Hermite–Fejer Polynomials as an Approximate Solution of Singular Integro-Differential Equations

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Abstract For full singular integro-differential equations with Gilbert kernel, the collocation method is justified. The approximate solution is sought in the form of Hermite–Fejer polynomial. The convergence of the method is proved and the rate of convergence is estimated.

1 Introduction

Algebraic interpolation polynomials with multiple nodes, known as Hermite polynomials, are well-investigated and are successfully used to solve a wide range of application-oriented problems. Their trigonometric analogue is investigated much less and many questions concerning the existence, uniqueness, and approximate properties of such polynomials still remain open.

Early studies of trigonometric interpolation polynomials with multiple nodes apparently began toward the 30th years of the 20th century. S. M. Lozinsky [1] considered the approximation of the complex-variable functions regular in a single circle, and continuous on its boundary, by the trigonometric interpolation polynomials with multiple nodes located on a single circle's border. He was the first to call such polynomials Hermite–Fejer polynomials.

E. O. Zeel [2, 3], generalizing the results of the predecessors [4–7], proved the existence of the trigonometrical interpolation polynomials of the arbitrary multiplicity w.r.t. the system of the equidistant nodes for the real-valued 2π - periodic functions. Moreover, he showed the explicite form of the corresponding fundamental polynomials and established the conditions of uniform convergence of such polymials to the interpolated function depending on the parity of its multiplicity and the smoothness of the interpolated function.

B. G. Gabdulkhayev [8] obtained in a convenient form the best, in the sense of an order, estimates of the speed of convergence of trigonometrical interpolation

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S. Pinelas et al. (eds.), *Differential and Difference Equations with Applications*, Springer Proceedings in Mathematics & Statistics 230,
https://doi.org/10.1007/978-3-319-75647-9_8

polynomials of the first multiplicity to continuously differentiable functions. Also, in this work he investigated the properties of the quadrature formulas for Gilbert's kernel singular integrals based on such polynomials. Relying on the results of [3] and using B. G. Gabdulkhayev [8] technique Yu. Soliyev [9, 10] investigated systematically quadrature formulas based on the interpolation polynomials of different multiplicity for singular integrals with Cauchy and Gilbert kernels.

In this paper the calculation scheme of the collocation method based on trigonometric interpolation polynomials with the multiple nodes for the full singular integro-differential equation in periodic case is constructed and justified. Convergence of the method is proved, and the errors of the approximate solution are estimated.

2 Statement of the Problem

Consider the singular integro-differential equation

$$\sum_{\nu=0}^{1}(a_\nu(t)x^{(\nu)}(t) + b_\nu(t)(Jx^{(\nu)})(t) + (J_0h_\nu x^{(\nu)})(t)) = y(t),\ t \in [0, 2\pi], \qquad (1)$$

where x is a required function, a_ν, b_ν, h_ν (by both variables), $\nu = 0, 1$, and y are known 2π-periodic functions, singular integrals

$$(Jx^{(\nu)})(t) = \frac{1}{2\pi}\int_0^{2\pi} x^{(\nu)}(\tau)\cot\frac{\tau - t}{2}d\tau,\ \nu = 0, 1,\ t \in [0, 2\pi],$$

are to be interpreted as the Cauchy–Lebesgues principal value, and

$$(J_0h_\nu x^{(\nu)})(t) = \frac{1}{2\pi}\int_0^{2\pi} h_\nu(t, \tau)x^{(\nu)}(\tau)d\tau,\ \nu = 0, 1,\ t \in [0, 2\pi],$$

are regular integrals.

3 Calculation Scheme

Let's denote $\mathbb{N}$ the set of natural numbers, $\mathbb{N}_0$ the set of natural numbers with zero added, $\mathbb{R}$ the set of real numbers $\mathbb{C}$ the set of complex numbers.

Let's fix the natural number $n \in \mathbb{N}$. An approximate solution of the Eq. (1) we seek as a Hermite–Fejer polynomial

$$x_n(t) = \frac{1}{n^2}\sum_{k=0}^{n-1}(x_{2k} + x'_{2k}\sin(t - t_{2k}))\frac{\sin^2\dfrac{n}{2}(t - t_{2k})}{\sin^2\dfrac{t - t_{2k}}{2}}, \; t \in [0, 2\pi], \qquad (2)$$

here t_{2k}, $k = 0, 1..., n-1$, are even numbered nodes of the mesh

$$t_k = \frac{\pi k}{n}, \quad k = 0, 1, ..., 2n-1. \qquad (3)$$

Unknown coefficients x_{2k}, x'_{2k}, $k = 0, 1..., n-1$, of the polynomial (2) we find out as a solution of the system of the algebraic equations

$$\sum_{\nu=0}^{1}(a_\nu(t_k)x_n^{(\nu)}(t_k) + b_\nu(t_k)(Jx_n^{(\nu)})(t_k) + (J_0 P_{2n}^{\tau}(h_\nu x_n^{(\nu)}))(t_k)) = \qquad (4)$$

$$= y(t_k), \quad k = 0, 1, ..., 2n-1,$$

where

$$P_{2n}^{\tau}(h_\nu x_n^{(\nu)})(t, \tau) = \frac{1}{2n}\sum_{k=0}^{2n-1} h_\nu(t, t_k)x_n^{(\nu)}(t_k)\frac{\sin n(\tau - t_k)\cos\dfrac{\tau - t_k}{2}}{\sin\dfrac{\tau - t_k}{2}},$$

$$\nu = 0, 1, \; t, \tau \in [0, 2\pi],$$

is a Lagrange interpolation operator w.r.t. the nodes (3) applied by the variable τ to the functions $h_\nu x_n^{(\nu)}$, $\nu = 0, 1$, and

$$(Jx_n)(t_k) = \frac{1}{n}\sum_{j=0}^{n-1}(\alpha_{0,k-2j}^0 x_{2j} + \alpha_{0,k-2j}^1 x'_{2j}), \quad k = 0, 1, ..., 2n-1,$$

$$\alpha_{0,r}^0 = \{-\cot\frac{r\pi}{2n} \quad \text{for} \quad r \neq 0, \quad 0 \quad \text{for} \quad r = 0\},$$

$$\alpha_{0,r}^1 = \{-\frac{1}{n} \quad \text{for} \quad r \neq 0, \quad 2 - \frac{1}{n} \quad \text{for} \quad r = 0\};$$

$$(Jx'_n)(t_{2k}) = \frac{1}{n}\sum_{j=0}^{n-1}(\alpha_{1,2k-2j}^0 x_{2j} + \alpha_{1,2k-2j}^1 x'_{2j}), \quad k = 0, 1, ..., n-1,$$

$$(Jx'_n)(t_{2k+1}) = \frac{1}{n}\sum_{j=0}^{n-1}\alpha_{1,2k-2j+1}^0 x_{2j}, \quad k = 0, 1, ..., n-1,$$

$$\alpha^0_{1,r} = \{\csc^2 \frac{r\pi}{2n} \quad \text{for} \quad r \neq 0, \quad -\frac{n^2-1}{3} \quad \text{for} \quad r = 0\},$$

$$\alpha^1_{1,r} = \{(-1)^r \csc \frac{r\pi}{2n} \quad \text{for} \quad r \neq 0, \quad 0 \quad \text{for} \quad r = 0\};$$

$$(J^0 P^\tau_{2n}(h_\nu x^{(\nu)}_n))(t_k) = \frac{1}{2n}\sum_{j=0}^{2n-1} h_\nu(t_k, t_j)x^{(\nu)}_n(t_j), \ \nu = 0, 1, \quad k = 0, 1, ..., 2n-1,$$

are the quadrature formulae.

4 Some Preliminaries

Let's denote C the space of continuous 2π-periodic functions with usual norm

$$\|f\|_{\mathrm{C}} = \sup_{t\in\mathbb{R}} \mid f(t) \mid, \ f \in \mathrm{C}.$$

For the fixed $m \in \mathbb{N}_0$ denote $\mathrm{C}^m \subset \mathrm{C}$ the set of the functions on $\mathbb{R}$ with continuous derivatives of order m ($\mathrm{C}^0 = \mathrm{C}$). The norm on C^m we define as follows:

$$\|f\|_{\mathrm{C}^m} = \max_{0\leq\nu\leq m} \|f^{(\nu)}\|_{\mathrm{C}}, \ f \in \mathrm{C}^m.$$

Let's denote H_α the set of Hölder continuous functions of order $\alpha \in \mathbb{R}, 0 < \alpha \leq 1$. For the function f of this set let's denote

$$H(f;\alpha) = \sup_{\substack{t\neq\tau \\ t,\tau\in\mathbb{R}}} \frac{\mid f(t) - f(\tau) \mid}{\mid t - \tau \mid^\alpha},$$

the smallest constant of Hölder condition of the function f. With the help of this constant we can now define the norm on the set H_α, namely,

$$\|f\|_{\mathrm{H}_\alpha} = \max\{\|f\|_{\mathrm{C}}, H(f;\alpha)\}.$$

From the set C^m, for the fixed $\alpha \in \mathbb{R}, \ 0 < \alpha \leq 1$, we can select the set of the functions H^m_α with derivatives of order m satisfying Hölder condition

$$\mid f^{(m)}(t) - f^{(m)}(\tau) \mid \leq H(f^{(m)};\alpha) \mid t - \tau \mid^\alpha, \ t, \tau \in \mathbb{R}.$$

The norm on the set H^m_α ($\mathrm{H}^0_\alpha = \mathrm{H}_\alpha$) we define as follows:

$$\|f\|_{\mathrm{H}^m_\alpha} = \max\{\|f\|_{\mathrm{C}^m}, H(f^{(m)};\alpha)\}.$$

Denote $\mathscr{T}_n$ the set of all trigonometric polynomials of order not higher than n. For the follows we need 2 lemmas from the paper [11].

Lemma 1 *Let the numbers* α, $\beta \in \mathbb{R}$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $m, r \in \mathbb{N}_0$, $m \leq r$, *are such that* $m + \beta \leq r + \alpha$. *Then for any* $n \in \mathbb{N}$ *and any function* $x \in \mathrm{H}_\alpha^r$ *the following estimate is valid* [1]*:*

$$\|x - T_n\|_{\mathrm{H}_\beta^m} \leq cn^{m-r-\alpha+\beta} H(x^{(r)}; \alpha),$$

where $T_n \in \mathscr{T}_n$ *is a polynomial of the best approximation of the function* x.

Lemma 2 *For any* $n \in \mathbb{N}$, $\beta \in \mathbb{R}$, $0 < \beta \leq 1$ *and arbitrary trigonomentric polynomial* $T_n \in \mathscr{T}_n$ *the following estimate is valid:*

$$\|T_n\|_{\mathrm{H}_\beta} \leq (1 + 2^{1-\beta} n^\beta) \|T_n\|_{\mathrm{C}}.$$

An operator P_{2n} is exact for any polynomial of order $n - 1$ and, as it is shown in [12, 13], has the following properties:

$$\|P_{2n}\|_{\mathrm{H}_\beta^m \to \mathrm{H}_\beta^m} \leq c \|P_{2n}\|_{\mathrm{C} \to \mathrm{C}} \leq c \ln n \tag{5}$$

for any $n \in \mathbb{N}$, $n \geq 2$, $\beta \in \mathbb{R}$, $0 < \beta \leq 1$, and arbitrary fixed number $m \in \mathbb{N}$.

5 Justification

Theorem 1 *Let the Eq. (1) and the calculation scheme (2)–(4) of the method satisfy the following conditions:*

A1 *functions* a_ν, b_ν, $\nu = 0, 1$, *and* y *belong to* H_α *for some* $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$*; functions* h_ν, $\nu = 0, 1$, *belong to* H_α *with the same* α *for each variable uniformly w.r.t. other variable,*

A2 $a_1^2(t) + b_1^2(t) \neq 0, \quad t \in [0, 2\pi]$,

A3 $\kappa = \mathrm{ind}(a_1 + ib_1) = 0$,

A4 *an Eq. (1) has a unique solution* $x^* \in \mathrm{H}_\beta^1$ *for each right-hand side* $y \in \mathrm{H}_\beta$, $0 < \beta < \alpha \leq 1$.

Then for n *large enough the system of equations (4) is uniquely solvable and approximate solutions* x_n^* *converge to the exact solution* x^* *of the Eq. (1) by the norm of the space* H_β^1

$$\|x^* - x_n^*\|_{\mathrm{H}_\beta^1} \leq cn^{-\alpha+\beta} \ln n, \quad 0 < \beta < \alpha \leq 1.$$

Proof Let's show first that the assumption **A4** of the Theorem 1 is not empty in the sense that there exist the equations of the class considered satisfying **A4**.

[1]Here and further c denotes generic real positive constants, independent from n.

In fact, consider an equation

$$a_1(t)(x'(t)+x(t))+b_1(t)((Jx')(t)+(Jx)(t))=y(t), \quad t\in[0,2\pi]. \tag{6}$$

It is known [14], that the characteristic operator

$$Bx\equiv a_1(t)x(t)+b_1(t)(Jx)(t), \quad B:H_\beta\to H_\beta,$$

of the Eq. (6) is invertable, and an inverse operator $B^{-1}:H_\beta\to H_\beta$ could be written explicitly. Now apply the operator B^{-1} to both sides of the Eq. (6). Then we'll get an equivalent equation

$$x'(t)+x(t)=(B^{-1}y)(t), \quad t\in[0,2\pi]. \tag{7}$$

In the couple of the spaces (H_β^1, H_β), an Eq. (7) is a Fredholm equation. Homogeneous equation

$$x'(t)+x(t)=0, \quad t\in[0,2\pi],$$

in the space of the real-valued functions has a solution $x(t)=ce^{-t},\ t\in[0,2\pi]$. However, this solution is not periodic for $c\neq 0$, so the only suitable value is $c=0$. It means that in the space of the periodic functions H_β^1 the homogeneous equation has the only zero solution $x(t)=0,\ t\in[0,2\pi]$, and it means that the Eq. (7), and thus the Eq. (6), are uniquely solvable for any right-hand side $y\in H_\beta, 0<\beta<\alpha\le 1$.

For the following part of the proof of the Theorem 1 we'll use the method described in [15, 16].

Let's fix $\beta\in\mathbb{R}, 0<\beta<\alpha\le 1$, and let $\mathrm{X}=\mathrm{H}_\beta^1$, $\mathrm{Y}=\mathrm{H}_\beta$. Then the Eq. (1) can be rewritten as an operator equation

$$Qx=y, \quad Q:\mathrm{X}\to\mathrm{Y}. \tag{8}$$

For each function $x\in\mathrm{X}$ we'll match the Cauchy integral

$$\Phi(z)=\Phi(x;z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{x(\tau)d\tau}{1-z\exp(-i\tau)},\ z\in\mathbb{C}.$$

Denote $x^+(t)$ $x^-(t)$ the limit values of the function $\Phi(z)$ as z trends to $\exp(it)$ by any ways inside and outside unit circle correspondently. For the functions x^+ and x^- the following Sokhotsky's formulae are valid means identical operator.

$$x^\pm(t)=\frac{1}{2}((\pm I-iJ)x)(t)+\frac{1}{2}J_0x,\ t\in\mathbb{R}. \tag{9}$$

Differentiating (9) and using known formulae

$$(x'(t))^{\pm} = (x^{\pm}(t))', \quad (Jx)'(t) = (Jx')(t),$$

we'll obtain

$$x'(t) = x'^{+}(t) - x'^{-}(t), \quad (Jx')(t) = i(x'^{+}(t) + x'^{-}(t)). \tag{10}$$

From the conditions **A2**, **A3**, according to [17] it follows

$$\frac{a_1 - ib_1}{a_1 + ib_1} = \frac{\psi^{+}}{\psi^{-}},$$

where

$$\psi(z) = e^{\theta(z)}, \quad \theta(z) = \Phi(u; z), \quad u = \ln \frac{a_1 - ib_1}{a_1 + ib_1}, \quad z \in \mathbb{C}.$$

Then, using (10), the characteristic operator of the Eq. (1) can be rewritten [14, 17] as

$$a_1(t)x'(t) + b_1(t)(Jx')(t) = \frac{(a_1(t) + ib_1(t))}{\psi^{-}(t)}(\psi^{-}(t)x'^{+}(t) - \psi^{+}(t)x'^{-}(t)).$$

The Eq. (1) or, in other notation, the Eq. (8) we rewrite as an equivalent operator equation

$$Kx \equiv Ux + Vx = f, \quad K : \mathrm{X} \to \mathrm{Y}, \tag{11}$$

where

$$Ux = \psi^{-}x'^{+} - \psi^{+}x'^{-}, \quad Vx = Ax + Bx + Wx,$$

$$Ax = v^{-1}a_0 x, \quad Bx = v^{-1}b_0 Jx, \quad Wx = v^{-1}\sum_{\nu=0}^{1} J^0 h_\nu x^{(\nu)},$$

$$f = v^{-1}y, \quad v = \frac{a_1 + ib_1}{\psi^{-}},$$

and according the condition **A2** of the Theorem 1, $v(t) \neq 0$, $t \in [0, 2\pi]$. An equivalence here means that the Eqs. (1) and (11) are both solvable or not solvable simultaneously and, if they are solvable, their solutions coincide.

Let $\mathrm{X}_n \subset \mathscr{T}_n$ be the set of trigonometrical polynomials of the form (2), and $\mathrm{Y}_n = P_{2n}\mathrm{Y} \subset \mathscr{T}_n$. Then the system of equations (4) is equivalent to the operator equation

$$K_n x_n \equiv U_n x_n + V_n x_n = f_n, \quad K_n : \mathrm{X}_n \to \mathrm{Y}_n, \tag{12}$$

where

$$U_n = P_{2n} U, \quad V_n x_n = P_{2n} A x_n + P_{2n} B x_n + W_n x_n,$$

$$W_n x_n = P_{2n} \sum_{\nu=0}^{1} J_0(P_{2n}^{\tau}(h_\nu x_n^{(\nu)})), \quad f_n = P_{2n} f.$$

Here an equivalence means that if the system of equations (4) has a solution x_{2k}^*, $x_{2k}'^*$, $k = 0, 1, ..., n-1$, then the Eq. (12) will also has a solution which coincide with the polynomial

$$x_n^*(t) = \frac{1}{n^2} \sum_{k=0}^{n-1} (x_{2k}^* + x_{2k}'^* \sin(t - t_{2k})) \frac{\sin^2 \frac{n}{2}(t - t_{2k})}{\sin^2 \frac{t - t_{2k}}{2}}, \; t \in \mathbb{R}.$$

Let's prove now that the operators K and K_n are close to each other on X_n.

For any $x_n \in \mathrm{X}_n$, using the polynomial of the best approximation $T_{n-1} \in \mathscr{T}_{n-1}$ for the function Ax_n, we'll have

$$\|Ax_n - P_{2n} A x_n\|_{\mathrm{Y}} \le (1 + \|P_{2n}\|_{\mathrm{Y} \to \mathrm{Y}}) \|Ax_n - T_{n-1}\|_{\mathrm{Y}}. \tag{13}$$

Now, taking into account the structural qualities of the function Ax_n, we can estimate

$$H(Ax_n; \alpha) \le c(\|x_n\|_C + \|x_n'\|_{\mathrm{C}}) \le c\|x_n\|_{\mathrm{X}}. \tag{14}$$

From (13), using Lemma 1, an estimation (5), and in view of (14) we have

$$\|Ax_n - P_{2n} A x_n\|_{\mathrm{Y}} \le c(n^{-\alpha+\beta} \ln n) \|x_n\|_{\mathrm{X}}. \tag{15}$$

In the same way, we obtain

$$\|Bx_n - P_{2n} B x_n\|_{\mathrm{Y}} \le c(n^{-\alpha+\beta} \ln n) \|x_n\|_{\mathrm{X}}. \tag{16}$$

Considering the trigonometrical degree of accuracy of the quadrature formulae for the regular integrals used in (4) we can write

$$\|Wx_n - W_n x_n\|_{\mathrm{Y}} \le \| \sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n} \sum_{\nu=0}^{1} J^0 P_{2n}^{\tau}(h_\nu x_n^{(\nu)}) \|_{\mathrm{Y}} \le \tag{17}$$

$$\le \| \sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n} \sum_{\nu=0}^{1} J^0 (h_\nu x_n^{(\nu)}) \|_{\mathrm{Y}} + \| P_{2n} \sum_{\nu=0}^{1} J^0 (x_n^{(\nu)} (h_\nu - P_{2n}^{\tau} h_\nu)) \|_{\mathrm{Y}}.$$

Now, using the polynomial of the best uniform approximation $T_{n-1} \in \mathscr{T}_{n-1}$ for the function $\sum\limits_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)}$, we get

$$\|\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)}) - P_{2n}\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)})\|_{\mathrm{Y}} \leq (1 + \|P_{2n}\|_{\mathrm{Y}\to\mathrm{Y}})\|\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - T_{n-1}\|_{\mathrm{Y}}. \tag{18}$$

Considering the structural qualities of the function $h_\nu(t, \tau)$ by the variable t, it is easy to show that

$$H(\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)}); \alpha) \leq c\sum_{\nu=0}^{1} \|x_n^{(\nu)}\|_{\mathrm{C}} \leq c\|x_n\|_{\mathrm{X}}. \tag{19}$$

From (18) and (19), using Lemma 1 and an estimation (5), we get

$$\|\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n}\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)}\|_{\mathrm{Y}} \leq c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \tag{20}$$

Further, taking into account the structural qualities of the functions $h_\nu(t, \tau)$ by the variable τ, error estimations of the quadrature formulae, and Lemma 2, for the second summand of the right-hand side of the estimate (17) we get

$$\|P_{2n}\sum_{\nu=0}^{1} J^0(x_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_{\mathrm{Y}} \leq \tag{21}$$

$$\leq c(n^\beta \ln n)\|\sum_{\nu=0}^{1} J^0(x_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_{\mathrm{C}} \leq c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}.$$

Finally, using the estimate (17), (20), and (21), we get

$$\|Wx_n - W_n x_n\|_{\mathrm{Y}} \leq c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \tag{22}$$

Let's denote $\psi_{n-1}(t) \in \mathscr{T}_{n-1}$ the polynomial of the best uniform approximation of the function $\psi(t)$. Using an auxiliary operator

$$\bar{U}_n : \mathrm{X}_n \to \mathrm{Y}_n, \quad \bar{U}_n x_n = \psi_{n-1}^- x_n^{\prime +} - \psi_{n-1}^+ x_n^{\prime -},$$

we get

$$\|Ux_n - U_n x_n\|_{\mathrm{Y}} \leq (1 + \|P_{2n}\|_{\mathrm{Y}\to\mathrm{Y}})\|Ux_n - \bar{U}_n x_n\|_{\mathrm{Y}}. \tag{23}$$

Futher, we have

$$\|Ux_n - \bar{U}_n x_n\|_{\mathrm{Y}} \leq \|(\psi^- - \psi^-_{n-1})x'^+_n\|_{\mathrm{Y}} + \|(\psi^+ - \psi^+_{n-1})x'^-_n\|_{\mathrm{Y}}. \tag{24}$$

Each summand of the right-hand side of (24) we estimate, using Lemma 1 as follows:

$$\|(\psi^\mp - \psi^\mp_{n-1})x'^\pm_n\|_{\mathrm{Y}} \leq \|\psi^\mp - \psi^\mp_{n-1}\|_{\mathrm{Y}}\|x'^\pm_n\|_{\mathrm{Y}} \leq cn^{-\alpha+\beta}\|x_n\|_{\mathrm{X}}. \tag{25}$$

Now by using (24), (25), and (5) we can rewrite inequality (23) as

$$\|Ux_n - U_n x_n\|_{\mathrm{Y}} \leq c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \tag{26}$$

And finally, using estimations (15), (16), (22), and (26), we get

$$\|K - K_n\|_{\mathrm{X}_n \to \mathrm{Y}} \leq cn^{-\alpha+\beta}\ln n.$$

As the operators Q and K are both invertable and the inverse operator Q^{-1} is bounded, then

$$\|K^{-1}\|_{\mathrm{Y}\to\mathrm{X}} \leq \|v\|_{\mathrm{Y}}\|Q^{-1}\|_{\mathrm{Y}\to\mathrm{X}} \leq c. \tag{27}$$

So there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$,

$$\|K^{-1}\|_{\mathrm{Y}\to\mathrm{X}}\|K - K_n\|_{\mathrm{X}_n\to\mathrm{Y}} \leq cn^{-\alpha+\beta}\ln n \leq \frac{1}{2}.$$

For such n according to the Theorem 1.1 of the paper [16] there exist the operators $K_n^{-1} : \mathrm{Y}_n \to \mathrm{X}_n$, and they are bounded. Moreover, for the right-hand sides of the Eqs. (11), (12), using the condition **A1** of the Theorem 1, Lemma 1 and estimation (5), we have

$$\|y - y_n\|_{\mathrm{Y}} = \|y - P_{2n}y\|_{\mathrm{Y}} \leq cn^{-\alpha+\beta}\ln n. \tag{28}$$

Now, using the corollary of the Theorem 1.2 [16], for the solutions x^* and x_n^* of the Eqs. (11), (12), taking into account (27), (28), we'll find

$$\|x^* - x_n^*\|_{\mathrm{X}} \leq cn^{-\alpha+\beta}\ln n.$$

The Theorem 1 is proved. □

Corollary 1 *If, in the conditions of the Theorem 1, the functions a_ν, b_ν, h_ν (by both variables), $\nu = 0, 1$, and y belong to H^r_α, $r \in \mathbb{N}$. Then the approximate solutions x_n^* converge to the exact solution x^* of the Eq. (1) as $n \to \infty$ by the norm of the space H^1_β as follows:*

$$\|x^* - x_n^*\|_{\mathrm{H}^1_\beta} \leq cn^{-r-\alpha+\beta}\ln n, \quad r + \alpha > \beta. \tag{29}$$

Proof Using the Theorem 6 from [15], we can write

$$\|x^* - x_n^*\|_X \leq (1 + \|K_n^{-1} P_{2n} K\|)\|x^* - \bar{x}_n\|_X + \|K_n^{-1}\| \|K_n \bar{x}_n - P_{2n} K \bar{x}_n\|_Y, \tag{30}$$

where $\bar{x}_n$ is an arbitrary element of the space X_n. Under corollary 1 conditions the solution x^* of the Eq. (1) is so, that $x^{*\prime} \in H_\alpha^r$ for $0 < \alpha < 1$ and $x^{*(r+1)} \in Z$ for $\alpha = 1$ (Z means Zigmund class of the functions). Then, taking for the $\bar{x}_n \in \mathscr{T}_n$ the polynomial of the best uniform approximation for the function x^* and using Lemma 1, for the first summand of the right-hand side of (30) we'll obtain

$$(1 + \|K_n^{-1} P_{2n} K\|)\|x^* - \bar{x}_n\|_X \leq cn^{-r-\alpha+\beta} \ln n. \tag{31}$$

Taking into account the structural qualities of the functions $h_\nu(t, \tau)$, $\nu = 0, 1$, by the variable τ, the error estimation of the quadrature formulae, using Lemma 2 and estimation (5) for the second summand of the right-hand side of the inequality (30), we get

$$\|K_n \bar{x}_n - P_{2n} K \bar{x}_n\|_Y = \|W_n \bar{x}_n - P_{2n} W \bar{x}_n\|_Y \leq \tag{32}$$

$$\leq \|P_{2n} \sum_{\nu=0}^{1} J_0(\bar{x}_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_Y \leq$$

$$\leq c(n^\beta \ln n)\| \sum_{\nu=0}^{1} J_0(\bar{x}_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_C \leq c(n^{-r-\alpha+\beta}) \ln n \|\bar{x}_n\|_X.$$

Now, substituting estimations (31) and (32) in (30), and taking into account, that

$$\|\bar{x}_n\|_X \leq \|x^*\|_X + \|x^* - \bar{x}_n\|_X \leq \|x^*\|_X + cn^{-r-\alpha+\beta},$$

we get an estimation (29). Corollary 1 is proved. □

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