we point out that the model developed in [48] is based on a local free energy density (in the bulk) of the form

$$\psi(\rho, \nabla\rho, \mu) = -\mu\, \rho + F(\rho) + \frac{1}{2}\, |\nabla\rho|^2. \tag{1.4}$$

From (1.4) one derives equations (1.1)–(1.2), which must be complemented with boundary and initial conditions. As far as the former are concerned, the standard boundary conditions for this class of problems are the homogeneous Neumann ones, namely

$$\partial_\nu \mu = \partial_\nu \rho = 0 \quad \text{on } \Sigma_\infty := \Gamma \times (0, +\infty), \tag{1.5}$$

where ∂_ν denotes the outward normal derivative. Combining now (1.1)–(1.2) with (1.5), we obtain a set of equations and conditions that is a variation of the celebrated Cahn–Hilliard system originally introduced in [1] and first studied mathematically in [31] (for an updated list of references on the Cahn–Hilliard system, see [42]). Nonetheless, an initial value problem for (1.1)–(1.2), (1.5) turns out to be strongly ill-posed (see [15, Subsect. 1.4], where an example is given): indeed, the related problem may have infinitely many smooth and even nonsmooth solutions. Then, two small regularizing parameters $\varepsilon > 0$ and $\delta > 0$ were introduced and considered in [12], which led to the regularized model equations

$$\big(\varepsilon + 2\rho\big)\,\partial_t\mu + \mu\,\partial_t\rho - \Delta\mu = 0\,, \tag{1.6}$$

$$\delta\,\partial_t\rho - \Delta\rho + F'(\rho) = \mu\,. \tag{1.7}$$

This regularized system has been deeply examined in [12], when both ε and δ are positive and fixed. In addition, let us underline that, while one can let ε tend to zero (see [16]) and obtain a solution to the limiting problem with $\varepsilon = 0$, it seems extremely difficult to pass to the limit as δ goes to 0. In fact, ill-posedness still holds for $\delta = 0$, even if ε is kept positive. Hence, one has to assume that δ is a fixed positive coefficient. Therefore, from now on, we take $\delta = 1$, without loss of generality. Let us point out that the long-time behavior of the solutions has been studied both with $\varepsilon > 0$ (cf. [12]) and $\varepsilon = 0$ (cf. [16]).

The system (1.6)–(1.7) constitutes a modification of the so-called *viscous* Cahn–Hilliard system (see [47] and the recent contributions[3, 20, 22] along with their references). We point out that (1.6)–(1.7) was analyzed, in the case of the boundary conditions (1.5), in the papers [12, 14, 18] concerning well-posedness, regularity, and optimal control. Later, the local free energy density (1.4) was generalized to the form

$$\psi(\rho, \nabla\rho, \mu) = -\mu\, g(\rho) + F(\rho) + \frac{1}{2}\,|\nabla\rho|^2, \tag{1.8}$$

thus putting $g(\rho)$ in place of ρ, where g is a nonnegative function on the domain of F. This leads to the system

$$\big(\varepsilon + 2g(\rho)\big)\,\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho - \Delta\mu = 0, \tag{1.9}$$

$$\partial_t\rho - \Delta\rho + F'(\rho) = \mu\, g'(\rho), \tag{1.10}$$

which is a generalization of (1.6)–(1.7) and has been studied in [13, 17] for the case $\varepsilon = 1$. Let us mention also the contribution [9] dealing with the time discretization of the problem and proving convergence results and error estimates. The related phase relaxation system (in which the diffusive term $-\Delta\rho$ disappears from (1.10)), has been dealt with in [10, 11, 19]. We also point out the recent papers [23–25], where a nonlocal version of (1.9)–(1.10)—based on the replacement of the diffusive term of (1.10) with a nonlocal operator acting on ρ—has been largely investigated, also from the side of optimal control.

Now, if we take $\varepsilon = 0$ in (1.9)–(1.10), we obtain

$$2g(\rho)\,\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho - \Delta\mu = 0 \qquad (1.11)$$

$$\partial_t\rho - \Delta\rho + F'(\rho) = \mu\, g'(\rho), \qquad (1.12)$$

which looks like a generalization of the viscous version of (1.1)–(1.2), where the affine function $\rho \mapsto \rho$ is replaced by a concave function $\rho \mapsto g(\rho)$, with g possessing suitable properties that are made precise in the later assumption (2.5). In particular, the new g may be symmetric and strictly concave: a possible simple choice of g satisfying (2.5) is

$$g(r) = 1 - r^2, \quad r \in [-1, 1]. \qquad (1.13)$$

Note that, if one collects (1.3) and (1.13) and assumes $\alpha_2 \neq 0$, the combined function

$$-\mu g(\rho) + F_{log}(\rho) \quad \text{(which is a part of } \psi) \qquad (1.14)$$

shows a global minimum in all cases, and it depends on the values of $(\alpha_1 - \mu)$ and α_2 which minimum actually occurs. Let us notice that the function in (1.14) turns out to be convex in the whole of $(-1, 1)$ for sufficiently large values of μ. On the other hand, the framework fixed by assumptions (2.5)–(2.8) allows for more general choices of g and F.

However, until now the boundary conditions (1.5), of Neumann type for both μ and ρ, have been considered in our discussion. Instead, in the present work we treat the dynamic boundary condition for ρ, i.e., we complement the above systems with

$$\partial_\nu\mu = 0 \quad \text{and} \quad \partial_\nu\rho + \partial_t\rho_\Gamma - \Delta_\Gamma\rho_\Gamma + F'_\Gamma(\rho_\Gamma) = 0 \quad \text{on } \Sigma_\infty, \qquad (1.15)$$

where ρ_Γ is the trace of ρ, Δ_Γ is the Laplace-Beltrami operator on the boundary, F'_Γ is the derivative of another potential F_Γ having more or less the same behavior as F, and the right-hand side of the dynamic boundary condition equals zero, just for simplicity. Indeed, one could consider a nonzero forcing term satisfying proper assumptions, as done in [26]. Once again, we have to add initial conditions.

Thus, we are concerned with a total free energy of the system which also includes a contribution on the boundary; in fact, we postulate that a phase transition phenomenon is occurring as well on the boundary, and the physical variable on the boundary is just the trace of the phase variable in the bulk. This corresponds to a total free energy functional of the form

$$\begin{aligned}\boldsymbol{\Psi}[\rho(t), \rho_\Gamma(t), \mu(t)] &= \int_\Omega \Big[-\mu\, g(\rho) + F(\rho) + \frac{1}{2}|\nabla\rho|^2 \Big](t) \\ &+ \int_\Gamma \Big[[-u_\Gamma\,\rho_\Gamma + F_\Gamma(\rho_\Gamma) + \frac{1}{2}|\nabla_\Gamma\rho_\Gamma|^2 \Big](t), \quad t \geq 0, \end{aligned} \qquad (1.16)$$

where ∇_Γ is the surface gradient and u_Γ may stand for the source term that exerts a (boundary) control on the system. From this expression of the total free energy, one recovers the PDE system resulting from equations (1.11)–(1.12) and the boundary conditions (1.15), with u_Γ in place of 0 in the right-hand side of the second condition. In relation to this, we would like to mention the contribution [27] dealing with the optimal boundary control problem for the system (1.6)–(1.7), (1.15) with $\varepsilon = 1$.

As for the dynamic boundary conditions, we would like to add some comments on the recent growing interest in the mathematical literature, either for the justification (see, e.g., [32, 33, 44]) or for the investigation of systems including dynamic boundary conditions. Without trying to be exhaustive, we point out at least the contributions [2, 4–8, 20–22, 28–30, 35–40, 43, 45, 46, 49, 50], which are concerned with various types of systems endowed with the dynamic boundary conditions for either some or all of the unknowns. Our citations mostly refer to phase-field models involving the Allen–Cahn and Cahn–Hilliard equations, whose structure is generally simpler than the one considered in the present paper.

Our aim here is investigating the long-time behavior of the full system in both the cases $\varepsilon > 0$ and $\varepsilon = 0$ (similar to [12, 16], in which the Neumann boundary conditions (1.5) were considered). More precisely, we show that the ω-limit of any trajectory in a suitable topology consists only of stationary solutions. In order to treat this problem also with $\varepsilon = 0$, we first study the asymptotics as ε tends to zero. To do that, we underline that the reasonable and somehow natural assumptions (2.5) for g along with the requirements (2.6)–(2.8) on F and F_Γ allow us to show that the variables ρ and ρ_Γ are strictly separated from the (singular) values ± 1. Indeed, we can prove this separation property and obtain the strict positivity of $g(\rho)$ as a consequence.

The paper is organized as follows: in the next section, we list our assumptions and notations and state our results, while the corresponding proofs are given in the last two sections. Precisely, in Sect. 3, we perform the asymptotic analysis as ε tends to zero and prove the well-posedness of the problem for $\varepsilon = 0$; in Sect. 4, we study the long-time behavior of the solution under the assumption $\varepsilon \geq 0$.

2 Statement of the Problem and Results

In this section, we state precise assumptions and notations and present our results. First of all, the set $\Omega \subset \mathbb{R}^3$ is assumed to be bounded, connected and smooth. As in the Introduction, ∂_ν and Δ_Γ stand for the outward normal derivative and the Laplace-Beltrami operator on the boundary Γ. Furthermore, we denote by ∇_Γ the surface gradient.

If X is a (real) Banach space, $\|\cdot\|_X$ denotes both its norm and the norm of X^3, X^* is its dual space, and ${}_{X^*}\langle\cdot,\cdot\rangle_X$ is the dual pairing between X^* and X. The only exception from this convention is given by the L^p spaces, $1 \le p \le \infty$, for which we use the abbreviating notation $\|\cdot\|_p$ for the norms in $L^p(\Omega)$. Furthermore, we put

$$H := L^2(\Omega), \quad V := H^1(\Omega) \quad \text{and} \quad W := \{v \in H^2(\Omega) : \partial_\nu v = 0\}, \tag{2.1}$$

$$H_\Gamma := L^2(\Gamma) \quad \text{and} \quad V_\Gamma := H^1(\Gamma), \tag{2.2}$$

$$\mathcal{H} := H \times H_\Gamma \quad \text{and} \quad \mathcal{V} := \{(v, v_\Gamma) \in V \times V_\Gamma : v_\Gamma = v_{|\Gamma}\}. \tag{2.3}$$

We also set, for convenience,

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for } 0 < t < +\infty,$$
$$Q_\infty := \Omega \times (0,+\infty) \quad \text{and} \quad \Sigma_\infty := \Gamma \times (0,+\infty), \tag{2.4}$$

and often use the shorter notations Q and Σ if $t = T$, a fixed final time $T \in (0,+\infty)$.

Now, we list our assumptions. For the structure of our system, we are given three functions $g \in C^2[-1,1]$ and $F,\ F_\Gamma \in C^2(-1,1)$ which satisfy

$$g \ge 0, \quad g'' \le 0, \quad g'(-1) > 0 \quad \text{and} \quad g'(1) < 0, \tag{2.5}$$

$$\lim_{r\searrow -1} F'(r) = \lim_{r\searrow -1} F'_\Gamma(r) = -\infty \quad \text{and} \quad \lim_{r\nearrow 1} F'(r) = \lim_{r\nearrow 1} F'_\Gamma(r) = +\infty, \tag{2.6}$$

$$F''(r) \ge -C \quad \text{and} \quad F''_\Gamma(r) \ge -C, \quad \text{for every } r \in (-1,1), \tag{2.7}$$

$$|F'(r)| \le \eta |F'_\Gamma(r)| + C \quad \text{for every } r \in (-1,1), \tag{2.8}$$

with some positive constants C and η.

For the initial data, we make rather strong assumptions in order to apply the results of [26] without any trouble. However, our first assumption on μ_0 could be replaced by $\mu_0 \in V$. Precisely, we assume that

$$\mu_0 \in W \quad \text{and} \quad \mu_0 \ge 0 \quad \text{in } \Omega\,; \tag{2.9}$$

$$\rho_0 \in H^2(\Omega), \quad \rho_{0|\Gamma} \in H^2(\Gamma), \quad \min \rho_0 > -1 \quad \text{and} \quad \max \rho_0 < 1\,. \tag{2.10}$$

At this point, we are ready to state our problem. For $\varepsilon \ge 0$, we look for a triplet (μ, ρ, ρ_Γ) satisfying the regularity requirements and solving the problem stated below. As for the regularity, we pretend that

$$\mu \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{2.11}$$

$$(\rho, \rho_\Gamma) \in W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;H^2(\Omega) \times H^2(\Gamma)), \tag{2.12}$$

$$\mu \ge 0, \quad -1 < \rho < 1 \quad \text{and} \quad (F'(\rho), F'_\Gamma(\rho_\Gamma)) \in L^\infty(0,T;\mathcal{H}), \tag{2.13}$$

for every finite $T > 0$, and the problem reads

$$\big(\varepsilon + 2g(\rho)\big)\partial_t\mu + \mu g'(\rho)\partial_t\rho - \Delta\mu = 0 \quad \text{a.e. in } Q_\infty\,, \tag{2.14}$$

$$\int_\Omega \partial_t\rho\, v + \int_\Gamma \partial_t\rho_\Gamma\, v_\Gamma + \int_\Omega \nabla\rho\cdot\nabla v + \int_\Gamma \nabla_\Gamma\rho_\Gamma\cdot\nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega F'(\rho)v + \int_\Gamma F'_\Gamma(\rho_\Gamma)v_\Gamma = \int_\Omega \mu g'(\rho)v$$
$$\text{a.e. in } (0,+\infty) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}\,, \tag{2.15}$$

$$\mu(0) = \mu_0 \quad \text{and} \quad \rho(0) = \rho_0 \quad \text{a.e. in } \Omega\,. \tag{2.16}$$

Notice that the Neumann boundary condition $\partial_\nu\mu = 0$ and the fact that ρ_Γ is the trace of ρ on Σ are contained in (2.11) and (2.12), respectively, due to the definitions (2.1)–(2.3) of the spaces involved. By accounting for the regularity conditions (2.11)–(2.13), it is clear that the variational problem (2.15) is equivalent to

$$\partial_t\rho - \Delta\rho + F'(\rho) = \mu g'(\rho) \quad \text{in } Q_\infty\,, \tag{2.17}$$

$$\partial_\nu\rho + \partial_t\rho_\Gamma - \Delta_\Gamma\rho_\Gamma + F'_\Gamma(\rho_\Gamma) = 0 \quad \text{on } \Sigma_\infty\,. \tag{2.18}$$

Moreover, it follows from standard embedding results (see, e.g., [51, Sect. 8, Cor. 4]) that $\rho \in C^0(\overline{Q})$ and thus also $\rho_\Gamma \in C^0(\overline{\Sigma})$.

Our starting point is the well-posedness result for $\varepsilon > 0$ that we state below and is already known. Indeed, recalling (2.6)–(2.7), we set

$$\widehat{\beta}(r) := F(r) - F(0) - F'(0)r + \frac{C}{2}r^2 \quad \text{for } r \in (-1,1) \quad \text{and} \quad \widehat{\pi} := F - \widehat{\beta},$$

and analogously introduce $\widehat{\beta}_\Gamma$ and $\widehat{\pi}_\Gamma$, starting from F_Γ. Then, we consider the convex and lower semicontinuous extensions of $\widehat{\beta}$ and $\widehat{\beta}_\Gamma$ to the whole of $\mathbb{R}$ and smooth extensions of $\widehat{\pi}$ and $\widehat{\pi}_\Gamma$ with bounded second derivatives. Therefore, the assumptions of [26, Thm. 2.1] are satisfied and the following well-posedness result holds true.

Theorem 1 *Assume* (2.5)–(2.8) *and* $\varepsilon > 0$ *for the structure and* (2.9)–(2.10) *for the initial data. Then problem* (2.14)–(2.16) *has a unique solution* $(\mu^\varepsilon, \rho^\varepsilon, \rho^\varepsilon_\Gamma)$ *satisfying the regularity properties* (2.11)–(2.13).

Our aim is the following: $i)$ by starting from the solution $(\mu^\varepsilon, \rho^\varepsilon, \rho^\varepsilon_\Gamma)$, we let ε tend to zero and prove that problem (2.14)–(2.16) with $\varepsilon = 0$ has a solution (μ, ρ, ρ_Γ); $ii)$ such a solution is unique; $iii)$ for $\varepsilon \geq 0$, we study the ω-limit of every trajectory.

Indeed, for i) and ii), we prove the following result in Sect. 3:

Theorem 2 *Assume* (2.5)–(2.8) *for the structure and* (2.9)–(2.10) *for the initial data. Then problem* (2.14)–(2.16) *with* $\varepsilon = 0$ *has a unique solution* (μ, ρ, ρ_Γ) *satisfying the regularity properties* (2.11)–(2.13)*. Moreover, for some constants* $\rho_*, \rho^* \in (-1, 1)$ *that depend only on the shape of the nonlinearities and on the initial data, both* (μ, ρ, ρ_Γ) *and the solution* $(\mu^\varepsilon, \rho^\varepsilon, \rho_\Gamma^\varepsilon)$ *given by Theorem 1 satisfy the separation property*

$$\rho_* \le \rho \le \rho^* \quad \textit{and} \quad \rho_* \le \rho^\varepsilon \le \rho^* \quad \textit{in } \overline{\Omega} \times [0, +\infty). \tag{2.19}$$

Finally, $(\mu^\varepsilon, \rho^\varepsilon, \rho_\Gamma^\varepsilon)$ *converges to* (μ, ρ, ρ_Γ) *in a proper topology.*

The last Sect. 4 is devoted to study the long-time behavior of the solution in both the cases $\varepsilon > 0$ and $\varepsilon = 0$. To this end, for a fixed $\varepsilon \ge 0$, we use the simpler symbol (μ, ρ, ρ_Γ) for the solution on $[0, +\infty)$ and observe that the regularity (2.11)–(2.13) on every finite time interval implies that (μ, ρ, ρ_Γ) is a continuous $(H \times \mathcal{V})$-valued function. In particular, it can be evaluated at every time t, and the following definition of ω-limit is completely meaningful:

$$\omega(\mu, \rho, \rho_\Gamma) := \Big\{ (\mu_\omega, \rho_\omega, \rho_{\omega\Gamma}) \in H \times \mathcal{V} : \ (\mu, \rho, \rho_\Gamma)(t_n) \to (\mu_\omega, \rho_\omega, \rho_{\omega\Gamma}) \\ \text{weakly in } H \times \mathcal{V} \text{ for some sequence } t_n \nearrow +\infty \Big\}. \tag{2.20}$$

Besides, we consider the stationary solutions. It is immediately seen that a stationary solution is a triplet $(\mu_s, \rho_s, \rho_{s_\Gamma})$ satisfying the following conditions: the first component μ_s is a constant, and $(\rho_s, \rho_{s_\Gamma}) \in \mathcal{V}$ is a solution to the system

$$\int_\Omega \nabla \rho_s \cdot \nabla v + \int_\Gamma \nabla_\Gamma \rho_{s_\Gamma} \cdot \nabla_\Gamma v_\Gamma + \int_\Omega F'(\rho_s) v + \int_\Gamma F'_\Gamma(\rho_{s_\Gamma}) v_\Gamma \\ = \int_\Omega \mu_s \, g'(\rho_s) v \qquad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \tag{2.21}$$

In terms of a boundary value problem, the conditions $(\rho_s, \rho_{s_\Gamma}) \in \mathcal{V}$ and (2.21) mean that

$$- \Delta \rho_s + F'(\rho_s) = \mu_s \, g'(\rho_s) \quad \text{in } \Omega,$$

$$\rho_{s_\Gamma} = \rho_{s|\Gamma} \quad \text{and} \quad \partial_\nu \rho_s - \Delta_\Gamma \rho_{s_\Gamma} + F'_\Gamma(\rho_{s_\Gamma}) = 0 \quad \text{on } \Gamma.$$

We prove the following result:

Theorem 3 *Assume* (2.5)–(2.8) *and* $\varepsilon \ge 0$ *for the structure and* (2.9)–(2.10) *for the initial data, and let* (μ, ρ, ρ_Γ) *be the unique solution to problem* (2.14)–(2.16) *satisfying the regularity requirements* (2.11)–(2.13)*. Then the* ω*-limit* (2.20) *is nonempty and consists only of stationary solutions. In particular, there exists a constant* μ_s *such that problem* (2.21) *has at least one solution* $(\rho_s, \rho_{s_\Gamma}) \in \mathcal{V}$*.*

Throughout the paper, we will repeatedly use the Young inequality

$$a\,b \le \delta\,a^2 + \frac{1}{4\delta}\,b^2 \quad \text{for all } a, b \in \mathbb{R} \text{ and } \delta > 0, \tag{2.22}$$

as well as the Hölder inequality and the continuity of the embedding $V \subset L^p(\Omega)$ for every $p \in [1, 6]$ (since Ω is three-dimensional, bounded and smooth). Besides, this embedding is compact for $p < 6$, and also the embedding $W \subset C^0(\overline{\Omega})$ is compact. In particular, we have the compactness inequality

$$\|v\|_4 \;\le\; \delta\,\|\nabla v\|_2 \;+\; \widetilde{C}_\delta\,\|v\|_2 \quad \text{for every } v \in H^1(\Omega) \text{ and } \delta > 0, \tag{2.23}$$

where $\widetilde{C}_\delta$ depends only on Ω and δ. We also recall some well-known estimates from trace theory and from the theory of elliptic equations we use in the sequel. For any v and v_Γ that make the right-hand sides meaningful, we have that

$$\|\partial_\nu v\|_{H^{-1/2}(\Gamma)} \le C_\Omega\big(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{L^2(\Omega)}\big)\,, \tag{2.24}$$

$$\|\partial_\nu v\|_{L^2(\Gamma)} \le C_\Omega\big(\|v\|_{H^{3/2}(\Omega)} + \|\Delta v\|_{L^2(\Omega)}\big)\,, \tag{2.25}$$

$$\|v\|_{H^2(\Omega)} \le C_\Omega\big(\|v_{|\Gamma}\|_{H^{3/2}(\Gamma)} + \|\Delta v\|_{L^2(\Omega)}\big)\,, \tag{2.26}$$

$$\|v\|_{H^2(\Omega)} \le C_\Omega\big(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{L^2(\Omega)}\big) \quad \text{if } \partial_\nu v = 0 \text{ on } \Gamma\,, \tag{2.27}$$

$$\|v_\Gamma\|_{H^2(\Gamma)} \le C_\Omega\big(\|v_\Gamma\|_{H^1(\Gamma)} + \|\Delta_\Gamma v_\Gamma\|_{L^2(\Gamma)}\big)\,, \tag{2.28}$$

$$\|v_\Gamma\|_{H^{3/2}(\Gamma)} \le C_\Omega\big(\|v_\Gamma\|_{H^1(\Gamma)} + \|\Delta_\Gamma v_\Gamma\|_{H^{-1/2}(\Gamma)}\big)\,, \tag{2.29}$$

with a constant $C_\Omega > 0$ that depends only on Ω.

We conclude this section by stating a general rule concerning the constants that appear in the estimates to be performed in the sequel. The small-case symbol c stands for a generic constant whose values might change from line to line and even within the same line and depends only on Ω, on the shape of the nonlinearities, and on the constants and the norms of the functions involved in the assumptions of our statements. In particular, the values of c do not depend on ε and T if the latter is considered. A small-case symbol with a subscript like c_δ (in particular, with $\delta = T$) indicates that the constant might depend on the parameter δ, in addition. On the contrary, we mark precise constants that we can refer to by using different symbols, like in (2.7)–(2.8) and (2.23)–(2.29).

3 Well-Posedness

This section is devoted to the proof of Theorem 2. First, we prove the separation properties (2.19). Then, we show uniqueness. Finally, we prove convergence for the family $\{(\mu^\varepsilon, \rho^\varepsilon, \rho_\Gamma^\varepsilon)\}$ and derive existence for the problem with $\varepsilon = 0$.

Separation We assume that $\varepsilon \geq 0$ and that (μ, ρ, ρ_Γ) is a solution to problem (2.14)–(2.16) satisfying (2.11)–(2.13). Recalling (2.10) and (2.5)–(2.7), we may choose $\rho_*, \rho^* \in (-1, 1)$ such that $\rho_* \leq \rho_0 \leq \rho^*$ and

$$g'(r) > 0 \quad \text{and} \quad F'(r) < 0 \quad \text{for } -1 < r \leq \rho_*,$$
$$g'(r) < 0 \quad \text{and} \quad F'(r) > 0 \quad \text{for } \rho^* \leq r < 1.$$

Now, we show that $\rho_* \leq \rho \leq \rho^*$, using the positivity of μ (see (2.13)). In fact, we prove just the upper inequality, since the proof of the other is similar. We test (2.15), written at the time s, by $((\rho - \rho^*)^+, (\rho_\Gamma - \rho^*)^+)(s)$ and integrate over $(0, t)$ with respect to s. We have

$$\frac{1}{2}\int_\Omega |(\rho(t) - \rho^*)^+|^2 + \frac{1}{2}\int_\Gamma |(\rho_\Gamma(t) - \rho^*)^+|^2$$
$$+ \int_{Q_t} |\nabla(\rho - \rho^*)^+|^2 + \int_{\Sigma_t} |\nabla_\Gamma(\rho_\Gamma - \rho^*)^+|^2$$
$$+ \int_{Q_t} F'(\rho)\,(\rho - \rho^*)^+ + \int_{\Sigma_t} F'_\Gamma(\rho_\Gamma)\,(\rho_\Gamma - \rho^*)^+ = \int_{Q_t} \mu g'(\rho)(\rho - \rho^*)^+ .$$

All of the terms on the left-hand side are nonnegative, while the right-hand side is nonpositive. We conclude that $(\rho(t) - \rho^*)^+ = 0$ in $\overline{\Omega}$ for every $t > 0$, i.e., our assertion.

Consequence Since g, F and F_Γ are smooth on $(-1, 1)$ and (2.5) implies that g is strictly positive on $(-1, 1)$, the separation inequalities (2.19) imply the bounds

$$g(\rho) \geq g_* > 0 \quad \text{and} \quad |\Phi(\rho)| \leq C^* \quad \text{in } \overline{Q}_\infty, \quad |\Phi_\Gamma(\rho_\Gamma)| \leq C^* \quad \text{on } \overline{\Sigma}_\infty, \tag{3.1}$$

for $\Phi \in \{g, g', g'', F, F', F''\}$ and $\Phi_\Gamma \in \{F_\Gamma, F'_\Gamma, F''_\Gamma\}$, and for some constants g_* and C^* that depend only on the shape of the nonlinearities and the initial datum ρ_0. In particular, they do not depend on ε.

Uniqueness We prove that the solution to problem (2.14)–(2.16) with $\varepsilon = 0$ is unique. To this end, we fix $T > 0$ and two solutions $(\mu_i, \rho_i, \rho_{i_\Gamma})$, $i = 1, 2$, and show that they coincide on $\overline{\Omega} \times [0, T]$. We set for convenience $\mu := \mu_1 - \mu_2$ and analogously define ρ and ρ_Γ. Then, we write (2.14) for both solutions and test the difference by μ. Using the identity

$$\{2g(\rho_1)\partial_t\mu_1 + \mu_1 g'(\rho_1)\partial_t\rho_1 - 2g(\rho_2)\partial_t\mu_2 - \mu_2 g'(\rho_2)\partial_t\rho_2\}\mu$$
$$= \partial_t\big(g(\rho_1)\,\mu^2\big) + 2\partial_t\mu_2\,\big(g(\rho_1) - g(\rho_2)\big)\,\mu + \mu_2\big(g'(\rho_1)\partial_t\rho_1 - g'(\rho_2)\partial_t\rho_2\big)\mu\,,$$

we obtain that

$$\int_\Omega g(\rho_1(t))\,|\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2$$
$$= -\int_{Q_t} 2\partial_t\mu_2\,\big(g(\rho_1) - g(\rho_2)\big)\,\mu - \int_{Q_t} \mu_2\big(g'(\rho_1)\partial_t\rho_1 - g'(\rho_2)\partial_t\rho_2\big)\mu\,. \tag{3.2}$$

Next, we write (2.15) at the time s for both solutions, test the difference by $\partial_t(\rho, \rho_\Gamma)(s)$, and integrate over $(0, t)$ with respect to s. Then, we add $\int_{Q_t} \rho\,\partial_t\rho + \int_{\Sigma_t} \rho_\Gamma\,\partial_t\rho_\Gamma$ to both sides. We get

$$\int_{Q_t} |\partial_t\rho|^2 + \int_{\Sigma_t} |\partial_t\rho_\Gamma|^2 + \frac{1}{2}\,\|\rho(t)\|_V^2 + \frac{1}{2}\,\|\rho_\Gamma(t)\|_{V_\Gamma}^2$$
$$= -\int_{Q_t} \big(F'(\rho_1) - F'(\rho_2)\big)\partial_t\rho - \int_{\Sigma_t} \big(F'_\Gamma(\rho_{1_\Gamma}) - F'_\Gamma(\rho_{2_\Gamma})\big)\partial_t\rho_\Gamma$$
$$+ \int_{Q_t} \big(\mu_1 g'(\rho_1) - \mu_2 g'(\rho_2)\big)\partial_t\rho + \int_{Q_t} \rho\,\partial_t\rho + \int_{\Sigma_t} \rho_\Gamma\,\partial_t\rho_\Gamma\,. \tag{3.3}$$

At this point, we add (3.2)–(3.3) to each other and use the separation property, the first inequality in (3.1) for ρ_1, and the boundedness and the Lipschitz continuity of the nonlinearities on $[\rho_*, \rho^*]$. We find that

$$g_*\int_\Omega |\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2 + \int_{Q_t} |\partial_t\rho|^2$$
$$+ \int_{\Sigma_t} |\partial_t\rho_\Gamma|^2 + \frac{1}{2}\,\|\rho(t)\|_V^2 + \frac{1}{2}\,\|\rho_\Gamma(t)\|_{V_\Gamma}^2$$
$$\le c\int_{Q_t} |\partial_t\mu_2|\,|\rho|\,|\mu| + c\int_{Q_t} \mu_2\big(|\partial_t\rho| + |\rho|\,|\partial_t\rho_2|\big)\,|\mu|$$
$$+ c\int_{Q_t} |\rho|\,|\partial_t\rho| + c\int_{\Sigma_t} |\rho_\Gamma|\,|\partial_t\rho_\Gamma| + c\int_{Q_t} \big(\mu_1|\rho| + |\mu|\big)\,|\partial_t\rho|\,. \tag{3.4}$$

Many integrals on the right-hand side can be dealt with just using the Hölder and Young inequalities. Thus, we consider just the terms that need some treatment. In the next lines, we owe to the continuous embeddings $V \subset L^p(\Omega)$ for $p \in [1, 6]$

and $W \subset C^0(\overline{\Omega})$, and δ is a positive parameter. We have

$$\int_{Q_t} |\partial_t \mu_2|\, |\rho|\, |\mu| \le \int_0^t \|\partial_t \mu_2(s)\|_2 \|\rho(s)\|_4 \|\mu(s)\|_4 \, ds$$
$$\le \delta \int_0^t \|\mu(s)\|_V^2 \, ds + c_\delta \int_0^t \|\partial_t \mu_2(s)\|_H^2 \|\rho(s)\|_V^2 \, ds \,,$$

and we notice that the function $s \mapsto \|\partial_t \mu_2(s)\|_H^2$ belongs to $L^1(0, T)$ by (2.11) for μ_2. We estimate the next integral as follows,

$$\int_{Q_t} \mu_2\big(|\partial_t \rho| + |\rho|\, |\partial_t \rho_2|\big)\, |\mu|$$
$$\le \int_0^t \|\mu_2(s)\|_\infty \|\partial_t \rho(s)\|_2 \|\mu(s)\|_2 \, ds$$
$$+ c \int_0^t \|\mu_2(s)\|_6 \|\rho(s)\|_6 \|\partial_t \rho_2(s)\|_6 \|\mu(s)\|_6 \, ds$$
$$\le \delta \int_0^t \|\partial_t \rho(s)\|_H^2 \, ds + c_\delta \int_0^t \|\mu_2(s)\|_W^2 \|\mu(s)\|_H^2 \, ds$$
$$+ \delta \int_0^t \|\mu(s)\|_V^2 \, ds + c_\delta \int_0^t \|\mu_2(s)\|_V^2 \|\partial_t \rho_2(s)\|_V^2 \|\rho(s)\|_V^2 \, ds \,,$$

and we point out that the functions $s \mapsto \|\mu_2(s)\|_W^2$, $s \mapsto \|\mu_2(s)\|_V^2$, and $s \mapsto \|\partial_t \rho_2(s)\|_V^2$, belong to $L^1(0, T)$, $L^\infty(0, T)$ and $L^1(0, T)$, respectively, due to (2.11)–(2.12) for μ_2 and ρ_2. Finally, we estimate one further term. We have that

$$\int_{Q_t} \mu_1 |\rho|\, |\partial_t \rho| \le \int_0^t \|\mu_1(s)\|_4 \|\rho(s)\|_4 \|\partial_t \rho(s)\|_2 \, ds$$
$$\le \delta \int_0^t \|\partial_t \rho\|_H^2 \, ds + c_\delta \int_0^t \|\mu_1(s)\|_V^2 \|\rho(s)\|_V^2 \, ds \,,$$

where the function $s \mapsto \|\mu_1(s)\|_V^2$ belongs to $L^\infty(0, T)$. Therefore, by choosing δ small enough and coming back to (3.4), we can apply the Gronwall lemma to conclude that (μ, ρ, ρ_Γ) vanishes on $\overline{\Omega} \times [0, T]$.

Now, we show the existence of a solution to problem (2.14)–(2.16) with $\varepsilon = 0$ and prove the last sentence of the statement of Theorem 2. To do that, it suffices to establish a number of a priori estimates on the solution $(\mu^\varepsilon, \rho^\varepsilon, \rho_\Gamma^\varepsilon)$ on an arbitrarily fixed time interval $[0, T]$ and to use proper compactness results. As the uniqueness of the solution to the limiting problem is already known, it follows that the convergence properties proved below for a subsequence actually hold for the whole family. In view of the asymptotic behavior that we aim to study in the next section, we distinguish in the notation the constants that may depend on T, as explained at the end of Sect. 2. Of course, we can assume $\varepsilon \le 1$. In order to keep the length of the paper reasonable, we perform some of the next estimates just formally.

First a Priori Estimate We observe that

$$\big\{(\varepsilon + 2g(\rho^\varepsilon))\partial_t \mu^\varepsilon + \mu^\varepsilon g'(\rho^\varepsilon)\partial_t \rho^\varepsilon\big\}\mu^\varepsilon = \partial_t\big(\big(\tfrac{\varepsilon}{2} + g(\rho^\varepsilon)\big)|\mu^\varepsilon|^2\big).$$

Hence, if we multiply (2.14) by μ^ε and integrate over Q_t, we obtain that

$$\frac{\varepsilon}{2}\int_\Omega |\mu^\varepsilon(t)|^2 + \int_\Omega g(\rho^\varepsilon(t))|\mu^\varepsilon(t)|^2 + \int_{Q_t} |\nabla\mu^\varepsilon|^2 = \frac{\varepsilon}{2}\int_\Omega \mu_0^2 + \int_\Omega g(\rho_0)\mu_0^2 .$$

By accounting for (2.19) and (3.1), we deduce, for every $t \ge 0$, the global estimate

$$g_*\int_\Omega |\mu^\varepsilon(t)|^2 + \int_{Q_t} |\nabla\mu^\varepsilon|^2 \le \frac{1}{2}\int_\Omega \mu_0^2 + \int_\Omega g(\rho_0)\mu_0^2 = c\,. \tag{3.5}$$

Second a Priori Estimate We write (2.15) at the time s and choose the test pair $(v, v_\Gamma) = (\partial_t\rho^\varepsilon, \partial_t\rho_\Gamma^\varepsilon)(s)$, which is allowed by the regularity (2.12). Then, we integrate over $(0, t)$. Thanks to the Schwarz and Young inequalities, we have

$$\begin{aligned}
&\int_{Q_t} |\partial_t\rho^\varepsilon|^2 + \int_{\Sigma_t} |\partial_t\rho_\Gamma^\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla\rho^\varepsilon(t)|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma\rho_\Gamma^\varepsilon(t)|^2 \\
&\quad + \int_\Omega F(\rho^\varepsilon(t)) + \int_\Gamma F_\Gamma(\rho_\Gamma^\varepsilon(t)) \\
&= \frac{1}{2}\int_\Omega |\nabla\rho_0|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma\rho_{0|\Gamma}|^2 \\
&\quad + \int_\Omega F(\rho_0) + \int_\Gamma F_\Gamma(\rho_{0|\Gamma}) + \int_{Q_t} \mu^\varepsilon g'(\rho^\varepsilon)\partial_t\rho^\varepsilon \\
&\le c + \frac{1}{2}\int_{Q_t} |\partial_t\rho^\varepsilon|^2 + c\int_{Q_t} |\mu^\varepsilon|^2.
\end{aligned}$$

Since $|\rho^\varepsilon| \le 1$, (3.5) holds, and (2.7) implies that F and F_Γ are bounded from below, we deduce that

$$\begin{aligned}
&\|(\rho^\varepsilon, \rho_\Gamma^\varepsilon)\|_{H^1(0,T;\mathcal{H})\cap L^\infty(0,T;\mathcal{V})} + \|F(\rho^\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \\
&\qquad + \|F_\Gamma(\rho_\Gamma^\varepsilon)\|_{L^\infty(0,T;L^1(\Gamma))} \le c_T\,.
\end{aligned} \tag{3.6}$$

Third a Priori Estimate By starting from (2.17)–(2.18) and accounting for (3.1) and (3.5)–(3.6), we successively deduce a number of estimates with the help of the inequalities (2.24)–(2.29), written with $v = \rho^\varepsilon(t)$ and $v_\Gamma = \rho_\Gamma^\varepsilon(t)$ and then squared and integrated over $(0, T)$. We have

$$\begin{aligned}
&\|\Delta\rho^\varepsilon\|_{L^2(0,T;H)} \le c_T \quad \text{from (2.17)},\\
&\|\partial_\nu\rho^\varepsilon\|_{L^2(0,T;H^{-1/2}(\Gamma))} \le c_T \quad \text{from (2.24)},\\
&\|\Delta_\Gamma\rho_\Gamma^\varepsilon\|_{L^2(0,T;H^{-1/2}(\Gamma))} \le c_T \quad \text{from (2.18)},\\
&\|\rho_\Gamma^\varepsilon\|_{L^2(0,T;H^{3/2}(\Gamma))} \le c_T \quad \text{from (2.29)},\\
&\|\rho^\varepsilon\|_{L^2(0,T;H^2(\Omega))} \le c_T \quad \text{from (2.26)},\\
&\|\partial_\nu\rho^\varepsilon\|_{L^2(0,T;H_\Gamma)} \le c_T \quad \text{from (2.25)},\\
&\|\Delta_\Gamma\rho_\Gamma^\varepsilon\|_{L^2(0,T;H_\Gamma)} \le c_T \quad \text{from (2.18)},\\
&\|\rho_\Gamma^\varepsilon\|_{L^2(0,T;H^2(\Gamma))} \le c_T \quad \text{from (2.28)}.
\end{aligned}$$

In conclusion, we have proved that

$$\|\rho^\varepsilon\|_{L^2(0,T;H^2(\Omega))} + \|\rho_\Gamma^\varepsilon\|_{L^2(0,T;H^2(\Gamma))} \le c_T\,. \tag{3.7}$$

Fourth a Priori Estimate We (formally) differentiate (2.15) with respect to time and set $\zeta := \partial_t\rho^\varepsilon$ and $\zeta_\Gamma := \partial_t\rho_\Gamma^\varepsilon$, for brevity. Then we write the variational equation we obtain at the time s and test it by $(\zeta, \zeta_\Gamma)(s)$. Finally, we integrate over $(0, t)$ and add $C\int_{Q_t}|\zeta|^2 + C\int_{\Sigma_t}|\zeta_\Gamma|^2$ to both sides, where C is the constant that appears in (2.7). We obtain the identity

$$\begin{aligned}
&\frac12\int_\Omega|\zeta(t)|^2 + \frac12\int_\Gamma|\zeta_\Gamma(t)|^2 + \int_{Q_t}|\nabla\zeta|^2 + \int_{\Sigma_t}|\nabla_\Gamma\zeta_\Gamma|^2\\
&\quad + \int_{Q_t}\bigl(F''(\rho^\varepsilon)+C\bigr)|\zeta|^2 + \int_{Q_t}\bigl(F_\Gamma''(\rho_\Gamma^\varepsilon)+C\bigr)|\zeta_\Gamma|^2\\
&= \frac12\int_\Omega|\zeta(0)|^2 + \frac12\int_\Gamma|\zeta_\Gamma(0)|^2 + \int_{Q_t}\partial_t\mu^\varepsilon\,g'(\rho^\varepsilon)\zeta + \int_{Q_t}\mu^\varepsilon g''(\rho^\varepsilon)|\zeta|^2\\
&\quad + C\int_{Q_t}|\zeta|^2 + C\int_{\Sigma_t}|\zeta_\Gamma|^2\,.
\end{aligned} \tag{3.8}$$

All of the terms on the left-hand side are nonnegative, while the second volume integral over Q_t on the right-hand side is nonpositive since $\mu^\varepsilon \ge 0$ and $g'' \le 0$. It remains to find bounds for the first volume integral over Q_t on the right-hand side and for the sum of the terms that involve the initial values. We handle the latter first. To this end, we write (2.15) at the time $t = 0$ and test it by $(v, v_\Gamma) = (\zeta, \zeta_\Gamma)(0)$.

We obtain

$$\int_\Omega |\zeta(0)|^2 + \int_\Gamma |\zeta_\Gamma(0)|^2 = -\int_\Omega \nabla\rho_0 \cdot \nabla\zeta(0) - \int_\Gamma \nabla_\Gamma \rho_{0|\Gamma} \cdot \nabla_\Gamma \zeta_\Gamma(0)$$
$$- \int_\Omega F'(\rho_0)\zeta(0) - \int_\Gamma F'_\Gamma(\rho_{0|\Gamma})\zeta_\Gamma(0) + \int_\Omega \mu_0 g'(\rho_0)\zeta(0) . \tag{3.9}$$

On account of (2.10), we have, using Young's inequality and (2.25),

$$- \int_\Omega \nabla\rho_0 \cdot \nabla\zeta(0) - \int_\Gamma \nabla_\Gamma \rho_{0|\Gamma} \cdot \nabla_\Gamma \zeta_\Gamma(0)$$
$$= \int_\Omega \Delta\rho_0\, \zeta(0) - \int_\Gamma \big(\partial_\nu \rho_0 - \Delta_\Gamma \rho_{0|\Gamma}\big)\zeta_\Gamma(0)$$
$$\le \frac{1}{4}\int_\Omega |\zeta(0)|^2 + \frac{1}{4}\int_\Gamma |\zeta_\Gamma(0)|^2 + c\,\|\rho_0\|^2_{H^2(\Omega)} + c\,\|\rho_{0|\Gamma}\|^2_{H^2(\Gamma)} .$$

Moreover, it follows from (2.9), (2.10), (3.1), and Young's inequality that the expression in the second line of (3.9) is bounded by

$$\frac{1}{4}\int_\Omega |\zeta(0)|^2 \,+\, \frac{1}{4}\int_\Gamma |\zeta_\Gamma(0)|^2 \,+\, c\,.$$

We thus have shown that

$$\int_\Omega |\zeta(0)|^2 + \int_\Gamma |\zeta_\Gamma(0)|^2 \,\le\, c\,. \tag{3.10}$$

It remains to bound the first volume integral over Q_t in (3.8), which we denote by I. This estimate requires more effort. At first, observe that (2.14) implies that

$$\partial_t \mu^\varepsilon \,=\, \frac{1}{\varepsilon + 2g(\rho^\varepsilon)}\,\Delta\mu^\varepsilon \,-\, \frac{g'(\rho^\varepsilon)}{\varepsilon + 2g(\rho^\varepsilon)}\,\zeta\,\mu^\varepsilon\,, \tag{3.11}$$

where, thanks to (3.1), $1/(\varepsilon + 2g(\rho^\varepsilon)) \le 1/(2g^*)$ for all $\varepsilon > 0$. Now, using (3.11), we find that

$$I \,= \int_{Q_t} \frac{g'(\rho^\varepsilon)\,\zeta}{\varepsilon + 2g(\rho^\varepsilon)}\,\Delta\mu^\varepsilon \,-\, \int_{Q_t} \mu^\varepsilon\, \frac{(g'(\rho^\varepsilon))^2}{\varepsilon + 2g(\rho^\varepsilon)}\,\zeta^2 \,=:\, I_1 + I_2\,, \tag{3.12}$$

with obvious notation. The second integral is easy to handle. In fact, thanks to (3.1), (2.23), and Hölder's and Young's inequalities, we infer that

$$I_2 \le c \int_0^t \|\mu^\varepsilon(s)\|_4 \, \|\zeta(s)\|_2 \, \|\zeta(s)\|_4 \, ds$$
$$\le \frac{1}{6} \int_{Q_t} |\nabla\zeta|^2 + c \int_0^t \Big(1 + \|\mu^\varepsilon(s)\|_V^2\Big) \|\zeta(s)\|_H^2 \, ds \,, \tag{3.13}$$

where we know from (3.5) that $\int_0^T \|\mu^\varepsilon(s)\|_V^2 \, ds \le c_T$. For the first integral, integration by parts and (3.1) yield that

$$I_1 = - \int_{Q_t} \nabla\mu^\varepsilon \cdot \nabla\Big(\frac{g'(\rho^\varepsilon)\,\zeta}{\varepsilon + 2g(\rho^\varepsilon)}\Big)$$
$$\le C_1 \int_{Q_t} |\nabla\mu^\varepsilon| \, |\nabla\zeta| + C_1 \int_{Q_t} |\nabla\mu^\varepsilon| \, |\nabla\rho^\varepsilon| \, |\zeta| =: C_1(I_{11} + I_{12}), \tag{3.14}$$

with obvious notation. Clearly, owing to (3.5) and Young's inequality, we find that

$$C_1 \, I_{11} \le \frac{1}{6} \int_{Q_t} |\nabla\zeta|^2 + c \,. \tag{3.15}$$

Moreover, invoking Hölder's and Young's inequalities, the compactness inequality (2.23), as well as the continuity of the embedding $H^2(\Omega) \subset W^{1,4}(\Omega)$, we infer that

$$C_1 \, I_{12} \le C_1 \int_0^t \|\nabla\mu^\varepsilon(s)\|_2 \, \|\nabla\rho^\varepsilon(s)\|_4 \, \|\zeta(s)\|_4 \, ds$$
$$\le \frac{1}{6} \int_{Q_t} |\nabla\zeta|^2 + c \int_{Q_t} |\zeta|^2 + c \int_0^t \|\nabla\mu^\varepsilon(s)\|_2^2 \, \|\rho^\varepsilon(s)\|_{H^2(\Omega)}^2 \, ds \,. \tag{3.16}$$

Notice that $\int_0^T \|\nabla\mu^\varepsilon(s)\|_2^2 \, ds \le c$ for every $T > 0$, by virtue of (3.5). We now aim to estimate $\|\rho^\varepsilon(s)\|_{H^2(\Omega)}$ in terms of ζ and ζ_Γ. To this end, we derive a chain of estimates which are each valid for almost every $s \in (0, T)$. To begin with, we deduce from (3.5) and (3.6) that

$$\|\Delta\rho^\varepsilon(s)\|_2 = \|\zeta(s) + F'(\rho^\varepsilon(s)) - \mu^\varepsilon(s) \, g'(\rho^\varepsilon(s))\|_2 \le c + \|\zeta(s)\|_2 \,. \tag{3.17}$$

Consequently, by (2.24) we have that

$$\|\partial_\nu\rho^\varepsilon(s)\|_{H^{-1/2}(\Gamma)} \le C_\Omega \left(\|\rho^\varepsilon(s)\|_V + \|\Delta\rho^\varepsilon(s)\|_2\right) \le c_T \, (1 + \|\zeta(s)\|_2) \,, \tag{3.18}$$

and (2.18), (3.1) and (3.6) imply that

$$\|\Delta_\Gamma \rho_\Gamma^\varepsilon(s)\|_{H^{-1/2}(\Gamma)} \le \|\partial_\nu \rho^\varepsilon(s) + F'_\Gamma(\rho_\Gamma^\varepsilon(s)) + \zeta_\Gamma(s)\|_{H^{-1/2}(\Gamma)}$$
$$\le c_T\,(1 + \|\zeta(s)\|_2 + \|\zeta_\Gamma(s)\|_{H_\Gamma})\,. \tag{3.19}$$

But then, thanks to (2.29) and (3.6), it is clear that

$$\|\rho_\Gamma^\varepsilon(s)\|_{H^{3/2}(\Gamma)} \le C_\Omega\left(\|\rho_\Gamma^\varepsilon(s)\|_{H^1(\Gamma)} + \|\Delta_\Gamma \rho_\Gamma^\varepsilon(s)\|_{H^{-1/2}(\Gamma)}\right)$$
$$\le c_T\,(1 + \|\zeta(s)\|_2 + \|\zeta_\Gamma(s)\|_{H_\Gamma})\,, \tag{3.20}$$

whence, owing to (2.26), we finally arrive at the estimate

$$\|\rho^\varepsilon(s)\|_{H^2(\Omega)} \;\le\; c_T\left(1 + \|\zeta(s)\|_H + \|\zeta_\Gamma(s)\|_{H_\Gamma}\right)\,. \tag{3.21}$$

We thus obtain from (3.16) that

$$C_1\, I_{12} \;\le\; \frac{1}{6}\int_{Q_t} |\nabla\zeta|^2 \,+\, c\int_{Q_t} |\zeta|^2 \,+\, c_T$$
$$+\; c_T \int_0^t \|\nabla\mu^\varepsilon(s)\|_2^2 \left(\|\zeta(s)\|_H^2 + \|\zeta_\Gamma(s)\|_{H_\Gamma}^2\right) ds\,. \tag{3.22}$$

Therefore, recalling (3.8) and invoking the estimates (3.10), (3.13)–(3.16), we can apply Gronwall's lemma and conclude that

$$\|(\partial_t \rho^\varepsilon, \partial_t \rho_\Gamma^\varepsilon)\|_{L^\infty(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{V})} \le c_T\,. \tag{3.23}$$

Fifth a Priori Estimate We now notice that (3.21) and (3.23) imply that

$$\|\rho^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \;\le\; c_T\,. \tag{3.24}$$

Then we may infer from (2.25), (2.18), (2.28), in this order, the estimates

$$\|\partial_\nu \rho^\varepsilon\|_{L^\infty(0,T;H_\Gamma)} \le c_T\,,$$
$$\|\Delta_\Gamma \rho_\Gamma^\varepsilon\|_{L^\infty(0,T;H_\Gamma)} \le c_T\,, \quad \|\rho_\Gamma^\varepsilon\|_{L^\infty(0,T;H^2(\Gamma))} \le c_T\,,$$

so that

$$\|(\rho^\varepsilon, \rho_\Gamma^\varepsilon)\|_{L^\infty(0,T;H^2(\Omega)\times H^2(\Gamma))} \le c_T\,. \tag{3.25}$$

Sixth a Priori Estimate At this point, we can multiply (2.14) by $\partial_t \mu^\varepsilon$ and integrate over Q_t. Then, we add $\int_{Q_t} \mu^\varepsilon \partial_t \mu^\varepsilon$ to both sides. By owing to the Hölder, Sobolev and Young inequalities, we obtain

$$\begin{aligned}
&\int_{Q_t} \bigl(\varepsilon + 2g(\rho^\varepsilon)\bigr)|\partial_t \mu^\varepsilon|^2 + \frac{1}{2}\,\|\mu^\varepsilon(t)\|_V^2 \\
&= \frac{1}{2}\,\|\mu_0\|_V^2 + \int_{Q_t} \mu^\varepsilon \partial_t \mu^\varepsilon - \int_{Q_t} \mu^\varepsilon g'(\rho^\varepsilon)\partial_t \rho^\varepsilon \partial_t \mu^\varepsilon \\
&\le c + \int_0^t \|\mu^\varepsilon(s)\|_2 \|\partial_t \mu^\varepsilon(s)\|_2 \,ds + c\int_0^t \|\mu^\varepsilon(s)\|_4 \|\partial_t \rho^\varepsilon(s)\|_4 \|\partial_t \mu^\varepsilon(s)\|_2 \,ds \\
&\le c + g_* \int_{Q_t} |\partial_t \mu^\varepsilon|^2 + c\,\|\mu^\varepsilon\|_{L^2(0,t;H)}^2 + c\int_0^t \|\partial_t \rho^\varepsilon(s)\|_V^2 \|\mu^\varepsilon(s)\|_V^2 \,ds\,,
\end{aligned}$$

where g_* is the constant introduced in (3.1). As $2g(\rho^\varepsilon) \ge 2g_*$, we may use (3.5), (3.23) and Gronwall's lemma to conclude that

$$\|\mu^\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c_T\,. \tag{3.26}$$

By comparison in (2.14), we estimate $\Delta\mu^\varepsilon$. Hence, by applying (2.27), we derive that

$$\|\mu^\varepsilon\|_{L^2(0,T;W)} \le c_T\,. \tag{3.27}$$

Conclusion If we collect all the previous estimates and use standard compactness results, then we have (in principle for a subsequence) that

$$\begin{aligned}
&\mu^\varepsilon \to \mu \quad \text{in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\,,\\
&(\rho^\varepsilon, \rho_\Gamma^\varepsilon) \to (\rho, \rho_\Gamma)\\
&\qquad \text{in } W^{1,\infty}(0,T;\mathcal{H})\cap H^1(0,T;\mathcal{V})\cap L^\infty(0,T;H^2(\Omega)\times H^2(\Gamma))\,,
\end{aligned}$$

as $\varepsilon \searrow 0$, the convergence being understood in the sense of the corresponding weak star topologies. Notice that the limiting triplet fulfills the regularity requirements (2.11)–(2.13). Next, by the compact embeddings $V \subset L^5(\Omega)$, $H^2(\Omega) \subset C^0(\overline{\Omega})$, and $H^2(\Gamma) \subset C^0(\Gamma)$, and using well-known strong compactness results (see, e.g., [51, Sect. 8, Cor. 4]), we deduce the useful strong convergence

$$\mu^\varepsilon \to \mu \quad \text{in } C^0([0,T];L^5(\Omega)), \quad (\rho^\varepsilon, \rho_\Gamma^\varepsilon) \to (\rho, \rho_\Gamma) \quad \text{in } C^0(\overline{Q})\times C^0(\overline{\Sigma}). \tag{3.28}$$

This allows us to deal with nonlinearities and to take the limits of the products that appear in the equations. Hence, we easily conclude that the triplet (μ, ρ, ρ_Γ) solves (2.14) and the time-integrated version of (2.15) on $(0, T)$ (which is equivalent to (2.15) itself) with $\varepsilon = 0$. Moreover, the initial conditions (2.16) easily pass to the limit in view of (3.28). This concludes the existence proof. By uniqueness, the whole family $\{(\mu^\varepsilon, \rho^\varepsilon, \rho_\Gamma^\varepsilon)\}$ converges to (μ, ρ, ρ_Γ) in the above topology as $\varepsilon \searrow 0$. $\square$

4 Long-Time Behavior

This section is devoted to the proof of Theorem 3. In the sequel, it is understood that $\varepsilon \in [0, 1]$ is fixed and that (μ, ρ, ρ_Γ) is the unique solution to problem (2.14)–(2.16) given by Theorems 1 and 2 in the two cases $\varepsilon > 0$ and $\varepsilon = 0$, respectively. First of all, we have to show that the ω-limit (2.20) is nonempty. This necessitates proper a priori estimates on the whole half-line $\{t \geq 0\}$.

First Global Estimate From (3.5), we immediately deduce that

$$\|\mu\|_{L^\infty(0,+\infty;H)} \leq c \quad \text{and} \quad \int_{Q_\infty} |\nabla\mu|^2 \leq c\,. \tag{4.1}$$

Second Global Estimate We start by rearranging (2.14) as follows:

$$\mu g'(\rho)\partial_t\rho = \partial_t\big((\varepsilon + 2g(\rho))\mu\big) - \Delta\mu\,. \tag{4.2}$$

Now, we test (2.15), written at the time s, by $\partial_t(\rho, \rho_\Gamma)(s)$, integrate over $(0, t)$ and replace the right-hand side with the help of (4.2). We obtain the identity

$$\begin{aligned}
&\int_{Q_t} |\partial_t\rho|^2 + \int_{\Sigma_t} |\partial_t\rho_\Gamma|^2 + \frac{1}{2}\int_\Omega |\nabla\rho(t)|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma\rho_\Gamma(t)|^2 \\
&\quad + \int_\Omega F(\rho(t)) + \int_\Gamma F_\Gamma(\rho_\Gamma(t)) \\
&= \frac{1}{2}\int_\Omega |\nabla\rho_0|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma\rho_{0|\Gamma}|^2 + \int_\Omega F(\rho_0) + \int_\Gamma F_\Gamma(\rho_{0|\Gamma}) + \int_{Q_t} \mu g'(\rho)\partial_t\rho \\
&= c + \int_\Omega \big(\varepsilon + 2g(\rho(t))\big)\mu(t) - \int_\Omega \big(\varepsilon + 2g(\rho_0)\big)\mu_0 - \int_{Q_t} \Delta\mu\,.
\end{aligned}$$

The last integral vanishes since $\partial_\nu\mu = 0$. By recalling that F and F_Γ are bounded from below and that $|\rho| \leq 1$, and using (4.1), we deduce that

$$\|(\rho, \rho_\Gamma)\|_{L^\infty(0,+\infty;\mathcal{V})} \leq c\,, \quad \int_{Q_\infty} |\partial_t\rho|^2 \leq c \quad \text{and} \quad \int_{\Sigma_\infty} |\partial_t\rho_\Gamma|^2 \leq c\,. \tag{4.3}$$

First Conclusion The first inequalities of (4.1) and (4.3), along with the continuity of (μ, ρ, ρ_Γ) from $[0, +\infty)$ to $H \times \mathcal{V}$, ensure that the ω-limit (2.20) is nonempty. Namely, every divergent sequence of times contains a subsequence $t_n \nearrow +\infty$ such that $(\mu, \rho, \rho_\Gamma)(t_n)$ converges weakly in $H \times \mathcal{V}$.

After establishing the first part of Theorem 3, we prove the second one. Thus, we pick any element $(\mu_\omega, \rho_\omega, \rho_{\omega\Gamma})$ of the ω-limit (2.20) and show that it is a stationary solution of our problem, i.e., that μ_ω is a constant μ_s and that the pair $(\rho_\omega, \rho_{\omega\Gamma})$ coincides with a solution $(\rho_s, \rho_{s_\Gamma})$ to problem (2.21). To this end, we fix a sequence $t_n \nearrow +\infty$ such that

$$(\mu, \rho, \rho_\Gamma)(t_n) \to (\mu_\omega, \rho_\omega, \rho_{\omega\Gamma}) \quad \text{weakly in } H \times \mathcal{V} \tag{4.4}$$

and study the behavior of the solution on the time interval $[t_n, t_n + T]$ with a fixed $T > 0$. For convenience, we shift everything to $[0, T]$ by introducing $(\mu^n, \rho^n, \rho^n_\Gamma) : [0, T] \to H \times \mathcal{V}$ as follows

$$\begin{aligned} &\mu^n(t) := \mu(t_n + t), \quad \rho^n(t) := \rho(t_n + t) \\ &\quad \text{and} \quad \rho^n_\Gamma(t) := \rho_\Gamma(t_n + t) \quad \text{for } t \in [0, T]\,. \end{aligned} \tag{4.5}$$

As T is fixed once and for all, we do not care on the dependence of the constants on T even in the notation, and write Q and Σ for Q_T and Σ_T, respectively. The inequalities (4.1) and (4.3) imply that

$$\|(\mu^n, \rho^n, \rho^n_\Gamma)\|_{L^\infty(0,T;H\times\mathcal{V})} \le c\,, \tag{4.6}$$

$$\lim_{n\to\infty} \Big(\int_Q |\nabla \mu^n|^2 + \int_Q |\partial_t \rho^n|^2 + \int_\Sigma |\partial_t \rho^n_\Gamma|^2 \Big) = 0\,. \tag{4.7}$$

The bound (4.6) yields a convergent subsequence in the weak star topology. If we still label it by the index n to simplify the notation, we have

$$(\mu^n, \rho^n, \rho^n_\Gamma) \to (\mu^\infty, \rho^\infty, \rho^\infty_\Gamma) \quad \text{weakly star in } L^\infty(0, T; H \times \mathcal{V}). \tag{4.8}$$

Now, we aim to improve the quality of the convergence. Thus, we derive further estimates.

First Auxiliary Estimate A partial use of (4.7) provides a bound, namely

$$\|\mu^n\|_{L^2(0,T;V)} + \|(\partial_t \rho^n, \partial_t \rho^n_\Gamma)\|_{L^2(0,T;\mathcal{H})} \le c\,. \tag{4.9}$$

Second Auxiliary Estimate We can repeat the argument that led to (3.7) and arrive at

$$\|(\rho^n, \rho^n_\Gamma)\|_{L^2(0,T;H^2(\Omega)\times H^2(\Gamma))} \le c\,. \tag{4.10}$$

Third Auxiliary Estimate We recall that μ^n and the space derivatives $D_i \rho^n$ and $D_i g(\rho^n) = g'(\rho^n) D_i \rho^n$ are bounded in

$$L^\infty(0, T; H) \cap L^2(0, T; L^6(\Omega)),$$

by (4.6), (4.9), (4.10), and the continuous embedding $V \subset L^6(\Omega)$. On the other hand, the continuous embedding

$$L^\infty(0, T; H) \cap L^2(0, T; L^6(\Omega)) \subset L^4(0, T; L^3(\Omega)) \cap L^6(0, T; L^{18/7}(\Omega))$$

holds true, by virtue of the interpolation inequalities. Therefore, we conclude that

$$\|\mu^n\|_{L^4(0,T;L^3(\Omega))} + \|\nabla \rho^n\|_{L^4(0,T;L^3(\Omega))} + \|\nabla g(\rho^n)\|_{L^6(0,T;L^{18/7}(\Omega))} \leq c\,. \tag{4.11}$$

Fourth Auxiliary Estimate We want to improve the convergence of μ^n. However, we cannot multiply (2.14) by $\partial_t \mu$ since we do not have any information on $\nabla \mu(t_n)$. Therefore, we derive an estimate for $\partial_t \mu^n$ in a dual space. By recalling that $g(\rho) \geq g_*$ (see (3.1)), we divide both sides of (2.14) by $\varepsilon + 2g(\rho)$. Then, we take an arbitrary test function $v \in L^4(0, T; V)$, multiply the equality we obtain by v, integrate over $\Omega \times (t_n, t_n + T)$ and rearrange. We get

$$\int_Q \partial_t \mu^n \, v = -\int_Q \frac{\mu^n g'(\rho^n) \partial_t \rho^n v}{\varepsilon + 2g(\rho^n)} + \int_Q \Delta \mu^n \, \frac{v}{\varepsilon + 2g(\rho^n)}\,,$$

and we now treat the terms on the right-hand side separately. The first one is handled using Hölder's inequality, namely,

$$-\int_Q \frac{\mu^n g'(\rho^n) \partial_t \rho^n v}{\varepsilon + 2g(\rho^n)} \leq c \|\mu^n\|_{L^4(0,T;L^3(\Omega))} \|\partial_t \rho^n\|_{L^2(0,T;L^2(\Omega))} \|v\|_{L^4(0,T;L^6(\Omega))}\,.$$

We integrate the other term by parts and use the Hölder, Sobolev and Young inequalities as follows:

$$\begin{aligned}
&\int_Q \Delta \mu^n \, \frac{v}{\varepsilon + 2g(\rho^n)} = -\int_Q \nabla \mu^n \cdot \frac{(\varepsilon + 2g(\rho^n)) \nabla v - 2v g'(\rho^n) \nabla \rho^n}{(\varepsilon + 2g(\rho^n))^2} \\
&\leq c \|\nabla \mu^n\|_{L^2(0,T;H)} \|v\|_{L^2(0,T;V)} + c \int_0^T \|\nabla \mu^n(s)\|_2 \|v(s)\|_6 \|\nabla \rho^n(s)\|_3 \, ds \\
&\leq c \|\nabla \mu^n\|_{L^2(0,T;H)} \bigl(\|v\|_{L^2(0,T;V)} + \|\nabla \rho^n\|_{L^4(0,T;L^3(\Omega))} \|v\|_{L^4(0,T;L^6(\Omega))} \bigr)\,.
\end{aligned}$$

Therefore, we have for every $v \in L^4(0, T; V)$

$$\int_Q \partial_t \mu^n \, v \le c \|\mu^n\|_{L^4(0,T;L^3(\Omega))} \|\partial_t \rho^n\|_{L^2(0,T;H)} \|v\|_{L^4(0,T;V)}$$
$$+ c \|\nabla \mu^n\|_{L^2(0,T;H)} \big(1 + \|\nabla \rho^n\|_{L^4(0,T;L^3(\Omega))}\big) \|v\|_{L^4(0,T;V)} \,.$$

Hence, on account of (4.1), (4.3) and (4.11), we conclude that

$$\|\partial_t \mu^n\|_{L^{4/3}(0,T;V^*)} \le c \,. \tag{4.12}$$

Conclusion By recalling the estimates (4.9)–(4.10) and (4.12), we see that the convergence (4.8) can be improved as follows:

$$\mu^n \to \mu^\infty \quad \text{in } W^{1,4/3}(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V) \,,$$
$$(\rho^n, \rho^n_\Gamma) \to (\rho^\infty, \rho^\infty_\Gamma)$$
$$\text{in } H^1(0, T; \mathcal{H}) \cap L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; H^2(\Omega) \times H^2(\Gamma)) \,,$$

all in the sense of the corresponding weak star topologies. Now, we prove that the limiting triple $(\mu^\infty, \rho^\infty, \rho^\infty_\Gamma)$ solves problem (2.14)–(2.15), the first equation being understood in a generalized sense. By [51, Sect. 8, Cor. 4] and the compact embeddings $H^2(\Omega) \subset V \subset H \subset V^*$ and $H^2(\Gamma) \subset V_\Gamma \subset H_\Gamma$, we also have (for a not relabeled subsequence)

$$\mu^n \to \mu^\infty \quad \text{strongly in } C^0([0, T]; V^*) \cap L^2(0, T; H) \text{ and a.e. in } Q, \tag{4.13}$$
$$(\rho^n, \rho^n_\Gamma) \to (\rho^\infty, \rho^\infty_\Gamma)$$
$$\text{strongly in } C^0([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \text{ and a.e. on } Q \times \Sigma, \tag{4.14}$$
$$\nabla g(\rho^n) = g'(\rho^n) \nabla \rho^n \to g'(\rho^\infty) \nabla \rho^\infty = \nabla g(\rho^\infty) \quad \text{a.e. in } Q \,. \tag{4.15}$$

It follows that $(F'(\rho^n), F'_\Gamma(\rho^n_\Gamma))$ strongly converges to $(F'(\rho^\infty), F'_\Gamma(\rho^\infty_\Gamma))$ in $L^\infty(0, T; \mathcal{H})$, just by Lipschitz continuity. This allows us to conclude that $(\rho^\infty, \rho^\infty_\Gamma)$ solves the time-integrated version of (2.15), thus equation (2.15) itself. As for (2.14), we recall (4.11) and notice that $4 < 6$ and $2 < 18/7$. Then, with the help of (4.15) and the Egorov theorem, we deduce that

$$\nabla g(\rho^n) \to \nabla g(\rho^\infty) \quad \text{strongly in } (L^4(0, T; L^2(\Omega)))^3, \quad \text{whence}$$
$$g(\rho^n) \to g(\rho^\infty) \quad \text{strongly in } L^4(0, T; V).$$

Therefore, if we assume that $v \in L^\infty(0, T; W^{1,\infty}(\Omega))$, we have that

$$g(\rho^n)v \to g(\rho^\infty)v \quad \text{strongly in } L^4(0, T; V), \quad \text{whence}$$
$$\int_Q \big(\varepsilon + 2g(\rho^n)\big)\partial_t\mu^n \, v \to {}_{L^{4/3}(0,T;V^*)}\langle \partial_t\mu^\infty, \big(\varepsilon + 2g(\rho^\infty)\big)v\rangle_{L^4(0,T;V)} .$$

On the other hand, from the convergence almost everywhere, we also have

$$g'(\rho^n) \to g'(\rho^\infty) \quad \text{strongly in } L^4(0, T; L^6(\Omega)),$$

since $g'(\rho^n)$ is bounded in $L^\infty(Q)$. Moreover, (4.11) implies that μ^n converges to μ^∞ weakly in $L^4(0, T; L^3(\Omega))$. On the other hand, (4.7) yields the strong convergence of $\partial_t\rho^n$ to 0 in $L^2(0, T; H)$ (by the way, 0 must coincide with $\partial_t\rho^\infty$). We deduce that

$$\mu^n g'(\rho^n)\partial_t\rho^n \to \mu^\infty g'(\rho^\infty)\partial_t\rho^\infty \quad \text{weakly in } L^1(Q) .$$

Therefore, we conclude that

$$\begin{aligned} &{}_{L^{4/3}(0,T;V^*)}\langle \partial_t\mu^\infty, \big(\varepsilon + 2g(\rho^\infty)\big)v\rangle_{L^4(0,T;V)} \\ &\quad + \int_Q \mu^\infty g'(\rho^\infty)\partial_t\rho^\infty \, v + \int_Q \nabla\mu^\infty \cdot \nabla v = 0 \end{aligned} \tag{4.16}$$

for every $v \in L^\infty(0, T; W^{1,\infty}(\Omega))$. On the other hand, we know that $\mu^\infty \in L^4(0, T; L^3(\Omega))$ by (4.11) and that $\partial_t\rho^\infty \in L^2(0, T; H)$. Since g' is bounded and the continuous embedding $V \subset L^6(\Omega)$ implies $L^{6/5}(\Omega) \subset V^*$, we also have that

$$\mu^\infty g'(\rho^\infty)\partial_t\rho^\infty \in L^{4/3}(0, T; L^{6/5}(\Omega)) \subset L^{4/3}(0, T; V^*) .$$

Hence, by a simple density argument, we see that the variational equation (4.16) also holds true for every $v \in L^4(0, T; V)$. At this point, we observe that (4.7) implies that

$$\nabla\mu^\infty = 0 , \quad \partial_t\rho^\infty = 0 \quad \text{and} \quad \partial_t\rho_\Gamma^\infty = 0 . \tag{4.17}$$

In particular, (4.16) reduces to

$${}_{L^{4/3}(0,T;V^*)}\langle \partial_t\mu^\infty, \big(\varepsilon + 2g(\rho^\infty)\big)v\rangle_{L^4(0,T;V)} = 0 \quad \text{for every } v \in L^4(0, T; V)$$

and we easily infer that $\partial_t\mu^\infty = 0$. Indeed, the inequality $g(\rho^n) \ge g_*$ for every n implies $g(\rho^\infty) \ge g_*$. Thus, every $\varphi \in C_c^\infty(Q)$ can be written as $\varphi = (\varepsilon + 2g(\rho^\infty))v$ for some $v \in L^4(0, T; V)$ since $\nabla\rho^\infty \in (L^4(0, T; H))^3$ by (4.11). Therefore, $\partial_t\mu^\infty$ actually vanishes and we conclude that μ^∞ takes a constant value μ_s.

From (4.17) we also deduce that $(\rho^\infty, \rho_\Gamma^\infty)$ is a time-independent pair $(\rho_s, \rho_{s_\Gamma})$, so that (2.15) reduces to (2.21). Finally, we show that $(\mu_s, \rho_s, \rho_{s_\Gamma}) = (\mu_\omega, \rho_\omega, \rho_{\omega_\Gamma})$. Indeed, (4.13) and (4.14) imply that

$$(\mu^n, (\rho^n, \rho_\Gamma^n)) \to (\mu^\infty, (\rho^\infty, \rho_\Gamma^\infty)) \quad \text{strongly in } C^0([0,T]; V^*) \times C^0([0,T]; \mathcal{H}),$$

and we infer that

$$(\mu, (\rho, \rho_\Gamma))(t_n) = (\mu^n, (\rho^n, \rho_\Gamma^n))(0) \to (\mu^\infty, (\rho^\infty, \rho_\Gamma^\infty))(0) = (\mu_s, (\rho_s, \rho_{s_\Gamma}))$$

weakly in $V^* \times \mathcal{H}$.

By comparing with (4.4), we conclude that $(\mu_s, \rho_s, \rho_{s_\Gamma}) = (\mu_\omega, \rho_\omega, \rho_{\omega_\Gamma})$, and the proof is complete. □

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